Design of Nearly Constant Velocity Track Filters for Tracking Maneuvering Targets

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***Abstract*—When tracking maneuvering targets with conventional algorithms, the process noise standard deviation used in the nearly constant velocity Kalman filter is selected vaguely in relation to the maximum acceleration of the target. The deterministic tracking index is introduced and used to develop a relationship between the maximum acceleration and the process noise variance that either minimizes the maximum mean squared error (MMSE) in position or weighted sum of the noise variance plus maneuver bias. For each case, the process noise standard deviation is expressed in terms of the maximum acceleration and deterministic tracking index for both piecewise constant and discretized continuous acceleration error models. A lower bound on the process noise variance is also expressed in terms of the maximum acceleration and deterministic tracking index. With the use of Monte Carlo simulations, the method for choosing process noise variance for tracking maneuvering targets is demonstrated.**

Keywords: Target Tracking, Maneuvering Targets, Track Filter Design.

I. Introduction

Due to the random modeling of deterministic-type maneuvers that are inherent in the maneuvering target tracking problem, the selection of the process noise variance is not straightforward and the error covariance is not always an accurate indicator of performance. When tracking maneuvering targets with conventional algorithms, the process noise variance used in the nearly constant velocity (NCV) Kalman filter is selected vaguely in relation with the maximum acceleration of the target [1]. Therefore, it is desirable to have a better expression that would enable one to find the process noise variance that minimizes the maximum mean squared error (MMSE) given the target's maximum acceleration. Toward this goal, the deterministic tracking index is introduced and used to develop a relationship between the maximum acceleration and the process noise variance that minimizes the MMSE in either position or velocity.

In order to develop a procedure for selecting the optimal process noise variance, an alpha-beta filter [2], which is a steady-state form of the nearly constant velocity filter, is used to characterize the performance in terms of the sensor-noise only (SNO) covariance matrix and the position and velocity lags [3] [4]. Decoupling the measurement noise effects from the maneuver effects on performance allows for the performance degradation due to deterministic-type maneuvers to characterized. Hence, knowledge of the SNO covariance matrix and the steady-state position and velocity lags give more insight into the choice of filter gains for deterministic-type maneuvers [5]. By introducing the deterministic tracking index, which unlike the tracking index for random maneuvers is dependent on the maximum acceleration of the target, a better means of characterizing deterministic-type maneuvers is possible.

For various design criteria, the optimal process noise variance is expressed in terms of the maximum acceleration and deterministic tracking index for both piecewise constant and discretized continuous acceleration error models. The design criteria include minimizing either the MMSE or a weighted sum of the SNO variance plus the maneuver lag squared. A lower bound on the process noise variance is also expressed in terms of the maximum acceleration and deterministic tracking index. The use of a process noise variance larger than the lower bound will ensure that the MMSE in position is less than the average error in the measurements. Since these design criteria are based on steady-state tracking during a maneuver, the number of measurements occupying the transient periods will be expressed as a function of the deterministic tracking index and used in the application of the design criteria. With the use of Monte Carlo simulations, the method for choosing process noise variance for tracking maneuvering targets will be shown to be optimal.

The Kalman filter is reviewed in Section II, and the alpha-beta filter is defined in Section III along with the SNO covariance and filter lags. Section IV develops the design criteria for the NCV filter with piecewise constant acceleration errors, and Section V develops the design criteria for the NCV filter with discretized continuous time acceleration errors [1]. Concluding remarks are given in Section VI.

II. Kalman Filter

A Kalman filter is often employed to filter the kinematic measurements for estimating the position, velocity, and acceleration of a target [1]. The kinematic model commonly assumed for a target in track is given by

$$X_{k+1} = F_k X_k + G_k v_k \tag{1}$$

where $v_k \sim N(0, Q_k)$ is the process noise that models the unknown target acceleration and F_k defines the linear dynamics. The target state vector X_k contains the position, velocity, and possibly acceleration of the target at time t_k, as well as other variables used to model the time-varying acceleration. The linear measurement model is given by

$$Y_k = H_k X_k + w_k \tag{2}$$

where Y_k is typically the measurement of the position of the target and $w_k \sim N(0, R_k)$ is the observation error. Both w_k and v_k are assumed to be independent "white" noise processes. When designing the Kalman filter, Q_k is selected such that the 65% to 95% confidence region about zero contains the maximum acceleration level of the target. However, when targets maneuver, the acceleration changes in a deterministic manner. Thus, the white noise assumption associated with w_k is violated and the filter develops a bias in the state estimates. If a larger Q_k is chosen, the bias in the state estimates is less during a maneuver, but then Q_k poorly characterizes the target motion when the target is not maneuvering and the filter performance is far from optimal. Furthermore, the error in modeling the two modes of flight (*i.e.,* nonmaneuvering and maneuvering) with a single model and the error in the white noise assumption for the process noise during maneuvers result in an inaccurate state error covariance that cannot be used reliably for performance prediction. While an Interacting Multiple Model (IMM) estimator [1] can be used to address this conflict in situations of demanding requirements, the focus of this work is on the filter design (*i.e.,* selection of Q_k) of a NCV Kalman filter when an IMM estimator is not warranted by the requirements or achievable given the computational limitations.

The Kalman filter is a predictor-corrector algorithm that is given in terms of a time update and measurement update. The $X_{k|j}$ denotes the state estimate at time t_k given measurements through time t_j and $P_{k|j}$ denotes the state error covariance at time t_k given measurements through time t_j. The K_k is referred to as the Kalman gain at time t_k.

III. Alpha-Beta Filter

The alpha-beta filter is based on the assumption that the target is moving with constant velocity plus zero-mean, white Gaussian acceleration errors. For the alpha-beta filter and NCV Kalman filter with piecewise constant acceleration errors, the state and measurement equations of (1) and (2) are defined by

$$X_k = \begin{bmatrix} x_k & \dot{x}_k \end{bmatrix}^T \tag{3}$$

$$F_k = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{4}$$

$$G_k = \begin{bmatrix} \frac{1}{2}T^2 \\ T \end{bmatrix} \tag{5}$$

$$H_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{6}$$

where T is the time interval between measurements. Also, $R_k = \sigma_w^2$ is the variance of the measurement errors in m^2, and $Q_k = \sigma_v^2$ is the variance of the "acceleration" errors in m^2/s^4. In order to simplify the example and permit analytical predictions of the filter performance, the motion of the target is defined in a single coordinate and the measurements are the positions of the target (*i.e.,* a linear function of the state).

The steady-state form of the NCV filter can be used for analytical predictions of filter performance. For a filter to achieve these steady-state conditions, the error processes v_k and w_k must be stationary and the data rate must be constant. While these conditions are seldom satisfied in practice, the steady-state form of the filter can be used to predict average or expect tracking performance. The alpha-beta filter is equivalent to the Kalman filter for this motion model in steady-state . For the alpha-beta filter, the steady-state gains that occur after the transients associated with filter initialization diminish are given by

$$K_k = \begin{bmatrix} \alpha & \frac{\beta}{T} \end{bmatrix}^T \tag{7}$$

where α and β are the optimal gains for piecewise constant acceleration model given in [1], [2] as

$$\Gamma_{PWC}^2 = \frac{\sigma_v^2 T^4}{\sigma_w^2} = \frac{\beta^2}{1-\alpha} \tag{8}$$

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha} \tag{9}$$

where Γ_{PWC} is the tracking index [2]. The steady-state error covariance of the alpha-beta filter covariance [1], [3] is given by

$$P_{k|k}^{\alpha\beta} = \sigma_w^2 \begin{bmatrix} \alpha & \frac{\alpha}{\beta T} \\ \frac{\alpha}{\beta T} & \frac{\beta(2\alpha-\beta)}{2(1-\alpha)T^2} \end{bmatrix} \tag{10}$$

For the tkinematic model with discretized continuous white acceleration errors, $G_k Q_k G_k^T$ takes on a different form resulting in a different relation between the optimal gains α and β. In this case,

$$G_k Q_k G_k^T = \tilde{q} \begin{bmatrix} \frac{1}{4}T^4 & \frac{1}{2}T^3 \\ \frac{1}{2}T^3 & T \end{bmatrix} \tag{11}$$

$$\tag{12}$$

where $\tilde{q}$ is the power spectral density of the white acceleration errors. The optimal α and β for discretized continuous-time, white acceleration is given in [1] as

$$\Gamma_{DCWA}^2 = \frac{\tilde{q}T^3}{\sigma_w^2} = \frac{\beta^2}{1-\alpha} \tag{13}$$

$$\beta = 3(2-\alpha) - \sqrt{3(\alpha^2 - 12\alpha + 12)} \tag{14}$$

where Γ_{DCWA} is the tracking index.

The covariance of the state estimate $X_{k|k}$ is given by

$$P_{k|k} = E[(X_{k|k} - \bar{X}_{k|k})(X_{k|k} - \bar{X}_{k|k})^T] \tag{15}$$

where $E[\cdot]$ denotes the expected value operator, and $\bar{X}_{k|k} = E[X_{k|k}]$. When $E[X_{k|k}] = X_k$, the true value, the estimator is unbiased and the state error covariance computed by the Kalman filter is a good predictor of performance. However, when the estimator is biased, the covariance is a poor predictor of performance. When a target undergoes a deterministic maneuver (i.e., a constant acceleration), the estimates are biased and the covariance matrix tends to be a biased estimate of track filter performance. When a target undergoes no maneuver (i.e., a zero acceleration), the covariance matrix tends to also be a biased estimate of track filter performance, because process noise is included in the filter for maneuver response. Thus, in order to address both conditions of the performance prediction, the mean-squared error (MSE) will be written in terms of a sensor-noise only (SNO) covariance for no maneuver and a bias or maneuver lag for the constant acceleration maneuver.

Let

$$B_{k|k} = \bar{X}_{k|k} - X_k \tag{16}$$

where $B_{k|k}$ denotes the filter bias. Thus,

$$E[(X_{k|k} - X_k)(X_{k|k} - X_k)^T] \quad = \quad P_{k|k} + B_{k|k}B_{k|k}^T \tag{17}$$

Considering the case of deterministic maneuvers of either zero acceleration or constant acceleration, the MSE of the filter is given by the SNO covariance when the acceleration is zero and the SNO covariance plus the bias error squared when the acceleration is a nonzero constant. Letting $S_{k|k}^{\alpha\beta}$ denote the SNO covariance of the alpha-beta filter and $B_{k|k}^{\alpha\beta}$ denote the bias due to an acceleration gives

$$E[(X_{k|k} - X_k)(X_{k|k} - X_k)^T] \quad = \quad S_{k|k}^{\alpha\beta} + B_{k|k}^{\alpha\beta}(B_{k|k}^{\alpha\beta})^T \tag{18}$$

Expressions for the SNO covariance and the bias are computed by representing the alpha-beta filter as a linear, time-invariant system with an input that can be expressed as a deterministic signal (*i.e.,* a constant acceleration rather than zero-mean white process noise) with white noise measurement errors. The input-output relationships between the measurements Y_k and state estimate $X_{k|k}$ can be expressed as a linear system that is given by

$$X_{k|k} = \bar{F}_k X_{k-1|k-1} + \bar{G}_k Y_k \tag{19}$$

where

$$\bar{F}_k \quad = \quad \begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\frac{\beta}{T} & 1-\beta \end{bmatrix} \tag{20}$$

$$\bar{G}_k \quad = \quad \begin{bmatrix} \alpha & \frac{\beta}{T} \end{bmatrix}^T \tag{21}$$

The error covariance of $X_{k|k}$ that results from the SNO is given in [3] and [4] for arbitrary α and β to be

$$S_{k|k}^{\alpha\beta} = \frac{\sigma_w^2}{\alpha(4-2\alpha-\beta)} \begin{bmatrix} 2\alpha^2 + \beta(2-3\alpha) & \frac{\beta}{T}(2\alpha-\beta) \\ \frac{\beta}{T}(2\alpha-\beta) & \frac{2\beta^2}{T^2} \end{bmatrix} \tag{22}$$

where T is the time period between consecutive measurements and σ_w^2 is the variance of measurement errors. Since (22) includes only the effects of sensor measurement errors, it is referred to as the SNO covariance matrix. For a maneuvering target, the bias or lag in the state estimate for arbitrary α and β is given by

$$B_{k|k}^{\alpha\beta} = \begin{bmatrix} (1-\alpha)\frac{T^2}{\beta} \\ (\frac{\alpha}{\beta} - 0.5)T \end{bmatrix} A_k \tag{23}$$

where A_k is the acceleration of the target at time t_k in the coordinate of interest. For an m-step (*i.e.,* measurement intervals) ahead prediction, the error covariance of the state estimate that results from the measurement errors only is given in [**?**] for an arbitrary α and β to be

$$S_{k|k+m}^{\alpha\beta} = \begin{bmatrix} 1 & mT \\ 0 & 1 \end{bmatrix} S_{k|k}^{\alpha\beta} \begin{bmatrix} 1 & 0 \\ mT & 1 \end{bmatrix} \tag{24}$$

For an m-step ahead prediction and a maneuvering target, the bias or lag in the state estimate for arbitrary α and β is given by [**?**] as

$$B_{k|k+m}^{\alpha\beta} = \begin{bmatrix} (1-\alpha+(\alpha-0.5\beta)m + 0.5\beta m^2)\frac{T^2}{\beta} \\ (\alpha + (m-0.5)\beta)\frac{T}{\beta} \end{bmatrix} A_k \tag{25}$$

The MSE in the position estimates of the alpha-beta filter for an m-step ahead prediction is given by

$$\begin{aligned} MSE^p(m) \quad &= \quad \frac{\sigma_w^2}{\alpha(4-2\alpha-\beta)}(2\alpha^2 + \beta(2-3\alpha) \\ &+ \quad 2m\beta(2\alpha-\beta) + 2m^2\beta^2) \\ &+ \quad (1-\alpha+(\alpha-0.5\beta)m + 0.5\beta m^2)^2 \frac{T^4}{\beta^2} A_k^2 \end{aligned} \tag{26}$$

The MSE in the velocity estimates can be expressed as

$$MSE^v(m) = \frac{2\sigma_w^2\beta^2}{\alpha(4-2\alpha-\beta)T^2} + ((\alpha + (m-0.5)\beta)\frac{T}{\beta})^2 A_k^2 \tag{27}$$

Let Γ_D denote the deterministic tracking index given by

$$\Gamma_D = \frac{A_{max}T^2}{\sigma_w} \tag{28}$$

where A_{max} denotes the maximum acceleration of the target. Then the maximum MSE (MMSE) in the position estimates of the alpha-beta filter for an m-step ahead prediction can be written as

$$\begin{aligned} MMSE^p(m) \quad &= \quad \sigma_w^2 \Big[\frac{1}{\alpha(4-2\alpha-\beta)}(2\alpha^2 + \beta(2-3\alpha) \\ &+ \quad 2m\beta(2\alpha-\beta) + 2m^2\beta^2) \\ &+ \quad (1-\alpha+(\alpha-0.5\beta)m + 0.5\beta m^2)^2 \frac{\Gamma_D^2}{\beta^2}\Big] \end{aligned} \tag{29}$$

IV. Design of NCV Filter with Piecewise Constant Acceleration Errors

Given the random tracking index Γ_{PWC} of (8) and the relationship between α and β of (9) [2], the steady-state Kalman gains for the NCV filter with piecewise constant acceleration errors are specified. The optimal gains are unique function of the random tracking index [1], [2] and shown in Figure 1. Utilizing the deterministic tracking index Γ_D of (28) and the relationship between α and β of (9), the steady-state gains for the NCV filter that minimize the MMSE of (29) with $m = 0$ are specified and plotted in Figure 2.

Utilizing the results in Figures 1 and 2 gives the relationship between Γ_{PWC} and Γ_D as shown in Figure 3. Thus, the random tracking index can be expressed in terms of the deterministic tracking index as

$$\Gamma_{PWC} = \kappa_1(\Gamma_D)\Gamma_D \tag{30}$$

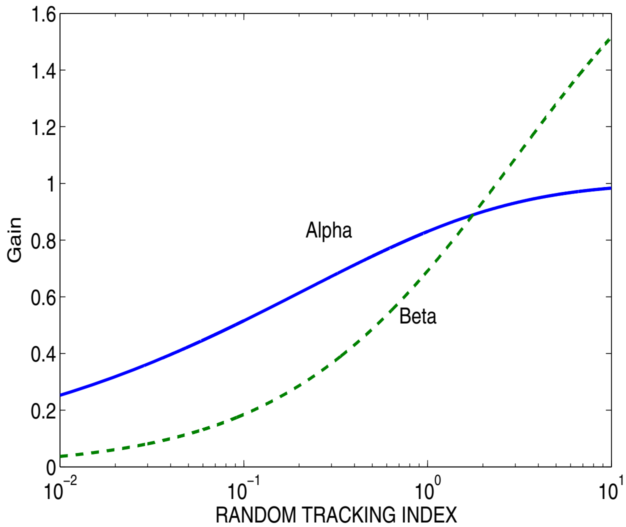

Figure 1. Steady-State Kalman Gains Versus the Random Tracking Index for the NCV Filter with Piecewise Constant Acceleration Errors

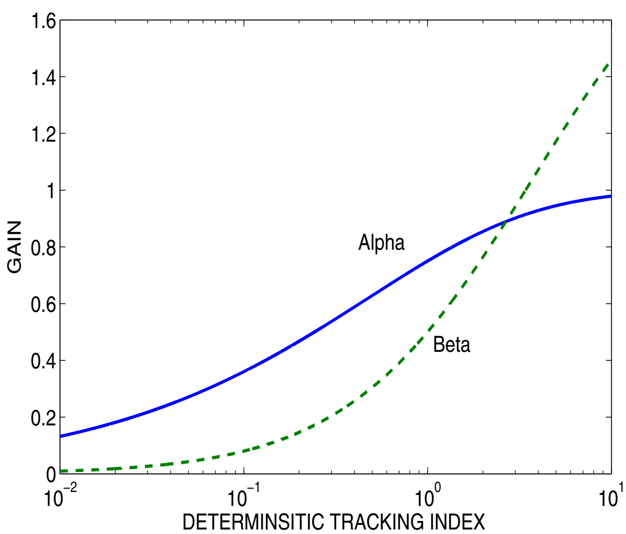

Figure 2. Steady-State Kalman Gains That Minimize the MMSE in Position for the NCV Filter with Piecewise Constant Acceleration Errors Versus the Deterministic Tracking Index

Utilizing the (8) and (28) in (30) gives

$$\sigma_v = \kappa_1(\Gamma_D)A_{max} \tag{31}$$

Setting $m = 0$ in (29) gives

$$MMSE^p = \sigma_w^2\Big[\frac{\lambda^2(2\alpha^2+\beta(2-3\alpha)}{\alpha(4-2\alpha-\beta)}+\frac{(1-\alpha)^2}{\beta^2}\Gamma_D^2\Big],\ \lambda = 1 \tag{32}$$

For a given value of Γ_D, there is a unique α and β that minimizes (32) in conjunction with (9). The α and β define a unique Γ_{PWC} that is related to Γ_D by κ_1. The κ_1 that corresponds to minimizing (32) is given versus Γ_D in Figure 4 and denoted by the line labeled with $\lambda = 1$ and given

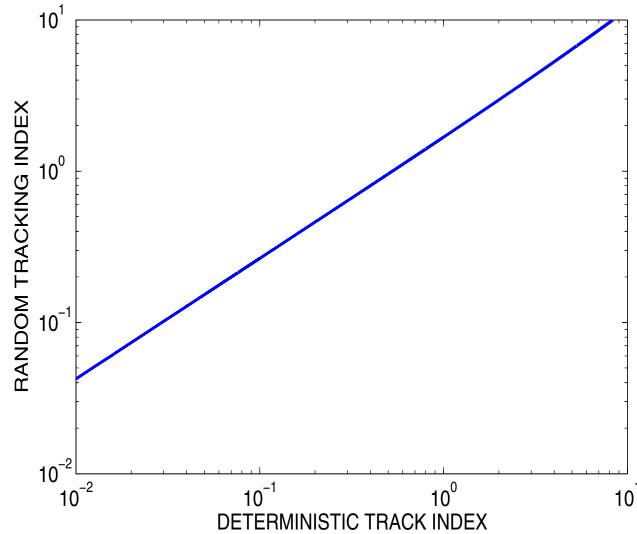

Figure 3. Random Tracking Index That Minimizes the MMSE Versus the Corresponding Deterministic Tracking Index

approximately by

$$\begin{aligned}\kappa_1^{m,\lambda}(\Gamma_D) &= \kappa_1^{0,1}(\Gamma_D) \\ &= 1.67 - 0.74\log(\Gamma_D) + 0.26(\log(\Gamma_D))^2 \\ &\qquad 0.01 \le \Gamma_D \le 10\end{aligned} \tag{33}$$

Thus, (33) along with (31) and the maximum acceleration of the target defines the process noise variance for the NCV filter. Using the constraint that the MMSE in the filtered position estimate should not exceed the measurement error variance gives that $MMSE^p \le \sigma_w^2$ or

$$\frac{(2\alpha^2+\beta(2-3\alpha)}{\alpha(4-2\alpha-\beta)}+\frac{(1-\alpha)^2}{\beta^2}\Gamma_D^2 \le 1 \tag{34}$$

For a given value of Γ_D, there is a unique α and β that satisfies (34) with equality. These α and β are the minimum gains and define the minimum Γ_{PWC} that is related to Γ_D by κ_1. The κ_1 that defines the minimum Γ_{PWC} is given versus Γ_D in Figure 4 and denoted by the line labeled with "MINIMUM" and given approximately by

$$\kappa_1^{min}(\Gamma_D) = 0.87 - 0.09\log(\Gamma_D) - 0.02(\log(\Gamma_D))^2 \tag{35}$$

Thus, Figure 4 shows that the constraint is most always satisfied if the process noise variance that is chosen to minimize the MMSE. However, Figure 4 shows that for $\Gamma_D < 1$, a wide range of values for κ_1 satisfy the constraint in (34). Introducing values other than 1 for λ into (32) to allow for an emphasis on noise reduction gives

$$\begin{aligned}C_1^{m,\lambda}(\Gamma_D) &= C_1^{0,\lambda}(\Gamma_D) \\ &= \sigma_w^2\Big[\frac{\lambda^2(2\alpha^2+\beta(2-3\alpha)}{\alpha(4-2\alpha-\beta)}+\frac{(1-\alpha)^2}{\beta^2}\Gamma_D^2\Big]\end{aligned} \tag{36}$$

The value of κ_1 that corresponds to minimizing (36) with $\lambda = 2$ is given versus Γ_D in Figure 4 and denoted by the line labeled with $\lambda = 2$ and given approximately by

$$\begin{aligned}\kappa_1^{m,\lambda}(\Gamma_D) &= \kappa_1^{0,2}(\Gamma_D) \\ &= 0.96 - 0.40\log(\Gamma_D) + 0.16(\log(\Gamma_D))^2 \\ &\quad \Gamma_D < 2\end{aligned} \tag{37}$$

Note that for $\Gamma_D > 2$, $\kappa_1^{0,2} = \kappa_1^{min}$. The value of κ_1 that corresponds to minimizing (36) with $\lambda = 3$ is given versus Γ_D in Figure 4 and denoted by the line labeled with $\lambda = 3$ and given approximately by

$$\kappa_1^{0,3}(\Gamma_D) = .72 - 0.24\log(\Gamma_D) + 0.14(\log(\Gamma_D))^2,\ \Gamma_D < 0.25 \tag{38}$$

Note that for $\Gamma_D > 0.25$, $\kappa_1^{0,3} = \kappa_1^{min}$.

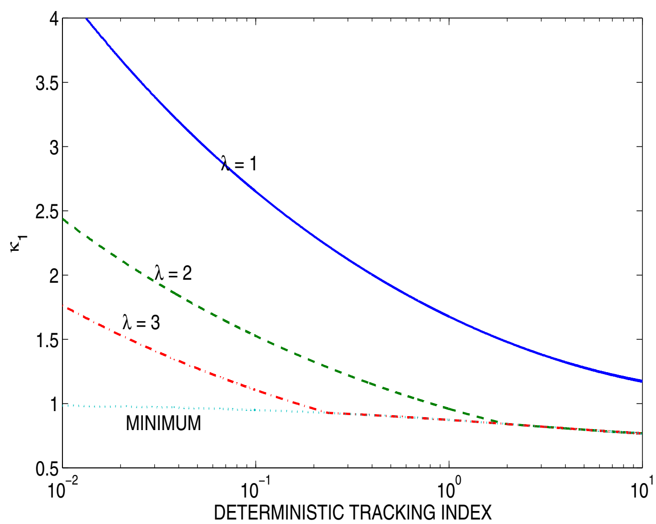

Figure 4. κ_1 for Defining the Process Noise Variance in Terms of the Maximum Acceleration

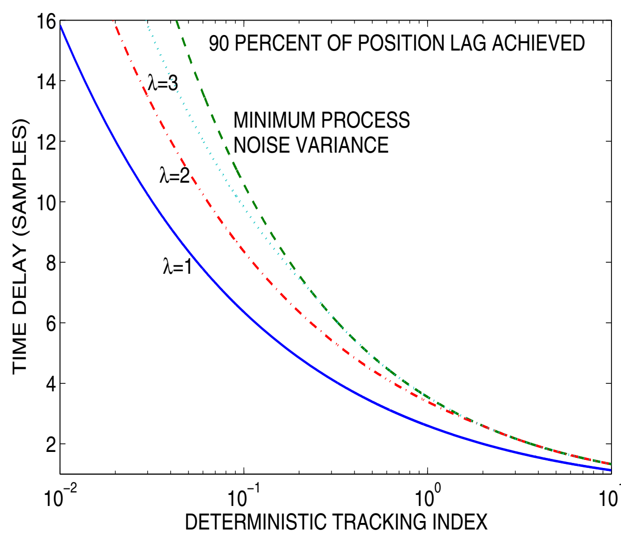

Figure 5. Approximate Time Delay in Filters Achieving 90% of Steady-State Lag in Position

When designing NCV filters for deterministic maneuvers, the question of duration of typical maneuvers arises. Figure 5 gives the approximate time period in measurement samples for the maneuver lag to achieve 90% of the steady-state lag in position [6]. Thus, for $\Gamma_D \approx 1$, the method presented here for selecting the process noise variance is valid for maneuvers lasting 3 measurements or longer. For $\Gamma_D \approx 0.1$, this method for selecting the process noise variance is valid for maneuvers lasting 6 measurements or longer. For targets with maximum maneuvers that are not sustained so that the maximum bias in the estimate is achieved, an alternate design procedure should be considered. One should note that the gains that minimize (32) with $\lambda = 2$ correspond to the gains that minimize (32) with $\lambda = 1$ (*i.e.*, the minimum MMSE) with bias or lag in position at one half of its full value.

For illustrating the filter design methodology, consider a target that maneuvers with 40 m/s^2 of acceleration from 40 to 60 s. The sensor measures the target position at a 1 Hz rate with errors defined by $\sigma_v = 120$ m. Thus, $\Gamma_D = 0.33$. Since the maneuver is sustained for more than 4 measurements, considering the maximum lag for design is acceptable. Then, (35) gives $\kappa_1^{min} = 0.91$ and $\sigma_v = 36.4$ as the minimum acceptable value for σ_v. Then, (32) with $\lambda = 1$ gives $\kappa_1^{0,1}(0.33) = 2.1$ from (33) and $\sigma_v = 83.3$ as the process noise standard deviation that minimizes the MSE. Then, (36) with $\lambda = 2$ gives $\kappa_1^{0,2}(0.33) = 1.2$ from (37) and $\sigma_v = 48$ as the process noise standard deviation that minimizes (36). Since $\Gamma_D > 0.3$, $\kappa_1^{0,3} = \kappa_1^{min}$. Figures 6 and 7 show the RMSE results of Monte Carlo simulations with 2000 experiments. Note that the RMSE in position for $\kappa_1 = 0.91$ and $\sigma_v = 36.4$ is closely matched to the standard deviation of the measurements of 120 m as anticipated. Also, note for $\kappa_1 = 2.1$ and $\sigma_v = 83.3$, the maximum RMSE in position is minimized and is only slightly larger than the RMSE in the absence of a maneuver.

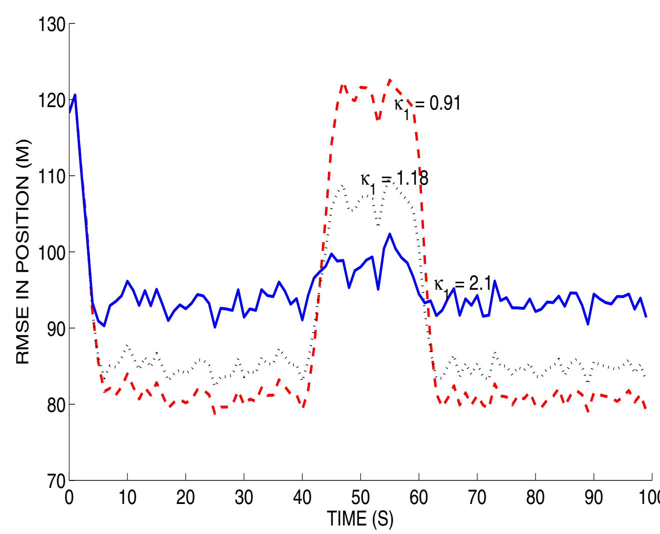

Figure 6. RMSE in Position Estimates of Three Filter Designs for Tracking Maneuvering Target

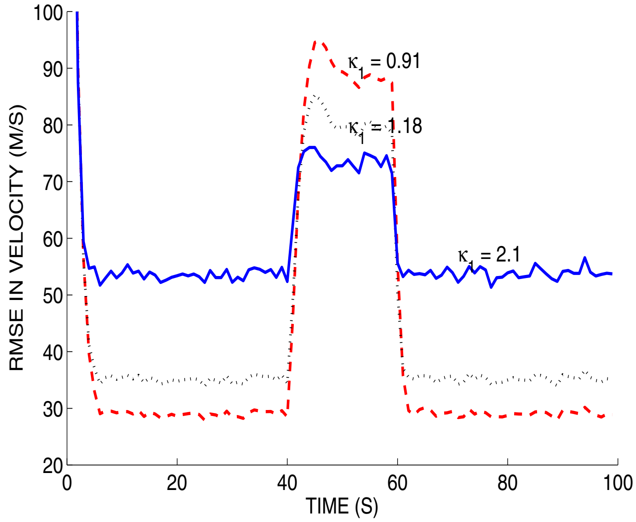

Figure 7. RMSE in Velocity Estimates of Three Filter Designs for Tracking Maneuvering Target

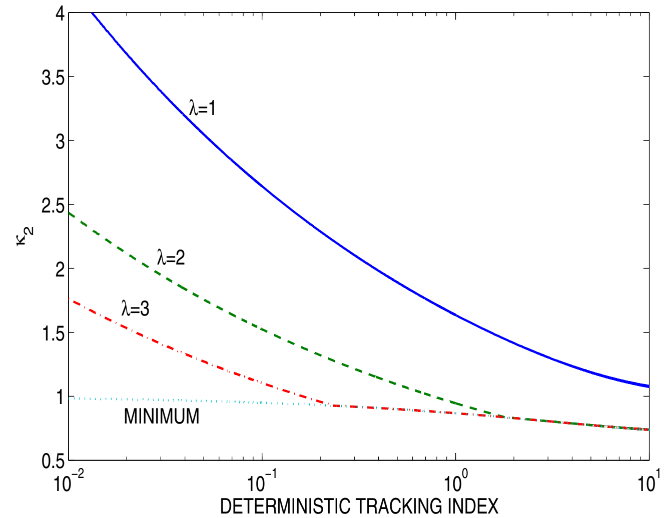

Figure 8. κ_2 for Defining the Process Noise Variance in Terms of the Power Spectral Density and Sample Period

V. NCV Filter with Discretized Continuous Time Acceleration Errors

Given the random tracking index Γ_{DCWA} of (13) and the relationship between α and β of (14), the steady-state Kalman gains for the NCV filter with discretized continuous time acceleration errors are specified. Utilizing the deterministic tracking index Γ_D of (28) and the relationship between α and β of (14), the steady-state gains for the NCV filter with discretized continuous time acceleration errors that minimize the MMSE of (29) with $m = 0$ are specified. Utilizing these results defines a relationship between Γ_{DCWA} and Γ_D. Thus, the random tracking index can be expressed in terms of the deterministic tracking index as

$$\Gamma_{DCWA} = \kappa_2(\Gamma_D)\Gamma_D \tag{39}$$

Utilizing the (13) and (28) in (39) gives

$$\sqrt{\frac{\tilde{q}}{T}} = \kappa_2(\Gamma_D)A_{max} \tag{40}$$

For a given value of Γ_D and the relationship between α and β of (14), there is a unique α and β that minimizes (32). The α and β define a unique Γ_{DCWA} that is related to Γ_D by κ_2. The κ_2 that minimizes (32) is given versus Γ_D in Figure 8 and denoted by the line labeled with $\lambda = 1$ and given approximately by

$$\begin{aligned} \kappa_2^{m,\lambda}(\Gamma_D) &= \kappa_2^{0,1}(\Gamma_D) \\ &= 1.62 - 0.79\log(\Gamma_D) + 0.24(\log(\Gamma_D))^2 \\ &\quad 0.01 \leq \Gamma_D \leq 10 \end{aligned} \tag{41}$$

Thus, (41) along with (40) and the maximum acceleration of the target defines the power spectral density $\tilde{q}$ for the NCV filter. Using the constraint that the MMSE in the filtered position estimate should not exceed the measurement error variance gives that $MMSE^p \leq \sigma_w^2$. For a given value of Γ_D, there is a unique α and β that satisfies (34) with equality. The α and β are the minimum gains and define the minimum Γ_{DCWA} that is related to Γ_D by κ_2. The κ_2 that defines the minimum Γ_{DCWA} is given versus Γ_D in Figure 8 and denoted by the line labeled with "MINIMUM" and given approximately by

$$\kappa_2^{min}(\Gamma_D) = 0.87 - 0.11\log(\Gamma_D) - 0.03(\log(\Gamma_D))^2 \tag{42}$$

Figure 8 shows that the constraint is most always satisfied if the process noise variance that is chosen to minimize the MMSE. Introducing values other than 1 for λ into (32) to allow for an added emphasis on noise reduction gives (36). The value of κ_2 that corresponds to minimizing (36) with $\lambda = 2$ is given versus Γ_D in Figure 8 and denoted by the line labeled with $\lambda = 2$ and given approximately by

$$\begin{aligned} \kappa_2^{m,\lambda}(\Gamma_D) &= \kappa_2^{0,2}(\Gamma_D) \\ &= .95 - 0.42\log(\Gamma_D) + 0.16(\log(\Gamma_D))^2 \\ &\quad \Gamma_D < 1.5 \end{aligned} \tag{43}$$

Note that for $\Gamma_D > 1.5$, $\kappa_2^{0,2} = \kappa_2^{min}$. The value of κ_2 that corresponds to minimizing (36) with $\lambda = 3$ is given versus Γ_D in Figure 8 and denoted by the line labeled with $\lambda = 3$ and given approximately by

$$\kappa_2^{0,3}(\Gamma_D) = 0.72 - 0.25\log(\Gamma_D) + 0.14(\log(\Gamma_D))^2, \ \Gamma_D < 0.25 \tag{44}$$

Note that for $\Gamma_D > 0.25$, $\kappa_2^{0,3} = \kappa_2^{min}$.

When designing NCV filters for deterministic maneuvers, the question of duration of typical maneuvers arises. Since $\kappa_1^{0,\lambda}(\Gamma_D) \approx \kappa_2^{0,\lambda}(\Gamma_D)$ for $\lambda = 1, 2$, and 3, Figure 5 also gives the approximate time period in number of measurement samples for the maneuver lag to achieve 90% of the steady-state lag in position for NCV filters with discretized continuous time acceleration errors.

VI. Conclusions

When given the parameters of a sensor, the measurement rate, and the maximum acceleration of the target, the variance of the piecewise constant acceleration errors or power spectral density of the discretized continuous time acceleration errors can be computed for optimal performance of track filters under steady-state conditions. However, the conditions of a fixed measurement rate and stationary measurement statistics are rarely satisfied in practice. Thus, nominal conditions can be used to select the process noise variance or power spectral density for the nominal operating point of the filter and the Kalman gains will adjust for the changes in the data rate and measurement variance. Note that the NCV with discretized continuous time acceleration errors is more robust to changes in the measurement period. Also, note that for $\Gamma_D \approx 1$, the method presented here for selecting the process noise variance is valid for maneuvers lasting 3 measurements or longer. For $\Gamma_D \approx 0.1$, this method for selecting the process noise variance is valid for maneuvers lasting 6 measurements or longer. Future work will involve the use of the techniques developed in this paper in the model selection and design for tracking maneuvering targets with the IMM estimator.

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