Lower bound for comparison based sorting algorithms

The problem of sorting can be viewed as following.

Input: A sequence of n numbers $<a_1, a_2, \ldots, a_n>$.
Output: A permutation (reordering) $<a'_1, a'_2, \ldots, a'_n>$ of the input sequence such that $a'_1 <= a'_2 \ldots.. <= a'_n$.

A sorting algorithm is comparison based if it uses comparison operators to find the order between two numbers. Comparison sorts can be viewed abstractly in terms of decision trees. A decision tree is a full binary tree that represents the comparisons between elements that are performed by a particular sorting algorithm operating on an input of a given size. The execution of the sorting algorithm corresponds to tracing a path from the root of the decision tree to a leaf. At each internal node, a comparison a_i a_j is

made. The left subtree then dictates subsequent comparisons for a_i  a_j, and the right subtree dictates subsequent comparisons for $a_i > a_j$. When we come to a leaf, the sorting algorithm has established the ordering. So we can say following about the decison tree.

1) Each of the *n*! permutations on *n* elements must appear as one of the leaves of the decision tree for the sorting algorithm to sort properly.

2) Let x be the maximum number of comparisons in a sorting algorithm. The maximum height of the decison tree would be x. A tree with maximum height x has at most 2^x leaves.

After combining the above two facts, we get following relation.

```
      n!  <= 2^x

 Taking Log on both sides.
      [Tex]\log_2n![/Tex]  <= x

 Since [Tex]\log_2n![/Tex]  = [Tex]\Theta(nLogn)[/Tex],  we can say
      x = [Tex]\Omega(nLog_2n)[/Tex]
```

Therefore, any comparison based sorting algorithm must make at least [Tex]\Omega(nLog_2n)[/Tex] comparisons to sort the input array, and Heapsort and merge sort are asymptotically optimal comparison sorts.

References:
Introduction to Algorithms, by Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein

Related Topics:

- Linearity of Expectation
- Iterative Tower of Hanoi
- Count possible ways to construct buildings
- Build Lowest Number by Removing n digits from a given number
- Set Cover Problem | Set 1 (Greedy Approximate Algorithm)
- Find number of days between two given dates
- How to print maximum number of A's using given four keys
- Write an iterative O(Log y) function for pow(x, y)

Writing code in comment? Please use **ideone.com** and share the link here.

Heta Vaishnani • 4 months ago
why does it need n! permutationz?
• Reply • Share ›

jayasuya_j Heta Vaishnani • 3 months ago
say for example three numbers : 1,2,3 permutation of them gives the possible arrangements i.e 123,213,231,312,, etc thus one among these must be the answer. So number of possibilities is n! out of which one is the answer. These possiblities form the leaves of the tree,where we choose only one leaf (which gives sorted sequence) by following a path from root to the leaf
1 • Reply • Share ›

sd • 7 months ago
Mention of Stirlings Approximation can be helpful.
• Reply • Share ›

GeeksforGeeks • 8 months ago
nice explaination
1 • Reply • Share ›

Kanhaiya Kumawat • a year ago
there is typo in the last line: "Therefore, any comparison based sorting algorithm must make at least comparisons to sort the input array, and Heapsort and merge sort are asymptotically optimal comparison sorts."
its nLogn rather than Logn.
• Reply • Share ›

GeeksforGeeks Mod Kanhaiya Kumawat • a year ago
Thanks for pointing this out. We have corrected the typo.
• Reply • Share ›

wgpshashank • 4 years ago
More Info ..

http://www.it-c.dk/courses/ITM/F2003/Sorting.pdf
• Reply • Share ›

lovocas • 4 years ago
" A decision tree is a full binary tree"
I am confused by this statement. :-<
A full binary tree is a tree has 2^(k+1)-1 nodes ?k is the heright,the root&#039s height is 0?.
I think desision tree is just a binary tree, whose nodes has either 2 or zero nodes?

• Reply • Share ›

GeeksforGeeks ➦ lovocas • 4 years ago

@lovocas: The statement looks correct. Please see the following Wiki definition of Full Binary Tree.

A full binary tree (sometimes proper binary tree or 2-tree or strictly binary tree) is a tree in which every node other than the leaves has two children.

To avoid confusion, we have updated the post and added the Wiki link for full binary tree.

2 • Reply • Share ›

Yang ➦ GeeksforGeeks • 2 years ago

What if I write a "really bad" algorithm and it asks the same question every time and gets the same answer every time. Then, the Decision tree will still be full binary tree?

• Reply • Share ›

lovocas ➦ GeeksforGeeks • 4 years ago

oh, thanks very much , got it!

• Reply • Share ›

- Interview Experiences
- Advanced Data Structures
- Dynamic Programming
- Greedy Algorithms
- Backtracking
- Pattern Searching
- Divide & Conquer
- Mathematical Algorithms

- Recursion
- Geometric Algorithms

Popular Posts

- All permutations of a given string
- Memory Layout of C Programs
- Understanding “extern” keyword in C
- Median of two sorted arrays
- Tree traversal without recursion and without stack!
- Structure Member Alignment, Padding and Data Packing
- Intersection point of two Linked Lists
- Lowest Common Ancestor in a BST.
- Check if a binary tree is BST or not
- Sorted Linked List to Balanced BST

Follow @GeeksforGeeks

Recent Comments

- lt_k

  i need help for coding this function in java...

  Java Programming Language · 1 hour ago

- Piyush

  What is the purpose of else if (recStack[*i])...

  Detect Cycle in a Directed Graph · 1 hour ago

- Andy Toh

  My compile-time solution, which agrees with the...

  Dynamic Programming | Set 16 (Floyd Warshall Algorithm) · 1 hour ago

- lucy

  because we first fill zero in first col and...

  Dynamic Programming | Set 29 (Longest Common Substring) · 2 hours ago

- lucy

  @GeeksforGeeks i don't n know what is this long...

  Dynamic Programming | Set 28 (Minimum insertions to form a palindrome) · 2 hours ago

- manish

  Because TAN is not a subsequence of RANT. ANT...

  Given two strings, find if first string is a subsequence of second · 2 hours ago