HASSE DIAGRAMS

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Abstract. In this report, we will study the topic of Hasse Diagrams. After reading this report, you will be able to:
(1) Solve problems with Hasse Diagrams.
(2) Concretely define divisibility.
(3) Define a poset.
Examples will be given diagram solutions and detailed descriptions. By the end of this report, you will be able to describe a Hasse Diagram with the same fluency.

1. Introduction

In order to understand Hasse diagrams, little background in number theory is needed. However, for our purposes we should be familiar with a couple topics. First, we must be familiar with the idea of a poset.

Definition 1.1. A poset is a partially ordered set, abbreviated to poset. The notation often used is $P = (P, \leq)$ where P is the poset and $\leq$ is a partial order, or a *relation* on to P.

For this report, the relation will always be given as $R = \{\}$ to help aid in understanding.

Definition 1.2. Divisibility, in math, is defined by $a = mb$ where $a, b, m \in \mathbb{Z}$.

I thank Professor Day for helping me with this report and Professor Robinson for providing a LaTeX template.

This equation is telling us that b divides a. We use the notation $b|a$ as shorthand. If we were to make a list, D, of all the numbers that divide a, b would be in that list and henceforth called a **divisor** of a.

Example 1.3. We can write $4 = 2 * 2 + 0$. Another example is $10 = 3 * 3 + 1$, but from this we know that $3 \nmid 10$ since there is a remainder, $r = 1$, in the equation.

Now, some notation we need to define is:

Definition 1.4. The $gcd(a, d)$ is the **greatest common divisor** of a, d. In other words, the greatest common divisor of a, d would be the largest number that divides both a and d.

Example 1.5. The $gcd(4, 8) = 4$ since $4|4$ and $4|8$. It is clear that there is no larger number that divides both 4 and 8, since 4 is its own largest divisor. Note that the $gcd(x, y)$ where x, y are prime will always be 1.

In the rest of this report, you will be introduced to multiple examples that build upon each other. As we progress through this report, the examples will become more complicated conceptually. By the end, you will be able to understand divisibility and its visual properties with fluency.

2. Analysis

In this section, we will be going through multiple examples of Hasse Diagrams, each example getting progressively harder as we go along. To start off, we have our first example here:

Example 1:

Given the relation $R = \{(a, b)|a \leq b\}$ and the set $S = \{1, 2, 3, 4\}$, generate the Hasse diagram:

This is a simple example of a Hasse diagram. Some things to take note of are:

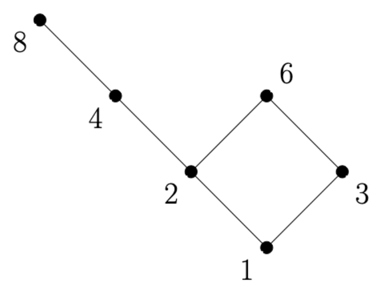
- The largest number will always be on top, and the smallest number will always be on the bottom.
- Hasse diagrams are **reflexive**. We know $1 \leq 1$, but do not need to draw a line from 1 back to itself.
- If $a \leq b$ and $b \leq c$, then $a \leq c$. In other words, **transitivity holds**.
- Hasse diagrams are **antisymmetric**. If $a \leq b$ and $b \leq a$, then $a = b$ and we only draw a. This was not illustrated in the example, but it will come up later.

As you may see in our example, $1 \leq 2 \leq 3 \leq 4$. Notice how we don't have to bother with drawing lines between $(1, 3)$ and $(1, 4)$ because we already have the line between $(1, 2)$ and know that Hasse diagrams are transitive. Also take note of how the diagram is vertical. It would be unclear what we are trying to say if the diagram were to be in any other orientation. If it were to be horizontal, the reader wouldn't be able to easily know what way the diagram is supposed to be read. So, for future examples, keep in mind that we will *always* read the diagram from top down.

Now, on to a more advanced example:

Example 2:

Given the relation $R = \{(a,b) \mid a|b\}$ and the set $S = \{1,2,3,4,6,8\}$, generate the Hasse diagram:



In this example, we can clearly see how all the numbers relate to each other. We can see that 2|8, 2|6, 2|4, 3|6, 4|8, and 1 divides all numbers. Additionally, we easily see how the $gcd(8,6) = 2$ since 2 is the largest number that is connected to both 8 and 6. One may wonder why our diagram was drawn with the 8 above and to the left of the 4 and not directly above, above and to the right, or any other orientation. This is because parallel lines in a Hasse diagram show relations. See how 4 is twice as large as 2, 8 is twice as large as 4, and 6 is twice as large as 3. Since these numbers are all related in the same way, the lines connecting them must be parallel. In 2D Hasse diagrams, this fact is very apparent. An alternative way to decipher the relations between the points is to try to create a quadrilateral.

Lemma 2.1. *There a, b, c, d, vertices in a Hasse diagram. Where $a < b$, and a is related to b through the operation defined on the poset, there exists an edge*

$$e_1 = ab$$

connecting the vertices. For $c < d$, c related to d through the operation defined on the poset, there exists an edge

$$e_2 = cd$$

connecting the vertices. Now, we create the edge

$$e_3 = bd$$

for $b < d$, b related to d through the operation defined on the post. We then create the last edge

$$e_4 = ac$$

for $a < c$, a related to c through the operation defined on the poset. After going through these steps, we can then conclude that the relation $bd = ac$ and the relation $ab = cd$.

Look in our example again. The quadrilateral created with the vertices $\{6, 3, 2, 1\}$ has this property. The relation between 1 and 2 is multiplication of 2, and the relation between 3 and 6 is multiplication of 2 as well. We can continue to see the relation between 1 and 3 is the same as the relation between 2 and 6 with it being multiplication of 3. This fact about the diagrams makes it so the most information can be relayed in the simplest way possible.

Practice:

At this point in the lab, it is suitable to try to solve a problem on your own. This homework problem is going to be very similar to Example 2, so reference it if you find yourself getting stuck. See the homework question here:

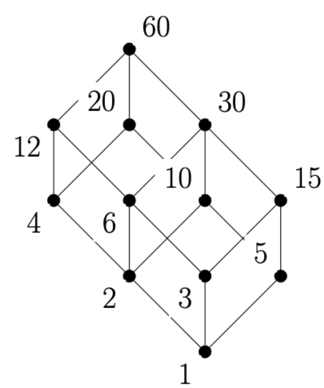
Homework 2.2. Given the relation $R = \{(a, b)|a|b\}$ on the set $S = \{1, 2, 3, 4, 7, 12, 21, 49\}$, draw the Hasse diagram.

The solution to this homework question will be at the end of this report.

Now we want to play with an example where our set is less defined. We will be given a number and a relation, divisibility, on that number. From there, we have to find all divisors and create the diagram ourselves. See this in action in our third example:

Example 3:

Given the relation R=$\{(a, b) \mid a|b\}$ where $b = 60$, generate the Hasse diagram:

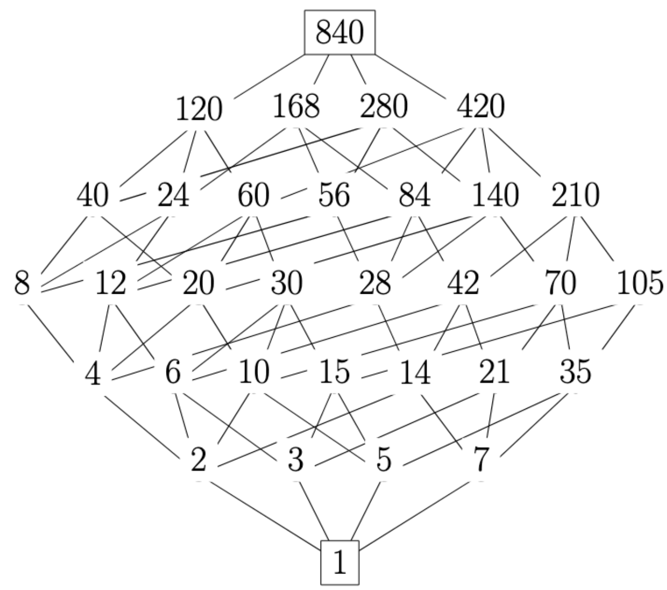


This example more clearly illustrates the way we can think of divisibility visually. Take note how we aren't drawing a line between $(1, 6)$, $(2, 12)$, $(5, 30)$, $(10, 60)$, and other points when it might make sense to. The reason we aren't drawing these lines is because of transitivity. We already know, for example, $5|30$ because they are connected via 15. Additionally, we can see relations we didn't originally plan on mapping out. Notice how we can easily see that the $gcd(20, 12)$ is 4 since that is the largest point that the two points share. Try to find the $gcd(20, 30)$ and $gcd(10, 4)$ to check understanding. Also, notice again the different quadrilaterals inside the diagram. Try to see three different quadrilaterals in the diagram and then derive the relations on the edges.

Now we want to connect our learning from Example 2 and Example 3 into one diagram. Again, we will simply be given a number and then have to derive its divisors and create the diagram ourselves. See this example:

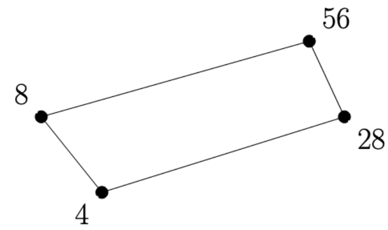
Example 4:

Given the relation $R = \{(a, b) \mid a|b\}$ where $b = 840$, generate the Hasse diagram:

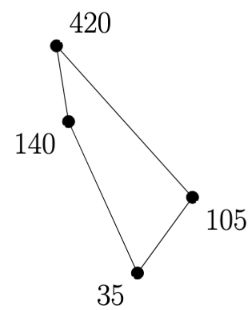


As one may observe, the creation of a Hasse diagram quickly becomes complicated. Notice how both example 2 and 3's Hasse diagrams appear fully on the inside of this larger Hasse diagram. This very clearly shows how divisibility is cumulative, in a sense. If need be, we could take apart the 4 divisors at the top of 840's diagram, create their Hasse diagrams separately, and then put them together while getting rid of redundant lines. Indeed, this would be a more intuitive way to create the diagram as opposed to a different approach, say, starting from the bottom and working your way to the top.

To make sure we can see the different relations and quadrilaterals in the Hasse diagram, see this quadrilateral that has been taken from our example above:



See how $28 = 4*7$ and $56 = 8*7$, and also $8 = 4*2$ and $56 = 28*2$. So, even though the lines aren't parallel to each other, the lines show the same relation, m, to each other. We can also see this same kind of pattern in this example:

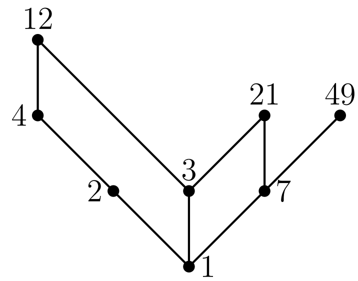


This holds because $420 = 140*3$ and $105 = 35*3$, and also $140 = 35*4$ and $420 = 105*4$. Notice how we are skipping over both 210 and 70 while making this quadrilateral. We are able to skip over these numbers because of the transitivity property.

Now, try to find other quadrilateral relations within example 4 to check if you understand the concept fully.

The solution to the practice problem:

Given the relation $R = \{(a, b)|a|b\}$ on the set $S = \{1, 2, 3, 4, 7, 12, 21, 49\}$:



3. Conclusion

Hasse diagrams are a very useful tool to start to understand the topic of divisibility and other relations on posets. Generally, this information is not needed for an undergraduate student. The lessons learned from creating and looking at different Hasse diagrams, such as the behavior of divisibility, is information that a student already learns while studying Discrete Mathematics and other higher-level undergraduate mathematics classes. However, the process of creating these diagrams forces a student to completely understand the concept of divisibility. It is not challenging to understand the division algorithm, but it is not an easy task trying to justify adding relations between points in a Hasse diagram. Each line and point needs to have a reason to be there, and the meticulous nature of solving these diagrams is a very useful skill for a student to have. Additionally, these diagrams would be very useful in helping people unacquainted with the topics of mathematics understand how numbers relate to each other. While more complicated Hasse diagrams, like example 4 and ones similar, would not be helpful to use as a tool of learning, all others would be a very good introduction to understanding divisibility on a higher level.

References

[1] *An Illustrated Theory of Numbers*, Martin H. Weissman http://illustratedtheoryofnumbers.com.

[2] *Hasse Diagrams — Discrete Mathematics*, Deepanshi Mittal, https://www.geeksforgeeks.org/discrete-mathematics-hasse-diagrams/?itm_source=auth&itm_medium=contributions&itm_campaign=articles.

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