Xyba Project

Matematika Dasar II

Pembahasan UTS 2017

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Thank you for your cooperation >v<

1. Kerjakan sesuai perintah!
   a. Jika $y = \arctan(\sin(x))$, carilah $\frac{dy}{dx}$
   b. Selesaikan $\int \frac{e^x}{e^{2x}-3e^x+2} dx$
   c. Buktikan $\int \cos^n x \, dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\, dx$, kemudian hitunglah:
$$\int_0^{\pi} \cos^8(x)\, dx$$

Jawab:

a. Diberikan $y = \arctan(\sin(x))$, maka:
$$\frac{dy}{dx} = \frac{d(\arctan(\sin(x)))}{d(\sin(x))} \cdot \frac{d(\sin(x))}{dx} = \frac{1}{1+\sin^2 x}(\cos x) = \frac{\cos x}{1+\sin^2 x}$$

b. Perhatikan:
$$\frac{1}{u^2-3u+2} = \frac{1}{(u-1)(u-2)} = \frac{A}{u-1} + \frac{B}{u-2}$$
$$\Leftrightarrow 1 = A(u-2) + B(u-1)$$
$u = 1 \to -A = 1 \to A = -1$
$u = 2 \to B = 1$
Sehingga:
$$\begin{aligned}\int \frac{e^x}{e^{2x}-3e^x+2} dx &= \int \frac{e^x}{(e^x-1)(e^x-2)} \frac{d(e^x)}{e^x} \\ &= \int \frac{1}{(e^x-1)(e^x-2)} d(e^x) \\ &= \int \left(\frac{-1}{e^x-1} + \frac{1}{e^x-2}\right) d(e^x) \\ &= -\int \frac{1}{e^x-1} d(e^x) + \int \frac{1}{e^x-2} d(e^x) \\ &= -\ln|e^x-1| + \ln|e^x-2| + C\end{aligned}$$

c. Akan dibuktikan:
$$\int \cos^n x \, dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\, dx$$
Perhatikan:
$$\frac{d(\cos^{n-1} x)}{dx} = \frac{d(\cos^{n-1} x)}{d(\cos x)}\frac{d(\cos x)}{dx} = (n-1)\cos^{n-2} x\,(-\sin x)$$
Maka:
$$\begin{aligned}\int \cos^n x\, dx &= \int (\cos^{n-1} x \cos x)\, dx \\ &= u.dv, \qquad u = \cos^{n-1} x \to du = (n-1)\cos^{n-2} x\,(-\sin x)dx \\ &\qquad\qquad dv = \cos x\, dx \to v = \sin x\end{aligned}$$

$$= uv - \int v\,du$$

$$= \cos^{n-1} x \sin x - \int \big(\sin x\,(n-1)\cos^{n-2} x\,(-\sin x)\big)\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (\sin^2 x)(\cos^{n-2} x)\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)(\cos^{n-2} x)\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int (\cos^{n-2} x - \cos^n x)\,dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

Sehingga:

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$\Leftrightarrow \int \cos^n x + (n-1)\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx$$

$$\Leftrightarrow \qquad n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx$$

$$\Leftrightarrow \qquad \int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Sehingga, terbukti bahwa:

$$\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

Sekarang, kita akan gunakan informasi ini untuk menentukan:

$$\int_0^\pi \cos^8(x)\,dx$$

Perhatikan:

$$\int \cos^8(x)\,dx = \frac{\cos^7(x)\sin(x)}{8} + \frac{7}{8}\int \cos^6(x)\,dx$$

$$= \frac{\cos^7(x)\sin(x)}{8} + \frac{7}{8}\left(\frac{\cos^5(x)\sin(x)}{6} + \frac{5}{6}\int \cos^4(x)\,dx\right)$$

$$= \frac{\cos^7(x)\sin(x)}{8} + \frac{7}{8}\left(\frac{\cos^5(x)\sin(x)}{6} + \frac{5}{6}\left(\frac{\cos^3(x)\sin(x)}{4} + \frac{3}{4}\int \cos^2 x\,dx\right)\right)$$

$$= \frac{\cos^7(x)\sin(x)}{8} + \frac{7}{8}\left(\frac{\cos^5(x)\sin(x)}{6} + \frac{5}{6}\left(\frac{\cos^3(x)\sin(x)}{4} + \frac{3}{4}\left(\frac{\cos(x)\sin(x)}{2} + \frac{1}{2}\int dx\right)\right)\right)$$

$$= \frac{\cos^7(x)\sin(x)}{8} + \frac{7}{8}\left(\frac{\cos^5(x)\sin(x)}{6} + \frac{5}{6}\left(\frac{\cos^3(x)\sin(x)}{4} + \frac{3}{4}\left(\frac{\cos(x)\sin(x)}{2} + \frac{1}{2}x\right)\right)\right) + C$$

Karena untuk $x = 0$ persamaan tersebut akan bernilai $0 + C$ dan karena $\sin(\pi) = 1$ serta $\cos(\pi) = 0$, maka:

$$\int_0^{\pi} \cos^8(x)\, dx = \left[\int \cos^8(x)\, dx\right]_0^{\pi} = \int \cos^8(x)\, dx|_{x=\pi} - 0$$

$$= \left(\frac{\cos^7(x)\sin(x)}{8} + \frac{7}{8}\left(\frac{\cos^5(x)\sin(x)}{6} + \frac{5}{6}\left(\frac{\cos^3(x)\sin(x)}{4} + \frac{3}{4}\left(\frac{\cos(x)\sin(x)}{2} + \frac{1}{2}x\right)\right)\right)\right)_{x=\pi}$$

$$= \frac{0^7 1}{8} + \frac{7}{8}\left(\frac{0^5 1}{6} + \frac{5}{6}\left(\frac{0^3 1}{4} + \frac{3}{4}\left(\frac{0.1}{2} + \frac{1}{2}\pi\right)\right)\right)$$

$$= 0 + \frac{7}{8}\left(0 + \frac{5}{6}\left(0 + \frac{3}{4}\left(0 + \frac{1}{2}\pi\right)\right)\right) = \frac{7}{8}\left(\frac{5}{6}\left(\frac{3}{4}\left(\frac{1}{2}\pi\right)\right)\right)$$

$$= \frac{105}{384}\pi$$

2. Diberikan grafik persamaan polar $r = 6\sin 2\theta$
   a. Selidiki kesimetrian grafik polar di atas dengan menggunakan uji-uji kesimetrian
   b. Sketsa grafik polar di atas
   c. Tentukan luas daerah yang dibatasi oleh grafik polar di atas yang terletak di kuadran satu

Jawab:

a. Akan diuji kesimetrian dari $r = 6\sin 2\theta$
   - $\theta = -\theta$

     $$r = 6\sin(2\theta) \overset{\theta=-\theta}{\rightarrow} 6\sin\big(2(-\theta)\big) = 6\sin(-2\theta) = -6\sin(2\theta) = -r$$

     Dengan kata lain, $(r, \theta) \overset{\theta=-\theta}{\rightarrow} (-r, -\theta)$. Artinya r simetri terhadap sumbu y.
   - $\theta = \pi - \theta$

     $$r = 6\sin(2\theta) \overset{\theta=\pi-\theta}{\rightarrow} 6\sin\big(2(\pi - \theta)\big) = 6\sin(2\pi - 2\theta) = -6\sin(2\theta) = -r$$

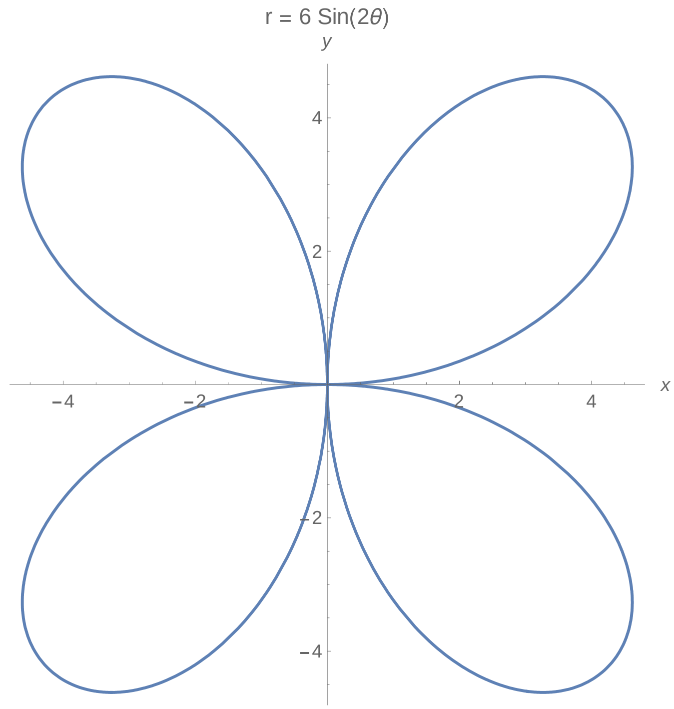
     Dengan kata lain, $(r, \theta) \overset{\theta=\pi-\theta}{\rightarrow} (-r, -\theta)$. Artinya r simetri terhadap sumbu x.
   - $\theta = \pi + \theta$

     $$r = 6\sin(2\theta) \overset{\theta=\pi+\theta}{\rightarrow} 6\sin\big(2(\pi + \theta)\big) = 6\sin(2\pi + 2\theta) = 6\sin(2\theta) = r$$

     Dengan kata lain, $(r, \theta) \overset{\theta=\pi-\theta}{\rightarrow} (-r, -\theta)$. Artinya r simetri terhadap titik asal.

   (Karena r simetri terhadap sumbu x dan y, sudah dapat dikatakan r simetri terhadap titik asal sehingga tidak perlu dicek jika tidak ingin)

b. $r = 6\sin(2\theta)$ akan berbentuk mawar dengan banyak bunga $2 \times 2 = 4$ karena 2 genap.



c. Karena kuadran 1 dibatasi oleh 0 dan $\frac{\pi}{2}$, maka luas daerah yang dibatasi oleh grafik polar di atas yang terletak di kuadran satu diberikan oleh:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 \, d\theta$$

Maka:

$$A = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} (6\sin(2\theta))^2 \, d\theta = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} (36\sin^2(2\theta)) \, d\theta = 18\int_{0}^{\frac{\pi}{2}} \sin^2(2\theta) \, d\theta$$

$$= 18\int_{0}^{\frac{\pi}{2}} \sin^2(2\theta) \, d\theta = 18\int_{0}^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos(4\theta)\right) d\theta = 9\int_{0}^{\frac{\pi}{2}} (1 - \cos(4\theta)) \, d\theta$$

$$= 9\left[\theta - \frac{1}{4}\sin(4\theta)\right]_{0}^{\frac{\pi}{2}} = 9\left[\left(\frac{\pi}{2} - \frac{1}{4}\sin(2\pi)\right) - \left(0 - \frac{1}{4}\sin(0)\right)\right] = 9\left(\frac{\pi}{2} - 0\right) = \frac{9}{2}\pi$$

3. Kerjakan sesuai perintah!

   a. Carilah $\lim\limits_{x\to 0}\left(2x+e^{\frac{x}{3}}\right)^{\frac{3}{x}}$

   b. Periksalah kekonvergenan dari $\int_{-\infty}^{\infty} e^{-|2x|}\,dx$

Jawab:

a. Akan ditentukan:

$$\lim_{x\to 0}\left(2x+e^{\frac{x}{3}}\right)^{\frac{3}{x}}$$

Definisikan:

$$y=\left(2x+e^{\frac{x}{3}}\right)^{\frac{3}{x}}$$

Perhatikan:

$$y=\left(2x+e^{\frac{x}{3}}\right)^{\frac{3}{x}}$$

$$\Leftrightarrow \ln y=\frac{3}{x}\ln\left(2x+e^{\frac{x}{3}}\right)$$

$$\Leftrightarrow \ln y=\frac{3\ln\left(2x+e^{\frac{x}{3}}\right)}{x}$$

Sehingga:

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{3\ln\left(2x+e^{\frac{x}{3}}\right)}{x}=\frac{0}{0}$$

$$\overset{L}{\cong}\lim_{x\to 0}\frac{\frac{3}{2x+e^{\frac{x}{3}}}\left(2+\frac{1}{3}e^{\frac{x}{3}}\right)}{1}=\lim_{x\to 0}\frac{6+e^{\frac{x}{3}}}{2x+e^{\frac{x}{3}}}=\frac{6+1}{0+1}=\frac{7}{1}=7$$

Sehingga:

$$\lim_{x\to 0}y=e^{\lim\limits_{x\to 0}\ln y}=e^{7}$$

b. Akan diperiksa kekonvergenan dari:

$$\int_{-\infty}^{\infty} e^{-|2x|}\,dx$$

Perhatikan integral tersebut merupakan integral tak wajar yang tidak terbatas di kedua batasnya maka:

$$\int_{-\infty}^{\infty} e^{-|2x|}\,dx=\lim_{a\to-\infty}\int_{a}^{0} e^{-(-2x)}\,dx+\lim_{b\to\infty}\int_{0}^{b} e^{-(2x)}\,dx$$

$$=\lim_{a\to-\infty}\int_{a}^{0} e^{2x}\,dx+\lim_{b\to\infty}\int_{0}^{b} e^{-2x}\,dx$$

$$
\begin{aligned}
&= \frac{1}{2}\lim_{a\to-\infty}[e^{2x}]_a^0 - \frac{1}{2}\lim_{b\to\infty}[e^{-2x}]_0^b \\
&= \frac{1}{2}\left(1 - \lim_{a\to-\infty} e^{2a}\right) - \frac{1}{2}\left(\lim_{b\to\infty} e^{-2b} - 1\right) \\
&= \frac{1}{2}(1-0) - \frac{1}{2}(0-1) \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}
$$

Artinya $\int_{-\infty}^{\infty} e^{-|2x|}\,dx$ konvergen.

4. Kerjakan sesuai perintah!
   a. Posisi sebuah titik pada saat t dinyatakan sebagai $x = \frac{1}{2}t^2, y = \frac{1}{9}(6t+9)^{3/2}$. Tentukan jarak yang ditempuh oleh titik tersebut dari $t = 0$ sampai $t = 4$.
   b. Tentukan luas permukaan yang terbentuk apabila kurva $x = y^3$ antara $y = 0$ dan $y = 1$ diputar mengelilingi sumbu y.

Jawab:

a. Jarak yang ditempuh di sini akan sama dengan dari panjang kurva tersebut, sehingga:

$$L = S(\alpha,\beta) = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Diberikan:

$$x = \frac{1}{2}t^2 \to \frac{dx}{dt} = \frac{1}{2}(2t) = t$$

$$y = \frac{1}{9}(6t+9)^{3/2} \to \frac{dy}{dt} = \left(\frac{1}{9}\right)\left(\frac{3}{2}\right)(6)(6t+9)^{\frac{1}{2}} = (6t+9)^{\frac{1}{2}} = \sqrt{6t+9}$$

Maka:

$$L = \int_0^4 \sqrt{(t)^2 + \left(\sqrt{6t+9}\right)^2}\,dt = \int_0^4 \sqrt{t^2+6t+9}\,dt = \int_0^4 \sqrt{(t+3)^2}\,dt = \int_0^4 (t+3)\,dt$$

$$= \left[\frac{1}{2}t^2 + 3t\right]_0^4 = [(8+12) - 0] = 20$$

b. Karena $x = g(y)$ diputar mengeliling sumbu y dengan $a \le y \le b$, maka luas permukaannya diberikan oleh:

$$A = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

Maka:

$$
\begin{aligned}
A &= 2\pi \int_0^1 y^3\sqrt{1+(3y^2)^2}\,dy = 2\pi \int_0^1 y^3\sqrt{1+9y^4}\,dy \\
&= 2\pi \int_{y=0}^{y=1} y^3\sqrt{1+9y^4}\,\frac{d(1+9y^4)}{36y^3} = \frac{\pi}{18}\int_{y=0}^{y=1} \sqrt{1+9y^4}\,d(1+9y^4) \\
&= \frac{\pi}{18}\left(\frac{2}{3}\left[(1+9y^4)^{\frac{3}{2}}\right]_{y=0}^{y=1}\right) = \frac{\pi}{27}\left[(1+9y^4)^{\frac{3}{2}}\right]_{y=0}^{y=1} \\
&= \frac{\pi}{27}\left[(10)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] = \frac{\pi}{27}\left(10\sqrt{10}-1\right)
\end{aligned}
$$

Afterword

Pembuatan dokumen ini dibantu oleh:

1. namora03, Matematika UI 2016.
2. rilo_chand, Matematika UI 2016.