PROBABILITY: Homework #2

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Problem 1

Denote C_1, C_2, C_3 as car in door #1, 2, 3; and E as host opens door #2

$$Pr(C_1|E) = \frac{Pr(E|C_1)Pr(C_1)}{\sum Pr(E|C_i)P(C_i)}$$

where

$$Pr(C_i) = \frac{1}{3}$$

In order to make $Pr(C_1|E) = Pr(C_3|E) = \frac{1}{2}$,

we need
$$\begin{aligned}\frac{1}{2} &= \frac{\frac{1}{3} \times Pr(E|C_1)}{\frac{1}{3} \times Pr(E|C_1) + 0 \times Pr(E|C_2) + \frac{1}{3} \times Pr(E|C_3)} \\ &= \frac{\frac{1}{3} \times Pr(E|C_1)}{\frac{1}{3} \times Pr(E|C_1) + \frac{1}{3} \times 1}\end{aligned}$$

So that, $Pr(E|C_1) = 1$
Therefore,

$$p = 1$$

Problem 2

(a)

E and *F* are independent events. Because owing a car will not influence whether the name listed in telephone book and neither will list in book influence owing a car.

(b)

E and *F* are not independent events. Because people's height and weight shows linear relationship to some extent.

(c)

E and F are not independent events. Because the United States is in the western hemisphere, which means E and F will happen on the same time.

(d)

E and F are not independent events. Because the weather of two adjacent days shows somewhat relationship. And we cannot say they are irrelevant.

Problem 3

(a)

In order to stop the game in 4th game, the only possible case is one win 1 game and the other win 3 games. And the one win 1 game must win in round 1st or 2nd.

$$\begin{aligned} P &= 2 \times p^3(1-p) + 2 \times (1-p)^3 p \\ &= 2p(1-p)(p^2 + (1-p)^2) \end{aligned}$$

(b)

Denote E as A wins, F as they played 2 games and game ends.

$$\begin{aligned} P(E) &= P(E \cap F) + P(E \cap F^c) \\ &= P(E \cap F) + P(E|F^c)P(F^c) \\ &= (p * p) + P(E) * 2p(1-p) \\ &= p^2 + P(E) * 2p(1-p) \end{aligned}$$

$$\Longrightarrow \ (1 - 2p + 2p^2)P(E) = p^2$$
$$\Longrightarrow \ P(E) = \frac{p^2}{(1 - 2p + 2p^2)}$$

Problem 4

Obviously, X can only be 1, 2, 3, 4, 5, 6.

$$P\{X = 1\} = C_5^1 * \frac{9!}{10!} = \frac{1}{2}$$

$$P\{X = 2\} = C_5^1 * C_5^1 * \frac{8!}{10!} = \frac{5}{18}$$

$$P\{X = 3\} = C_5^2 * C_5^1 * 2! * \frac{7!}{10!} = \frac{5}{36}$$

$$P\{X = 4\} = C_5^3 * C_5^1 * 3! * \frac{6!}{10!} = \frac{5}{84}$$

$$P\{X = 5\} = C_5^4 * C_5^1 * 4! * \frac{5!}{10!} = \frac{5}{252}$$

$$P\{X = 6\} = C_5^5 * C_5^1 * 5! * \frac{4!}{10!} = \frac{1}{252}$$

$$P\{X = i\} = C_5^{i-1} \times C_5^1 \times (i-1)! \times \frac{(10-i)!}{10!}, \qquad i = 1, 2, 3, 4, 5, 6$$

Problem 5

$$\text{when } x \notin \{0,\ 1,\ 2,\ 3,\ 3.5\}, \quad f(x) = F(x) - \lim_{y \to x^-} F(y) = 0$$

$$f(0) = F(0) - \lim_{y \to 0^-} F(y) = \frac{1}{2} - 0 = \frac{1}{2}$$

$$f(1) = F(1) - \lim_{y \to 1^-} F(y) = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}$$

$$f(2) = F(2) - \lim_{y \to 2^-} F(y) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$

$$f(3) = F(3) - \lim_{y \to 3^-} F(y) = \frac{9}{10} - \frac{4}{5} = \frac{1}{10}$$

$$f(3.5) = F(3.5) - \lim_{y \to 3.5^-} F(y) = 1 - \frac{9}{10} = \frac{1}{10}$$

Problem 6

(a)

$$
\begin{aligned}
P(X=1) &= F(1) - \lim_{y\to 1^-} F(y) = \frac{1}{2} + \frac{b-1}{4} - \frac{b}{4} = \frac{1}{4} \\
P(X=\frac{1}{2}) &= F(\frac{1}{2}) - \lim_{y\to \frac{1}{2}^-} F(y) = \frac{1/2}{4} - \frac{1/2}{4} = 0 \\
P(X=3) &= F(3) - \lim_{y\to 3^-} F(y) = 1 - \frac{11}{12} = \frac{1}{12}
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(\frac{1}{2} < X < \frac{3}{2}) &= P(\frac{1}{2} < X < 1) + P(1 \le X < \frac{3}{2}) \\
&= \lim_{y\to 1^-} F(y) - F(\frac{1}{2}) + F(\frac{3}{2}) - F(1) \\
&= (\frac{1}{4} - \frac{1/2}{4}) + (\frac{1}{2} + \frac{3/2-1}{4} - \frac{1}{2}) \\
&= \frac{1}{4}
\end{aligned}
$$

Problem 7

(a)

$$
\begin{aligned}
f(2n) &= (1-p_1)^n(1-p_2)^{(n-1)} * p_2 \qquad n \in \mathbb{N} \\
f(2n+1) &= (1-p_1)^n(1-p_2)^n * p_1 \qquad n \in \mathbb{Z}^+
\end{aligned}
$$

(b)

Let player wins, then they must play $2n+1$ times.

$$
\begin{aligned}
P &= \sum_{n=0}^{\infty} f(2n+1) \\
&= \sum_{n=0}^{\infty} (1-p_1)^n (1-p_2)^n * p_1 \\
&= \frac{p_1}{1-(1-p_1)(1-p_2)}
\end{aligned}
$$

Problem 8

(a)

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 f(x)dx = \int_0^1 cx^2 dx = \frac{1}{3}cx^3\Big|_0^1 = \frac{1}{3}c$$

So c should be 3.

(b)

When $x \in [0,\ 1]$,

$$F(x) = \int_0^x f(x)dx = \int_0^1 3x^2 dx = x^3\Big|_0^x = x^3$$

When $x \in [-\infty,\ 0)$, $F(x) = 0$

When $x \in (1, \infty]$, $F(x) = 1$

(c)

$$P(.1 \le X < .5) = F(.5) - F(.1) = .125 - .001 = .124$$

Problem 9

(a)

	1	2	3	4	Y
1	.10	.05	.02	.02	**.19**
2	.05	.20	.05	.02	**.32**
3	.02	.05	.20	.04	**.31**
4	.02	.02	.04	.10	**.18**
X	**.19**	**.32**	**.31**	**.18**	

(b)

$$P(X=1;Y=1)=.01 \qquad P(X=1)=P(Y=1)=.19$$
$$P(X=1;Y=1)\neq P(X=1)\times P(Y=1)$$

Therefore, X and Y are not independent.

(c)

$$f(x|Y=1)=\frac{P(X=x;Y=1)}{P(Y=1)}=\begin{cases}\frac{.10}{.19}=\frac{10}{19}, & x=1\\ \frac{.05}{.19}=\frac{5}{19}, & x=2\\ \frac{.02}{.19}=\frac{2}{19}, & x=3\\ \frac{.02}{.19}=\frac{2}{19}, & x=4\end{cases}$$

$$f(y|X=1)=\frac{P(Y=y;X=1)}{P(X=1)}=\begin{cases}\frac{.10}{.19}=\frac{10}{19}, & y=1\\ \frac{.05}{.19}=\frac{5}{19}, & y=2\\ \frac{.02}{.19}=\frac{2}{19}, & y=3\\ \frac{.02}{.19}=\frac{2}{19}, & y=4\end{cases}$$

Problem 10

Since the point is chosen uniformly, we can get the joint pdf easily:

$$f(x, y) = C \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

In order to get the value of C, we need to integral

$$\begin{aligned} 1 &= \int_{-a}^{a} \int_{-\sqrt{b^2 - \frac{b^2x^2}{a^2}}}^{\sqrt{b^2 - \frac{b^2x^2}{a^2}}} C \ dydx \\ &= C \int_{-a}^{a} \int_{-\sqrt{b^2 - \frac{b^2x^2}{a^2}}}^{\sqrt{b^2 - \frac{b^2x^2}{a^2}}} \ dydx \end{aligned}$$

This integral is equal to C times the area of ellipse, which is πab. Thus, $C = \frac{1}{ab}$

Now we compute the marginal density of X:

$$\begin{aligned} f_X(x) &= \int_{-\sqrt{b^2 - \frac{b^2x^2}{a^2}}}^{\sqrt{b^2 - \frac{b^2x^2}{a^2}}} \frac{1}{ab} \ dy \\ &= \frac{2}{ab} \sqrt{b^2 - \frac{b^2x^2}{a^2}} \\ &= 2\sqrt{\frac{1}{a^2} - \frac{x^2}{a^4}} \qquad x \in [-a, a] \end{aligned}$$

Then we compute the marginal density of Y:

$$\begin{aligned} f_Y(y) &= \int_{-\sqrt{a^2 - \frac{a^2y^2}{b^2}}}^{\sqrt{a^2 - \frac{a^2y^2}{b^2}}} \frac{1}{ab} \ dx \\ &= \frac{2}{ab} \sqrt{a^2 - \frac{a^2y^2}{b^2}} \\ &= 2\sqrt{\frac{1}{b^2} - \frac{y^2}{b^4}} \qquad y \in [-b, b] \end{aligned}$$

Problem 11

(a)

First, we can get the joint pdf is

$$f_{XY}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \alpha\beta e^{-\alpha x - \beta y}$$

Then we compute the marginal density of X:

$$f_X(x) = \int_0^\infty \alpha\beta e^{-\alpha x - \beta y}\ dy = \alpha e^{-\alpha x} \qquad x \geq 0$$

And now we compute the marginal density of Y:

$$f_Y(y) = \int_0^\infty \alpha\beta e^{-\alpha x - \beta y}\ dx = \beta e^{-\beta x} \qquad y \geq 0$$

Since $f(x,y) = f_X(x)f_Y(y)$, X and Y are independent.

(b)

In the previous part we have already get that

$$f_{XY}(x,y) = \alpha\beta e^{-\alpha x - \beta y} \qquad x \geq 0, \quad y \geq 0$$
$$f_X(x) = \alpha e^{-\alpha x} \qquad x \geq 0$$
$$f_Y(y) = \beta e^{-\beta x} \qquad y \geq 0$$

Problem 12

(a)

$$\begin{aligned}
1 =& \int_0^\infty \int_{-x}^{x} c(x^2 - y^2)e^{-x}\ dydx \\
=& \int_0^\infty (2c - \frac{2c}{3})x^3 e^{-x}\ dx \\
=& (2c - \frac{2c}{3})(-x^3 - 3x^2 - 6x - 6)e^{-x}\Big|_0^\infty \\
=& 8c
\end{aligned}$$

So $c = \dfrac{1}{8}$

(b)

First, we compute the marginal density of X:

$$\begin{aligned} f_X(x) &= \int_{-x}^{x} \frac{1}{8}(x^2 - y^2)e^{-x}\ dy \\ &= \frac{1}{6}x^3 e^{-x} \qquad x \geq 0 \end{aligned}$$

Then now we compute the marginal density of Y:

$$\begin{aligned} f_Y(y) &= \int_{|y|}^{\infty} \frac{1}{8}(x^2 - y^2)e^{-x}\ dx \\ &= \frac{1}{8}(-x^2 - 2x - 2 + y^2)e^{-x}\Big|_{|y|}^{\infty} \\ &= \frac{|y|+1}{4e^{|y|}} \qquad -x \leq y \leq x, \quad 0 \leq x < \infty \end{aligned}$$

Since $f(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

(c)

From the previous part, we already computed the marginal densities:

$$f_X(x) = \frac{1}{6}x^3 e^{-x} \qquad x \geq 0$$

$$f_Y(y) = \frac{|y|+1}{4e^{|y|}} \qquad -x \leq y \leq x, \quad 0 \leq x < \infty$$

(d)

Given y:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \frac{\frac{1}{8}(x^2 - y^2)e^{-x}}{\frac{|y|+1}{4e^{|y|}}} \\ &= \frac{(x^2 - y^2)e^{-x+|y|}}{2(|y|+1)} \qquad -x \leq y \leq x, \quad 0 \leq x < \infty \end{aligned}$$

Given x:

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\
&= \frac{\frac{1}{8}(x^2-y^2)e^{-x}}{\frac{1}{6}(x^3e^{-x})} \\
&= \frac{3(x^2-y^2)}{4x^3} \qquad -x \le y \le x
\end{aligned}
$$

Problem 13

$Y = g(X) = aX + b,\ g(X)$ is a monotone transformation. $g^{-1}(Y) = \frac{Y-b}{a}$. Then

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{\partial g^{-1}(y)}{\partial y}\Big| \\
&= f_X(\frac{y-b}{a})\Big|\frac{\partial \frac{y-b}{a}}{\partial y}\Big| \\
&= f_X(\frac{y-b}{a})\Big|\frac{1}{a}\Big| \\
&= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)
\end{aligned}
$$

Problem 14

$$
\begin{aligned}
P(Y=k) &= P\left(F(k-1) < U \le F(k)\right) \\
&= P\left(U \le F(k)\right) - P\left(U \le F(k-1)\right) \\
&= F(k) - F(k-1)
\end{aligned}
$$

Which means Y has cdf F.

Problem 15

(a)

Suppose that $P(X = x) = c(x+1)(8-x)$, where c is a constant number.

Then we have

$$\begin{aligned}
1 &= \sum_{x=0}^{7} f(x) \\
&= \sum_{x=0}^{7} c(x+1)(8-x) \\
&= c(8+14+18+20+20+18+14+8) \\
&= 120c
\end{aligned}$$

Thus, $c = \frac{1}{120}$

$$f(x) = \frac{1}{120}(x+1)(8-x) \qquad x = 0, ..., 7$$

(b)

$$\begin{aligned}
P(X \geq 5) &= \sum_{x=5}^{7} f(x) \\
&= \sum_{x=5}^{7} \frac{1}{120}(x+1)(8-x) \\
&= \frac{1}{120}(18+14+8) \\
&= \frac{1}{3}
\end{aligned}$$

Problem 16

(a)

$$\begin{aligned}
P(X \leq t) &= \int_{-\infty}^{t} \frac{1}{8}x \, dx \\
&= \int_{0}^{t} \frac{1}{8}x \, dx \\
&= \frac{t^2}{16} = \frac{1}{4}
\end{aligned}$$

Thus, $t = 2 \quad (\text{when } t = -2,\ P(X \leq t) = 0)$

(b)

$$\begin{aligned}
P(X \geq t) &= \int_t^{\infty} \frac{1}{8}x \ dx \\
&= \int_t^{4} \frac{1}{8}x \ dx \\
&= 1 - \frac{t^2}{16} = \frac{1}{2}
\end{aligned}$$

Thus, $t = 2\sqrt{2} \quad \big(\text{when } t = -2\sqrt{2},\ P(X \geq t) = 1\big)$

Problem 17

(a)

$$\begin{aligned}
f_{XY}(x, y) &= \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{1}{156}(3x^2 + 2y) \\
P(1 \leq X \leq 2 \text{ and } 1 \leq Y \leq 2) &= \int_1^2 \int_1^2 \frac{1}{156}(3x^2 + 2y) \ dydx \\
&= \int_1^2 \frac{1}{156} \left[(3x^2 y + y^2) \Big|_{y=1}^{y=2} \right] \ dx \\
&= \int_1^2 \frac{1}{156}(3x^2 + 3) \ dx \\
&= \frac{1}{156}(x^3 + 3x) \Big|_1^2 \\
&= \frac{5}{78}
\end{aligned}$$

(b)

$$\begin{aligned}P(2 \le X \le 4 \text{ and } 2 \le Y \le 4) =& P(2 \le X \le 3 \text{ and } 2 \le Y \le 4)\\ =& \int_2^3 \int_2^4 \frac{1}{156}(3x^2+2y)\ dydx \\ =& \int_2^3 \frac{1}{156}\left[(3x^2y+y^2)\Big|_{y=2}^{y=4}\right]\ dx \\ =& \int_2^3 \frac{1}{156}(6x^2+12)\ dx \\ =& \frac{1}{156}(2x^3+12x)\Big|_2^3 \\ =& \frac{25}{78}\end{aligned}$$

(c)

$$\begin{aligned}F_Y(y) =& \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{XY}(x,y)\ dxdy \\ =& \int_0^y \int_0^3 \frac{1}{156}(3x^2+2y)\ dxdy \\ =& \int_0^y \frac{1}{156}\left[(x^3+2yx)\Big|_{x=0}^{x=3}\right]\ dxdy \\ =& \int_0^y \frac{1}{156}(27+6y)\ dy \\ =& \frac{1}{156}(27y+3y^2)\Big|_0^y \\ =& \frac{9}{52}y+\frac{1}{52}y^2 \qquad y \in [0,4]\end{aligned}$$

When $y \le 0,\ F_Y(y) = 0$

When $y \ge 4,\ F_Y(y) = 1$

(d)

From part (a), we already computed the joint p.d.f:

$$f_{XY}(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{1}{156}(3x^2+2y)$$

(e)

$$\begin{aligned}
P(Y \leq X) &= \int_0^3 \int_0^x \frac{1}{156}(3x^2 + 2y)\ dydx \\
&= \int_0^3 \frac{1}{156}\left[(3x^2y + y^2)\Big|_{y=0}^{y=x}\right]\ dx \\
&= \int_0^3 \frac{1}{156}(3x^3 + x^2)\ dx \\
&= \frac{1}{156}(\frac{3}{4}x^4 + \frac{1}{3}x^3)\Big|_0^3 \\
&= \frac{93}{208}
\end{aligned}$$