(a) Suppose that the loss of an asset L, is normal with mean μ and variance σ^2. Show that the expected shortfall is given by $ES(\alpha) = \mu + \sigma\phi(z_\alpha)/\alpha$, where z_α is the $(1-\alpha)$-th quantile of the standard normal distribution and $\phi(\cdot)$ is the density function of N(0,1).

z_α is the $(1-\alpha)$-th quantile of the standard normal distribution and L is the loss of an asset, so

$$VaR(s) = \sigma z_s + \mu$$

$$ES(\alpha) = -\frac{1}{\alpha}\int_{-\infty}^{VaR(\alpha)} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

Since $\frac{x-\mu}{\sigma} \sim N(0,1)$, let $x = s\sigma + \mu$,then

$$\begin{aligned}
ES(1-\alpha) &= -\frac{1}{\alpha}\int_{-\infty}^{VaR(\alpha)} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&= -\frac{1}{\alpha}\int_{-\infty}^{\frac{VaR(\alpha)-\mu}{\sigma}} \frac{s\sigma+\mu}{\sqrt{2\pi}\sigma} e^{-\frac{s^2}{2}}\, d(s\sigma+\mu) \\
&= -\frac{1}{\alpha}\int_{-\infty}^{z_\alpha} \frac{s\sigma+\mu}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}\, ds \\
&= -\frac{\sigma}{\alpha}\int_{-\infty}^{z_\alpha} \frac{s}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}\, ds - \frac{\mu}{\alpha}\int_{-\infty}^{z_\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}\, ds \\
&= -\frac{\sigma}{\alpha}\phi(z_\alpha) - \frac{\mu}{\alpha}\alpha \\
&= -\frac{\sigma}{\alpha}\phi(z_\alpha) - \mu \\
ES(\alpha) &= \frac{\sigma\phi(z_\alpha)}{\alpha} + \mu
\end{aligned}$$

(b) Suppose that the yearly return on a stock is normally distributed with mean 0.04 and standard deviation 0.18. If one purchases $100,000 worth of this stock, what is ES(0.05) with T equal to one year?

the **loss** on this stock is normally distributed with mean **-0.04** and standard deviation 0.18
$z_{0.05} = -1.64485$ and $\phi(z_{0.05}) = 0.1031356.$

$$\begin{aligned}
ES(\alpha) &= 100000 \times [\mu + \sigma\phi(z_\alpha)/\alpha] \\
&= 100000 \times [-0.04 + 0.18 \times \phi(z_{0.05})/0.05] \\
&= 100000 \times [(-0.04 + 0.18 \times 0.1031356/0.05] \\
&= 100000 \times 0.3312883 \\
&= 33128.83
\end{aligned}$$

(c) Suppose that the yearly returns on stocks A and B are both normally distributed with mean 0.04 and standard deviation 0.18. Assume further that they are jointly normal with correlation 0.2. Consider a portfolio of investing $50,000 in A and $50,000 in B, what is ES(0.05) with T equal to one year? Compare the result with (b).

A rank statistic is a statistic that depends on the data only through the ranks. A key property of ranks is that they are unchanged by strictly monotonic transformations of the variables.
Spearman rank correlation and Kendall's tau are both rank correlation. In particular, the ranks are unchanged by transforming each variable by its CDF, so the distribution of any rank statistic depends only on the copula of the observations, not on the univariate marginal distributions. Therefore, Spearman rank correlation and Kendall's tau between X and Y are both 1.

While Pearson correlation needs the linear relationship between the two variables and needs them to be approximates normal distribution. In this case X and Y can not satisfy the assumption of Pearson correlation, so it is less than 1.

Problem 4

(Optional) Recall that the bivariate Gumbel copula takes the form

$$C_\alpha(u,v) = \exp\{-[(-\log u)^\alpha + (-\log v)^\alpha]^{\frac{1}{\alpha}}\},$$

where $\alpha \in [1,\infty)$***. Show that, as*** $\alpha \to \infty$***,*** $C_\alpha(u,v) \to C^+(u,v) = \min(u,v)$***.***

Denote $a = -\log u \in (0,\infty)$ and $b = -\log v \in (0,\infty)$
Without loss of generality, let's suppose $u \leq v$, then $a \geq b > 0$
Let's consider $-[(-\log u)^\alpha + (-\log v)^\alpha]^{\frac{1}{\alpha}}$ which is $-(a^\alpha + b^\alpha)^{\frac{1}{\alpha}}$ first.
Since $a \geq b$, we have

$$\begin{aligned}
-(a^\alpha + a^\alpha)^{\frac{1}{\alpha}} &\leq -(a^\alpha + b^\alpha)^{\frac{1}{\alpha}} &&< -(a^\alpha)^{\frac{1}{\alpha}} \\
-(2a^\alpha)^{\frac{1}{\alpha}} &\leq -(a^\alpha + b^\alpha)^{\frac{1}{\alpha}} &&< -(a^\alpha)^{\frac{1}{\alpha}} \\
\lim_{\alpha\to\infty} -(2a^\alpha)^{\frac{1}{\alpha}} &\leq \lim_{\alpha\to\infty} -(a^\alpha + b^\alpha)^{\frac{1}{\alpha}} &&< \lim_{\alpha\to\infty} -(a^\alpha)^{\frac{1}{\alpha}} \\
-a &\leq \lim_{\alpha\to\infty} -(a^\alpha + b^\alpha)^{\frac{1}{\alpha}} &&< -a
\end{aligned}$$

Therefore, $\lim_{\alpha\to\infty} -(a^\alpha + b^\alpha)^{\frac{1}{\alpha}} = -a$
which is $\lim_{\alpha\to\infty} -[(-\log u)^\alpha + (-\log v)^\alpha]^{\frac{1}{\alpha}} = \log u$
So, $\lim_{\alpha\to\infty} \exp\{-[(-\log u)^\alpha + (-\log v)^\alpha]^{\frac{1}{\alpha}}\} = \exp(\log u) = u$
Under our assumption $u \leq v$, so, $\lim_{\alpha\to\infty} C_\alpha(u,v) = \min(u,v) = C^+(u,v)$

Problem 5

Suppose that we have two bonds A and B. Denote by T_A and T_B their respective default times (in year). Suppose that T_A follows exponential distribution with hazard $\lambda_A = 0.01$ (i.e. $P(T_A \geq t) = e^{-\lambda_A t}$) and T_B follows exponential with hazard $\lambda_B = 0.02$. Suppose that jointly they satisfy the Gumbel copula with $\alpha = 2$. Find the probabilities that
(i) both will default by the end of the first year;
(ii) at least one will default by the end of the first year.

(i)

$$g_\alpha^{-1}(u) = \exp(-u^{1/\alpha}) = \exp(-\sqrt{u}) \tag{6.1}$$

$$P(\tau_A \leq T, \tau_B \leq T) = C(F_A(T), F_B(T)) \tag{6.2}$$

$$\begin{aligned}
F_A(T) &= P(T_A \leq T) \\
&= 1 - P(T_A \geq T) \\
&= 1 - e^{-\lambda_A T} \\
&= 1 - e^{-0.01} \\
F_B(T) &= P(T_B \leq T) \\
&= 1 - P(T_B \geq T) \\
&= 1 - e^{-\lambda_B T} \\
&= 1 - e^{-0.02}
\end{aligned}$$

$$\begin{aligned}
P(\tau_A \leq 1, \tau_B \leq 1) &= C_g(F_A(1), F_B(1)) \\
&= g_\alpha^{-1}(g_\alpha(F_A(1)) + g_\alpha(F_A(1))) \\
&= g_\alpha^{-1}(g_\alpha(1 - e^{-0.01}) + g_\alpha(1 - e^{-0.02}))
\end{aligned}$$

$$\begin{aligned}
g_\alpha(1 - e^{-0.01}) &= (-\log\{1 - e^{-0.01}\})^2 \\
&= (\log\{1 - e^{-0.01}\})^2 \\
&= 21.25363 \\
g_\alpha(1 - e^{-0.02}) &= (-\log\{1 - e^{-0.02}\})^2 \\
&= (\log\{1 - e^{-0.02}\})^2 \\
&= 15.38213
\end{aligned}$$

$$\begin{aligned}
P(\tau_A \leq 1, \tau_B \leq 1) &= g_\alpha^{-1}(21.25363 + 15.38213) \\
&= g_\alpha^{-1}(36.63576) \\
&= \exp(-\sqrt{36.63576}) \\
&= 0.00235
\end{aligned}$$

Solution to Practice Questions

1 Interest Rate

1. Let r be the yearly compounded rate. Expected present value of all the payments is

$$\begin{aligned}&p\times 0+(1-p)p\frac{C}{1+r}+(1-p)^2p\sum_{t=1}^{2}\frac{C}{(1+r)^t}+(1-p)^3p\sum_{t=1}^{3}\frac{C}{(1+r)^t}\\&+(1-p)^4p\sum_{t=1}^{4}\frac{C}{(1+r)^t}+\mathbb{P}(X\geq 6)\Bigg(\sum_{t=1}^{5}\frac{C}{(1+r)^t}+\frac{\text{PAR}}{(1+r)^5}\Bigg),\end{aligned}$$

which equals 785.979.

2. (a)

$$\text{return}=\frac{\text{Payment}-800}{800}=\frac{1000\cdot I(\text{no deafult})-800}{800},$$

where $I(\cdot)$ is the indicator function. Therefore,

$$\mathbb{E}(\text{return})=\frac{1000\cdot 0.9-800}{800}=\frac{1}{8}$$

and

$$\text{SD}(\text{return})=\sqrt{\left(\frac{1000}{800}\right)^2\cdot 0.1(0.9)}=\frac{3}{8}.$$

(I am using the formulae for expected value and variance of a Bernoulli random variable)

(b)

$$\text{return}=\frac{1000(I(\text{A no default})+I(\text{B no default}))-1600}{1600}.$$

As in (a),

$$\mathbb{E}(\text{return})=\frac{1}{8}.$$

Since the two default events are mutually exclusive (note that this does not mean they are independent), $I(\text{A default})I(\text{B default})=0$ and thus

$$\begin{aligned}&\text{Cov}(I(\text{A no default}),I(\text{B no default}))\\=\;&\mathbb{E}(I(\text{A no default})I(\text{B no default}))-\mathbb{E}(I(\text{A no default}))\mathbb{E}(I(\text{B no default}))\\=\;&\mathbb{E}(1-I(\text{A default}))-I(\text{B default}))+I(\text{A default})I(\text{B default}))-0.9(0.9)\\=\;&1-0.1-0.1-0.9(0.9)=-0.01.\end{aligned}$$

Hence,

$$\begin{aligned}&\text{Var}(I(\text{A no default})+I(\text{B no default}))\\=\;&\text{Var}(I(\text{A no default}))+\text{Var}(I(\text{B no default}))+2\text{Cov}(I(\text{A no default}),I(\text{B no default}))\\=\;&0.1(0.9)+0.1(0.9)+2(-0.01)=0.16\end{aligned}$$

and

$$\text{SD}(\text{return})=\sqrt{\left(\frac{1000}{1600}\right)^2\cdot 0.16}=0.25.$$

3. (a) Let s be the 5-year spot rate.

$$\begin{aligned} s &= \frac{1}{5}\int_0^5 r(t)dt = \frac{1}{5}\int_0^5 0.03 + 0.001t + 0.0002t^2 dt \\ &= \frac{1}{5}\left[0.03t + \frac{0.001}{2}t^2 + \frac{0.0002}{3}t^3\right]_0^5 \\ &= 0.03417. \end{aligned}$$

(b) Price $= \text{PAR}e^{-5s} = 0.843\text{PAR}$.

4. WLOG, assume PAR = 1.

$$\begin{aligned} P_0 &= e^{-\int_0^8 0.04+0.001t\, dt} = 0.703 \\ P_{0.5} &= e^{-\int_0^{7.5} 0.03+0.013t\, dt} = 0.5539809 \\ \text{Return} &= \frac{P_{0.5} - P_0}{P_0} = -0.212. \end{aligned}$$

2 Portfolio Theory

1. Suppose we assign weight w to asset 1 and $1 - w$ to asset 2.

$$\begin{aligned} Var(wR_1 + (1-w)R_2) =& w^2 Var(R_1) + 2w(1-w)Cov(R_1, R_2) + (1-w)^2 Var(R_2) \\ =& w^2 20^2 + 2w(1-w)20 \times 10 \times \rho + (1-w)^2 10^2. \end{aligned}$$

Taking derivative w.r.t. w, we get

$$w = \frac{1 - 2\rho}{5 - 4\rho}.$$

Plugging in $\rho = 0, 0.3, -0.3$ we get the answer.

2. Use equation (16.5) of the book (page 471)

$$w_T = \frac{V_1\sigma_2^2 - V_2\rho_{12}\sigma_1\sigma_2}{V_1\sigma_2^2 + V_2\sigma_1^2 - (V_1 + V_2)\rho_{12}\sigma_1\sigma 2},$$

where $V_1 = \mu_1 - \mu_f$, $V_2 = \mu_2 - \mu_f$.

3. (a) $\mu_R = \mu_f + (\mu_M - \mu_f) \times \frac{\sigma_R}{\sigma_M} = 0.023 + (0.1 - 0.023) \times 0.05/0.12 = 0.05508$.

(b) $Cov(R_A, R_M) = \beta Var(R_M) = \beta \times (0.12)^2 = 0.004$ so $\beta = 0.004/0.12^2 = 0.27$.

(c) β of the portfolio is $(1.5 + 1.8)/2 = 1.65$, so the expected return is

$$\mu_f + \beta(\mu_M - \mu_f) = 0.023 + 1.65 \times (0.1 - 0.023) = 0.15.$$

The σ_ϵ for the portfolio is $\sigma_\epsilon = \sqrt{\frac{1}{2^2}(0.08^2 + 0.10^2)} = 0.064$, therefore the standard deviation of the return of the portfolio is

$$\sqrt{\beta^2\sigma_M^2 + \sigma_\epsilon^2} = \sqrt{1.65^2 \times 0.12^2 + 0.064^2} = 0.208.$$

3 Copula

1. Denote $W = U + V$. Using definition of Kendall's τ.

$$
\begin{aligned}
\tau =& E\left[\operatorname{sign}\left((U_t - U_s)(W_t - W_s)\right)\right] \\
=& 1 - 2P((U_t - U_s)(W_t - W_s) < 0) \\
=& 1 - 2P(U_t > U_s, W_t < W_s) - 2P(U_t < U_s, W_t > W_s) \\
=& 1 - 4P(U_t > U_s, W_t < W_s) \\
(*) =& 1 - 4P(U_t > U_s, V_t = 0, V_s = 1) \\
(**) =& 1 - 4P(U_t > U_s)P(V_t = 0)P(V_s = 1) \\
=& 1 - 4 \times 0.5 \times (1-p)p = 2p^2 - 2p + 1
\end{aligned}
$$

$(*)$ holds because that $U_t > U_s, W_t < W_s$ can only happen when $V_t = 0, V_s = 1$ (note V_t and V_s are binary). $(**)$ holds because U_t, U_s, V_t, V_s are all independent of each other. Note that this holds as long as U is a continuous variable with support in $(0, 1)$.

For Spearman's ρ, from definition we have

$$
\rho = \operatorname{Corr}\left(F_U(U), F_W(W)\right) = \frac{E(F_U(U)F_W(W)) - E(F_U(U)) \times E(F_W(W))}{SD(F_U(U)) \times SD(F_W(W))},
$$

where F_U and F_W are the CDF of U and W. Note that $F_U(U)$ and $F_V(V)$ both follows uniform distribution (this is the property of CDF), so we have

$$
\rho = \frac{E(F_U(U)F_W(W)) - \frac{1}{2} \times \frac{1}{2}}{\frac{1}{12}}.
$$

Now we only need to compute $E(F_U(U)F_W(W))$. We have $F_U(U) = U$ (U follows uniform distribution). Computing F_W is slightly trickier, we have, for $x \in [0, 1]$

$$
F_W(x) = P(W = U + V < x) = P(U < x, V = 0) = P(V = 0)P(U < x) = (1-p)x.
$$

For $x \in [1, 2]$

$$
F_W(x) = P(W = U + V < x) = P(V = 0) + P(U < x - 1, V = 1) = (1-p) + p \times (x - 1).
$$

Therefore we have

$$
\begin{aligned}
E(F_U(U)F_W(W)) =& E(UF_W(U + V)) \\
=& P(V = 0)E(UF_W(U)|V = 0) + P(V = 1)E(UF_W(U + 1)) \\
=& (1-p)E((1-p)U^2) + pE(U[(1-p) + pU]) \\
=& \frac{(1-p)^2}{3} + \frac{p(1-p)}{2} + \frac{p^2}{3}.
\end{aligned}
$$

Therefore we have

$$
\rho = \frac{\frac{(1-p)^2}{3} + \frac{p(1-p)}{2} + \frac{p^2}{3} - \frac{1}{2} \times \frac{1}{2}}{\frac{1}{12}} = 2p^2 - 2p + 1.
$$

Now Pearson's correlation is much more straightforward, Here we have

$$\begin{aligned}
Corr(U,W) =& \frac{Cov(U,U+V)}{SD(U)\times SD(W)}\\
=& \frac{Cov(U,U)+Cov(U,V)}{\sqrt{var(U)}\sqrt{var(U)+var(V)}}\\
=& \frac{\frac{1}{12}}{\sqrt{\frac{1}{12}}\sqrt{\frac{1}{12}+p(1-p)}}\\
=& \frac{1}{\sqrt{1+12p(1-p)}}.
\end{aligned}$$

2. (a) $F_X(x) = F_\theta(x,\infty) = 1-e^{-x}$. X has an exponential distribution with parameter 1. The same holds for Y.

   (b) By definition,

$$\begin{aligned}
C_\theta(x,y) &= \mathbb{P}(F_X(X)\le x, F_Y(Y)\le y)\\
&= \mathbb{P}(X\le F_X^{-1}(x), Y\le F_Y^{-1}(y))\\
&= \mathbb{P}(X\le -\log(1-x), Y\le -\log(1-y))\\
&= F_\theta(-\log(1-x), -\log(1-y)).
\end{aligned}$$

3. Same as Problem 6 in Homework 5, just plug in different numbers.

4 Risk Management

1. First we compute the normalizing constant

$$Z = \int_{-\infty}^{+\infty} \exp(-\lambda|x|)dx = 2\int_0^{+\infty}\exp(-\lambda|x|)dx = \frac{2}{\lambda}.$$

For $l>0$, we have

$$P(\mathcal{L}>l) = \frac{1}{Z}\int_l^{+\infty}\exp(-\lambda|x|)dx = \frac{\lambda}{2}\times\frac{\exp(-\lambda l)}{\lambda} = \frac{\exp(-\lambda l)}{2}.$$

Solving $P(\mathcal{L}>\text{VaR}(0.05)) = \frac{\exp(-\lambda\text{VaR}(0.05))}{2} = 0.05$, we get

$$\text{VaR}(0.05) = -\frac{\log(0.1)}{\lambda} = \frac{\log(10)}{\lambda}.$$

We also have, for $l>0$

$$\text{ES}(0.05) = E(\mathcal{L}|\mathcal{L}>l) = \frac{\frac{1}{Z}\int_l^{+\infty}x\exp(-\lambda|x|)dx}{P(\mathcal{L}>l)},$$

where

$$\int_l^{+\infty}x\exp(-\lambda|x|)dx = \frac{\exp(-\lambda l)(1+\lambda l)}{\lambda^2}.$$

Plut in $l = \mathrm{VaR}(0.05) = \frac{\log(10)}{\lambda}$, we have

$$\mathrm{ES}(0.05) = \frac{\lambda}{2} \times \frac{0.1(1+\log(10))}{\lambda^2}\frac{1}{0.05} = \frac{\log(10)}{\lambda} + \frac{1}{\lambda}.$$

Note (ignore this if you are not interested): the interesting pattern that $\mathrm{ES}(0.05) = \mathrm{VaR}(0.05) + \frac{1}{\lambda}$. For the double exponential distribution, this would actually hold in general. i.e.

$$\mathrm{ES}(\alpha) = \mathrm{VaR}(\alpha) + \frac{1}{\lambda}$$

for all $\alpha < 0.5$. This is related to the memoryless property of exponential distribution.

2. (a) We first compute the mean and variance of the portfolio return. We have $\mu_1 = 1/7$, $\mu_2 = 2/7$ and $\mu_3 = 4/7$, so $\mu_P = \sum_{i=1}^3 w_i\mu_i = 0.044286$ and

$$\sigma_P^2 = \sum_{i=1}^3 w_i^2\sigma_i^2 + \sum_{i\neq j} w_i w_j \sigma_{ij} = \sum_{i=1}^3 w_i^2\sigma_i^2 + 2\sum_{i<j} w_i w_j \sigma_{ij} = 0.195876,$$

where σ_{ij} denotes the covariance between the returns of stocks i and j. Using the formula of normal VaR, we have $\mathrm{VaR}(0.05) = -350000[\mu_P + \sigma_P(-z_{0.05})] = 97265$.

(b) $\mathrm{ES}(0.05) = 350000(-\mu_P + \frac{\sigma_P}{0.05}\phi(z_{0.05})) = 125913$.

3. Recall that $\mathrm{VaR}(\alpha)$ is defined such that $\mathbb{P}(L > \mathrm{VaR}(\alpha)) = \alpha$. Hence, $\mathrm{VaR}(s) = F_L^{-1}(1-s)$, where F_L is the cdf of L. Note that

$$\mathrm{ES}(\alpha) = \frac{\int_0^\alpha \mathrm{VaR}(s)ds}{\alpha} = \frac{\int_\infty^{\mathrm{VaR}(\alpha)} -lf_L(l)dl}{\alpha} = \frac{\int_{\mathrm{VaR}(\alpha)}^\infty lf_L(l)dl}{P(L > \mathrm{VaR}(\alpha))} = \mathbb{E}(L|L > \mathrm{VaR}(\alpha)) = \mathrm{CTE}(\alpha).$$

The second equality follows by using the change of variable $l = \mathrm{VaR}(s) = F_L^{-1}(1-s)$, which gives $F_L(l) = 1 - s$ and so $ds = -f_L(l)$.