Chapter 7: Differentiation

Objective

The objective of this chapter is to

1. understand the concept of limits leading to the derivative.
2. understand derivative as a slope or a rate of change.
3. find the derivatives of various functions using formulae.
4. use Chain, Product and Quotient rules.
5. find the derivatives of implicit functions.
6. find the higher derivatives of functions.

Content

7.1 Introduction

Differentiation is a mathematical technique for analyzing the way in which functions change. In particular, it determines how rapidly a function is changing at any specific point. As the function in question may represent the internet traffic, growth in technology penetration, the temperature of a chemical mix, etc., it is important to know how quickly these quantities change.

7.2 The Derivative: Rate of change, Slope and Tangent Line

7.2.1 Slope/Gradient and Tangent line of a Curve

The gradient of a straight line is a constant. It is equal to the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ between any two points (x_1, y_1) and (x_2, y_2) on the line.

On a curve, however, the gradient is changing from one point to another. We define the gradient at any point on a curve to be the gradient of the tangent line to the curve at that point.

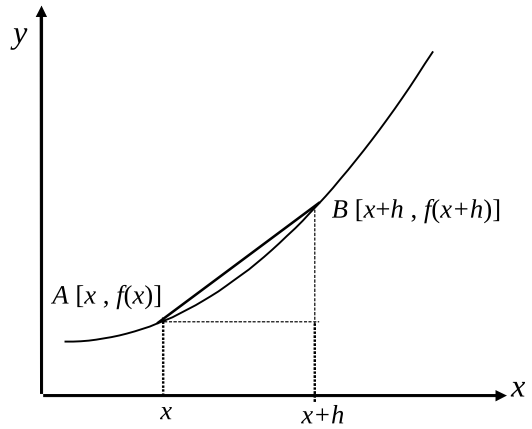


Figure 1

Figure 1 shows a curve $y = f(x)$ with points A and B *very close* to one another. Let A be a general point $[x\ ,\ f(x)]$. Now consider a small change in x. This small change is denoted by h. The original x together with the small change, h, means we have moved to a new point B with the horizontal value $x+h$. The *y*-coordinate of B is $y = f(x+h)$. Hence, B is the point $[x+h\ ,\ f(x+h)]$.

The gradient of the chord AB = $\dfrac{f(x+h) - f(x)}{h}$

If A is fixed and we move B closer and closer to A, the chord gets nearer to the curve. Eventually, as B reaches A, the chord resembles a tangent line to the curve at A. This is shown in Figure 2 below. As B approaches A, the changes, h gets smaller and smaller. Symbolically this is written as $h \to 0$, meaning that h has become very small and is almost zero.

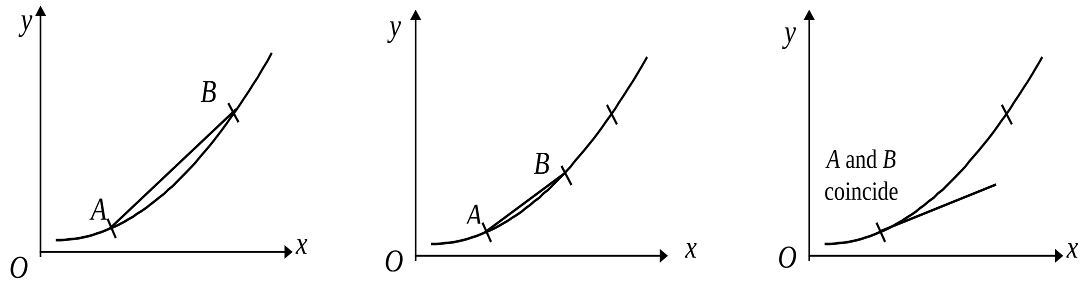


Figure 2

The limiting value of $\frac{f(x+h)-f(x)}{h}$ as h tends to zero is defined as the gradient of the tangent line to the graph at A (or the gradient of the curve at A).

That is, gradient of the curve at point A = Limit of $\frac{f(x+h)-f(x)}{h}$ as h tends to zero.

In notation: $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

or $$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Note: $f'(x)$ (read as '*f* prime of *x*')

$\frac{dy}{dx}$ (read as 'dee *y*, dee *x*')

7.2.2 Differentiation notation $\frac{dy}{dx}$

$f'(x)$ or $\frac{dy}{dx}$ is called the **derivative** or the differential coefficient of *y* with respect to *x.*

$f'(x)$ or $\frac{dy}{dx}$ has various interpretations, via:

(i) the differentiation of y with respect to x,
(ii) the rate of change of y with respect to x,
(iii) the first derivative,
(vi) the gradient (slope) of a graph,
(v) the gradient (slope) of a tangent to a graph.

Example 1

Find the tangent line of the graph $y = x^2$ at $x = 3$.

$y = x^2$

$\frac{dy}{dx} = 2x$

At $x = 3$ the gradient of the tangent line is given by

$m = \frac{dy}{dx} = 2(3) = 6$

At $x = 3$ and $y = 3^2 = 9$

Hence the tangent line touches the graph $y = x^2$ at $(3,9)$

Thus the equation of the tangent line

$\frac{y - y_1}{x - x_1} = m$

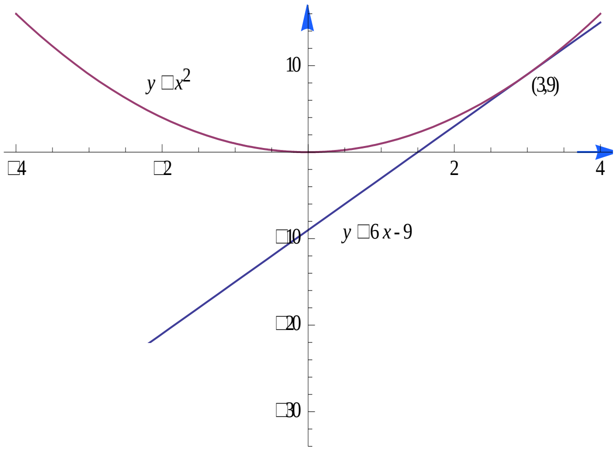
Substituting the point $(3,9)$ into (x_1, y_1)

we get the equation of the tangent line:

$\frac{y - 9}{x - 3)} = 6$

$y - 9 = 6(x - 3)$

$\underline{\underline{y = 6x - 9}}$



7.3 Techniques of Differentiation

It would be tedious and time-consuming to use the limit definition every time we wanted to compute a derivative. Fortunately, we have a set of rules that simplify the process of differentiation.

7.3.1 The Constant Rule

For any constant c, $\boxed{\frac{d}{dx}(c) = 0}$

That is, the derivative of a constant is zero.

Example 2

(a) Given $y = 3$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 0$$

(b) Given $y = -\frac{2}{9}$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 0$$

7.3.2 The Power Rule

For any real number n, $\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}}$

Example 3

(a) Given $f(x) = x^3$, find $f'(x)$.

$$\begin{aligned} f'(x) &= 3x^{3-1} \\ &= 3x^2 \end{aligned}$$

(b) Given $y = x^{\frac{2}{3}}$, find $\frac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{2}{3}x^{\frac{2}{3}-1} \\ &= \frac{2}{3}x^{-\frac{1}{3}} \end{aligned}$$

Exercise 7.2

Differentiate the followings with respect to x:

(a) x^5

(b) $x^{\frac{3}{2}}$

(c) $\sqrt{x^3}$

(d) $\frac{7}{x^3}$

7.3.3 The Constant Multiple Rule

If c is a constant and $f(x)$ is differentiable, then

$$\boxed{\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right]}$$

Example 4

1	$y = 100x^3$	$\frac{dy}{dx} = 3(100x^2) = 300x^2$
2	$r = 7t^2$	$\frac{dr}{dt} = 2(7t) = 14t$

Exercise 7.3

Differentiate the followings with respect to x:

(a) $3x$

(b) $-5x^4$

(c) $\frac{1}{2x}$

7.3.4 The Sum/Difference Rule

If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$$

Example 5

1	$y = 10x + 20x^2 - 30x^3$	$\frac{dy}{dx} = 10 + 2(20x) - 3(30x^2)$ $= 10 + 40x - 90x^2$
2	$r = 7t^2 + 9t - 50t^{-3}$	$\frac{dr}{dt} = 2(7t) + 9 - (-3)50t^{-4}$ $= 14t + 9 + 150t^{-4}$

Exercise 7.4

(1) Find $\frac{dy}{dx}$ if $y = 3x^5 - 2x^2 + 3x - 10$.

(2) Differentiate the following with respect to *x*.

(a) $5x^{-1} - 4\sqrt{x} + \frac{1}{x^2}$ (b) $\frac{x^2 + 5x + 4}{x}$

(3) Find the gradient of the curve $y = 2x + \frac{1}{x}$ at the point $(1, 3)$.

7.4 The Product and Quotient Rules

7.4.1 The Product Rule

If $f(x)$ and $g(x)$ are differentiable at x, then

$$\boxed{\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}}$$

or

$$\boxed{(fg)' = fg' + gf'}$$

Note: This rule is also applicable to products with more than two factors.

Example 6

	Function	Differentiation
(a)	$y = x^2(x+1)$	$\frac{dy}{dx} = (x+1)\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(x+1)$ $= (x+1)(2\text{x}) + x^2$ $= 2x^2 + 2x + x^2$ $= 3x^2 + 2x$ $= x(3x+2)$
(b)	$y = (x+2)(x+1)$	$\frac{dy}{dx} = (x+1)\frac{d}{dx}(x+2) + (x+2)\frac{d}{dx}(x+1)$ $= (x+1) + (x+2)$ $= 2x+3$

Exercise 7.5

Differentiate the following with respect to x:

(a) $\left(x^2+1\right)\left(1-2x\right)$

(b) $\left(3x^2+1\right)\left(6x^3+x^2+2\right)$

7.4.2 The Quotient Rule

If $f(x)$ and $g(x)$ are differentiable at x, then

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{df}{dx}-f(x)\frac{dg}{dx}}{\left[g(x)\right]^2}}$$

or equivalently $$\boxed{\left(\frac{f}{g}\right)'=\frac{gf'-fg'}{g^2}}$$

Example 7

(a)	$y=\frac{x}{x+1}$	$\frac{dy}{dx}=\frac{(x+1)\frac{d}{dx}(x)-x\frac{d}{dx}(x+1)}{(x+1)^2}$ $=\frac{(x+1)-x}{(x+1)^2}$ $=\frac{1}{(x+1)^2}$
(b)	$y=\frac{x^2}{x+1}$	$\frac{dy}{dx}=\frac{(x+1)\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(x+1)}{(x+1)^2}$ $=\frac{(x+1)(2\,\mathrm{x})-x^2}{(x+1)^2}$ $=\frac{x\left[2(x+1)-x\right]}{(x+1)^2}$ $=\frac{x(x+2)}{(x+1)^2}$

Exercise 7.6

Differentiate the following with respect to x:

(a) $\dfrac{2x+1}{x^2+1}$

(b) $\dfrac{x^2+3x+7}{x-1}$

7.5 The Chain Rule

Suppose y is a function of u and u is a function of x, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

This is known as the ***chain rule***.

Example 8

Given $y = (2x+1)^5$, find $\frac{dy}{dx}$.

Let $u = 2x+1$,

Hence by substituting u into equation y, we will get $y = u^5$,

$$\frac{dy}{du} = 5u^4$$

Since $u = 2x+1$,

$$\frac{du}{dx} = 2.$$

Using chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 5u^4 \times 2 \\ &= 10\left(2x+1\right)^4 \end{aligned}$$

Exercise 7.7

(1) Find $\frac{dy}{dx}$ if $y = (x^3 - 4x + 5)^2$

(2) If $y = r^5$ and $r = 3t^2 + 5t + 7$, find $\frac{dy}{dt}$.

7.5.1 The General Power Rule

In general, for any real number n and differentiable function $f(x)$,

$$\boxed{\frac{d}{dx}\left[f(x)\right]^n = n\left[f(x)\right]^{n-1}\frac{d}{dx}\left[f(x)\right]}$$

Example 9

(a)	$f(x)=\sqrt{(2x^3+x^2+9)^3}$	$f(x)=\sqrt{(2x^3+x^2+9)^3}$ $=(2x^3+x^2+9)^{\frac{3}{2}}$ $f'(x)=\frac{3}{2}(2x^3+x^2+9)^{\frac{1}{2}}\frac{d}{dx}(2x^3+x^2+9)$ $=\frac{3}{2}(2x^3+x^2+9)^{\frac{1}{2}}(6x^2+2x)$ $=\frac{3}{2}(2x)(3x+1)(2x^3+x^2+9)^{\frac{1}{2}}$ $=3x(3x+1)(2x^3+x^2+9)^{\frac{1}{2}}$
(b)	$f(x)=7(2x^3+x^2+9)^3$	$f'(x)=7(3)(2x^3+x^2+9)^2\frac{d}{dx}(2x^3+x^2+9)$ $=21(2x^3+x^2+9)^2(6x^2+2x)$ $=21(2x)(3x+1)(2x^3+x^2+9)^2$ $=42x(3x+1)(2x^3+x^2+9)^2$

Exercise 7.8

(1) Differentiate the followings with respect to x:

(a) $\left(3x^2-x+1\right)^7$

(b) $\dfrac{2}{\left(x^2+1\right)^3}$

(2) Find $f'(x)$ given that

(a) $f(x) = \dfrac{x^3}{(1+2x)^2}$ (b) $f(x) = \dfrac{\left(1+x^2\right)^2}{(2-x)^3}$

(3) Find $f'(1)$ given that $f(x) = \sqrt{x}\,(x+1)^3$.

(4) Find the gradient of the curve $h(x) = \dfrac{1}{\sqrt{2x-1}}$ at the point where $x = 9$.

(5) Calculate the gradient of the curve $y = x\sqrt{x+3}$ at the point where x = 1. Find also the coordinates of the points when $\dfrac{dy}{dx} = 0$.

7.5.2 Derivatives of Exponential Functions

$$\boxed{\frac{d}{dx}\left(e^x\right) = e^x}$$

If $u(x)$ is a function of x, then by Chain Rule:

$$\boxed{\frac{d}{dx}\left(e^u\right) = e^u \frac{du}{dx}}$$

Example 10

	$f(x)$	$f^{'}(x)$
(a)	e^{2x}	$e^{2x}\frac{d}{dx}(2x)$ $= e^{2x}(2)$ $= 2e^{2x}$
(b)	$e^{x^3+5x^2+7}$	$e^{x^3+5x^2+7}\frac{d}{dx}(x^3+5x^2+7)$ $= e^{x^3+5x^2+7}(3x^2+10x)$ $= (3x^2+10x)e^{x^3+5x^2+7}$

Exercise 7.9

(1) Differentiate with respect to x:

(a) e^{3x} (b) e^{2x-3} (c) $e^{\frac{1}{2}x+1}$

(2) Find the gradient of the curve $g(x) = xe^{2x}$ at the point where $x = 1$.

7.5.3 Derivatives of Logarithmic Functions

$$\boxed{\frac{d}{dx}\left(\ln x\right)=\frac{1}{x}}$$

If $u(x)$ is a function of x, then by the chain rule:

$$\boxed{\frac{d}{dx}\left(\ln u\right)=\frac{1}{u}\frac{du}{dx}}$$

Rules of Logarithmic function

1. $\ln MN = \ln M + \ln N$

2. $\ln \frac{M}{N} = \ln M - \ln N$

3. $\ln M^n = n \ln M$

4. $\ln e^a = a$

5. $e^{\ln a} = a$

Example 11

	$f(x)$	$f'(x)$
(a)	$\ln(2x)$	$\left(\frac{1}{2x}\right)\frac{d}{dx}(2x)$ $=\left(\frac{1}{2x}\right)(2)$ $=\frac{1}{x}$
(b)	$\ln(x^3+3)^2$	$\left(\frac{1}{(x^3+3)^2}\right)\frac{d}{dx}(x^3+3)^2$ $=\left(\frac{1}{(x^3+3)^2}\right)2(x^3+3)\frac{d}{dx}(x^3)$ $=\left(\frac{2}{(x^3+3)}\right)(3x^2)$ $=\frac{6x^2}{(x^3+3)}$

Exercise 7.10

(1) Differentiate with respect to x :

(a) $\ln 5x$ (b) $\ln(3x+2)$ (c) $\ln(x^2+e^{3x})$

(2) Find $f'(x)$ given that:

(a) $f(x)=x\ln x$ (b) $\ln\sqrt{\frac{1+x}{1-x}}$

(3) Differentiate with respect to x :

(a) $\frac{\ln x^3}{e^{3x}}$ (b) $\left(e^{x^2}+\ln x\right)^5$

(4) Find the equation of the tangent line to the curve $h(x)=\ln(2x-1)$ at the point where $x=1$.

7.5.4 Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x$$
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\cos ec\ x = -\cos ec\ x \cot x$$
$$\frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\cot x = -\cos ec^2\ x$$

Example 12

Differentiate the following with respect to x:
a) $x + 2\,cos\,x$, b) $x^2\ sin\,x$ and c) $\frac{\tan x}{x}$.

a) $$\frac{d}{dx}\left(x + 2\cos x\right) = \frac{d}{dx}x + 2\frac{d}{dx}\cos x$$
$$= 1 - 2\sin x$$

b) $$\frac{d}{dx}\left(x^2 \sin x\right) = \sin x \frac{d}{dx}x^2 + x^2 \frac{d}{dx}\sin x$$
$$= \sin x\left(2x\right) + x^2 \cos x$$
$$= x\left(2\sin x + x\cos x\right)$$

c) $$\frac{d}{dx}\frac{\tan x}{x} = \frac{x\frac{d}{dx}\tan x - \tan x\frac{d}{dx}x}{x^2}$$
$$= \frac{x\sec^2 x - \tan x}{x^2}$$

7.6 Implicit Differentiation and Related Rates

7.6.1 Implicit Differentiation

Explicit Function

The functions that we have encountered so far are explicit, i.e. one variable can be expressed clearly in terms of other variable. For example, $y = x^5 - x^2 + 1$.

Implicit Function

Sometimes, it is difficult or impossible to express one variable in terms of other variable (or express y in terms of x). For example, $y^3 = 2 + yx - y^2$. Such functions are called **implicit functions.**
To differentiate such functions, we will need to differentiate both sides of the equation w.r.t x (or whichever independent variable). This technique is called **implicit differentiation**.

<u>**Example 13**</u>

(a)	$y = x^2 + y + y^2$	$\frac{dy}{dx} = 2x + \frac{dy}{dx} + 2y\frac{dy}{dx}$ $\Rightarrow -2y\frac{dy}{dx} = 2x$ $\Rightarrow \frac{dy}{dx} = \frac{2x}{-2y}$ $\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$
(b)	$y = x^2 + xy$	$\frac{dy}{dx} = 2x + \left[x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right]$ product rule $\Rightarrow \frac{dy}{dx} = 2x + \left[x\frac{dy}{dx} + y(1)\right]$ $\Rightarrow \frac{dy}{dx}(1-x) = 2x + y$ $\Rightarrow \frac{dy}{dx} = \frac{2x+y}{(1-x)}$

Exercise 7.11

(1) Find $\frac{dy}{dx}$ for the following:

(a) $x^2 + y^3 = x + 9$

(b) $f(x) = 3x^2 + 7y + xy^2$

(c) $y = x^2 + y^2$

(d) $y = (x^3 + 2)(\mathrm{y}+1)$

(2) Find the equation of the tangent line to the graph $2x^2 + y^2 = 4$ at the point $(0,2)$.

7.6.2 Related Rates

Let's look at some examples on rate of change.

Example 14

Annual profits, $P(t)$ million of dollars, of a new computer software company are given by the equation

$$P(t) = -1 + 0.8t - 0.01t^2$$

where *t* represents the time in years, since the company began. Find the rate at which the company's profit is changing when $t = 6$.

Solution

$P'(t) = 0.8 - 2(0.01)t = 0.8 - 0.02t$

At $t = 6$, the rate of change is

$P'(6) = 0.8 - (0.02)(6)$

$= 0.8 - 0.12$

$= 0.68$ million/year

Example 15

The total weekly cost C for the ABC Printer Company to produce x printers is given by

$$C = 4000 + 50x + 10\sqrt{x}$$

If the production of the units is increasing at a rate of 12 units per week, how fast are the costs changing when $x = 400$.

Solution

Given $C = 4000 + 50x + 10\sqrt{x}$,

$\Rightarrow \dfrac{dC}{dx} = 50 + 10\left[\dfrac{1}{2}(x)^{-\frac{1}{2}}\right] = 50 + \dfrac{5}{\sqrt{x}}$

Production of the units is increasing at a rate of 12 units per week:

$\Rightarrow \dfrac{dx}{dt} = 12$ units / week

Therefore, $\dfrac{dC}{dt} = \dfrac{dC}{dx} \times \dfrac{dx}{dt} = \left(50 + \dfrac{5}{\sqrt{x}}\right) \times (12)$

$= \left(50 + \dfrac{5}{\sqrt{400}}\right) \times (12)$

$= \$603$ / week

Exercise 7.12

(1) The price of a radiation test equipment is expected to change according to the equation

$$P(t)=\frac{20000}{5+t^2} \text{ dollars}$$

where t is the time in month. Find the rate at which the price is changing after 2 months.

(2) A manufacturer is willing to supply x hundred units of a certain products if the price of the product is p dollars per unit and $3p^2-x^2=25$.
How fast is the supply changing when the price is $\$5$ per unit and is increasing at the rate of 50 cents per week?

7.7 Higher Order Derivatives

After taking the derivative of a function y, we may then take the derivative of the derivative.

The symbols used are:

	Symbol	
Function	$y, f(x)$	
First derivative	$\frac{dy}{dx}, f'(x)$	Differentiation the function
Second derivative	$\frac{d^2y}{dx^2}, f''(x)$	Differentiating the first derivative $\frac{dy}{dx}$
Third derivative	$\frac{d^3y}{dx^3}, f'''(x)$	Differentiating the second derivative $\frac{d^2y}{dx^2}$

We can then go on to find fourth and higher derivatives.

Example 16

	Function	1st order	2nd order	3rd order
(a)	$y = 100x^3$	$\frac{dy}{dx} = 300x^2$	$\frac{d^2y}{dx^2} = 600x$	$\frac{d^3y}{dx^3} = 600$
(b)	$u = 100t^3 + 7t^2 + t$	$\frac{du}{dt} = 300t^2 + 14t + 1$	$\frac{d^2u}{dt^2} = 600t + 14$	$\frac{d^3u}{dt^3} = 600$

Exercise 7.13

(1) Find the second and third derivative of $y = 2x^3 + 5x^2 - 3x$.

(2) Find the second derivative of $y = x\left(x^2 + 2\right)$.

Tutorial 7-1

1. Find the derivatives of the following functions:

(a) $y=1$ (b) $f(x)=\pi$ (c) $g(t)=e^2$

(d) $y=x^3$ (e) $y=20x^{\frac{3}{4}}$ (f) $f(x)=\sqrt{x}$

(g) $f(x)=3\sqrt[3]{x}$ (h) $g(t)=\dfrac{1}{t^2}$ (i) $g(t)=\dfrac{5}{t^3}$

(j) $h(s)=\dfrac{2}{\sqrt{s}}$ (k) $h(s)=\dfrac{1}{3\sqrt[3]{s}}$

2. Differentiate the following with respect to x :

(a) $4x^2+3x+1$ (b) $\dfrac{6}{\sqrt[3]{x}}-\dfrac{4}{\sqrt{x}}$ (c) $5x^4+\dfrac{4}{x}-\pi$

(d) $3x+2\sqrt{x}-3$ (e) $\dfrac{2x^2+4x}{x}$ (f) $x(x+4)$

(g) $4x^2\sqrt{x}-\dfrac{6}{\sqrt{x}}$ (h) $\dfrac{(1-x)(x-2)}{x}$

3. Differentiate the following functions with respect to x :

(a) $y=2x^3-4x^2+x+3$ (b) $y=\dfrac{1}{2x}-\dfrac{5}{x^2}$

(c) $g(x)=\dfrac{x^3+x^2-2x}{x^4}$ (d) $f(t)=6\sqrt{t}-\dfrac{1}{\sqrt{t}}$

4. Find the slope of the tangent to the curve $y=\dfrac{3x^2-4x}{\sqrt{x}}$ at $x=1$. Hence, find the equation of the tangent to the curve at $x=1$.

5. Differentiate the following functions with respect to x by product rule.

(a) $y=(x+7)^{10}(x^2+2)^{-7}$ (b) $y=\dfrac{\sqrt{t}}{t^2+4}$ (c) $y=\dfrac{3u-5}{3u^2+7}$

6. Differentiate the following functions with respect to x by quotient rule:

(a) $y=(x+7)^{10}(x^2+2)^{-7}$ (b) $y=\dfrac{\sqrt{t}}{t^2+4}$ (c) $y=\dfrac{3u-5}{3u^2+7}$

7*. Differentiate the following functions with respect to x by product or quotient rule:

(a) $y=(2ax+b)^5(5x^2-ab)^6$ (b) $y=\dfrac{2ax^2+bx}{bx^3-cx}$

Answers(7-1)

1. (a) $y' = 0$ (b) $f'(x) = 0$ (c) $g'(t) = 0$

(d) $\frac{dy}{dx} = 3x^2$ (e) $\frac{dy}{dx} = 15x^{-\frac{1}{4}}$ (f) $\frac{df(x)}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

(g) $\frac{d}{dx}f(x) = x^{-\frac{2}{3}}$ (h) $\frac{d}{dt}g(t) = -2t^{-3}$ (i) $\frac{d}{dt}g(t) = -15t^{-4}$

(j) $\frac{d}{ds}h(s) = -s^{-\frac{3}{2}}$ (k) $\frac{d}{ds}h(s) = -\frac{1}{9}s^{-\frac{4}{3}}$

2. (a) $8x+3$ (b) $-2x^{-\frac{4}{3}} + 2x^{-\frac{3}{2}} = 2x^{-\frac{3}{2}}\left(-x^{\frac{1}{6}} + 1\right)$

(c) $20x^3 - 4x^{-2} = 4x^{-2}\left(5x^5 - 1\right)$ (d) $3 + x^{-\frac{1}{2}}$ (e) 2

(f) $2x+4$ (g) $10x^{\frac{3}{2}} + 3x^{-\frac{3}{2}} = x^{-\frac{3}{2}}\left(10x^3 + 3\right)$ (h) $\frac{2}{x^2} - 1$

3. (a) $6x^2 - 8x + 1$ (b) $-\frac{1}{2x^2} + \frac{10}{x^3}$ (c) $-\frac{1}{x^2} - \frac{2}{x^3} + \frac{6}{x^4}$ (d) $\frac{3}{\sqrt{t}} + \frac{1}{2\sqrt{t^3}}$

4. Gradient of the slope $\frac{5}{2}$. Equation of tangent line $2y = 5x - 7$

5. and 6.

(a) $-2\left(x^2+2\right)^{-8}\left(x+7\right)^9\left(2x^2+49x-10\right) = \frac{-2\left(x+7\right)^9\left[2x^2+49x-10\right]}{\left(x^2+2\right)^8}$

(b) $\frac{4-3t^2}{2\sqrt{t}\left(t^2+4\right)^2}$

(c) $\frac{3\left(7+10u-3u^2\right)}{\left(3u^2+7\right)^2}$

7. (a) $\left(2ax+b\right)^4\left(5x^2-ab\right)^5\left(170ax^2+60bx-10a^2b\right)$

(b) $\frac{-2x^2\left(abx^2+b^2x+ac\right)}{\left(bx^3-cx\right)^2}$

Tutorial 7-2

1. Differentiate the following functions with respect to x by chain rule.

   (a) $y=\left(2x^3+7\right)^6$ (b) $y=\sqrt{2x+5}$ (c) $y=\dfrac{2}{3\sqrt{x^2-5x}}$

2. Differentiate the following functions with respect to x:

   (a) $y=5e^x$ (b) $y=e^{x^2+7}$ (c) $P(t)=\ln(5t-2)$

   (d) $y=\ln\sqrt{5x^3-4}$ (e) $h(t)=t^3e^{5t}$ (f) $y=\ln\left(t^2+e^{t^2}\right)$

3. Given that $y^2=2x^3+y+7$, obtain $\dfrac{dy}{dx}$ in terms of x and y.

4. Calculate the value of $\dfrac{dy}{dx}$ at the point $(1,1)$ on the curve $9x^2+16y^2=25$.

5. Find $\dfrac{dy}{dx}$ given that $x^2-4xy=y^2-5$.

6. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ if

   (a) $y=5x^4+3x^2-x+2$ (b) $y=\dfrac{1}{x^3}+\dfrac{2}{\sqrt{x}}$

7. Find $f'(x)$ and $f''(x)$ if

   (a) $f(x)=(1-x)^2$ (b) $f(x)=\dfrac{1}{x^2}+\dfrac{2}{x^3}+\dfrac{1}{x^4}$

8. Given $s=\left(3t^2-t\right)(t-2)$, find $s'(1)$ and $s''(1)$.

9. Given that $y=\sqrt{2-3x^2}$, show that $y^3\dfrac{d^2y}{dx^2}+6=0$.

Answers(7-2)

1. (a) $\dfrac{dy}{dx} = 36x^2\left(2x^3+7\right)^5$ (b) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2x+5}}$ (c) $\dfrac{dy}{dx} = \dfrac{5-2x}{3\left(x^2-5x\right)^{\frac{3}{2}}}$

2. (a) $\dfrac{dy}{dx} = 5e^x$ (b) $\dfrac{dy}{dx} = 2xe^{x^2+7}$ (c) $P'(t) = \dfrac{5}{5t-2}$

(d) $\dfrac{dy}{dx} = \dfrac{15x^2}{2\left(5x^3-4\right)}$ (e) $h'(t) = t^2e^{5t}\left(3+5t\right)$ (f) $\dfrac{dy}{dt} = \dfrac{2t\left(1+e^{t^2}\right)}{t^2+e^{t^2}}$

3. $\dfrac{dy}{dx} = \dfrac{6x^2}{2y-1}$

4. $\dfrac{dy}{dx} = -\dfrac{9}{16}$

5. $\dfrac{dy}{dx} = \dfrac{x-2y}{y+2x}$

6. (a) $\dfrac{dy}{dx} = 20x^3+6x-1$ $\dfrac{d^2y}{dx^2} = 60x^2+6 = 6\left(10x^2+1\right)$

(b) $\dfrac{dy}{dx} = -\left(\dfrac{3}{x^4}+\dfrac{1}{x^{3/2}}\right)$ $\dfrac{d^2y}{dx^2} = \dfrac{12}{x^5}+\dfrac{3}{2x^{5/2}}$

7. (a) $f'(x) = -2(1-x)$ $f''(x) = 2$

(b) $f'(x) = -2\left(\dfrac{1}{x^3}+\dfrac{3}{x^4}+\dfrac{2}{x^5}\right)$ $f''(x) = 2\left(\dfrac{3}{x^4}+\dfrac{12}{x^5}+\dfrac{10}{x^6}\right)$

8. $s'(1) = -3$ $s''(1) = 4$

Questions from Past Year Examination papers

1) 2012/13S1 EG1740 Sem Exam– Q4

Find the derivatives of the following functions:

(a) $y = \ln(x-5)^2$ (2 marks)

(b) $y = (3-x)\sqrt{(2x+1)}$ (3 marks)

(c) $y = \dfrac{e^{-x}}{(x^2-1)}$ (3 marks)

2) 2013/14S2 EG1740 Sem Exam– Q5a

Differentiate $f(x) = \dfrac{x^{-3}}{2} + \ln|x| + e^{2x} + 5$. (4 marks)

3) 2013/14S2 EG1740 Sem Exam– Q7b

Consider the curve with equation $y = 3xe^{-2x} + 7$ where $x > 0$.

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. (6 marks)

4) 2012/13S2 EG1740 Sem Exam– Q5a

Differentiate $f(x) = 2x^{-5} + \ln x$. (4 marks)

5) 2014/15S1 IT1101/1501/1561/1751/1621 Sem Exam– Q7

(a) Differentiate the following:

(i) $y = \dfrac{1}{3}x^5 - \dfrac{2}{x^3} + e^{-7x} + \ln x$ (4 marks)

(ii) $y = \dfrac{3x^2}{5x-2}$ (4 marks)

(b) Use implicit differentiation to find $\dfrac{dy}{dx}$ for the following:

$(y-3)^4 - y^2 + x^5 = 8$

(7 marks)

Answers

1a) $$\frac{dy}{dx}=\frac{2}{x-5}$$

b) $$\frac{dy}{dx}=\frac{2-3x}{\sqrt{2x+1}}$$

c) $$\frac{dy}{dx}=\frac{-e^{-x}\left(x^2+2x-1\right)}{\left(x^2-1\right)^2}$$

2 $$f'(x)=\frac{-3x^{-4}}{2}+\frac{1}{x}+2e^{2x}$$

3 $$\frac{dy}{dx}=3e^{-2x}(-2x+1)$$
$$\frac{d^2y}{dx^2}=12e^{-2x}(x-1)$$

4. $$f'(x)=\frac{-10}{x^6}+\frac{1}{x}=\frac{x^5-10}{x^6}$$

5a) i) $$\frac{dy}{dx}=\frac{5}{3}x^4+6x^{-4}-7e^{-7x}+\frac{1}{x}$$

5a) ii) $$\frac{dy}{dx}=\frac{3x(5x-4)}{(5x-2)^2}$$

5b) $$\frac{dy}{dx}=-\frac{5x^4}{\left(4(y-3)^3-2y\right)}$$