Course	:	Diploma in Electronic Systems Diploma in Telematics & Media Technology Diploma in Aerospace Systems & Management Diploma in Electrical Engineering with Eco-Design Diploma in Mechatronics Engineering Diploma in Digital & Precision Engineering Diploma in Aeronautical & Aerospace Technology Diploma in Biomedical Engineering Diploma in Nanotechnology & Materials Science Diploma in Engineering with Business Diploma in Information Technology Diploma in Financial Informatics Diploma in Cybersecurity & Forensics Diploma in Infocomm & Security Diploma in Chemical & Pharmaceutical Technology Diploma in Biologics & Process Technology Diploma in Chemical & Green Technology		
Module	:	Engineering Mathematics 2B / Mathematics 2B/ Computing Mathematics 2	–	EG1761/2008/2681/2916/2961 EGB/D/F/H/J/M207 IT1201/1531/1631/1761 CLB/C/G201

Topic 6 : Binomial and Poisson Distributions

Objectives :

At the end of this lesson, the student should be able to:

1 list the conditions of a binomial experiment

2 explain the binomial probability function $P(x) = {}^nC_x p^x q^{n-x}$

3 calculate the mean, variance and standard deviation of a binomial distribution

4 use the binomial probability distribution in problem solving

5 define Poisson random variable

6 list the conditions of the Poisson distribution

7 explain the Poisson probability function $P(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

8 calculate the mean and variance of a Poisson distribution

9 use the Poisson probability distribution in problem solving

Topic 6: Binomial and Poisson Distributions

6.1 Binomial Distribution

- Suppose we throw a die ten times and we want to create a probability distribution for the number of times "6" appear. It is reasonable to make the following assumptions:

  (a) Each trial has only two outcomes: success or failure (a "6" or not a "6").

  (b) The outcome of each trial is independent of other trials.
  (The next throw is not affected by previous throws.)

  (c) The probability of success, p, for each trial is the same
  (e.g. $p = P(\text{get a "6"}) = \frac{1}{6}$).

- In general if we conduct an experiment for n trials and the experiment satisfies the conditions (a) to (c), then we can model the number of successes using a **<u>Binomial Distribution</u>**.

- Notation**:**
  Let X be denote the number of successes in a Binomial experiment with n trials and p = probability of success for each trial. Then

$$X \sim B(n, p)$$

$$P(X = k) = {}_nC_k\, p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n$$

****** Note that X is a **<u>discrete</u>** random variable.**

Example 6.1-1

A certain surgical procedure has an 85 % chance of success. A doctor performs the procedure on eight patients. The random variable X represents the number of successful surgeries. State the distribution of X .

Solution:

Let X be the number of successful surgeries out of 8 patients. $n = 8$, $p = 0.85$

$$X \sim$$

Example 6.1-2

A jar contains 5 red marbles, 9 blue marbles, and 6 green marbles. You randomly select 3 marbles from the jar, without replacement. The random variable X represents the number of red marbles. Explain whether X is a Binomial random variable.

Solution:

P (first marble is red) =

Since the probability of obtaining a red marble for each draw is

____

X is NOT a Binomial random variable.

Example 6.1-3

Microfracture knee surgery has a 75 % chance of success on patients with degenerative knees. The surgery is performed on three patients. Find the probability of the surgery being successful on exactly two patients.

Solution:

Let X be the number of successful surgery out of 3 patients.

$$X \sim B(\qquad , \qquad)$$

$P(X = 2) =$

Example 6.1-4

Childhood asthma is a public health problem in country A. It is known that one out of 10 children in country A has asthmatic problems. In a randomly chosen group of 14 children from the population, what is the probability that

(a) 3 has asthmatic problems?

(b) 1 or less has asthmatic problems?

(c) more than 1 has asthmatic problems?

Solution:

Let X be the number of children with asthmatic problems out of 14 children.

$$X \sim B(\qquad , \qquad)$$

(a) $P(X = 3) =$

(b) $P(X \leq 1) = P(X = 0) + P(X = 1) =$

(c) $P(X > 1) = 1 - P(X \leq 1) =$

6.2 Mean and Variance of a Binomial Distribution

- For a Binomial random variable, $X \sim B(n, p)$

Expectation or population mean, $\mu = np$

Population Variance, $\sigma^2 = np(1-p)$

Example 6.2-1

5 % of workers at construction sites are known to suffer from hearing impaired problem due to the unhealthy noise level. If we randomly select 28 workers from construction sites, find

(a) the probability that exactly 4 of them suffer from hearing impaired problem ,

(b) the mean and standard deviation of the number of workers suffering from hearing impaired problem.

Solution:

Let X be the number of workers with hearing impaired problems out of 28 workers.

$$X \sim B(\qquad , \qquad)$$

(a) $P(X = 4)$ =

(b) $E(X) =$

$Var(X), \quad \sigma^2 =$

Standard deviation, σ =

Example 6.2-2

The random variable X which follows a Binomial distribution is such that the mean is 2 and variance is $\frac{24}{13}$. Find the values of *n* and *p*.

Solution:

$$X \sim B(\qquad , \qquad)$$

$E(X) = np = 2$ --- (1) $\qquad$ $Var(X) = np(1-p) = \frac{24}{13}$ --- (2)

6.3 Poisson Distribution

- Suppose in a country it is known that a cyclone will arrive at a rate of 1.5 times every 2 years. We want to create a probability distribution on the number of times a cyclone arrives in a specific time period. It is reasonable to make the following assumptions:

  (a) The **mean rate** of events, μ, occurring in an unit interval / region is the **same** for every other unit interval / region.
  (E.g. Mean rate of cyclones arriving is the same across any interval of 2 years.).

  (b) Events occurring in an interval / region are **independent** of events occurring in other non-overlapping intervals/ regions.
  (E.g. The number of cyclones in year 2013 to 2014 is independent of the number of cyclones in year 2011 to 2012.)

  (c) No two events can occur at the same time.
  (E.g. we assume that no two cyclones can happen together.)

- In general if we are counting the number of events occurring in an interval / region and conditions (a) to (c) are satisfied, we can model the number of events occurring using a **Poisson Distribution**.

- Notation**:**
  Let X be denote the number of events occurring in an interval / region with a mean rate μ. Then

$$X \sim Po(\mu)$$

$$P(X = k) = e^{-\mu}\frac{\mu^k}{k!}, \quad k = 0,1,2,\ldots$$

** Note that X is a **discrete** random variable.**

Example 6.3-1

The mean number of accidents per month at a certain intersection is 3. What is the probability that in any given month,

(a) 4 accidents will occur at this intersection?

(b) more than 1 accidents will occur at this intersection?

Solution:

Let X be the number of accidents **<u>per month</u>**.

$$X \sim Po(\ 3\ )$$

(a) $P(X=4)=e^{-3}\dfrac{3^4}{4!}=$

(b) $P(X>1)=1-P(X=0)-P(X=1)=$

Example 6.3-2

2000 brown trout are introduced into a small lake. The lake has a volume of 20000 cubic meters. Find the probability that

(a) 3 brown trout are found on any given cubic meter of the lake.

(b) less than 2 brown trout are found on any 10 cubic meters of the lake.

Solution:

Let X be the number of trouts **<u>per cubic meter</u>** of lake.

$$X \sim Po(\qquad)$$

(a) $P(X=3)=$

(b) Let Y be the number of trouts **<u>per 10 cubic meters</u>** of lake.

$$Y \sim Po(\qquad)$$

$$P(Y<2)=P(Y=0)+P(Y=1)=$$

6.4 Mean and Variance of Poisson Distribution

- For a Poisson random variable, $X \sim P_o(\mu)$

Expectation or population mean $= \mu$
Population Variance, $\sigma^2 = \mu$

Example 6.4-1

A school "Lost and Found" department receives an average of 3.7 reports per week of lost student ID cards.

(a) Find the probability that at most 2 such reports will be received during a given week by this department.

(b) Find the probability that there will be 1 to 3 (inclusive) such reports received during a given week by this department.

(c) Find the variance and standard deviation of the probability distribution.

Solution:

Let X be the number of reports **per week**.

$$X \sim Po(\quad\quad)$$

(a) $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) =$

(b) $P(1 \le X \le 3) = P(X = 1) + P(X = 2) + P(X = 3) =$

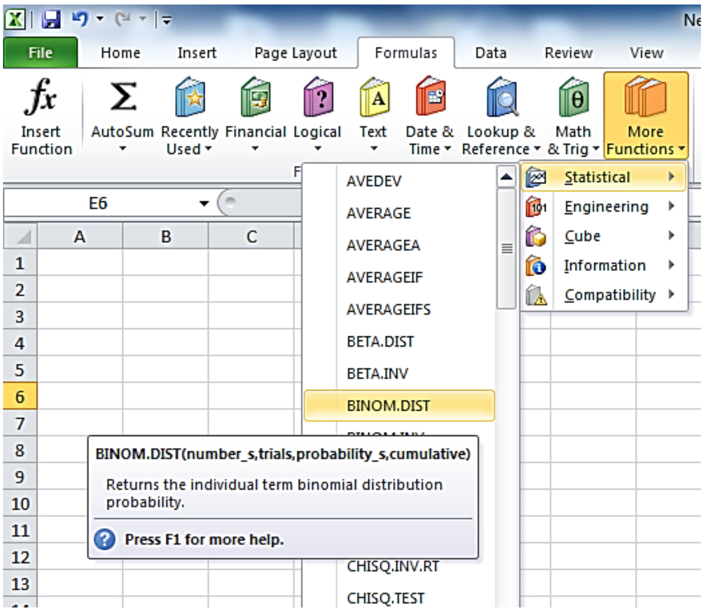
(c) $E(X) = Var(X) =$

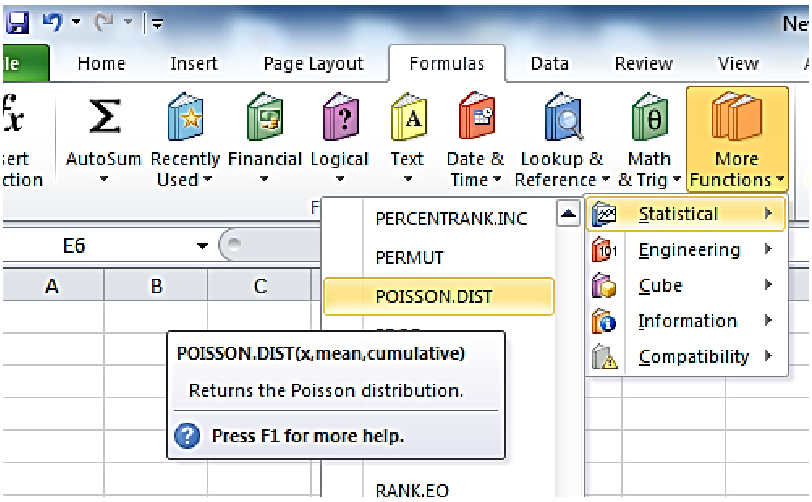
Appendix Binomial and Poisson Distributions using Excel

- Under the tab “Formulas” → “More Functions” → “Statistical” there are 2 options to calculate probabilities for Binomial and Poisson distributions.

  (a) BINOM.DIST: For Binomial distribution.

  (b) POISSON.DIST: For Poisson distribution.

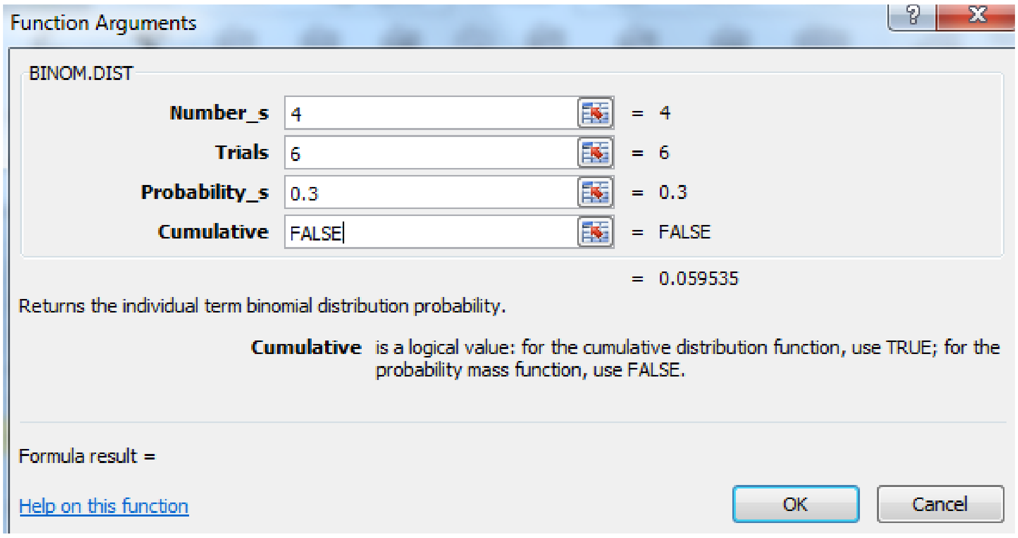


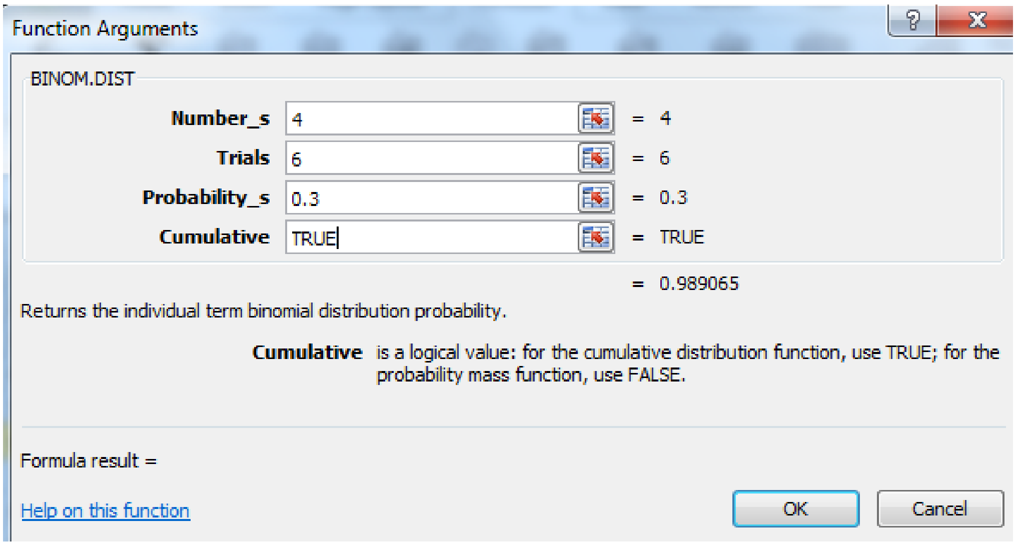


- To compute $P(X=4)$ and $P(X \leq 4)$, given that $X \sim B(6, 0.3)$. Select BINOM.DIST

  For $P(X=4)$: Key "FALSE" under the "CUMULATIVE" option.

  For $P(X \leq 4)$: Key "TRUE" under the "CUMULATIVE" option.





$P(X=4)=0.0595$, $P(X \leq 4)=0.989$

- To compute $P(X=4)$ and $P(X \leq 4)$, given that $X \sim Po(1.8)$. Select POISSON.DIST Under the option "MEAN", enter 1.8. The rest are the same as the Binomial distribution shown above.

Tutorial 6: Binomial and Poisson Distributions

A Self Practice Questions

A.1 Binomial Distribution

1 Given that $X \sim B(10, 0.3)$, calculate the following:

(i) mean, μ (ii) variance, σ^2

(iii) $P(X = 4)$ (iv) $P(X \le 2)$

(v) $P(X < 2)$ (vi) $P(X \ge 2)$

(vii) $P(X > 8)$

2 If $X \sim B\left(n, \frac{4}{5}\right)$ and $P\left(X = 0\right) = \frac{1}{15625}$,

(a) How many outcomes are there in each trial?

(b) How many trials are there?

(c) How many possible values that X can take.

(d) Find the mean and standard deviation of this distribution.

A.2 Poisson Distribution

3 Given that $X \sim Po(3)$, calculate the following:

(i) mean, μ (ii) variance, σ^2

(iii) $P(X = 4)$ (iv) $P(X \le 2)$

(v) $P(X \ge 3)$

4 The number of calls arriving, X, is Poisson distributed with a rate of 2 per hour. Write the distribution of the number of calls arriving in

(i) 3 hours, (ii) 45 minutes.

B Discussion Questions

1 10% of drivers do not wear seat-belts. Find the probability that, in the next 10 cars to pass, less than 2 drivers will not be wearing seat-belts.

2 A telephone enquiry service is so busy that only 80% of calls to it are successfully connected. It may be assumed that all calls are independent. Twelve calls are made at random to the service. Find the probability that at least 10 are successfully connected.

3 The probability of a patient recovering from a heart operation is 0.85. In a particular hospital, 10 patients went through such an operation in a particular month. What is the probability that

(i) exactly 4 survive the operation?
(ii) the actual number of survivors is more than the expected value?
(iii) exactly 2 do not survive the operation?

4 Coach *A* has four wheels and equipped with two spare tires, and coach *B* has six wheels and equipped with three spare tires. These coaches travel from town A to town B independently. The probability that a tire needed to be replaced during the journey is 0.1.

(i) State an assumption required for the Binomial distribution to be a suitable model.

(ii) Determine whether coach *A* or coach *B* has the higher probability for a successful journey.

5 On average, a household receives 1.8 junk mails per day. Using the Poisson formula, find the probability that a randomly selected household receives

(a) exactly 3 junk mails on a certain day,

(b) at most 2 junk mails on a certain day.

6 A budget airline receives an average of 9.7 complaints per day from its passengers. Using the Poisson formula, find the probability that on a certain day this airline will receive

(a) exactly 5 complaints.

(b) at least 3 complaints.

7 A customer service department receives an average of 1.6 telephone calls in any 10-minute interval. Find the probability that the department receives

(a) no calls in any 10-minute interval.

(b) at most 1 calls in any 5-minute interval.

(c) more than 2 calls in any 15-minute interval.

Answers

A1	i	3,	ii	2.1,	iii	0.200,	iv	0.383,	v	0.149	
	vi	0.851	vii	0.000144							
A 2	a	2	b	6	c	7	d	4.8, 0.980			
A 3	i	3	ii	3	iii	0.168	iv	0.423	v	0.577	
A 4	i	$X \sim Po(6)$	ii	$X \sim Po\left(\frac{6}{4}\right)$							
B 1	0.736	**B2**	0.558								
B 3	i	0.00125	ii	0.544	iii	0.276					
B 4	i	independent	ii	Coach *B*							
B 5	a	0.161	b	0.731							
B 6	a	0.0439	b	0.996							
B 7	a	0.202	b	0.809	c	0.430					