Chapter 6: Some Common Functions, Graphs and Limits

Objective

The objective of this chapter is to
1. describe the properties of the linear, quadratic, exponential, logarithmic and trigonometric functions.
2. find limits of various functions.
3. define a continuous function and identify the discontinuity of functions.

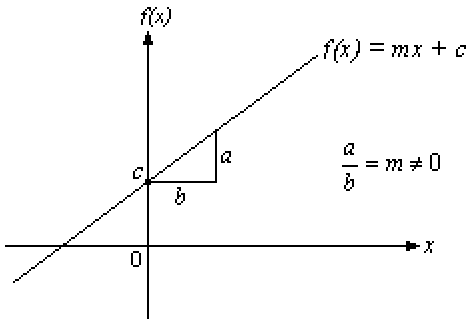
Content

6.1 Introduction

Graphs play a major role in our daily lives. It provides a quick visualization of many aspects of technical and data analysis & reporting.

6.2 Linear Functions

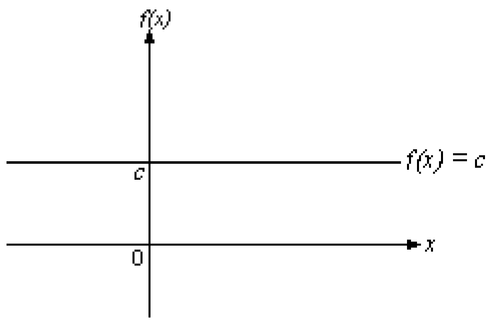
$f(x) = mx + c$ where *m* and *c* are constants



Characteristics:
m represents the **gradient** of the graph
c represents the **intercept** on the $f(x)$ axis

positive m will slope **upwards from left to right**
negative m will slope **downwards from left to right**

Constant Function (Horizontal Lines)
$f(x) = c$ where *c* is a constant



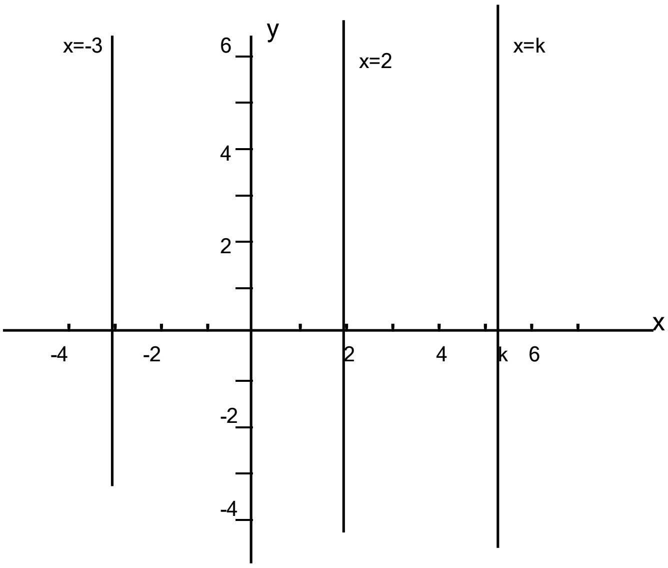
Characteristics:
The line $f(x) = c$ is always a **horizontal** straight line.

The gradient of this function is always **zero**

Vertical Line
Vertical lines do not have slopes but they do have equations
The equation of any vertical line can be put in the form

$$x = k$$

where k is a constant.



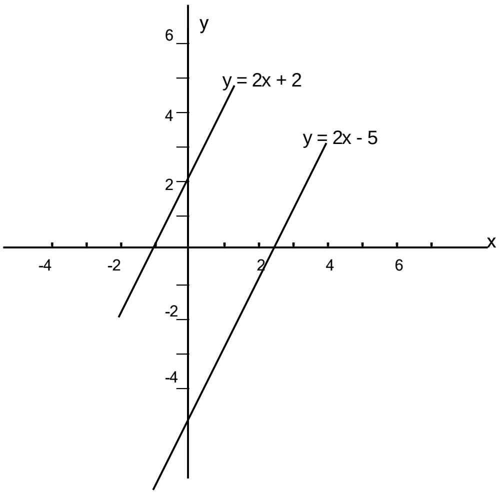
Parallel Lines

If two lines have the same slope, they are parallel. Thus,

$$y = 2x + 2$$

$$y = 2x - 5$$

represent **parallel** lines; both have a slope of 2. The second line is 7 units below the first for every value of x.



Perpendicular Lines

Two lines are perpendicular if and only if their slopes are negative reciprocals of each other.

ie the lines

$$y = m_1 x + c_1$$

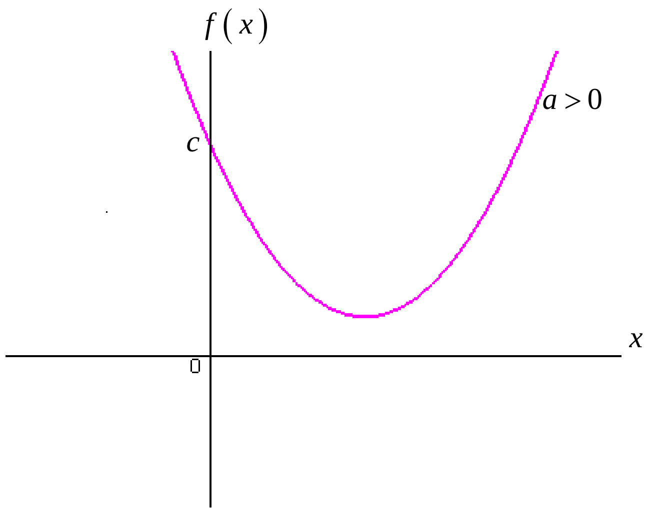
$$y = m_2 x + c_2$$

are perpendicular if and only if $m_1 = -\dfrac{1}{m_2}$ or $m_2 = -\dfrac{1}{m_1}$ or $m_1 m_2 = -1$.

6.3 Quadratic Functions

$f(x) = ax^2 + bx + c$ where *a*, *b* and *c* are constants

(i) $f(x) = ax^2 + bx + c$ with $a > 0$

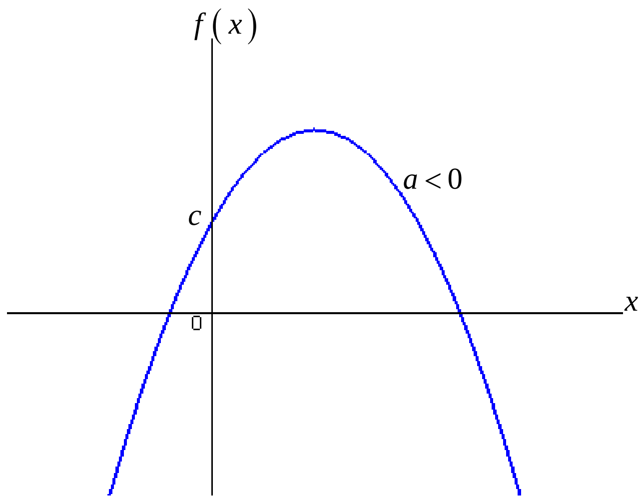


Characteristics:
When $a > 0$, the function yields a **U**-shaped graph.

Notice that the graph cuts the y-axis at c . This value can be found by substituting $x = 0$ into $f(x)$.

The graph turns at the point where $x = -\dfrac{b}{2a}$. This point can be easily found using the technique of differentiation, a topic which you will learn later.

(ii) $f(x) = ax^2 + bx + c$ with $a < 0$



Characteristics:
When $a < 0$, the function yields an **inverted U**-shaped graph.

6.4 Exponential Functions

An exponential function is of the form

$$f(x) = b^x$$

where the base, *b* is a constant ($b > 0$, $b \neq 1$)
and the power or the exponent, *x* is can be any real number.

The following are exponential functions:

$y = 4^x$ $y = 10^x$ $y = e^t$ $y = 7^{t-2}$ $y(t) = e^{-3t}$

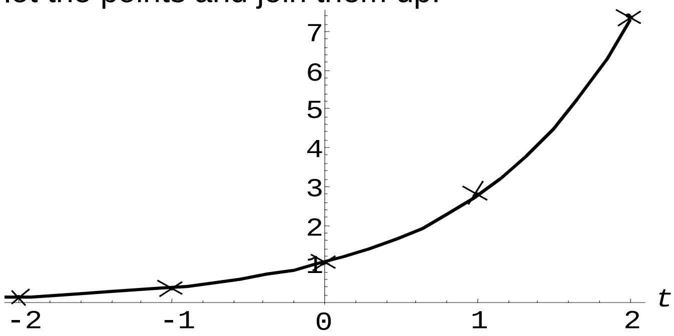
Notice the variable *t* can be used instead of *x*.
Note also that *e* is the natural number 2.718281828459045235360287471 3…

6.4.1 Plotting the Exponential Function

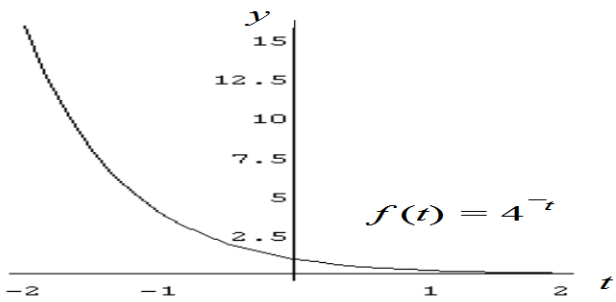
To plot $f(t) = e^t$, for $-2 < t < 2$ we construct a table :

t	-2	-1	0	1	2
e^t	0.135	0.368	1.000	2.718	7.389

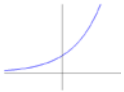
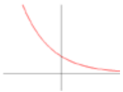
Plot the points and join them up:

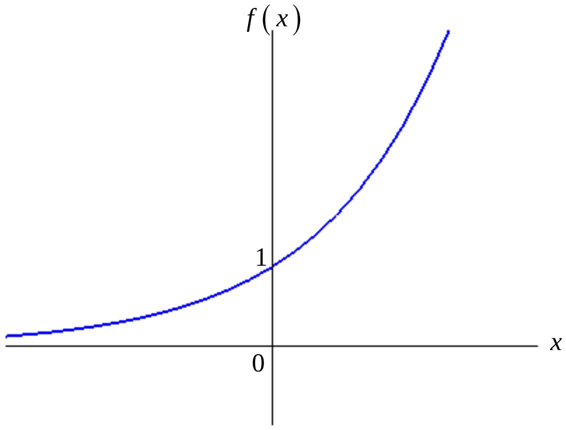


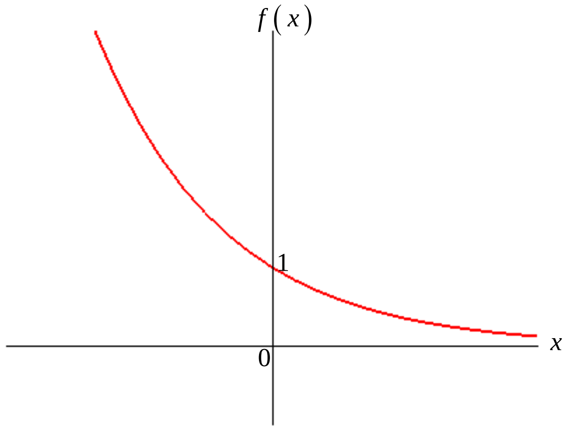
In a similar way we can plot $f(t) = 4^{-t}$.



All exponential functions have ‘more or less’ the same shape. Whether it curves this way

 or  depends on the sign of the exponent.





Characteristics:

k^{ax}

When the value of $k^a > 1$, the exponential graph is growing as x increases

When the value of $k^a < 1$, the exponential graph is decreasing as x increases

6.5 Logarithmic Functions

A logarithmic function is of the form

$$f(x) = \log_a x$$

where *a* (a positive constant) is the base of the logarithm and $x > 0$.

(The two commonly used bases are base 10 and base *e*, represented by log *x* and ln *x* respectively)

6.5.1 Natural Number *e* and Natural logarithm

In science and engineering the most frequently used base is the famous number
e = 2.71828182845904523536028747135 2662 ...

We say the natural logarithm of *x*, written ln *x*, is :
ln *x* = $\log_e x = c$ which means $e^c = x$

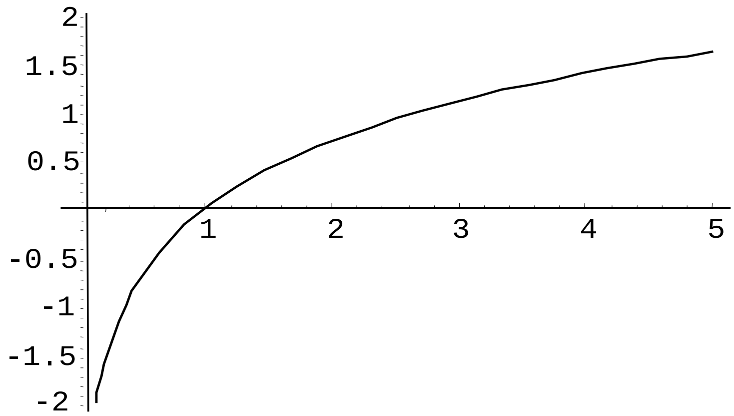
Note that $e^{\ln x} = x$ and $\ln e^x = x$

5.5.2 Plotting the Logarithmic Function

To plot *f* (*t*) = ln(*t*) for 0 < *t* < 5:

t	0	1	2	3	4	5
ln(*t*)	-∞	0	0.693	1.099	1.386	1.609

Plot the points and join them up:



Characteristics:
The basic logarithmic function cuts the $x-\text{axis}$ at zero.

The function has an asymptote and it is the line $y = 0$.

6.6 Trigonometric Functions

6.6.1 Measuring angles in radians

You are already familiar with angles measured in degrees. There is another unit of angle measurement called radian.

We know that 360^0 makes a complete round of a circle. In terms of radians, 2π radians make a complete round of a circle.

To convert degrees to radians and vice-versa, remember that

$$\boxed{180^{o} = \pi \text{ radians}}$$

which means $1° = \dfrac{\pi}{180}\text{rad}$ and $1 \text{ rad} = \dfrac{180°}{\pi}$.

Therefore to convert an angle from degree to radian, we multiply by $\dfrac{\pi}{180}$ and to convert an angle from radian to degree, we multiply by $\dfrac{180}{\pi}$.

Exercise 6.1

Express the following angles in radians: 36^0, 212^0.

Exercise 6.2

Express in degrees: $\dfrac{5\pi}{6}$, $\dfrac{\pi}{10}$.

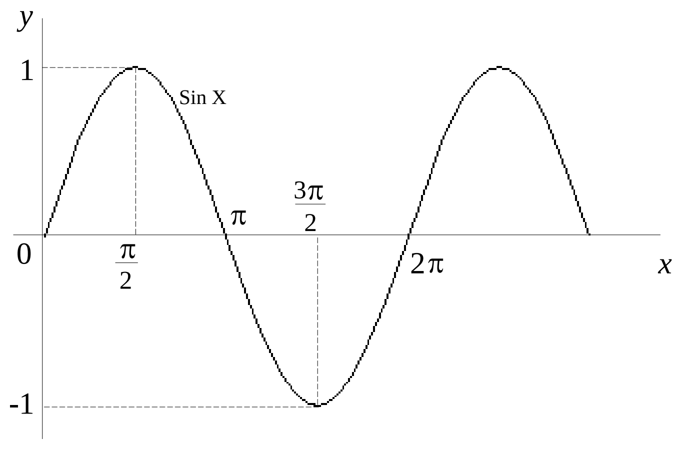
Remember the following values:

$$30^\circ = \frac{\pi}{6}\text{ rad}$$
$$45^\circ = \frac{\pi}{4}\text{ rad}$$
$$60^\circ = \frac{\pi}{3}\text{ rad}$$
$$90^\circ = \frac{\pi}{2}\text{ rad}$$
$$180^\circ = \pi\text{ rad}$$
$$270^\circ = \frac{3\pi}{2}\text{ rad}$$
$$360^\circ = 2\pi\text{ rad}$$

6.6.2 The Sine Function

One method of graphing the function $y = f(x) = \sin x$ is to tabulate values of *y* for different values of the angle *x* and then plot the resulting table of pairs of points.

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π	$5\pi/2$	3π
y	0	0.7	1	0.7	0	–0.7	–1	–0.7	0	1	0



The above graph shows two important aspects of the sine function:

1. The function is **periodic**, that is, the curve repeats itself at a regular interval. The **period** of the sine function is 2π, so the graph of the sine function looks exactly the same every 2π units. The graph of the function through one period is called a **cycle**.

The **frequency** of a periodic function is the number of cycles that will fit into one unit(degree, radian or second) along the *x*-axis. It is the reciprocal of the period:

$$\text{frequency} = \frac{1}{\text{period}}$$

2. The amplitude of the sine function is 1. The **amplitude** represents the maximum variation of the curve from the *x*-axis.

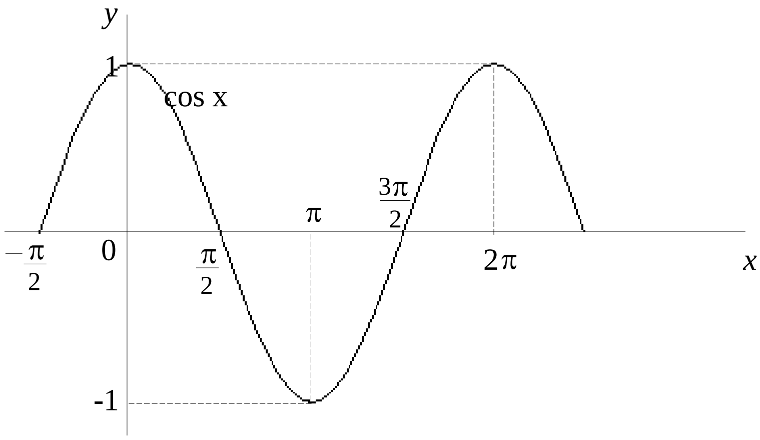
Other observations on the sine function include:

3. The graph crosses the *x*-axis at the initial point, endpoint and midpoint of a period or cycle.

4. The maximum occurs midway between the first half of the period and the minimum occurs midway between the second half of the period.

6.6.3 The Cosine Function

The $y = f(x) = \cos x$ cosine graph can be drawn in a similar way.



The graph of the cosine function is also periodic with period 2π and has amplitude 1. Other observations of the cosine function include:

1. The maximum occurs at the initial point and the endpoint of a cycle. A minimum occurs midway in the cycle.

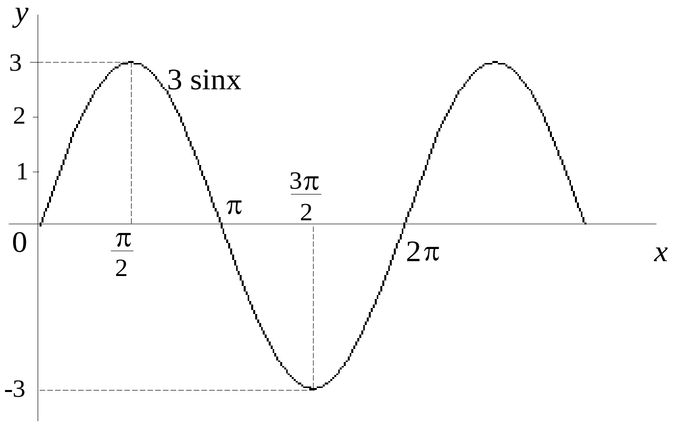
2. The cosine function crosses the *x*-axis at the midpoint of the first half of the cycle and at the midpoint of the second half of the cycle.

6.6.4 Graph of *y* = *a* sin *x* and *y* = *a* cos *x*

To examine the effect of a constant multiplier on the sine and cosine functions, graph $y = 3\sin x$ through one period with the help of the following table values:

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$y = 3\sin x$	0	2.1	3	2.1	0	–2.1	–3	–2.1	0

Observe that the 'shape' of the sine graph remains unchanged and that the amplitude is 'stretched' to 3.



We should see that multiplying the sine or cosine function by a constant changes only the amplitude. The **period** of the function **remains unchanged**.

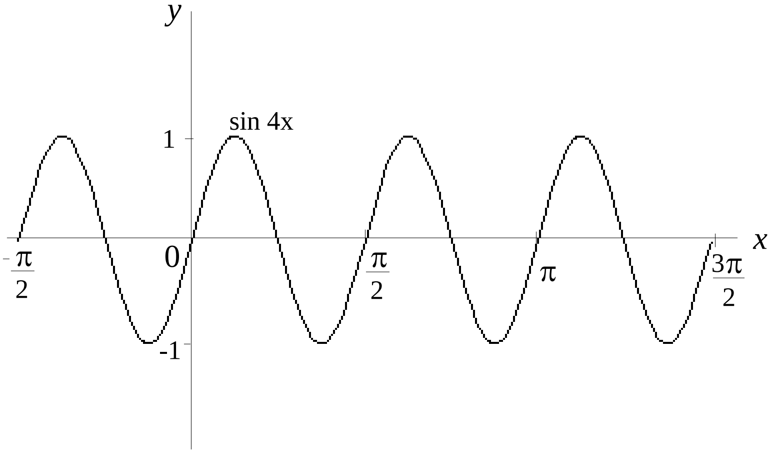
In general, the graphs of ***y* = *a* sin *x*** or ***y* = *a* cos *x*** will have a maximum value of '*a*' and a minimum value of '-*a*'.

<u>Exercise 6.3</u>

Sketch the graph of $y = 4\cos x$ through one period (without a table of values)

6.6.5 Graph of $y = \sin bx$ and $y = \cos bx$

Now let's graph $y = \sin 4x$.



The graph is interesting as it looks like a sine wave with an amplitude of 1, but it does not have a period of 2π. Instead its period is $\pi/2$. Note that the maximum and minimum still occurs at the endpoints and midpoints, respectively, of the period and these locations are at 0, $\pi/4$ and $\pi/2$ for the graph of $y = \sin 4x$.

For graphs of the form ***y* = *a* sin *bx*** or ***y* = *a* cos *bx***, the **period** is given by $\frac{2\pi}{b}$

<u>Exercise 6.4</u>

Sketch the graph of $y = \sin \frac{x}{2}$ through one period.

6.6.6 Graph of $y = a\sin bx$ and $y = a\cos bx$

From the previous discussions, we know the following:

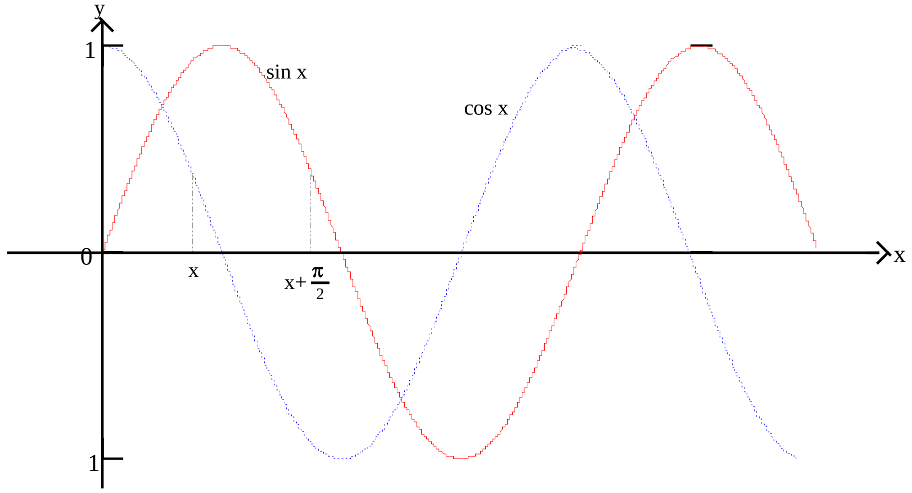
1. The amplitude is 'a'.
2. The period is $\frac{2\pi}{b}$.
3. A negative sign in front of the function inverts the graph of the function about the x-axis.
4. For $y = a \cos bx$, the maximum occurs at the endpoints of the period, which are at 0 and $2\pi/b$. The minimum occurs at the midpoint of the period, i.e. at π/b. The curve crosses the x-axis midway between each maximum and minimum.
5. For $y = a \sin bx$, the curve crosses the x-axis at the endpoints and midpoints of the period at 0, π/b, and $2\pi/b$. The maximum occurs midway between 0 and π/b whilst the minimum occurs midway between π/b and $2\pi/b$.

<u>Exercise 6.5</u>

Sketch the graph of $y = -3 \sin \frac{2}{3}x$ through one period.

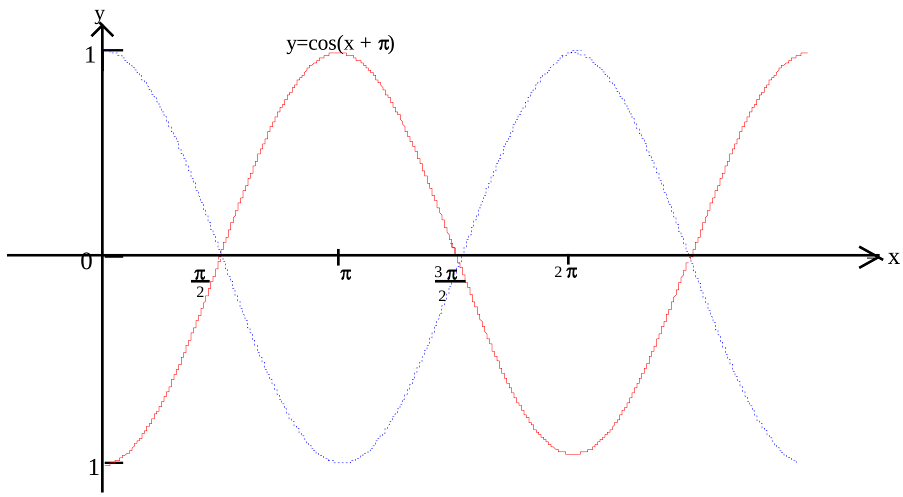
6.6.7 Graph of $y = \sin(x + c)$ and $y = \cos(x + c)$

The graphs of the *y* = sin *x* and *y* = cos *x* drawn on the same axes.



Note that the graphs would be identical if the graph of the sine function is shifted π/2 units to the left. For this reason, the cosine function is said to **lead** the sine function by π/2. This horizontal movement along the x-axis is called the **phase shift**.

The figure below shows the graph of *y* = cos(*x* + π) through one period. Note that the graph can be obtained by shifting the graph of *y* = cos *x* along the axis π units to the left.



The difference between the graphs of *y* = cos *x* and *y* = cos(*x* + *c*) is a horizontal shift of *c* units to the left or right depending on whether c is positive or negative. The angle *c* is called the **phase shift** or **phase angle.** Similar behaviour applies to *y* = sin(*x* + *c*).

Exercise 6.6

Sketch the graph of $y = \sin\left(x + \frac{\pi}{4}\right)$ and $y = \sin\left(x - \frac{\pi}{4}\right)$.

6.6.8 Graph of $y = a\sin(bx + c)$ and $y = a\cos(bx + c)$

The graph of $y = a \sin (bx + c)$ or $y = a \cos(bx + c)$ has the following characteristics:

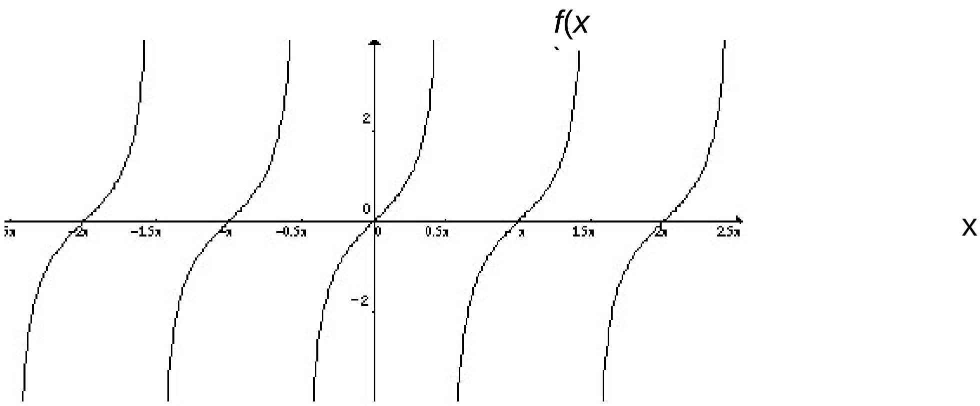
1. The amplitude is 'a'.
2. The period is $\dfrac{2\pi}{b}$
3. The phase shift is $\dfrac{c}{b}$. The shift is to the left if c is positive and to the right if c is negative.

Exercise 6.7

Sketch the graph of $y = -2\sin\left(2x + \dfrac{\pi}{2}\right)$ through one cycle.

6.6.9 The Tangent Function

$f(x) = \tan x$



Graph of y = tan (x)

Characteristics:

Similar to the sine and cosine functions, the tangent graph is periodic. But for the tangent function, the period is π instead of 2π.

Notice that each cycle is bounded horizontally by two asymptotes. The function actually tends to positive infinity at one end and negative infinity at the other.

6.7 Limits and Continuity

6.7.1 Introduction

The limit of a function $f(x)$ is the value which *f* approaches as *x* approaches a given value, *a*. We write this as :

$$\boxed{\lim_{x \to a} f(x) = L}$$

L is known as the limit of the function *f* as $x \to a$.

Note : The symbol "$\to$" denotes 'approaches'
When we say $x \to a$, it means *x* approaches a value close to *a* but *x* may not be equal to *a*.

Consider the function *f* (*x*) = 2*x* + 1. We will illustrate the finding of $\lim_{x \to 0} f(x)$.
Let's examine the behavior of *f* for values of *x* that are close to 0.

x	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
f (x)	0.8	0.98	0.998	0.9998		1.0002	1.002	1.02	1.2

We see that $f(x)$ approaches 1 as *x* approaches 0.

Hence $\lim_{x \to 0} f(x) = 1$.

We note also that *f*(0) = 2(0) + 1 = 1. i.e. $\lim_{x \to 0} f(x) = f(0)$.

In general, for function of this type, $\boxed{\lim_{x \to a} f(x) = f(a)}$

Exercise 6.8

Evaluate the following:

(a) $\lim_{x \to 0} 6x^3$ (b) $\lim_{x \to 10} (2x - 1)$

6.7.2 Algebraic Properties of Limits

$$\lim_{x\to c}\left[f(x)\pm g(x)\right]=\lim_{x\to c}f(x)\pm\lim_{x\to c}g(x)$$

$\lim_{x\to c}\left[kf(x)\right]=k\lim_{x\to c}f(x)$ for any constant *k*

$$\lim_{x\to c}\left[f(x)g(x)\right]=\left(\lim_{x\to c}f(x)\right)\left(\lim_{x\to c}g(x)\right)$$

$\lim_{x\to c}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}$ if $\lim_{x\to c}g(x)\neq 0$

$\lim_{x\to c}\left[f(x)\right]^p=\left[\lim_{x\to c}f(x)\right]^p$ if $\left[\lim_{x\to c}f(x)\right]^p$ exists

Limits of Two Linear Functions

$\lim_{x\to c}k=k$ $\lim_{x\to c}x=c$

where *k* is a constant

<u>Exercise 6.9</u>

(a) $\lim_{x\to -1}(3x^2-4x+8)=$

(b) $\lim_{x\to 1}\dfrac{4x^2+x-9}{x-3}=$

6.7.3 Limits for discontinuous functions

Let's look at how $\lim_{x\to 3}\dfrac{x^2-9}{x-3}$ is evaluated. Can we use $\lim_{x\to a}f\left(x\right)=f\left(a\right)$?

Let $f(x)=\dfrac{x^2-9}{x-3}$

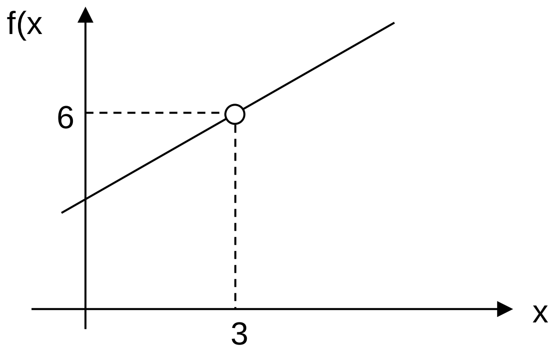
$f(3)=\dfrac{3^2-9}{3-3}=\dfrac{0}{0}$ which is indeterminate.

We see that in this case the limit cannot be determined using $f(3)$.

Let's look at how the limit is evaluated:

we observe that $x^2-9=(x-3)(x+3)$, so $f(x)=\dfrac{(x-3)(x+3)}{x-3}$, which can be simplified to *f*(*x*) = (*x* + 3), so $\lim_{x\to3}\dfrac{x^2-9}{x-3}=\lim_{x\to3}(x+3)=6$.

The graph of $f(x)=\dfrac{x^2-9}{x-3}$ is a line with a "hole" at $x=3$. $f(x)=\dfrac{x^2-9}{x-3}$ is discontinuous at $x=3$. From the graph, we see that as *x* approaches 3, $\dfrac{x^2-9}{x-3}$ approaches 6. ($x\to3$, $\dfrac{x^2-9}{x-3}\to6$). It is important to note that limits describe the behavior of a function near a particular point, not necessarily at the point itself.



Exercise 6.10

Evaluate the following:

(a) $\lim_{x\to2}\dfrac{x^2-4}{x-2}$

(b) $\lim_{x\to1}\dfrac{x^2-1}{x^2-3x+2}=$

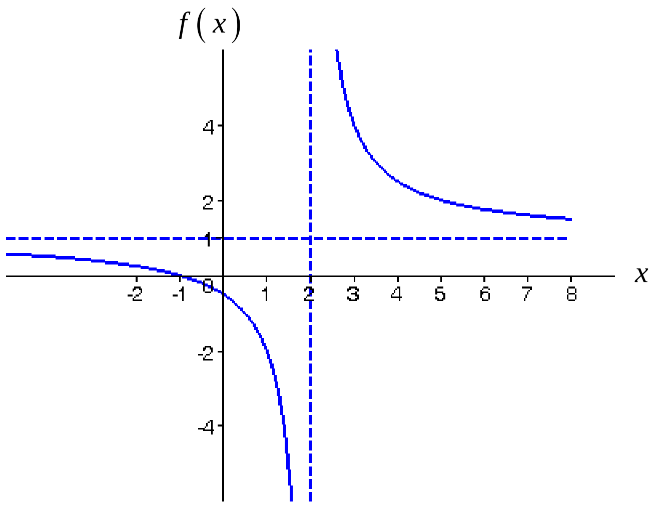
Now, let's look at another case, $\lim_{x \to 2} \frac{x+1}{x-2}$. Can we use $\lim_{x \to a} f(x) = f(a)$?

Let $f(x) = \frac{x+1}{x-2}$

$f(2) = \frac{3}{0}$ which is indeterminate.

In this case, the denominator approaches 0 and the numerator does not. When this happens, you can conclude that the **limit doest not exist**.

We can see this clearly from the graph of $f(x) = \frac{x+1}{x-2}$



<u>Exercise 6.11</u>

$\lim_{x \to 3} \frac{x+2}{x-3}$