Cooley–Tukey FFT algorithm

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The **Cooley–Tukey algorithm**, named after J.W. Cooley and John Tukey, is the most common fast Fourier transform (FFT) algorithm. It re-expresses the discrete Fourier transform (DFT) of an arbitrary composite size $N = N_1N_2$ in terms of smaller DFTs of sizes N_1 and N_2, recursively, in order to reduce the computation time to O(N log N) for highly composite N (smooth numbers). Because of the algorithm's importance, specific variants and implementation styles have become known by their own names, as described below.

Because the Cooley-Tukey algorithm breaks the DFT into smaller DFTs, it can be combined arbitrarily with any other algorithm for the DFT. For example, Rader's or Bluestein's algorithm can be used to handle large prime factors that cannot be decomposed by Cooley–Tukey, or the prime-factor algorithm can be exploited for greater efficiency in separating out relatively prime factors.

The algorithm, along with its recursive application, was invented by Carl Friedrich Gauss. Cooley and Tukey independently rediscovered and popularized it 160 years later.

See also the fast Fourier transform for information on other FFT algorithms, specializations for real and/or symmetric data, and accuracy in the face of finite floating-point precision.

Contents

History

This algorithm, including its recursive application, was invented around 1805 by Carl Friedrich Gauss, who used it to interpolate the trajectories of the asteroids Pallas and Juno, but his work was not widely recognized (being published only posthumously and in neo-Latin).[1][2] Gauss did not analyze the asymptotic computational time, however. Various limited forms were also rediscovered several times throughout the 19th and early 20th centuries.[2] FFTs became popular after James Cooley of IBM and John Tukey of Princeton published a paper in 1965 reinventing the algorithm and describing how to perform it conveniently on a computer.[3]

Tukey reportedly came up with the idea during a meeting of a US presidential advisory committee discussing ways to detect nuclear-weapon tests in the Soviet Union.[4][5] Another participant at that meeting, Richard Garwin of IBM, recognized the potential of the method and put Tukey in touch with Cooley, who implemented it for a different (and less-classified) problem: analyzing 3d crystallographic data (see also: multidimensional FFTs). Cooley and Tukey subsequently published their joint paper, and wide adoption quickly followed.

The fact that Gauss had described the same algorithm (albeit without analyzing its asymptotic cost) was not realized until several years after Cooley and Tukey's 1965 paper.[2] Their paper cited as inspiration only work by I. J. Good on what is now called the prime-factor FFT algorithm (PFA);[3] although Good's algorithm was initially mistakenly thought to be equivalent to the Cooley–Tukey algorithm, it was quickly realized that PFA is a quite different algorithm (only working for sizes that have relatively prime factors and relying on the Chinese Remainder Theorem, unlike the support for any composite size in Cooley–Tukey).[6]

The radix-2 DIT case

A **radix-2** decimation-in-time (**DIT**) FFT is the simplest and most common form of the Cooley–Tukey algorithm, although highly optimized Cooley–Tukey implementations typically use other forms of the algorithm as described below. Radix-2 DIT divides a DFT of size *N* into two interleaved DFTs (hence the name "radix-2") of size *N*/2 with each recursive stage.

The discrete Fourier transform (DFT) is defined by the formula:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} nk},$$

where k is an integer ranging from 0 to $N-1$.

Radix-2 DIT first computes the DFTs of the even-indexed inputs $(x_{2m} = x_0, x_2, \ldots, x_{N-2})$ and of the odd-indexed inputs $(x_{2m+1} = x_1, x_3, \ldots, x_{N-1})$, and then combines those two results to produce the DFT of the whole sequence. This idea can then be performed recursively to reduce the overall runtime to O(*N* log *N*). This simplified form assumes that *N* is a power of two; since the number of sample points *N* can usually be chosen freely by the application, this is often not an important restriction.

The Radix-2 DIT algorithm rearranges the DFT of the function x_n into two parts: a sum over the even-numbered indices $n = 2m$ and a sum over the odd-numbered indices $n = 2m+1$:

$$X_k \quad = \quad \sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N}(2m)k} + \sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N}(2m+1)k}$$

One can factor a common multiplier $e^{-\frac{2\pi i}{N}k}$ out of the second sum, as shown in the equation below. It is then clear that the two sums are the DFT of the even-indexed part x_{2m} and the DFT of odd-indexed part x_{2m+1} of the function x_n. Denote the DFT of the ***E***ven-indexed inputs x_{2m} by E_k and the DFT of the ***O***dd-indexed inputs x_{2m+1} by O_k and we obtain:

$$X_k = \underbrace{\sum_{m=0}^{N/2-1} x_{2m} e^{-\frac{2\pi i}{N/2} mk}}_{\mathrm{DFT\ of\ even-indexed\ part\ of\ } x_m} + e^{-\frac{2\pi i}{N}k} \underbrace{\sum_{m=0}^{N/2-1} x_{2m+1} e^{-\frac{2\pi i}{N/2} mk}}_{\mathrm{DFT\ of\ odd-indexed\ part\ of\ } x_m} = E_k + e^{-\frac{2\pi i}{N}k} O_k.$$

Thanks to the periodicity of the DFT, we know that

$$E_{k+\frac{N}{2}} = E_k$$

and

$O_{k+\frac{N}{2}} = O_k$. Therefore, we can rewrite the above equation as

$$X_k = \begin{cases} E_k + e^{-\frac{2\pi i}{N}k} O_k & \text{for } 0 \le k < N/2 \\ \\ E_{k-N/2} + e^{-\frac{2\pi i}{N}k} O_{k-N/2} & \text{for } N/2 \le k < N. \end{cases}$$

We also know that the twiddle factor $e^{-2\pi i k/N}$ obeys the following relation:

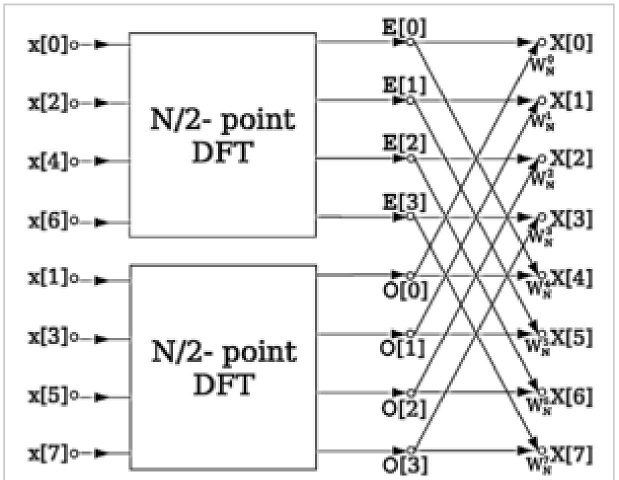
$$\begin{aligned} e^{\frac{-2\pi i}{N}(k+N/2)} &= e^{\frac{-2\pi i k}{N} - \pi i} \\ &= e^{-\pi i} e^{\frac{-2\pi i k}{N}} \\ &= -e^{\frac{-2\pi i k}{N}} \end{aligned}$$

This allows us to cut the number of "twiddle factor" calculations in half also. For $0 \le k < \frac{N}{2}$, we have

$$\begin{aligned} X_k &= E_k + e^{-\frac{2\pi i}{N}k} O_k \\ X_{k+\frac{N}{2}} &= E_k - e^{-\frac{2\pi i}{N}k} O_k \end{aligned}$$

This result, expressing the DFT of length *N* recursively in terms of two DFTs of size *N*/2, is the core of the radix-2 DIT fast Fourier transform. The algorithm gains its speed by re-using the results of intermediate computations to compute multiple DFT outputs. Note that final outputs are obtained by a +/− combination of E_k and $O_k \exp(-2\pi i k/N)$, which is simply a size-2 DFT (sometimes called a butterfly in this context); when this is generalized to larger radices below, the size-2 DFT is replaced by a larger DFT (which itself can be evaluated with an FFT).

This process is an example of the general technique of divide and conquer algorithms; in many traditional implementations, however, the explicit recursion is avoided, and instead one traverses the computational tree in breadth-first fashion.



Data flow diagram for *N*=8: a decimation-in-time radix-2 FFT breaks a length-*N* DFT into two length-*N*/2 DFTs followed by a combining stage consisting of many size-2 DFTs called "butterfly" operations (so-called because of the shape of the data-flow diagrams).

The above re-expression of a size-*N* DFT as two size-*N*/2 DFTs is sometimes called the **Danielson–Lanczos** lemma, since the identity was noted by those two authors in 1942[7] (influenced by Runge's 1903 work[2]). They applied their lemma in a "backwards" recursive fashion, repeatedly *doubling* the DFT size until the transform spectrum converged (although they apparently didn't realize the linearithmic [i.e., order *N* log *N*] asymptotic complexity they had achieved). The Danielson–Lanczos work predated widespread availability of computers and required hand calculation (possibly with mechanical aids such as adding machines); they reported a computation time of 140 minutes for a size-64 DFT operating on real inputs to 3–5 significant digits. Cooley and Tukey's 1965 paper reported a running time of 0.02 minutes for a size-2048 complex DFT on an IBM 7094 (probably in 36-bit single precision, ~8 digits).[3]

Rescaling the time by the number of operations, this corresponds roughly to a speedup factor of around 800,000. (To put the time for the hand calculation in perspective, 140 minutes for size 64 corresponds to an average of at most 16 seconds per floating-point operation, around 20% of which are multiplications.)

Pseudocode

In pseudocode, the below procedure could be written:[8]

```
$X_{0,\dots,N-1}$ ← ditfft2(x, N, s):            DFT of ($x_0$, $x_s$, $x_{2s}$, ..., $x_{(N-1)s}$):
    if N = 1 then
        $X_0$ ← $x_0$                                trivial size-1 DFT base case
    else
        $X_{0,\dots,N/2-1}$ ← ditfft2(x, N/2, 2s)           DFT of ($x_0$, $x_{2s}$, $x_{4s}$, ...)
        $X_{N/2,\dots,N-1}$ ← ditfft2(x+s, N/2, 2s)         DFT of ($x_s$, $x_{s+2s}$, $x_{s+4s}$, ...)
        for k = 0 to N/2-1                           combine DFTs of two halves into full DFT:
            t ← $X_k$
            $X_k$ ← t + exp(−2πi k/N) $X_{k+N/2}$
            $X_{k+N/2}$ ← t − exp(−2πi k/N) $X_{k+N/2}$
        endfor
    endif
```

Here, **ditfft2**(x,N,1), computes X=DFT(x) out-of-place by a radix-2 DIT FFT, where N is an integer power of 2 and s=1 is the stride of the input x array. $x+s$ denotes the array starting with x_s.

(The results are in the correct order in X and no further bit-reversal permutation is required; the often-mentioned necessity of a separate bit-reversal stage only arises for certain in-place algorithms, as described below.)

High-performance FFT implementations make many modifications to the implementation of such an algorithm compared to this simple pseudocode. For example, one can use a larger base case than N=1 to amortize the overhead of recursion, the twiddle factors $\exp[-2\pi ik/N]$ can be precomputed, and larger radices are often used for cache reasons; these and other optimizations together can improve the performance by an order of magnitude or more.[8] (In many textbook implementations the depth-first recursion is eliminated entirely in favor of a nonrecursive breadth-first approach, although depth-first recursion has been argued to have better memory locality.[8][9]) Several of these ideas are described in further detail below.

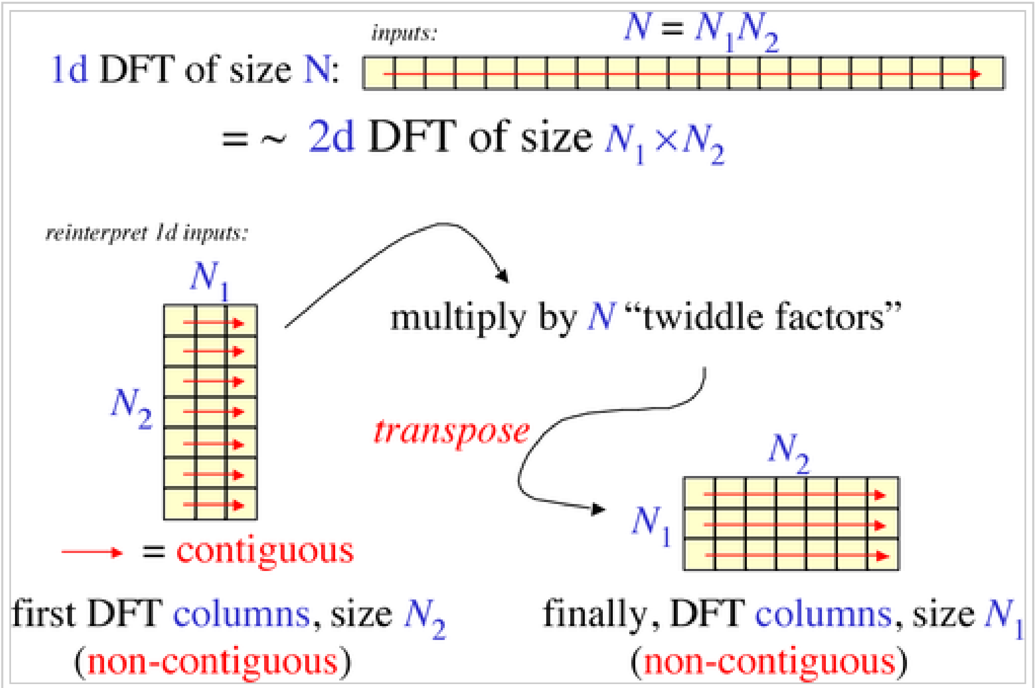
General factorizations

More generally, Cooley–Tukey algorithms recursively re-express a DFT of a composite size $N = N_1N_2$ as:[10]

1. Perform N_1 DFTs of size N_2.
2. Multiply by complex roots of unity called twiddle factors.
3. Perform N_2 DFTs of size N_1.

Typically, either N_1 or N_2 is a small factor (*not* necessarily prime), called the **radix** (which can differ between stages of the recursion). If N_1 is the radix, it is called a **decimation in time** (DIT) algorithm, whereas if N_2 is the radix, it is **decimation in frequency** (DIF, also called the Sande-Tukey algorithm). The version presented above was a radix-2 DIT algorithm; in the final expression, the phase multiplying the odd transform is the twiddle factor, and the +/- combination (*butterfly*) of the even and odd transforms is a size-2 DFT. (The radix's small DFT is sometimes known as a butterfly, so-called because of the shape of the dataflow diagram for the radix-2 case.)

There are many other variations on the Cooley–Tukey algorithm. **Mixed-radix** implementations handle composite sizes with a variety of (typically small) factors in addition to two, usually (but not always) employing the O(N^2) algorithm for the prime base cases of the recursion (it is also possible to employ an $N \log N$ algorithm for the prime base cases, such as Rader's or Bluestein's algorithm). Split radix merges radices 2 and 4, exploiting the fact that the first transform of radix 2 requires no twiddle factor, in order to achieve what was long the lowest known arithmetic operation count for power-of-two sizes,[10] although recent variations achieve an even lower count.[11][12] (On present-day computers, performance is determined more by cache and CPU pipeline considerations than by strict operation counts; well-optimized FFT implementations often employ larger radices and/or hard-coded base-case transforms of significant size.[13]) Another way of looking at the Cooley–Tukey algorithm is that it re-expresses a size N one-dimensional DFT as an N_1 by N_2 two-dimensional DFT (plus twiddles), where the output matrix is transposed. The net result of all of these transpositions, for a radix-2 algorithm, corresponds to a bit reversal of the input (DIF) or output (DIT) indices. If, instead of using a small radix, one employs a radix of roughly $\sqrt{N}$ and explicit input/output matrix transpositions, it is called a **four-step** algorithm (or *six-step*, depending on the number of transpositions), initially proposed to improve memory locality,[14][15] e.g. for cache optimization or out-of-core operation, and was later shown to be an optimal cache-oblivious algorithm.[16]



The basic step of the Cooley–Tukey FFT for general factorizations can be viewed as re-interpreting a 1d DFT as something like a 2d DFT. The 1d input array of length $N = N_1N_2$ is reinterpreted as a 2d $N_1 \times N_2$ matrix stored in column-major order. One performs smaller 1d DFTs along the N_2 direction (the non-contiguous direction), then multiplies by phase factors (twiddle factors), and finally performs 1d DFTs along the N_1 direction. The transposition step can be performed in the middle, as shown here, or at the beginning or end. This is done recursively for the smaller transforms.

The general Cooley–Tukey factorization rewrites the indices k and n as $k = N_2 k_1 + k_2$ and $n = N_1 n_2 + n_1$, respectively, where the indices k_a and n_a run from 0..N_a-1 (for a of 1 or 2). That is, it re-indexes the input (n) and output (k) as N_1 by N_2 two-dimensional arrays in column-major and row-major order, respectively; the difference between these indexings is a transposition, as mentioned above. When this re-indexing is substituted into the DFT formula for nk, the $N_1 n_2 N_2 k_1$ cross term vanishes (its exponential is unity), and the remaining terms give

$$X_{N_2k_1+k_2} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_{N_1n_2+n_1} e^{-\frac{2\pi i}{N_1N_2}\cdot(N_1n_2+n_1)\cdot(N_2k_1+k_2)}$$

$$= \sum_{n_1=0}^{N_1-1} \left[e^{-\frac{2\pi i}{N}n_1k_2}\right] \left(\sum_{n_2=0}^{N_2-1} x_{N_1n_2+n_1} e^{-\frac{2\pi i}{N_2}n_2k_2}\right) e^{-\frac{2\pi i}{N_1}n_1k_1}$$

where each inner sum is a DFT of size N_2, each outer sum is a DFT of size N_1, and the [...] bracketed term is the twiddle factor.

An arbitrary radix r (as well as mixed radices) can be employed, as was shown by both Cooley and Tukey[3] as well as Gauss (who gave examples of radix-3 and radix-6 steps).[2] Cooley and Tukey originally assumed that the radix butterfly required $O(r^2)$ work and hence reckoned the complexity for a radix r to be $O(r^2 N/r \log_r N) = O(N \log_2(N)\ r/\log_2 r)$; from calculation of values of $r/\log_2 r$ for integer values of r from 2 to 12 the optimal radix is found to be 3 (the closest integer to e, which minimizes $r/\log_2 r$).[3][17] This analysis was erroneous, however: the radix-butterfly is also a DFT and can be performed via an FFT algorithm in $O(r \log r)$ operations, hence the radix r actually cancels in the complexity $O(r \log(r)\ N/r \log_r N)$, and the optimal r is determined by more complicated considerations. In practice, quite large r (32 or 64) are important in order to effectively exploit e.g. the large number of processor registers on modern processors,[13] and even an unbounded radix $r=\sqrt{N}$ also achieves $O(N \log N)$ complexity and has theoretical and practical advantages for large N as mentioned above.[14][15][16]

Data reordering, bit reversal, and in-place algorithms

Although the abstract Cooley–Tukey factorization of the DFT, above, applies in some form to all implementations of the algorithm, much greater diversity exists in the techniques for ordering and accessing the data at each stage of the FFT. Of special interest is the problem of devising an in-place algorithm that overwrites its input with its output data using only O(1) auxiliary storage.

The most well-known reordering technique involves explicit **bit reversal** for in-place radix-2 algorithms. Bit reversal is the permutation where the data at an index n, written in binary with digits $b_4b_3b_2b_1b_0$ (e.g. 5 digits for N=32 inputs), is transferred to the index with reversed digits $b_0b_1b_2b_3b_4$. Consider the last stage of a radix-2 DIT algorithm like the one presented above, where the output is written in-place over the input: when E_k and O_k are combined with a size-2 DFT, those two values are overwritten by the outputs. However, the two output values should go in the first and second *halves* of the output array, corresponding to the *most* significant bit b_4 (for N=32); whereas the two inputs E_k and O_k are interleaved in the even and odd elements, corresponding to the *least* significant bit b_0. Thus, in order to get the output in the correct place, b_0 should take the place of b_4 and the index becomes $b_0b_4b_3b_2b_1$. And for next recursive stage, those 4 least significant bits will become $b_1b_4b_3b_2$, If you include all of the recursive stages of a radix-2 DIT algorithm, *all* the bits must be reversed and thus one must pre-process the input (*or* post-process the output) with a bit reversal to get in-order output. (If each size-N/2 subtransform is to operate on contiguous data, the DIT *input* is pre-processed by bit-reversal.) Correspondingly, if you perform all of the steps in reverse order, you obtain a radix-2 DIF algorithm with bit reversal in post-processing (or pre-processing, respectively). Alternatively, some applications (such as convolution) work equally well on bit-reversed data, so one can perform forward transforms, processing, and then inverse transforms all without bit reversal to produce final results in the natural order.

Many FFT users, however, prefer natural-order outputs, and a separate, explicit bit-reversal stage can have a non-negligible impact on the computation time,[13] even though bit reversal can be done in $O(N)$ time and has been the subject of much research.[18][19][20] Also, while the permutation is a bit reversal in the radix-2 case, it is more generally an arbitrary (mixed-base) digit reversal for the mixed-radix case, and the permutation algorithms become more complicated to implement. Moreover, it is desirable on many hardware architectures to re-order intermediate stages of the FFT algorithm so that they operate on consecutive (or at least more localized) data elements. To these ends, a number of alternative implementation schemes have been devised for the Cooley–Tukey algorithm that do not require separate bit reversal and/or involve additional permutations at intermediate stages.

The problem is greatly simplified if it is **out-of-place**: the output array is distinct from the input array or, equivalently, an equal-size auxiliary array is available. The **Stockham auto-sort** algorithm[21][22] performs every stage of the FFT out-of-place, typically writing back and forth between two arrays, transposing one "digit" of the indices with each stage, and has been especially popular on SIMD architectures.[22][23] Even greater potential SIMD advantages (more consecutive accesses) have been proposed for the **Pease** algorithm,[24] which also reorders out-of-place with each stage, but this method requires separate bit/digit reversal and O(N log N) storage. One can also directly apply the Cooley–Tukey factorization definition with explicit (depth-first) recursion and small radices, which produces natural-order out-of-place output with no separate permutation step (as in the pseudocode above) and can be argued to have cache-oblivious locality benefits on systems with hierarchical memory.[9][13][25]

A typical strategy for in-place algorithms without auxiliary storage and without separate digit-reversal passes involves small matrix transpositions (which swap individual pairs of digits) at intermediate stages, which can be combined with the radix butterflies to reduce the number of passes over the data.[13][26][27][28][29]

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External links

- a simple, pedagogical radix-2 Cooley–Tukey FFT algorithm in C++. (http://www.librow.com/articles/article-10)
- KISSFFT (http://sourceforge.net/projects/kissfft/): a simple mixed-radix Cooley–Tukey implementation in C (open source)

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