STATISTICAL METHODS IN FINANCE

HW1

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Problem 1 (Page 15 Prob 09)

Prob 09. In this simulation, what are the mean and standard deviation of the log-returns for 1 year?

$$R_t(20) = 1 + \frac{100-97}{97} = \frac{100}{97}$$

$$1 + R_t(253) = \exp\{\sum_{i=0}^{252} \tilde{R}_{t-i}\}$$

$$\sum_{i=0}^{252} \tilde{R}_{t-i} = \log(1 + R_t(253))$$

Since the daily log returns on a stock are independent and normally distributed. So, $\sum_{i=0}^{252} \tilde{R}_{t-i} \sim N(0.05/253 \times 253, 0.2/\sqrt{253} \times \sqrt{253})$

Answer. In this simulation, the mean is $0.05/253 * 253 = 0.05$, and standard deviation is $0.2/\sqrt{253} * \sqrt{253} = 0.2$

Problem 2 (Page 15 Prob 11)

Prob 11. Explain what the code `c(120,120*exp(cumsum(logr)))` ***does.***

This code gives a vector of price of the asset. The index represents the time. For example, the first element of this vector is the initial price of asset while the i^{th} elemtent is the $(i-1)^{th}$ trading day.

Since $P_t = P_0(1 + R_t(t)) = P_0 e^{\tilde{R}_t}$, where $\tilde{R}_t$ is the t^{th} trading day log-return. And for this case, $P_0 = 120$ and time series of $\tilde{R}$ is computed by cumsum(logr).

Problem 3 (Page 15 Prob 12)

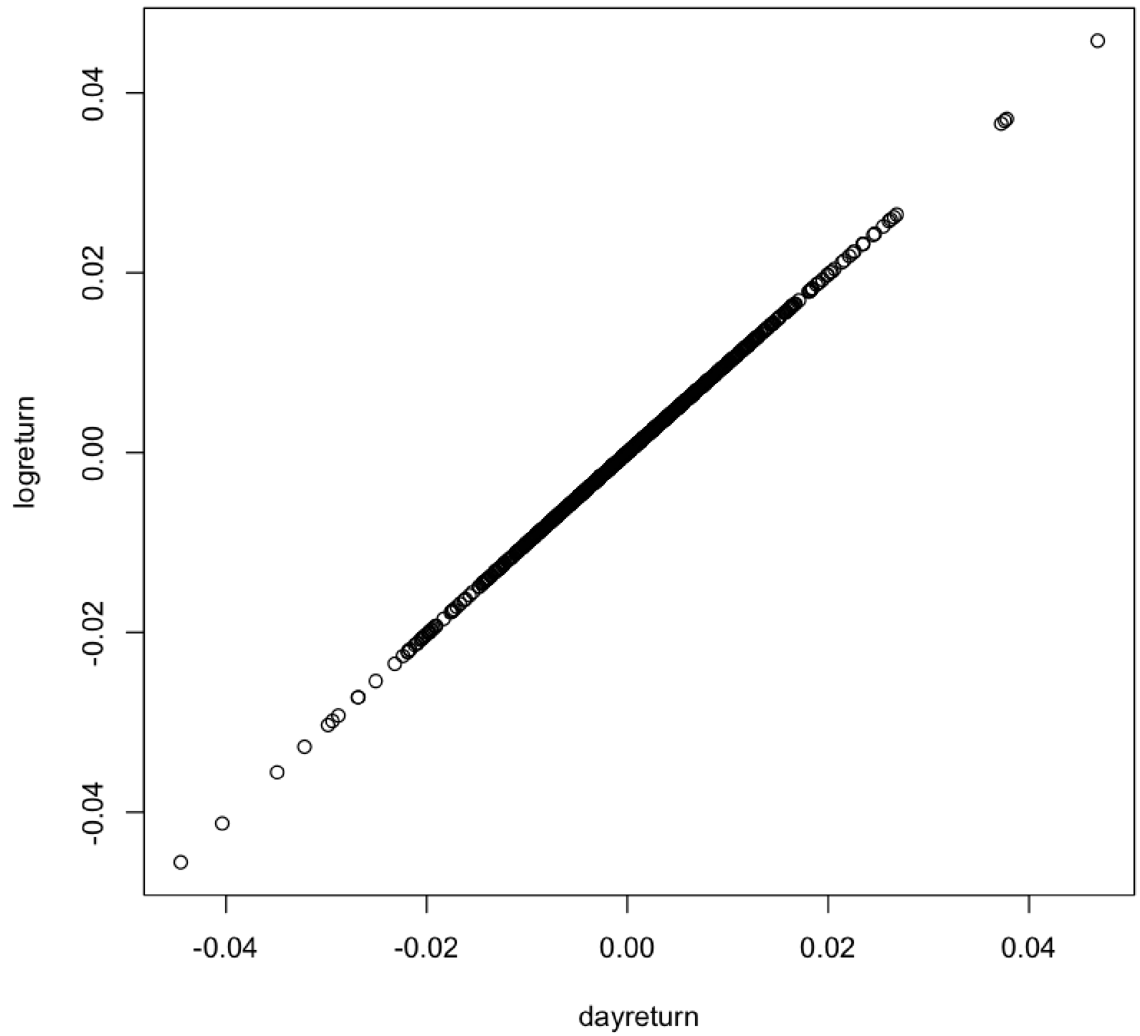
Prob 12. Compute the returns and log returns and plot them against each other. As discussed in Sect. 2.1.3, does it seem reasonable that the two types of daily returns are approximately equal?

In [2]:
```
data = read.csv("./datasets/MCD_PriceDaily.csv")
head(data)
adjPrice = data[, 7]
```

Date	Open	High	Low	Close	Volume	Adj.Close
1/4/2010	62.63	63.07	62.31	62.78	5839300	53.99
1/5/2010	62.66	62.75	62.19	62.30	7099000	53.58
1/6/2010	62.20	62.41	61.06	61.45	10551300	52.85
1/7/2010	61.25	62.34	61.11	61.90	7517700	53.24
1/8/2010	62.27	62.41	61.60	61.84	6107300	53.19
1/11/2010	62.02	62.43	61.85	62.32	6081300	53.60

Compute returns and log returns.

In [9]:
```
dayreturn = adjPrice[-1] / adjPrice[-length(adjPrice)] -1
logreturn = log(dayreturn+1)
plot(dayreturn,logreturn)
```



From the plot above, we can see that the day return and log return are almost equal. As we discussed in 2.1.3 that $\log(1 + x) \approx x$, it is reasonable.

Problem 4 (Page 16 Exec 1)

Suppose that the daily log returns on a stock are independent and normally distributed with mean 0.001 and standard deviation 0.015. Suppose you buy $1,000 worth of this stock.

> ***(a)*** *What is the probability that after one trading day your investment is worth less than $990? (Note: The R function `pnorm()` will compute a normal CDF, so, for example, `pnorm(0.3, mean = 0.1, sd = 0.2)` is the normal CDF with mean 0.1 and standard deviation 0.2 evaluated at 0.3.)*
> ***(b)*** *What is the probability that after five trading days your investment is worth less than $990?*

(a).

$$\begin{aligned}\tilde{R}_t &= \log(1+R_t) = \log(1+\frac{990-1000}{1000}) \\ &= \log 0.99\end{aligned}$$

In [15]:
```
pnorm(log(0.99),0.001,0.015)
```

0.230655731554758

The probability that after one trading day your investment is worth less than $990 is 0.23065573.

(b).

$$\begin{aligned}R_t(5) = 1 + \frac{990-1000}{1000} &= 0.99 \\ 1 + R_t(5) &= \exp\{\tilde{R}_t + \tilde{R}_{t-1} + \tilde{R}_{t-2} + \tilde{R}_{t-3} + \tilde{R}_{t-4}\} \\ \tilde{R}_t + \tilde{R}_{t-1} + \tilde{R}_{t-2} + \tilde{R}_{t-3} + \tilde{R}_{t-4} &= \log(1+R_t(5)) = \log(0.99)\end{aligned}$$

Since the daily log returns on a stock are independent and normally distributed. So, $\sum_{i=0}^{4} \tilde{R}_{t-i} \sim N(0.001 \times 5, 0.015 \times \sqrt{5})$

In [3]:
```
pnorm(log(0.99),0.001*5,0.015*sqrt(5))
```

0.326818876324785

The probability that after five trading day your investment is worth less than $990 is 0.326818876.

Problem 5 (Page 16 Exec 3)

The yearly log returns on a stock are normally distributed with mean 0.08 and standard deviation 0.15. The stock is selling at $80 today. What is the probability that 2 years from now it is selling at $90 or more?

$$R_t(2) = 1 + \frac{90-80}{80} = \frac{9}{8}$$
$$1 + R_t(2) = \exp\{\tilde{R}_t + \tilde{R}_{t-1}\}$$
$$\tilde{R}_t + \tilde{R}_{t-1} = \log(1 + R_t(2)) = \log(\frac{9}{8})$$

Since the daily log returns on a stock are independent and normally distributed. So,
$\sum_{i=0}^{1} \tilde{R}_{t-i} \sim N(0.08 \times 2, 0.15 \times \sqrt{2})$

In [45]:
```
1-pnorm(log(9/8),0.08*2,0.15*sqrt(2))
```

0.578873586504166

The probability that before 2 trading years your investment is worth at $90 is 0.57887.

Problem 6 (Page 17 Exec 10)

The daily log returns on a stock are normally distributed with mean 0.0002 and standard deviation 0.03. The stock price is now $97. What is the probability that it will exceed $100 after 20 trading days?

$$R_t(20) = 1 + \frac{100-97}{97} = \frac{100}{97}$$
$$1 + R_t(20) = \exp\{\sum_{i=0}^{19} \tilde{R}_{t-i}\}$$
$$\sum_{i=0}^{19} \tilde{R}_{t-i} = \log(1 + R_t(20)) = \log(\frac{100}{97})$$

Since the daily log returns on a stock are independent and normally distributed. So,
$\sum_{i=0}^{19} \tilde{R}_{t-i} \sim N(0.0002 \times 20, 0.03 \times \sqrt{20})$

In [15]:
```
1-pnorm(log(100/97),0.0002*20,0.03*sqrt(20))
```

0.421829533513185

The probability that before 20 trading years your investment exceed \$100 is 0.4218295.

Problem 7 (Page 40 Exec 1)

Suppose that the forward rate is r(t) = 0.028 + 0.00042t.

> (a) What is the yield to maturity of a bond maturing in 20 years?
> (b) What is the price of a par \$1,000 zero-coupon bond maturing in 15 years?

(a).

$$\begin{aligned} y(t) &= \frac{1}{t}\int_0^t r(s)ds \\ &= \frac{1}{20}\int_0^{20} 0.028 + 0.00042s\ ds \\ &= \frac{0.644}{20} \\ &= 0.0322 \end{aligned}$$

(b).

$$\begin{aligned} P(t) &= PAR \times e^{-\int_0^t r(s)ds} \\ &= 1000 \times e^{-\int_0^{15} 0.028+0.00042s\ ds} \\ &= 1000 \times e^{-0.46725} \\ &= 626.723 \end{aligned}$$

Problem 8 (Page 40 Exec 3)

A coupon bond has a coupon rate of 3% and a current yield of 2.8%.

> (a) Is the bond selling above or below par? Why or why not?
> (b) Is the yield to maturity above or below 2.8%? Why or why not?

(a).

$$
\begin{aligned}
C &= \text{PAR} \times \text{coupon rate} = \text{PAR} \times 0.03 \\
C &= \text{Price} \times \text{current yield} = \text{Price} \times 0.028 \\
\text{so, Price} &> \text{PAR}
\end{aligned}
$$

The bond selling is above par.

(b).

The bond is selling above par value, then the coupon rate is greater than the current yield because the bond sells above par value, and the current yield is greater than the yield to maturity because the yield to maturity accounts for the loss of capital when at the maturity date you get back only the par value, not the entire investment. In summary,

coupon rate > current yield > yield to maturity.

Problem 9 (Page 40 Exec 8)

A par \$1,000 zero-coupon bond that matures in 5 years sells for \$828. Assume that there is a constant continuously compounded forward rate r.

> (a) What is r?
> (b) Suppose that 1 year later the forward rate r is still constant but has changed to be 0.042. Now what is the price of the bond?
> (c) If you bought the bond for the original price of \$828 and sold it 1 year later for the price computed in part (b), then what is the net return?

(a).

$$
\begin{aligned}
P(t) &= \text{PAR} \times e^{-\int_0^t r(s)ds} \\
&= \text{PAR} \times e^{-rt} \\
r &= \frac{-\log(\frac{P(t)}{\text{PAR}})}{t} = \frac{-\log(\frac{828}{1000})}{5} \\
&= 0.037748
\end{aligned}
$$

(b).

Suppose r_0 is the original r and r_1 is the forward rate after 1 year.

$$\begin{aligned} P(t) &= \text{PAR} \times e^{-\int_1^5 r_1 ds} \\ &= \text{PAR} \times e^{-4r_1} \\ &= 1000 \times e^{-4*0.042} \\ &= 845.3538 \end{aligned}$$

(c).

$$\begin{aligned} R_t &= \frac{P_t - P_{t-5}}{P_{t-5}} \\ &= \frac{845.3538 - 828}{828} \\ &= 0.0209587 \end{aligned}$$

Problem 10 (Page 41 Exec 11)

Suppose that the continuous forward rate is $r(t) = 0.033 + 0.0012t$. What is the current value of a par \$100 zero-coupon bond with a maturity of 15 years?.

$$\begin{aligned} P(t) &= \text{PAR} \times e^{-\int_0^t r(s)ds} \\ &= 100 \times e^{-\int_0^{15} 0.033+0.0012s\, ds} \\ &= 100 \times 0.532592 \\ &= 53.2592 \end{aligned}$$

Problem 11 (Page 41 Exec 12)

Suppose the continuous forward rate is $r(t) = 0.04 + 0.001t$ when a 8-year zero coupon bond is purchased. Six months later the forward rate is $r(t) = 0.03 + 0.0013t$and bond is sold. What is the return?

$$
\begin{aligned}
P_1(t) &= \text{PAR} \times e^{-\int_0^t r(s)ds} \\
&= \text{PAR} \times e^{-\int_0^8 \ 0.04+0.001s\ ds} \\
&= \text{PAR} \times 0.70328 \\
P_2(t) &= \text{PAR} \times e^{-\int_0^t r(s)ds} \\
&= \text{PAR} \times e^{-\int_0^{7.5} \ 0.03+0.0013s\ ds} \\
&= \text{PAR} \times 0.769848 \\
R_t &= \frac{P_2(t) - P_1(t)}{P_1(t)} \\
&= \frac{0.769848 - 0.70328}{0.70328} \\
&= 0.0946536
\end{aligned}
$$

Problem 12 (Page 42 Exec 16)

Assume that the yield curve is $Y_T = 0.04 + 0.001T$ ***.***

> (a) What is the price of a par-$1,000 zero-coupon bond with a maturity of 10 years?
 (b) Suppose you buy this bond. If 1year later the yield curve is $Y_T = 0.042 + 0.001T$, then what will be the net return on the bond

(a).

$$
\begin{aligned}
\text{Price} &= \text{PAR} \times e^{-TY_t} \\
&= 1000 \times e^{10\times(-0.04-0.001\times 10)} \\
&= 606.5306597
\end{aligned}
$$

(b).

$$\begin{aligned}\text{Price}_1 &= \text{PAR} \times e^{-TY_t} \\ &= 1000 \times e^{(10-1)\times(-0.042-0.001\times(10-1))} \\ &= 631.9152 \\ R_t &= \frac{\text{Price}_1 - \text{Price}}{\text{Price}} \\ &= \frac{631.9152 - 606.5306597}{606.5306597} \\ &= 0.041852\end{aligned}$$

Problem 13 (Page 43 Exec 22)

A coupon bond matures in 4 years. Its par is $1,000 and it makes eight coupon payments of $21, one every one-half year. The continuously compounded forward rate is

$$r(t) = 0.022 + 0.005t - 0.004t^2 + 0.0003t^3.$$

> (a) Find the price of the bond.
> (b) Find the duration of this bond.

(a).

First we need to calculate the every coupon payments.

$$\begin{aligned}P(1) &= 21 \times e^{-\int_0^t r(s)ds} \\ &= 21 \times e^{-\int_0^{0.5} r(s)ds} \\ p(n) &= 21 \times e^{-\int_0^{0.5n} r(s)ds} \\ p(8) &= (1000 + 21) \times e^{-\int_0^{0.5\times 8} r(s)ds}\end{aligned}$$

```
In [40]: intg = c()
for (i in 1:7){
    intg[i] = 21*exp( -integrate(
        function(x) 0.022+0.005*x-0.004*(x^2)+0.0003*x^3,0,0.5*i
        )$value)
    }
intg[8] = 1021*exp(-integrate(
        function(x) 0.022+0.005*x-0.004*(x^2)+0.0003*x^3,0,0.5*8
        )$value)
sum(intg)
```

1100.86966145191

So, the price of the bond is 1100.86966.

(b).

$$\text{DUR} = \frac{1}{\text{Price}}\left(\sum_{i=1}^{8}\frac{C}{e^{rt_i}}t_i + \frac{\text{PAR}}{e^{rT}}T\right)$$

```
In [44]: T = c(1:8)/2
sum(intg*T) / sum(intg)
```

3.74141886139914

So, the duration of this bond is 3.74141886.

Other questions

Problem 14 $Q(1)$

If X is a continuous random variable with a strictly increasing distribution function F, find the distribution of U = F(X).

$$
\begin{aligned}
\text{when } u \le 0,\ Pr(U < u) &= 0 \\
\text{when } u \ge 1,\ Pr(U < u) &= 1 \\
\text{when } 0 < u < 1,\ Pr(U < u) &= Pr(F(X) < u) \\
&= Pr\left(X < F^{-1}(u)\right) \quad \text{because F is continuous} \\
&= F\left(F^{-1}(u)\right) \\
&= u
\end{aligned}
$$

Problem 15 $Q(2)$

Let X have a normal distribution with mean μ and variance σ^2 and let $Y = e^X$. Y is said to have a lognormal distribution with parameters μ and σ^2 (since $X = \log Y$ has a normal distribution).

> (a) Find the density f_Y . (Hint: compute $F_Y(y) = P(Y \le y)$).
> (b) Find the mean and the variance of Y. (Hint: if $X \sim N(\mu, \sigma^2)$,then$E(e^{tX}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$).

(a).

$$
\begin{aligned}
F_Y(y) &= Pr(Y \le y) = Pr(e^X \le y) \\
&= Pr(X \le \log y) \\
&= F_X(\log y) \\
f_Y(y) &= \frac{d\,F_Y(y)}{d\,y} = \frac{d\,F_X(\log y)}{d\,y} \\
&= \frac{1}{y} f_X(\log y) \\
&= \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(\log y - \mu)^2}{2\sigma^2} \right\}
\end{aligned}
$$

(b).

$$
\begin{aligned}
E(e^{tX}) &= e^{\mu t+\frac{1}{2}\sigma^2 t^2} \\
\text{let } t=1, \text{then } E(Y) &= E(e^{X}) = e^{\mu+\frac{1}{2}\sigma^2} \\
\text{let } t=2, \text{then } E(Y^2) &= E(e^{2X}) = e^{2\mu+2\sigma^2} \\
Var(Y) &= E(Y^2) - E^2(Y) \\
&= e^{2\mu+2\sigma^2} - e^{2\mu+\frac{2}{2}\sigma^2} \\
&= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}
\end{aligned}
$$

Problem 16 $_{Q(3)}$

Find the largest eigenvalue of the following matrices:

(i) $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.

(ii) $\begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(i).

Denote A as the matrix.

$$
\begin{aligned}
|A-\lambda I| &= \begin{vmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 - \rho^2 \\
&= (1-\lambda+\rho)(1-\lambda-\rho) \\
\text{so, } \lambda &= 1+\rho, \ \ 1-\rho. \\
\text{if } \rho > 0, \ \text{then } \lambda_{\max} &= 1+\rho \\
\text{otherwise, } \lambda_{\max} &= 1-\rho
\end{aligned}
$$

(ii).

Denote A as the matrix.

$$
\begin{aligned}
|A-\lambda I| &= \begin{vmatrix} 1-\lambda & \rho & 0 \\ \rho & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\left[(1-\lambda)^2-\rho^2\right] \\
&= (1-\lambda)(1-\lambda+\rho)(1-\lambda-\rho) \\
\text{so, } \lambda &= 1,\ \ 1+\rho,\ \ 1-\rho. \\
\text{if } \rho>0,\ \text{then } \lambda_{\max} &= 1+\rho \\
\text{if } \rho<0,\ \text{then } \lambda_{\max} &= 1-\rho \\
\text{if } \rho=0,\ \text{then } \lambda_{\max} &= 1
\end{aligned}
$$

Problem 17 $_{Q(4)}$

Suppose that X and Y are independent and identically distributed (iid) exponentially distributed random variables with $Pr(X \geq t) = P(Y \geq t) = e^{-t}$. Are $X+Y$ and $X-Y$ uncorrelated? Are $X+Y$ and $X-Y$ independent? Explain your answer

uncorrelated

$$
\begin{aligned}
cov(X+Y, X-Y) &= cov(X, X-Y) + cov(Y, X-Y) \\
&= cov(X,X) - cov(X,Y) + cov(Y,X) - cov(Y,Y) \\
&= Var(X) - Var(Y)
\end{aligned}
$$

Since X and Y are independent and identically distributed, $Var(X) = Var(Y)$

So $cov(X+Y, X-Y) = 0$, $X+Y$ and $X-Y$ are uncorrelated.

not independent

$f_X(t) = f_Y(t) = e^{-t}$,
$U = X + Y, V = X - Y$,
X and Y are iid, so $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) = e^{-x-y}$,

$$\begin{aligned} f_{X,Y}(x,y) &= f_{U,V}(u,v) \cdot \left| \det \begin{pmatrix} \partial u / \partial x & \partial u / \partial y \\ \partial v / \partial x & \partial v / \partial y \end{pmatrix} \right| \\ &= f_{U,V}(u,v) \cdot \left| \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| \\ &= 2 \cdot f_{U,V}(u,v) \end{aligned}$$

$f_{U,V}(u,v) = \frac{1}{2} f_{X,Y}(x,y) = \frac{1}{2} e^{-x-y} = \frac{1}{2} e^{-u},\ (u > |v|)$,
We can easily get that
$f_U(u) = ue^{-u},\ (u > 0)$,
$f_V(v) = \frac{1}{2} e^{|v|},\ (-u < v < u)$,
Since $f_{U,V}(u,v) \neq f_U(u) \cdot f_V(v)$.
Therefore, $X + Y$ and $X - Y$ are not independent.