Chapter 3: Logic

Objective

The objective of this chapter is to

1. understand and use the different types of Logic and its terminology.
2. deduce the truth table for different logic combinations
3. apply the laws of the algebra of propositions to prove identities.

Content

3.1 Introduction

The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. In addition to its importance in understanding mathematical reasoning, logic has numerous applications to computer science. These rules are used in the design of computer circuits, the construction of computer programmes, the verification of the correctness of programme, etc.

In Logic, a **statement** or **proposition** is a declarative sentence that is either true or false, but not both. In this chapter, we will use the lowercase letters $p, q, r, ...$ to denote statements.

Example 1

Which of the following are statements?

(a) The sun rises in the east.
(b) $1+2=4$
(c) $5-x=3$
(d) Do you speak French?
(e) Do your homework now.
(f) The temperature on the Sun is 1000 degrees Celsius.
(g) It will rain tomorrow.

Solution
(a) is a statement, and it happens to be true.
(b) is also a statement, but it happens to be false.
(c) is not a statement, because it may be true or false depending on the value of *x*.
(d) is not a statement because it is a question.
(e) is not a statement because it is an order.
(f) is a statement and it happens to be true.
(g) is a statement and it could be true.

Statements such as (a), (b), (f) and (g) are examples of simple statements.

A compound statement is formed by combining two or more simple statements, called components.

Example 2

The statement "Beijing is the capital of China and $1+2=5$" is a compound statement. Its components are "Beijing is the capital of China" and "$1+2=5$".

3.2 Truth Tables

Truth tables are used to list the different possible outcomes. This is particularly important for software development in terms of planning as well as documentation for the purpose in software maintenance.

A truth table with a single proposition, for example p , is shown below:

p	**Outcome**
T	
F	

A truth table with a two propositions, for example p and q, is shown below:

p	q	**Outcome**
T	T	
T	F	
F	T	
F	F	

A truth table with a three propositions, for example p, q and r , is shown below

p	q	r	**Outcome**
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

In the preceding chapter, we discussed several methods of combining sets to construct new sets. In this section, we shall learn how to combine statements to make new statements. We shall also discuss the truth values of these new statements. The words **not**, **and**, **or**, **if . . . then**, and **if and only if**, are used to connect statements to form compound statements. Accordingly, they are called **connectives**.

3.2.1 Negation

If p is a statement, the negation of p is the statement **not** $\boldsymbol{p}$, denoted by $\sim \boldsymbol{p}$ (read "not p"). Thus $\sim \boldsymbol{p}$ is the statement "it is not the case that p".

Example 3

Write the negation of the statement "Today is Friday".

Solution

The negation is "It is not the case that today is Friday".

Example 4

Consider the following statement:

p : All cats can fly.

Then $\sim p$: It is not the case that all cats can fly.

From the definition of $\sim p$, it follows that if p is true, then $\sim p$ is false, and if p is false, then $\sim p$ is true. The truth values of $\sim p$ relative to p is given in the table below. Such a table, giving the truth values of a compound statement in terms of its component parts, is called a **Truth Table**. In this table, we represent True by 'T' and False by 'F'.

p	$\sim p$
T	F
F	T

3.2.2 Conjunction

Let p and q be any two statements. Any two statements p and q can be combined by the word 'and' to form a compound statement called the **conjunction of the original statements**, denoted by $\boldsymbol{p \wedge q}$, and read "p **and** q".

Example 5
Let p and q be the following statements:

p : Today is Sunday
q : All cats can fly

Then the conjunction of p and q is $p \wedge q$:
A student is allowed to take the semester exams if $p \wedge q$ is True.

When is the statement $p \wedge q$ true and when is the statement $p \wedge q$ false?
If p is true and q is true, then $p \wedge q$ is True.
If p is true and q is false, then $p \wedge q$ is False.
If p is false and q is true, then $p \wedge q$ is False.
If p is false and q is false, then $p \wedge q$ is False.

The truth table for $p \wedge q$ is given as follows:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Hence the outcome is True only if both p and q is True. Otherwise it is False.

3.2.3 Disjunction

Let p and q be any two statements. Any two statements p and q can be combined by the word 'or' to form a compound statement called the **disjunction of the original statements**, denoted by $p \vee q$, and read "p **or** q".

Example 6

Consider the following statements:

p : I passed my Mathematics exam.
q : I passed my Chinese exam.

Then the disjunction of p and q is $p \vee q$:
I passed my Mathematics exam or I passed my Chinese exam.

Exclusive/Inclusive disjunction

In the English language, the word "or" can be used in two different ways. Sometimes it means "p or q or both" and at other times it means "p or q but not both".

For example, if we say "I went to Penang by plane or I went to Penang by ship", this means that I either went to Penang by plane or by ship, but not both. This type of disjunction is called the **exclusive disjunction** or the **exclusive or.**

In example 6, we have the disjunction "I passed my Mathematics exam or I passed my Chinese exam". This means that I either passed my Mathematics exam, or I passed my Chinese exam, or both. This type of disjunction is called the **inclusive disjunction** or **the inclusive or.**

Thus in the English language, the word "or" can sometime be taken as the exclusive disjunction or the inclusive disjunction, depending on the context of the sentence.

Note: However, in Mathematical Logic, the disjunction $p \vee q$ **will always mean** the **inclusive disjunction** of p and q, ie. ***it will always mean "p or q or both***".

When is the statement $p \vee q$ true and when is the statement $p \vee q$ false?
If p is true and q is true, then $p \vee q$ is true.
If p is true and q is false, then $p \vee q$ is true.
If p is false and q is true, then $p \vee q$ is true.
If p is false and q is false, then $p \vee q$ is false.
If either p or q is true, $p \vee q$ is true. Otherwise, it is false

The truth table for $p \vee q$ is given as follows:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exercise 3.1

(1) Consider the following statements:

p : A cat is an animal.
q : All dogs can fly.

Determine the truth value of each of the following compound statements:
(a) $p \wedge q$ (b) $p \vee q$ (c) $\sim p \wedge q$ (d) $\sim p \vee \sim q$ (e) $\sim \left(p \wedge q \right)$

(2) Consider the following statements:

p : NYP graduates do well in the industry.
q : NYP students have intention to further their studies after they graduate.

Write the compound sentences corresponding to each of the following:
(a) $p \vee q$

(b) $p \wedge q$

(c) $p \wedge \sim q$

(d) $\sim p \vee \sim q$

(3) Construct the truth table for the proposition $\sim \left(p \wedge \sim q \right)$.

p	q	$\sim q$	$p \wedge \sim q$	$\sim \left(p \wedge \sim q \right)$

(4) Construct a truth table for $(p \vee q) \wedge \left[(p \vee r) \wedge \sim r \right]$.

p	q	r	$p \vee r$	$\sim r$	$(p \vee r) \wedge \sim r$	$p \vee q$	$(p \vee q) \wedge \left[(p \vee r) \wedge \sim r \right]$

3.2.4 Tautologies and Contradictions

A proposition that is always true, no matter what the truth values of the variables that occur in it, is called a **tautology**. In the truth table for a tautology, the last column of the table must **all be true** or "1".

A proposition that is always false, no matter what the truth values of the variables that occur in it, is called a **contradiction**. In the truth table for a contradiction, the last column of the table **must all be false** or "0".

Exercise 3.2

(1) Use the truth table to show that $p \vee \sim p$ is a tautology and that $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \vee \sim p$

p	$\sim p$	$p \wedge \sim p$

(2) Use the truth table to show that $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

p	q	$p \vee q$	$\sim (p \vee q)$	$p \wedge q$	$(p \wedge q) \wedge \sim (p \vee q)$

3.3 Conditional and Biconditional Statements

3.3.1 Conditional Statements

Many statements in Mathematics are of the form "**if p then q**". Such a statement is called a **Conditional** or an **Implication statement** and is denoted by $p \to q$. In the conditional statement $p \to q$, the statement p is called the **hypothesis** or **antecedent** or the **sufficient condition**, and the statement q is called the **conclusion** or **consequent** or the **necessary condition**.

Example 7

Consider the following statements:

p : Rafi has a driving license.
q : Rafi is above 17 years old .

Then we have $p \to q$:
If Rafi has a driving license, then Rafi is above 17.

The implication $p \to q$ can also be read in any of the following ways:

1. p implies q
2. If p, then q
3. q if p
4. p only if q
5. p is sufficient for q
6. q is necessary for p

Example 8

Consider the statements p and q.

p : $x = 10$.
q : $x^2 = 100$.

Then the implication $p \to q$ can be read in any of the following ways:

1. $x = 10$ implies that $x^2 = 100$.
2. If $x = 10$, then $x^2 = 100$.
3. $x^2 = 100$ if $x = 10$.
4. $x = 10$ only if $x^2 = 100$.
5. $x = 10$ is sufficient for $x^2 = 100$.
6. $x^2 = 100$ is necessary for $x = 10$.

Let us now consider the truth values of the conditional statement $p \to q$.

Since the statements p and q can be true or false, there are four cases to be discussed, as before. Let us look at example 7 to discuss the truth values of $p \to q$.

p : Rafi has a driving license.
q : Rafi is above 17 years old .

Then we have $p \to q$:
If Rafi has a driving license, then Rafi is above 17 years of age.

We can regard an implication as a conditional – the implication is false if and only if the statement p is violated.

If Rafi has a driving license, he is above 17 years of age.
If Rafi does not have a driving license, then he may or may not be above 17 years of age.

What if Rafi does not have a driving license **(p false)?** Then he may not be 17 years of age (q true or false). In either case, the conditional promise *p* has not been tested and hence was not violated. Consequently, the implication $p \to q$ is not false, and is therefore true. In other words, if p is false, the implication $p \to q$ **is true regardless of the truth values of q**.

In summary then,
If p is true and q is true, then $p \to q$ is true.
If p is true and q is false, then $p \to q$ is false.
If p is false and q is true, then $p \to q$ is true.
If p is false and q is false, then $p \to q$ is true.

The truth table for $p \to q$ is given below:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note:
In mathematical logic, the use of the implication $p \to q$ **differs from** the English language. In English language, we may say “If it rains, the football match will be postponed”. This indicates a cause and effect relationship between the hypothesis and conclusion.

However, in logic, the implication $p \to q$ may connect any two statements p and q, **regardless of whether there is any relationship between them**, as shown in the following example.

Example 9

Consider the following statements:

p : $1+1=5$
q : All dogs can fly

Then we have $p \rightarrow q$:
If $1+1=5$, then all dogs can fly.

Since the statement "$1+1=5$" is false, the implication "If $1+1=5$, then all dogs can fly" is true, according to the truth table above.

Note:
The statements p and q have no relationship to each other at all. In Mathematical Logic, we are not concerned with the meaning or structure of the statements but only with their truth values.

3.3.2 Biconditional Statements

Another common statement in Mathematics is of the form "$\boldsymbol{p}$ **if and only if** $\boldsymbol{q}$". Such statements are called **Bi-conditional or Equivalent statement** and are denoted by $p \leftrightarrow q$.

Example 10

Consider the following statements:

p: x is an odd integer.
q: The last digit of x is odd.

Then we have $p \leftrightarrow q$:
x is an odd integer if and only if the last digit of x is odd.

The bi-conditional statement $p \leftrightarrow q$ can also be read in any of the following ways:

1. p if and only if q
2. If p then q, and if q, then p
3. p implies q and q implies p
4. p is necessary and sufficient for q

Example 11

Consider the same example above.

p : x is an odd integer.
q : The last digit of x is odd.

Then the bi-conditional statement $p \leftrightarrow q$ can be read in any of the following ways:

1. x is an odd integer if and only if the last digit of x is odd.
2. If x is an odd integer, then the last digit of x is odd, and if the last digit of x is odd, then x is an odd integer.
3. x is an odd integer implies that the last digit of x is odd, and the last digit of x is odd implies that x is an odd integer.
4. x is an odd integer is necessary and sufficient for the last digit of x to be odd.

If p and q have the same truth value, then $p \leftrightarrow q$ is true. If p and q have opposite truth values, then $p \leftrightarrow q$ is false.

In summary,

If p is true and q is true, then $p \leftrightarrow q$ is true.
If p is true and q is false, then $p \leftrightarrow q$ is false.
If p is false and q is true, then $p \leftrightarrow q$ is false.
If p is false and q is false, then $p \leftrightarrow q$ is true.

The truth table for $p \leftrightarrow q$ is given below:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

3.4 Logical Equivalence: Algebra of Propositions

Two propositions are said to be **logically equivalent**, or simply equivalent or equal, denoted by $\equiv$ or $\Leftrightarrow$, if they have the same truth table.

Example 12

Consider the truth table of $\sim(p \wedge q)$ and $\sim p \vee \sim q$.

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Since both $\sim(p \wedge q)$ and $\sim p \vee \sim q$ have the same truth table, then these two propositions are said to be logically equivalent.

Exercise 3.3

(1) Is $[(p \to q) \wedge (q \to r)] \to (p \to r)$ a tautology or a contradiction?

(2) Use the truth table to show that the propositions $p \to q$ and $\sim p \vee q$ are logically equivalent.

p	q	$\sim p$	$\sim p \vee q$	$p \to q$

3.5 Logical Implication

A proposition P is said to logically imply a proposition Q, written $P \Rightarrow Q$, if Q is True whenever P is True.

Example 13

Use the truth table to show that p logically implies $p \vee q$ *, i.e.* $p \Rightarrow (p \vee q)$.

P		Q
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

As the truth table above shows whenever proposition P is True, proposition Q is True.

Thus proposition P implies proposition Q. $P \Rightarrow Q$.

Exercise 3.5

Use the truth table to show that $p \wedge q$ logically implies $p \leftrightarrow q$. i.e. $(p \wedge q) \Rightarrow (p \leftrightarrow q)$.

3.6 Laws of the Algebra of Propositions

The following are the laws of the algebra of propositions.

1	Idempotent Laws	$p \vee p \equiv p$	$p \wedge p \equiv p$
2	Associative Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
3	Commutative Laws	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
4	Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
5	Identity Laws	$p \vee F \equiv p$ $p \vee T \equiv T$	$p \wedge T \equiv p$ $p \wedge F \equiv F$
6	Complement Laws	$p \vee \sim p \equiv T$ $\sim T \equiv F$	$p \wedge \sim p \equiv F$ $\sim F \equiv T$
7	Involution Law	$\sim(\sim p) \equiv p$	
8	De Morgan's Laws	$\sim(p \vee q) \equiv \sim p \wedge \sim q$	$\sim(p \wedge q) \equiv \sim p \vee \sim q$

Example 14

(1) $p \wedge (p \vee \sim p)$

$\equiv p \wedge T$ Complement Laws

$\equiv p$ Identity Laws

(2) $\sim (p \vee q) \vee (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ De Morgan's Law

$\equiv \sim p \wedge (\sim q \vee q)$ Distributive Law

$\equiv \sim p \wedge T$ Complement Law

$\equiv \sim p$ Identity Law

Exercise 3.6

(1) Simplify each proposition by using laws of algebra of propositions.

(a) $p \vee (p \wedge q)$

(b) $\sim (p \vee q) \vee (\sim p \wedge q)$

(2) Use the laws of algebra of propositions to show that $\sim (p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent.

Tutorial 3

1. Which of the following are statements?

   (a) Sally is the class president.
   (b) Have you finished your homework?
   (c) $x + y > 10$
   (d) $2 + 8 = 7$
   (e) Please be quiet.

2. Let p be the proposition "The system is ready" and q "The light is on".

   (1) If the system is not ready, then the light is not on.
   (2) If the light is on, then the system is not ready.
   (3) If the light is not on, then the system is not ready.
   (4) If the system is ready, then the light is on.
   (5) Either the system is not ready or the light is not on.
   (6) Either the system is not ready or the light is on.
   (7) Either the light is not on or the system is ready.
   (8) If the system is ready, then the light is not on.
   (9) If the light is on, then the system is ready.

   (a) Write each of the above propositions in symbolic notations.

   (b) The above propositions can be arranged into three groups so that each member of a group is logically equivalent to the other two. Hence or otherwise, state and explain your answer by listing the three groups. (Hint: Use truth table to prove)

3. Let p and q be the propositions
   p : Ah Beng has long hair.
   q : Ah Beng is punished by the discipline master.

   Write the following propositions using p and q and logical connectives.

   (a) Ah Beng does not have long hair.
   (b) Ah Beng has long hair, but Ah Beng is not punished by the discipline master.
   (c) Ah Beng will be punished by the discipline master if he has long hair.
   (d) If Ah Beng does not have long hair, then he will not be punished by the discipline master.

4. Construct truth tables for the following propositions;
   (a) $\sim\left[(p \wedge r) \vee q\right]$ (b) $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$

5. By using laws of algebra of propositions, show that
   (a) $\sim p \vee (\sim q \vee p) \equiv 1$ (b) $\sim (p \wedge \sim q) \wedge (\sim p \vee \sim q) \equiv \sim p$

6. By using laws of algebra of propositions, simplify $\sim\left(p \wedge\left(\sim p \vee \sim q\right)\right)$.

7*. Rewrite the following statements without using the conditional.

(a) If it is hot, he wears a hat.
(b) If productivity decreases, then wages fall.

8*. The following exercises involve the logical operators NAND and NOR. The proposition p NAND q is true when either p or q, or both, are false; and it is false when both p and q are true. The proposition p NOR q is true when both p and q are false, and it is false otherwise. The propositions p NAND q and p NOR q are denoted by $p \mid q$ and $p \downarrow q$ respectively.

(a) Construct a truth table for the logical operator NAND
(b) Show that $p \mid q$ is logically equivalent to $\sim(p \wedge q)$
(c) Construct a truth table for the logical operator NOR.
(d) Show that $p \downarrow q$ is logically equivalent to $\sim(p \vee q)$

9*. Assuming that p and r are false and that q is true, find the truth value of the proposition $\left(p \rightarrow q\right) \wedge \left(q \rightarrow r\right)$.

Answers

1. (a) Y (b) N (c) N (d) Y (e) N

2. (a) Group 1: 1, 7 and 9, Group 2: 2, 5 and 8, and Group 3: 3, 4 and 6

3. (a) $\sim p$ (b) $p \wedge \sim q$ (c) $p \rightarrow q$ (d) $\sim p \rightarrow \sim q$

6. $\sim p \vee q$

7*. (a) It is not hot or he wears a hat.
(b) Productivity does not decrease or wages fall.

9*. 0

Questions from past year examination papers

1) 2013/14S1 IT1101/1501/1561/1751/1621 Sem Exam– Q5

(a) Copy and complete the following truth table to determine whether the compound statement $\sim p \vee (r \rightarrow q)$ is a tautology, contradiction or neither.

(8 marks)

p	q	r	$\sim p$	$r \rightarrow q$	$\sim p \vee (r \rightarrow q)$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

(b) By using the Laws of Algebra of Propositions, show that $\sim (p \vee \sim q) \vee p \equiv q \vee p$.

(7 marks)

2) 2013/14S2 EG1740Sem Exam– Q3

(a) Construct the truth table for the proposition $p \wedge \sim (p \leftrightarrow q)$. (4 marks)

(b) Use the Laws of Algebra of Sets to show that $(A \cap B) \cap \left(A^C \cup B\right)^C = \varnothing$.

(4 marks)

3) 2013/14S2 EG1740Sem Exam– Q6a

Using the Laws of Algebra of Propositions, determine whether the following proposition is a tautology, a contradiction or neither.

(i) $(p \wedge \sim p) \vee p$. (2 marks)

(ii) $(p \vee q) \vee \sim (\sim p \wedge q)$. (5 marks)

Answers

1a)

p	q	r	$\sim p$	$r \to q$	$\sim p \vee (r \to q)$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

$\sim p \vee (r \to q)$ is neither a tautology nor contradiction.

b) $q \vee p$

2a)

(p)	(q)	$(p \leftrightarrow q)$	$\sim (p \leftrightarrow q)$	$p \wedge \sim (p \leftrightarrow q)$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	F
F	F	T	F	F

b) $\varnothing$

3i) This proposition is neither a tautology nor contradiction.

ii) This proposition is a tautology