Chapter 8: Application of Differentiation

Objective

The objective of this chapter is to

1. use derivatives to identify increasing and decreasing functions.
2. using first/second derivatives to find local maxima or minima.
3. solve application problems involving differentiation.

Content

8.1 Increasing and Decreasing Functions

We use the derivative of a function to understand its graph and its ‘behavior’. From its derivative we can find out where the function is increasing/decreasing, where it is ‘turning’ and where their maximum/minimum are.

Increasing function

A function $f(x)$ is increasing if its derivative $f'(x) > 0$. For example:

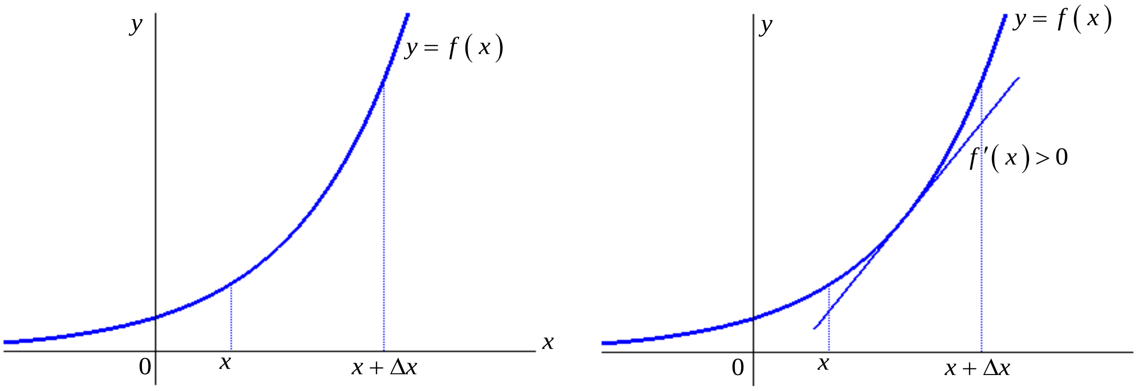


Figure 8.1 : Graph of an increasing function

Note that $f'(x) > 0$ for the interval between *x* and *x* + Δ*x*. In this particular case, $f'(x) > 0$ for the domain shown.

Decreasing function

A function $f(x)$ is decreasing if its derivative $f'(x) < 0$. For example

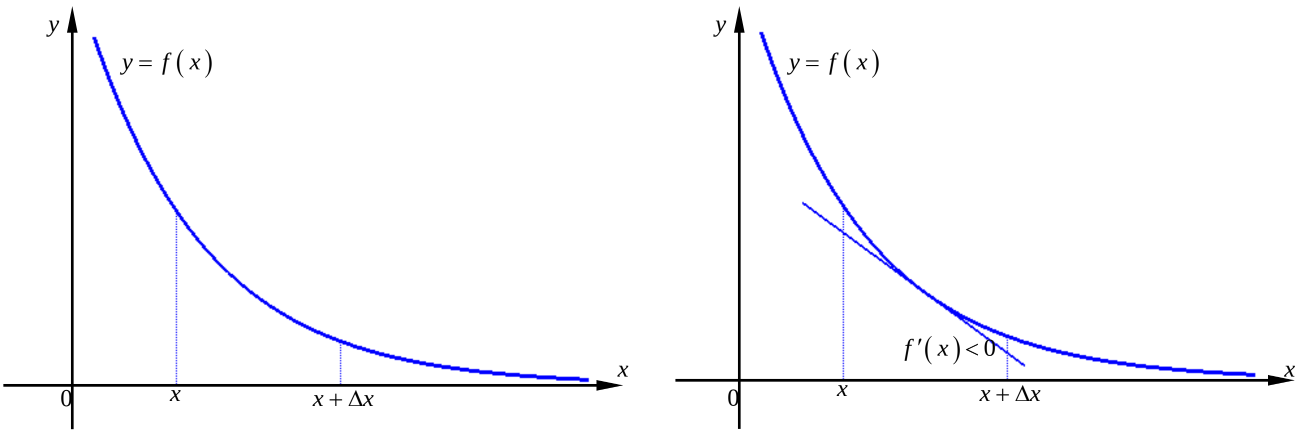


Figure 8.2 : Graph of a decreasing function

Note that $f'(x) < 0$ for the interval between *x* and *x* + Δ*x*. In this particular case, $f'(x) < 0$ for the domain shown.

So the sign of the derivative $f'(x)$ is important in that it tells us if the function $f(x)$ is increasing or decreasing over an interval of *x* (or at a particular point of $f(x)$.

$f(x)$ is increasing on an interval where $f'(x) > 0$
$f(x)$ is decreasing on an interval where $f'(x) < 0$

To know if the $f(x) = x^3 - 9x^2 - 48x + 50$ is increasing or decreasing at a given interval of x, we differentiate $f(x)$. Thus $f'(x) = 3x^2 - 18x - 48$. Now if we want to know if $f(x)$ is increasing or decreasing at $x = 1$, we evaluate $f'(x)$ at $x = 1$. Hence, $f'(1) = 3(1)^2 - 18(1) - 48 = -63$, i.e. $f'(1) < 0$ at $x = 1$.

So $f(x)$ is decreasing at $x = 1$.

8.1.1 Critical Points

If the $f'(x) = 0$ at a particular point, then the function $f(x)$ is said to be ***critical*** at that point. Sometimes a *critical* point is also called a *stationary* point.

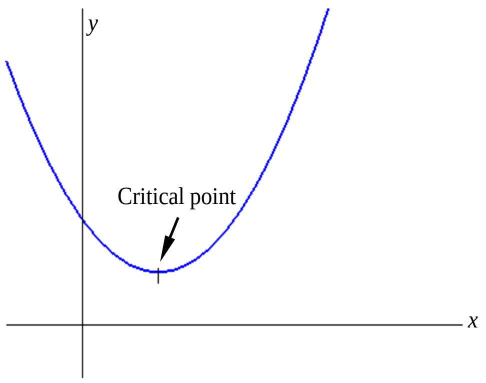


Figure 8.3a: One critical point

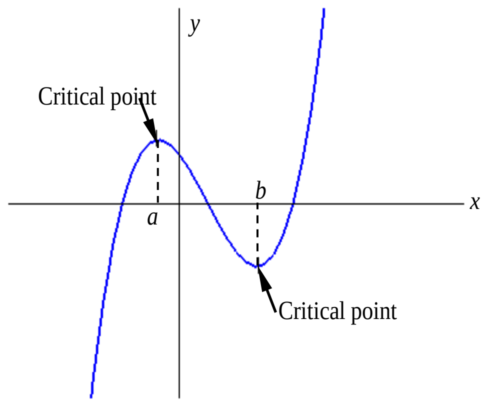


Figure 8.3b : Two critical points

At a *critical* point where $f'(x) = 0$ the tangent line to the graph is horizontal. The points where the derivative $f'(x)$ is 0 divide the domain of $f(x)$ into intervals on which the sign of $f'(x)$ stays the same, either + or –.

So in Figure 8.3b, the function $f(x)$ has critical points at $x = a$ and $x = b$. The critical points divide the domain into 3 parts.

For $x < a$, $f'(x) > 0$
For $a < x < b$, $f'(x) < 0$
For $x > b$, $f'(x) > 0$

Exercise 8.1

(1) Suppose $y = x^2 - 4x + 5$. Determine for which interval of x for which the values of y is increasing, decreasing or stationary.

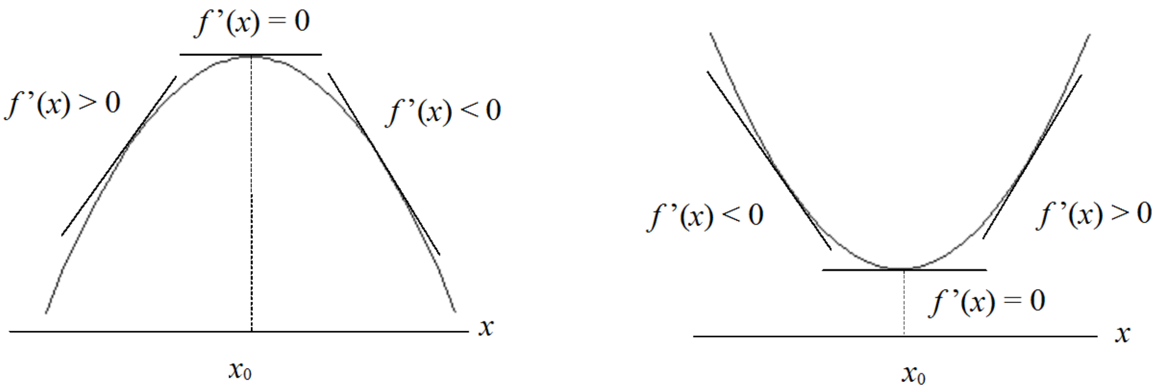
(2) Suppose $y = 3x^2 - 18x + 20$. Determine for which interval of x for which the values of y is increasing, decreasing or stationary.

(3) For what values of x is the function $f(x) = x^3 - 3x^2 + 4$ increasing, decreasing or stationary.

8.2 Maxima and Minima

8.2.1 Local Maxima and Minima

From the previous section, we see that if $f'(x)$ has different signs on either side of a *critical* point, then the graph 'turns'.



(a) Maximum at $x = x_0$ (b) Minimum at $x = x_0$

Figure 8.4: Turning points of graphs

At a turning point, $f'(x) = 0$ and $f'(x)$ *changes sign* as x increases through the point.

How to decide which Critical Points are Local Maxima/Minima?

First Derivative Test for Maximum/Minimum Points

- If the sign of $f'(x)$ changes from + to – as x increases through the point $x = x_0$, then $f(x)$ has a local maximum at $x = x_0$.
- If the sign of $f'(x)$ changes from – to + as x increases through the point $x = x_0$, then $f(x)$ has a local minimum at $x = x_0$

We say the maximum/minimum are 'local' because we are describing only what happens near the turning point.

Warning!
Not all *critical* points are turning points. Sometimes the sign of $f'(x)$ changes from + to 0 and to + again. For example $f(x) = x^3$. The sign of $f'(x)$ *did not* change.

Exercise 8.2

Find the turning points of $y = f(x) = x^3 - 3x$. Determine if they are max/min.

8.2.2 Global Maxima and Minima

The local maxima/minima tell us where a function is *locally* largest/smallest. Often we are interested in finding out where the function is absolutely largest/smallest in a given domain, i.e. the global maximum/minimum.

To find the global max/min of a continuous function compare values of the function at all the critical points.

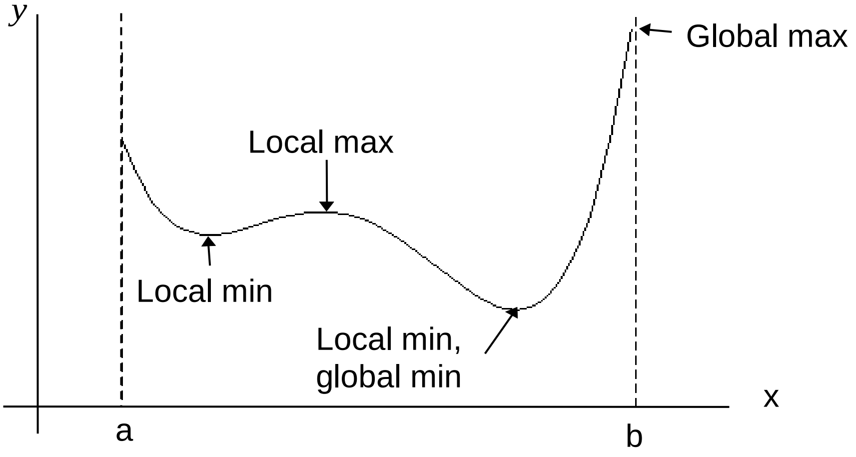


Figure 8.5 : Maximum and minimum on a closed interval $a \leq x \leq b$

For a closed interval, you have to check the end-points.
For an open interval or on the entire real line, you need to sketch a graph to ascertain the max/min points.

For example in the following figure 8.6, there is a global max but no global minimum in the open interval.

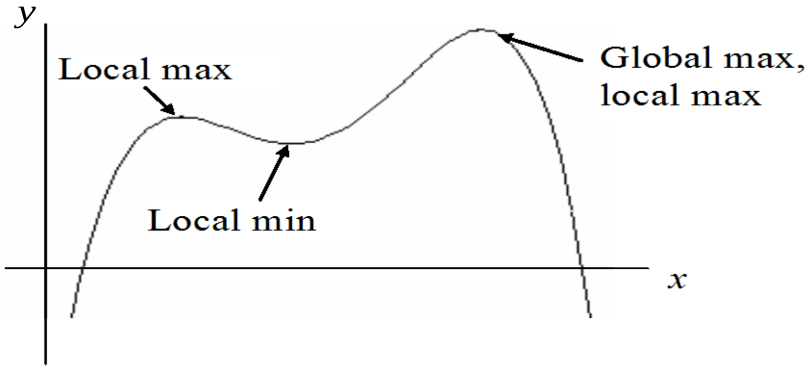


Figure 8.6 : Global max and no global min.

8.2.3 Second Derivative Test

It is useful to know if a function is concave up or down over an interval.

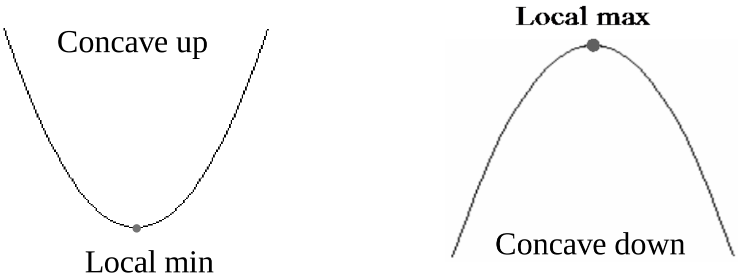


Figure 8.7 : Local max/min and concavity

We need to find the second derivative of the function $f(x)$.

If $f''(x) > 0$ on an interval, then the graph of $f(x)$ concaves up on the interval; and
if $f''(x) < 0$ on an interval, then the graph of $f(x)$ concaves down on the interval.

Second Derivative Test for Local Maxima/Minima

Knowing the concavity can be useful in testing if a critical point is a local max/min.

Suppose $f'(x) = 0$ at $x = x_0$, then the point where $x = x_0$ is a :
- maximum point of the function if $f''(x) < 0$ and is a
- minimum point of the function $f''(x) > 0$.

Note if $f''(x) = 0$ then the test fails, i.e. we don't know if the point is a max/min.

Points of Inflection

A point at which the graph of a function changes concavity is called an *inflection point* of the function.

Exercise 8.3

Find the turning points on the curve $f(x) = 2x^3 + 3x^2 - 12x - 5$ and determine whether it is a maximum or minimum.

8.3 Optimization Problems

Everywhere in business and industry today, we see people struggling to minimize waste and maximize productivity. We are now in the position to bring the power of calculus to bear on problems involving finding a maximum and minimum.

Exercise 8.4

(1) A certain country club had a membership of $f(x) = 100(2x^3 - 45x^2 + 264x)$, where *x* is the number of years since the opening of the club. At what time was the membership of the club smallest? What was the membership at that time?

(2) An automobile manufacturer, in testing a new engine on one of its models, found that the efficiency η of the engine as function of the speed v of the car was given by

$$\eta = 0.768v - 0.00004v^3$$

where η is given in percent and v is given in km/h.

What would be the maximum efficiency of the engine and at what speed would this be achieved?

(3) In designing airplanes, an important feature to take into consideration is the retarding force exerted on the plane by the air. The retarding force $F(v)$ N is related to the speed v mph as follows:

$$F(v) = av^2 + \frac{b}{v^2}$$

where a and b are positive constants.

If the force is minimized when $v = 160$ mph, find the ratio of $\frac{b}{a}$.

Tutorial 8

1. Find the following for each of the functions given

   (i) the critical points

   (ii) the interval(s) where the function is increasing or decreasing.

   (iii) determine whether they are a relative maxima or minima

   (a) $y = x^2 - 5x + 1$ (b) $y = x^2 + 12x - 8$

   (c) $y = 3x^3 - 12x^2 - 7$ (d) $f(x) = \frac{2}{3}x^3 - x^2 - 4x + 2$

   (e) $f(x) = 2x^3 - 3x^2 - 72x + 15$

2. A certain company manufactures x number of chairs a year. The revenue the company received for selling x chairs that year is: $R(x) = 200x - 0.15x^2$ dollars and the cost incurred by the company for manufacturing x chairs is: $C(x) = 4000 + 6x - 0.001x^2$ dollars. Suppose that the profit for that year is $P(x) = R(x) - C(x)$, then find the production level (or the number of chairs x that should be produced) to maximise the yearly profit.

Answers

1 (a) Critical point: $\left(2\frac{1}{2},-5\frac{1}{4}\right)$, minimum. y is increasing on the interval $x>2\frac{1}{2}$ and decreasing on the interval $x<2\frac{1}{2}$.

(b) Critical point: $(-6,-44)$, minimum. y is increasing on the interval $x>-6$ and decreasing on the interval $x<-6$.

(c) Critical points: $(0,-7)$, maximum; $\left(2\frac{2}{3},-35\frac{4}{9}\right)$, minimum; y is increasing on the intervals $x<0)$ and $x>2\frac{2}{3}$; and decreasing on the interval $0<x<2\frac{2}{3}$.

(d) Critical points: $\left(-1,4\frac{1}{3}\right)$, maximum; $\left(2,-4\frac{2}{3}\right)$, minimum; y is increasing on the intervals $x<-1$ and $x>2$; and decreasing on the interval $-1<x<2$

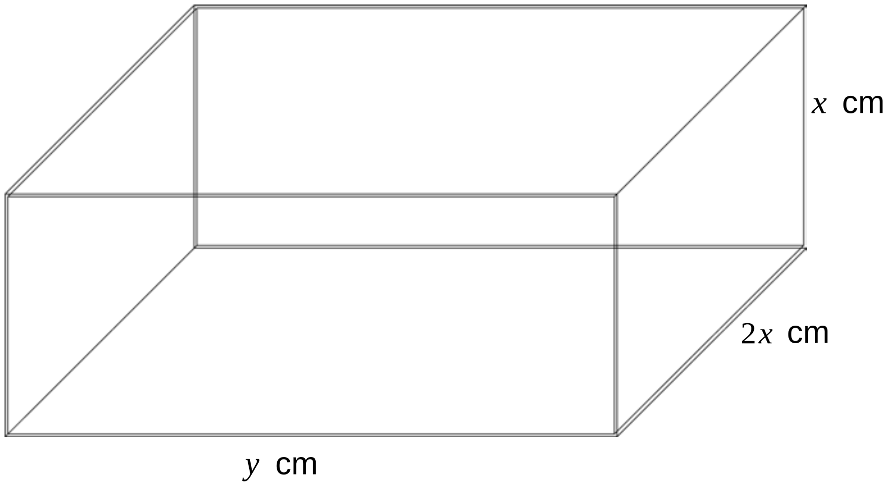
(e) Critical points: $(-3,150)$, maximum; $(4,-193)$, minimum; y is increasing on the intervals $x<-3$ and $x>4$; and decreasing on the interval $-3<x<4$

2. $x=651$ (correct to 3 sig fig)

Questions from Past Year Examination papers

1) 2012/13S1 EG1740 Sem Exam– Q9a

(a) The diagram shows a rectangle block that has a volume of 72 cm^3.



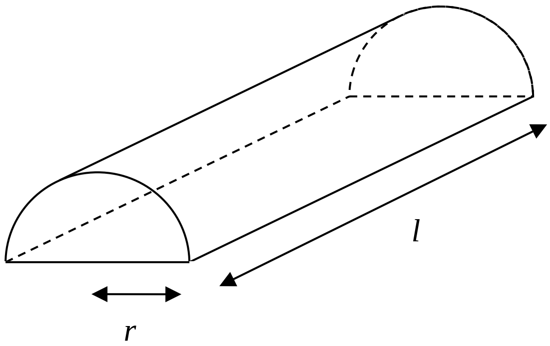
(i) Show that the total surface area, A cm^2, of the block is given by $A = 4x^2 + \frac{216}{x}$.

(3 marks)

(ii) Hence find the value of x such that the value of A is a minimum.

(7 marks)

2) 2013/14S1 IT1101/1501/1561/1751/1621 Sem Exam– Q7



The figure above shows a roof top with a horizontal rectangular base. The vertical semicircle and the curved roof are made of aluminium. The radius of each semicircle is r metres (m) and the length of the roof is l metres (m).
(Note that area of circle is πr^2 while the circumference of circle is $2\pi r$.)

(a) Given that 1200 m^2 of aluminium is used for the roof top, show that the volume, V m^3, of the roof is given by $V = 600r - \frac{\pi r^3}{2}$. (6 marks)

(b) Given that r may vary, find the value of r for which V has a stationary value. Find this value of V, leaving your answer to the nearest **m³** and determine whether it is a maximum or minimum value. (9 marks)

3) 2010/11S1 IT1101/1501/1561/1751/1621 Sem Exam– Q9

(a) Given that the gradient of the tangent to the curve $y = ax^3 + bx^2 + 3$ at the point $(1, 4)$ is 7, calculate the values of a and b. (6 marks)

(b) Given the function $y = -3(x+1)^{4/3}$, find

(i) the critical value of x. (2 marks)

(ii) the intervals where the function is increasing and decreasing.

(2 marks)

Answers

1ai) $4x^2 + \dfrac{216}{x}$

ii) $x = 3$

2a)

$$\pi r^2 + \frac{1}{2}\left(2\pi r\right)l = 1200$$
$$\pi rl = 1200 - \pi r^2$$
$$l = \frac{1}{\pi r}(1200 - \pi r^2)$$

$$V = \frac{1}{2}\left(\pi r^2\right)l$$
$$= \frac{\pi r^2}{2} \cdot \frac{1}{\pi r}\left(1200 - \pi r^2\right)$$
$$= \frac{r}{2}\left(1200 - \pi r^2\right)$$
$$= 600r - \frac{\pi r^3}{2}$$

b) $V \approx 4514$ m^3 maximum

3a) $a = 5$, $b = -4$

bi) $x = -1$

bii) The function is increasing for $x < -1$ and decreasing for $x > -1$.