GR5261/GU4261 Statistical Methods in Finance
Homework 5 (due on March 1, 2018; online submission only)

1. Show that the following inequalities are always true for any copula function C:

$$C^-(u,v) \le C(u,v) \le C^+(u,v).$$

Recall that $C^-(u,v) = \max(u+v-1,0)$ and $C^+(u,v) = \min(u,v)$.

2. Kendall's tau rank correlation between X and Y is 0.55. Both X and Y are strictly positive random variables. What is Kendall's tau between X and 1/Y ? What is the Kendall's tau between 1/X and 1/Y ?

3. Suppose that X is Uniform(0,1) and $Y = X^2$. Then the Spearman rank correlation and the Kendall's tau between X and Y will both equal 1, but the Pearson correlation between X and Y will be less than 1. Explain why this is the case.

4.* (Optional) Recall that the bivariate Gumbel copula takes the form

$$C_\alpha(u,v) = \exp\{-[(-\log u)^\alpha + (-\log v)^\alpha]^{\frac{1}{\alpha}}\},$$

where $\alpha \in [1,\infty)$. Show that, as $\alpha \to \infty$, $C_\alpha(u,v) \to C^+(u,v) = \min(u,v)$.

5. We have two B-rated bonds, with one-year default probability at 3.46%. Suppose that the interest rate is 4%, and their default times satisfy the Gaussian copula with $\rho = 0.5$, calculate the following expected present values, i.e. fair prices.

   (a) What is the fair price of a first-to-default swap which pays $1,000,000 if at least one of them defaults by the end of the first year?

   (b) How about a second-to-default swap which pays $1,000,000 if both default in the first year?

6. Suppose that we have two bonds A and B. Denote by T_A and T_B their respective default times (in year). Suppose that T_A follows exponential distribution with hazard $\lambda_A = 0.01$ (i.e. $P(T_A \ge t) = e^{-\lambda_A t}$) and T_B follows exponential with hazard $\lambda_B = 0.02$. Suppose that jointly they satisfy the Gumbel copula with $\alpha = 2$. Find the probabilities that (i) both will default by the end of the first year; (ii) at least one will default by the end of the first year.

7. Continue from the preceding problem. Suppose that a policy pays 1 million dollars if both A and B default by the end of the first year. If the interest rate is 0, what would be fair value of this policy? What if $\alpha = 1$ instead of 2?