Chapter 1: Number Systems

Objective

The objective of this chapter is to
1. understand and appreciate the different number systems
2. use the arithmetic operators (+,-,/,*) in different number systems
3. use complements to do subtraction
4. convert one number system to another

Content

1.1 Introduction

Natural numbers

The natural numbers are $1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad \ldots$

Integers

The integers $\ldots, \quad -3, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad \ldots$ are the whole numbers, including zero and negative values.

Rational and irrational numbers

The rational numbers include the integers and all other numbers that can be expressed as the quotient of two integers, examples

$$\frac{1}{2}, \ -\frac{3}{2}, \ \frac{22}{7}, \ 7 \text{ and } -4$$

Numbers cannot be expressed as quotients of two integers are called irrational numbers. Some irrational numbers are

$$\sqrt{2}, \quad \sqrt[3]{5}, \quad \pi, \quad e$$

Real numbers

The rational and irrational numbers together make up the real numbers.
Numbers such as $\sqrt{-4}$ do not belong to the real number system. They are called imaginary or complex numbers.

Complex numbers

These are numbers of the form $a \ + \ b\sqrt{-1}$, where a and b are real numbers. Normally the complex numbers are written as $a \ + \ jb$ where $j = \sqrt{-1}$.

Different number systems

Number system	Range of a single place value	Example
Decimal	0 to 9	$903,159.75$
Binary	0 and 1	1101.11
Octal	0 to 7	721
Hexadecimal	0 to $F(15)$	$FF01$, $10A$, $B9$

Number system	Decimal (Base 10)	Binary (Base 2)	Octal (Base 8)	Hexadecimal (Base 16)
Representation	0	0	0	0
	1	1	1	1
	2	10	2	2
	3	11	3	3
	4	100	4	4
	5	101	5	5
	6	110	6	6
	7	111	7	7
	8	1000	10	8
	9	1001	11	9
	10	1010	12	A
	11	1011	13	B
	12	1100	14	C
	13	1101	15	D
	14	1110	16	E
	15	1111	17	F
	16	10000	20	10
	17	10001	21	11

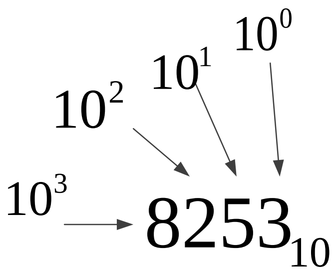
1.2 Decimal Number System

A decimal number can be expressed as a sum of powers of 10.

For example, $8253 = (8\times10^3)+(2\times10^2)+(5\times10^1)+(3\times10^0)$

This is called the expanded notation and the powers of ten are called **the place values** of the digit.

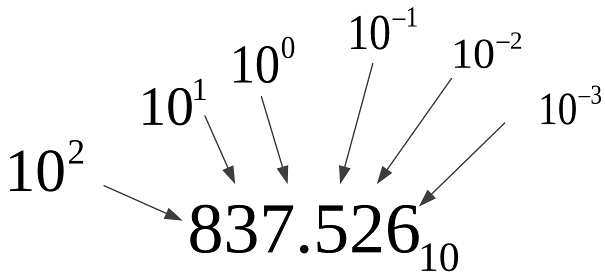
Place Values



Another example is the fractional form 837.526

$837.526 = (8\times10^2)+(3\times10^1)+(7\times10^0)+(5\times10^{-1})+(2\times10^{-2})+(6\times10^{-3})$

Place Values



1.3 Binary Number System

The binary number system is very important to computer studies because:

1. computer electronic components are bi-stable in nature, denoted by 0 and 1,
2. all data stored in computers are represented as bytes, which is denoted by 0 and 1,
3. data are transmitted and received as bits and bytes in any input/output devices such as printers and mouse and
4. information transmitted through network by software protocol such as TCP/IP, X.25, and Net-bios, are all in certain binary representation formats.

All computer data representations are in terms of bits and bytes. These are represented by two digits: 0 and 1. Thus, computer data representations are expressed using the binary system that has a base of 2.

The place values of the integral part of a binary number are

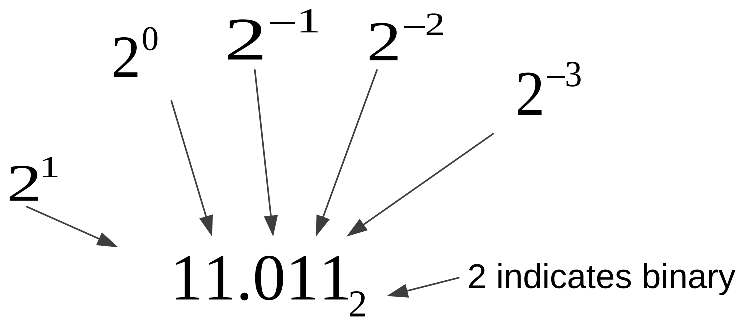
..., 2^5, 2^4, 2^3, 2^2, 2^1, and 2^0

and the place values of the fractional part of a binary number are

2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, ...

An example of such an expression is as follows:

Place Values



1.3.1 Binary Addition

Binary addition basics

		0	+	0	=	0
		0	+	1	=	1
		1	+	0	=	1
		1	+	1	=	10
1	+	1	+	1	=	11

Note that there is no digit 2 or higher used in binary. Once the digit is two or more in value, it is carried over to the next higher place value.

Example 1

$$\begin{array}{r} 1110 \\ \underline{+1010} \\ \underline{11000} \end{array}$$

Exercise 1.1

(1) Evaluate the binary sum $111+101$.

(2) Calculate the following binary sum:

(a) $1100\ 1101+1011\ 0111$ (b) $1001+1101+1101+1011$

(c) $1101.01+1010.1101$

1.3.2 Binary Subtraction

Binary subtraction basics

$$\boxed{\begin{array}{ccccc} 0 & - & 0 & = & 0 \\ 1 & - & 0 & = & 1 \\ 1 & - & 1 & = & 0 \\ 10 & - & 1 & = & 1 \end{array}}$$

Example 2

$$\begin{array}{r} 1110 \\ \underline{-1010} \\ \underline{100} \end{array}$$

Exercise 1.2

(1) Evaluate the binary difference $1101 - 1011$.

(2) Evaluate the binary difference $1100 - 1001$.

(3) Calculate the difference for the following:

(a) $1100\,1001 - 1001\,1100$ (b) $1101.101 - 11.1011$

1.3.3 Binary Multiplication

Binary Multiplication basics

$$\begin{array}{|ccccc|}\hline 0 & \times & 0 & = & 0 \\ 0 & \times & 1 & = & 0 \\ 1 & \times & 0 & = & 0 \\ 1 & \times & 1 & = & 1 \\ \hline\end{array}$$

Example 3

$$\begin{array}{r} 1010 \\ \times\quad 11 \\ \hline 1010 \\ 1010 \\ \hline 11110 \end{array}$$

Exercise 1.3

(1) Calculate the binary product 1101×1011.

(2) Calculate the binary product 11.01×101.1

(3) Calculate the binary product $110\,1011 \text{ x } 10101$

1.3.4 Binary division

Binary division is similar to the method used in decimal number system.

Example 4

$$\begin{array}{r l}
 & 1001 \\
11 & \overline{)11011} \\
 & \underline{11} \\
 & 11 \\
 & \underline{11} \\
 & \underline{00}
\end{array}$$

Exercise 1.4

(1) Evaluate $11011 \div 11$.

(2) Evaluate (a) $110111 \div 1001$, (b) $111.0000 \div 1.01$

1.3.5 Binary Complements

It is possible to reserve a bit to denote the sign of a number, '0' for '+' and '1' for '–', but most computers store negative numbers in the form of their arithmetic complements.

Complements are also use in the operation of subtraction to reduce subtraction to addition, in order to avoid the possible borrowing '1' from one column to another.

There two types of complements, namely radix-minus-one complement and radix complement. They are known as ones complement and twos complement.

For example, the ones complement of a binary number is obtained by inverting each digit of the binary number, i.e. '0' is replaced by '1', and '1' is replaced by '0', and twos complements of this binary number is its ones complement plus '1'.

Example 5

Binary Number	Ones Complement	Twos Complement
$1111\,1000\,1111_2$	$0000\,0111\,0000_2$	$0000\,0111\,0001_2$
$1100\,0100\,1100_2$	$0011\,1011\,0011_2$	$0011\,1011\,0100_2$
$1110\,1011\,1000_2$	$0001\,0100\,0111_2$	$0001\,0100\,1000_2$

Binary subtraction using complements

Let A and B be two binary numbers with the same number of 4 bits. Then:

Step 1: Find twos complement of A

Step 2: B + twos complement of A

Step 3: (a) If $A < B$, ignore overflow to find $B - A$.

(b) If $A > B$, no overflow, add negative sign to twos complement of result in Step 2 to find $B - A$.

Exercise 1.5

(1) Find the ones and twos complement of $1100\ 1100$.

(2) Evaluate $1111_2 - 1100_2$ using twos complement.

(3) Evaluate $1111\ 0000_2 - 1000\ 1100_2$ using twos complement.

(4) Evaluate $10\ 1011_2 - 11\ 0100_2$ using twos complement.

1.4 Octal Number System

The octal system has a positional numeration system of base 8. The eight hexadecimal digits are $0,1,2,3,4,5,6$ and 7.

Since $8 = 2^3$, each octal digit has a unique 3-bit representation shown below:

Octal Digit	Binary Equivalent
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

1.4.1 Octal Arithmetic

Octal Addition

The addition of two octal digits can be obtained by:

(a) finding their decimal sum and
(b) modifying the decimal sum, if exceeds 7, by subtracting 8 and carry 1 to next column

Exercise 1.6

(1) Evaluate the following octal sum.

(a) $3_8 + 2_8$ (b) $6_8 + 3_8$ (c) $3_8 + 5_8 + 7_8$

(2) Evaluate $7516_8 + 2703_8$

Octal Subtraction

During the octal subtraction, we may need to borrow 1 from the next column.

Exercise 1.7

(1) Evaluate $5501_8 - 1722_8$.

(2) Evaluate $671354_8 - 213604_8$

1.5 Hexadecimal Number System

The hexadecimal system has a positional numeration system of base 16. The sixteen hexadecimal digits are

$$0, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad A, \quad B, \quad C, \quad D, \quad E, \quad \text{and} \quad F$$

Since $16 = 2^4$, each hexadecimal digit has a unique 4-bit representation shown below:

Hexadecimal Digit	Binary Equivalent
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
A	1010
B	1011
C	1100
D	1101
E	1110
F	1111

1.5.1 Hexadecimal Arithmetic

<u>Hexadecimal Addition</u>

The sum of two hexadecimal numbers is obtained through

1. finding their decimal sum, each hexadecimal letter is changed to decimal form when finding the decimal sum,
2. modifying the decimal sum, if it exceeds 15, by subtracting 16 and carrying 1 to the next column, change .

We need to change each decimal sum greater than nine to hexadecimal number during the hexadecimal addition. You may find it useful to be familiar with the following hexadecimal equivalent:

$A = 10$, $\quad B = 11$, $\quad C = 12$, $\quad D = 13$, $\quad E = 14$, $\quad F = 15$

<u>Exercise 1.8</u>

(1) Evaluate the following hexadecimal sums

(a) $6_{16}+2_{16}$ (b) $F_{16}+E_{16}$ (c) $B_{16}+A_{16}+4_{16}$

(2) Evaluate $C89A_{16}+72D6_{16}$

Hexadecimal Subtraction

The hexadecimal difference is obtained through finding their decimal difference, each hexadecimal letter is changed to decimal form when finding the decimal difference, and we may need to borrow 1 from the next column. We need to change each decimal difference greater than nine to hexadecimal number during the hexadecimal subtraction.

Exercise 1.9

(1) Evaluate $9B36_{16}-4E85_{16}$

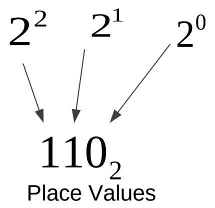
(2) Evaluate $A57913_{16}-64EE00_{16}$

1.6 Converting to Decimal number system

1.6.1 Binary to decimal conversion

This conversion to decimal number system is done is by multiplying each digit with its place values.

<u>Example 6</u>



$$\begin{aligned} 110_2 &= (1\times 2^2)+(1\times 2^1)+(0\times 2^0) \\ &= 4+2 \\ &= \underline{\underline{6}} \end{aligned}$$

$$\begin{aligned} 101_2 &= (1\times 2^2)+(0\times 2^1)+(1\times 2^0) \\ &= 4+1 \\ &= \underline{\underline{5}} \end{aligned}$$

$$\begin{aligned} 0.1_2 &= (1\times 2^{-1}) \\ &= \frac{1}{2} \\ &= \underline{\underline{0.5}} \end{aligned}$$

$$\begin{aligned} 0.11_2 &= (1\times 2^{-1})+(1\times 2^{-2}) \\ &= \frac{1}{2}+\frac{1}{2^2} \\ &= \frac{3}{4} = \underline{\underline{0.75}} \end{aligned}$$

<u>Exercise 1.10</u>

Convert the following to its decimal equivalent:

(1) 1100

(2) 101.1101

1.6.2 Octal to decimal conversion

The conversion techniques from octal to decimal number is similar to that in binary-decimal conversion, except that the factor of multiplication is replaced by 8.

Example 7

$$\begin{aligned}13_8 &= (1\times 8^1)+(3\times 8^0)\\ &= 8+3=\underline{\underline{11}}\end{aligned}$$

$$\begin{aligned}76_8 &= (7\times 8^1)+(6\times 8^0)\\ &= 56+6=\underline{\underline{62}}\end{aligned}$$

$$\begin{aligned}74.53_8 &= (7\times 8^1)+(4\times 8^0)+(5\times 8^{-1})+\left(3\times 8^{-2}\right)\\ &= 56+4+\frac{5}{8}+\frac{3}{64}=\underline{\underline{60\frac{43}{64}}}\end{aligned}$$

Exercise 1.11

Convert the following octal into decimal number system:

(a) 135_8

(b) 35.25_8

1.6.3 Hexadecimal to decimal conversion

The conversion technique from hexadecimal to decimal number is similar to that in binary-decimal conversion, except that the factor of multiplication is replaced by 16.

Example 8

$$\begin{aligned} F9_{16} &= (15\times16^1)+(9\times16^0) \\ &= 240+9=\underline{\underline{249}} \end{aligned}$$

$$\begin{aligned} 3A_{16} &= (3\times16^1)+(10\times16^0) \\ &= 48+10=\underline{\underline{58}} \end{aligned}$$

$$\begin{aligned} 3A.56_{16} &= (3\times16^1)+(10\times16^0)+(5\times16^{-1})+(6\times16^{-2}) \\ &= 48+10+\frac{5}{16}+\frac{6}{256}=\underline{\underline{58\frac{43}{128}}} \end{aligned}$$

Exercise 1.12

Convert the following hexadecimal into decimal form:

(a) 19_{16}

(b) $3F.\mathrm{A}_{16}$

1.7 Converting from Decimal to other number systems

1.7.1 Decimal to binary conversion

This conversion is done through

1 Division of the integer value of the decimal number by 2, and
2 Multiplication of the fractional value of the decimal number by 2.

The remainders recorded after each division is taken as the binary equivalent to the decimal number.

Example 9

Convert 25_{10} to its binary equivalent

$$
\begin{array}{r|l l}
 & & \text{remainder} \\
2 & \underline{25} & \\
2 & \underline{12} & 1 \\
2 & \underline{6} & 0 \\
2 & \underline{3} & 0 \\
2 & \underline{1} & 1 \\
 & 0 & 1 \uparrow
\end{array}
$$

Answer: $25_{10} = 11001_2$

Example 10

Convert 0.125_{10} to its binary equivalent

$$
\begin{aligned}
0.125 \times 2 &= 0.250 \\
0.250 \times 2 &= 0.5 \\
0.5 \times 2 &= 1.0 \downarrow
\end{aligned}
$$

Answer: $0.125_{10} = 0.001_2$

Example 11

Convert 25.125_{10} to its binary equivalent

$$
\begin{array}{l|ll}
 & & \text{remainder} \\
2 & 25 & \\
2 & 12 & 1 \\
2 & 6 & 0 \\
2 & 3 & 0 \\
2 & 1 & 1 \\
 & 0 & 1
\end{array}
\uparrow
$$

$$
\begin{array}{l|l}
0.125 \times 2 = & 0.250 \\
0.250 \times 2 = & 0.5 \\
0.5 \times 2 \quad = & 1.0
\end{array}
\downarrow
$$

Answer $25.125_{10} = 11001.001_2$

Exercise 1.13

(1) Convert the decimal number 109.78125_{10} to its binary equivalent.

(2) Convert the decimal number 13.6875_{10} to its binary equivalent.

The binary equivalent of a terminating decimal does not always terminate.

(3) Convert the decimal number 0.6 to binary:

1.7.2 Decimal to octal conversion

This conversion is done through

1. Division of the integer value of the decimal number by 8, and
2. Multiplication of the fractional value of the decimal number by 8.

The remainders recorded after each division is taken as the octal equivalent to the decimal number.

Example 12

Convert 20_{10} to its octal equivalent.

$$\begin{array}{r|l l l} & & & \text{remainder} \\ 8 & 20 & & \\ 8 & 2 & \rightarrow & 4 \uparrow \\ & 0 & \rightarrow & 2 \end{array}$$

Answer: $20_{10} = 24_8$

Example 13

Convert 0.5625_{10} to its octal equivalent.

$$\begin{array}{l l l} 0.5625 \text{ x } 8 = 4.5 & \rightarrow & 4 \\ 0.5 \text{ x } 8 = 4.0 & \rightarrow & 4 \end{array} \downarrow \quad \text{Integral part}$$

Answer: $0.5625_{10} = 0.44_8$

Example 14

Convert 20.5625_{10} to its octal equivalent

Based on example 12 and 13, $20.5625_{10} = 24.44_8$

Exercise 1.14

Convert 684.90625_{10} to its octal equivalent

1.7.3 Decimal to hexadecimal conversion

This conversion is done through

1. Division of the integer value of the decimal number by 16, and
2. Multiplication of the fractional value of the decimal number by 16.

The remainders recorded after each division is taken as the hexadecimal equivalent to the decimal number.

Example 15

Convert 9718_{10} to its hexadecimal equivalent.

$$
\begin{array}{r|rcl}
16 & 9718 & & \\
16 & 607 & \rightarrow & 6 \\
16 & 37 & \rightarrow & 15 = F \\
16 & 2 & \rightarrow & 5 \\
 & 0 & \rightarrow & 2
\end{array}
$$

Answer: $9718_{10} = 25F6_{16}$

Example 16

Convert 0.78125_{10} to its hexadecimal equivalent.

$$
\begin{array}{lcl}
0.78125 \text{ x } 16 = 12.5 & \rightarrow & 12 = C \\
0.5 \text{ x } 16 = 8.0 & \rightarrow & 8
\end{array}
$$

Integral part

Answer: $0.78125_{10} = 0.C8_{16}$

Example 17

Convert 9718.78125_{10} to its hexadecimal equivalent

Based on example 15 & 16, $9718.78125_{10} = 25F6.C8_{16}$

Exercise 1.15

Convert 229.34375_{10} to its hexadecimal equivalent

1.8 Conversion between Binary, Octal and Hexadecimal number system

1.8.1 Binary to octal conversion

When converting the binary number to octal number, group the bits into sets of three starting at the binary point, adding zeros as needed to fill out the groups for fractional value of the binary number. Then assign each 3-bit group with the appropriate octal digit.

Octal Digit	Binary Equivalent
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

Example 18

Convert $11101\ 0111\ 1011_2$ to its octal system.

$11101\ 0111\ 1011_2 = 16573_8$

Exercise 1.16

Convert $1100\ 1110.0101011_2$ to its octal system.

1.8.2 Octal to binary conversion

The conversion from octal number to binary number is to convert octal number to a 3 bit binary number.

Example 19

Covert 27.72_8 to its binary equivalent.

$27.72_8 = 10111.111010_2$

Exercise 1.17

Convert 672.534_8 to its binary equivalent.

1.8.3 Binary to hexadecimal conversion

When converting the binary number to hexadecimal number, group the bits into sets of four starting at the binary point, adding zeros as needed to fill out the groups for fractional value of the binary number. Then assign each 4-bit group with the appropriate hexadecimal digit.

Hexadecimal Digit	Binary Equivalent
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
A	1010
B	1011
C	1100
D	1101
E	1110
F	1111

Example 20

Convert the binary number $0001\,1010.0110\,1110_2$ into hexadecimal number

$0001\,1010.0110\,1110_2 = 1A.6E_{16}$

Exercise 1.18

Convert the binary number $1100.1011\ 0110\ 11_2$ into hexadecimal number.

1.8.4 Hexadecimal to binary conversion

The conversion from hexadecimal number to binary number is to convert hexadecimal number to a 4 bit binary number.

Example 21

Convert the hexadecimal $27.A3C_{16}$ into binary.

$27.A3C_{16} = 0010\ 0111.1010\ 0011\,1100_2$

Exercise 1.19

Convert $39.B8_{16}$ to its binary equivalent

Tutorial 1

1. State the place value of each underlined bit:

   (a) $10\underline{1}010_2$ (b) $1011.1\underline{0}01_2$ (c) $4\underline{7}.21_{10}$ (d) $A65.E3\underline{B}_{16}$ (e) $\underline{3}57.16_8$

2. Find X by performing the following conversions:

   (a) 110.11_2 to X_{10} (b) 809.625_{10} to X_2 (c) $23E.9A_{16}$ to X_2
   (d) 255.125_{10} to X_{16} (e) 1100.100100_2 to X_{10} (f) 163_{10} to X_2
   (g) 27.72_8 to X_{10} (h) $CE9.D5_{16}$ to X_{10} (i) 7546_{10} to X_8

3. Evaluate the following arithmetic:

   (a) $1101.0111_2 + 11011.1011_2$ (b) $1101011_2 + 1100101_2 + 10011_2$
   (c) 1101.011_2 x 1.01_2 (d) $11001.01101_2 - 1101.10111_2$
   (e) $1001.011_2 \div 10.1_2$ (f) $75.432_8 + 4.446_8$
   (g) $134.63_8 - 77.572_8$ (h) $AD.1B7_{16} + 1E.8D_{16}$
   (i) $F3.D2_{16} - 68.ACE_{16}$

4. Find the binary difference of $1110101_2 - 1101001_2$ using 2s complements.

5*. Find the binary difference of $10101_2 - 11011_2$ using 2s complements.

6*. Find Y_2, given that the twos complement of Y_2 is $1001\,1001_2$.

7. Evaluate giving your answer in hexadecimal
   $2\mathrm{F}.4_{16} + (1101.1_2 \times 1000.1_2)$.

Answers

1. (a) 2^3 (b) 2^{-2} (c) 10^0 (d) 16^{-3} (e) 8^2

2. (a) 6.75_{10} (b) 1100101001.101_2 (c) $0010\ 0011\ 1110.1001\ 1010_2$
   (d) $FF.2_{16}$ (e) 12.5625_{10} (f) 10100011_2
   (g) 23.90625_{10} (h) $3{,}305.8320_{10}$ (i) 16572_8

3. (a) 101001.0010_2 (b) 11100011_2 (c) 10000.10111_2
   (d) 1011.10110_2 (e) 11.11_2 (f) 102.100_8
   (g) 35.036_8 (h) $CB.A87_{16}$ (i) $8B.252_{16}$

4. 1100 5*. -00110 6*. $0110\ 0111_2$ 7. $\mathrm{A}2_{16}$

Questions from past year examination Papers

1) 2010/11S1 IT1101/1501/1561/1751/1621 Sem Exam– Q1

(a) Find the product of the two binary numbers, $\mathbf{1101_2 \times 101_2}$. (2 marks)

(b) Convert $\mathbf{467.5_8}$ into its hexadecimal equivalent. (4 marks)

(c) Evaluate $Y = B - A$, using complements method, where $\boldsymbol{A} = \mathbf{10001_2}$ and $\boldsymbol{B} = \mathbf{11011_2}$. (4 marks)

2) 2013/14S2 EG1740 Sem Exam– Q1

(a) Evaluate $101.1_2 + A9_{16}$ and leave your answer in decimal form. (4 marks)

(b) Evaluate $1001\,1101_2 - 0101\,1001_2$ using the twos complement. (4 marks)

3) 2012/13S2 EG1740 Sem Exam– Q1

(a) Convert 6.75_{10} to its binary equivalent. (3 marks)

(b) Given $X_8 = 153_8$ and $Y_{16} = 2D_{16}$, evaluate $X_8 + Y_{16}$ and leave your answer in binary form. (5 marks)

Answers

1a) $100\ 0001$
b) $467.5_8 = \; = 137.A_{16}$
c) 1010_2

2a) 174.5_{10}
b) $0100\ 0100$

3a) $6.75_{10} = 110.11_2$
b) 1001 1000