Name: Lj Gonzales

Assignment: HW 12

Due Date: Friday December 1

Problem 1.

Solution. $\square$

Problem 2. Let $I = [0, 1]$. Do the following:

- Let $f_n(x) = \frac{x^n}{n}$ on I. Show that $\{f_n(x)\}$ converges uniformly to a differentiable function f on I by finding f. Then show that $f'(1) \neq \lim_{n\to\infty} f_n'(1)$.
- For g a Riemann integrable (hence bounded) function on I, find $\lim_{n\to\infty} \int_0^1 \frac{g(x)}{n} dx$.

Solution. We claim that $\{f_n(x)\} \to 0$. To see this, see that

$$|f_n(x) - 0| = |\frac{x^n}{n}| \leq |\frac{1}{n}|.$$

If x is in $[0, 1]$. Now for any $\epsilon > 0$, we can choose any $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Then, for all $n > N$, we have

$$|f_n(x) - f(x)| < \epsilon.$$

and since x was chosen arbitrarily in $[0, 1]$, the convergence is uniform. We then see that $\lim_{n\to\infty} f_n'(x) = x^{n-1}$, which, evaluated at $x = 1$, gives a value of 1. This is not equal to $(0)'|_{x=1} = 0$, as sought. $\square$

Solution. We will try to estimate the expression $|\lim_{n\to\infty} \int_0^1 \frac{g(x)}{n} dx|$. If we can bound this above by 0 (since it is trivially bounded below by 0), we can use a sandwich argument to conclude that $\lim_{n\to\infty} \int_0^1 \frac{g(x)}{n} dx = 0$.
Note that $g(x)$ is bounded, in other words, $|g(x)| < M$ for some real M. We use the triangle inequality for integrals and theorem 6.2.4, which states that $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$ if $\{f_n\}$ converges to f uniformly. See the first part of the exercise for a proof.

$$|\lim_{n\to\infty} \int_0^1 \frac{g(x)}{n} dx| \leq \lim_{n\to\infty} \int_0^1 \frac{M}{|n|} dx = M \lim_{n\to\infty} \int_0^1 \frac{1}{n} dx = 0.$$

Because its absolute value is sandwiched between 0 and 0 we have $\lim_{n\to\infty} \int_0^1 \frac{g(x)}{n} dx$=0, as sought. $\square$

Problem 3. Find an example of a sequence of continuous functions on $(0,1)$ that converges pointwise to a continuous function on $(0,1)$, but the convergence is not uniform.

Solution. Consider the sequence of functions defined by $f_n(x) = \frac{1}{x+\frac{1}{n}}$. The sequence converges to $f(x) = \frac{1}{x}$.
The sequence converges pointwise. To see this, we write

$$|\frac{1}{x+\frac{1}{n}} - \frac{1}{x}| = |\frac{x-x-\frac{1}{n}}{x(x+\frac{1}{n})}| = \frac{1}{n}|\frac{1}{x(x+\frac{1}{n})}|.$$

By choosing $N > \max(\frac{2x^2}{\epsilon}, \frac{1}{x})$, we can further estimate this:

$$< \frac{1}{n}|\frac{1}{x(x+x)}| = \frac{1}{n}|\frac{1}{2x^2}| < \epsilon.$$

Hence, the function converges pointwise. However as we have seen in class, it does not converge uniformly. □

Problem 4. Suppose $f_n : [a, b] \to \mathbb{R}$ is a sequence of continuous functions that converges pointwise to a continuous $f : [a, b] \to \mathbb{R}$. Suppose that for every $x \in [a, b]$, the sequence $\{|f_n(x) - f(x)|\}$ is monotone. Show that the sequence $\{f_n\}$ converges uniformly.

Solution. Be given $\epsilon > 0$, and consider the function $g : x \in [a, b] \to \mathbb{N}$ $|f_n(x) - f(x)| < \epsilon$ is satisfied for all $n \geq N$. We are guaranteed such an x by the definition of pointwise convergence.
Consider now the function $|f_N(x) - f(x)|$ over the closed bounded interval $x \in [a, b]$. It is continuous, because f_N, f are. By the Extreme Value Theorem, it must attain its maximum on some $x_0 \in [a, b]$. If $|f_N(x_0) - f(x_0)| \geq \epsilon$, we can repeat step 1 of this procedure with $x = x_0$ to get an N_2 such that for all $n > N_2$, $|f_{N_2}(x_0) - f(x_0)| < \epsilon$. Because the sequence $\{|f_n(x) - f(x)|\}$ is monotonically decreasing for all x $\square$

Problem 5. Find a sequence of Lipschitz continuous functions on $[0, 1]$ whose uniform limit is $\sqrt{x}$, which is a non-Lipschitz function.

Solution. Consider the sequence given by

$$f_n(x) = \sqrt{x + \frac{1}{n}}.$$

Since $\sqrt{x}$ fails Lipschitz continuity when its argument is less than or equal to 0, all elements of the sequence are Lipschitz continuous everywhere on $[0, 1]$. In the limit however, we get the same problem studied in class where the slope is undefined at 0. $\square$