P40-43.

1. (a) $y_T = \frac{1}{T}\int_0^T r(t)\,dt$.

$y_{20} = \frac{1}{20}\int_0^{20} (0.028+0.00042t)\,dt$

$= \frac{1}{20}\cdot\left[0.028t + 0.00042\, t^2/2\right]\Big|_0^{20} = 0.0322$

(b) $P = 1000\times D(15) = 1000\cdot e^{-\int_0^{15} 0.028+0.00042t\,dt}$

$= 1000\cdot e^{-\left[0.028t+0.00042t^2/2\right]\Big|_0^{15}} = 626.72$

3. (a) The bond is selling above par, because the coupon rate > current yield.

~~由~~ current yield $= \frac{\text{coupon}}{\text{Price}}$, coupon rate $= \frac{\text{coupon}}{\text{Par}}$.

we have $\frac{\text{coupon}}{\text{Price}} < \frac{\text{coupon}}{\text{Par}} \Rightarrow \text{Price} > \text{Par}$.

(b). The yield to maturity is below 2.8%.

$\text{Price} = \sum_{t=1}^{T} \frac{\text{Coupon}}{(1+YTM)^t} + \frac{\text{Par}}{(1+YTM)^T}$

$\Rightarrow \frac{\text{Coupon}}{\text{Price}} = \frac{1 - \frac{1}{P}\cdot\frac{\text{Par}}{(1+YTM)^T}}{\sum_{t=1}^{T}\frac{1}{(1+YTM)^t}}$

$\sum_{t=1}^{T}\frac{1}{(1+YTM)^t} = \frac{\frac{1}{1+YTM}\left(1-\frac{1}{(1+YTM)^T}\right)}{1-\frac{1}{1+YTM}} = \frac{1-\frac{1}{(1+YTM)^T}}{YTM}$

$\Rightarrow \frac{\text{Coupon}}{\text{Price}} = \frac{1-\frac{\text{Par}}{P}\cdot\frac{1}{(1+YTM)^T}}{1-\frac{1}{(1+YTM)^T}}\cdot YTM$

$\Rightarrow$ current yield $> YTM$ when $\text{Par} < \text{Price}$.

8. (a) $PV = \frac{\text{Par}}{e^{rT}}$ $\quad 828 = 1000\cdot e^{-r\times 5} \Rightarrow r = 0.0377$

(b) $PV = 1000\cdot e^{-0.042\times 4} = 845$

(c) net return $= \frac{845-828}{828} = 0.0205$

11. $PV = 100\times D(15) = 100\cdot e^{-\int_0^{15} 0.033+0.0012t\,dt} = 100\cdot e^{-(0.033t+0.0012\,t^2/2)\big|_0^{15}}$

$= 53.26$

12. $PV_1 = \text{Par}\; e^{-\int_0^{8} 0.04+0.001t\,dt} = 0.7033\,\text{Par}$.

$PV_2 = \text{Par}\; e^{-\int_0^{7.5} 0.03+0.0013t} = 0.7698\,\text{Par}$. $\quad$ return $= \frac{0.7698-0.7033}{0.7033} = 0.0946$

16. (a) $PV_1 = 1000 \times D(10) = 1000 \times e^{-10\times(0.04+0.001\times 10)} = 606.53$

(b) $PV_2 = 1000 \times e^{-9\times(0.042+0.001\times 9)} = 631.92$

$$\text{net return} = \frac{PV_2 - PV_1}{PV_1} = \frac{631.92-606.53}{606.53} = 0.04185$$

22. (a) $P_1 = 21 \times e^{-\int_0^{0.5} 0.022+0.005t-0.004t^2+0.0003t^3\,dt} = 20.76$

$P_2 = 21 \times e^{-\int_0^{1} 0.022+0.005t-0.004t^2+0.0003t^3\,dt} = 20.52$

$P_3 = 21 \times e^{-\int_0^{1.5} 0.022+0.005t-0.004t^2+0.0003t^3\,dt} = 20.29$

$P_4 = 21 \times e^{-\int_0^{2} 0.022+0.005t-0.004t^2+0.0003t^3\,dt} = 20.09$

$P_5 = 21 \times e^{-\int_0^{2.5} 0.022+0.005t-0.004t^2+0.0003t^3\,dt} = 19.92$

$P_6 = 21 \times e^{-\int_0^{3} 0.022+0.005t-0.004t^2+0.0003t^3\,dt} = 19.81$

$P_7 = 21 \times e^{-\int_0^{3.5} 0.022+0.005t-0.004t^2+0.0003t^3\,dt} = 19.74$

$P_8 = 1021 \times e^{-\int_0^{4} 0.022+0.005t-0.004t^2+0.0003t^3\,dt} = 959.75$

$PV = \sum_{i=1}^{8} P_i = 1100.88$

(b) $DUR = \frac{1}{Price}\left(\sum_{i=1}^{8} \frac{C_i}{e^{r t_i}} t_i + \frac{Par}{e^{rT}} T\right)$

$$= \frac{1}{1100.88}\left(\frac{21}{e^{r(0.5)\cdot 0.5}}0.5 + \frac{21}{e^{r(1)\cdot 1}}\cdot 1 + \frac{21}{e^{r(1.5)\cdot 1.5}}1.5 + \frac{21}{e^{r(2)\cdot 2}}\cdot 2 + \frac{21}{e^{r(2.5)\cdot 2.5}}\cdot 2.5 + \frac{21}{e^{r(3)\cdot 3}}\cdot 3 + \frac{21}{e^{r(3.5)\cdot 3.5}}\cdot 3.5 + \frac{1021}{e^{r(4)\cdot 4}}\cdot 4\right)$$

$$= \frac{4415.34}{1100.88} = 4.01$$

(1) $P(U \le u) = P(F(X) \le u) = P(X \le F^{-1}(u)) = F(F^{-1}(u)) = u,\ u \in (0,1) \Rightarrow U \sim Unif(0,1)$.

(2) (a) $F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\ln y} \exp\left(-\frac{(\ln y - \mu)^2}{2\sigma^2}\right) d\ln y$

$$f_Y = \frac{\partial F_Y(y)}{\partial y} = \frac{\partial F_Y(y)}{\partial \ln y}\cdot\frac{\partial \ln y}{\partial y} = \frac{1}{y\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln y-\mu)^2}{2\sigma^2}\right)$$

(b) $E(e^{Xt}) = \int_{-\infty}^{\infty} e^{xt}\cdot\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2-2x\mu+\mu^2-2x\sigma^2 t}{2\sigma^2}\right)dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-(\mu+\sigma^2 t))^2 - 2\mu\sigma^2 t - \sigma^4 t^2}{2\sigma^2}\right)dx$$

$$= \exp\left(\mu t + \tfrac{1}{2}\sigma^2 t^2\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\right)dx$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$E(e^X) = e^{\mu+\frac{1}{2}\sigma^2}$

$Var(Y) = Var(e^X) = E(e^X)^2 - (E(e^X))^2 = E(e^{2X}) - (E(e^X))^2$

$= e^{2\mu+2\sigma^2} - (e^{\mu+\sigma^2/2})^2 = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$

(3). (i) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - \rho^2 = 0. \Rightarrow 1-\lambda = \pm\rho. \Rightarrow \lambda = 1\pm\rho.$

the largest eigenvalue is $1+|\rho|$.

(ii) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & \rho & 0 \\ \rho & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)[(1-\lambda)^2-\rho^2] = (1-\lambda)^3 - (1-\lambda)\rho^2 = 0.$

$\Rightarrow \lambda = 1\pm\rho, 1$ $\Rightarrow$ the largest eigenvalue is $1+|\rho|$.

(4) X and Y are iid, so we have $Var(X) = Var(Y)$ and $Cov(X,Y)=0$.

$Cov(X+Y, X-Y) = Var(X) - Var(Y) - 2Cov(X,Y) = 0,$

Thus, X+Y and X−Y uncorrelated.

Let $U = X+Y$, $V = X-Y$. $\Rightarrow X = \frac{U+V}{2}$, $Y = \frac{U-V}{2}$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}.$$

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)|J| = \frac{e^{-u}}{2}\mathbb{1}(-u<v<u).$$

Thus, they cannot be independent.