INFERENCE: Homework #4

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Problem 1 (#1 on page 375)

$$Pr(sum\ is\ 7) = \frac{6}{36} = \frac{1}{6}$$

$$Pr(X = x) = \binom{120}{x}(\frac{1}{6})^x(\frac{5}{6})^{120-x}$$

X follows $Bin(120, \frac{1}{6})$, so $E(X) = 120 \times \frac{1}{6} = 20$ and

$$Var(X) = 120 \times \frac{1}{6} \times \frac{5}{6} = \frac{50}{3}$$

Using Central Limits Theorem,

$$\frac{X - 20}{\sqrt{50/3}} \sim N(0, 1)$$

So, let

$$Pr\left(\frac{|X - 20|}{\sqrt{50/3}} \leq \frac{k}{\sqrt{50/3}}\right) = 0.95$$

then, $\frac{k}{\sqrt{50/3}} = 1.96$, which is k=8

Problem 2 (#3 on page 375)

Since X has the Poisson distribution with mean 10, the variance is also 10 too.

$$\frac{X - 10}{\sqrt{10}} \sim N(0, 1)$$

$$\begin{aligned}
&Pr\,(8 \leq X \leq 12)\\
=&Pr\left(\frac{8-10}{\sqrt{10}} \leq \frac{X-10}{\sqrt{10}} \leq \frac{12-10}{\sqrt{10}}\right)\\
=&\Phi(\frac{12-10}{\sqrt{10}}) - \Phi(\frac{8-10}{\sqrt{10}})\\
=&0.4729107
\end{aligned}$$

Using Poisson table:

$$\begin{aligned}
&Pr\,(8 \leq X \leq 12)\\
=&0.7915565 - 0.3328197\\
=&0.4587368
\end{aligned}$$

Problem 3 (#1 on page 461)

(a)

$$P(\theta) = \theta \qquad p(\theta) = 1 \qquad 0 \leq \theta \leq 1$$
$$\begin{aligned} p(\theta|x) \propto p(x|\theta)p(\theta) &= \binom{25}{10}\theta^{10}(1-\theta)^{15} \\ &= \binom{25}{10}\theta^{10}(1-\theta)^{15} \end{aligned}$$

This is a Beta distribution with $\alpha = 11$ and $\beta = 16$

(b)

In order to get squared error loss, the estimator should be mean of the posterior distribution.

$$\hat{\theta} = E(\theta|x) = \frac{\alpha}{\alpha+\beta} = \frac{11}{27}$$

Problem 4 (#3 on page 461)

$$\begin{aligned} p(\theta|x) \propto p(x|\theta)\xi(\theta) &= \binom{10}{3}\theta^{3}(1-\theta)^{7} \times 60\theta^2(1-\theta)^3 \\ &= \binom{10}{3}\theta^{5}(1-\theta)^{10} \end{aligned}$$

This is a Beta distribution with $\alpha = 6$ and $\beta = 11$

In order to get squared error loss, the estimator should be mean of the posterior distribution.

$$\hat{\theta} = E(\theta|x) = \frac{\alpha}{\alpha+\beta} = \frac{6}{17}$$

Problem 5 (#4 on page 461)

Likelihood function is

$$L(\underset{\sim}{x}|\theta) = \prod f(x_i|\theta) = (\frac{1}{\theta})^n \qquad \theta \le X_1\ X_2 ... X_n \le 2\theta$$

Because $\theta \le X_1\ X_2 ... X_n \le 2\theta$, the support for θ is $X_{(n)}/2 \le \theta \le X_{(1)}$
What' s more, likelihood function is a monotone decreasing function, when θ equals to its minimum value, likelihood function reach its maximum value. Therefore,

$$\hat{\theta}_{\text{MLE}} = \frac{X_{(n)}}{2}$$

Problem 6 (#7 on page 462)

(a)

Likelihood function is:

$$\begin{aligned} L(\underset{\sim}{x}|\theta) &= \prod f(x_i|\theta) \\ &= \frac{1}{\theta} e^{-X_1/\theta} \times \frac{1}{2\theta} e^{-X_2/2\theta} \times \frac{1}{3\theta} e^{-X_3/3\theta} \\ &= \frac{1}{6\theta^3} e^{-(X_1+X_2/2+X_3/3)/\theta} \end{aligned}$$

Log-likelihood function is:

$$\begin{aligned} l(\underset{\sim}{x}|\theta) &= log\ L(\underset{\sim}{x}|\theta) \\ &= -(X_1 + X_2/2 + X_3/3)/\theta - 3log\ \theta - log\ 6 \\ l^{'}(\underset{\sim}{x}|\theta) &= \frac{(X_1 + X_2/2 + X_3/3)}{\theta^2} - \frac{3}{\theta} \end{aligned}$$

Let $l^{'}(\underset{\sim}{x}|\theta) = 0$, we have $\theta = \dfrac{X_1 + X_2/2 + X_3/3}{3}$

so, $\hat{\theta}_{\text{MLE}} = \dfrac{X_1 + X_2/2 + X_3/3}{3}$

(b)

$$\xi(\psi) \propto \psi^{\alpha-1} e^{-\beta\psi}$$

$$L(\underset{\sim}{x}|\theta) = \frac{1}{6\theta^3} e^{-(X_1+X_2/2+X_3/3)/\theta}$$

$$\text{so, } L(\underset{\sim}{x}|\psi) = \frac{1}{6}\psi^3 e^{-(X_1+X_2/2+X_3/3)\psi}$$

then,

$$\begin{aligned} p(\psi|\underset{\sim}{x}) &\propto L(\underset{\sim}{x}|\psi)\xi(\psi) \\ &= \psi^3 e^{-(X_1+2X_2+3X_3)\psi}\psi^{\alpha-1}e^{-\beta\psi} \\ &= \psi^{\alpha+2}e^{-(X_1+X_2/2+X_3/3+\beta)\psi} \end{aligned}$$

This is a Gamma distribution with $\alpha_{\text{new}} = \alpha + 3$ and $\beta_{\text{new}} = X_1 + \dfrac{X_2}{2} + \dfrac{X_3}{3} + \beta$

Problem 7(#9 on page 462)

(a)

$$\begin{aligned} p(\theta|x) &\propto f(x|\theta)\xi(\theta) \\ &= e^{-\theta} \qquad \theta > x \end{aligned}$$

$$\int_x^\infty e^{-\theta}\ d\theta = e^{-x}$$

So, $p(\theta|x) = e^{x-\theta} \qquad \theta > x$

$$E(\theta|x) = \int_x^\infty \theta p(\theta|x)\ d\theta = \int_x^\infty \theta e^{x-\theta}\ d\theta = x + 1$$

(b)

Let $\displaystyle\int_x^{\hat{\theta}} p(\theta|x)\ d\theta = \int_x^{\hat{\theta}} e^{x-\theta}\ d\theta = \frac{1}{2}$

which is $1 - e^{x-\hat{\theta}} = \dfrac{1}{2}$

so, $\hat{\theta} = x + log\ 2$

Problem 8 (#10 on page 462)

$$
\begin{aligned}
L(\underset{\sim}{x}|\theta) &= \prod f(x_i|\theta) \\
&= \prod (\frac{1}{3}(1+\beta))^{x_i}(1-\frac{1}{3}(1+\beta))^{1-x_i} \\
&= (\frac{1}{3}(1+\beta))^{\sum x_i}(1-\frac{1}{3}(1+\beta))^{n-\sum x_i}
\end{aligned}
$$

As we know in class that the MLE for $\theta = (1/3)(1+\beta)$ is $\bar{X}_n$, and likelihood function increases before $\bar{X}_n$ and decreases after $\bar{X}_n$.
so, the MLE for β should be $3\bar{X}_n - 1$

While the support of β is $[0, 1]$

if $\bar{X}_n < \frac{1}{3}$, $\hat{\beta}_{\text{MLE}} = 0$

if $\frac{1}{3} \leq \bar{X}_n \leq \frac{2}{3}$, $\hat{\beta}_{\text{MLE}} = \bar{X}_n$

if $\bar{X}_n > \frac{2}{3}$, $\hat{\beta}_{\text{MLE}} = 1$

Problem 9 (#14 on page 462)

The likelihood function is

$$
\begin{aligned}
L(\underset{\sim}{x}|\beta,\theta) &= \prod f(x_i|\beta,\theta) \\
&= \beta^n e^{-\beta(\sum x_i)+n\beta\theta} 1_{\{x_{(1)}\geq\theta\}}
\end{aligned}
$$

Let $k_1(u(\underset{\sim}{x}),\beta,\theta) = \beta^n e^{-\beta(\sum x_i)+n\beta\theta} 1_{\{x_{(1)}\geq\theta\}}$ and $k_2(\underset{\sim}{x}) = 1$

Then we know $(\sum x_i, x_{(1)})$ is a pair of jointly sufficient statistics.

Problem 10 (#15 on page 462)

The likelihood function is

$$
\begin{aligned}
L(\underset{\sim}{x}|x_0,\alpha) &= \prod f(x_i|x_0,\alpha) \\
&= \frac{\alpha^n {x_0}^{n\alpha}}{(\prod x_i)^{\alpha+1}}
\end{aligned}
$$

The likelihood function increases when x_0 increases. While the support of x_0 is $x_0 \le x_{(1)}$, so when $x_0 = x_{(1)}$ likelihood function reaches maximum value. Therefore,

$$\hat{x}_0 = x_{(1)}$$

Problem 11 (#16 on page 462)

While the MLE of x_0 is already the minimal possible value for x_0, we only need to show it is a sufficient statistic:

$$\begin{aligned} L(\underset{\sim}{x}|x_0,\alpha) &= \prod f(x_i|x_0,\alpha) \\ &= \frac{\alpha^n {x_0}^{n\alpha}}{(\prod x_i)^{\alpha+1}} \end{aligned}$$

Let $k_1(u(\underset{\sim}{x}), x_0, \alpha) = x_0^{n\alpha} 1_{\{x_0 \le x_{(1)}\}}$ and $k_2(\underset{\sim}{x}|\alpha) = \frac{\alpha^n}{(\prod x_i)^{\alpha+1}}$

Then we know $x_{(1)}$ is a sufficient statistic and now it is minimal sufficient statistic.

Problem 12 (#17 on page 462)

Likelihood function is:

$$\begin{aligned} L(\underset{\sim}{x}|x_0,\alpha) &= \prod f(x_i|x_0,\alpha) \\ &= \frac{\alpha^n {x_0}^{n\alpha}}{(\prod x_i)^{\alpha+1}} \ 1_{\{x_0 \le x_{(1)}\}} \end{aligned}$$

Log-likelihood function is:

$$l(\underset{\sim}{x}|x_0,\alpha) = nlog\alpha + n\alpha logx_0 - (\alpha+1)\sum logx_i$$

$$\frac{\partial l(\underset{\sim}{x}|x_0,\alpha)}{\partial \alpha} = \frac{n}{\alpha} + nlogx_0 - \sum logx_i$$

When $\alpha = \frac{n}{-nlogx_0 + \sum logx_i}$, the likelihood gets maximum value.

Therefore, the MLEs are

$$\begin{aligned} \hat{x}_0 &= x_{(1)} \\ \hat{\alpha} &= \frac{n}{-nlogx_0 + \sum logx_i} \end{aligned}$$

Problem 13 (#18 on page 462)

$$
\begin{aligned}
L(\underset{\sim}{x}|x_0,\alpha) &= \prod f(x_i|x_0,\alpha) \\
&= \frac{\alpha^n {x_0}^{n\alpha}}{(\prod x_i)^{\alpha+1}}\ 1_{\{x_0 \le x_{(1)}\}}
\end{aligned}
$$

From the likelihood function we see $(\prod x_i, x_{(1)})$ is a pair of jointly sufficient statistics. While $\hat{x}_0 = x_{(1)}$ and $\hat{\alpha} = \dfrac{n}{-nlogx_0 + \sum logx_i}$ is an one-to-one transformation from $(\prod x_i, x_{(1)})$, so $\hat{x}_0 = x_{(1)}$ and $\hat{\alpha} = \dfrac{n}{-nlogx_0 + \sum logx_i}$ is a pair of sufficient statistics. Then they are minimal sufficient statistics.