Xyba Project

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1st Kyu Past Test Section 1

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1. Solve the following system of equations for real numbers x, y, and z, giving all solutions.

$$\begin{cases} xy^2z^3 = \dfrac{12}{7} \\ x^3yz^2 = -\dfrac{7}{11} \\ x^2y^3z = -\dfrac{11}{12} \end{cases}$$

Answer:
Let $a \coloneqq xyz$, then the system becomes:

$$\begin{cases} ayz^2 = \dfrac{12}{7} & \dots(1) \\ ax^2z = -\dfrac{7}{11} & \dots(2) \\ axy^2 = -\dfrac{11}{12} & \dots(3) \end{cases}$$

Notice that if we multiply (1), (2), and (3), we get:

$$ayz^2 \cdot ax^2z \cdot axy^2 = \frac{12}{7} \cdot -\frac{7}{11} \cdot -\frac{11}{12} \Leftrightarrow a^3x^3y^3z^3 = 1 \Leftrightarrow a^6 = 1 \Leftrightarrow a^6 - 1 = 0$$

It is obvious that 1 and -1 are solutions for a. We can find the other roots using Horner but only $a = xyz = \pm 1$ belongs to $\mathbb{R}$. We now can use this information to solve the system.

We find the relations:

$$xy^2z^3 = \frac{12}{7} \quad \Leftrightarrow \quad (xyz)^2\frac{z}{x} = \frac{12}{7} \quad \Leftrightarrow \quad \frac{z}{x} = \frac{12}{7} \quad \Leftrightarrow \quad z = \frac{12}{7}x$$

$$x^3yz^2 = -\frac{7}{11} \quad \Leftrightarrow \quad (xyz)^2\frac{x}{y} = -\frac{7}{11} \quad \Leftrightarrow \quad \frac{x}{y} = -\frac{7}{11} \quad \Leftrightarrow \quad y = -\frac{11}{7}x$$

Substitute these relations to the first equation and we find that:

$$xy^2z^3 = \frac{12}{7} \Leftrightarrow x\left(-\frac{11}{7}x\right)^2\left(\frac{12}{7}x\right)^3 = \frac{12}{7} \Leftrightarrow x^6 \cdot \frac{11^2 \cdot 12^3}{7^2 \cdot 7^3} = \frac{12}{7} \Leftrightarrow x^6 = \frac{12}{7} \cdot \frac{7^5}{11^2 \cdot 12^3}$$

$$\Leftrightarrow x^6 = \frac{7^6}{7^2 \cdot 11^2 \cdot 12^2} \Leftrightarrow x^6 = \sqrt[6]{\frac{7^6}{(7 \cdot 11 \cdot 12)^2}} \Leftrightarrow x = \pm\frac{7}{\sqrt[3]{7 \cdot 11 \cdot 12}} = \pm\frac{7}{\sqrt[3]{924}}$$

Simply substitute the result to the relations.

$$y = -\frac{11}{7}x = \mp\frac{11}{\sqrt[3]{924}}, \qquad z = \frac{12}{7}x = \pm\frac{12}{\sqrt[3]{924}}$$

$\therefore$ The solution of the system for $x, y, z \in \mathbb{R}$ are:

$$(x, y, z) = \left(\pm\frac{7}{\sqrt[3]{924}}, \mp\frac{11}{\sqrt[3]{924}}, \pm\frac{12}{\sqrt[3]{924}}\right)$$

with respective corresponding signs.

2. Simplify the following expression. Note that i represents the imaginary unit.

$$\frac{(1-i)^{11}}{\left(-\sqrt{3}+i\right)^6}$$

Answer:

$$\frac{(1-i)^{11}}{\left(-\sqrt{3}+i\right)^6}=\frac{((1-i)^2)^5(1-i)}{\left(\left(-\sqrt{3}+i\right)^2\right)^3}=\frac{(1-2i-1)^5(1-i)}{\left(3-2i\sqrt{3}-1\right)^3}=\frac{(-2i)^5(1-i)}{\left(2-2i\sqrt{3}\right)^3}=\frac{-32i^5(1-i)}{8\left(1-i\sqrt{3}\right)^3}$$

$$=-4\frac{i(1-i)}{\left(1-i\sqrt{3}\right)^2\left(1-i\sqrt{3}\right)}=-4\frac{i+1}{\left(1-2i\sqrt{3}-3\right)\left(1-i\sqrt{3}\right)}$$

$$=-4\frac{1+i}{-2\left(1+i\sqrt{3}\right)\left(1-i\sqrt{3}\right)}=2\frac{1+i}{1+3}=\frac{1+i}{2}$$

$\therefore \dfrac{(1-i)^{11}}{\left(-\sqrt{3}+i\right)^6}$ can be simplied to $\dfrac{1+i}{2}$

3. Find the inverse matrix of the following matrix.

$$\begin{pmatrix} 1 & 0 & 4 \\ 3 & -1 & 0 \\ -2 & 1 & -1 \end{pmatrix}$$

Answer:

Let A be the matrix given. We can find the inverse matrix by performing elementary row operations such that $[A|I]$ becomes $[I|A^{-1}]$. Notice that:

$$[A|I]=\begin{bmatrix} 1 & 0 & 4 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} b_2-3b_1 \\ \sim \\ b_3+2b_1 \end{matrix} \begin{bmatrix} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & -1 & -12 & -3 & 1 & 0 \\ 0 & 1 & 7 & 2 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} -b_2 \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 12 & 3 & -1 & 0 \\ 0 & 1 & 7 & 2 & 0 & 1 \end{bmatrix} \begin{matrix} b_3-b_2 \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 12 & 3 & -1 & 0 \\ 0 & 0 & -5 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{matrix} -\frac{1}{5}b_3 \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 12 & 3 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \end{bmatrix} \begin{matrix} b_1-4b_3 \\ \sim \\ b_1-12b_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{5} & \frac{4}{5} & \frac{4}{5} \\ 0 & 1 & 0 & \frac{3}{5} & \frac{7}{5} & \frac{12}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \end{bmatrix}=[I|A^{-1}]$$

$$\therefore A^{-1}=\begin{pmatrix} \frac{1}{5} & \frac{4}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{7}{5} & \frac{12}{5} \\ \frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \end{pmatrix}=\frac{1}{5}\begin{pmatrix} 1 & 4 & 4 \\ 3 & 7 & 12 \\ 1 & -1 & -1 \end{pmatrix}$$

4. For a surface $z = \arctan\frac{y}{x}$, where $x \neq 0$ and $-\frac{\pi}{2} < z < \frac{\pi}{2}$, answer the following.
   a. Compute $\frac{\partial z}{\partial x}$.
   b. Find the equation of the tangent plane to the above surface at the point
$$(x, y, z) = \left(1, -1, -\frac{\pi}{4}\right)$$

Answer:

a. Given $z = \arctan\left(\frac{y}{x}\right)$, where $x \neq 0$ and $-\frac{\pi}{2} < z < \frac{\pi}{2}$, then:

$$\frac{\partial z}{\partial x} = \frac{\partial\left(\arctan\left(\frac{y}{x}\right)\right)}{\partial x} = \frac{\partial\left(\arctan\left(\frac{y}{x}\right)\right)}{\partial\left(\frac{y}{x}\right)} \cdot \frac{\partial\left(\frac{y}{x}\right)}{\partial x}$$

$$= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot -\frac{y}{x^2} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot -\frac{y}{x^2} = \frac{x^2}{x^2 + y^2} \cdot -\frac{y}{x^2} = -\frac{y}{x^2 + y^2}$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{y}{x^2 + y^2}$$

b. Let $z = f(x, y) = \arctan\left(\frac{y}{x}\right)$, where $x \neq 0$ and $-\frac{\pi}{2} < z < \frac{\pi}{2}$.

The equation of the tangent plane at the point $\vec{p} = (x_0, y_0, z_0) = \left(1, -1, -\frac{\pi}{4}\right)$ is given by:

$$z - z_0 = f_x(\vec{p})(x - x_0) + f_y(\vec{p})(y - y_0)$$

We already have $f_x = \frac{\partial z}{\partial x} = -\frac{y}{x^2 + y^2}$, so now we find that $f_y = \frac{\partial z}{\partial y}$ is given by:

$$\frac{\partial z}{\partial y} = \frac{\partial\left(\arctan\left(\frac{y}{x}\right)\right)}{\partial y} = \frac{\partial\left(\arctan\left(\frac{y}{x}\right)\right)}{\partial\left(\frac{y}{x}\right)} \cdot \frac{\partial\left(\frac{y}{x}\right)}{\partial y}$$

$$= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x^2}{x^2 + y^2} . \frac{1}{x} = \frac{x}{x^2 + y^2}$$

Hence, the equation of the tangent plane at the point $\vec{p} = \left(1, -1, -\frac{\pi}{4}\right)$ is given by:

$$\begin{aligned} z - z_0 = f_x(\vec{p})(x - x_0) + f_y(\vec{p})(y - y_0) &\Leftrightarrow z + \frac{\pi}{4} = \frac{1}{1+1}(x - 1) + \frac{1}{1+1}(y + 1) \\ &\Leftrightarrow z + \frac{\pi}{4} = \frac{1}{2}(x - 1) + \frac{1}{2}(y + 1) \\ &\Leftrightarrow z + \frac{\pi}{4} = \frac{1}{2}(x - 1) + \frac{1}{2}(y + 1) \\ &\Leftrightarrow z = \frac{1}{2}x + \frac{1}{2}y - \frac{\pi}{4} \end{aligned}$$

$\therefore$ The equation of the tangent plane at the point $(x, y, z) = \left(1, -1, -\frac{\pi}{4}\right)$ is given by:

$$z = \frac{1}{2}x + \frac{1}{2}y - \frac{\pi}{4} \quad \text{or} \quad 2x + 2y - 4z = \pi$$

5. A bag contains 8 white balls and 2 red balls. One ball is drawn from the bag at random on the 1st trial. If the ball is red, this is the end of the trial. If the ball is white, replace the ball in the bag and draw a ball at random on the 2nd trial. Repeat the trials until a red ball is drawn. Suppose that a red ball is drawn on the Xth trial.
   a. Find the expected value of X.
   b. Find the expected value of X^2.

Answer:

Since we keep redoing the Bernoulli trials until what we want to observe comes out, this implies that this is a Geometric Distribution. Because there are 2 red balls out of a total of 10 balls, then we have $p = 2/10 = 1/5$, hence $X \sim \text{Geometric}(1/5)$.

a. The expected value of X is given by:

$$E(X) = \frac{1}{p} = \frac{1}{1/5} = 5$$

b. The expected value of X^2 is given by:

$$E(X^2) = \text{Var}(X) + \big(E(X)\big)^2 = \frac{1-p}{p^2} + \frac{1}{p^2} = \frac{2-p}{p^2} = \frac{2-1/5}{(1/5)^2} = \frac{9/5}{1/25} = 45$$

6. Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Answer:

Let A be the given matrix. We can find the eigenvalues of A by solving:

$$|A-\lambda I| = 0 \Leftrightarrow \begin{vmatrix} -\lambda & 0 & 0 & 0 & -2 \\ 0 & -\lambda & 0 & -1 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & -\lambda & 0 \\ 2 & 0 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

for λs.

Notice that:

$$\begin{aligned}
& \begin{vmatrix} -\lambda & 0 & 0 & 0 & -2 \\ 0 & -\lambda & 0 & -1 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 1 & 0 & -\lambda & 0 \\ 2 & 0 & 0 & 0 & -\lambda \end{vmatrix} \\
\overset{\text{expand on } 3^{\text{rd}} \text{ row}}{=} \quad & -\lambda \begin{vmatrix} -\lambda & 0 & 0 & -2 \\ 0 & -\lambda & -1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 2 & 0 & 0 & -\lambda \end{vmatrix} \\
\overset{\text{expand on } 1^{\text{st}} \text{ row}}{=} \quad & -\lambda \cdot \left(-\lambda \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \\ 2 & 0 & 0 \end{vmatrix} \right) \\
\overset{\text{expand on } 3^{\text{rd}} \text{ row}}{=} \quad & -\lambda \cdot \left(-\lambda \left(-\lambda \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \right) + 2 \left(2 \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \right) \right) \\
= \quad & -\lambda\left(\lambda^2(\lambda^2+1) + 4(\lambda^2+1)\right) \\
= \quad & -\lambda(\lambda^2+1)(\lambda^2+4)
\end{aligned}$$

Hence, we find that:

$$|A-\lambda I| = 0 \Leftrightarrow -\lambda(\lambda^2+1)(\lambda^2+4) = 0$$

This equality yields the solutions:

$$-\lambda = 0 \quad \text{or} \quad \lambda^2+1=0 \quad \text{or} \quad \lambda^2+4=0$$

For $-\lambda = 0$ then $\lambda = 0$,
For $\lambda^2+1=0$ then $\lambda = \pm\sqrt{-1} = \pm\, i$,
For $\lambda^2+4=0$ then $\lambda = \pm\sqrt{-4} = \pm\, 2i$

$\therefore$ The eigenvalues of the given matrix are $0, \pm i$, and $\pm 2i$.

7. Solve the differential equation

$$(3x - y)\frac{dy}{dx} = 2x$$

under the initial condition "$y = \frac{1}{2}$ when $x = 0$".

Answer:

Rewrite the equation as:

$$(3x - y)\frac{dy}{dx} = 2x \Leftrightarrow \frac{dy}{dx} = \frac{2x}{3x - y} = \frac{2}{3 - \frac{y}{x}}$$

Let $v := \frac{y}{x} \Leftrightarrow y := vx$, then we will have:

$$\frac{dy}{dx} = \frac{d(vx)}{dx} = v + x\frac{dv}{dx}$$

Hence the equation becomes:

$$\frac{dy}{dx} = \frac{2}{3 - \frac{y}{x}} \Leftrightarrow v + x\frac{dv}{dx} = \frac{2}{3 - v} \Leftrightarrow x\frac{dv}{dx} = \frac{2}{3 - v} - v \Leftrightarrow x\frac{dv}{dx} = \frac{2 - (3v - v^2)}{3 - v}$$

$$\Leftrightarrow \frac{dv}{dx} = \frac{v^2 - 3v + 2}{x(3 - v)} \Leftrightarrow \frac{3 - v}{v^2 - 3v + 2}\frac{dv}{dx} = \frac{1}{x} \Leftrightarrow \frac{3 - v}{v^2 - 3v + 2}\frac{dv}{dx} - \frac{1}{x} = 0$$

Note that:

$$\begin{aligned}
\frac{3 - v}{v^2 - 3v + 2}\frac{dv}{dx} &= \frac{dv}{dx}\frac{d}{dv}\left(\int \frac{3 - v}{v^2 - 3v + 2}dv\right) = \frac{d}{dx}\left(\int \frac{-(2v - 3) + 3}{2(v^2 - 3v + 2)}dv\right) \\
&= \frac{d}{dx}\left(-\frac{1}{2}\int \frac{2v - 3}{v^2 - 3v + 2}\frac{d(v^2 - 3v + 2)}{2v - 3} + \frac{3}{2}\int \frac{1}{v^2 - 3v + 2}dv\right) \\
&= \frac{d}{dx}\left(-\frac{1}{2}\int \frac{d(v^2 - 3v + 2)}{v^2 - 3v + 2} + \frac{3}{2}\int \frac{1}{\left(v - \frac{3}{2}\right)^2 - \frac{1}{4}}\frac{d\left(v - \frac{3}{2}\right)}{1}\right) \\
&= \frac{d}{dx}\left(-\frac{1}{2}\int \frac{d(v^2 - 3v + 2)}{v^2 - 3v + 2} - 6\int \frac{1}{1 - 4\left(v - \frac{3}{2}\right)^2}d\left(v - \frac{3}{2}\right)\right) \\
&= \frac{d}{dx}\left(-\frac{1}{2}\int \frac{d(v^2 - 3v + 2)}{v^2 - 3v + 2} - 6\int \frac{1}{1 - (2v - 3)^2}\frac{d(2v - 3)}{2}\right) \\
&= \frac{d}{dx}\left(-\frac{1}{2}\ln|v^2 - 3v + 2| - 3\,\mathrm{arctanh}(2v - 3) + C_0\right)
\end{aligned}$$

Let us restrict v to $-1 < 2v - 3 < 1 \Leftrightarrow 2 < 2v < 4 \Leftrightarrow 1 < v < 2$. Hence, this continues as:

$$\begin{aligned}
&= \frac{d}{dx}\left(-\frac{1}{2}\ln|v^2 - 3v + 2| - \frac{3}{2}\ln\left|\frac{1 + (2v - 3)}{1 - (2v - 3)}\right| + C_0\right) \\
&= \frac{d}{dx}\left(-\frac{1}{2}\ln|v^2 - 3v + 2| - \frac{3}{2}\ln\left|\frac{2v - 2}{-2v + 4}\right| + C_0\right)
\end{aligned}$$

$$= \frac{d}{dx}\left(-\frac{1}{2}\ln|v^2-3v+2|-\frac{3}{2}\ln\left|\frac{v-1}{-v+2}\right|+C_0\right)$$
$$= \frac{d}{dx}\left(-\frac{1}{2}\ln|v-2|-\frac{1}{2}\ln|v-1|-\frac{3}{2}\ln|v-1|+\frac{3}{2}\ln|-v+2|+C_0\right)$$
$$= \frac{d}{dx}\left(-\frac{1}{2}\ln|v-2|-\frac{1}{2}\ln|v-1|-\frac{3}{2}\ln|v-1|+\frac{3}{2}\ln|-v+2|+C_0\right)$$
$$= \frac{d}{dx}\left(-\frac{1}{2}\ln|2-v|+\frac{3}{2}\ln|2-v|-\frac{1}{2}\ln|1-v|-\frac{3}{2}\ln|1-v|+C_0\right)$$
$$= \frac{d}{dx}(\ln|2-v|-2\ln|1-v|+C_0)$$
$$= \frac{d}{dx}(\ln|2-v|-2\ln|1-v|)$$

Thus, the equation now becomes:

$$\frac{3-v}{v^2-3v+2}\frac{dv}{dx}-\frac{1}{x}=0 \Leftrightarrow \frac{d}{dx}(\ln|2-v|-2\ln|1-v|)-\frac{d}{dx}(\ln|x|)=0$$
$$\Leftrightarrow \frac{d}{dx}(\ln|2-v|-2\ln|1-v|-\ln|x|)=0$$
$$\Leftrightarrow \ln|2-v|-2\ln|1-v|-\ln|x|=C_1$$
$$\Leftrightarrow \ln\left|\frac{2-v}{x(1-v)^2}\right|=C_1$$
$$\Leftrightarrow \frac{2-v}{x(1-v)^2}=C, \qquad C=e^{C_1}$$
$$\Leftrightarrow \frac{2-\frac{y}{x}}{x\left(1-\frac{y}{x}\right)^2}=C$$
$$\Leftrightarrow \frac{2x-y}{x^2\left(\frac{x-y}{x}\right)^2}=C$$
$$\Leftrightarrow \frac{2x-y}{(x-y)^2}=C$$

We now apply the initial condition "$y=\frac{1}{2}$ when $x=0$", hence we find that:

$$\frac{2\cdot 0-\frac{1}{2}}{\left(0-\frac{1}{2}\right)^2}=C \Leftrightarrow C=\frac{-\frac{1}{2}}{\frac{1}{4}}=-2$$

Substitute C into the equation and we have found our solution to the differential equation.

$$\frac{2x-y}{(x-y)^2}=-2 \Leftrightarrow y-2x=2(y-x)^2$$

∴ The solution to the ordinary differential equation $(3x-y)\frac{dy}{dx}=2x$ under the initial condition "$y=\frac{1}{2}$ when $x=0$" is $y-2x=2(y-x)^2$.

Afterword

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