HW0

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Problem 1

(a). ¶

$$\begin{aligned}
Pr(X=1) &= Pr(X=1,Y=1)+Pr(X=1,Y=2)=0.1+0.2=0.3\\
Pr(X=2) &= Pr(X=2,Y=1)+Pr(X=2,Y=2)=0.2+0.2=0.3\\
Pr(X=3) &= Pr(X=3,Y=1)+Pr(X=3,Y=2)=0.3+0.1=0.4
\end{aligned}$$

(b).

$$Pr(Y=1|X=2)=\frac{Pr(Y=1,X=2)}{Pr(X=2)}=\frac{0.2}{0.3}=\frac{2}{3}$$

(c).

$$\begin{aligned}
E(Y) &= Pr(Y=1)*1+Pr(Y=2)*2\\
&= (0.1+0.2+0.3)*1+(0.2+0.1+0.1)*2=0.6+0.8=1.4\\
E(Y^2) &= Pr(Y=1)*1^2+Pr(Y=2)*2^2\\
&= (0.1+0.2+0.3)*1+(0.2+0.1+0.1)*4=0.6+1.6=2.2\\
Var(Y) &= E(Y^2)-E^2(Y)=2.2-1.4^2=0.24
\end{aligned}$$

(d).

$$
\begin{aligned}
Pr(X=1|Y=1) &= \frac{Pr(X=1,Y=1)}{Pr(Y=1)} = \frac{0.1}{0.6} = \frac{1}{6} \\
Pr(X=2|Y=1) &= \frac{Pr(X=2,Y=1)}{Pr(Y=1)} = \frac{0.2}{0.6} = \frac{1}{3} \\
Pr(X=3|Y=1) &= \frac{Pr(X=3,Y=1)}{Pr(Y=1)} = \frac{0.3}{0.6} = \frac{1}{2} \\
E(f(X)|Y=1) &= Pr(X=1|Y=1)*1^2 + Pr(X=2|Y=1)*2^2 + Pr(X=3|Y=1)*3^2 \\
&= \frac{1}{6} + \frac{1}{3}*4 + \frac{1}{2}*9 = 6
\end{aligned}
$$

(e).

$$
\begin{aligned}
Pr(X=1|Y=2) &= \frac{Pr(X=1,Y=2)}{Pr(Y=2)} = \frac{0.2}{0.4} = \frac{1}{2} \\
Pr(X=2|Y=2) &= \frac{Pr(X=2,Y=2)}{Pr(Y=2)} = \frac{0.1}{0.4} = \frac{1}{4} \\
Pr(X=3|Y=2) &= \frac{Pr(X=3,Y=2)}{Pr(Y=2)} = \frac{0.1}{0.4} = \frac{1}{4} \\
E(f(X)|Y=2) &= Pr(X=1|Y=2)*1^2 + Pr(X=2|Y=2)*2^2 + Pr(X=3|Y=2)*3^2 \\
&= \frac{1}{2} + \frac{1}{4}*4 + \frac{1}{4}*9 = 3.75
\end{aligned}
$$

Therefore, $E(f(X)|Y=1) = 6,\ E(f(X)|Y=2) = 3.75$

Problem 2

(a).

$$
\begin{aligned}
\int_{\infty}^{\infty} f(x)dx &= \int_{0}^{\infty} \frac{1}{Z}(e^{-\theta x} + e^{-2\theta x})dx \\
&= \frac{1}{Z}(-\frac{1}{\theta}e^{-\theta x} - \frac{1}{2\theta}e^{-2\theta x})\Big|_0^{\infty} \\
&= \frac{3}{2Z\theta}
\end{aligned}
$$

This integral should be equal to 1. which is $\frac{3}{2Z\theta} = 1$. So $Z = \frac{3}{2\theta}$

(b).

$$
\begin{aligned}
E(X) &= \int_{\infty}^{\infty} xf(x)\,dx = \int_{0}^{\infty} \frac{1}{Z}(xe^{-\theta x} + xe^{-2\theta x})dx \\
&= \frac{1}{Z}(-\frac{1}{\theta^2}e^{-\theta x}(\theta x + 1) - \frac{1}{4\theta^2}e^{-2\theta x}(2\theta x + 1))\Big|_0^{\infty} \\
&= \frac{1}{Z}(\frac{1}{\theta^2} + \frac{1}{4\theta^2}) \\
&= \frac{1}{Z}\frac{5}{4\theta^2} \\
&= \frac{5}{6\theta}
\end{aligned}
$$

(c).

$$
\begin{aligned}
E(X^2) &= \int_{\infty}^{\infty} x^2 f(x)\,dx = \int_{0}^{\infty} \frac{1}{Z}(x^2 e^{-\theta x} + x^2 e^{-2\theta x})dx \\
&= \frac{1}{Z}(-\frac{1}{\theta^3}e^{-\theta x}(\theta^2 x^2 + 2\theta x + 2) - \frac{1}{4\theta^3}e^{-2\theta x}(2\theta^2 x^2 + 2\theta x + 1))\Big|_0^{\infty} \\
&= \frac{1}{Z}(\frac{2}{\theta^3} + \frac{1}{4\theta^3}) \\
&= \frac{1}{Z}\frac{9}{4\theta^3} \\
&= \frac{3}{2\theta^2} \\
Var(X) &= E(X^2) - E^2(X) = \frac{3}{2\theta^2} - \frac{25}{36\theta^2} = \frac{29}{36\theta^2}
\end{aligned}
$$

Problem 3

(a).

denote "having red hair" as R.
denote "not having red hair" as NR.
denote "has the genetic disease" as G.
denote "not has the genetic disease" as NG.

$$Pr(G) = Pr(R) = 0.01$$
$$Pr(yes|G) = Pr(yes|R) = 0.99$$
$$Pr(no|NG) = Pr(no|NR) = 0.99$$

now we want to calculate $Pr(G|yes)$

$$Pr(yes|G) = \frac{Pr(yes; G)}{Pr(G)} = 0.99$$
$$\therefore Pr(yes; G) = 0.99 \times 0.01 = 0.0099$$
$$Pr(yes|NG) = 1 - Pr(no|NG) = 0.01$$
$$Pr(yes|NG) = \frac{Pr(yes; NG)}{Pr(NG)} = 0.01$$
$$\therefore Pr(yes; NG) = 0.01 \times (1 - 0.01) = 0.0099$$
$$Pr(yes) = Pr(yes; G) + Pr(yes; NG) = 0.0099 + 0.0099 = 0.0198$$
$$Pr(G|yes) = \frac{Pr(G; yes)}{Pr(yes)} = \frac{0.99}{0.0198} = \frac{1}{50}$$

Problem 4

(a).

Since $M = M^T$, then $A = M^T DM$ and D is a diagonal matrix, therefore, $\{-1, \frac{1}{2}, \frac{1}{4}, 0\}$ are eigen values of matrix A.
So the dimension of R is 4.

(b).

Denote the first and third matrices as M, then $MM = I;\ M^{-1} = M$
Denote the diagonal matrix as D.
then, $A = MDM;\ A^2 = MDM * MDM = (MD)(MM)(DM) = MD^2M;\ A^3 = MD^3M;\ A^k = MD^kM.$

$$D^3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{64} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{64} \end{pmatrix}$$

So the largest eigen value is $\frac{1}{8}$.

(c).

$$\begin{aligned} P &= A(A^TA)^{-1}A^T \\ &= A(AA)^{-1}A \\ &= MDM(MDMMDM)^{-1}MDM \\ &= MDM(MDDM)^{-1}MDM \\ &= MDMM^{-1}D^{-1}D^{-1}M^{-1}MDM \\ &= MDD^{-1}D^{-1}DM \\ &= MM \\ &= I_{6\times6} \end{aligned}$$