Xyba Project

Persamaan Differensial Biasa

Short Summary for UTS

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C. This document is created by Xyba, Student of Mathematics University of Indonesia Batch 2016

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Thank you for your cooperation >v<

A. PDB Orde 1

1. PDB Eksak

   $Mdx + Ndy = 0$ merupakan PDB Eksak jika $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

2. PDB Non-Eksak

   Apabila $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

   Faktor Integrasi:

   $R(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ atau $R(y) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$

   $F = e^{\int R(x)dx} = e^{\int R(y)dy}$

   $Mdx + Ndy = 0 \Rightarrow FMdx + FNdy = 0$

3. PDB Linier

   Bentuk Umum:

   $$\frac{dy}{dx} + P(x)y = Q(x)$$

   Faktor Integrasi: $F(x) = e^{\int P\,dx}$

   Solusi: $yF = \int QF\,dx + C$

4. PDB Bernoulli

   $\frac{dy}{dx} + P\,y = Q\,y^n$

5. PDB Riccati

   $\frac{dy}{dx} = P + Qy + Ry^2$

   Jika y_1 merupakan solusi maka $y = y_1 + \frac{1}{V}$

   $\frac{dV}{dx} + (Q + 2R\,y_1)V = -R$

B. Sifat-sifat

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$$

$$d\left(\frac{x}{y}\right) = \frac{x\,dy - y\,dx}{y^2}$$

$$d\left(\ln\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{xy}$$

$$d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

$$d(\ln(xy)) = \frac{x\,dy + y\,dx}{xy}$$

$$\frac{1}{2}d(\ln(x^2 + y^2)) = \frac{x\,dx + y\,dy}{x^2 + y^2}$$

C. PDB Orde 2

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = g(x)$$

Solusi: $y(x) = \text{Solusi Homogen} + \text{Solusi Khusus}$

- Solusi Homogen: $y'' + a_1 y' + a_0 y = 0$

  Persamaan Karakteristik: $\lambda^2 + a_1\lambda + a_0 = 0$Jika $\lambda_1 \neq \lambda_2$, maka $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
  1. Jika $\lambda_1 \neq \lambda_2$, maka $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
  2. Jika $\lambda_1 = \lambda_2 = \lambda$, maka $y = c_1 e^{\lambda_1 x} + x c_1 e^{\lambda_1 x}$
  3. Jika $\lambda = p \pm qi$, maka $y = e^{px}(c_1 \cos qx + c_2 \sin qx)$

- Solusi Khusus
  1) Metode Koefisien Tak Tentu

     $y'' + a_1 y' + a_0 y = g(x)$

$g(x)$	Pemisalan
x^2	$y = Ax^2 + Bx + C$
e^{mx}	$y = Ae^{mx}$
$\cos mx$	$y = A\cos mx + B\sin mx$
$\sin mx$	

  2) Metode Variasi Parameter

$$y'' + a_1 y' + a_0 y = g(x)$$

Misal

Solusi Homogen: $y = c_1 y_1 + c_2 y_2$

$\therefore$ Solusi Khusus: $y = u y_1 + v y_2$

Dengan syarat:

$$u' y_1 + v' y_2 = 0$$
$$u' y_1' + v' y_2' = g(x)$$

D. Transformasi Laplace

$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s),$ ada jika konvergen

$$\mathcal{L}[1] = \frac{1}{s}, s > 0$$
$$\mathcal{L}[e^{at}] = \frac{1}{s-a}, s > a$$
$$\mathcal{L}[t] = \frac{1}{s^2}, s > 0$$
$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$
$$\mathcal{L}[\cos bt] = \frac{s}{s^2 + b^2}$$
$$\mathcal{L}[\sin bt] = \frac{b}{s^2 + b^2}$$

Sifat-sifat:

1. $\mathcal{L}[a\, f(t) + b\, g(t)] = a\, \mathcal{L}[f(t)] + b\, \mathcal{L}[g(t)]$
2. $\mathcal{L}[f'(t)] = s\, \mathcal{L}[f(t)] - f(0)$

$$\mathcal{L}[f''(t)] = s^2 \mathcal{L}[f(t)] - sf(0) - f'(0)$$
$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n \mathcal{L}[f(t)] - \sum_{i=0}^{n-1} s^{n-i-1} f^{(i)}(0)$$

Sifat Translasi

$$\mathcal{L}[e^{at} f(t)] = F(s-a)$$

$\mathcal{L}[t\, f(t)] = -F'(s)$
$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

E. Fungsi Heaviside

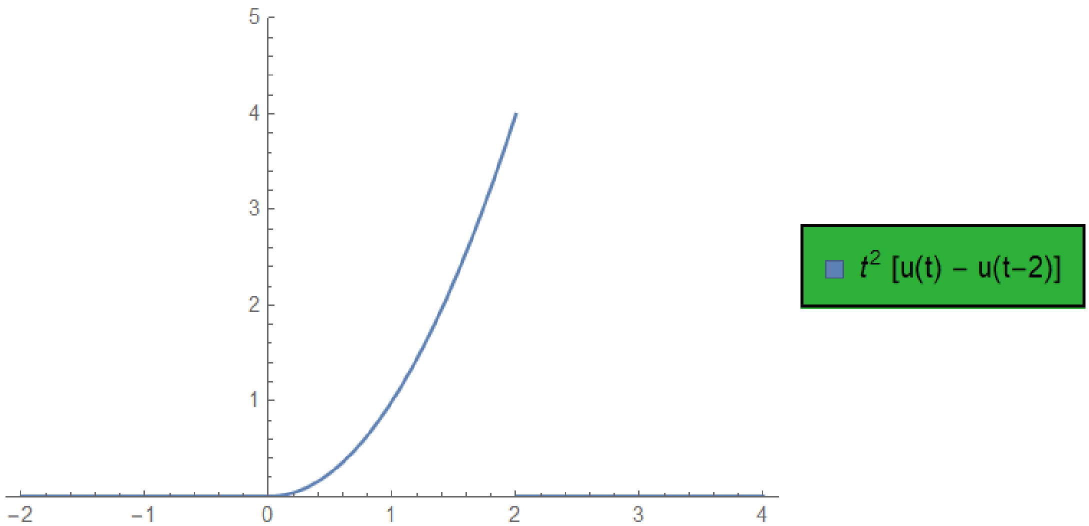
$u(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$

$\mathcal{L}[u(t-a)] = \frac{1}{s} e^{-as}$

$\mathcal{L}[u(t-a) - u(t-b)] = \frac{1}{s}(e^{-as} - e^{-bs})$

Fungsi Filter:
$f(t) = u(t-a) - u(t-b)$
$\mathcal{L}[f(t)] = \frac{1}{s}(e^{-as} - e^{-bs})$

e.g.:



Translasi t

$$f(t-a)\, u(t-a) = \begin{cases} 0, & t \le a \\ f(t-a), & t > a \end{cases}$$

$\mathcal{L}[f(t-a)\, u(t-a)] = e^{-as} F(s)$

F. Konvolusi

$f(t) * g(t) = \int_0^t f(\omega)\, g(t-\omega) d\omega$

$$g(t) * f(t) = \int_0^t f(t-\omega)\, g(\omega) d\omega$$

($*$: operator konvolusi)

$f(t) * g(t) = g(t) * f(t)$

$\mathcal{L}[f(t) * g(t)] = F(s)G(s)$

e.g.:

1. $H(s) = \frac{1}{s}\frac{1}{s^2+1}$

   Ambil $F(s) = \dfrac{1}{s} \rightarrow f(t) = 1$

   Ambil $G(s) = \dfrac{1}{s^2+1} \rightarrow g(t) = \sin t$

   $f(t) * g(t) = \displaystyle\int_0^t \cos(t-\omega)\, d\omega = -\int_0^t \sin(t-\omega)\, d(t-\omega) = -\cos t$

2. $H(s) = \frac{1}{s+1}\frac{s}{s^2+4}$

   Ambil $F(s) = \dfrac{1}{s+1} \rightarrow f(t) = e^{-t}$

   Ambil $G(s) = \dfrac{s}{s^2+4} \rightarrow g(t) = \cos 2t$

   $\mathcal{L}^{-1}[H(s)] = \displaystyle\int_0^t e^{-t} \cos(2t-2\omega)\, d\omega = -\frac{1}{2} e^{-t} \int_{2t}^0 \cos(2t-2\omega)\, d(2t-2\omega)$

   $\mathcal{L}^{-1}[H(s)] = \dfrac{1}{2} e^{-t} \sin 2t$

G. Fungsi Delta Dirac

$$f_\epsilon = \begin{cases} \dfrac{1}{\varepsilon}, a < t < a+\varepsilon \\ 0, \qquad\quad t \text{ lain} \end{cases}$$

$$\delta(t-a) = \lim_{\varepsilon \to 0} f(\varepsilon) = 0, \qquad t \neq a$$

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1$$

$\mathcal{L}[\delta(t-a)] = e^{-as}$

$\mathcal{L}[\delta(t)] = 1$

H. Invers Transformasi Laplace (Fungsi Rasional)

$$F(s)=\frac{P(s)}{Q(s)}\Rightarrow F(s)=\frac{P_1(s)}{Q_1(s)}+\frac{P_2(s)}{Q_2(s)}$$

1) Jika $Q(s)=(s-a)Q_2(s)$

$$F(s)=\frac{P(s)}{Q(s)}=\frac{A}{s-a}+\frac{P_2(s)}{Q_2(s)}$$

-------------------------------------- $\times (s-a)$

$$(s-a)\frac{P(s)}{Q(s)}=A+\frac{P_2(s)}{Q_2(s)}(s-a)$$

$$W(s)=\frac{P(s)}{Q_2(s)}=A+P_2(s)\frac{s-a}{Q_2(s)}$$

$$W(a)=\frac{P(a)}{Q_2(a)}=A$$

2) Jika $Q(s)=(s-a)^2Q_2(s)$

$$F(s)=\frac{P(s)}{Q(s)}=\frac{A}{(s-a)^2}+\frac{B}{s-a}+\frac{P_2(s)}{Q_2(s)}$$

-- $\times (s-a)^2$

$$(s-a)^2\frac{P(s)}{Q(s)}=A+B(s-a)+\frac{P_2(s)}{Q_2(s)}(s-a)^2$$

$$W(s)=\frac{P(s)}{Q_2(s)}=A+B(s-a)+P_2(s)\frac{(s-a)^2}{Q_2(s)}$$

$$W(a)=A$$

$$W'(b)=B$$

3) Jika $Q(s)=[(s-a)^2+b^2]Q_2(s)$

$$F(s)=\frac{P(s)}{Q(s)}=\frac{A(s-a)+Bb}{(s-a)^2+b^2}+\frac{P_2(s)}{Q_2(s)}$$

-- $\times ((s-a)^2+b^2)$

$$W(s)=A(s-a)+Bb+H(s)[(s-a)^2+b^2]$$

$$W(a+bi)=Abi+Bb$$

$$A=\frac{1}{b}\Im\big(W(a+bi)\big)$$

$$B=\frac{1}{b}\Re\big(W(a+bi)\big)$$

e.g.:

Misal $F(s) = \frac{s^2}{[(s-2)^2+9](s-4)}$, tentukan $f(t)$

$$F(s) = \frac{A(s-2)+3B}{(s-2)^2+9} + \frac{C}{s-4}$$

- Mencari C

  $W(s) = \frac{s^2}{(s-2)^2+9}$

  $C = W(4) = \frac{16}{4+9} = \frac{16}{13}$

- Mencari A dan B

  $W(s) = \frac{s^2}{s-4}$

  $W(2+3i) = \frac{4+12i-9}{2+3i-4} = \frac{12i-5}{3i-2}\frac{3i+2}{3i+2} = \frac{-36+9i-10}{-9-4} = \frac{-46+9i}{-11} = \frac{46}{11} - \frac{9}{11}i$

  $A = \frac{1}{3}\Im\big(W(2+3i)\big) = \frac{1}{3}\left(-\frac{9}{11}\right) = -\frac{3}{11}$

  $B = \frac{1}{3}\Re\big(W(2+3i)\big) = \frac{1}{3}\left(\frac{46}{11}\right) = \frac{46}{33}$

$$F(s) = \frac{-\frac{3}{11}(s-2)+3\left(\frac{46}{33}\right)}{(s-2)^2+9} + \frac{\frac{16}{13}}{s-4}$$

$$\therefore f(t) = -\frac{3}{11}e^{2t}\cos 3t + \frac{46}{33}e^{2t}\sin 3t + \frac{16}{13}e^{4t}$$