Math 251 Fall 2024 **Name: Aby Soumare**
Report 1: Linear Iteration
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Introduction:

In this report, we are going to explore the topic of linear iteration. In order to fully understand the topic, there are some things that we need to get straight before we continue:

- What is the definition of a sequence?
- What is an iteration sequence?
- How do we define linear equations?
- What is the definition of a linear iteration sequence?

We will continue by explaining each of these topics in the order in which they are listed, and we will give an example of each one. After that, we will introduce the main purpose of this report.

A **Sequence**, in simple terms, is nothing more than an infinite list of numbers written in a specific order. We often use the notation

$$\{a_n\} = \{a_1, a_2, ...\}$$

to showcase what a sequence is. A more concrete example of a sequence would be:

$$a_0 = 1, a_1 = 2, a_2 = 3 \text{ ... and so on.}$$

To put this back into a general form, we get that:

$$\{a_n\} = \{1, 2, ...\} \text{ for } n = 0, 1, 2, ...$$

In this example $a_n = n + 1$ is called the general term of the sequence.

Next, we need to define an **iteration sequence**. We can think of iteration as a process to produce a sequence. What that means is that we are going to take an initial value x_0 and a given function $f(x)$ and repeatedly apply $f(x)$ to x_0 and then to each succeeding output of $f(x)$. Also note that $f(x)$ stands for a "function of x" which means x is the variable we are changing in our equation. Try to follow along with this example:

Example 0.1 $f(x) = x^2$ *and* $x_0 = 2$,
$x_1 = 4$,
$x_2 = 16$,
$x_3 = 256$
.
.
.

and so on. You may notice that the result of the former $f(x)$ is going to be the x in the next equation. To rephrase, we are taking an initial value x_0 and then *iterating* it via the function $f(x)$.

Next, we will specialize to the case when $f(x)$ is a **linear function**. A linear function is a function of the following form, where a, b are real numbers:

$$f(x) = ax + b.$$

When the $f(x) = ax + b$ function is graphed on the Cartesian plane, it will create a straight line.

The a in our equation is commonly known as the "slope" of $f(x) = ax + b$. This value is the number that we are going to multiply by our x in order to modify it. For this laboratory, our a can be any real number.

Finally, the b in our equation is sometimes referred to as the y-intercept. The b will simply tell us where to start our line when graphing it. For the most part, the a is the variable doing the most heavy-lifting in a linear equation on the y-axis.

Linear iteration sequences are specific cases of iteration sequences. Recall our definition of an iteration sequence and consider the case where $f(x)$ is of the form $f(x) = ax + b$. For example, suppose:

Example 0.2 $f(x) = 3x + 5$ *and* $x_0 = 1$,
and then we generate the terms,
$x_1 = 3(1) + 5$,
$x_2 = 3(8) + 5$,
$x_3 = 3(29) + 5$
and so on.
So, $\{x_n\} = \{1, 8, 29, ...\}$.

Again, this is very similar to general iteration sequences, but with the special condition that the $f(x)$ can *only* be a linear equation of the form $f(x) = ax + b$.

Finally, in the rest of this report, we will continue to explore linear iteration sequences. We will be exploring the general term of linear iteration sequences. There we will look at and walk through the proof for the general form. We will go through both the algebraic analysis for this problem and the geometric visualization. Finally, we will also want to determine all the various types of behaviors that can occur in linear iteration sequences, and figure out for which a, b, and x_0 exactly each behavior occurs.

Different Types of Behavior:

First, the two broadest types of behavior we observed in our investigation were convergence and divergence. Let's start by defining **convergent sequences**. A convergent sequence is a sequence for which there exists a real number L called the limit of the sequence. A limit in this case is how we describe the behavior of a sequence as the n in x_n gets larger.

Definition 0.3 *A sequence $\{x_n\}$ converges if there exists a real number, L, such that for all $\epsilon > 0$ there exists an N for which:*
$|x_n - L| < \epsilon$ whenever $n > N$.

Look at this example to try and understand this definition in more detail:

Example 0.4 $\{x_n\} = \{(\frac{1}{2})^n\}$ *and* $\epsilon = \frac{1}{3}$.
We intuitively know that the limit, L, of this function is going to be 0 since all numbers are getting smaller and smaller. So, we need to find when x_n are all within $\frac{1}{3}$ of 0.
Consider:
Find N such that $|x_n - 0| < \frac{1}{3}, n > N$.
$|(\frac{1}{2}x)^n - 0| < \frac{1}{3}$,
$(\frac{1}{2}x)^n < \frac{1}{3}$,
$(\frac{1}{2})^n < \frac{1}{3}$,
$ln(\frac{1}{2})^n < ln(\frac{1}{3})$,
$-nln(2) < -ln(3)$,
$n > \frac{ln(3)}{ln(2)}$.
So this means that for every n larger than $\frac{ln(3)}{ln(2)}$, $|x_n - 0| < \frac{1}{3}$.

Now let us look at all the different types of convergence we are going to be using throughout this report:

Example 0.5 *When $f(x) = \frac{1}{2}x$, and $x_0 = 1$,*
we generate:
$x_1 = \frac{1}{2}$,
$x_2 = \frac{1}{4}$,
$x_3 = \frac{1}{8}$,
$x_4 = \frac{1}{16}$,
and so on.
So, $\{x_n\} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\}$.

As you may observe, the results are approaching 0 the larger n gets. So, the limit of example 0.5 seems to be 0. For example 0.5 and sequences with the same behavior as it, we are going to refer to them as exhibiting "decreasing convergence".

Definition 0.6 *A Sequence $\{x_n\}$ exhibits decreasing convergence if $x_n > x_{n+1}$ for all $n \geq 0$, and also approaches a limit, L.*

Notice how in example 0.5, that number was 0 since our results will continue to get smaller but never become negative. Now let us get to know other forms of convergent sequences. Look at this example:

Example 0.7 *When* $f(x) = \frac{1}{2}x$, *and* $x_0 = -2$,
we generate:
$x_1 = -1$,
$x_2 = -\frac{1}{2}$,
$x_3 = -\frac{1}{4}$,
$x_4 = -\frac{1}{8}$,
and so on.
So, $\{x_n\} = \{-1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}...\}$.

Notice how as n gets larger, our result will increasingly approach 0. The example 0.7 and sequences with the same behavior we will refer to as increasing convergent sequences.

Definition 0.8 *A sequence* $\{x_n\}$ *exhibits increasing convergence if* $x_n < x_{n+1}$ *for all* $n \geq 0$, *and also* $\{x_n\}$ *approaches a limit,* L.

This behavior means that the terms of our sequence will be increasingly approaching a limit, in the case of example 0.7, again, it was 0. Now let's look at another example of a convergent sequence:

Example 0.9 *When* $f(x) = 0x + 1$, *and* $x_0 = 15$,
we generate:
$x_1 = 1$,
$x_2 = 1$,
$x_3 = 1$,
$x_4 = 1$,
and so on.
So, $\{x_n\} = \{15, 1, 1, 1, 1, ...\}$.

Notice that no matter what number we have as our x_0, our result will always come out to be 1. This is what we are going to define as constant convergence: when the terms of a sequence is only one number.

Definition 0.10 *A sequence* $\{x_n\}$ *exhibits constant convergence if* $x_n = x_{n+1}$ *for all* $n \geq N$. *The limit,* L *of this function is the constant value* x_n *for* $n \geq N$.

As you may observe, in example 0.9, the limit, L, is 1. Now, for the final convergent sequence, take a look at this example:

Example 0.11 *When* $f(x) = -\frac{1}{2}x + 1$, *and* $x_0 = 3$,
we generate:
$x_1 = -\frac{1}{2}$,
$x_2 = \frac{5}{4}$,
$x_3 = \frac{3}{8}$,
$x_4 = \frac{13}{16}$,
$x_5 = \frac{19}{32}$
and so on.
So, $\{x_n\} = \{3, -\frac{1}{2}, \frac{3}{2}, \frac{3}{8}, \frac{13}{16}, \frac{19}{32}, ...\}$.

Notice how this sequence seems to be oscillating between larger and smaller numbers. The results of this sequence will continue to get closer to $\frac{2}{3}$ the larger our n gets, however a special behavior we observe is just *how* they will approach $\frac{2}{3}$, and that is by oscillating between larger and smaller numbers. So, as a formal definition for this behavior is as follows:

Definition 0.12 *A sequence* $\{x_n\}$ *exhibits oscillating convergence if:*
$x_1 < x_3 < ... < L < ... < x_2 < x_0$,
or $x_0 < x_2 < ... < L < ... < x_3 < x_1$,
for all $n \geq 0$.

These 4 kinds of convergence are the only kinds of convergence we found while investigating linear iteration sequences.

Recall our definition of a convergent sequence above. A **divergent sequence** is a sequence x_n for which the limit, L, does not exist. Now we discuss the different types of divergent sequences that appear in linear iteration sequences. Look at this example:

Example 0.13 *When* $f(x) = x + 10$*, and* $x_0 = 1$,
we generate:
$x_1 = 11$,
$x_2 = 21$,
$x_3 = 31$,
$x_4 = 41$,
and so on.
So, $\{x_n\} = \{1, 11, 21, 31, 41, ...\}$.

As you may see, the terms of this sequence will continue to grow with no bound as x_n grows larger, and there is no way for us to discern what, say, x_{132} would be without actually computing all the terms. A divergent sequence that is *increasing* without bound, we are going to call an increasing divergent sequence.

Definition 0.14 *A sequence* $\{x_n\}$ *exhibits increasing divergence if* $x_n < x_{n+1}$ *for all* $n \geq 0$*, and the limit* L *does not exist.*

Look at an example of a decreasing divergent sequence here:

Example 0.15 *When* $f(x) = x - 1$*, and* $x_0 = 1$,
we generate:
$x_1 = 0$,
$x_2 = -1$,
$x_3 = -2$,
$x_4 = -3$,
and so on.
So, $\{x_n\} = \{1, 0, -1, -2, -3, ...\}$.

You may notice how this sequence is getting smaller and smaller without bound. This behavior is what we are going to refer as decreasing divergence: when our sequence is *decreasing* without any lower bound.

Definition 0.16 *A sequence $\{x_n\}$ exhibits decreasing divergence if $x_n > x_{n+1}$ for all $n \geq 0$, and the limit L does not exist.*

Now take a look at this example of a 2-cycle divergent sequence:

Example 0.17 *When $f(x) = -x + 3$, and $x_0 = 10$,*
we generate:
$x_1 = -7$,
$x_2 = 10$,
$x_3 = -7$,
$x_4 = 10$,
and so on.
So, $\{x_n\} = \{10, -7, 10, -7, 10, ...\}$.

You may notice that the results of our function will only ever be 10 or -7. No matter what x_0 we have, given $x_0 \neq 3$, we will see this pattern with this function. Example 0.17 and sequences with the same behavior we are going to refer to as 2-cycle divergent sequences. In other words, 2-cycle divergent sequences are sequences with only two possible alternating values for their terms.

Definition 0.18 *A sequence $\{x_n\}$ exhibits 2-cycle divergence if $\{x_n\} = \{a, b, a, b, ...\}$ where $a \neq b$.*

Example 0.19 *When $f(x) = -2x + 1$, and $x_0 = 6$,*
we generate:
$x_1 = -11$,
$x_2 = 23$,
$x_3 = -45$,
$x_4 = 91$,
and so on.
So, $\{x_n\} = \{6, -11, 23, -45, 91, ...\}$.

Recall our definition of oscillating convergence; this behavior is a very similar kind of behavior, however the results are not converging to a limit. Since our terms are alternately increasing into very large numbers and decreasing into very small numbers without bound. We are going to define that kind of behavior as oscillating divergence.

Definition 0.20 *A Sequence $\{x_n\}$ exhibits oscillating divergence if the terms of the sequence are alternately growing larger and larger and smaller and smaller, so that $|x_n|$ grows without bound.*
$x_1 > x_3 > ... > x_2 > x_0$,
or $x_0 > x_2 > ... > x_3 > x_1$,
for all $n \geq 0$.

These 8 different types of behaviors noted here are the only behaviors we are going to be observing as this lab progresses.

Analysis:

Let us explore linear iteration sequences in more depth. Remember that in our definition of linear iteration, we were examining the function $f(x) = 3x + 5$ with $x_0 = 1$ up until x_3. If we were to transform that example into a general form, we would get something that looks like this expression:

$$x_0$$
$$x_1 = a * x_0 + b$$
$$x_2 = a * (a * x_0 + b) + b = a^2 * x_0 + b(1 + a), \text{ and so on.}$$

What we are doing here is the same as what we were doing before: we are inputting the output of the first x_0 back into the equation to get x_1, just now we are using a general form so we can examine the behavior more generally. One may notice how future terms, say, x_3 would be produced. Also, for x_n, we will prove the following closed form expression when $a \neq 1$:

$$x_n = a^n(x_0 - \tfrac{b}{1-a}) + \tfrac{b}{1-a}.$$

This x_n equation is especially important, as it will help us with future problems that we will come across. It is not something you necessarily need to understand right now, as we will break down each step as we continue in this report. First, let us start with a proposition that will lay down the building blocks for a future theorem:

Proposition 0.21 *If $f(x) = ax + b$ and a, b, x_0 are real numbers, then the nth iterate of the sequence $\{x_n\}$ has the recursive form:*

$$x_n = a^n x_0 + b(1 + a + ... + a^{n-1})$$

for all $n \geq 1$.

Proof 0.22 *We will prove this proposition via induction on n. First consider our base case when $n = 1$. In this case, we know by definition that $x_1 = f(x_0) = ax_0 + b$. Since this is precisely what our base case is looking for, then our base case holds for $n = 1$.*
Now, assume that our proposition holds for $n = k \geq 1$. We assume:

$$x_k = a^k x_0 + b(1 + ... + a^{k-1})$$
where $k \geq 1$.

We need to show that the proposition holds for $n = k + 1$. Consider x_{k+1} via the definition of iteration we know that:

$$x_{k+1} = f(x_k).$$

Thus, we know

$$x_{k+1} = ax_k + b.$$

Now, using our inductive assumption on x_k, we have:

$$x_{k+1} = a(a^k x_0 + b(1 + ... + a^{k-1}) + b.$$

Multiply the a in to the two factors:

$$x_{k+1} = a^{k+1}x_0 + ab(1 + ... + a^{k-1}) + b,$$

and with some algebra, we get:

$$x_{k+1} = a^{k+1}x_0 + b(1 + ... + a^k).$$

Hence the proposition holds for $n = k + 1$. Therefore, we have proved our proposition via induction on n.

This proposition is just one part to allow us to understand the concept of our larger theorem. To understand our theorem, we must first this lemma:

Lemma 0.23 *If a is any real number, then:*

$$1 + a + a^2 + ... + a^{n-1} = \begin{cases} \frac{1-a^n}{1-a} & \text{for } a \neq 1 \\ \\ n & \text{for } a = 1. \end{cases}$$

Proof 0.24 *If $a = 1$, this statement is clear.*
If $a \neq 1$ then we prove this statement by induction on n:
First consider the base case $n = 1$. In this case, we need to show the left-hand side is equal to the right-hand side. If $n = 1$ then $1 + a + ... + a^{n-1} = 1$, and

$$1 = \tfrac{1-a^n}{1-a} \text{ given } a \neq 1.$$

Now, consider our inductive hypothesis:

$$1 + a + a^2 + ... + a^{n-1} = \tfrac{1-a^n}{1-a} \text{ for } n \geq 1.$$

We need to show that

$$1 + a + a^2 + ... + a^{n-1} + a^n = \tfrac{1-a^{n+1}}{1-a}.$$

to prove our statement. Consider:

$$1 + a + a^2 + ... + a^{n-1} + a^n.$$

Now substitute our inductive hypothesis for the 1st n terms to the sum above to get:

$$1 + a + ... + a^{n-1} + a^n = \tfrac{1-a^n}{1-a} + a^n.$$

We need to show that:

$$\tfrac{1-a^n}{1-a} + a^n = \tfrac{1-a^{n+1}}{1-a}.$$

But this step follows from a little algebra. First, we set the terms over a common divisor as:

$$\frac{1-a^n+a^n(1-a)}{1-a}.$$

Then we distribute and simplify to get:

$$\frac{1-a^n+a^n-a^{n+1}}{1-a}.$$

Finally, we have shown that:

$$1+a+...+a^n = \frac{1-a^{n+1}}{1-a} = \frac{1-a^{n+1}}{1-a}.$$

Therefore, we have proven our statement by induction on n in the case that $a \neq 1$ and our lemma holds for all a.

Now, we will use both the lemma and propositions proved to prove a closed form expression theorem here:

Theorem 0.25 *If $f(x) = ax + b$ and a, b, x_0 are real numbers, then the nth iterate of the sequence $\{x_n\}$ has closed form expression:*

$$x_n = \begin{cases} a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a} & \text{for } a \neq 1 \\ \\ x_0 + bn & \text{for } a = 1. \end{cases}$$

Proof 0.26 *Suppose $a = 1$, then, by our proposition,*

$$x_n = a^n x_0 + b(1 + ... + a^{n-1}).$$

Since $a = 1$, then, by our lemma,

$$x_n = x_0 + bn.$$

Therefore, our theorem holds for $a = 1$. Now we suppose $a \neq 1$. Then, by our proposition,

$$x_n = a^n x_0 + b(1 + ... + a^{n+1})$$

for $n \geq 1$. Since $a \neq 1$, then, by our lemma,

$$x_n = a^n x_0 + b(\frac{1-a^n}{1-a})$$

for $n \geq 1$. Now, we must simplify our equation to have our desired form:

$$x_n = a^n x_0 + \frac{b}{1-a} - \frac{a^n b}{1-a}.$$

Now, we will group all the factors of a^n together:

$$x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}.$$

Therefore, we have proved our theorem using our some algebra, our proposition, and our lemma.

This theorem then produces 7 corollaries, each describing what happens with the equation when a changes. Observe these corollaries:

Corollary 0.27 *Given the hypothesis of our theorem and supposing $a < -1$ then the iteration sequence $\{x_n\}$ can exhibit two behaviors:*

- *If $x_0 = \frac{b}{1-a}$ then the iteration sequence will constantly converge to $\frac{b}{1-a}$.*
- *If $x_0 \neq \frac{b}{1-a}$ then the iteration sequence will have oscillating divergent behavior.*

Proof 0.28 *We will prove this corollary via a proof by cases. We have two cases:*

1) if $x_0 = \frac{b}{1-a}$,
2) and $x_0 \neq \frac{b}{1-a}$.

Let's start with case 1:

If $x_0 = \frac{b}{1-a}$ where $a < -1$, then by theorem 0.25:
$$x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a} = \frac{b}{1-n} \text{ for } n \geq 1.$$
The sequence will exhibit constant convergence to $\frac{b}{1-a}$.

Now for case 2:

If $x_0 \neq \frac{b}{1-a}$ where $a < -1$, then:
$$x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}.$$
Since a^n is growing without bound and oscillating in sign, then this sequence will exhibit 2-cycle divergence.

Therefore, we have proven our corollary via proof by cases.

Corollary 0.29 *Given the hypothesis of our theorem and supposing $a = -1$ then the iteration sequence $\{x_n\}$ will exhibit two behaviors:*

- *If $x_0 = \frac{b}{2}$ then $\{x_n\}$ exhibits constant convergence to $\frac{b}{2}$.*
- *If $x_0 \neq \frac{b}{2}$ then $\{x_n\}$ will have 2-cycle divergence where $\{x_n\} = \{x_0, -x_0 + b, x_0, -x_0 + b, ...\}$.*

Proof 0.30 *We will prove this corollary via proof by cases. We have two cases:*

1) if $x_0 = \frac{b}{2}$
2) if $x_0 \neq \frac{b}{2}$

Case 1:

If $x_0 = \frac{b}{1-a}$ where $a = -1$ then:
$$x_n = a^n(0) + \frac{b}{2} = \frac{b}{2}.$$
Therefore, the sequence $\{x_n\}$ will constantly converge to $\frac{b}{2}$.

Case 2:

If $x_0 \neq \frac{b}{1-a}$ and $a = -1$ then consider the first 4 terms of $\{x_n\}$:
$$x_0 = x_0,$$
$$x_1 = -x_0 + b,$$
$$x_2 = x_0,$$
and so on.
As you can see, a^n where $a = -1$ will exhibit oscillating divergence.

Therefore, we have proven our corollary via proof by cases.

Corollary 0.31 *Given the hypothesis from our theorem and supposing $-1 < a < 0$ then the iteration sequence $\{x_n\}$ will exhibit two behaviors:*

- *If $x_0 = \frac{b}{1-a}$, x_n exhibits constant convergence to $\frac{b}{1-a}$,*
- *If $x_0 \neq \frac{b}{1-a}$ then x_n exhibits oscillating convergence to $\frac{b}{1-a}$.*

Proof 0.32 *We will prove this corollary via proof by cases. We have two cases:*

1) if $x_0 = \frac{b}{1-a}$,
2) and $x_0 \neq \frac{b}{1-a}$.

Let's start with case 1:

If $x_0 = \frac{b}{1-a}$ where $-1 < a < 0$, then:
$$x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a},$$
$$= a^n(0) + \frac{b}{1-a},$$
$$= 0 + \frac{b}{1-a},$$
Therefore, the sequence will constantly converge to $\frac{b}{1-a}$.

Now case 2:

If $x_0 \neq \frac{b}{1-a}$ where $-1 < a < 0$, then by our theorem, $|a^n|$ approaches 0 by alternates in sign, so this sequence will exhibit oscillating divergent behavior.

Therefore, we have proven our corollary via proof by cases.

Corollary 0.33 *Given the hypothesis of our theorem 0.25 and assuming $a = 0$, $\{x_n\}$ exhibits only one behavior, which is constant convergence to b.*

Proof 0.34 *By theorem 0.25, if $a = 0$ then $x_n = 0^n(x_0 - \frac{b}{1-0}) + \frac{b}{1-0}$ and $n \geq 1$, so $\{x_n\} = \{x_0, b, b, ...\}$.*

Corollary 0.35 *Given the hypothesis of our theorem and assuming $0 < a < 1$ then $\{x_n\}$ can exhibit three behaviors:*

- *If $x_0 < \frac{b}{1-a}$ then $\{x_n\}$ will exhibit increasing convergence to $\frac{b}{1-a}$,*
- *If $x_0 = \frac{b}{1-a}$ then $\{x_n\}$ will exhibit constant convergence to $\frac{b}{1-a}$,*

- *If* $x_0 > \frac{b}{1-a}$ *then* $\{x_n\}$ *will exhibit decreasing convergence to* $\frac{b}{1-a}$.

Proof 0.36 *We will prove this corollary via proof by cases. We have three cases:*

1) if $x_0 > \frac{b}{1-a}$
2) if $x_0 = \frac{b}{1-a}$
3) if $x_0 < \frac{b}{1-a}$.

Case 1:

If $x_0 > \frac{b}{1-a}$ *and* $0 < a < 1$*, then we know that*
a^n *will exhibit decreasing convergence to* 0.

Case 2:

If $x_0 = \frac{b}{1-a}$ *and* $0 < a < 1$*, then consider:*

$$= a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a},$$
$$= a^n(0) - \frac{b}{1-a},$$
$$= 0 - \frac{b}{1-a},$$
$$= \frac{b}{1-a}.$$

Therefore, $\{x_n\}$ *will constantly converge to* $\frac{b}{1-a}$.

Case 3:

If $x_0 < \frac{b}{1-a}$ *and* $0 < a < 1$*, then we know that*
a^n *will exhibit decreasing convergence.*

Therefore, we have proven our corollary via proof by cases.

Corollary 0.37 *Given the hypothesis of our theorem and assuming* $a = 1$ *then the iteration sequence* $\{x_n\}$ *exhibits three behaviors:*

- *If* $b > 0$ *then the sequence will exhibit increasing divergence,*
- *If* $b < 0$ *then the sequence will exhibit decreasing divergence,*
- *If* $b = 0$ *then the sequence will exhibit constant convergence to* x_0.

Proof 0.38 *We will proceed by proof by cases to prove this corollary. First, note that* $\{x_n\} = x_0 + bn$ *by our theorem when* $a = 1$*. We have three cases:*

1) If $b > 0$,
2) if $b < 0$,
3) if $b = 0$.

Case 1:

If $b > 0$*, and* $a = 1$,
$x_n = x_0 + bn$.

Thus we know that this sequence will exhibit increasing divergence as the terms x_n *grow in steps of* b *without upper bound as* n *increases.*
Case 2:

If $b < 0$, and $a = 1$,
$x_0 - bn$.
Thus we know that this sequence will exhibit decreasing divergence as the terms x_n decrease in steps of b without lower bound as n increases.

Case 3:

If $b = 0$, and $a = 1$,
$x_0 = x_0 + 0n = x_0$,
Therefore, this sequence will exhibit constant convergence to x_0.

Therefore, we have proven our corollary via proof by cases.

Corollary 0.39 *Given the hypothesis of our theorem and assuming $a > 1$ then the iteration sequence $\{x_n\}$ exhibits three behaviors:*

- *If $x_0 = \frac{b}{1-a}$ then $\{x_n\}$ exhibits constant convergence to $\frac{b}{1-a}$,*
- *If $x_0 < \frac{b}{1-a}$ then $\{x_n\}$ exhibits decreasing divergence,*
- *If $x_0 > \frac{b}{1-a}$ then $\{x_n\}$ exhibits increasing divergence.*

Proof 0.40 *We will prove this corollary via proof by cases. We have three cases:*

1) if $x_0 > \frac{b}{1-a}$
2) if $x_0 = \frac{b}{1-a}$
3) if $x_0 < \frac{b}{1-a}$.

Case 1:

If $x_0 > \frac{b}{1-a}$ and $a > 1$, then
let $x_0 - \frac{b}{1-a} = c$.
In this case, $x_n = a^n(c) + \frac{b}{1-a}$ where c is positive.
Thus, our sequence will exhibit increasing divergence since c is a positive number and a^n is getting larger without bound.

Case 2:

If $x_0 = \frac{b}{1-a}$ and $a > 1$, then
$x_n = a^n(0) + \frac{b}{1-a} = \frac{b}{1-a}$.
Therefore, our sequence will constantly converge to $\frac{b}{1-a}$.

Case 3:

If $x_0 < \frac{b}{1-a}$ and $a > 1$, then
let $x_0 - \frac{b}{1-a} = c$,
$a^n(c) + \frac{b}{1-a}$ where c is negative,
then our sequence will exhibit decreasing divergence since c is a negative number and a^n is getting larger without bound.

Therefore, we have proven our corollary via proof by cases.

Conclusion:

There are multiple things that could be further explored in another paper. One item not fully explained in this paper is the role of $\frac{b}{1-a}$ within our function. In a couple of the corollaries, it was mentioned that $\frac{b}{1-a}$ acted as a horizon line for the function to converge to or as a value for x_n to constantly converge to. One may see now $\frac{b}{1-a}$ plays a pivotal role in the behavior of the linear iteration sequence, especially when thinking geometrically and how $\frac{b}{1-a}$ relates to the graphed line on a Cartesian plane. However, the most important element to describe the behavior of the sequence x_n within $x_n = a^n(x_0 - \frac{b}{1-a}) + \frac{b}{1-a}$ is $f(x) = ax + b$ the value of a. As we have pointed out throughout this paper, the value of a determines if the function will be convergent, or divergent, and what type of convergence or divergence the sequence exhibits. So, when looking at a function, you can predict the behavior of the function before you graph it just by understanding what the a is doing. Another topic not touched on in this paper is what is known as a "fixed point". A fixed point of an iteration sequence is a value x for which $f(x) = x$. Look at this example:

Example 0.41 *When $f(x) = -x + 3$, and $x_0 = 1.5$, we generate:*
$x_1 = 1.5$,
$x_2 = 1.5$,
$x_3 = 1.5$,
$x_4 = 1.5$,
$x_5 = 1.5$,
and so on.
So, $\{x_n\} = \{1.5, 1.5, 1.5, 1.5, ...\}$.

As you may notice, $x_0 = 1.5$ acts as a fixed point for this function. If we were to choose a different x_0, then this function would be exhibiting 2-cycle divergence. Also note that 1.5 is a value x where $f(x) = x$. For any function $f(x)$ that particular function's fixed point. Geometrically, the fixed point for a function will be the point at which the line created by graphing the function intersects with the line $b = x$.