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TP 3 : Hasting - Metropolis and Gibbs samplers

Exercice 2 : Multiplicative Hasting - Metropolis

f density function on $]-1, 1[$.
X be the current state of the Markov Chain.
(i) Sample ε from f.
and $B \sim \text{Bernoulli}(1/2)$
(ii) If $B = 1$, $Y = \varepsilon X$
otherwise $Y = \dfrac{X}{\varepsilon}$
And we accept Y with proba. $\alpha(X, Y)$

1) The proposal density or jumping distribution is

$$q(x \to y) = \frac{1}{2} f\left(\frac{Y}{X}\right) \underbrace{\frac{1}{|X|}}_{\text{normalization}} \mathbb{1}_{\{|Y| < |X|\}} + \frac{1}{2} f\left(\frac{X}{Y}\right) \underbrace{\frac{|X|}{Y^2}}_{\text{normalization}} \mathbb{1}_{\{|Y| > |X|\}}$$

2) The acceptation ratio α:

$$\alpha(x, y) = \min\left(1, \frac{\pi(y)}{\pi(x)} \frac{q(y, x)}{q(x, y)}\right)$$

And
$$\frac{q(y,x)}{q(x,y)} = \frac{\frac{1}{2} f(y/x) \times \frac{|y|}{x^2}}{\frac{1}{2} f\left(\frac{y}{x}\right)\frac{1}{|x|}} \mathbb{1}_{\{|y| < |x|\}} + \frac{\frac{1}{2} f\left(\frac{x}{y}\right)\frac{1}{|y|}}{\frac{1}{2} f\left(\frac{x}{y}\right)\frac{|x|}{y^2}} \mathbb{1}_{\{|y| > |x|\}}$$

$$= \frac{|y|}{|x|} \mathbb{1}_{\{|y| < |x|\}} + \frac{|y|}{|x|} \mathbb{1}_{\{|y| > |x|\}}.$$

Therefore, $\alpha(x, y) = \min\left(1, \dfrac{\pi(y)}{\pi(x)} \times \dfrac{|y|}{|x|}\right)$

$$\boxed{\alpha(x,y) = \min\left(1, \frac{\pi(y)}{\pi(x)} \times |\varepsilon| \times \mathbb{1}_{\{B=1\}} + \frac{\pi(y)}{\pi(x)} \times \frac{1}{|\varepsilon|} \mathbb{1}_{\{B=0\}}\right)}$$

Exercice 3 : Data Augmentation

Let $f : (x,y) \in \mathbb{R}^p \times \mathbb{R}^q \longrightarrow f(x,y) \in \mathbb{R}^+$

1) For $F_n = \sigma((X_0, Y_0), \dots, (X_n, Y_n))$

For $A \subset \mathbb{R}^p \times \mathbb{R}^q$, we have $P((X_n, Y_n) \in A \mid F_{n-1})$

$$= \mathbb{P}(\underbrace{X_n \in \Pi_{\mathbb{R}^p}(A)}_{\text{depend on } Y_{n-1}}, \underbrace{Y_n \in \Pi_{\mathbb{R}^q}(A)}_{\text{depend on } X_n} \mid F_{n-1})$$

$$= \mathbb{P}((X_n, Y_n) \in A \mid (\cancel{X_{n-1}}, Y_{n-1}))$$

Therefore $\{(X_n, Y_n), n \geq 0\}$ is a Markov Chain

Transition Kernel Let $A \subset \mathbb{R}^p \times \mathbb{R}^q$
$= A_1 \times A_2$

$$P((x,y), A) = \int_{(x',y') \in A} f_{X|Y}(x',y') \times f_{Y|X}(x',y')\, dx' dy'$$

$$= \int_{x' \in A_1} f_{X|Y}(x',y') \left(\int_{y' \in A_2} f_{Y|X}(x',y')\, dy' \right) dx'$$

2) $F_{Y_n} = \sigma(Y_0, \dots, Y_n)$

$B \in \mathbb{R}^q$.

$$\mathbb{P}(Y_n \in B \mid F_{Y_n}) = \mathbb{P}(Y_n \in B \mid \sigma(Y_{n-1}, \dots, Y_0))$$

we have $Y_n \sim f_{Y|X}(X_n, \cdot)$ and $X_n \sim f_{X|Y}(\cdot, Y_{n-1})$

So $\mathbb{P}(Y_n \in B \mid \sigma(Y_{n-1}, \dots, Y_0)) = \mathbb{P}(Y_n \in B \mid Y_{n-1})$.

So $\{Y_n, n \geq 0\}$ is a Markov chain.

Transition kernel : Let $B' \subset \mathbb{R}^q$.

we have $P(y, B') = \int_{y' \in B'} \int_{x \in \mathbb{R}^p} f_{Y|X}(x,y') \times f_{X|Y}(x,y)\, dx\, dy'$

- $f_Y(y)\, dy$ is invariant for kernel :

$$\int_{y \in \mathbb{R}^q} P(y, B') \times f_Y(y)\, dy$$

$$= \int_{y \in \mathbb{R}^q} \int_{y' \in B'} \int_{x \in \mathbb{R}^p} f_{Y|X}(x, y') \times f_{X|Y}(x, y) \times f_Y(y)\, dx\, dy\, dy'$$

$f_{X|Y}(x,y) \times f_Y(y) = f(x,y)$

$$= \int_{y \in \mathbb{R}^q} \int_{y' \in B'} \int_{x \in \mathbb{R}^p} f_{Y|X}(x, y') \times f(x,y)\, dx\, dy\, dy'$$

$\int_{y \in \mathbb{R}^q} f(x,y)\, dy = f_X(x)$

$$= \int_{y' \in B'} \underbrace{\int_{x \in \mathbb{R}^p} f_{Y|X}(x, y')\, f_X(x)\, dx}_{= f_Y(y')}\, dy'$$

$$= \int_{y' \in B'} f_Y(y')\, dy'$$

Therefore, We have shown that $f_Y(y)\, dy$ is invariant for this kernel

3) We consider $f(x,y) = \frac{4}{\sqrt{2\pi}} y^{3/2} \exp\left[-y\left(\frac{x^2}{2} + 2\right)\right] \mathbb{1}_{\mathbb{R}^+}(y)$

Given (x_0, y_0) in $\mathbb{R} \times \mathbb{R}^+$ and $N \in \mathbb{N}$

for $n = 1$ to N do.

$x_n \sim f_{X|Y}(\cdot, y_{n-1})$

$y_n \sim f_{Y|X}(x_n, \cdot)$

end

return $\{(x_n, y_n) ; 0 \leq n \leq N\}$.

Where $x \sim f_{X|Y}(\cdot, y) \propto \frac{4}{\sqrt{2\pi}} y^{3/2} \exp\left(-y\left(\frac{x^2}{2} + 2\right)\right]$.
$y > 0$

$$\propto \exp\left(\frac{-x^2}{2\left(\frac{1}{y}\right)}\right)$$

So it is a Centered Gaussian with variance $\left(\frac{1}{y}\right)$

And We have

$$y \sim f_{Y|X}(x,\cdot) \propto y^{3/2} \exp\left(-\left(\frac{x^2}{2}+2\right) y\right)$$

So it is a Gamma distribution with $\alpha = \frac{5}{2}$

and $\beta = \frac{x^2}{2}+2$

as $\text{Gamma}(\alpha,\beta) \propto x^{\alpha-1} \exp(-\beta x)$

4) Let H be a bounded function on $\mathbb{R}$,

we would like to approximate $\int_{\mathbb{R}} \frac{H(x)}{(4+x^2)^{5/2}}\, dx$

from the output $\{(X_n, Y_n);\ 0 \leq n \leq N\}$.

Let's consider $\int_{\mathbb{R}\times\mathbb{R}^+} H(x) f(x,y)\, dx \times dy$

$$= \int_{\mathbb{R}\times\mathbb{R}^+} H(x) \frac{4}{\sqrt{2\pi}} y^{3/2} \exp\left[-y\left(\frac{x^2}{2}+2\right)\right] dx\, dy$$

we do the change of variable $y' = y\left(\frac{x^2}{2}+2\right)$

$y = 0 \Rightarrow y' = 0$, $y \to +\infty \Rightarrow y' \to +\infty$

So that: $$= \frac{4}{\sqrt{2\pi}} \int_{\mathbb{R}\times\mathbb{R}^+} H(x) \frac{y'^{3/2}}{\left(\frac{x^2}{2}+2\right)^{3/2}} \exp(-y')\, dx\, \frac{dy'}{\left(\frac{x^2}{2}+2\right)}$$

$$= \frac{4}{\sqrt{2\pi}} \times 2^{5/2} \times \left[\int_{\mathbb{R}} \frac{H(x)}{(x^2+4)^{5/2}}\, dx\right] \underbrace{\int_{\mathbb{R}^+} y'^{3/2} e^{-y'}\, dy'}_{\Gamma(5/2)}$$

with $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt$. The Gamma function

As property of Γ, we know that $\Gamma(z)\,\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$

$z = 2$

So $$\Gamma\left(\frac{5}{2}\right) = \frac{2^{1-2\times 2}\sqrt{\pi}\,\Gamma(2\times 2)}{\Gamma(2)}$$

$\Gamma(n+1) = n! \Rightarrow \Gamma(2) = 1$ and $\Gamma(4) = 3! = 6$

$$\Rightarrow \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}.$$

$$\Rightarrow 1\cancel{8} \int_{\mathbb{R}\times\mathbb{R}^+} \frac{H(x)}{(4+x^2)^{5/2}}\, dx = \int_{\mathbb{R}\times\mathbb{R}^+} H(x)\, f(x,y)\, dx\, dy$$

Therefore, to approximate $\int_{\mathbb{R}\times\mathbb{R}^+} \frac{H(x)}{(4+x^2)^{5/2}}\, dx$

We compute $\boxed{\frac{1}{12} \sum_{n=0}^{N} H(X_n) \cancel{f(X_n, Y_n)}}$

Monté Carlo Integration.