PROBABILITY: Homework $\#3$

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Problem 1

$$p(u_1, u_2) = p(u_1)p(u_2) = 1, \qquad u_1, u_2 \in [0, 1]$$

when $s \geq 2,\ P(S < s) = 1,\ p(s) = 0,$

when $1 \leq s \leq 2,$

$$\begin{aligned} P(S \leq s) = 1 - P(S > s) &= 1 - \int_{s-1}^{1} \int_{s-u_1}^{1} 1\ du_2 du_1 \\ &= 1 - \int_{s-1}^{1} (1 - s + u_1)\ du_1 \\ &= (2-s)(s-1) + \frac{(s-1)^2}{2} + \frac{1}{2} \end{aligned}$$

$$\begin{aligned} p(s) &= \frac{d\ \{(2-s)(s-1) + \frac{(s-1)^2}{2} + \frac{1}{2}\}}{d\ s} \\ &= -s + 2 \end{aligned}$$

when $0 \leq s \leq 1,$

$$\begin{aligned} P(S \leq s) &= \int_0^1 \int_0^{s-u_1} 1\ du_2 du_1 \\ &= \int_0^1 (s - u_1)\ du_1 \\ &= s - \frac{1}{2} \end{aligned}$$

$$\begin{aligned} p(s) &= \frac{d\ (2-s)}{d\ s} \\ &= -1 \end{aligned}$$

So $p(s) = \begin{cases} 0 & s < 0 \\ -1 & 0 \leq s \leq 1 \\ s - \dfrac{1}{2} & 1 \leq s \leq 2 \\ 2 - s & s > 2 \end{cases}$

Problem 2

$$E(X) = \int_{-1}^{1} x \frac{1 + \alpha x}{2}\ dx = \frac{\alpha}{3}$$

$$E(X^2) = \int_{-1}^{1} x^2 \frac{1 + \alpha x}{2}\ dx = \frac{1}{3}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{1}{3} - \frac{\alpha^2}{9}$$

Problem 3

$$f_1(u) = n(1 - F_u(u))^{n-1} f_u(u) = \frac{1}{b-a} n(1 - \frac{u-a}{b-a})^{n-1}, \qquad a \le u \le b$$
$$f_n(u) = nF_u(u)^{n-1} f_u(u) = \frac{1}{b-a} n(\frac{u-a}{b-a})^{n-1}, \qquad a \le u \le b$$

$$\begin{aligned}
E(U_{(n)}) &= \int_a^b \frac{1}{b-a} nu(\frac{u-a}{b-a})^{n-1} du \\
&= \int_a^b u \ d(\frac{u-a}{b-a})^n \\
&= u \ (\frac{u-a}{b-a})^n \Big|_a^b - \int_a^b (\frac{u-a}{b-a})^n \ du \\
&= u \ (\frac{u-a}{b-a})^n \Big|_a^b - \frac{b-a}{n+1} (\frac{u-a}{b-a})^{n+1} \Big|_a^b \\
&= b(\frac{b-a}{b-a})^n - a(\frac{a-a}{b-a})^n - \frac{b-a}{n+1}(\frac{b-a}{b-a})^{n+1} + \frac{b-a}{n+1}(\frac{a-a}{b-a})^{n+1} \\
&= b - \frac{b-a}{n+1}
\end{aligned}$$

$$\begin{aligned}
E(U_{(1)}) &= \int_a^b \frac{1}{b-a} nu(1 - \frac{u-a}{b-a})^{n-1} du \\
&= \int_a^b -u \ d(1 - \frac{u-a}{b-a})^n \\
&= -u \ (1 - \frac{u-a}{b-a})^n \Big|_a^b + \int_a^b (1 - \frac{u-a}{b-a})^n \ du \\
&= -u \ (1 - \frac{u-a}{b-a})^n \Big|_a^b - \frac{b-a}{n+1}(1 - \frac{u-a}{b-a})^{n+1} \Big|_a^b \\
&= -b(1 - \frac{b-a}{b-a})^n + a(1 - \frac{a-a}{b-a})^n - \frac{b-a}{n+1}(1 - \frac{b-a}{b-a})^{n+1} + \frac{b-a}{n+1}(1 - \frac{a-a}{b-a})^{n+1} \\
&= a + \frac{b-a}{n+1}
\end{aligned}$$

So

$$\begin{aligned}E(U_{(n)} - U_{(1)}) &= E(U_{(n)}) - E(U_{(1)}) \\ &= b - a - 2\frac{b-a}{n+1} \\ &= \frac{n-1}{n+1}(b-a)\end{aligned}$$

Suppose $U_1, U_2, ..., U_n$ i.i.d follow $U(0,1)$, then

$$E(U_{(n)} - U_{(1)}) = \frac{n-1}{n+1}$$

Problem 4

$f(x) = 1 \qquad 0 \le x \le 1$

$E(X^2) = \int_0^1 x^2 \; dx = \frac{1}{3}$

Expected area is 1/3.

Problem 5

$$\begin{aligned}E[1/(X+1)] &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{1}{k+1} \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k+1)!} \\ &= \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1} e^{-\lambda}}{(k+1)!} \\ &= 1 + \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k+1} e^{-\lambda}}{(k+1)!} \\ &= 1 + \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\ &= 1 + \frac{1}{\lambda}\end{aligned}$$

Problem 6

(a)

$$
\begin{aligned}
f(x,y) &= e^{-y} \qquad 0 \le x \le y \\
f_x(x) &= \int_x^{\infty} e^{-y}\ dy = e^{-x} \qquad x \ge 0 \\
f_y(y) &= \int_0^{y} e^{-y}\ dx = ye^{-y} \qquad y \ge 0 \\
E(X) &= \int_0^{\infty} xe^{-x}\ dx = -(x+1)e^{-x}\Big|_0^{\infty} = 1 \\
E(X^2) &= \int_0^{\infty} x^2e^{-x}\ dx = -(x^2+2x+2)e^{-x}\Big|_0^{\infty} = 2 \\
Var(X) &= E(X^2) - E(X)^2 = 1 \\
E(Y) &= \int_0^{\infty} y^2e^{-y}\ dx = -(y^2+2y+2)e^{-y}\Big|_0^{\infty} = 2 \\
E(Y^2) &= \int_0^{\infty} y^3e^{-y}\ dx = -(y^3+3y^2+6y+6)e^{-y}\Big|_0^{\infty} = 6 \\
Var(Y) &= E(Y^2) - E(Y)^2 = 2 \\
E(XY) &= \int_0^{\infty}\int_0^{y} xye^{-y}\ dxdy = \int_0^{\infty} \frac{1}{2}y^3e^{-y}\ dy \\
&= -\frac{1}{2}(y^3+3y^2+6y+6)e^{-y}\Big|_0^{\infty} = 3 \\
Cov(X,Y) &= E(XY) - E(X)E(Y) = 3 - 2 = 1 \\
Corr(X,Y) &= \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\sqrt{2}}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
f(x|y) &= \frac{f(x,y)}{f_y(y)} = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y} \\
f(y|x) &= \frac{f(x,y)}{f_x(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y} \\
E(X|Y=y) &= \int_0^y xf(x|y)\ dx = \int_0^y \frac{x}{y}\ dx = \frac{y}{2}, \quad \text{for a given y} \\
E(Y|X=x) &= \int_x^\infty yf(y|x)\ dy = \int_x^\infty ye^{x-y}\ dx \\
&= e^x(-y-1)e^{-y}\Big|_x^\infty \\
&= x+1, \quad \text{for a given x}
\end{aligned}
$$

(c)

$$
E(X|Y) = \int_0^y xf(x|y)\ dx = \int_0^y \frac{x}{y}\ dx = \frac{y}{2}
$$

$$
P(E(X|Y) < z) = P(\frac{y}{2} < z) = P(y < 2z) = \int_0^{2z} ye^{-y}\ dy = -(2z+1)e^{-2z} + 1
$$

density for $E(X|Y)$ is

$$
p(z) = \frac{d\ P(E(X|Y) < z)}{d\ z} = 4ze^{-2z} \qquad z \geq 0
$$

$$
E(Y|X) = \int_x^\infty yf(y|x)\ dy = x+1
$$

$$
\begin{aligned}
P(E(Y|X) < z) &= P(x+1 < z) = P(x < z-1) = \int_0^{z-1} e^{-x}\ dx \\
&= -(2z+1)e^{-2z} + 1 = 1 - e^{-2z}
\end{aligned}
$$

density for $E(Y|X)$ is

$$
p_{E(Y|X)}(z) = \frac{d\ P(E(Y|X) < z)}{d\ z} = 2e^{-2z} \qquad z \geq 1
$$

Problem 7

(a)

$$Cov(X_i, X_j) = 0 \quad i \neq j$$

$$\begin{aligned}
&Cor(X_1 + 2X_2, X_2 + X_3 + 1) \\
&= \frac{Cov(X_1 + 2X_2, X_2 + X_3 + 1)}{\sqrt{Var(X_1 + 2X_2)Var(X_2 + X_3 + 1)}} \\
&= \frac{Cov(X_1, X_2) + Cov(X_1, X_3) + 2Cov(X_2, X_2) + 2Cov(X_2, X_3)}{\sqrt{(Var(X_1) + 4Var(X_2))(Var(X_2) + Var(X_3))}} \\
&= \frac{2 * 1}{\sqrt{(1+4)(1+1)}} \\
&= \frac{2}{\sqrt{10}}
\end{aligned}$$

(b)

$$\begin{aligned}
&Cor(X_1 + X_2 + 3, 5X_3 + X_4) \\
&= \frac{Cov(X_1 + X_2 + 3, 5X_3 + X_4)}{\sqrt{Var(X_1 + X_2 + 3)Var(5X_3 + X_4)}} \\
&= \frac{5Cov(X_1, X_3) + Cov(X_1, X_4) + 5Cov(X_2, X_3) + Cov(X_2, X_4)}{\sqrt{Var(X_1 + X_2 + 3)Var(5X_3 + X_4)}} \\
&= \frac{0}{\sqrt{Var(X_1 + X_2 + 3)Var(5X_3 + X_4)}} \\
&= 0
\end{aligned}$$

Problem 8

(a)

$$P(N = n) = p^{n-1}(1-p) + (1-p)^{n-1}p \qquad n \geq 2$$

$$
\begin{aligned}
E(N) &= \sum_{n=2}^{\infty} nP(N=n) \\
&= \sum_{n=2}^{\infty} [np^{n-1}(1-p) + n(1-p)^{n-1}p] \\
&= -1 + \sum_{n=1}^{\infty} np^{n-1}(1-p) + \sum_{n=1}^{\infty} n(1-p)^{n-1}p
\end{aligned}
$$

As we already know $\sum_{n=1}^{\infty} np^{n-1} = \frac{1}{1-p}$

So

$$
\begin{aligned}
E(N) &= -1 + \sum_{n=1}^{\infty} np^{n-1}(1-p) + \sum_{n=1}^{\infty} n(1-p)^{n-1}p \\
&= -1 + (1-p)\frac{1}{1-p} + +p\frac{1}{1-(1-p)} \\
&= 1
\end{aligned}
$$

(b)

Suppose we flip *i* times and stop, then $P(\text{last head at } i) = (1-p)^{i-1}p$

Then

$$
\begin{aligned}
P(\text{last head}) &= \sum_{i=2}^{\infty} (1-p)^{i-1}p \\
&= p(1-p)\frac{1}{1-(1-p)} \\
&= 1-p
\end{aligned}
$$

Problem 9

Number in the sample space is $\binom{30}{12}$

$$
P(X=x) = \frac{\binom{10}{x}\binom{20}{12-x}}{\binom{30}{12}} \qquad x = 0, 1, 2..., 10
$$

$$P(Y=y)=\frac{\binom{8}{y}\binom{22}{12-y}}{\binom{30}{12}} \qquad y=0,1,2,...,8$$

$$P(X=x,Y=y)=\frac{\binom{10}{x}\binom{8}{y}\binom{12}{12-x-y}}{\binom{30}{12}} \qquad x+y\leq 12$$

$$E(X)=\sum_{x=0}^{10} xP(X=x)=\sum_{x=0}^{10} x\frac{\binom{10}{x}\binom{20}{12-x}}{\binom{30}{12}}$$

$$E(Y)=\sum_{y=0}^{8} yP(Y=y)=\sum_{y=0}^{8} y\frac{\binom{8}{y}\binom{22}{12-y}}{\binom{30}{12}}$$

$$E(XY)=\sum_{y=0}^{8}\sum_{x=0}^{12-y} xyP(X=x,Y=y)=\sum_{y=0}^{8}\sum_{x=0}^{12-y} xy\frac{\binom{10}{x}\binom{8}{y}\binom{12}{12-x-y}}{\binom{30}{12}}$$

```
> X=0:10
> Y=0:8
> EX=sum(X*choose(10,X)*choose(20,12-X)/choose(30,12))
> EY=sum(Y*choose(8,Y)*choose(22,12-Y)/choose(30,12))
> EXY=0
> for (X in 0:10){
+   for (Y in 0:8){
+     EXY=EXY+X*Y*choose(10,X)*choose(8,Y)*choose(12,12-X-Y)/choose(30,12)
+   }
+ }
> print(c(EX,EY,EXY))
[1]  4.00000  3.20000 12.13793
```

So $E(X)=4, \quad E(Y)=3.2, \quad E(XY)=12.13793$

$$Cov(X,Y)=E(XY)-E(X)E(Y)=12.13793-4\times 3.2=-0.662069$$

Problem 10 (#8 on page 167)

(a)

$$
\begin{aligned}
p(x, y_1, y_2, ..., y_n) &= g(y_1, y_2, ..., y_n|x)f(x) \\
&= g(y_1|x)g(y_2|x)...g(y_n|x)f(x) \\
&= \frac{1}{n!}x^n e^{-x}\frac{1}{x^n} \\
&= \frac{1}{n!}e^{-x} \qquad 0 < y_1, y_2, ..., y_n < x \\
p_Y(y_1, y_2, ..., y_n) &= \int_y^\infty \; p(x, y_1, y_2, ..., y_n)\; dx = \int_y^\infty \; \frac{1}{n!}e^{-x}\; dx \\
&= -\frac{1}{n!}e^{-x}\Big|_y^\infty \\
&= \frac{1}{n!}e^{-y} \qquad 0 < y_1, y_2, ..., y_n \leq y
\end{aligned}
$$

(b)

$$
\begin{aligned}
p(x|y_1, y_2, ..., y_n) &= \frac{p(x, y_1, y_2, ..., y_n)}{p_Y(y_1, y_2, ..., y_n)} \\
&= \frac{\frac{1}{n!}e^{-x}}{\frac{1}{n!}e^{-y}} \\
&= e^{y-x} \qquad 0 < Y_1, Y_2, ..., Y_n \leq y < x
\end{aligned}
$$

Problem 11(#3 on page 174)

$$
\begin{aligned}
P(Y \leq y) &= P(X(2-X) \leq y) \\
&= P(X \geq 1 + \sqrt{1-y}) + P(X \leq 1 - \sqrt{1-y}) \\
&= \int_{1+\sqrt{1-y}}^{2} \; \frac{1}{2}x\; dx + \int_0^{1-\sqrt{1-y}} \; \frac{1}{2}x\; dx \\
&= 1 - \sqrt{1-y} \\
&(0 < y < 1)
\end{aligned}
$$

$$\begin{aligned} p(y) &= \frac{d\ P(Y \leq y)}{d\ y} \\ &= \frac{d\ 1 - \sqrt{1-y}}{d\ y} \\ &= \frac{1}{2\sqrt{1-y}} \\ &(0 < y < 1) \end{aligned}$$

Problem 12(#4 on page 187)

Let $u = x_1 \quad v = x_1 x_2$, which means $x_1 = u \quad x_2 = v/u \qquad 0 < v < u < 1$
Then the Jacobian is

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix} = \frac{1}{u}$$

$$\begin{aligned} p(v) &= \int_{-\infty}^{\infty} f(u, v/u)|J|\ du \\ &= \int_v^1 (u + v/u)(1/u)\ du \\ &= \int_v^1 1 + v/u^2\ du \\ &= (u - v/u)\Big|_v^1 \\ &= 2 - 2v \qquad 0 < v < 1 \end{aligned}$$

so, $p(y) = 2 - 2y \qquad 0 < y < 1$

Problem 13(#10 on page 240)

$$\begin{aligned} \psi_Z(t) &= E(e^{t(2X-3Y+4)}) = E(e^{2tX}e^{-3tY}e^{4t}) \\ &= E(e^{2tX})E(e^{-3tY})E(e^{4t}) \\ &= E(e^{2tX})E(e^{-3tY})E(e^{4t}) \\ &= (\psi(2t))(\psi(-3t)))(e^{4t}) \\ &= e^{13t^2+t} \end{aligned}$$

Problem 14(#11 on page 240)

Since

$$\begin{aligned}\psi(t) = E(e^{tX}) &= \sum_k P(X=k)e^{tk} \\ &= \frac{1}{5}e^{t} + \frac{2}{5}e^{4t} + \frac{2}{5}e^{8t}\end{aligned}$$

Intuitively we can say $k = 1, 4, 8$ with probability

$$P(X=1) = \frac{1}{5};\quad P(X=4) = \frac{2}{5};\quad P(X=8) = \frac{2}{5}.$$

Problem 15(#5 on page 247)

(a)

$$\begin{aligned}E(X) &= \int_0^1 \frac{1}{2}xf(x)\ dx + \int_2^4 \frac{1}{2}xg(x)\ dx \\ &= \frac{1}{2}E_f(X) + \frac{1}{2}E_g(X)\end{aligned}$$

(b)

$$\begin{aligned}1 &= \int_0^1 \frac{1}{2}f(x)\ dx + \int_2^4 \frac{1}{2}g(x)\ dx \\ &= \frac{1}{2}\int_0^1 f(x)\ dx + \frac{1}{2}\int_2^4 g(x)\ dx\end{aligned}$$

We also know that $\int_0^1 f(x)\ dx = 1$ and $\int_2^4 g(x)\ dx = 1$

So if we take any value $1 \le x \le 2$

$$\begin{aligned}P(X=x) &= \int_{-\infty}^{\infty} \frac{1}{2}f(x)\ dx + \int_{-\infty}^{\infty} \frac{1}{2}g(x)\ dx \\ &= \frac{1}{2}\int_0^1 f(x)\ dx \\ &= \frac{1}{2}\end{aligned}$$

this value of x is the median.

Problem 16(#12 on page 255)

$$f_X(x) = \int_0^2 \frac{1}{3}(x+y)\ dy = \frac{2x+2}{3} \qquad 0 \le x \le 1$$
$$f_Y(y) = \int_0^1 \frac{1}{3}(x+y)\ dx = \frac{1+2y}{6} \qquad 0 \le y \le 2$$
$$E(X) = \int_0^1 x\frac{2x+2}{3}\ dx = \frac{5}{9}$$
$$E(X^2) = \int_0^1 x^2\frac{2x+2}{3}\ dx = \frac{7}{18}$$
$$E(Y) = \int_0^2 y\frac{2y+1}{6}\ dx = \frac{11}{9}$$
$$E(Y^2) = \int_0^2 y^2\frac{2y+1}{6}\ dx = \frac{16}{9}$$
$$\begin{aligned} E(XY) &= \int_0^1 \int_0^2 \frac{1}{3}xy(x+y)\ dydx \\ &= \int_0^1 \frac{2x^2}{3} + \frac{8x}{9}\ dx \\ &= \frac{2}{3} \end{aligned}$$
$$\begin{aligned} &Var(2X - 3Y + 8) \\ &= 4Var(X) + 9Var(Y) - 12Cov(X,Y) \\ &= 4(E(X^2) - E(X)^2) + 9(E(Y^2) - E(Y)^2) - 12(E(XY) - E(X)E(Y)) \\ &= \frac{245}{81} \end{aligned}$$