Chapter 4: Matrices

Objective

The objective of this chapter is to

1. identify and use the various type of matrices
2. perform arithmetic operation on matrices
3. learn to solve determinants
4. use inverse matrix to solve simultaneous equations
5. use determinants to solve simultaneous equations

Content

4.1 Introduction

An $m \times n$ matrix A is a rectangular array of numbers in the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The pair of numbers $m \times n$ is called the size of the matrix where m represents the number of rows and n the number of columns.

<u>Example 1</u>

The rectangular array $\begin{bmatrix} 1 & -3 & 4 \\ 0 & 5 & -2 \end{bmatrix}$ is a 2×3 matrix.

Row and Column vector

A matrix with only one row is called a row vector and a matrix with only one column is called a column vector.

<u>Example 2</u>

Row vector: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a $1 \text{ x } 3$ matrix.

Column vector: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a $3 \text{ x } 1$ matrix.

Zero matrix

A matrix whose entries are all zero is called a zero matrix and is denoted by 0.

<u>Example 3</u>

The 2×4 zero matrix is $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Equality of two matrices

Two matrices A and B are equal, denoted by $A = B$, if they have the same size and the same corresponding element

<u>Exercise 4.1</u>

Given that $\begin{bmatrix} x+y & 2z+2 \\ x-y & z-w \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$, solve for *x*, *y*, *z* and *w*.

4.2 Arithmetic operations on matrices

4.2.1 Matrix Addition/Subtraction and Scalar Multiplication

Two matrices A and B of the same size can be added (or subtracted) by adding (or subtracting) their corresponding elements:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \pm \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \dots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \dots & a_{2n} \pm b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \dots & a_{mn} \pm b_{mn} \end{bmatrix}$$

The product of a scalar k and a matrix A, written as kA or Ak, is obtained by multiplying each element of A by k:

$$k \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}.$$

A matrix defined as $-A$ is called the negative of A.

As with ordinary addition, the communicative law and the associative laws apply to matrix addition.

(a) Commutative Law: $A+B \quad = \quad B+A$

(b) Associative Law: $A+(B+C) \quad = \quad (A+B)+C$

Exercise 4.2

(1) Evaluate: $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & -6 \\ 2 & -3 & 1 \end{bmatrix}$.

(2) Evaluate: $\begin{bmatrix} 9 & -4 & 1 \\ 7 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 8 & 0 & -4 \\ 2 & -7 & 5 \end{bmatrix}$

(3) Evaluate: $3\begin{bmatrix} 1 & -2 & 0 \\ 4 & 3 & -5 \end{bmatrix}$.

4.2.2 Matrix Multiplication

Two matrices A and B can be multiplied together if and only if the number of columns in A is the same as the number of rows in B.

$$A_{\boxed{m} \times p} \qquad\qquad B_{p \times \boxed{n}} \leftarrow \text{no. of cols in product } AB$$

$$\underset{\text{no. of rows in } AB}{\downarrow} \qquad \text{no. of cols in } A \;=\; \text{no. of rows in } B$$

Thus, $A_{m\times p}$ and $B_{p\times n}$ can be multiplied together to give a new matrix C of size $m \times n$.

Example 4

$$A = \begin{pmatrix} -1 & 3 \\ 6 & 0 \\ 5 & 1 \\ -2 & -4 \end{pmatrix}_{4\times 2} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & -3 & 7 \\ 0 & 8 & -6 \end{pmatrix}_{2\times 3}$$

We can find the product AB, because the number of columns in A (2) equals the number of rows in B (2). The product is a 4×3 matrix and it is formed as follows:

$$\begin{pmatrix} \boxed{-1 \quad 3} \\ 6 \quad 0 \\ 5 \quad 1 \\ \boxed{-2 \quad -4} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix}2\\0\end{matrix}} & \boxed{\begin{matrix}-3\\8\end{matrix}} & \begin{matrix}7\\-6\end{matrix} \end{pmatrix} = \begin{pmatrix} \boxed{(-1)(2)+(3)(0)} & (-1)(-3)+(3)(8) & (-1)(7)+(3)(-6) \\ (6)(2)+(0)(0) & (6)(-3)+(0)(8) & (6)(7)+(0)(-6) \\ (5)(2)+(1)(0) & (5)(-3)+(1)(8) & (5)(7)+(1)(-6) \\ (-2)(2)+(-4)(0) & \boxed{(-2)(-3)+(-4)(8)} & (-2)(7)+(-4)(-6) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 27 & -25 \\ 12 & -18 & 42 \\ 10 & 0 & 29 \\ -4 & -26 & 10 \end{pmatrix}$$

Note that we cannot find the product BA because the number of columns in B (3) does not equal the number of rows in A (4).

Exercise 4.3

(1) Let $(R \times S)$ denote an $R \times S$ matrix. Determine if the matrix product exists and find the matrix dimension of the product if it exists.

(a) (2 x 3)(3 x 4)

(b) (1 x 2)(3 x 1)

(c) (4 x 4)(3 x 3)

(d) (2 x 2)(2 x 4)

(2) Given: $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$, find AB.

4.3 Properties of matrix multiplication

1. Matrix multiplication is associative:
   $A(BC) \;=\; (AB)C$ and $(kA)B \;=\; k(AB) \;=\; A(kB)$

2. Matrix multiplication is distributive over addition:
   $A(B+C) \;= AB+AC$ and $(B+C)A \;=\; BA+CA$

3. Matrix multiplication is not communicative.
   In general, $AB \neq BA$.

4. $AB=0$ does not necessarily imply $A=0$ or $B=0$.

5. For a matrix product AB, we say that B is pre-multiplied by A or A is post-multiplied by B.

4.4 Transpose Matrix

The transpose matrix of A, denoted by A^T is obtained by interchanging the rows and columns of A.

Example 5

Given that $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then $A^T=\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Properties of A^T

1. $(A+B)^T \;=\; A^T+B^T$

2. $\left(A^T\right)^T \;=\; A$

3. $(kA)^T \;=\; kA^T$, $(k \text{ is a scalar})$

4. $(AB)^T \;=\; B^T A^T$

Exercise 4.4

(1) Given that $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{bmatrix}$, find A^T, AA^T and $A^T A$

(2) Given that $B = \begin{bmatrix} 6 & 3 \\ -1 & 0 \\ 2 & 5 \end{bmatrix}$, find B^T, BB^T and $\mathrm{B}^T B$.

4.5 Square Matrices

A matrix with the same number of rows and columns is called a square matrix.

Example 6

The following matrix is a square matrix of order 3: $\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -1 \\ 5 & 3 & 2 \end{bmatrix}$.

Below is a list of some special types of square matrices that are commonly seen.

(a) Identity matrix

If the main diagonal of a square matrix is filled with 1's and 0's elsewhere, it is called an identity matrix, denoted by I.

Example 7

Below shows the identity matrix of order 2 and 3 respectively.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2nd order identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3rd order identity matrix

For any square matrix $A, AI = IA = A.$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Exercise 4.5

Given that $BI = \begin{bmatrix} 1 & 9 & 5 \\ 7 & 6 & 0 \\ 2 & 4 & 3 \end{bmatrix}$, find the matrix B.

(b) Diagonal matrix

A diagonal matrix is a square matrix with all its non-diagonal elements being zero.

Example 8

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(c) Upper triangular and lower triangular matrix

An upper triangular matrix is a square matrix with all its elements below the main diagonal being zero. A lower triangular matrix is a square matrix with all its elements above the main diagonal being zero.

Example 9

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

upper triangular lower triangular

(d) Symmetric, skew-symmetric and orthogonal matrix

A symmetric matrix is defined by $A^T = A$.

A skew-symmetric matrix is defined by $A^T = -A$.

An orthogonal matrix is defined by $A^T = A^{-1}$,
That is, $A^T A = AA^T = I$.

Exercise 4.6

Find x and the matrix B if B is symmetric, where $B = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$.

(e) Powers of a square matrix

We form powers of a square matrix A as follows:

$A^0 = I$, $A^1 = A$, $A^2 = AA$, $A^3 = AAA$, and so on.

Matrices and polynomials are very closely related to one another. Normally, we form polynomials as follows:

$$f(x) = a_o + a_1 x + a_2 x^2 + ... + a_n x^n$$

This could be transformed into matrices as follows:

$$f(X) = a_o + a_1 X + a_2 X^2 + ... + a_n X^n$$

<u>Exercise 4.7</u>

(1) Let $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$. Find A^2 , A^3 and $f(A)$ where $f(x) = 2x^3 - 4x + 5$.

4.6 Inverse Matrix

A square matrix A is said to be invertible if there exists a matrix B with the property that

$$AB = BA = I$$

If A is invertible, then B is called the inverse of A and is denoted by A^{-1}.

For a 2×2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is given by

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

A square matrix is invertible if and only if it has a nonzero determinant.

Example 10

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 4.8

(1) Find the inverse of $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$.

(2) Find the inverse of $B = \begin{bmatrix} 2 & 5 \\ -1 & -\frac{3}{2} \end{bmatrix}$.

4.6.1 Solving a set of equations by matrix inversion

A system of equations could be represented by matrices. For example, the system

$$2x + 3y = 5$$
$$3x + 5y = 9$$

would be represented by three matrices A, X and K, where $AX = K$. To do this, we let

(a) A be the matrix formed by the coefficients of the variables,
(b) X be a column vector of the variables and
(c) K be a column vector of the constants.

Thus, for this system, we have

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad K = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Now that we can invert a matrix, we can find A^{-1}. Since $A^{-1}A = I$, then

$$A^{-1}(AX) = \left(A^{-1}A\right)X = IX = X$$

With this knowledge, we can find the solution to our system of equations.

$$AX = K$$
$$A^{-1}(AX) = A^{-1}K$$
$$X = A^{-1}K$$

<u>Exercise 4.9</u>

Use the inverse of the coefficient matrix to solve the linear system.

$$8x - 6y = -27$$
$$-6x + 4y = 19$$

4.7 Determinant

The determinant of an $n \text{ x } n$ square matrix A is denoted by *det*(A) or $|A|$ and is known as the determinant of order n.

For a determinant of order 2, we evaluate its value as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Minor

If a determinant $|A|$ of order n, we delete the *i*th row and the *j*th column and form a determinant from all the remaining elements, we shall have a new determinant of order $(n-1)$. This new determinant is called the minor of the element a_{ij}.

<u>Example 11</u>

Let $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ be a determinant of order 3, then the minor of a_{12} is $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

Cofactor

If we multiply the minor by it sign according to the positional pattern shown below, the result so obtained is called the cofactor.

For a determinant of order 3, the sign pattern is as follows:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

For a determinant of any size, we always start with a + in the top left position and from this position work both horizontally and vertically downwards with alternating signs.

$$\begin{matrix} + & \rightarrow & - & \rightarrow & + \\ \downarrow & & & & \\ - & & + & & - \\ \downarrow & & & & \\ + & & - & & + \end{matrix}$$

<u>**Example 12**</u>

Given $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, the cofactor of a_{12}, written as A_{12} is $A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

Expansion and evaluation of determinants

We can evaluate a determinant of any order by expanding it along any one row or any one column.

For a determinant of order 2,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

For a determinant of order 3, expanding it along row 1, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

The rules for a determinant of higher order are exactly the same, eg.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{23} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}$$

$$+a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

<u>Exercise 4.10</u>

(1) Evaluate $\begin{vmatrix} 0 & 3 \\ 5 & 2 \end{vmatrix}$

(2) Evaluate $\begin{vmatrix} 2x+5 & 4-3x \\ 3-4x & 2+3x \end{vmatrix}$

(3) Evaluate $\begin{vmatrix} 1 & -1 & 3 \\ 0 & 2 & 5 \\ -2 & 1 & 6 \end{vmatrix}$

Sarrus' Rule for 3 x 3 determinants

There is a simple rule for evaluating 3 x 3 determinants only, known as the Sarrus' Rule. To find 3 x 3 determinant, repeat the first two columns and multiply the diagonals with 3 elements. Arrows pointing down give positive multiplication and arrows pointing up give negative multiplication.

The following steps is using Sarrus' rule to find the determinant $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 6 & -1 \\ 3 & 5 & 4 \end{vmatrix}$:

Step 1: Replicate column 1 and 2 to the right

$$\left|\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 2 & 6 & -1 & 2 & 6 \\ 3 & 5 & 4 & 3 & 5 \end{array}\right.$$

Step 2: Circle the diagonals as shown below

$$\left|\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 2 & 6 & -1 & 2 & 6 \\ 3 & 5 & 4 & 3 & 5 \end{array}\right. \qquad \left|\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 2 & 6 & -1 & 2 & 6 \\ 3 & 5 & 4 & 3 & 5 \end{array}\right.$$

Step3: Add the totals of the multiplication of the numbers in the circled diagonals for both the left and right separately.

$$\begin{aligned} &1(6)(4)+(2)(-1)(3)+(0)(2)(5) \\ &=24-6+0 \\ &=18 \end{aligned} \qquad \begin{aligned} &3(6)(0)+(5)(-1)(1)+(4)(2)(2) \\ &=0-5+16 \\ &=11 \end{aligned}$$

$$\Rightarrow 18-11=7$$

Step4: Minus the right total with the one on the left and the answer of the 3X3 determinant is 7

Exercise 4.9

Evaluate $\begin{vmatrix} 2 & 1 & 3 \\ 4 & 6 & -1 \\ 5 & 1 & 0 \end{vmatrix}$ using Sarrus' rule

4.8 Cramer's Rule

Cramer's Rule is a method used to solve simultaneous equations.

(a) Consider the set of 2 x 2 simultaneous equations

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

The solution is given by $x = \dfrac{D_x}{D}, \quad y = \dfrac{D_y}{D}$

where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad \mathrm{D}_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Thus $x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \quad y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

This is known as Cramer's rule for solving a set of 2 x 2 simultaneous equations.

(b) For the set of 3 x 3 simultaneous equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Cramer's rule for solving them gives

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}$$

where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\mathrm{D}_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\mathrm{D}_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Exercise 4.11

(1) Solve the system of equations using Cramer's rule:

$$3x - 2y = 7$$

$$4x + 5y = 2$$

(2) Solve the system of equations using Cramer's rule:

$$x + 2y + 3z = 14$$

$$2x + y + 2z = 10$$

$$3x + 4y - 3z = 2$$

Tutorial 4

1. Express the following as a single matrix:

(a) $\begin{pmatrix} 7 & 11 \end{pmatrix} + \begin{pmatrix} 9 & 5 \end{pmatrix}$

(b) $3\begin{pmatrix} 1 & 2 \\ -5 & 6 \end{pmatrix} - \begin{pmatrix} -5 & 6 \\ 3 & -4 \end{pmatrix} + 2\begin{pmatrix} 1 & 3 \\ -2 & 0 \end{pmatrix}$

2. Evaluate (wherever possible) the following multiplications and explain why if it is not possible:

(a) $\begin{pmatrix} 3 & -1 \\ -2 & 3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -1 \\ -2 & 3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

3. Find the values of x, y and z given that $2\begin{pmatrix} x & 3 \\ -2 & y \end{pmatrix} - \begin{pmatrix} -5 & z \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ -7 & -6 \end{pmatrix}$.

4. Find the transpose of $\begin{pmatrix} 15 & 12 & -1 \\ -3 & 4 & 17 \end{pmatrix}$.

5. Evaluate the following determinants:

(a) $\begin{vmatrix} -\frac{1}{3} & \frac{1}{5} \\ -\frac{2}{3} & \frac{8}{5} \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 5 & 0 & 2 \end{vmatrix}$

6. Find the inverse (wherever possible) of the following matrices:

(a) $\begin{pmatrix} 3 & -2 \\ 14 & -7 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}$

7. It is given that $A = \begin{pmatrix} 10 & 6 \\ 3 & -2 \end{pmatrix}$, find A^{-1}.

Hence solve the system of equations :

$$\begin{aligned} 10x + 6y &= 5 \\ 3x - 2y &= 11 \end{aligned}$$

8. Solve the following system of equations using Cramer's Rule :

$$\begin{aligned} x + y + z &= 12 \\ x - y &= 2 \\ x - z &= 4 \end{aligned}$$

9*. Find the values of *k* such that $\begin{vmatrix} k & 1 & 0 \\ 1 & k & 1 \\ 0 & 1 & k \end{vmatrix} = 0$.

10. At a soccer game, the number of student tickets, adult tickets and children's tickets (below 7 years old) sold is represented by x , y and z respectively.

The number of adult tickets was four times more than the children tickets sold. Also, the total number of children's tickets and adult tickets was half the number of student tickets sold. The total number of tickets sold was 150.

(a) Write down a system of three linear equations in terms of x, y and z **.**
(b) Write the equations obtained in (i) in a form of a matrix.

(c) Using Cramer's rule, determine how many of each type of tickets were sold.

Answers

1. (a) $\begin{pmatrix} 16 & 16 \end{pmatrix}$ (b) $\begin{pmatrix} 10 & 6 \\ -22 & 22 \end{pmatrix}$

2. (a) The product is undefined because the number of columns in the first matrix is not equal to the number of rows in the second matrix.

   (b) $\begin{pmatrix} 7 \\ -7 \\ -3 \end{pmatrix}$

3. $x = 1$, $y = 0$ and $z = 4$ 4. $\begin{pmatrix} 15 & -3 \\ 12 & 4 \\ -1 & 17 \end{pmatrix}$ 5. (a) $-\frac{2}{5}$ (b)

45

6. (a) $\begin{pmatrix} -1 & \frac{2}{7} \\ -2 & \frac{3}{7} \end{pmatrix}$ (b) inverse does not exist

7. $x = 2$ and $y = -\frac{5}{2}$ 8. $x = 6$, $y = 4$ and $z = 2$

9*. 0, $\pm\sqrt{2}$

10.

(a) $-y + 4z = 0....(1)$

$-x + 2y + 2z = 0....(2)$

$x + y + z = 150....(3)$

(b) $\begin{bmatrix} 0 & -1 & 4 \\ -1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 150 \end{bmatrix}$

(c) $x = 100,\ y = 40,\ z = 10$

Questions from past year examination papers

1) 2013/14S1 IT1101/1501/1561/1751/1621 Sem Exam– Q6

a) Given the matrices $X = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$, $Y = \begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix}$, $Z = \begin{bmatrix} 3 & 9 \\ 2 & 7 \end{bmatrix}$.

If $2Y - Z = \alpha X$, find the value of α. (4 marks)

(b) An alloy used in the medical field is composed of three metals A, B and C. The percentage of each metal are given by the following system of equations

$$\begin{cases} A + B + C = 100 \\ A - 2B = 0 \\ -4A + C = 0 \end{cases}$$

Determine the percentage (answers correct to 2 decimal places) of each metal in the alloy by using the Cramer's rule. (11 marks)

2) 2013/14S2 EG1740 Sem Exam– Q4

Given the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find

(a) matrix B, (4 marks)

(b) the values of x and y if $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \end{bmatrix}$. (4 marks)

3) 2013/14S2 EG1740 Sem Exam– Q8

(a) A point (x, y) is transformed into an image (p, q) by the following matrix equation.

$$\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2x+1 \\ x+2y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

Find the point (x,y) that has an image $(23,5)$. (6 marks)

(b) The tensions x, y and z in a wireframe structure is given by the equations:

$$2x + 3y + z = 12$$
$$3x + 5y = 11$$
$$4x + 2y = 18$$

(i) Write a matrix equation, involving x, y and z which represents the above equations. (2 marks)

(ii) Use Cramer's rule to find the values of x, y and z. (12 marks)

4) 2012/13S1 IT1101/1501/1561/1751/1621 Sem Exam– Q6

(a) A company manufactures three types of products, A, B and C. Each product requires three processes – assembly, finishing and packaging. The number of minutes required for each product in different processes is given in the following table. Let a, b and c be the number of A, B and C products manufactured per week respectively. Due to resource constraints, the company can only operate the assembly process for 1676 minutes; the finishing process for 900 minutes and the packaging process for 1228 minutes in a week.

	Number of minutes		
	Assembly	Finishing	Packaging
Product A	8	4	5
Product B	9	5	7
Product C	4	2	3

(i) Write a matrix equation, involving a, b and c. (2 marks)

(ii) Using Cramer's Rule, find a, b and c. (8 marks)

(b) Given matrix $Y = \begin{bmatrix} p & \frac{2}{5} \\ q & \frac{1}{5} \end{bmatrix}$ and $YA = B$ where $A = \begin{bmatrix} 2 & -4 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$.

Determine p and q to find the matrix Y. (5 marks)

5) 2014/15S1 IT1101/1501/1561/1751/1621 Sem Exam– Q6

A baker was commissioned on National Day to make 3 types of pastries: muffins, scones and croissants. The baker worked 5 hours to produce the pastries and on average he took 1 minute to make a muffin, 1 minute for a scone and 2 minutes for a croissant. The baker charged $3 for a muffin, $5 for a scone and $6 for a croissant and total cost was $1000. The baker used 3.3 kilograms of flour for baking for which 11 grams was used for a muffin, 14 grams for a scone and 20 grams for a croissant.

(a) Write down a system of three linear equations using the variables *x* for the number of muffins produced, *y* for the number of scones produced and *z* for the number of croissants produced.

(3 marks)

(b) Use Cramer's rule to find the total quantity of all the pastries produced.

(12 marks)

Answers

1a) $\alpha = 3$

b) The alloy consists of 18.18% metal A, 9.09% metal B, and 72.73% metal C.

2a) $\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$

b) $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

3a) $x = 2$, $y = 1$

3bi) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 0 \\ 4 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \\ 18 \end{bmatrix}$

3bii) $x = \frac{D_x}{D} = \frac{-68}{-14} = \frac{34}{7} = 4.857$

$y = \frac{D_y}{D} = \frac{10}{-14} = \frac{5}{-7} = -0.714$

$z = \frac{D_z}{D} = \frac{-62}{-14} = \frac{31}{7} = 4.429$

4ai) $\begin{pmatrix} 8 & 9 & 4 \\ 4 & 5 & 2 \\ 5 & 7 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1676 \\ 900 \\ 1228 \end{pmatrix}$

ii) $a = \frac{120}{2} = 60,\quad b = \frac{248}{2} = 124,$

$c = \frac{40}{2} = 20$

b) $p = -\frac{3}{5}, q = -\frac{4}{5}$

$$Y = \begin{bmatrix} -\frac{3}{5} & \frac{2}{5} \\ -\frac{4}{5} & \frac{1}{5} \end{bmatrix}$$

5a)
$$\begin{aligned} x + y + 2z &= 300 \\ 3x + 5y + 6z &= 1000 \\ 11x + 14y + 20z &= 3300 \end{aligned}$$

(b) $x + y + z = 225$