AS.110.420 Problem Set 7

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Problem 1 Given X has pdf $\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}$, we have that $Y = cX$ has pdf $\frac{(cx)^{\alpha-1}e^{-(cx)/\beta}}{\beta^\alpha\Gamma(\alpha)} = \frac{c^{\alpha-1}}{c^\alpha}\frac{x^{\alpha-1}e^{-x(\frac{c}{\beta})}}{\frac{\beta}{c}^\alpha\Gamma(\alpha)} = \frac{1}{c}\text{Gamma}(\alpha, \frac{\beta}{c})$.
On the other hand, the event $Z = cX + d$ does not have a Gamma distribution, because expanding the numerator $(cx)^{\alpha-1}$ in the binomial sense will give the first term looking like a Gamma distribution, but for $\alpha \neq 1$ there will be additional terms which cannot be reduced in the same way.

Problem 2 $P(X > x) = 1 - P(X \leq x) = 1 - \int_0^x \tau^{\alpha-1}e^{\frac{-\tau}{\beta}}$. Because the limit of the quotient is the quotient of the limit, we can first compute $lim_{x\to\infty}m(x) = lim_{x\to\infty}\frac{1-\int_0^x \tau^{\alpha-1}e^{\frac{-\tau}{\beta}}}{x^{\alpha-1}e^{-\alpha/\beta}}$. Applying L'Hopital's rule we this limit is equivalently $lim_{x\to\infty}\frac{-x^{\alpha-1}e^{-\alpha/\beta}}{(\alpha-1)e^{-\alpha/\beta}+x^{\alpha-2}+x^{\alpha-1}(\frac{1}{\beta})e^{-x/\beta}} = lim_{x\to\infty}\frac{x^{\alpha-1}e^{-x/\beta}(-1)}{x^{\alpha-1}e^{-x/\beta}\left[(\alpha-1)x^{-1}-\frac{1}{\beta}\right]}$. Multiplying both numerator and denominator by x we arrive at the result $m(x) = lim_{x\to\infty}\frac{-x}{(\alpha-1)-\frac{x}{\beta}} = \lim_{x\to\infty}\frac{x}{\frac{x}{\beta}-(\alpha-1)}$. The result does hold.

Problem 3

1. $y(\theta) := ln(M(\theta))$. We can apply the chain rule with $f(\theta) = \ln(\theta), g(\theta) = M(\theta)$ such that $y'(\theta) = \frac{M'(\theta)}{M(\theta)}$, where $M(0) = 1, M'(0) = \mu$. It does follow that $y'(0) = \mu$. Likewise, using the quotient rule on the newfound y', we have $y''(\theta) = \frac{M(\theta)M''(\theta)-M'(\theta)M'(\theta)}{M(\theta)^2} = \frac{E(X^2)-E(X)^2}{1^2} = Var(X)$, as sought.

2. We can derive the moment generating function of X to be $M(\theta) = (1 - \beta\theta)^{-\alpha}$, and y is defined as above $y(\theta) = ln[(1-\beta\theta^{-\alpha})] = -\alpha ln(1-\beta\theta)$. By iterative application of the chain rule we have $y'(\theta) = -\alpha\frac{1}{1-\beta\theta}(-\beta)$, $y''(\theta) = \alpha\beta\frac{d}{d\theta}(\frac{1}{1-\beta\theta}) = \alpha\beta(\frac{\beta}{(1-\beta\theta)^2})$. We indeed have $y'(0) = \alpha\beta$, $y''(0) = \alpha\beta^2$, as sought.

Problem 4 X unif$(a, b) = \frac{1}{b-a}$ and has pdf $\int_a^x \frac{1}{b-a}$ for $a \geq x < b$. $P(Y < y) = P(cX + d < y) = P(X < \frac{y-d}{c})$, for c¿0. For c¡0, we will have $P(X > \frac{y-d}{c})$, the complement of the previous. We then have $\int_a^{\frac{y-d}{c}} \frac{1}{b-a}dx = \left[\frac{x}{b-a}\right|_a^{\frac{y-d}{c}} = \frac{y-d}{c(b-a)} - \frac{a}{b-a}$, which is the pdf of Y. The result would be 1 minus that if c¡0.

Problem 5 We have $P(X \leq x) = F(X)$, and $P(Y \leq y) = P(F(X) \leq y)$. At this point we should check that $F(X)$ has an inverse if we want to write

$P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y$. F is a CDF: because we cannot have a probability of an outcome being two numbers at once, F is surjective. Hence it has an inverse, and the above is valid. Note that a uniform(1,0) distribution has CDF $\int_0^y \frac{1}{1-0} dx = y - 0 = y$. This is the same as found previously, thus Y has the same uniform distribution.

Problem 6 $E(X) = \int x f(x) dx = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta-1} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \text{Beta}(\alpha+1, \beta)$, where we noticed that the integrand looked like a Beta distribution with parameters alpha+1, beta, and only needed to multiply and divide by its constant parameter to make it go to 1. Simplifying, we have $E(X) = \frac{\Gamma(\alpha+\beta)(\alpha+1)\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta)(\alpha+\beta+1)\Gamma(\alpha+\beta)} = \frac{(\alpha+1)}{(\alpha+\beta+1)}$.

Likewise, $E(X^2) = \int_0^1 x^2 f(x) dx = \int_0^1 x^{\alpha+1}(1-x)^{\beta} dx$. The integrand looks close to a Beta(α+2, β) distribution, if we just multiply and divide by $\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta)}$. Making the integrand go to one as in part a, we have $E(X^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{(\alpha+2)(\alpha+1)\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta+2)(\alpha+\beta+1)\Gamma(\alpha+\beta)} = \frac{(\alpha+1)(\alpha+2)}{(\alpha+\beta+2)(\alpha+\beta+1)}$.

Problem 7 $P(Y = y) = P(-ln(X) = y) = P(X = e^{-y}) = \frac{1}{e^{-y}-0} dx = \frac{1}{e^{-y}} = e^y = \frac{1}{x}$.

Problem 8

1. $P(Z < 1.27) = 0.8980$
2. $P(Z > 2.17) = 1 - P(Z < 2.17) = 0.015$
3. $P(Z > -1.27) = 1 - P(Z < -1.27) = 0.898$
4. $= P(Z \leq 2.17) - P(Z \leq 1.27) = 0.087$
5. $= P(Z < 2.17) - P(Z < -1.27) = 0.748$
6. For the remaining exercises, we let Y be the event $\frac{X-20}{100}$, which is normalized to have a distro $N(0,1)$. We'll then have $P(X = x) = P(Y = \frac{x-20}{100})$, by a property of the normal distribution. $P(Y \leq 0.127) \approx P(Y \leq 0.13)$. If precision is important, we could also linearly extrapolate from the table, but ≈ 0.5517.
7. $P(Y > \frac{21.7}{100}) \approx 1 - P(Y < 0.22) = 0.4129$.
8. $P(Y > \frac{-21.7}{100}) \approx 1 - P(Y < -0.02) = 0.508$.
9. $= P(X \leq 41.7) - P(X \leq 32.7) = 0.0354$.
10. $= P(X \leq 41.7) - P(X \leq 32.7) = 0.0354$.