Course	:	Diploma in Electronic Systems Diploma in Telematics & Media Technology Diploma in Aerospace Systems & Management Diploma in Electrical Engineering with Eco-Design Diploma in Mechatronics Engineering Diploma in Digital & Precision Engineering Diploma in Aeronautical & Aerospace Technology Diploma in Biomedical Engineering Diploma in Nanotechnology & Materials Science Diploma in Engineering with Business Diploma in Information Technology Diploma in Financial Informatics Diploma in Cybersecurity & Forensics Diploma in Infocomm & Security Diploma in Chemical & Pharmaceutical Technology Diploma in Biologics & Process Technology Diploma in Chemical & Green Technology

Module	:	Engineering Mathematics 2B / Mathematics 2B/ Computing Mathematics 2	–	EG1761/2008/2681/2916/2961 EGB/D/F/H/J/M207 IT1201/1531/1631/1761 CLB/C/G201

Topic 7 : Normal Distribution

Objectives :

At the end of this lesson, the student should be able to:

1 describe the characteristics of a normal distribution including its shape and the relationship among its mean, median and mode

2 define normal random variable and standard normal random variable

3 compute normal probabilities using standard normal tables

4 use the normal probability distribution to approximate the binomial probabilities (including correction for continuity)

Topic 7: Normal Distribution

7.1 Introduction

- Many **<u>continuous</u>** random variables can be modelled using the **<u>normal distribution</u>**. Examples include:
  (a) Students' examination scores,
  (b) Height and weight of people.

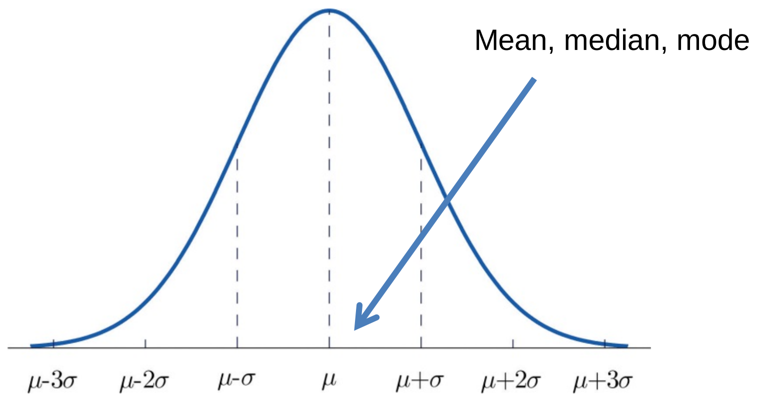
- The normal distribution can be described using two features: **<u>mean</u>** μ and **<u>variance</u>** σ^2 . Notation: $X \sim N(\mu, \sigma^2)$.

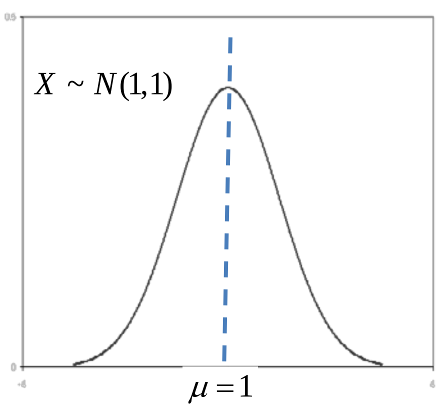
- The normal distribution can be represented using a **<u>bell – shaped</u>** curve with the properties:

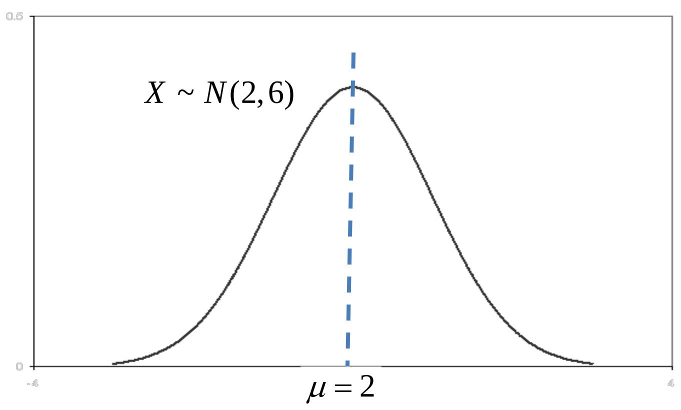
o The curve is symmetrical about the mean.

o The mean, median, mode are the same.

  o Approximately 95% of the distribution lies within 2 standard deviations of the mean. This is sometimes known as the '2σ rule'.

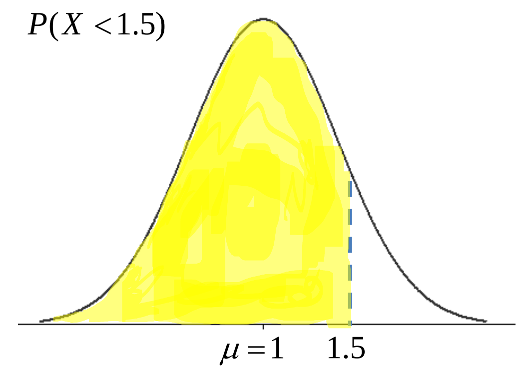


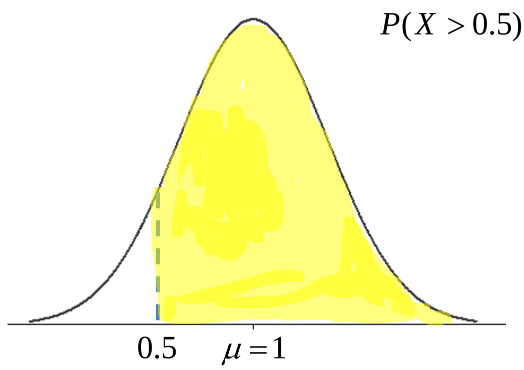


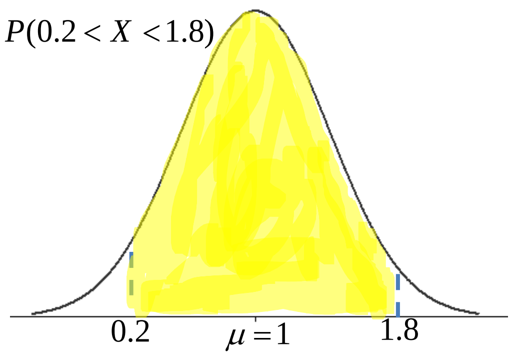


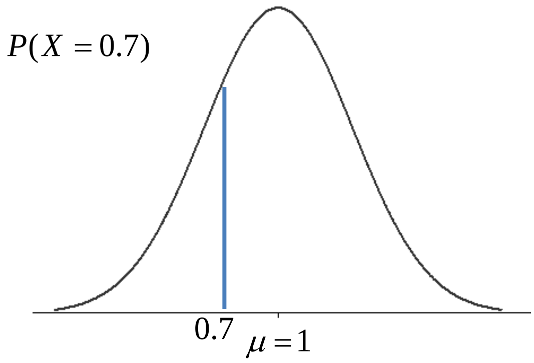
7.2 Probability for normal distribution

- For normal distribution, the probability is interpreted as **area under the curve**.
  For example $X \sim N(1,2)$:









Note that $P(X = k) = 0$. Hence for normal distribution, $P(X \le k) = P(X < k)$.

7.3 Standard normal random variable

- A normal random variable can have many different values of mean and variance. When the **mean is 0 and variance is 1**, we call it a **standard normal random variable, denoted as Z.**

$$Z \sim N(0,1)$$

- To convert from $X \sim N(\mu, \sigma^2)$ to $Z \sim N(0,1)$, we apply the formula:

$$Z = \frac{X - \mu}{\sigma}$$

  This procedure is also known as standardization.

Example 7.3-1

Given that $X \sim N(2,5)$, rewrite the following probabilities in the form $P(Z \le k)$.

(a) $P(X \le 3)$, (b) $P(X \ge 1.5)$, (c) $P(1.5 < X < 3)$

Solution:

(a) $P(X \le 3) = P\left(\frac{X-2}{\sqrt{5}} \le \frac{3-2}{\sqrt{5}}\right) = P\left(Z \le 0.45\right)$

(b) $P(X \ge 1.5) =$

(c) $P(1.5 < X < 3) = P(X < 3) - P(X \le 1.5) =$

7.4 Standard Normal Table

- To calculate probabilities involving normal distribution, we will obtain the probability value via the **standard normal table** (on pages 112 and 113).

  Step 1: Apply standardization from $X \sim N(\mu, \sigma^2)$ to $Z \sim N(0,1)$.

  Step 2: Ensure the probability is expressed in the form $P(Z \le k)$.

  Step 3: Obtain the required probabilities' value from the standard normal table.

- The following example illustrates how the standard normal table is to be read:

  (a) Suppose we want to find $P(Z \le 0.52)$:

2nd decimal place

z	.00	.01	.02	.03
0.0	.5000	.5040	.5080	.5120
0.1	.5398	.5438	.5478	.5517
0.2	.5793	.5832	.5871	.5910
0.3	.6179	.6217	.6255	.6293
0.4	.6554	.6591	.6628	.6664
0.5	.6915	.6950	.6985	.7019

1st decimal place

Probability value

$P(Z \le 0.52) = 0.6985$

(b) Suppose we want to find the value of k such that $P(Z \le k) = 0.0020$:

2nd decimal place

z	.09	.08	.07	.06
−3.4	.0002	.0003	.0003	.0003
−3.3	.0003	.0004	.0004	.0004
−3.2	.0005	.0005	.0005	.0006
−3.1	.0007	.0007	.0008	.0008
−3.0	.0010	.0010	.0011	.0011
−2.9	.0014	.0014	.0015	.0015
−2.8	.0019	.0020	.0021	.0021

1st decimal place

Probability value

$k = -2.88$

Example 7.4-1

Let $Z \sim N(0,1)$. Use the standard normal table on pages 112 and 113 to evaluate the following probabilities:

(a) $P(Z < -0.99)$, (b) $P(Z > 1.06)$, (c) $P(-1.5 < Z < -1.25)$

Solution:

(a)

(b)

(c)

Example 7.4-2

Given the normally distributed variable X with mean 20 and standard deviation 4, find

(a) $P(X > 28)$

(b) $P(17.5 < X < 22.5)$

(c) the value of k such that $P(X > k) = 0.1539$

Solution:

(a) Step 1: Switch the inequality sign to $"<"$ or $"\leq"$

$$P(X > 28) = 1 - P\left(X \leq 28\right)$$

Step 2: Convert to standard normal random variable, Z

$$P(X > 28) = 1 - P\left(X \leq 28\right) = 1 - P\left(\frac{X-20}{4} \leq \frac{28-20}{4}\right) = 1 - P(Z \leq 2)$$

Step 3: Obtain the probability value from standard normal table

$$P(X > 28) = 1 - P\left(X \leq 28\right) = 1 - P\left(\frac{X-20}{4} \leq \frac{28-20}{4}\right) = 1 - P(Z \leq 2)$$

$$= 1 - 0.9772 = 0.0228$$

(b) $P(17.5 < X < 22.5) = P(X < 22.5) - P(X \leq 17.5) =$

(c) $P(X > k) = 0.1539 \Rightarrow 1 - P(X \leq k) = 0.1539 \Rightarrow P(X \leq k) = 0.8461$

$$\Rightarrow P\left(Z \leq \frac{k-20}{4}\right) = 0.8461$$

From standard normal table,

$$\frac{k-20}{4} = 1.02 \Rightarrow k = 24.08$$

Example 7.4-3

The serum cholesterol levels of a certain population of 40-year-olds male adults follow approximately a normal distribution with mean 185 mg/dl and standard deviation 36 mg/dl. If a 40-year-old male adult is chosen at random from this population, what is the probability that he has serum cholesterol level

(a) greater than 195 mg/dl ?

(b) less than 178 mg/dl ?

(c) between 178 and 195 mg/dl ?

Solution:

Let X be the cholesterol levels of a 40 year old male

$$X \sim N(185, 36^2)$$

(a) $P(X > 195) = 1 - P\left(X \le 195\right) = 1 - P\left(Z \le \frac{195-185}{36}\right)$

$=$

(b) $P(X < 178) = P\left(Z \le \frac{178-185}{36}\right)$

$=$

(c) $P(178 \le X \le 195) = P(X \le 195) - P(X < 178) =$

Example 7.4-4

The weights of a certain batch of obese male recruits are approximately normally distributed with mean 88 kg and standard deviation 9. The lightest 15% of the recruits receive a classification of A whilst the heaviest 12.5% receive a classification of F. Find

(i) the minimum weight required to obtain a classification of F,

(ii) the weight of the heaviest recruit in classification A.

Solution:

Let X be the weight of a obese male recruit.

$$X \sim N(88, 9^2)$$

(i) Let m be the minimum weight to be in classification F.

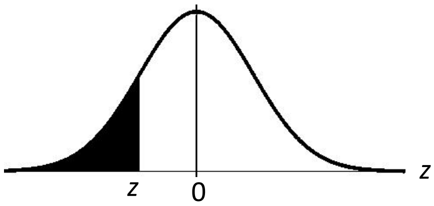
$P(X \geq m) = 0.125 \Rightarrow P(X < m) = 0.875$

$\Rightarrow$

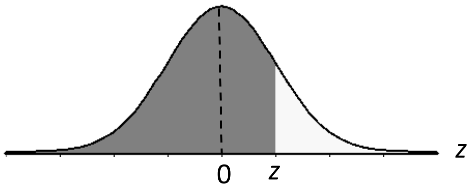
(ii) Let k be the largest weight to be in classification A.

$P(X \leq k) = 0.15 \Rightarrow$

Standard Normal Table



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Appendix A Normal approximation to Binomial

- Suppose $X \sim B(100, 0.45)$ and we wish to calculate $P(X \le 60)$. If we use the Binomial distribution then we have to calculate $P(X = k), \quad k = 0,1,2,\ldots,60$ before adding up the probabilities. This is quite tedious and hence we may want to use some approximation methods instead.

- Given $X \sim B(n, p)$, if $n > 30, \quad np > 5, \quad n(1-p) > 5$ then

$$X \sim N(np, np(1-p))$$ **approximately**.

- Since Binomial random variable is discrete but a normal random variable is continuous, we will need to do some adjustment to the calculation of the probabilities known as **continuity correction**.

  Step 1: Rewrite the probabilities into the form $P(X \le k)$, when $X \sim B(n, p)$.

  Step 2: Using the approximate normal distribution, calculate the probability when we **add** 0.5 to k, i.e. using $X \sim N(np, np(1-p))$, calculate $P(X \le k + 0.5)$.

Example A-1

A Binomial random variable is given by $X \sim B(50, 0.45)$.

(a) State reasons why normal distribution can be used as an approximation.

(b) Find

(i) $P(X \le 14)$

(ii) $P(X > 26)$

(iii) $P(15 \le X \le 26)$

Solution:

(a) Since $n = 50$ is large, $np = 50*0.45 = 22.5 > 5$, $n\left(1-p\right) = 50*0.55 = 27.5 > 5$,

X can be approximated using normal distribution.

(bi) mean of $X = np = 22.5$, variance of $X = np(1 - p) = 12.375$

$$X \sim N\left(22.5, 12.375\right) \textbf{ approximately}$$

$$P\left(X \le 14\right) \overset{c.c}{\approx} P\left(X < 14.5\right) = P\left(Z < \frac{14.5-22.5}{\sqrt{12.375}}\right) =$$

(bii) $P\left(X > 26\right) = 1 - P\left(X \le 26\right) \overset{c.c}{\approx} 1 - P\left(X < 26.5\right) =$

(biii) $P\left(15 \le X \le 26\right) = P(X \le 26) - P(X \le 14) =$

Ans: (bi) 0.0116, (bii) 0.1271, (biii) 0.01155

Example A-2

Sing-Chip produces computer chips. On average, 2% of all computer chips produced are defective. In a sample of 500 chips, the quality-control inspector accepts the batch if less than 1% of the chips tested are defective.

(i) Explain why the number of defective computer chips, X can be approximated by a normal distribution. Hence determine the mean and standard deviation of X

(ii) Use the normal approximation of X, find the probability that a batch is accepted.

Solution:

(i) Let X be the number of defective chips.

$$X \sim B(500, 0.02)$$

Since $n = 50$ is large, $np = 500*0.02 = 10 > 5$, $n(1-p) = 500*0.98 = 490 > 5$,

X can be approximated using normal distribution.

Mean = $np = 500*0.02 = 10$, variance = $np(1-p) = 9.8 \Rightarrow$

standard deviation = $\sqrt{9.8}$

(ii) $X \sim N(10, 9.8)$ approx.

Batch is accepted if there are less than $1\% * 500 = 5$ defects.

$P(X < 5) = P(X \le 4) \overset{c.c}{\approx} P(X < 4.5)$ =

Ans: (i) $\mu = 10,\quad \sigma = 3.13$, (ii) 0.0392

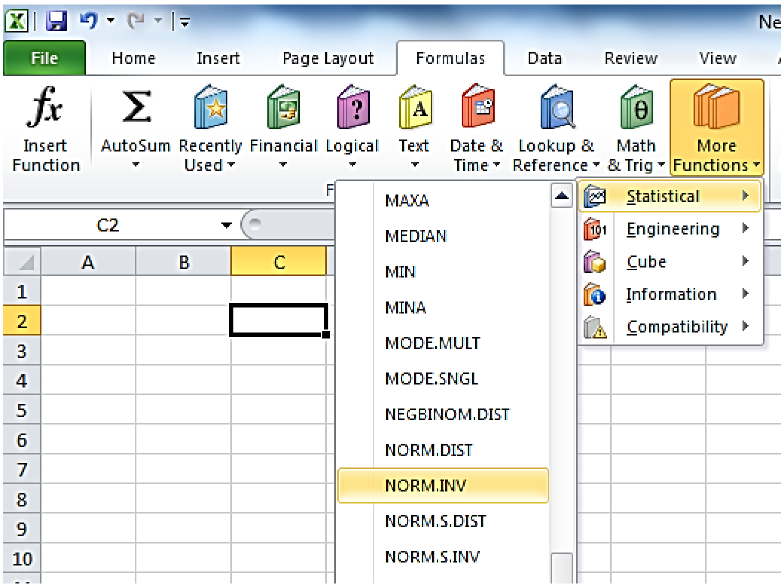
Appendix B Normal Distribution using Excel

- Under the tab "Formulas" → "More Functions" → "Statistical" there are 2 options related to normal distribution.

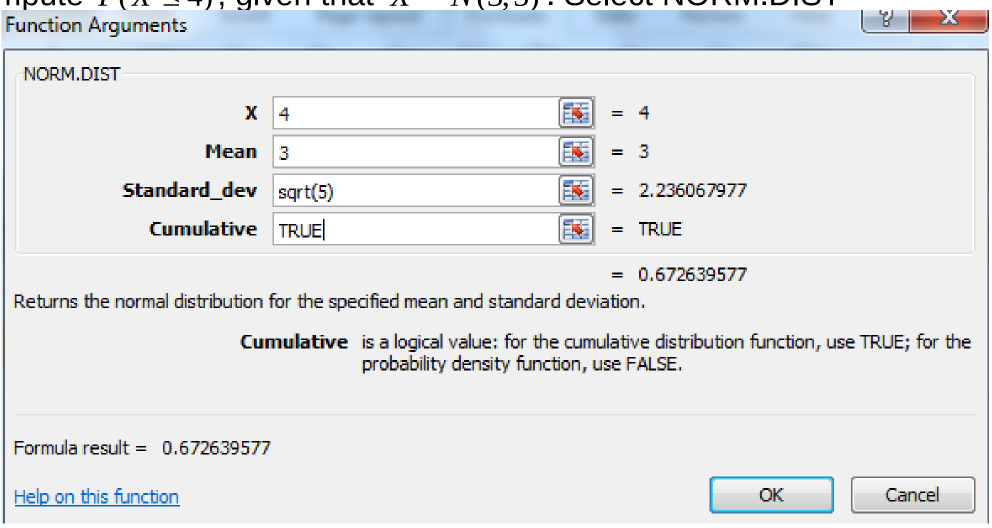
  When $X \sim N(\mu, \sigma^2)$:

  (a) NORM.DIST: Calculate probability value $P(X \leq k)$, with k known.

  (b) NORM.INV: Given the value of the probability $P(X \leq k)$, find k.

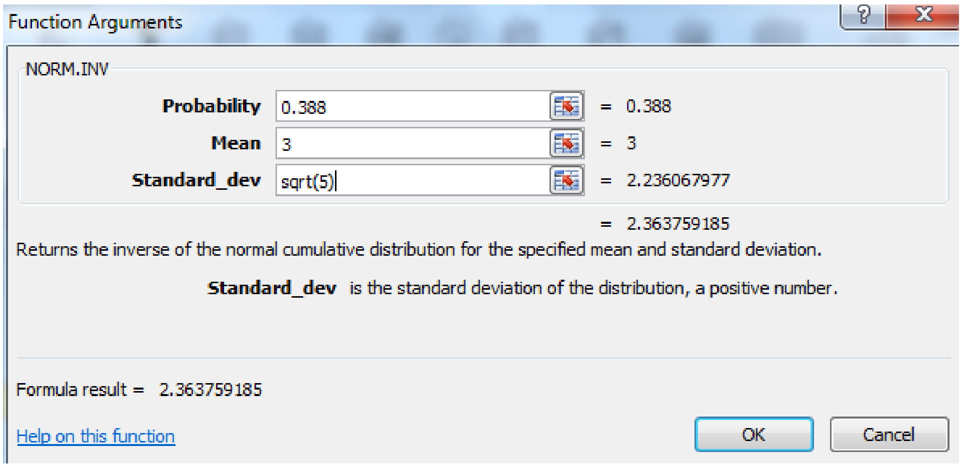


- To compute $P(X \leq 4)$, given that $X \sim N(3,5)$. Select NORM.DIST



$P(X \leq 4) = 0.673$

- To find the value of k such that $X \sim N(3,5)$ and $P(X \leq k) = 0.388$, select NORM.INV.



$k = 2.36$

Tutorial 7: Normal Distribution

A Self Practice Questions

1 Let *Z* be a standard normal random variable. Use the normal table provided to find:

(i) $P(Z < 2.11)$ (ii) $P(Z < -0.35)$

(iii) $P(Z > 1.02)$ (iv) $P(Z > -0.99)$

(v) $P(-0.35 < Z < 2.11)$ (vi) $P(Z > 1.02$ or $Z < -0.35)$

2 Given that $X \sim N(3,4)$, use the normal table provided to find:

(i) $P(X < 1)$ (ii) $P(X \leq 4)$

(iii) $P(X > 0.5)$ (iv) $P(X \geq 3.5)$

(v) $P(1 \leq X < 4)$ (vi) $P(X < 1$ or $X \geq 3.5)$

3 Let *Z* be a standard normal random variable, find m such that

(i) $P(Z < m) = 0.9082$ (ii) $P(Z > m) = 0.0096$

4 Given that $X \sim N(3,4)$, find m such that:

(i) $P(X \leq m) = 0.6217$ (ii) $P(X > m) = 0.7734$

B Discussion Questions

1 The brain weights of a certain population of 18-year olds follow a normal distribution with mean 1380 gm and standard deviation 80 gm. Suppose an 18-year old is chosen at random, find the probability that the person's brain weight is

(i) less than 1300 gm,

(ii) more than 1400 gm,

(iii) between 1320 and 1420 gm.

2 The random variable X has the distribution $N(1, 20)$. Find a such that $P(X < a) = 2P(X > a)$.

3 The masses of articles are normally distributed such that 4.36% are under 30 kg and 6.3% are over 60 kg. Calculate the mean and standard deviation of the distribution.

4 A recent survey on a group of adults shows that the average daily calories intake of an adult is normally distributed with mean 1380 calories and standard deviation 320 calories.

(i) Find the probability that an adult chosen at random from this group consumes less than 1000 calories per day.

(ii) What should be the recommended daily caloric intake if 90% of the group has average daily calories below this recommended daily intake?

(iii) If 12 000 adults participated in the survey, find the expected number of people, to the nearest integer, to consume more than 1200 calories per day?

5 The mass, in kilograms, of an apple sold in a supermarket has a normal distribution with mean 0.15 and standard deviation 0.03. Suppose the apples are sold at $9 per kilogram, find

(i) the probability that a single apple cost between $1.30 and $1.50;

(ii) the minimum price set for an apple such that the probability of an apple being sold for less than this minimum price is at least 0.9.

Answers

A1	i	0.9826	ii	0.3632	iii	0.1539	iv	0.8389
	v	0.6194	vi	0.5171				
A2	i	0.1587	ii	0.6915	iii	0.8944	iv	0.4013
	v	0.5328	vi	0.560				
A3	i	1.33	ii	2.34				
A4	i	3.62	ii	1.50				

B1	i	0.1587	ii	0.4013	iii	0.4649
B2	$a = 2.92$					
B3	$\mu = 45.8, \quad \sigma = 9.26$					
B4	i	0.1170	ii	1789.6	iii	8548
B5	i	0.2876	ii	\$1.70		