Chapter 9: Integration

Objective

The objective of this chapter is to:

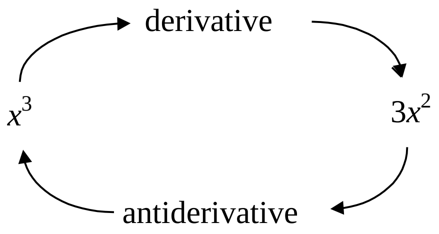
1. understand that integration is the reverse process of differentiation.
2. find the integration of functions using formulae.
3. find the integration of functions by the method of substitution.
4. evaluate the value of a definite integral.
5. solve application problems involving integration.

Content

9.1 Introduction

Integration is the inverse of differentiation.

For example:



The *antiderivative* of $3x^2$ is x^3. The derivative of x^3 is $3x^2$.

There are two types of integration

i) The indefinite integral $\int f(x)dx$

ii) The definite integral $\int_a^b f(x)dx$

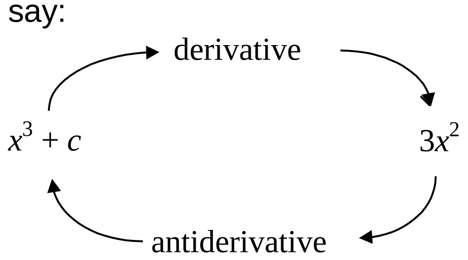
9.2 Antiderivative: The indefinite integral

9.2.1 The constant of integration

The *antiderivative* of $3x^2$ is x^3. The derivative of x^3 is $3x^2$.

However the derivative of x^3+2 is also $3x^2$. In fact the derivative of x^3 + (*any constant*) is also $3x^2$. This constant, called the constant of integration, must be included when we find the antiderivative of a function.

Thus it is more accurate to say:



where c is the constant of integration.

The antiderivative is also called the *indefinite integral*.

9.2.2 Notation for antiderivatives

In differentiation we denote the derivative of $f(x)$ as $f'(x)$ or $\dfrac{df(x)}{dx} = \dfrac{d}{dx} f(x)$.

In integration the antiderivative or indefinite integral of $f(x)$ is denoted by:

$$\int f(x)\,dx = F(x) + c \quad \text{where } f(x) = F'(x) \ .$$

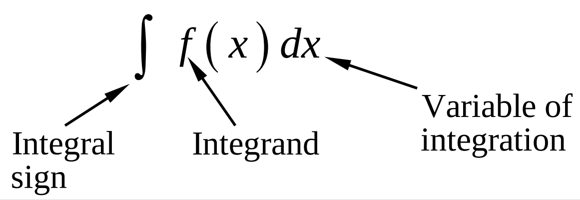
Note: $\int$ is called the ***integral sign***.

$f(x)$ is called the ***integrand*** of the indefinite integral.

dx indicates the ***variable of integration.***

Thus the integration of $3x^2$ can be represented by $\int 3x^2\,dx = x^3 + c$

Components of the Indefinite Integral



9.2.3 Integral of x^n

Integral of a Constant

$$\int k\,dx = kx + c \qquad \text{where } k \text{ is a constant}$$

Integral of x^n

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \qquad \text{where } n \neq -1$$

Let *f* and *g* both have antiderivatives and *k* is constants. Then

$$\int k\,f(x)\,dx = k\int f(x)\,dx$$

$$\int \left[f(x) \pm g(x) \right] dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example 1

		Answer
(a)	$\int x^3 \ dx$	$\frac{x^{3+1}}{3+1}+C=\frac{x^4}{4}+C$
(b)	$\int x^{-3} \ dx$	$\frac{x^{-3+1}}{-3+1}+C=\frac{x^{-2}}{-2}+C=-\frac{1}{2x^2}+C$
(c)	$\int 8x^2+4x^3 dx$	$\frac{8x^3}{3}+\frac{4x^4}{4}+C=\frac{8}{3}x^3+x^4+C$

Exercise 9.1

(1) Find: (a) $\int x^3 dx$ (b) $\int 3x^5 dx$ (c) $\int\left(\sqrt{x}+\frac{2}{x^2}-5\right)dx$

(2) Find (a) $\int\left(x^2+1\right)^2 dx$. (b) $\int \frac{x^4-3x^3+2x^2}{x} \ dx$

9.2.4 Integrals of the Exponential Functions

Integral of e^x $\boxed{\int e^x \, dx = e^x + C}$

Integral of e^{ax+b} $\boxed{\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C}$

Example 2

		Answer
(a)	$\int e^{3x} \, dx$	$\frac{1}{3} e^{3x} + C$
(b)	$\int e^{2x+15} \, dx$	$\frac{1}{2} e^{2x+15} + C$

Exercise 9.2

(a) $\int \left(e^x + 1\right) dx$

(b) $\int e^{4x-1} dx$

(c) $\int e^{-3x} dx$

(d) $\int \frac{1}{e^{2t}} dt$

(e) $\int \frac{3 - e^{2u}}{e^u} du$

9.2.5 Integrals leading to the Logarithmic Functions

Integral of $\frac{1}{x}$

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C \qquad \text{where } x \neq 0}$$

Integral of $\frac{f'(x)}{f(x)}$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

Example 3

(a)	$\int \frac{2x}{x^2}\,dx$	$\ln\left\|x^2\right\| + C$
(b)	$\int \frac{9x^2+2x}{3x^3+x^2}\,dx$	$\ln\left\|3x^3+x^2\right\| + C$

Exercise 9.3

(a) $\int \frac{1}{2x}dx$

(b) $\int \frac{x^3-2x+1}{2x}\,dx$

(c) $\int \frac{1}{3x+4}dx$

(d) $\int \frac{1}{2t-3}dt$

(e) $\int \frac{10-4u}{3+5u-u^2}du$

(f) $\int \frac{e^{3x}+1}{e^{3x}+3x-1}dx$

9.3 Integration by Substitution

Recall: **Integral of** x^n

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1}$$

Note: This formula is only applicable if the base of the function x^n is the same as the variable of integration dx .

I can also write the above formula using a different variable, u :

$$\boxed{\int u^n \, du = \frac{u^{n+1}}{n+1} + c \qquad n \neq -1}$$

Using the above formula,

(a)
$$\begin{aligned}\int x^3 \, dx &= \frac{x^{3+1}}{3+1} + C \\ &= \frac{x^4}{4} + C\end{aligned}$$

(b)
$$\begin{aligned}\int u^7 \, du &= \frac{u^{7+1}}{7+1} + C \\ &= \frac{u^8}{8} + C\end{aligned}$$

What if the base of the function is the different from the variable of integration?
For example, $\int (x+1)^3 dx$. Notice that the base of the function $(x+1)^3$ is different from the variable of integration dx. In this case, we cannot use the formula directly.

We can solve the question by using **INTEGRATION by SUBSTITUTION**, which attempts to make the base of the function the same as the new variable of integration, u .

INTEGRATION by SUBSITUTION

Step 1: Define u .
Step 2: Differentiate u with respect to x .
Step 3: Express dx in terms of du .
Step 4: Rewrite the original integral in terms of u and du .
Step 5: Integrate according to the relevant formula.
Step 6: Replace u with the original expression of x in Step 1.

Example 4

(a)	$\int (x-7)^2\,dx$	Let $u = x-7 \Rightarrow \frac{du}{dx} = 1 \Rightarrow dx = du$ $\int (x-7)^2\,dx$ $= \int u^2\ du$ (must only be in u to integrate wrt to u) $= \frac{u^3}{3} + C$ $= \frac{(x-7)^3}{3} + C$
(b)	$\int x(3x^2-7)^2\,dx$	Let $u = 3x^2-7 \Rightarrow \frac{du}{dx} = 6x \Rightarrow dx = \frac{du}{6x}$ $\int x(3x^2-7)^2\,dx$ $= \int xu^2\ \frac{du}{6x}$ (must only be in u to integrate wrt to u) $= \frac{1}{6}\int u^2\ du$ $= \frac{1}{6}\left(\frac{u^3}{3}\right) + C$ $= \frac{(3x-7)^3}{18} + C$

Exercise 9.4

(1) Find $\int (2x-7)^3\,dx$

(2) Evaluate $\int (x^2+5x-1)^3(2x+5)\,dx$

(3) Evaluate $\int x\sqrt{1-x^2}\,dx$

(4) Evaluate $\int \frac{t^2+1}{(t^3+3t)^2}\,dt$

(5) Find $\int x\,e^{-x^2}\,dx$

(6) Find $\int \frac{\ln x}{x} dx$

(7) Find $\int \frac{2x \ln(x^2+1)}{x^2+1} dx$

9.4 The Definite Integral

9.4.1 Area between curves

The definite integral can be interpreted as an area.

When $f(x)$ is positive and $a < b$:

Area under graph of $f(x)$ = $\int_a^b f(x)\,dx$

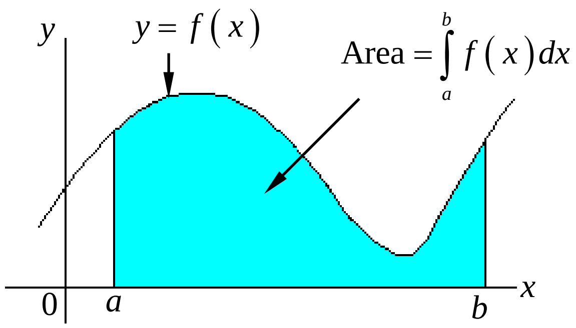


Figure 1: The definite integral $\int_a^b f(x)\,dx$

The Fundamental Theorem of Calculus

If $f(x)$ is the derivative of F, that is $f(x) = F'(x)$, and if $f(x)$ is continuous, Then

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a)$$

The theorem shows us how to evaluate definite integrals.

<u>Graphical Interpretation</u>

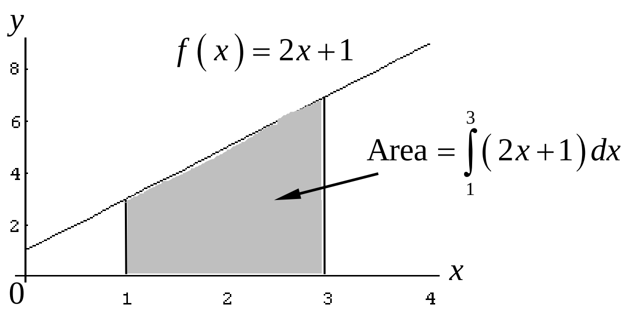


Figure 2 : Area under graph of $f(x) = 2x + 1$ between $x = 1$ and $x = 3$ is given by the definite integral $\int_1^3 (2x+1)\,dx$

Note:

When $f(x)$ **is**

So far the functions we have dealt lies above the x-axis. If the graph of a function lies below the x-*axis*, then each value of $f(x)$ is negative, so the area gets counted negatively. In other words, the definite integral of a function which is *under* the x-axis is negative.

Example 5

(a)	$\int_1^3 x^3 dx$	$\left.\frac{x^4}{4}\right\|_1^3 = \left(\frac{3^4}{4}\right) - \left(\frac{1^4}{4}\right) = \frac{80}{4} = 20$
(b)	$\int_{-5}^{-3} x^3 dx$	$\left.\frac{x^4}{4}\right\|_{-5}^{-3} = \left(\frac{(-3)^4}{4}\right) - \left(\frac{(-5)^4}{4}\right) = \frac{81-625}{4} = -136$
(c)	$\int_{-1}^{5} (x^3 - 3x^2)dx$	$\left.\left(\frac{x^4}{4} - x^3\right)\right\|_{-1}^{5} = \left(\frac{5^4}{4} - 5^3\right) - \left(\frac{(-1)^4}{4} - (-1)^3\right) = 31.25 - 1.25 = 30$

Exercise 9.5

(1) Evaluate the following:

(a) $\int_1^3 (2x+1)\, dx$.

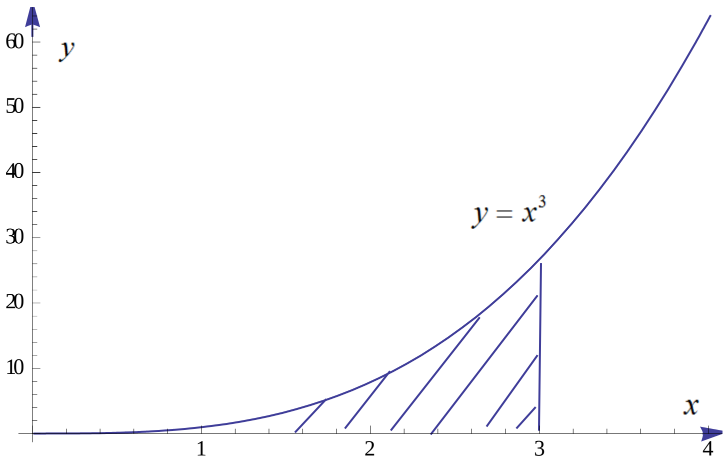
(b) $\int_1^2 (x^3 + 5)\, dx$

(c) $\int_0^1 (\sqrt{x} - x^2)\ dx$

(d) $\int_1^3 \frac{2}{x} dx$

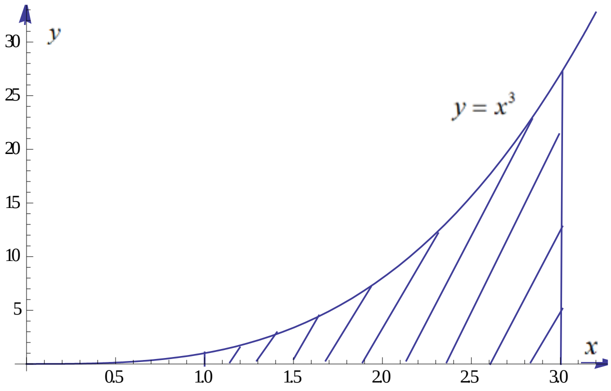
(e) $\int_{-1}^1 (e^{-x} + x)\, dx$

(2) Find the area bounded by $y = x^3$ from the y-axis to $x = 3$ as shown in the graph below.

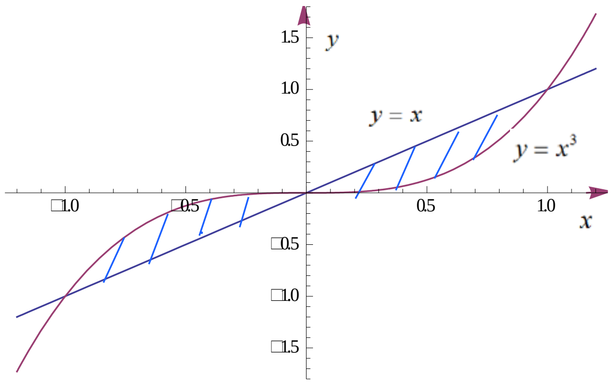


y
60
50
40
30
20
10
$y = x^3$
x
1
2
3
4

(3) Find the area bounded by $y = x^3$ from $x = 1$ to 3 as shown in the graph below.



(4) Find the area bounded by $y = x^3$ and $y = x$ as shown in the graph below



9.5 Application Problems

Example 6

Find the equation of the curve whose gradient function is given by $f'(x) = x+1$ and the curve passes through the point $(2,1)$.

$$f'(x) = x+1$$
$$f(x)$$
$$= \int (x+1)\,dx$$
$$= \frac{x^2}{2} + x + C$$

As the curve passes through the point $(2,1)$

$$f(2) = 1$$
$$\Rightarrow \frac{2^2}{2} + 2 + C = 1$$
$$\Rightarrow 2 + 2 + C = 1$$
$$\Rightarrow C = -3$$
$$f(x) = \frac{x^2}{2} + x - 3$$

Exercise 9.7

(1) Find the function $g(x)$ whose tangent has slope e^{1-2x} for each value of x and whose graph passes through the point $(0.5, 1.5)$.

(2) It is estimated that t years from now, the population of a certain town will grow at rate of $3t+5\sqrt{t}$ per year. If the current population is 4000, find the population of the town 5 years from now.

(3) Biologists are studying the effects of bacteria contamination on leftover food. The level of contamination is changing at the rate of $\frac{dN}{dt} = 60\left(1 + e^{-\frac{t}{5}}\right)$ bacteria/h, where N is the level of contamination and t is the time in hours.

(a) Find an expression for N if $N = 0$ when $t = 0$.
(b) Find the level of contamination after 4 hours.

(4) The body temperature, T °C, of a patient varies with the amount of a certain drug, x mg, administrated to the patient. If the rate of change in the patient's temperature with respect to the dosage is given by:

$$T'(x) = -\frac{2}{(1.2x + 0.72)^2},$$

Find an expression to describe how the body temperature of the patient changes with the dosage administrated. Assume that the patient's temperature was 39 °C initially.

Tutorial 9-1

1. Find the following integrals:

(a) $\int \frac{2}{3}\,dx$ (b) $\int \pi^2 dx$ (c) $\int x^8 dx$ (d) $\int x^{-\frac{2}{3}} dx$

(e) $\int \frac{1}{x^4}\,dx$ (f) $\int \sqrt[3]{x}\,dx$ (g) $\int \sqrt[3]{x^2}\,dx$ (h) $\int 6x^3 dx$

(i) $\int \frac{1}{3\pi} t\,dt$ (j) $\int \frac{\pi^2}{3} t^2 dt$ (k) $\int \frac{3}{x^2}\,dx$ (l) $\int 5\sqrt{x}\,dx$

(m) $\int \frac{1}{\sqrt{x}}\,dx$ (n) $\int \frac{1}{\sqrt[5]{t^2}}dt$

2. Find the following indefinite integrals:

(a) $\int \left(x^2 + x + 3\right)dx$ (b) $\int \left(7 - 5x - 3x^2\right)dx$ (c) $\int \left(4t^2 + 3t - 2\right)dt$

(d) $\int (x+3)^2 dx$ (e) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)dx$ (f) $\int \left(\frac{2}{x^2} - 3x^2 + 4\right) dx$

(g) $\int \left(\frac{5}{x^2} - \frac{3}{x^4}\right)dx$ (h) $\int \left(8\sqrt{x} + \frac{1}{4\sqrt{x}}\right)dx$ (i) $\int \left(2x^3 + \frac{1}{\sqrt{x}} - \frac{2}{x^2}\right) dx$

3. Evaluate the following definite integrals:

(a) $\int_0^2 \left(4t^2 - t\right)dt$ (b) $\int_1^2 \frac{2t^2 + 1}{t^2}\,dt$ (c) $\int_1^3 2r(r-2)dr$

4. Find the following Indefinite Integrals:

(a) $\int (-6x+1)\,dx$ (b) $\int \left(x^3 + 6\sqrt{x} - \frac{1}{x^2}\right)dx$ (c) $\int \left(\frac{x^4 + 7x}{x^3}\right)dx$

(d) $\int (2-3x)^2\,dx$ (e) $\int e^{-3x} dx$ (f) $\int \left(7e^{7t+1} + \frac{2}{t}\right)dt$

5. Find the following definite integrals.

(a) $\int_1^4 (x+1)(2x+1)\,dx$ (b) $\int_0^1 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$ (c) $\int_0^2 4e^{-2t}\,dt$

Answers

1.(a) $\frac{2}{3}x+C$ (b) $\pi^2 x+C$ (c) $\frac{1}{9}x^9+C$ (d) $3x^{\frac{1}{3}}+C$

(e) $-\frac{1}{3x^3}+C$ (f) $\frac{3}{4}x^{\frac{4}{3}}+C$ (g) $\frac{3}{5}x^{\frac{5}{3}}+C$ (h) $\frac{3}{2}x^4+C$

(i) $\frac{1}{6\pi}t^2+C$ (j) $\frac{\pi^2}{9}t^3+C$ (k) $-\frac{3}{x}+C$ (l) $\frac{10}{3}x^{\frac{3}{2}}+C$

(m) $2x^{\frac{1}{2}}+C$ (n) $\frac{5}{3}t^{\frac{3}{5}}+C$

2(a) $\frac{x^3}{3}+\frac{x^2}{2}+3x+C$ (b) $7x-\frac{5}{2}x^2-x^3+C$ (c) $\frac{4}{3}t^3+\frac{3}{2}t^2-2t+C$

(d) $\frac{x^3}{3}+3x^2+9x+C$ (e) $\frac{2}{3}x^{\frac{3}{2}}-2x^{\frac{1}{2}}+C$ (f)
$-\frac{2}{x}-x^3+4x+C$

(g) $-\frac{5}{x}+\frac{1}{x^3}+C$ (h) $\frac{16}{3}x^{\frac{3}{2}}+\frac{1}{2}x^{\frac{1}{2}}+C$ (i) $\frac{1}{2}x^4+2x^{\frac{1}{2}}+\frac{2}{x}+C$

3(a) $8\frac{2}{3}$ (b) $2\frac{1}{2}$ (c) $1\frac{1}{3}$

4(a) $-3x^2+x+C$ (b) $\frac{x^4}{4}+4x^{\frac{3}{2}}+\frac{1}{x}+C$ (c) $\frac{x^2}{2}-\frac{7}{x}+C$

(d) $4x-6x^2+3x^3+C$ (e) $-\frac{1}{3}e^{-3x}+C$ (f) $e^{7t+1}+2\ln|t|+C$

5(a) $67\frac{1}{2}$ (b) $2\frac{2}{3}$ (c) $2-2e^{-4}\approx 1.96$

Tutorial 9-2

1. Evaluate the followings:

a) $\int (1+x)^3 dx$ b) $\int (3x-2)^4 dx$ c) $\int 2x(x^2+2)^4 dx$

d) $\int x^2(x^3+4)^8 dx$ e) $\int \left(2x^2-1\right)(2x^3-3x+9)^{\frac{1}{3}} dx$

f) $\int \frac{x}{\sqrt{3-5x^2}} dx$ g) $\int x\sqrt{1-x^2}\, dx$

2. Evaluate the following indefinite integrals:

a) $\int x^2-2x+3-x^{-1} dx$ b) $\int 5-x^{-1}+3x^{-2} dx$

c) $\int (\frac{3+x}{x})dx$ d) $\int \frac{2x+1}{x}dx$ e) $\int \frac{1}{x+1}dx$

f) $\int \frac{1}{4x+3}dx$ g) $\int \frac{4}{4-x}dx$ h) $\int \left(\frac{1}{x}+\frac{2}{2x-1}\right)dx$

i) $\int \frac{2x}{x^2+2}dx$ j) $\int \frac{2x+2}{x^2+2x+7}dx$ k) $\int \frac{x^2}{x^3+2}dx$

l) $\int \frac{-3x^2}{1-x^3}dx$ m) $\int \frac{3x^2-2x-1}{x^3-x^2-x+10}dx$

3. Find the following indefinite integrals.

a) $\int \left(e^x+3\right)dx$ b) $\int e^{2x}\, dx$ c) $\int e^{-3x} dx$

d) $\int e^{2x+1} dx$ e) $\int 3e^{5x+4} dx$ f) $\int \left(2-3e^x\right)^2 dx$

g) $\int \left(3e^x+\frac{3}{e^{3x}}\right)dx$ h) $\int x\, e^{-x^2}\, dx$ i) $\int \left(x+1\right)e^{x^2+2x-1}\, dx$

4. Evaluate, by using the formula $\int \frac{f'(x)}{f(x)}dx = \ln\left|f(x)\right| + c$

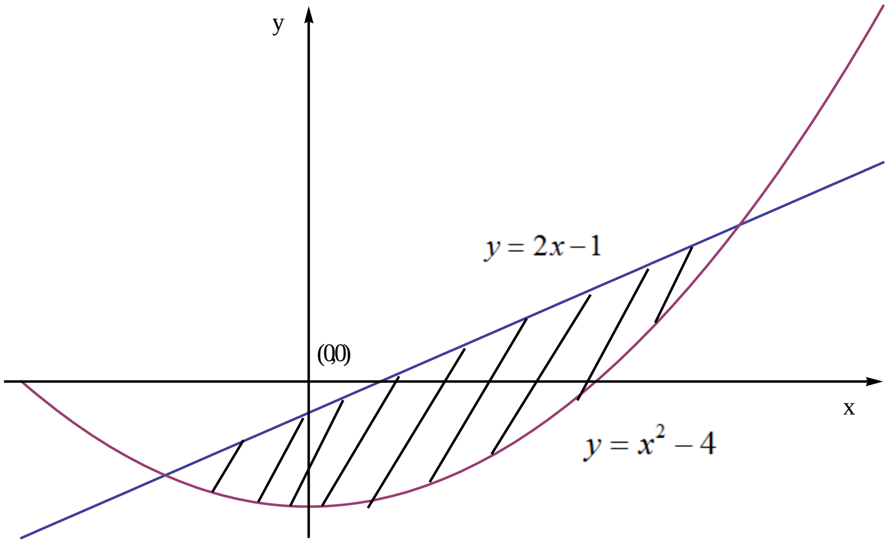
(a) $\int \frac{3x^2}{1-x^3}dx$ (b) $\int \frac{8e^x}{e^x+2}dx$

5*. Find the following indefinite integrals by substitution method.

(a) $\int 4x\left(x^2+3\right)^4 dx$ (b) $\int \frac{5x}{\sqrt{7-2x^2}}dx$

6. (i) Show that the points of intersection of the curves $y = 2x - 1$ and $y = x^2 - 4$ are $(-1, -3)$ and $(3, 5)$.

(ii) Hence, find the shaded area bounded by the curves in the figure below.



Answers

1. a) $\dfrac{(1+x)^4}{4}+C$ b) $\dfrac{(3x-2)^5}{15}+C$ c) $\dfrac{(x^2+2)^5}{5}+C$

   d) $\dfrac{(x^3+4)^9}{27}+C$ e) $\dfrac{(2x^3-3x+9)^{\frac{4}{3}}}{4}+C$

   f) $\dfrac{-1}{5}\sqrt{3-5x^2}$ +C g) $-\dfrac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}+C$

2 a) $\dfrac{x^3}{3}-x^2+3x-\ln|x|+C$ b) $5x-\ln|x|-3x^{-1}+C$

   c) $3\ln|x|+x+C$ d) $2x+\ln|x|+C$ e) $\ln|x+1|+C$

   f) $\dfrac{1}{4}\ln|4x+3|+C$ g) $-4\ln|4-x|+C$

   h) $\ln|x|+\ln|2x-1|+C$ i) $\ln\left|x^2+2\right|+C$ j) $\ln\left|x^2+2x+7\right|+C$

   k) $\dfrac{1}{3}\ln\left|x^3+2\right|+C$ l) $\ln\left|1-x^3\right|+C$

3 a) e^x+3x+C b) $\dfrac{e^{2x}}{2}+C$ c) $-\dfrac{e^{-3x}}{3}+C$

   d) $\dfrac{e^{2x+1}}{2}+C$ e) $\dfrac{3e^{5x+4}}{5}+C$

   f) $4x-12e^x+\dfrac{9}{2}e^{2x}+C$ g) $3e^x-e^{-3x}+C$ h) $-\dfrac{1}{2}e^{-x^2}+C$

   i) $\dfrac{1}{2}e^{x^2+2x-1}+C$

4. (a) $-\ln\left|1-x^3\right|+C$ (b) $8\ln\left|e^x+2\right|+C$

5* (a) $\dfrac{2}{5}\left(x^2+3\right)^5+C$ (b) $-\dfrac{5}{2}\left(7-2x^2\right)^{\frac{1}{2}}+C$

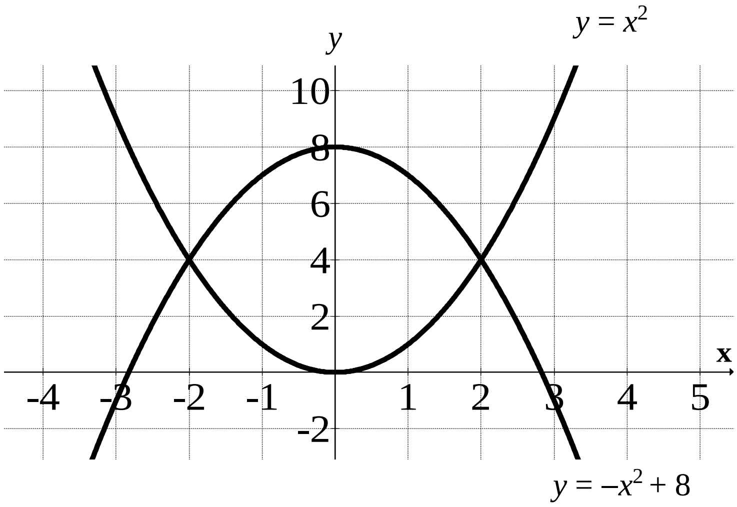
6. (ii) $10\dfrac{2}{3}$ units^2

Questions from past year examination papers

1) 2013/14S1 IT1101/1501/1561/1751/1621 Sem Exam– Q8

(a) Two curves $y = x^2$ and $y = -x^2 + 8$ intersect at $(-2,4)$ and $(2,4)$ as shown in the figure below. Find the enclosed area between the two intersection points, leaving your answer to one decimal place.

(7 marks)



(b) Evaluate the integral $\int \dfrac{3\sqrt{x} - x + 4}{2x}\, dx$. (4 marks)

(c) Use the substitution $u = 3 - 2x^4$ to evaluate the integral $\int \dfrac{2x^3}{\sqrt{3-2x^4}}\, dx$.

(4 marks)

2) 2013/14S2 EG1740 Sem Exam– Q5b

Evaluate $\int_1^3 \left(x^3 - x^{-1} + 3\right) dx$. (4 marks)

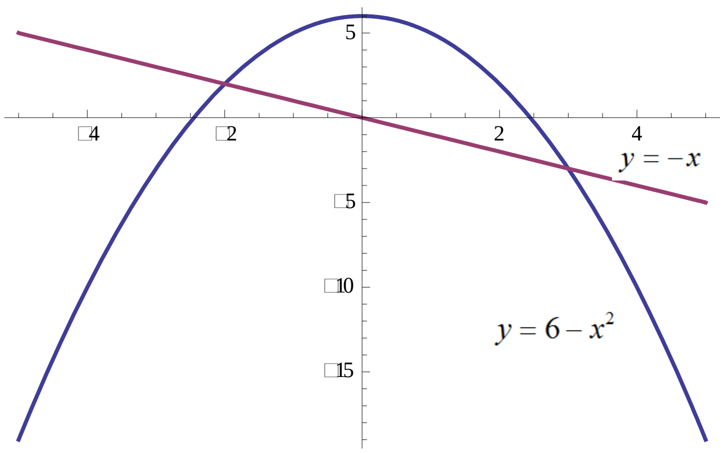
3) 2013/14S2 EG1740 Sem Exam– Q9a

The diagram below shows the curve $y = 6 - x^2$ and the line $y = -x$

(i) Find the points of intersection of the curve and the line.

(4 marks)

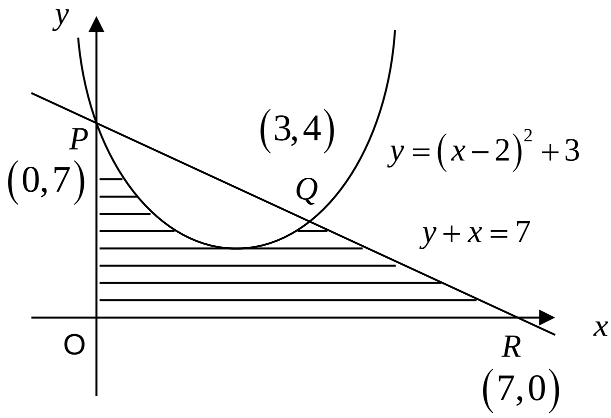
(ii) Hence, find the area bounded by the line $y = -x$ and the curve $y = 6 - x^2$.



(6 marks)

4) 2012/13S2 EG1740 Sem Exam– Q9b

The figure shows the curve $y=(x-2)^2+3$ and the line $y+x=7$. Both the curve and the line intersect at points P and Q. Given the points $P(0,7)$, $Q(3,4)$ and $R(7,0)$, find the area of the shaded region. (10 marks)



Answers

1a) $21.3 \quad units^2$

b) $\frac{1}{2}\left(6\sqrt{x} - x + 4\ln x)\right) + c$

c) $\frac{1}{-2}\sqrt{3-2x^4} + C$

2 24.901

3i) $\therefore x = -2, y = 2 \text{ and } x = 3, y = -3$

$or\ P = (-2, 2), Q(3, -3)$

ii) $20\frac{5}{6}unit^2$ or $20.83unit^2$

4 The area of the shaded region $= 20$ units^2