Chapter 5: Relation and Function

Objective

The objective of this chapter is to

1. understand the concepts of relations and functions.
2. identify the different types of relations.
3. find the inverse of a relation/function.
4. find the composition of relations/functions.
5. show that a function is injective or surjective.

Content

5.1 Introduction

In software development, functions and relations are commonly found in relational databases. We can say that relationships between elements of sets occur in many contexts. Hence in this chapter, a detailed study is done on these concepts.

5.1.1 Ordered pair

An ordered pair consists of two elements, of which one is designated as the first element and the other as the second element. Such an ordered pair is written as (a,b) where a is the first element and b is the second element. Two ordered pairs (a,b) and (c,d) are equal if and only if $a=c$ and $b=d$. In an ordered pair, the order is important. Thus (a,b) and (b,a) are not the same unless the value of a is the same as the value of b.

Consider two arbitrary sets A and B. The set of all ordered pairs (a,b) where $a\in A$ and $b\in B$ is called the cartesian product of A and B, denoted by $A \times B$. By definition,

$$A \times B = \{(a,b) : a\in A \text{ and } b\in B\}$$

__Example 1__

Let $A=\{1,2,3\}$ and $B=\{a,b\}$. Then

$$A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$$

Binary Relation

Let A and B be sets. A binary relation R from A to B is a subset of $A \times B$.

__Example 2__

Let $A=\{0,1,2\}$ and $B=\{a,b\}$. Then

$R=\{(0,a),(0,b),(1,a),(2,b)\}$ is a relation from A to B.

A **binary** relation from a set A to a set B is a set of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

We use aRb to denote that $(a,b)\in R$ and $a\not{R}b$ to denote that $(a,b)\notin R$. In the Example 2 above, $(0,a)\in R$ and $(3,a)\notin R$, $(a,2)\notin R$.

A relation from a set A to the same set A is called a relation on A.

__Exercise 5.1__

(1) Let A be the set $\{1,2,3,4\}$. Which ordered pairs are in the relation $R=\{(a,b): a>b\}$?

(2) Consider the following relations on the set of integers:

$R1=\{(a,b): a<b\}$,

$R2=\{(a,b): a>b\}$,

$R3=\{(a,b): a=b \text{ or } a=-b\}$,

$R4=\{(a,b): a=b\}$,

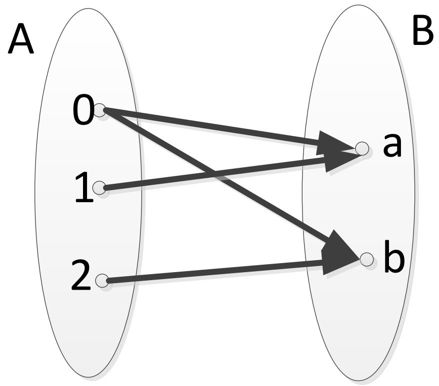
$R5=\{(a,b): a=b+1\}$ and

$R6=\{(a,b): a+b\leq 3\}$.

Associate the following ordered pairs $(1,1),(1,2),(2,1),(1,-1)$ and $(2,2)$ to the respective relations. The ordered pairs may be in one or more relations.

5.1.2 Arrow Diagram

Let $A = \{0,1,2\}$ and $B = \{a,b\}$. Then $\{(0,a),(0,b),(1,a),(2,b)\}$ is a relation from A to B since it is the subset of $A \times B$. Relations can be represented in arrow diagrams as shown below, using arrows to represent ordered pairs.

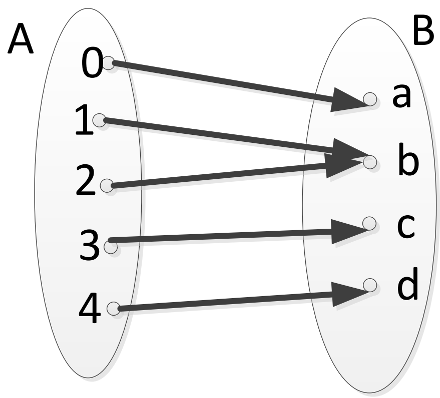


Exercise 5.2

(1) List all the ordered pairs in the relation $R = \{(a,b) : a > b\}$ on the set $\{1,2,3,4\}$. Display this relation diagrammatically

2) The relation R is displayed graphically below:

Write down the relation in a form of ordered pairs.



5.1.3 Matrix representation of relation

Let $A=\{0,1,2\}$ and $B=\{a,b\}$. Then $\{(0,a),(0,b),(1,a),(2,b)\}$ is a relation from A to B since it is the subset of A x B. Relations can be represented in a matrix as shown below.

$(0,a)$ $(0,b)$

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

$(1,a)$

$(2,b)$

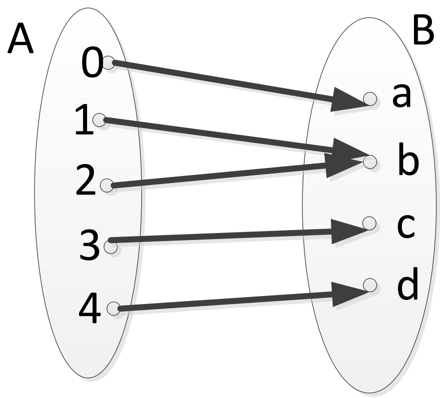
The rows denote set A and the columns denotes set B. If a relation exist between the row and columns, it is represented by 1 in the matrix.

Exercise 5.3

(1) List all the ordered pairs in the relation $R=\{(a,b): a>b\}$ on the set $\{1,2,3,4\}$. Display this relation in a form of a matrix.

2) The relation R is displayed graphically below:

Write down the relation in a form of matrix.



In summary, a relation R can be represented by:

(a) Ordered pairs

(b) Arrow diagram

(c) Matrix

5.2 Properties of Relation

5.2.1 Reflexive relation

In some relations, **an element is always related to itself**. For example, let $\boldsymbol{R}$ be the relation on the set of all people consisting of pairs (x, y) where x and y have the same father and same mother. Then (x, x) belongs to the relation for every person x.

Definition

> A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called **reflexive** if $(a,a) \in R$ for **every element** $a \in A$.

Example 3

Let R, S and T be the relation on $A = \{1,2,3\}$ defined by

$$R = \{(1,1),(2,2),(3,3)\}$$
$$S = \{(1,2),(2,1),(3,3)\}$$
$$T = \{(1,2),(2,3),(1,3)\}$$

Determine which of these relations are reflexive.

Solution

S is not reflexive since $1 \in A$, but $(1,1) \notin S$

T is not reflexive since $(a,a) \notin T$ for all $a = 1,2$ and 3

R is reflexive since $(a,a) \in R$ for $a = 1,2$ and 3

Exercise 5.4

(1) Consider the relations $R1, R2, R3, R4, R5$ and $R6$ on the set $\{1,2,3,4\}$:

$$R1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$
$$R2 = \{(1,1),(1,2),(2,1)\}$$
$$R3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$$
$$R4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$$
$$R5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$$
$$R6 = \{(3,4)\}$$

Which of these relations are reflexive?

5.2.2 Symmetric relation

In some relations, an element is related to a second element if and only if the second element is also related to the first element. For example, if a relation is defined as “is the brother of”, then (Sam, Mathew) and (Mathew, Sam) are both in the relation. We say that this relation as the symmetric property.

Definition

> A relation on a set A is called **symmetric** if $(b,a) \in R$ whenever $(a,b) \in R$, for $a,b \in A$.

Important Notes:

(1) The ordered pair $(\boldsymbol{a,a})$ is said to be **symmetric to itself**.

(2) To show that a relation is not symmetric, we need only to single out one case in which (a,b) **belongs** to $\boldsymbol{R}$, but (b,a) **does not belong** to $\boldsymbol{R}$.

<u>Example 4</u>

$R_1 = \{(1,1),(1,2),(2,1)\}$ is a symmetric relation while $R_2 = \{(1,1),(2,1)\}$ is not a symmetric relation

<u>Exercise 5.5</u>

(1) Consider the relations $R1, R2, R3, R4, R5$ and $R6$ on the set $\{1,2,3,4\}$:

$$R1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$
$$R2 = \{(1,1),(1,2),(2,1)\}$$
$$R3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$$
$$R4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$$
$$R5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$$
$$R6 = \{(3,4)\}$$

Which of the relations are symmetric?

(2) Let $A = \{0,1,2,3\}$ and the relations R, S and T on A is as follows:

$$R = \{(0,0),(0,1),(0,3),(1,0),(1,1),(2,2),(3,0),(3,3)\}$$
$$S = \{(0,0),(0,2),(0,3),(2,3)\}$$
$$T = \{(0,1),(2,3)\}$$

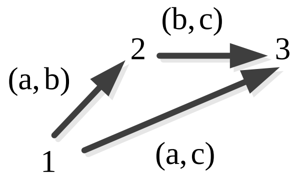
Which of these relations are symmetric?

5.2.3 Transitive Relation

Let R be the relation consisting of all pairs (x, y) of students at Nanyang Polytechnic where x has obtained more distinctions than y in the final examination. Suppose that x is related to y and y is related to z. This means that x has more distinctions than y and y has more distinctions than z. We can conclude that x has more distinctions than z, so that x is related to z. We have shown that R has the transitive property, which is defined as follow.

Definition

A relation $\boldsymbol{R}$ on a set A is called **transitive** if whenever $(a,b)\in R$ and $(b,c)\in R$, then $(a,c)\in R$, for $a,b,c\in A$.



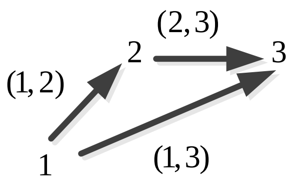
So when is a relation R in a set A **not transitive?**

Answer

R is not transitive if there exist elements a, b and $c\in A$, but not necessarily distinct, such that $(a,b)\in R$, $(b,c)\in R$ but $(a,c)\notin R$.

<u>Example 5</u>

$R_1=\{(1,1),(1,2),(2,3),(1,3)\}$ is a transitive relation while
$R_2=\{(1,1),(1,2),(2,3)\}$ is not a transitive relation because $(1,3)$ is not in R_2.



Exercise 5.6

(1) Consider the relations $R1, R2, R3, R4, R5$ and $R6$ on the set $\{1,2,3,4\}$:

$$R1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$
$$R2 = \{(1,1),(1,2),(2,1)\}$$
$$R3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$$
$$R4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$$
$$R5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$$
$$R6 = \{(3,4)\}$$

Which of the relations are transitive?

(2) Let $A = \{0,1,2,3\}$ and the relations R, S and T on A is as follows:

$$R = \{(0,0),(0,1),(0,3),(1,0),(1,1),(2,2),(3,0),(3,3)\}$$
$$S = \{(0,0),(0,2),(0,3),(2,3)\}$$
$$T = \{(0,1),(2,3)\}$$

Which are the transitive relations?

(3) The relations $R1, R2, R3, R4, R5$ and $R6$ on the set $\mathrm{A} = \{1,2,3,4\}$ is given as:

$$R1 = \{(a,b) : a \le b\}$$
$$R2 = \{(a,b) : a > b\}$$
$$R3 = \{(a,b) : a = b \text{ or } a = -b\}$$
$$R4 = \{(a,b) : a = b\}$$
$R5 = \{(a,b) : a = b + 1\}$ and
$$R6 = \{(a,b) : a + b \le 3\}$$

Which of the relations are transitive?

5.2.4 Equivalence Relation

Definition

> A relation on a set A is called an **equivalence relation** if the relation is reflexive, symmetric, and transitive.

<u>Example 6</u>

If $R=\{(1,\ 1),(2,\ 2),(1,\ 3),(3,\ 1),(3,\ 3)\}$ and is a relation on $A=\{1,2,3\}$.
R is :

reflexive as $(1,\ 1),(2,\ 2),(3,\ 3)\in R$

symmetric as $(1,\ 3),(3,\ 1)\in R$

transitive as $(1,\ 3),(3,\ 1),(1,\ 1)\in R$

Thus R is an equivalence relation

<u>Exercise 5.7</u>

Let $R=\{(1,1),(1,3),(3,1),(3,3)\}$. Is R an equivalence relation on $A=\{1,2,3\}$ and on $B=\{1,3\}$?

5.2.5 Inverse relation

Let R be any relation from A to B. The inverse of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which when reversed belong to R :

$$R^{-1} = \{(b,a) : (a,b) \in R\}$$

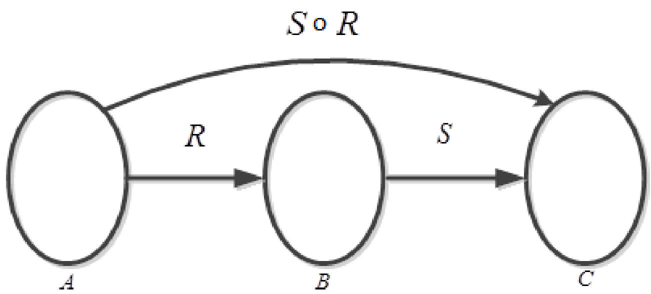
Exercise 5.8

Let R be a relation from $\{1,2,3\}$ to $\{a,b,c,d\}$ defined by $R = \{(1,a),(1,b),(2,c),(3,d)\}$. What is the inverse relation R^{-1}?

5.3 Composition of Relations

Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite relation of R and S, denoted by $(S \circ R)$. The relation $(S \circ R)$ consisting of ordered pairs (a,c) where $a \in A,\ c \in C$ and for which there exists an element $b \in B$ such that

$$(a,b) \in R \text{ and } (b,c) \in S$$



Example 7

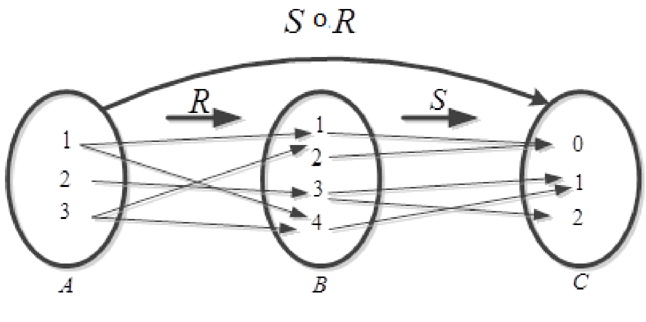
Let R be a relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ defined by

$$R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$$

Let S be a relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ defined by

$$S = \{(1,0),(2,0),(3,1),(3,2),(4,1)\}$$

The composite relation $S \circ R$ is as shown:



Taking the starting values from set A and ending values from set C, we get

$$S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$$

5.4 Introduction to Function

A function *f* defined on a set X to a set Y, is a relationship between the elements of X and elements of Y such that each member in X is **uniquely related** to each member in Y.

X is called the **domain** of *f* and Y is called the **co-domain** of f.
For some cases the co-domain is also the **range** (see below).

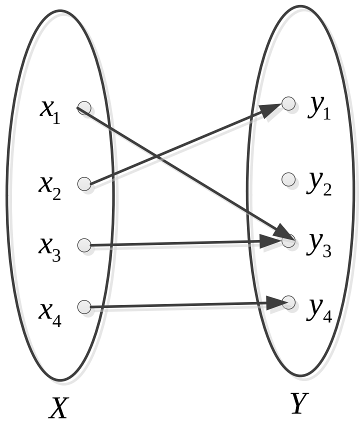
So one can think of $x \in X$ as an input and $y \in Y$ the related output. One can also say that $\boldsymbol{f}$ **sends** $\boldsymbol{x}$ **to** $\boldsymbol{y}$. Some authors say $\boldsymbol{f}$ **maps** $\boldsymbol{x}$ **to** $\boldsymbol{y}$.

The unique element y to which f sends x is denoted by $f(x)$ and is called

1. f of x or
2. the value of f at x or
3. the image of x under f.

The **set of all values** of f taken together **is called the range** of f or the image of X under f.

Example 8



Domain=$\{x_1, x_2, x_3, x_4\}$
Co-domain= $\{y_1, y_2, y_3, y_4\}$
Range(Image of x)=$\{y_1, y_3, y_4\}$

Note that a function has the following properties

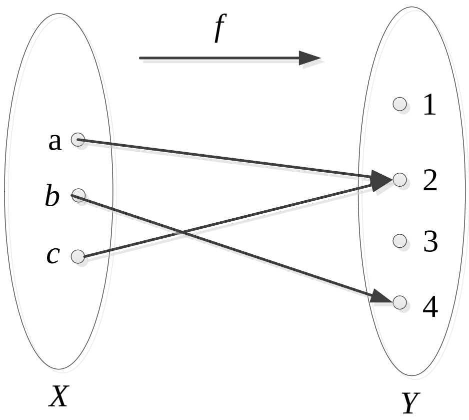
(1) **Every element** of X has **an arrow pointing** to an element in Y.
(2) **No element** in X has **2 arrows that point to 2 different** elements of Y.

Example 9

Let f be the function from Z^+ to Z^+ that assigns the square of an integer to this integer, ie $f(x) = x^2$.
The domain and co-domain of f is the set $Z^+ = \{1, 2, 3, 4.....\}$.
The range of f is the set $\{1, 4, 9, 16.......\}$

Exercise 5.9

1) Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. A function f from X to Y is defined below by the arrow diagrams.



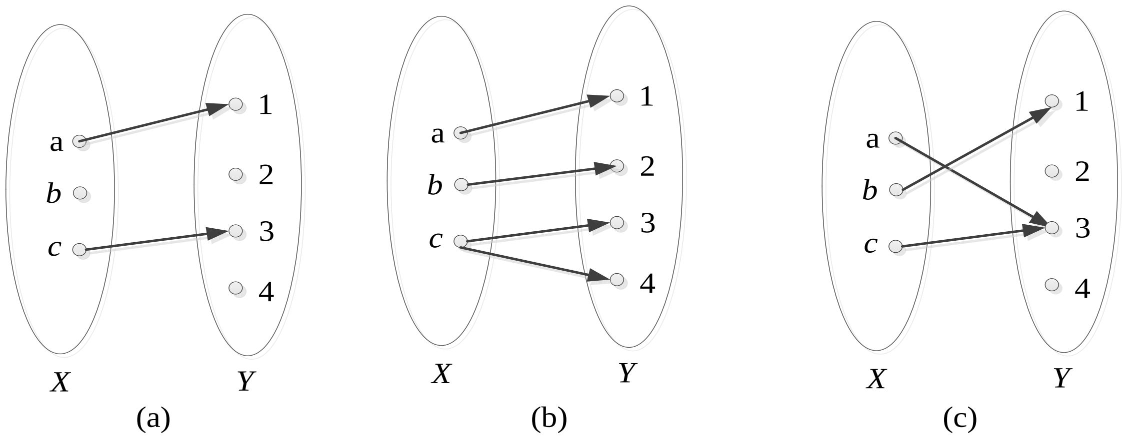
a) Write the domain and co-domain of f.

b) Find $f(a)$, $f(b)$ and $f(c)$.

c) What is the range of f ?

d) Represent f as a set of ordered pairs.

2) Which of the following defines a function from $X = \{a, b, c\}$ to $Y = \{1, 2, 3, 4\}$?



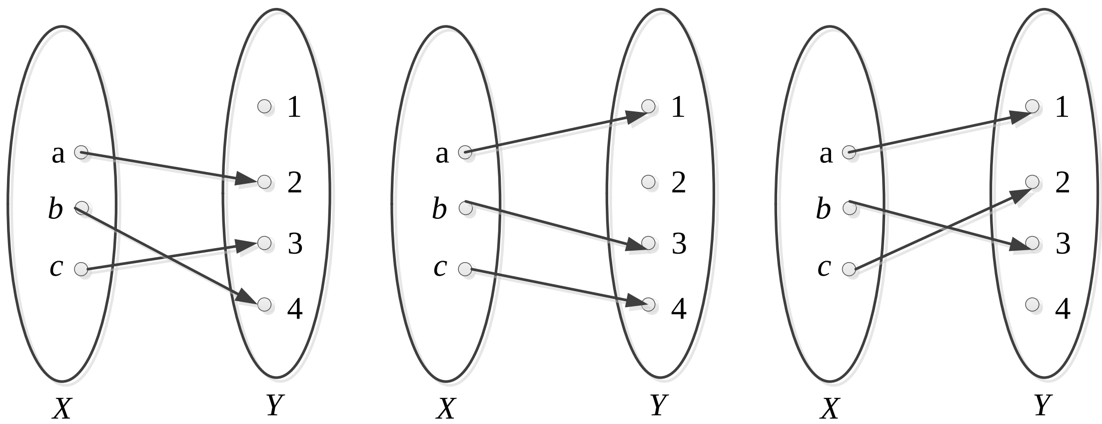
5.5 Injective and Surjective

5.5.1 Injective (One to One)

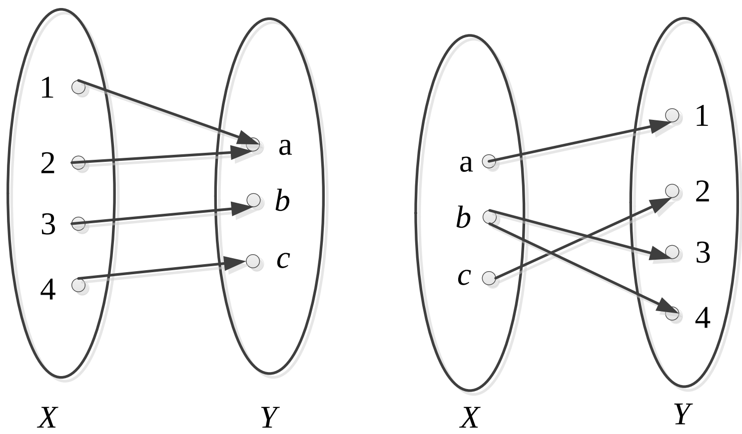
Let f be a function from a set X to a set Y. f is one-to-one (or **Injective**) if and only if, for all elements x_1 and x_2 in X ,if $f(x_1) = f(x_2)$, implies $x_1 = x_2$

Example 10

Below are some examples of One to one functions

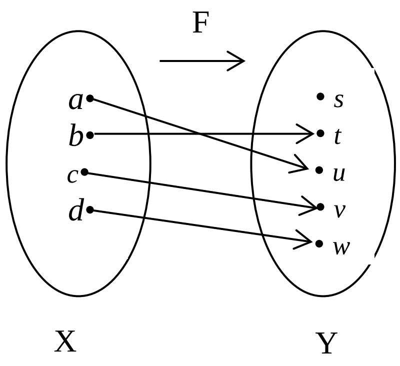


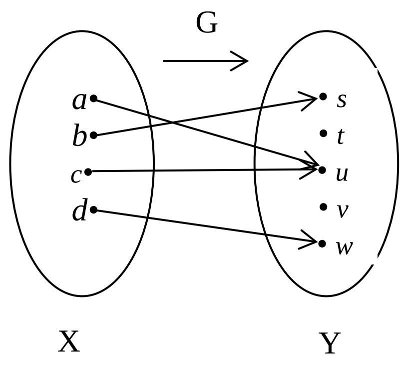
Below are examples which are **NOT** one to one functions



Exercise 5.10

(1) Which of the arrow diagrams define one-to-one function?





(2) Determine whether the function f from $\{a,b,c,d\}$ to $\{1,2,3,4,5\}$ defined by $f(a)=4$, $f(b)=5$, $f(c)=1$ and $f(d)=3$ is one-to-one.

(3) Determine whether the function $f(x)=x^2$ from the set of integers to the set of integers is one-to-one.

5.5.2 Surjective

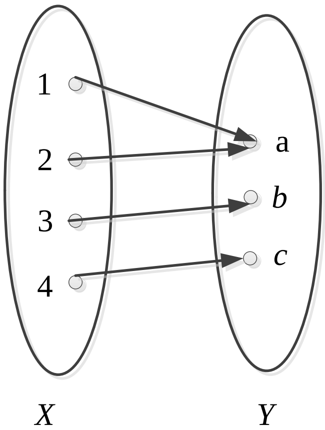
Let f be a function from a set X to a set Y. f is onto (or **surjective**) if, and only if, given any element y in Y it is possible to find an element x in X with the property that $y = f(x)$. Symbolically:

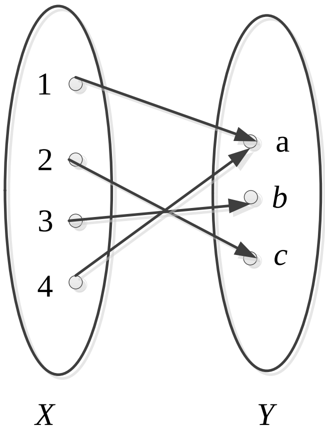
$$f: X \to Y \text{ is onto} \Leftrightarrow \forall y \in Y, \ \exists x \in X \text{ such that } y = f(x)$$

Note that: $\forall$ means "for all"; $\in$ means "belong to"; $\exists$ means " there exists".

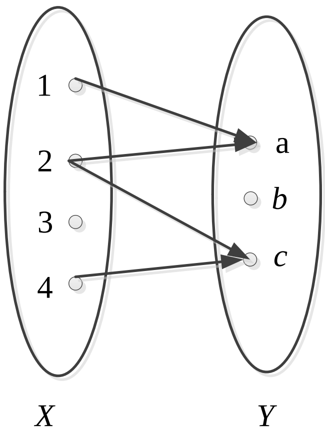
<u>Example 11</u>

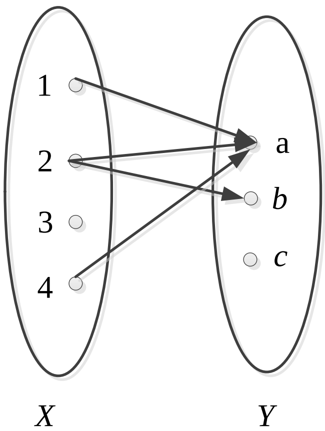
Below are some examples of surjective functions





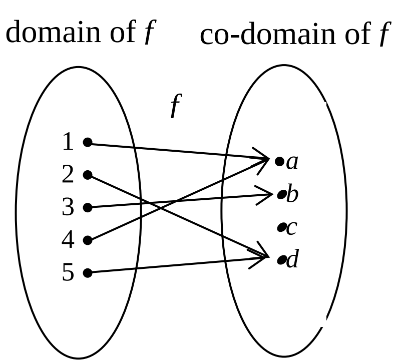
Below are some examples which are **NOT** surjective functions

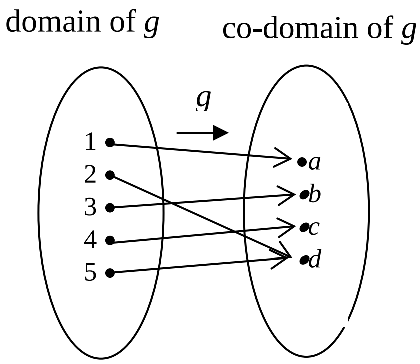




Exercise 5.11

(1) Which of the following function is an onto function?





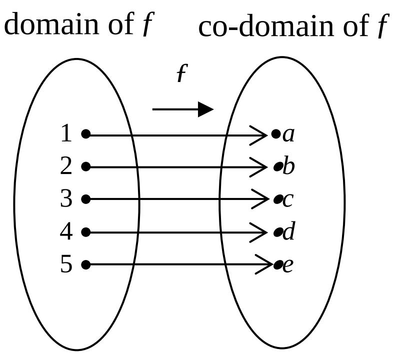
(2) Let f be the function from $\{a,b,c,d\}$ to $\{1,2,3\}$ defined by

$f(a)=3$, $f(b)=2$, $f(c)=1$, $f(d)=3$. Is f an onto function?

(3) Is the function $f(x)=x^2$ from the set of integers to the set of integers onto?

5.5.3 One-to-One Correspondence (Bijection)

Consider a function $f : X \to Y$ that is **both one-to-one and onto**. Given any element x in X, there is a unique corresponding element $y = f(x)$ in Y (f is a function). Also given any element *y* in Y, there is an element x in X such that *f* (*x*) = *y* (onto) and there is only one such x (one-to-one). Thus, a function that is **one-to-one and onto sets up a pairing** between the elements of X and the elements of Y that pairs each element of X with exactly one element of Y and each element of Y with exactly one element of X. Such a pairing is called a **One-to-One correspondence or Bijection** and is illustrated in the arrow diagram below:



An arrow diagram for a one-to-one correspondence

Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function.

$f : X \to Y$ that is both one-to-one and onto.

Exercise 5.12

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 1$, $f(c) = 1$ and $f(d) = 3$. Is f a bijection?

5.6 Invertible Functions

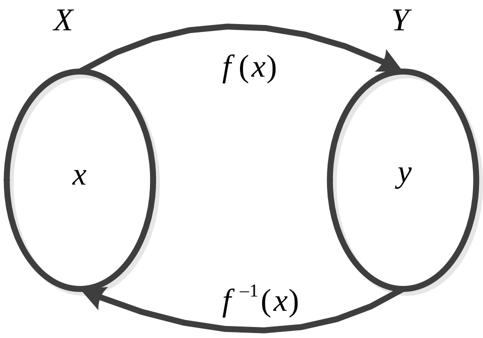
If f is a one-to-one correspondence from a set X to a set Y, then there exist a function from Y to X that "**undo**" the action of f; that is, it **sends each element of** Y **back to the element of** X that it came from. This function is called **inverse function**.

Definition

Let f be a one-to-one correspondence function from the set X to the set Y. The inverse function of f is the function that assigns to an element y belonging to Y the unique element x in X such $y = f(x)$. The inverse function of *f* is denoted by f^{-1}. Hence $f^{-1}(y) = x$ when $f(x) = y$.

Notation: $f^{-1}(y) = x \Leftrightarrow y = f(x)$.

The diagram below illustrates the fact that **an inverse function sends each element back to where it came from.**

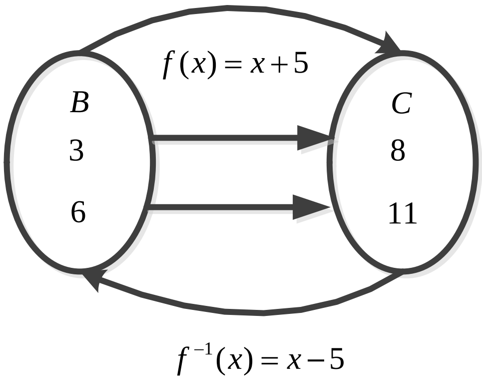


<u>**Important Note**</u>:

We **cannot define an inverse function** $f^{-1}(x)$, if a function $f(x)$ is **not a one-to-one correspondence (Bijection).**

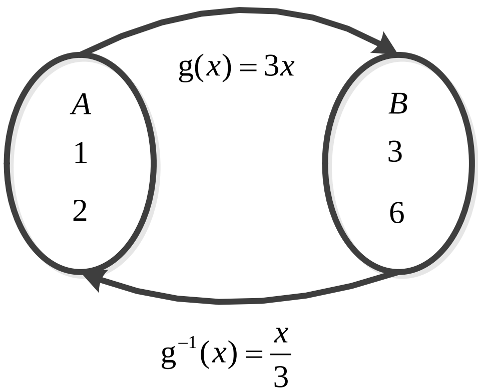
When *f* is not a Bijection, it is either not one-to-one or it is not onto.

Example 12



$$f(x) = x + 5$$
$$y = x + 5$$
$$x = y - 5$$
$$f^{-1}(x) = x - 5$$

Example 13



$$\mathrm{g}(x) = 3x$$
$$y = 3x$$
$$x = \frac{y}{3}$$
$$g^{-1}(x) = \frac{x}{3}$$

Exercise 5.13

1) Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a)=2$, $f(b)=3$, and $f(c)=1$. Is f invertible? If it is, what is its inverse?

2) Let f be the function from the set on integers to the set of integers such that $f(x)=x+1$. Is f invertible? If it is, what is its inverse?

3) Let f be the function from the set on integers to the set of integers such that $f(x)=9x+5$. Is f invertible? If it is, what is its inverse?

5.7 Composite function

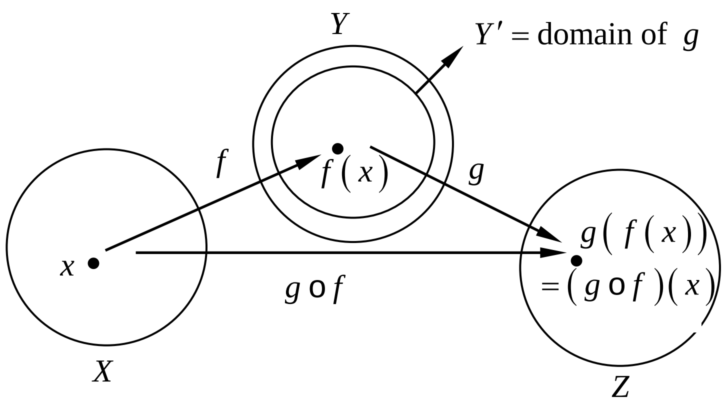
Definition

Let $f : X \to Y'$ and $g : Y' \to Z$ be functions with the property that the range of f is a subset of the domain of g. Define a new function $g \circ f : X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \qquad \text{for all } x \in X$$

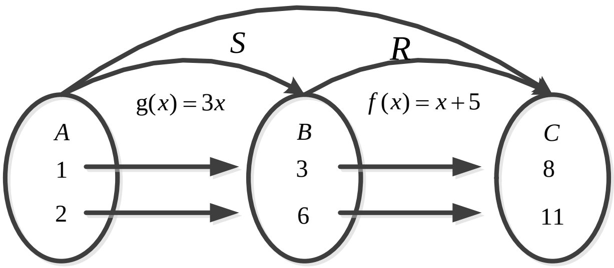
where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x". The function $g \circ f$ is called the composition of f and g. (We put the f first when we say "the composition of f and g" because an element x is first acted upon by f and then by g.)

This definition is shown diagrammatically below:



Example 14

$(f \circ g)(x)$



If relation S is a function $g(x) = 3x$ and relation R is a function $f(x) = x + 5$ as shown above.

Then the composite function

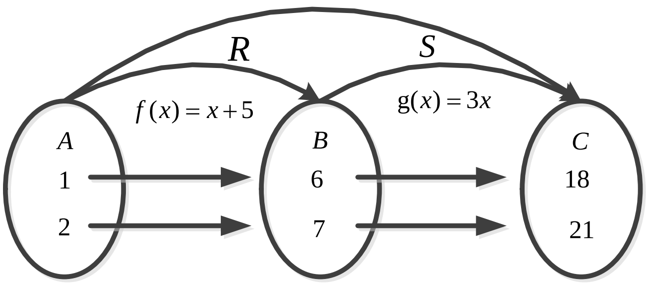
$$\begin{aligned} &(f \circ g)(x) \\ &= f\left(g(x)\right) \\ &= f(3x) \\ &= 3x + 5 \end{aligned}$$

When $x = 1$ $\quad (f \circ g)(1) = 3x + 5 = 3(1) + 5 = \underline{\underline{8}}$

When $x = 2$ $\quad (f \circ g)(2) = 3x + 5 = 3(2) + 5 = \underline{\underline{11}}$

Example 15

$(g \circ f)(x)$



If relation S is a function $g(x) = 3x$ and relation R is a function $f(x) = x + 5$ as shown above.

Then the composite function

$$\begin{aligned} &(g \circ f)(x) \\ &= g\left(\mathrm{f}(x)\right) \\ &= g(x + 5) \\ &= 3(x + 5) \end{aligned}$$

When $x = 1$ $\quad (f \circ g)(1) = 3(x + 5) = 3(1 + 5) = \underline{\underline{18}}$

When $x = 2$ $\quad (f \circ g)(2) = 3(x + 5) = 3(2 + 5) = \underline{\underline{21}}$

Exercise 5.14

1) Let g be the function from the set $\{a,b,c\}$ to itself such that $g(a)=b$, $g(b)=c$ and $g(c)=a$. Let f be the function from the set $\{a,b,c\}$ to the set $\{1,2,3\}$ such that $f(a)=3$, $f(b)=2$, and $f(c)=1$. What is the composition of g and f given by the symbol $(f \circ g)$? What is the composition of f and g given by the symbol $(g \circ f)$?

2) Let f and g be the functions from the set of integers to the set of integers defined by $f(x)=2x+3$ and $g(x)=3x+2$. What is the composition of f and g? What is the composition of g and f?

Tutorial 5-1

1. Let R be the relation on set A where $R = \{(a,b) \mid a = 2b\}$. List all the possible order paired of relation R if set $A = \{1,2,3,4\}$.

2. The relation R onto a set $A = \{1,2,3\}$ is given by $R = \{(1,1),(1,2),(3,2)\}$
   Show the relation R in the form of a matrix and arrow diagram.

3. The matrix below represents the relation S onto a set $B = \{a,b,c\}$. State and give the reasons on whether the relation S is reflexive, symmetric and/or transitive. State whether it is an equivalence relation.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

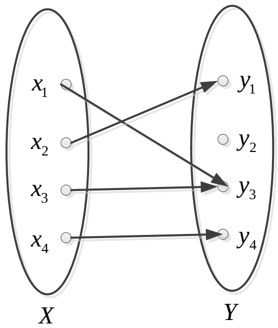
4. The matrix below represents the relation R onto a set $A = \{a,b,c\}$. State and give the reasons on whether the relation R is reflexive, symmetric and/or transitive. State whether it is an equivalence relation.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

5. The matrix below represents the relation S onto a set $B = \{a,b,c\}$. State and give the reasons on whether the relation S is reflexive, symmetric and/or transitive. State whether it is an equivalence relation.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

6. The relation S from set X to set Y is shown in the arrow diagram below. Show the relation S in a form of matrix and ordered pairs.

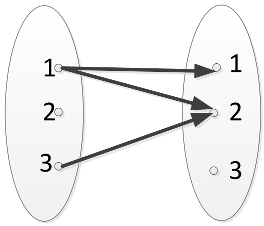


Answers (Tutorial 5-1)

1. $R = \{(2,1),(4,2)\}$

2.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



3. S is not a reflexive and not a transitive relation. S is a symmetric reltation. Thus S is not an equivalence relation.

4. Relation R is an equivalence relation as it is reflexive, symmetric and transitive.

5. Relation S is not an equivalence relation as it is not reflexive, not symmetric and not transitive.

6.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\{(x_1, \mathrm{y}_3),(x_2, \mathrm{y}_1),(x_3, \mathrm{y}_3),(x_4, \mathrm{y}_4)\}$

Tutorial 5-2

1. Determine whether each of the following sets of ordered pairs is a function from the given domain to the given codomain.

   (a) Domain $=\{1,2,3\}$, Codomain $=\{a,b,c,d,e\}$, $R=\{(1,a),(2,b),(3,b)\}$

   (b) Domain $=\{1,2,3,4\}$, Codomain $=\{a,b,c,d\}$, $S=\{(2,d),(3,a),(4,d)\}$

   (c) Domain $=\{1,2,3\}$, Codomain $=\{a,b,c,d\}$, $T=\{(1,b),(2,c),(3,a),(3,d)\}$

   (d) Domain $=\{1,2,3,4\}$, Codomain $=\{a,b\}$, $T=\{(1,b),(2,b),(3,b),(4,b)\}$

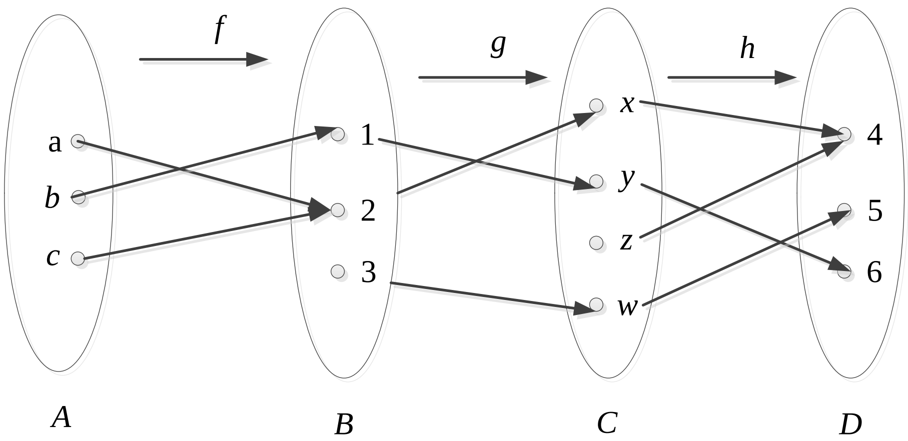
2. The function f is defined by $f(x)=x^3+3$ for all real values of x. Evaluate the values of the following:

   (a) $f(1)$ (b) $f(-1)$ (c) $f(2a)$

3. Let the function $f:A\to B$, $g:B\to C$, $h:C\to D$ be defined by the figure below. Determine which of the functions are:

   (i) injective (one-to-one) and/or

   (ii) surjective (onto).



4. The functions *f* and *g* are defined for all real values of *x* as follows:

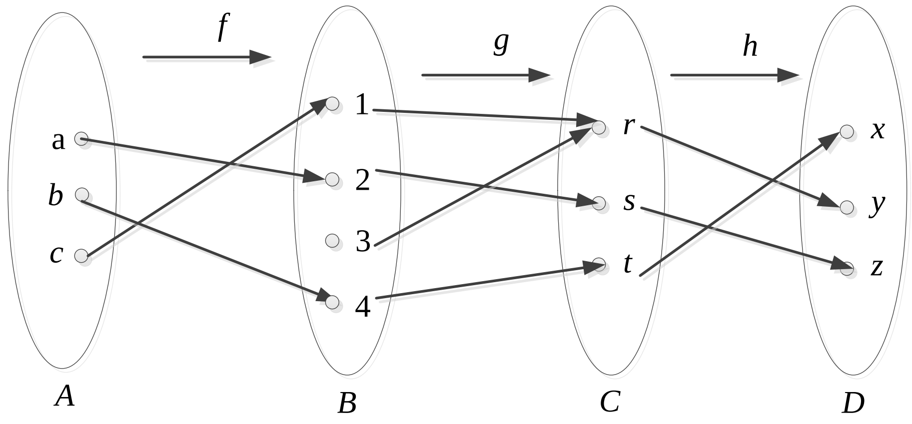
$$f(x)=2x-1 \text{ and } g(x)=\frac{2x+3}{x-1},\ x\neq 1$$

   (a) Find the composite functions $(f\circ g)(x)$ and $(g\circ f)(x)$.

   (b) Find the inverse functions $f^{-1}(x)$ and $g^{-1}(x)$.

   (c) Evaluate $f\circ g^{-1}(1)$ and $g^{-1}\circ f^{-1}(-2)$.

5. Let the function $f : A \to B$, $g : B \to C$, $h : C \to D$ be defined by the figure below. Determine which of the functions are is invertible, and, if it is, find its inverse.



6. Consider the two functions $f : R \to R$ and $g : R \to R$, where R is the set of all real values of *x*, as $f(x) = x + 2$ and $g(x) = x^2$. State with reason, which of these two functions is invertible.

7. The functions f and g are defined for all real values of x as follows:

$$f(x) = \frac{8}{x-3}, x \neq 3 \text{ and } g(x) = 2x - 3$$

(a) Find the expressions for f^{-1}, $(f \circ g)(x)$ and $(g \circ f)(x)$.

(b) Find the value of x for which $(f \circ g)(x) = (g \circ f)(x)$.

8*. The functions *f* and *g* are defined by $f(x) = x - 3$ and $g(x) = x^2$ respectively. Find another function *h* such that $hgf(x) = x^2 - 6x + 3$.

9*. The functions *f* and *g* are defined for all real values of *x* as follows:

$$f(x) = x^2 - 1 \text{ and } g(x) = (x-1)^2$$

(a) If $4f(x) + 3 = f(kx)$, find the value(s) of *k*.
(b) Express $g(2x+1)$ in terms of $f(x)$.
(c) Find a function *h* such that $f(x) = g(x) + 2h(x)$.

Answers (Tutorial 5-2)

1. (a) Yes (b) No (c) No (d) Yes

2. (a) 4 (b) 2 (c) $8a^3+3$

3. f is both not injective and not surjective.
g is injective but not surjective.
h is not injective but surjective.

4. (a) $(f \circ g)(x)=\dfrac{3x+7}{x-1}, x\neq 1;$ $(g \circ f)(x)=\dfrac{4x+1}{2x-2}, x\neq 1$
(b) $f^{-1}(x)=\dfrac{1}{2}(x+1),$ $g^{-1}(x)=\dfrac{3+x}{x-2}, x\neq 2$
(c) -9, -1

5. f is injective but is not surjective. Thus, f is not invertible.
g is not injective but is surjective. Thus, g is not invertible.
h is both injective and surjective. Thus, h is invertible.
Hence, $h^{-1}=\{(x,f),(y,r),(z,s)\}$

6. f is invertible, whereas g is not.

7. (a) $f^{-1}(x)=\dfrac{8+3x}{x}, x\neq 0;$ $f \circ g(x)=\dfrac{4}{x-3}, x\neq 3;$ $g \circ f(x)=\dfrac{25-3x}{x-3}, x\neq 3$
(b) 7

8*. $h(x)=x-6$

9*. (a) ± 2 (b) $4f(x)+4$ (c) $h(x)=x-1$

Questions from past year examination papers

1) 2013/14S1 IT1101/1501/1561/1751/1621 Sem Exam– Q3

Given the relation R from A to B and relation S from B to C where $A=\{\ a,b,c\ \}$, $B=\{\ 1,2,3\ \}$ and $C=\{\ x,y,z\ \}$.

$$R=\{(a,1),(a,3),(c,2)\}$$
$$S=\{(1,y),(2,x),(3,y),(3,z)\}$$

(a) Find R^{-1}, leaving your answer in matrix form. (2 marks)

(b) Draw an arrow diagram showing the sets and elements of A, B, C and the relations R and S. (4 marks)

(c) Find the composite relation $S \circ R$. (2 marks)

(d) State the domain and range of $S \circ R$. (2 marks)

2) 2013/14S1 IT1101/1501/1561/1751/1621 Sem Exam– Q4

(a) Given the function $f(x)=x^2+1$ for the set $R=\{\ \textit{real numbers}\ \}$ and $f: R \to R$.

(i) Determine if $f(x)$ is injective and/or surjective. (4 marks)

(ii) Explain when is a function bijective and conclude if $f(x)$ is bijective. (2 marks)

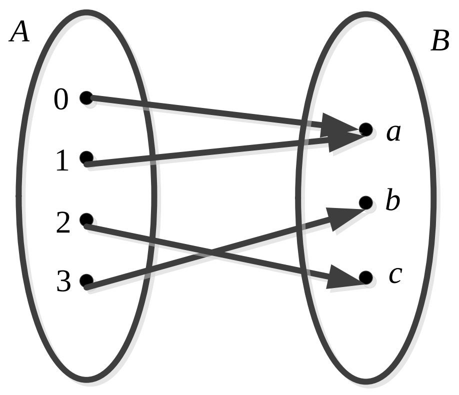
(b) Given $g(x)=x^3+5$ and $h(x)=4x-5$,

(i) Find $g^{-1}(x)$ (2 marks)

(ii) Find $(h \circ g)(x)$ (2 marks)

3) 2013/14S2 EG1740 Sem Exam– Q2

(a) List the ordered pairs in the relation given by the arrow diagram below. (4 marks)



(b) The matrix of relation R defined on the set $A = \{1, 2, 3, 4\}$ is

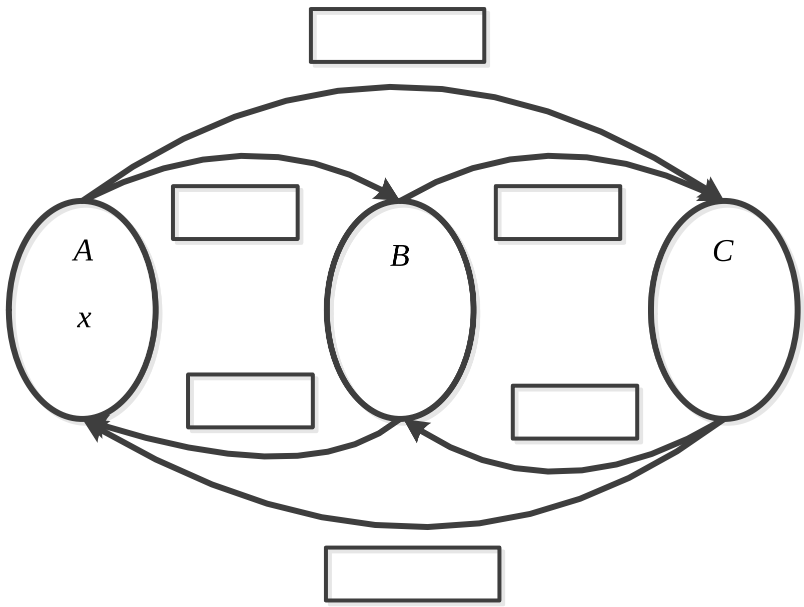
$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Determine if the relation R is reflexive, symmetric and/or transitive.

4) 2013/14S2 EG1740 Sem Exam– Q7a

Given that $f(x) = \log x$ for $x > 0$ and $g(x) = x + 5$.

(i) Copy and complete the arrow diagram below using $f(x),\ g(x),\ f^{-1}(x),\ g^{-1}(x),\ (f \circ g)(x)$ and $(f \circ g)^{-1}(x)$.



(6 marks)

(ii) Find $(f \circ g)(x)$ and $(f \circ g)^{-1}(x)$. (5 marks)

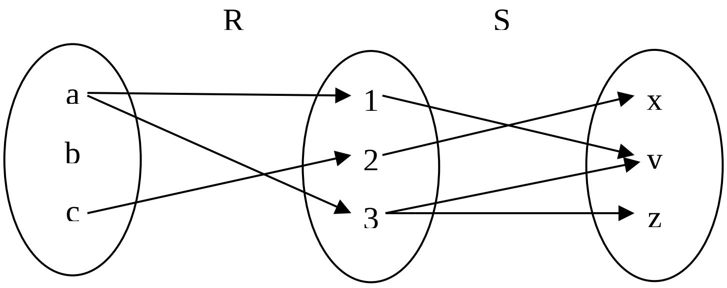
(iii) Given that $(f \circ g)^{-1}(x) = 5$, find the value of x. (3 marks)

Answers

1a

$$\text{Matrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

b)



c) $S \circ R = \{(a,y),(a,z),(c,x)\}$

d) domain $= \{a,c\}$

range $= \{x,y,z\}$

2ai) f is not injective

f is not surjective

ii) $f(x)$ is not bijective.

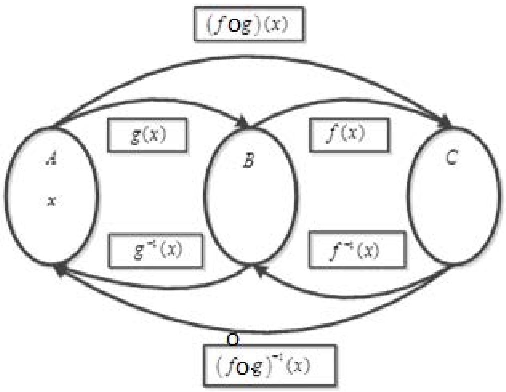
bi) $g^{-1}(x) = \sqrt[3]{x-5}$

ii) $(h \circ g)(x) = 4x^3 + 15$

3a) $\{(0,a),(1,a),(2,c),(3,b)\}$

b) Reflexive and Symmetric

4i)



ii) $(f \circ g)(x) = \log(x+5)$

$(f \circ g)^{-1}(x) = 10^x - 5$