

Course :	Diploma in Electronic Systems
	Diploma in Telematics & Media Technology
	Diploma in Aerospace Systems & Management
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Topic 11 : Hypothesis Testing with Two Samples

Objectives :

At the end of this lesson, the student should be able to:

1. to distinguish between independent and dependent samples
2. compare the means of 2 independent samples using the hypothesis testing approach
3. compare the means of 2 dependent samples using the hypothesis testing approach

TOPIC 11: Hypothesis Testing with Two Samples

11.1 Introduction

- Oftentimes, we hear people say ‘The kids these days are taller than before’. In general, teenagers do seem taller than the adults who are now in their 30s. How can we justify this assumed comparison with sufficient evidence?
- We know that it is both impractical and non-economical to measure the heights of all teenagers and the adults in their 30s to make the comparison. The more sensible approach to compare the difference in their heights is to take random samples from the two different populations and compare their sample means.
- In reality, it is very common to make comparison between two or more distinct populations. In this module, we will be exploring the comparison of the sample means of two populations, although in other situations, it might be necessary to compare other parameters such as the standard deviation and shape of the distributions.

11.2 Independent and Dependent Samples

- In comparing two means, we want to see how different is one mean (let’s call this \bar{x}_1) from the other (let’s call this \bar{x}_2), so the most natural thing to do is to observe the difference between the two means, $\bar{x}_1 - \bar{x}_2$.
- The hypothesis testing approach in the comparison of two means allows us to test and see if there is enough evidence to conclude that
 - two means differ from each other.
 - one mean is greater/less than the other.
- The approach differs for comparison between independent and dependent samples.

Definitions:

1. Two samples are **independent** if members of one sample are **unrelated** to members of the other sample.
 2. Two samples are **dependent** when each member of one sample is **related** to the other sample. Dependent samples are also called **paired** or **matched** samples.
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Example 11.2-1

Classify each pair of samples as independent or dependent. Justify your answer.

- (a) Sample 1: Resting heart rates of 35 individuals before drinking coffee.
Sample 2: Resting heart rates of the same individuals after drinking two cups of coffee.
- (b) Sample 1: Test scores for 35 statistics students.
Sample 2: Test scores for 42 biology students who do not study statistics.

Solution:

11.3 Hypothesis Testing for Two Independent Samples

- The hypothesis test of two independent samples follows the same 6 steps as the hypothesis test of one sample. The difference lies in step 3, which requires us to know the distribution of the difference in sample means $\overline{x}_1 - \overline{x}_2$.
- General Steps for a Hypothesis Test between 2 Independent Samples are:
Step 1: State the claim mathematically. Identify the **null**, H_0 and **alternative**, H_a **hypotheses**. The possible hypotheses are:

$$\begin{array}{lll}
H_0 : \mu_1 \geq \mu_2 & H_0 : \mu_1 \leq \mu_2 & H_0 : \mu_1 = \mu_2 \\
H_a : \mu_1 < \mu_2 & H_a : \mu_1 > \mu_2 & H_a : \mu_1 \neq \mu_2
\end{array}$$

Regardless of which hypothesis, we always assume that the population means are the same, i.e. $\mu_1 = \mu_2 \Rightarrow \mu_1 - \mu_2 = 0$.

Step 2: Identify (a) the **type of test** and (b) the **level of significance**, α of the hypothesis test.

Step 3: State the type of **distribution of the difference in sample means** $\bar{x}_1 - \bar{x}_2$ follows.

(i) The sampling distribution of $\bar{x}_1 - \bar{x}_2$ follows a normal distribution;

$$\bar{x}_1 - \bar{x}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right) \text{ if three conditions are met:}$$

- (a) The samples must be randomly selected;
- (b) The samples must be independent;
- (c) Each sample size must be large ($n \geq 30$) or each population follows a normal distribution with known

$$\text{standard deviation, } \sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} ; \quad \text{or}$$

$$\text{unknown standard deviation, } \sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} .$$

(ii) The sampling distribution of $\bar{x}_1 - \bar{x}_2$ follows a *t*-distribution if

- (a) each sample size is small ($n < 30$);
- (b) equal but unknown population variance; then

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \hat{\sigma} \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \quad \text{with d.f.} = n_1 + n_2 - 2 \quad \text{and}$$

$$\hat{\sigma} = \sqrt{\frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2}} ; \text{ or}$$

- (c) unknown and unequal population variance; then

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \text{ with d.f. = smaller of } n_1 - 1 \text{ or } n_2 - 1$$

Step 4: Determine the rejection criteria using the **rejection region**

Step 5: Find the **standardized test statistics** = $\frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{X}_1 - \bar{X}_2}}$ with $\mu_1 - \mu_2 = 0$
since $\mu_1 = \mu_2$.

Step 6: Decide whether to reject or fail to reject H_0 and interpret the decision in the context of the original claim.

Example 11.3-1

121 boys and 144 girls sat for the PSLE in 2013. The mean PSLE scores for the boys and girls are 237 and 240 respectively. Assuming a common population standard deviation score of 12, test whether the results provide significant evidence, at the 1% level, that the academic standard of boys is inferior to that of the girls.

Solution:

Step 1: $H_0 : \mu_B \geq \mu_G$
 $H_a : \mu_B < \mu_G$ (claim)

Step 2: It is a left-tailed test and $\alpha = 0.01$

Step 3: Since $n_B = 121 > 30$, $n_G = 144 > 30$ with known σ_B, σ_G , $\bar{X}_B - \bar{X}_G$ follows a normal distribution with $\mu_{\bar{X}_B - \bar{X}_G} = 0$ and

$$\sigma_{\bar{X}_B - \bar{X}_G} = \sqrt{\frac{\sigma_B^2}{n_B} + \frac{\sigma_G^2}{n_G}} = \sqrt{\frac{12^2}{121} + \frac{12^2}{144}} = \sqrt{\frac{265}{121}}$$

Step 4: Reject H_0 if $z < z_\alpha = -2.33$.

Step 5:: Standardized test statistic,

$$z = \frac{(\bar{x}_B - \bar{x}_G) - (\mu_B - \mu_G)}{\sigma_{\bar{X}_B - \bar{X}_G}} = \frac{(237 - 240) - 0}{\sqrt{\frac{265}{121}}} = -2.03$$

Step 6: Since $z = -2.03 > z_\alpha = -2.33$, we do not reject H_0 .

At $\alpha = 0.01$, there is not enough evidence to support the claim that the academic standard of boys is inferior to that of the girls.

Example 11.3-2

The braking distances of 8 Volkswagen GTIs and 10 Ford Focuses were tested when travelling at 60 miles per hour on dry pavement. The results are shown below.

GTI	$\bar{x}_1 = 134$	$s_1 = 6.9$	$n_1 = 8$
FOCUS	$\bar{x}_2 = 143$	$s_2 = 2.6$	$n_2 = 10$

Can you conclude that there is a difference in the mean braking distances of the two types of cars? Use $\alpha = 0.01$. Assume the populations are normally distributed and the population variances are not equal.

Solution:

Example 11.3-3

A study sought to find out if playing soft classical music to plants helps in plant growth. 40 plants grown from the same batch of seeds are divided equally into two samples, A and B. Sample A is grown for a month under the sound of soft classical music while sample B acts as the control group. The mean growth (in mm) and standard deviation (in mm) of both samples are shown below:

Sample A:

36	38	33	39	31	34	40	33	36	35	35	34
36	38	33	32	39	45	41	34				

Sample B:

32	36	31	38	29	32	38	31	34	33	33	32
34	36	31	31	37	39	29	32				

Assume the populations are normally distributed and the population variances are equal, test at the 5% level if music has indeed helped in plant growth.

Solution:

By calculator:

Sample A	$\bar{x}_A =$	$s_A =$	$n_A =$
Sample B	$\bar{x}_B =$	$s_B =$	$n_B =$

11.4 Hypothesis Testing for Two Dependent Samples

- In the hypothesis test of 2 dependent samples or paired data, we are interested in the difference between the 2 values within each paired data (x_1, x_2) . The difference denoted by d is defined as $d = x_1 - x_2$. The mean of the differences between paired data entries in the dependent samples is calculated using, $\bar{d} = \frac{\sum d}{n}$, where n is the number of data pairs.

- DISTRIBUTION OF SAMPLE MEAN OF THE DIFFERENCE:
 \bar{d} follows approximately a ***t-distribution*** with degrees of freedom $n-1$, if the following conditions are satisfied:
 - the samples are randomly selected
 - the samples are dependent (paired)
 - both populations are normally distributed.

- General Steps for a Hypothesis Test between 2 Dependent Samples:

Step 1: State the claim mathematically. Identify the **null**, H_0 and **alternative**, H_a **hypotheses**. The possible hypotheses are:

$$\begin{array}{lll} H_0 : \mu_d \geq k & H_0 : \mu_d \leq k & H_0 : \mu_d = k \\ H_a : \mu_d < k & H_a : \mu_d > k & H_a : \mu_d \neq k \end{array}$$

Step 2: Identify (a) the **type of test** and (b) the **level of significance**, α of the hypothesis test.

Step 3: State that \bar{d} follows *t* – distribution with d.f. = $n-1$.

Step 4: Determine the rejection criteria using **rejection region**.

Step 5: Find the **standardized test statistic**, $t = \frac{\bar{d} - \mu_d}{\frac{s_d}{\sqrt{n}}}$

Step 6: Decide whether to reject or fail to reject H_0 and interpret the decision in the context of the original claim.

Example 11.4-1

An advertisement states that a particular lymphatic massage program will help participants lose weight after one month. The table shows the weights of 12 adults before and after the participating in the program. At $\alpha = 0.10$, can you conclude that the massage program helps participants lose weight? Assume the weights are normally distributed.

Subject	1	2	3	4	5	6	7	8	9	10	11	12
Weight (Before)	157	185	120	212	230	165	207	251	196	140	137	172
Weight (After)	150	181	121	206	215	169	210	232	188	138	145	172

Solution:

Subject	1	2	3	4	5	6	7	8	9	10	11	12
D (after - before)	-7	-4	1	-6	-15	4	3	-19	-8	-2	8	0

By calculator,

$$\bar{d} = -3.75 \quad \& \quad s_d = 7.8407$$

Step 1: $H_0 : \mu_d \geq 0$
 $H_a : \mu_d < 0 \quad (\text{claim})$

Step 2: It is a left-tailed test and $\alpha = 0.10$

Step 3: \bar{d} follows t – distribution with d.f. = $12 - 1 = 11$, $\mu_{\bar{d}} = 0$ and

$$\sigma_{\bar{d}} = \frac{s_d}{\sqrt{n}} = \frac{7.8407}{\sqrt{12}} = 2.2634$$

Step 4: Reject H_0 if $t < t_{0.10} = -1.363$.

Step 5: Standardized test statistic,

$$t = \frac{\bar{d} - \mu_{\bar{d}}}{\sigma_{\bar{d}}} = \frac{-3.75 - 0}{2.2634} = -1.657$$

Step 6: Since $t = -1.657 < t_{0.10} = -1.363$, we reject H_0 .

At $\alpha = 0.10$, there is enough evidence to support the claim that massage program helps participants lose weight.

Example 11.4-2

The table gives the blood pressures (in mm Hg) of seven adults before and after the completion of a special dietary plan.

Individual	1	2	3	4	5	6	7
Before	210	180	195	220	231	199	224
After	193	186	186	223	220	183	233

Let μ_d be the mean of the differences between the systolic blood pressures before and after completing this special dietary plan for the population of all adults. Using the 5% significance level, can you conclude that the mean of the paired difference μ_d is different from zero? Assume the blood pressures are normally distributed.

Solution:

Individual	1	2	3	4	5	6	7
d							

By calculator,

Tutorial 11: Hypothesis Testing with Two Populations

A Independent Samples

QUESTION 1

A study was designed to investigate the effect of a calcium-deficient diet on lead consumption in rats. One hundred rats were randomly divided into 2 groups of 50 each. One group served as a control group and the other was the experimental, or calcium-deficient group. The response record was the amount of lead consumed per rat. The results were summarized by:

CONTROL	$\bar{x}_1 = 5.2$	$s_1 = 1.1$	$n_1 = 50$
EXPERIMENTAL	$\bar{x}_2 = 5.6$	$s_2 = 1.3$	$n_2 = 50$

At $\alpha = 0.05$, is there sufficient evidence to suggest that the calcium deficient diet results in increased lead consumption in rats?

QUESTION 2

A study was conducted to assess whether teenage boys worry more than teenage girls. A scale called the Anxiety Scale was used to measure the level of anxiety experienced by an individual. A higher value on the Anxiety Scale corresponds to a higher level of anxiety. The results obtained are summarized in the table below:

	Sample size	Sample Mean	Sample Standard Deviation
Boys	102	66.78	9.2
Girls	76	65.33	9.3

Is there sufficient evidence at the 5% level that teenage boys score higher on the Anxiety Scale than the teenage girls?

QUESTION 3

An insurance company wants to know if the average speed at which men drive cars is higher than that of women drivers. The company took a random sample of 20 cars driven by men on an expressway and found the mean speed to be 89 km/h with a standard deviation of 3 km/h. Another sample of 18 cars driven by women on the same expressway gave a mean speed of 86 km/h with a standard deviation of 2.5 km/h. Assume that the speeds at which all men and all women drive cars on this expressway are normally distributed with unequal population standard deviations. Test at the 10% significance level whether the mean speed of cars driven by all men drivers on this expressway is higher than that of cars driven by all women drivers.

B Dependent Samples

QUESTION 1

Triglyceride is a type of fat found in fatty tissue. Individuals found with high level of triglyceride in their blood have a higher risk of contracting heart diseases. To determine if regular exercise can reduce triglyceride levels, researchers measured the triglyceride level of 8 individuals with mild high cholesterol before and after attending 3 months of intensive aerobics exercise program.

Individual	1	2	3	4	5	6	7	8
Before	200	226	218	246	195	278	254	237
After	135	206	146	172	175	224	233	192

Test, at the 5% significance level, if the aerobics exercise program has been effective in reducing triglyceride level in blood serum. Assume triglyceride levels are normally distributed.

QUESTION 2

A dietitian wishes to see if a person's cholesterol level (in mg/dL) will change if the diet is supplemented by a certain mineral. Six subjects were pretested and then they took the mineral supplement for a six-week period. The results are shown in the table below.

Subject	1	2	3	4	5	6
Before	210	235	208	190	172	244
After	190	170	210	188	173	228

- State the underlying assumptions needed to perform a hypothesis testing in this context.
- Test, at the 10% significance level, whether there is a change in cholesterol level when the mineral supplements the diet.

QUESTION 3

Susan, the receiving clerk of a chemical distributor, is faced with a continuing problem of broken glassware which includes test tubes, petri dishes and flasks. Susan imposed some additional shipping precautions which she believes can prevent further breakage on these types of glassware. After a month of implementing the precautionary measures, she requested the purchasing clerk to provide her the information on the average number of broken items per shipment. Data from eight different suppliers given to the purchasing clerk are given below.

Supplier	1	2	3	4	5	6	7	8
Before	16	12	18	7	14	19	6	17
After	14	13	12	6	9	15	8	15

Does the data indicate, at $\alpha = 0.05$, that the new measures have lowered the average number of broken items? Assuming the number of broken glassware is normally distributed.

Answers

A1 Reject H_0

A2 Do not reject H_0

A3 Reject H_0

B1 Reject H_0

B2 Do not reject H_0

B3 Reject H_0