problem 6

$(\Rightarrow)$ given $\hat{\beta} = \hat{\beta}_{(i)}$

$$\hat{\underset{\sim}{\beta}} = \arg\min_{\underset{\sim}{\beta}} \sum_{k=1}^{n} (y_k - \underset{\sim}{\beta}\underset{\sim}{X}_k)^2$$

$$\hat{\underset{\sim}{\beta}}_{(i)} = \arg\min_{\underset{\sim}{\beta}} \sum_{k\neq i}^{n} (y_k - \underset{\sim}{\beta}\underset{\sim}{X}_k)^2$$

Since $\hat{\beta} = \hat{\beta}_{(i)}$, $\sum_{k=1}^{n} (y_k - \hat{\underset{\sim}{\beta}}\underset{\sim}{X}_k)^2 = \sum_{k\neq i}^{n} (y_k - \hat{\underset{\sim}{\beta}}\underset{\sim}{X}_k)^2$

$$\Rightarrow \quad y_i - \hat{\underset{\sim}{\beta}}\underset{\sim}{X}_i = 0$$

Therefore observation $i$ lies precisely on the fitted regression line.

$(\Leftarrow)$ given $y_i - \hat{\underset{\sim}{\beta}}\underset{\sim}{X}_i = 0$

$$\hat{\underset{\sim}{\beta}} = \arg\min_{\underset{\sim}{\beta}} \sum_{k=1}^{n} (y_k - \beta \underset{\sim}{X}_k)^2$$

because $y_i - \hat{\underset{\sim}{\beta}} X_i = 0$

$$\min_{\underset{\sim}{\beta}} \sum_{k=1}^{n} (y_k - \underset{\sim}{\beta}\underset{\sim}{X}_k)^2 = \min_{\underset{\sim}{\beta}} \sum_{k\neq i}^{n} (y_k - \underset{\sim}{\beta}\underset{\sim}{X}_k)^2$$

and $\hat{\underset{\sim}{\beta}}_{(i)} = \arg\min_{\underset{\sim}{\beta}} \sum_{k\neq i}^{n} (y_k - \underset{\sim}{\beta}\underset{\sim}{X}_k)^2$

Therefore $\hat{\underset{\sim}{\beta}} = \arg\min_{\underset{\sim}{\beta}} \sum_{k=1}^{n} (y_k - \underset{\sim}{\beta}\underset{\sim}{X}_k)^2 = \arg\min_{\beta} \sum_{k\neq i}^{n} (y_k - \underset{\sim}{\beta}\underset{\sim}{X}_k)^2 = \hat{\beta}_{(i)}$

$$\Rightarrow \quad \hat{\underset{\sim}{\beta}} = \hat{\underset{\sim}{\beta}}_{(i)}$$