AS.110.420 Homework 4

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Problem 1 Writing the event $X = X_i \leq x$ in shorthand, by law of total probability $P(X) = P(X \cap A_1) + P(X \cap A_2) = P(A_1)P(X|A_1) + P(A_2)P(X|A_2)$. However, notice that $P(X|A_1)$ is exactly how we defined $P(X_1)$, and likewise with 2. We then have $F(x) = aF_1(x) + (1-a)F_2(x)$. In other words the "weighted percentage" rule we used to compute direct probability will also apply for distributions

Problem 2

1. $g_1(x)$ is positive on its entire support if c_1 is chosen positive: check. We also need $\int_{-1}^{1} c_1(x-1)^2 = 1$ for the function to qualify as a pdf. By u-substitution on $x-1$ we find that $c_1(0+\frac{8}{3}) = 1$, in other words $c_1 = \frac{3}{8}$. This makes g_1 valid as a pdf.

2. $\text{sign}(c_2(-1)) \neq \text{sign}(c_2(1))$ for any choice of c_2. Because we cannot allow negative density values inside the pdf, g_2 cannot be valid.

3. For choice of negative c_3, g_3 has the potential to be a pdf. We need $\int_{-1}^{1} c_3(x-1)dx = 1$. Meaning, $c_3(0-2) = 1$, or $c_3 = \frac{-1}{2}$. With this choice g_3 is a valid pdf.

Problem 3

1. Because $\alpha > 0$, $x > 0$, the fraction is greater than 0 for all x in the support. This satisfies our first condition for a pdf. The second is we need $\alpha \int_{1}^{\infty} \frac{1}{x^{\alpha+1}} = 1$. We can safely include 1 in the integral because the variable is continuous, so the probability at any particular point is irrelevant. We then $\int_{1}^{\infty} x^{-\alpha-1}dx = \alpha\big[\frac{-1}{\alpha}x^{-\alpha}\big|_{1}^{\infty} = \alpha(\frac{1}{\alpha}) = 1$. The claim checks out.

2. The CDF $P(X \leq x)$ can be computed directly from the integral $\int_{1}^{x} \frac{\alpha}{x^{\alpha+1}}dx = \frac{-\alpha}{\alpha}\big[x^{-\alpha}\big|_{1}^{x} = -x^{\alpha} + 1$.

3. We can compute $E(X)$ directly by $\int_{0}^{\infty} x\frac{\alpha}{x^{\alpha+1}}dx = \int_{0}^{\infty} \alpha x^{-\alpha} = \frac{\alpha}{-\alpha+1}\big[x^{-\alpha+1}\big|_{0}^{\infty} = \frac{\alpha}{-\alpha+1}(0-1) = \frac{\alpha}{\alpha-1}$. This expression will only evaluate to a finite value if $\alpha > 1$, (because for alpha less than or equal to 1 the integrand will be a positive power or a natural log, both of which do not converge).

4. We can first compute $E(X^2) = \alpha\int_{0}^{\infty} x^2x^{-\alpha-1}dx = \alpha\int_{0}^{\infty} x^{-\alpha+1}dx = \alpha\big[\frac{1}{-\alpha+2}x^{-\alpha+2}\big|_{0}^{\infty} = \frac{\alpha}{\alpha-2}$. For this to be true we also need to ask that

$-\alpha+1<-1$, or $\alpha>2$. If this is true, then $Var(X)=E(X^2)-E(X)^2=\frac{\alpha}{\alpha-2}-(\frac{\alpha}{\alpha-1})^2=$

Problem 4 1. We note that the Bernoulli distribution has a distribution $p^x(1-p)^{1-x}$ for $x=0,1$. It follows that $E((\frac{X-\mu}{\sigma})^3)=\frac{1}{\sigma^3}E((X-\mu)^3)=\frac{1}{\sigma^3}\left[E(X^3)-3E(X^2)E(X)+3E(X)E(X)^2-E(X)^3\right]$. However the kth moment of a Bernoulli distribution is given by $\sum_{x=0}^{1}x^kp^x(1-p)^{1-x}$, which is p for all k. From this, it follows that σ^2 is given by $E(X^2)-E(X)^2=p-p^2=p(1-p)$. We can then simplify our expression to $\frac{p-3p^2+3p^3-p^3}{p(1-p)(p(1-p))^{1/2}}=\frac{2p^3-3p^2+p}{p(1-p)(p(1-p))^{1/2}}=\frac{p(2p-1)(p-1)}{p(p-1)(p(1-p))^{1/2}}=\frac{(2p-1)}{(p(1-p))^{1/2}}$. The denominator is always positive for any choice of p between 0 and 1. We note that for $p>\frac{1}{2}$ this is positive (and the distribution is negatively skewed) and negative for $p<\frac{1}{2}$, where it is positively skewed. at $p=\frac{1}{2}$, the distribution has no skewness.

2. By a similar method as above, $E((\frac{X-\mu}{\sigma})^4)=\frac{1}{p^2(1-p)^2}(E(X^4)-4E(X^3)E(X)+6E(X^2)E(X)^2)-4E(X)E(X)^3+E(X)^4)=\frac{p-4p^2+6p^3-4p^4+p^4}{(p^2(1-p)^2)}$. We note that numerator(0)=numerator(1)=0 by inspection so we can reduce to $\frac{-p(p-1)(3p^2-3p+1)}{p^2(1-p)^2}=\frac{3p^2-3p+1}{p(1-p)}$. We want this ≥ 3 so $3p^2-3p+1\geq 3p(1-p)$. Completing the square on $6p^2-6p+1\geq 0$, we get $6(p-\frac{1}{2})^2\geq\frac{1}{2}$, or $(p-\frac{1}{2})^2\geq\frac{1}{12}$. The square root function is monotonically increasing on its domain so we can write $|p-\frac{1}{2}|\geq\sqrt{\frac{1}{12}}$ for our condition. This is equivalent to $p-\frac{1}{2}\geq\sqrt{\frac{1}{12}}$, $p-\frac{1}{2}\leq-\sqrt{\frac{1}{12}}$. It follows that $p\geq\frac{1}{2}+\sqrt{\frac{1}{12}}$, $p\leq\frac{1}{2}-\sqrt{\frac{1}{12}}$ are the two bounds which satisfy this.

Problem 5 1. We write $P(X>t)=1-P(X\leq t)$. $exp(\lambda)$ has distribution $\lambda e^{-\lambda x}$, so it follows that the CDF is $\int_0^x\lambda e^{-\lambda y}dy=-e^{-\lambda y}\Big|_0^x=-e^{-\lambda x}+1$. It follows that $P(X>t)=1-(1-e^{-\lambda x})=e^{-lambdax}$, as sought.

2. By direct application of the conditional probability expression $P(X>s+t|X>s)=\frac{P(X>s+t\cap X>s)}{P(X>s)}$. Notice that for $t>0$, $X>s+t\subset X>s$, thus $=\frac{P(X>s+t)}{P(X>s)}=\frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}=e^{-\lambda t}$. Of course this is identically $P(X>t)$, as we sought to prove.

Problem 6 We just have to make sure that the expected value of Prof. Torcaso's loss is equal to the expected value of his gain. We know that a binomial distribution has expected value np, so the average amount of throws made is $10*0.25=2.5$. The expected value of throws missed must then be 7.5. We impose $7.5*5=2.5*c$, such that c must be 15$ for the game to be fair.

Problem 7 $\int_0^\infty\frac{e^{-x/2}}{\sqrt{x}}dx=\int_0^\infty x^{-1/2}e^{-x/2}dx$. We apply a change of bases with $u=\frac{x}{2}$ such that $dx=2du$, $x^{-1/2}=2*2^{-1/2}u^{-1/2}=\sqrt{2}u^{-1/2}$ and the integral is $\sqrt{2}\int_0^\infty u^{-1/2}e^{-u}$. At this point we notice that the integrand is in the form of the gamma function, specifically, $\Gamma(\frac{1}{2})=\sqrt{\pi}$. We then have $\sqrt{2\pi}$.

Problem 8

1. I would expect $E(X^4) = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\beta^4$, and by the same token $E(X^k) = \frac{(\alpha+k-1)!}{(\alpha-1)!}\beta^k$.

2. $\frac{1}{X} = X^{-1}$ so by blind application of the formula above, $E(Y)$ would equal $\frac{(\alpha-2)!}{(\alpha-1)}\beta^{-1} = \frac{1}{\beta(\alpha-1)}$. To do this however, we need to have $\alpha \geq 2$, since we have not defined the factorial operation for negative numbers, and also $\beta \neq 0$, because we don't like division by 0.

Problem 9 As suggested, we can write $E(X) = \sum_0^\infty xP(X=x) = \sum_{\sigma=0}^x \sum_{x=0}^\infty)P(X = x)$. We can now apply a change of bounds in the calculus sense, where if x goes from 0 to infinity, and sigma goes from x to infinity, this is the same as sigma going from 0 to infinity, and x going from sigma to infinity. $\sum_{\sigma=0}^\infty \sum_{x=\sigma}^\infty P(X = x)$.We notice that the inner integrand is identically $P(X > \sigma)$. Simplifying to $\sum_{\sigma=0}^\infty P(X > \sigma)$. Keeping in mind that $P(X > n) = P(X \geq n)$ for continuous variable, we have shown the result.