PROBABILITY: Homework #2

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Problem 1

Denote C_1, C_2, C_3 as car in door #1, 2, 3; and E as host opens door #2

$$Pr(C_1|E) = \frac{Pr(E|C_1)Pr(C_1)}{Pr(E)}$$

where

$$Pr(C_1) = \frac{1}{3} \text{ and } Pr(E) = \frac{1}{2}$$

In order to make $Pr(C_1|E) = Pr(C_3|E) = \frac{1}{2}$,

we need
$$\frac{1}{2} = \frac{\frac{1}{3} \times Pr(E|C_1)}{\frac{1}{2}}$$

So that, $Pr(E|C_1) = \frac{3}{4}$

Therefore,

$$p = \frac{3}{4}$$

Problem 2

(a)

E and *F* are independent events. Because owing a car will not influence whether the name listed in telephone book and neither will listed in book influence owing a car.

(b)

E and *F* are not independent events. Because people's height and weight shows linear relationship to some extent.

(c)

E and *F* are not independent events. Because the United States is in the western hemisphere, which means E and F will happen on the same time.

(d)

E and *F* are not independent events. Because the weather of two adjacent days shows somewhat relationship. And we cannot say they are irrelevant.

Problem 3

(a)

In order to stop the game in 4^{th} game, the only possible case is one win 1 game and the other win 3 games. And the one win 1 game must win in round 1^{st} or 2^{nd}.

$$\begin{aligned} P &= 2 \times p^3(1-p) + 2 \times (1-p)^3 p \\ &= 2p(1-p)(p^2 + (1-p)^2) \end{aligned}$$

(b)

Denote *A, B* as the games player A, B win. Since *A*-*B*=2, *A*+*B* must be an even number. Denote *P*(*n*) as the probability that player A wins in n games, and n must be even.

When A wins in n games, the n^{th} game winner must be A. And in $(n-1)^{\text{th}}$ game, $A - B = 1$. Supposing A wins in $n+2$ games, then at the n^{th} game winner must be B, and at $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$ game winner must be A. So:

$$\begin{aligned} P(n+2) &= \frac{P(n)}{p} * (1-p) * p^2 \\ &= P(n) * (1-p)p \\ &= P(n-2) * ((1-p)p)^2 \\ &= \ldots \\ &= P(4) * ((1-p)p)^{\frac{n-4}{2}} \end{aligned}$$

Which means $P(2n) = P(4) * ((1-p)p)^{n-3}$

$$\sum_{n=1}^{\infty} P(2n) = \sum_{n=1}^{\infty} P(4) * ((1-p)p)^{n-3} \rightarrow \frac{P(4) * ((1-p)p)^{-2}}{1-(1-p)p}$$

which is

$$\frac{2p(1-p)(p^2+(1-p)^2) * (1-p)^{-2}p^{-2}}{1-(1-p)p} = \frac{2(p^2+(1-p)^2)}{p(1-p)[1-(1-p)p]}$$

Problem 4

Obviously, X can only be 1, 2, 3, 4, 5, 6.

$$P\{X=1\} = C_5^1 * \frac{9!}{10!} = \frac{1}{2}$$

$$P\{X=2\} = C_5^1 * C_5^1 * \frac{8!}{10!} = \frac{5}{18}$$

$$P\{X=3\} = C_5^2 * C_5^1 * 2! * \frac{7!}{10!} = \frac{5}{36}$$

$$P\{X=4\} = C_5^3 * C_5^1 * 3! * \frac{6!}{10!} = \frac{5}{84}$$

$$P\{X=5\} = C_5^4 * C_5^1 * 4! * \frac{5!}{10!} = \frac{5}{252}$$

$$P\{X=6\} = C_5^5 * C_5^1 * 5! * \frac{4!}{10!} = \frac{1}{252}$$

$$P\{X=i\} = C_5^{i-1} \times C_5^1 \times (i-1)! \times \frac{(10-i)!}{10!}, \qquad i = 1,2,3,4,5,6$$

Problem 5

$$\text{when } x \notin \{0,\ 1,\ 2,\ 3,\ 3.5\}, \quad f(x) = F(x) - \lim_{y \to x^-} F(y) = 0$$

$$
\begin{aligned}
f(0) &= F(0) - \lim_{y\to 0^-} F(y) = \frac{1}{2} - 0 = \frac{1}{2} \\
f(1) &= F(1) - \lim_{y\to 1^-} F(y) = \frac{3}{5} - \frac{1}{2} = \frac{1}{10} \\
f(2) &= F(2) - \lim_{y\to 2^-} F(y) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5} \\
f(3) &= F(3) - \lim_{y\to 3^-} F(y) = \frac{9}{10} - \frac{4}{5} = \frac{1}{10} \\
f(3.5) &= F(3.5) - \lim_{y\to 3.5^-} F(y) = 1 - \frac{9}{10} = \frac{1}{10}
\end{aligned}
$$

Problem 6

(a)

$$
\begin{aligned}
P(X=1) &= F(1) - \lim_{y\to 1^-} F(y) = \frac{1}{2} + \frac{b-1}{4} - \frac{b}{4} = \frac{1}{4} \\
P(X=\frac{1}{2}) &= F(\frac{1}{2}) - \lim_{y\to \frac{1}{2}^-} F(y) = \frac{1/2}{4} - \frac{1/2}{4} = 0 \\
P(X=3) &= F(3) - \lim_{y\to 3^-} F(y) = 1 - \frac{11}{12} = \frac{1}{12}
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(\frac{1}{2} < X < \frac{3}{2}) &= P(\frac{1}{2} < X < 1) + P(1 \le X < \frac{3}{2}) \\
&= \lim_{y\to 1^-} F(y) - F(\frac{1}{2}) + F(\frac{3}{2}) - F(1) \\
&= (\frac{1}{4} - \frac{1/2}{4}) + (\frac{1}{2} + \frac{3/2-1}{4} - \frac{1}{2}) \\
&= \frac{1}{4}
\end{aligned}
$$

Problem 7

(a)

$$f(2n) = (1-p_1)^n(1-p_2)^{(n-1)} * p_2 \qquad n \in \mathbb{N}$$
$$f(2n+1) = (1-p_1)^n(1-p_2)^n * p_1 \qquad n \in \mathbb{Z}^+$$

(b)

Let player wins, then they must play $2n+1$ times.

$$\begin{aligned} P &= \sum_{n=0}^{\infty} f(2n+1) \\ &= \sum_{n=0}^{\infty}(1-p_1)^n(1-p_2)^n * p_1 \\ &= \frac{p_1}{1-(1-p_1)(1-p_2)} \end{aligned}$$

Problem 8

(a)

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 f(x)dx = \int_0^1 cx^2dx = \frac{1}{3}cx^3\Big|_0^1 = \frac{1}{3}c$$

So *c* should be 3.

(b)

When $x \in [0,\ 1]$,

$$F(x) = \int_0^x f(x)dx = \int_0^1 3x^2dx = x^3\Big|_0^x = x^3$$

When $x \in [-\infty,\ 0)$, $F(x) = 0$

When $x \in (1, \infty]$, $F(x) = 1$

Problem 9

(a)

	1	2	3	4	Y
1	.10	.05	.02	.02	**.19**
2	.05	.20	.05	.02	**.32**
3	.02	.05	.20	.04	**.31**
4	.02	.02	.04	.10	**.18**
X	**.19**	**.32**	**.31**	**.18**	

(b)

$$P(X = 1; Y = 1) = .01 \qquad P(X = 1) = P(Y = 1) = .19$$
$$P(X = 1; Y = 1) \neq P(X = 1) \times P(Y = 1)$$

Therefore, X and Y are not independent.

(c)

$$f(x|Y = 1) = \frac{P(X = x; Y = 1)}{P(Y = 1)} = \begin{cases} \frac{.10}{.19} = \frac{10}{19}, & x = 1 \\ \frac{.05}{.19} = \frac{5}{19}, & x = 2 \\ \frac{.02}{.19} = \frac{2}{19}, & x = 3 \\ \frac{.02}{.19} = \frac{2}{19}, & x = 4 \end{cases}$$

$$f(y|X = 1) = \frac{P(Y = y; X = 1)}{P(X = 1)} = \begin{cases} \frac{.10}{.19} = \frac{10}{19}, & y = 1 \\ \frac{.05}{.19} = \frac{5}{19}, & y = 2 \\ \frac{.02}{.19} = \frac{2}{19}, & y = 3 \\ \frac{.02}{.19} = \frac{2}{19}, & y = 4 \end{cases}$$

Problem 10

Since the point is chosen uniformly, we can get the joint pdf easily:

$$f(x,y) = C \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

In order to get the value of C, we need to integral

$$\begin{aligned} 1 &= \int_{-a}^{a} \int_{-\sqrt{b^2 - \frac{b^2x^2}{a^2}}}^{\sqrt{b^2 - \frac{b^2x^2}{a^2}}} C \; dydx \\ &= C \int_{-a}^{a} \int_{-\sqrt{b^2 - \frac{b^2x^2}{a^2}}}^{\sqrt{b^2 - \frac{b^2x^2}{a^2}}} \; dydx \end{aligned}$$

This integral is equal to C times the area of ellipse, which is πab. Thus, $C = \frac{1}{ab}$

Now we compute the marginal density of X:

$$\begin{aligned} f_X(x) &= \int_{-\sqrt{b^2 - \frac{b^2x^2}{a^2}}}^{\sqrt{b^2 - \frac{b^2x^2}{a^2}}} \frac{1}{ab} \; dy \\ &= \frac{2}{ab}\sqrt{b^2 - \frac{b^2x^2}{a^2}} \\ &= 2\sqrt{\frac{1}{a^2} - \frac{x^2}{a^4}} \qquad x \in [-a, a] \end{aligned}$$

Then we compute the marginal density of Y:

$$\begin{aligned} f_Y(y) &= \int_{-\sqrt{a^2 - \frac{a^2y^2}{b^2}}}^{\sqrt{a^2 - \frac{a^2y^2}{b^2}}} \frac{1}{ab} \; dy \\ &= \frac{2}{ab}\sqrt{a^2 - \frac{a^2y^2}{b^2}} \\ &= 2\sqrt{\frac{1}{b^2} - \frac{y^2}{b^4}} \qquad y \in [-b, b] \end{aligned}$$

Problem 11

(a)

First, we can get the joint pdf is

$$f_{XY}(x,y)=\frac{\partial^2 F(x,y)}{\partial x \partial y}=\alpha\beta e^{-\alpha x-\beta y}$$

Then we compute the marginal density of X:

$$f_X(x)=\int_0^\infty \alpha\beta e^{-\alpha x-\beta y}\,dy=\alpha e^{-\alpha x} \qquad x\geq 0$$

And now we compute the marginal density of Y:

$$f_Y(y)=\int_0^\infty \alpha\beta e^{-\alpha x-\beta y}\,dx=\beta e^{-\beta x} \qquad y\geq 0$$

Since $f(x,y)=f_X(x)f_Y(y)$, X and Y are independent.

(b)

In the previous part we have already get that

$$f_{XY}(x,y)=\alpha\beta e^{-\alpha x-\beta y} \qquad x\geq 0,\quad y\geq 0$$
$$f_X(x)=\alpha e^{-\alpha x} \qquad x\geq 0$$
$$f_Y(y)=\beta e^{-\beta x} \qquad y\geq 0$$

Problem 12

(a)

$$\begin{aligned}1=&\int_0^\infty\int_{-x}^{x} c(x^2-y^2)e^{-x}\,dydx\\=&\int_0^\infty (2c-\frac{2c}{3})x^3e^{-x}\,dx\\=&(2c-\frac{2c}{3})(-x^3-3x^2-6x-6)e^{-x}\Big|_0^\infty\\=&8c\end{aligned}$$

批注 [杨帆1]:

So $c = \dfrac{1}{8}$

(b)

First, we compute the marginal density of X:

$$\begin{aligned} f_X(x) &= \int_{-x}^{x} \frac{1}{8}(x^2 - y^2)e^{-x}\ dy \\ &= \frac{1}{6}x^3 e^{-x} \qquad x \geq 0 \end{aligned}$$

Then now we compute the marginal density of Y:

$$\begin{aligned} f_Y(y) &= \int_{0}^{\infty} \frac{1}{8}(x^2 - y^2)e^{-x}\ dx \\ &= \frac{1}{8}(-x^2 - 2x - 2 + y^2)e^{-x}\Big|_0^\infty \\ &= \frac{1}{8}(2 - y^2) \qquad -x \leq y \leq x \end{aligned}$$

Since $f(x, y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

(c)

From the previous part, we already computed the marginal densities:

$$f_X(x) = \frac{1}{6}x^3 e^{-x} \qquad x \geq 0$$

$$f_Y(y) = \frac{1}{8}(2 - y^2) \qquad -x \leq y \leq x$$

(d)

Given y:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x, y)}{f_Y(y)} \\ &= \frac{\frac{1}{8}(x^2 - y^2)e^{-x}}{\frac{1}{8}(2 - y^2)} \\ &= \frac{(x^2 - y^2)e^{-x}}{2 - y^2} \qquad x \geq 0 \end{aligned}$$

Given x:

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\
&= \frac{\frac{1}{8}(x^2-y^2)e^{-x}}{\frac{1}{6}(x^3e^{-x})} \\
&= \frac{3(x^2-y^2)}{4x^3} \qquad -x \le y \le x
\end{aligned}
$$

Problem 13

$Y = g(X) = aX + b$, $g(X)$ is a monotone transformation. $g^{-1}(Y) = \frac{Y-b}{a}$. Then

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{\partial g^{-1}(y)}{\partial y}\Big| \\
&= f_X(\frac{y-b}{a})\Big|\frac{\partial \frac{y-b}{a}}{\partial y}\Big| \\
&= f_X(\frac{y-b}{a})\Big|\frac{1}{a}\Big| \\
&= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)
\end{aligned}
$$

Problem 14

$$
\begin{aligned}
P(Y=k) &= P\left(F(k-1) < U \le F(k)\right) \\
&= P\left(U \le F(k)\right) - P\left(U \le F(k-1)\right) \\
&= F(k) - F(k-1)
\end{aligned}
$$

Which means Y has cdf F.

Problem 15

(a)

Suppose that $P(X = x) = c(x+1)(8-x)$, where c is a constant number.

Then we have

$$\begin{aligned}1 &= \sum_{x=0}^{7} f(x)\\ &= \sum_{x=0}^{7} c(x+1)(8-x)\\ &= c(8+14+18+20+20+18+14+8)\\ &= 120c\end{aligned}$$

Thus, $c = \frac{1}{120}$

$$f(x) = \frac{1}{120}(x+1)(8-x) \qquad x = 0, ..., 7$$

(b)

$$\begin{aligned}P(X \geq 5) &= \sum_{x=5}^{7} f(x)\\ &= \sum_{x=5}^{7} \frac{1}{120}(x+1)(8-x)\\ &= \frac{1}{120}(18+14+8)\\ &= \frac{1}{3}\end{aligned}$$