Problem A
Find the roots of $f(x) = (e^x - e^\pi)(e^x - \pi)$ where e denotes Euler's number.

Answer:
We denote the set of real numbers and integers by $\mathbb{R}$ and $\mathbb{Z}$ respectively.
Since it is not mentioned, we assume that we are not only finding $x \in \mathbb{R}$ but all $x \ni f(x) = 0$.

The roots of $f(x)$ are x's such that $f(x) = 0 \Leftrightarrow (e^x - e^\pi)(e^x - \pi) = 0$.
This implies that either $e^x - e^\pi = 0$ or $e^x - \pi = 0$.

Let $x = p + iq$ where $p, q \in \mathbb{R}$ and i is the imaginary unit.

If $e^x - e^\pi = 0$, then we have:

$$e^x = e^\pi \Leftrightarrow e^{p+iq} = e^\pi \Leftrightarrow e^p e^{iq} = e^\pi$$

Since $q \in \mathbb{R}$, then by Euler's formula, $e^{iq} = \cos q + i \sin q$, hence:

$$e^p e^{iq} = e^\pi \Leftrightarrow e^p(\cos q + i \sin q) = e^\pi \Leftrightarrow e^p \cos q + ie^p \sin q = e^\pi$$

This implies that $e^p \cos q = e^\pi$ and $ie^p \sin q = 0$ simultaneously.
Since $e^p \neq 0$ for an arbitrary $p \in \mathbb{R}$, then we find that $\sin q = 0 \Leftrightarrow q = 2n\pi, n \in \mathbb{Z}$.
This means that $\cos q = \cos 2n\pi = 1$. So, $e^p \cos q = e^\pi \Leftrightarrow e^p = e^\pi \Leftrightarrow p = \pi$.
Hence, for $e^x - e^\pi = 0$, we have $x = p + iq = \pi + i \cdot 2n\pi = \pi + 2in\pi, n \in \mathbb{Z}$.

If $e^x - \pi = 0$, then we have:

$$e^x = \pi \Leftrightarrow e^{p+iq} = \pi \Leftrightarrow e^p e^{iq} = \pi$$

Since $q \in \mathbb{R}$, then by Euler's formula, $e^{iq} = \cos q + i \sin q$, hence:

$$e^p e^{iq} = \pi \Leftrightarrow e^p(\cos q + i \sin q) = \pi \Leftrightarrow e^p \cos q + ie^p \sin q = \pi$$

This implies that $e^p \cos q = \pi$ and $ie^p \sin q = 0$ simultaneously.
Since $e^p \neq 0$ for an arbitrary $p \in \mathbb{R}$, then we find that $\sin q = 0 \Leftrightarrow q = 2k\pi, k \in \mathbb{Z}$.
This means that $\cos q = \cos 2k\pi = 1$. So, $e^p \cos q = \pi \Leftrightarrow e^p = \pi \Leftrightarrow p = \ln \pi$.
Hence, for $e^x - \pi = 0$, we have $x = p + iq = \ln \pi + i \cdot 2k\pi = \ln \pi + 2ik\pi, k \in \mathbb{Z}$.

This means the roots of $f(x) = (e^x - e^\pi)(e^x - \pi)$ are the sets:

$$\{\pi + 2in\pi : n \in \mathbb{Z}\} \text{ and } \{\ln \pi + 2ik\pi : k \in \mathbb{Z}\}$$

$\therefore$ The roots of $f(x) = (e^x - e^\pi)(e^x - \pi)$ are given by the set:

$$\{x : f(x) = 0\} = \{\pi + 2in\pi : n \in \mathbb{Z}\} \cup \{\ln \pi + 2ik\pi : k \in \mathbb{Z}\}$$

Problem B
Show that $n^4 - n^3 + n^2 - n$ is divisible by 2 for all positive integers n.

Answer:
We denote the set of positive integers as $\mathbb{N}$ and extensively for $\mathbb{N}$ with 0 as $\mathbb{N}_0$.
Note that:
$$n^4 - n^3 + n^2 - n = n(n^3 - n^2 + n - 1) = n(n-1)(n^2+1)$$

Since n and $n-1$ have different parities for any given $n \in \mathbb{N}$, i.e. if n is odd then $n-1$ is even and if n is even then $n-1$ is odd, then $n(n-1)(n^2+1)$ is even, meaning it is divisible by 2.

If n is odd, then we can write $n = 2k+1 \Leftrightarrow n-1 = 2k, k \in \mathbb{N}_0$. Hence:
$$n^4 - n^3 + n^2 - n = n(n-1)(n^2+1) = 2(nk(n^2+1)), k \in \mathbb{N}_0$$
Therefore, it is divisible by 2.

If n is even, then we can write $n = 2m, m \in \mathbb{N}$. Hence:
$$n^4 - n^3 + n^2 - n = n(n-1)(n^2+1) = 2(m(n-1)(n^2+1)), m \in \mathbb{N}$$
Therefore, it is divisible by 2.

$\therefore$ It is shown that $n^4 - n^3 + n^2 - n$ is divisible by 2 for all positive integers n.

Problem C
You have given a sphere with a volume of π^3. What is the radius of this sphere?
Explain whether or not it is possible to build such a sphere in reality?

Answer:
A sphere with radius r has a volume of $\frac{4}{3}\pi r^3$. Hence, we have:

$$\frac{4}{3}\pi r^3 = \pi^3 \Leftrightarrow r^3 = \frac{3}{4}\pi^2 = 0.75\pi^2 \Leftrightarrow r = \sqrt[3]{0.75}\pi^{\frac{2}{3}}$$

It is impossible to build a sphere in reality with radius π because π is an irrational number, meaning we cannot pinpoint the exact measurements, i.e. there is bound to be an error in the building process. Since it is impossible to build a sphere in reality with radius π, it is also impossible to build a sphere with radius $\sqrt[3]{0.75}\pi^{\frac{2}{3}}$.

$\therefore$ The radius of a sphere with volume π^3 is $\sqrt[3]{0.75}\pi^{\frac{2}{3}}$ and in reality, it is impossible to build such a sphere.

Problem D
Find the numerical value of the following expression without the use of a calculator.

$$\log_2(2^2 + 5 \cdot 2^2 \cdot 3) \cdot (2\log_3 2 + \log_3(7 - \frac{1}{4})) + \frac{(\log_2 128 - 2)^3}{3+2} + (-1)^{32+\pi^0}$$

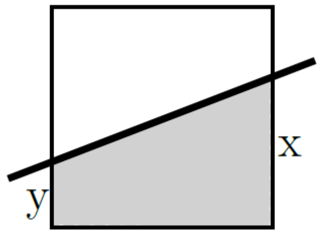
Answer:
By logarithm properties, we find that:

$$\begin{aligned}
&\log_2(2^2 + 5 \cdot 2^2 \cdot 3) \cdot (2\log_3 2 + \log_3(7 - \frac{1}{4})) + \frac{(\log_2 128 - 2)^3}{3+2} + (-1)^{32+\pi^0} \\
&= \log_2(4 + 60) \cdot (\log_3 4 + \log_3(\frac{27}{4})) + \frac{(7-2)^3}{5} + (-1)^3 \\
&= \log_2(64) \cdot \log_3(27) + 5^2 + ((-1)^2)^{16}(-1) \\
&= 6 \cdot 3 + 25 - 1 \\
&= 18 + 24 \\
&= 42
\end{aligned}$$

$$\therefore \log_2(2^2 + 5 \cdot 2^2 \cdot 3) \cdot (2\log_3 2 + \log_3(7 - \frac{1}{4})) + \frac{(\log_2 128 - 2)^3}{3+2} + (-1)^{32+\pi^0} = 42$$

Problem E

The square below has an edge length of a. A line intersects the square at a height of x and y. Find an expression for the surface area $A(x, y)$ below the line (gray area).



Answer:
Notice that $A(x, y)$ is a trapezoid.

The area of a trapezoid with lower base p, upper base q, and height h is given by:

$$\frac{h}{2} \cdot (p + q)$$

Hence,

$$A(x, y) = \frac{a}{2} \cdot (x + y)$$

Alternatively, we can view the area $A(x, y)$ as the area of the square subtracted by the area of the white trapezoid, i.e.:

$$A(x, y) = a^2 - \frac{a}{2} \cdot ((a - x) + (a - y)) = a(a - \frac{2a - (x + y)}{2}) = a(\frac{x + y}{2} = \frac{a}{2}(x + y)$$

which yields the same result as it should.

$\therefore$ The surface area $A(x, y)$ below the line is given by the expression:

$$A(x, y) = \frac{a}{2}(x + y)$$