An Attempt to Generalize the Edge Irregularity Strength of $m \times n \times l$ Grid Graphs

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ABSTRACT

The edge irregularity strength of a simple graph G, denoted as $es(G)$, is defined to be the minimum k such that the graph G has an edge irregular k-labeling. The vertex labeling $\phi : V(G) \to \{1, 2, ..., k\}$ is called a vertex k-labeling for G. For any edge xy in G, its weight is $w_\phi(xy) = \phi(x) + \phi(y)$. If all the edge weights are distinct, then ϕ is called an edge irregular k-labeling of G.

Consider the cartesian product $P_n \Box P_m \Box P_l$ for $n, m, l \geq 2$. The formula for the edge irregularity strength for $n, m \geq 2$ and $l = 2$ has been established however the authors have yet to establish the cases when $l \geq 3$ and posed it as an open problem. In this paper, we attempt to generalize these cases.
Keywords: Cartesian product, Edge irregularity strength, Grid graph, Vertex labeling

1. Introduction

A graph is a collection of points and lines connecting some (possibly empty) subset of them. We write a graph G as $G = (V, E)$ where $V = V(G)$ is the non-empty set of vertices of G and $E = E(G)$ is the set of edges of G.

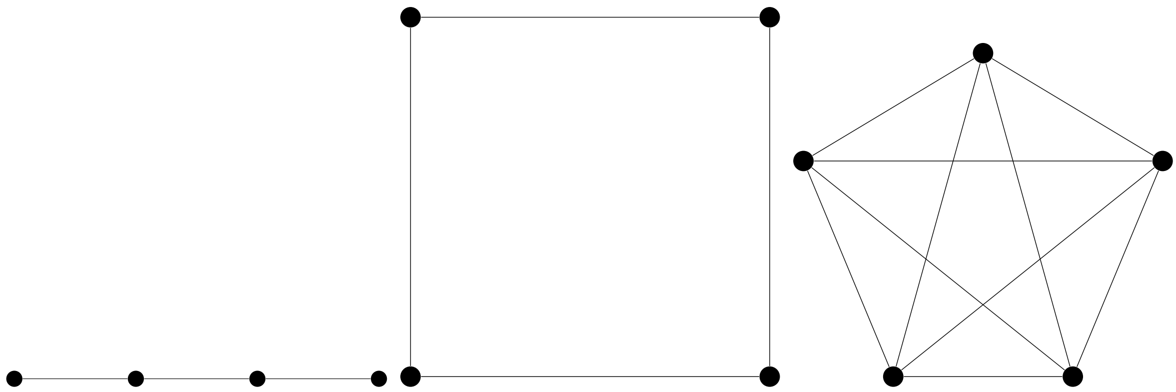
Definition 1. Simple Graph
A simple graph is an unweighted and undirected graph containing no loops or multiple edges.

Definition 2. Connected Graph
A connected graph is a graph which has a path connecting any point to any other point in the graph.

Definition 3. Vertex Degree
The degree of a vertex v of a graph G is the number of edges that are connected to v. We denote the degree of a vertex v as $\rho(v)$.

Definition 4. Maximum Vertex Degree
The maximum vertex degree of a graph G is denoted as $\Delta(G)$. This value can be 0 if $|E(G)|$ is 0, i.e. there are no edges in G.

Definition 5. Path Graph
A path graph is a graph that can be drawn such that all of its vertices and edges lie on a straight line. The two outermost vertices have degree 1 and the other $n-2$ vertices have degree 2. The graph is denoted as P_n.



(a) Path Graph P_4 (b) Cycle Graph C_4 (c) Complete Graph K_5

Figure 1: Some basic graphs

Figure 1 shows some examples of simple and connected graphs.

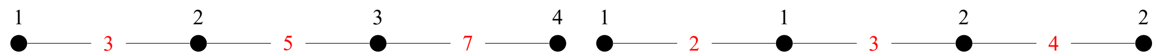
Definition 6. Vertex Labeling

A vertex labeling of a graph G is an assignment of integers onto every vertex of G.

Remark. The labeling function does not necessarily have to be one-to-one or onto.

Definition 7. ([1]) Edge Irregularity Strength

Let $G = (V, E)$ be a simple-connected graph. The edge irregular k-labeling of a graph G is defined to be the labeling of the vertices of G, $\phi : V(G) \to \{1, 2, ..., k\}$ such that the edge weights $w_\phi(vu) = \phi(v) + \phi(u)$ are distinct for every edge in G. The minimum k that satisfies the graph G having an edge irregular k-labeling is called the edge irregularity strength of G, denoted as $es(G)$.



(a) Edge irregular 4-labeling of P_4 (b) Edge irregular 2-labeling of P_4

Figure 2: Two ways to edge irregularly label the vertices of path graph P_4

We can label the vertices of a graph in more than one way given there are at least two vertices. Figure 2 shows two ways of labeling P_4 such that it is an edge irregular labeling. The red text shows the respective edge weights of the graph. A straightforward way to label the vertices so that it is edge irregular is simply to label all the vertices different positive integers which automatically makes it an edge irregular labeling, given in the figure by (a). A better way to label them so that less integers are used is given in the figure by (b). In fact, this is the minimum k that satisfies edge irregularity as labeling every vertex 1 does not, implying that $es(P_4) = 2$.

The following theorem formulates the lower bound for the edge irregularity strength for an arbitrary simple graph G.

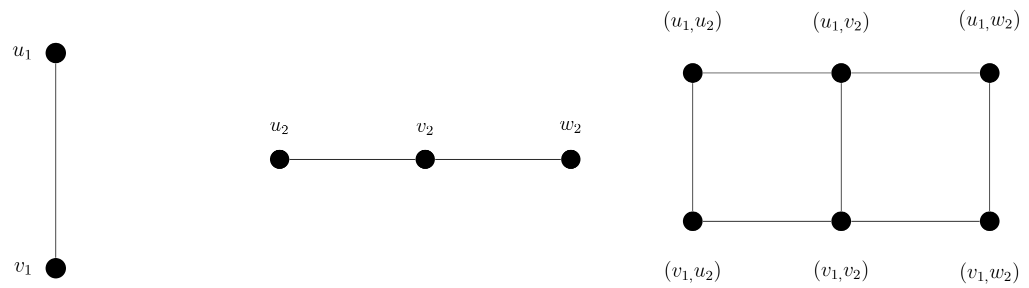
Theorem 1. ([1])

Let $G = (V, E)$ be a simple graph with maximum degree $\Delta = \Delta(G)$. Then,

$$es(G) \geq \max\left\{\left\lceil\frac{|E(G)|+1}{2}\right\rceil, \Delta(G)\right\}$$

Definition 8. Graph Cartesian Product

Let there be two graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where V_1 and V_2 are disjoint. The cartesian product $G = G_1 \square G_2$, sometimes denoted as $G = G_1 \times G_2$, is a graph with $V(G) = V_1 \times V_2$ and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever [$u_1 = v_1$ and u_2 adj v_2] or [$u_2 = v_2$ and u_1 adj v_1].



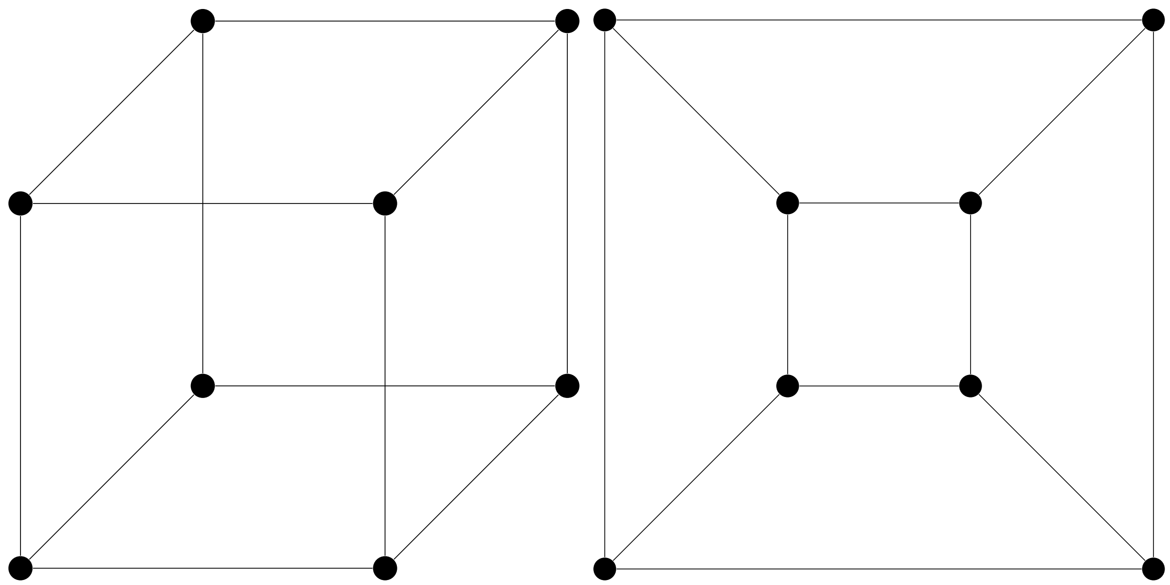
(a) Path Graph P_2 (b) Path Graph P_3 (c) Cartesian Product $P_2 \square P_3$

Figure 3: The domino graph $P_2 \square P_3$

The cartesian product was introduced in 1950 by Gert Sabidussi along with strong and weak multiplication graph operations [3].

2. $m \times n \times l$ grid graph

Consider the cartesian product $P_n \square P_m \square P_l$ for $m, n, l \geq 2$. When we draw this, we will attain an $m \times n \times l$ grid graph.



(a) (b)

Figure 4: (a) Cubical Graph $Q_3 = P_2 \square P_2 \square P_2$, (b) The same simpler-drawn graph

Figure 4 shows an example of the $m \times n \times l$ grid graph with $m = n = l = 2$.

3. Edge irregularity strength of $m \times n \times l$ grid graph

The following theorem establishes the edge irregularity strength of the $m \times n \times l$ grid graph, i.e. the cartesian product $P_n \Box P_m \Box P_l$ for $n, m \geq 2$ and a fixed $l = 2$.

Theorem 2. ([5])
Let $G = P_n \Box P_m \Box P_2$ where $m, n \geq 2$. Then,

$$es(G) = \left\lceil \frac{5mn - 2m - 2n + 1}{2} \right\rceil$$

The above theorem is achievable using the following labeling function defined by the authors.

$$\phi_3(x_i y_j z_r) = \begin{cases} \frac{i-1}{2}(5m-2) + \left\lfloor \frac{r}{2} \right\rfloor (m-1) + \left\lceil \frac{j+r-1}{2} \right\rceil, & \text{if } i \text{ is odd} \\ \frac{i-1}{2}(5m-2) + 4m + r - \left\lfloor \frac{j-r+1}{2} \right\rfloor - 2\left\lfloor \frac{j+r-2}{2} \right\rfloor - 3, & \text{if } i \text{ is even} \end{cases}$$

where $V(G) = \{(x_i, y_j, z_r) : 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq r \leq 2\}$ and $E(G) = \{(x_i, y_j, z_r)(x_{i+1}, y_j, z_r) : 1 \leq i \leq n-1, 1 \leq j \leq m, 1 \leq r \leq l\} \cup \{(x_i, y_j, z_r)(x_i, y_{j+1}, z_r) : 1 \leq i \leq n, 1 \leq j \leq m-1, 1 \leq r \leq l\}\{(x_i, y_j, z_1)(x_i, y_j, z_2) : 1 \leq i \leq n, 1 \leq j \leq m\}$.

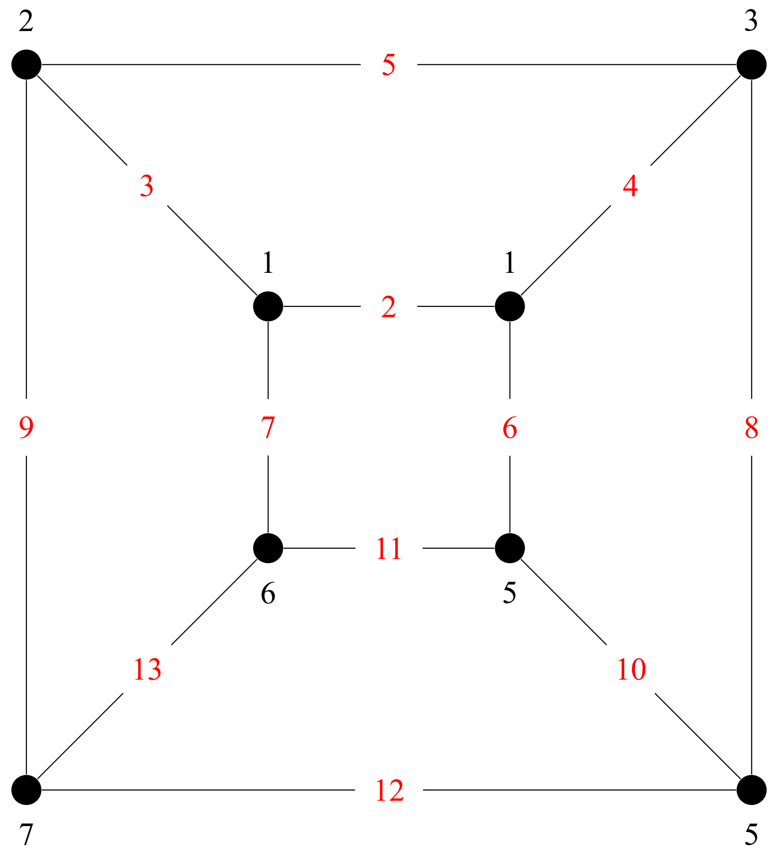


Figure 5: Edge Irregular 7-labeling of $P_2 \Box P_2 \Box P_2$

Using Theorem 2, we find that $es(P_2 \Box P_2 \Box P_2) = 7$. Figure 5 shows the edge irregular 7-labeling of $P_2 \Box P_2 \Box P_2$ achieved using the ϕ_3 labeling function. The red text shows the respective edge weights of the graph.

The same paper by I. Tarawneh, R. Hasni, and A. Ahmad, that established the above theorem posed the following open problem, "Determine the exact value of $es(P_n \Box P_m \Box P_l)$ for $m, n \geq 2$ and $l \geq 3$."

To generalize the cases where $l \geq 3$, we need to establish a lower bound for $P_n \square P_m \square P_l$ for $m, n, l \geq 3$.

Theorem 3.

Let $G = P_n \square P_m \square P_l$ where $m, n, l \geq 3$. Then,

$$es(G) \geq \left\lceil \frac{3lmn - mn - lm - ln + 1}{2} \right\rceil$$

Proof.

Let $m, n, l \geq 3$ and $G = P_n \square P_m \square P_l$. For each of the graphs P_n, P_m, P_l, it is clear that $\Delta(P_n) = \Delta(P_m) = \Delta(P_l) = 2$ by Definition 5. The cartesian product operation will cause $\Delta(P_n \square P_m) = \Delta(P_n) + \Delta(P_m) = 4$ because every degree of a vertex increases by a maximum of 2. This implies that $\Delta(G) = \Delta(P_n) + \Delta(P_m) + \Delta(P_l) = 6$.

Counting the bases, we find $l(n(m-1) + m(n-1))$ edges and counting the sides, we find $mn(l-1)$ edges. Hence, $|E(G)| = l(n(m-1)+m(n-1))+mn(l-1) = 3lmn-mn-lm-ln$.

By Theorem 1 we know that,

$$es(G) \geq \max\left\{ \left\lceil \frac{3lmn - mn - lm - ln + 1}{2} \right\rceil, 6 \right\}$$

We can find the lowermost value for $es(G)$ by considering the case when $m = n = l = 3$. Notice that $\left\lceil \frac{3lmn-mn-lm-ln+1}{2} \right\rceil$ will evaluate to 28. This implies that for any $m, n, l \geq 3$, $\left\lceil \frac{3lmn-mn-lm-ln+1}{2} \right\rceil$ will always be higher than 6. Hence,

$$es(G) \geq \left\lceil \frac{3lmn - mn - lm - ln + 1}{2} \right\rceil$$

This completes the proof. $\square$

This result is useful, because it helps us know the lower bound of any $P_n \square P_m \square P_l$ graph where $m, n, l \geq 3$.

Corollary.

Let $G = P_n \square P_m \square P_3$ where $m, n \geq 3$. Then,

$$es(G) \geq \left\lceil \frac{8mn - 3m - 3n + 1}{2} \right\rceil$$

Useful as it is, this result isn't the exact edge irregularity strength of $P_n \square P_m \square P_l$ for any $m, n, l \geq 3$. Therefore, we propose a conjecture to solve this.

Conjecture.

Let $G = P_n \square P_m \square P_l$ where $m, n, l \geq 2$. Then,

$$es(G) = \left\lceil \frac{3lmn - mn - lm - ln + 1}{2} \right\rceil$$

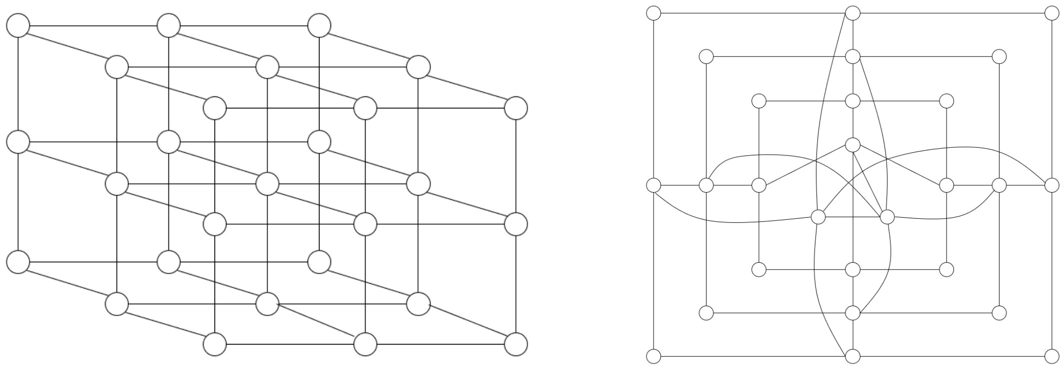
This simply means that we believe that the edge irregularity strength of the graph $P_n \square P_m \square P_l$ where $m, n, l \geq 2$ is exactly the lower bound. We have yet to attain the proof to this.

In the case where $m, n \geq 2$ and $l = 2$, our conjecture means that:

$$es(G) = \left\lceil \frac{5mn - 2m - 2n + 1}{2} \right\rceil$$

which is exactly the result of Theorem 2.

In the case where $m = n = l = 3$, our conjecture means that $es(P_3 \square P_3 \square P_3) = 28$. But so far, we have managed an edge irregular 29-labeling, which is still one off.



(a) (b)

Figure 6: (a) Cubical Graph $Q_4 = P_3 \square P_3 \square P_3$, (b) The same simpler-drawn graph

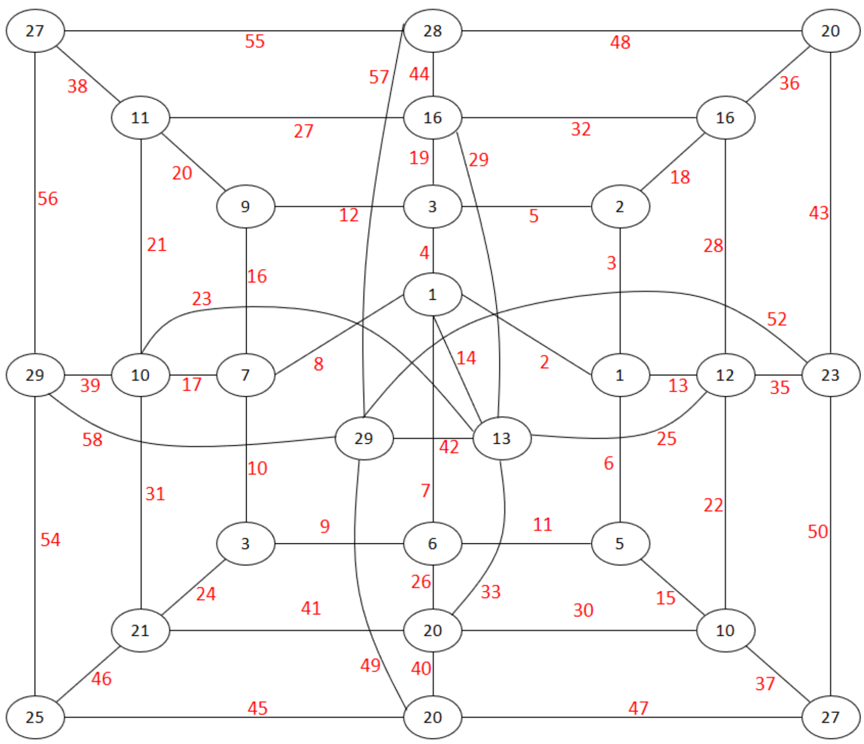


Figure 7: Edge Irregular 29-labeling of $P_3 \square P_3 \square P_3$

Figure 6 shows the graph $P_3 \square P_3 \square P_3$ and Figure 7 shows the edge irregular 29-labeling of $P_3 \square P_3 \square P_3$ that we have managed. The red text shows the respective edge weights of the graph. Since an edge irregular 29-labeling of $P_3 \square P_3 \square P_3$ is possible and because by Theorem 3, $es(P_3 \square P_3 \square P_3) \geq 28$, this implies that $es(P_3 \square P_3 \square P_3)$ is either 28 or 29. We still believe that an edge irregular 28-labeling of $P_3 \square P_3 \square P_3$ is possible because notice that up to the highest edge weight 58, the edge weights $34, 51,$ and 53 are not used in Figure 7.

4. Conclusion

In this paper, we discussed the $m \times n \times l$ grid graph, i.e. the cartesian product $P_n \square P_m \square P_l$ where $m, n, l \geq 2$ and its edge irregularity strength. We managed to obtain the lower bound of $es(P_n \square P_m \square P_l)$ for any $m, n, l \geq 3$.

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