Scalable Underapproximative Verification of Stochastic LTI Systems using Convexity and Compactness

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April 11, 2018

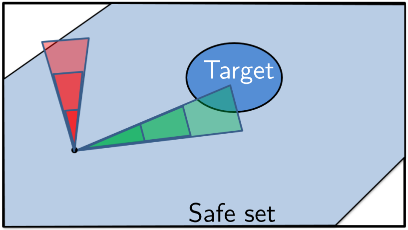
HSCC 2018 — Porto, Portugal

Motivation

- Real world problems — stochastic and high-dimensional
  - Motion planning in stochastic environments — transportation
  - Human-automation collaboration systems — biomedical systems
- Need for probabilistic guarantees of safety and performance

Motivation



- Stochastic reach-avoid problem — viability and reachability

$$\begin{array}{ll} \text{maximize} & \mathbb{P}\{\textit{stay}\text{ safe and }\textit{reach}\text{ target at the time horizon}\} \\ \text{subject to} & \text{dynamics, initial state, policy constraints} \end{array}$$

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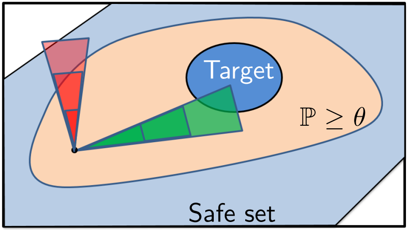

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- High computation costs and lacks scalability

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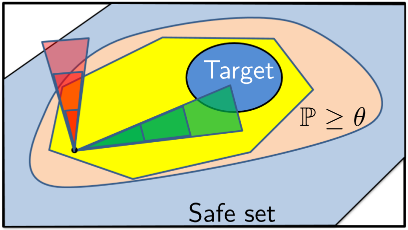

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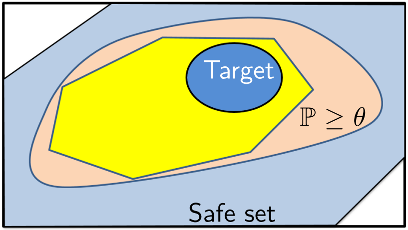
$\boxed{\text{Objective: Compute a polytopic underapproximation of } \mathbb{P} \geq \theta}$

Main contributions



- Polytopic underapproximation of stochastic reach-avoid set
  - Open-loop underapproximation
  - Convex compact sets
- Sufficient conditions for closed, compact, and convex
  - Stochastic reach-avoid set
  - Open-loop underapproximation
- Admittance of optimal bang-bang Markov policies

Related work

Backward stochastic reachability
Abate, Amin, Prandini, Lygeros, & Sastry (2007); Abate, Prandini, Lygeros, & Sastry (2008); Summers, & Lygeros (2010)

Approximation techniques
Lesser, Oishi, & Erwin (2013); Manganini, Pirotta, Restelli, Piroddi, & Prandini (2015); Kariotoglou, Kamgarpour, Summers, & Lygeros (2017); Gleason, Vinod, & Oishi (2017); Vinod & Oishi (2017)

Forward stochastic reachability
Lasota & Mackey (1985); Althoff, Stursberg, & Buss (2009); Vinod, HomChaudhuri, & Oishi (2017); HomChaudhuri, Vinod, & Oishi (2017)

Uncertain system reachability (discrete time)
Bertsekas and Rhodes (1971); Girard (2005); Kurzhanskiy & Varaiya (2006); Kvasnica, Takács, Holaza, & Ingole (2015); Althoff (2015); Bak & Duggirala (2017)

Uncertain system reachability (continuous time)
Tomlin, Mitchell, Bayen, & Oishi (2003); Bokanowski, Forcadel, & Zidani (2010); Huang, Ding, Zhang, & Tomlin (2015); Chen, Herbert, & Tomlin (2017)

System formulation

- Discrete-time LTI system (time horizon N and initial state $\overline{x}_0 \in \mathcal{X}$)

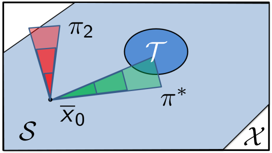
$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\overline{u}_k + \boldsymbol{w}_k \qquad \begin{aligned} &\boldsymbol{x}_k \in \mathcal{X} = \mathbb{R}^n, \overline{u}_k \in \mathcal{U} \subseteq \mathbb{R}^m \\ &\boldsymbol{w}_k \in \mathcal{W} \subseteq \mathbb{R}^p, \boldsymbol{w}_k \sim \psi_{\boldsymbol{w}} \end{aligned}$$

- ($\equiv$) Continuous state Markov decision process (when IID noise)

$$Q(d\overline{y}|\overline{x}, \overline{u}) = \psi_{\boldsymbol{w}}(\overline{y} - A\overline{x} - B\overline{u})d\overline{y}$$

- $\boldsymbol{X} = [\boldsymbol{x}_1^\top \ \ldots \ \boldsymbol{x}_N^\top]^\top \in \mathcal{X}^N$ random vector with probability measure
  - $\mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0} \leftarrow$ Markov (closed-loop) policy $\pi : \mathbb{N}_{[0,N-1]} \times \mathcal{X} \to \mathcal{U}$, $\pi \in \mathcal{M}$
  - $\mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0} \leftarrow$ Open-loop policy $\rho : \mathcal{X} \to \mathcal{U}^N$, $\rho(\overline{x}_0) \notin \mathcal{M}$

Stochastic reach-avoid problem and its solution



- Borel $\mathcal{S}, \mathcal{T} \subseteq \mathcal{X}$
- Terminal stochastic reach-avoid problem

$$V_0^*(\overline{x}_0) = \underset{\pi}{\max.} \;\; \mathbb{P}_{\boldsymbol{X}}^{\pi, \overline{x}_0} \left\{ \boldsymbol{X} \in \left(\mathcal{S}^{N-1} \times \mathcal{T} \right) \right\}$$
$$\text{s. t.} \quad \pi \in \mathcal{M}$$

  - Solution via dynamic programming

$$V_N^*(\overline{x}) = 1_{\mathcal{T}}(\overline{x})$$
$$V_k^*(\overline{x}) = \sup_{\overline{u} \in \mathcal{U}} 1_{\mathcal{S}}(\overline{x}) \int_{\mathcal{X}} V_{k+1}^*(\overline{y}) Q(d\overline{y} | \overline{x}, \overline{u})$$

$$\overline{x}_0 \in \mathcal{S}$$
$$\boldsymbol{x}_1 \in \mathcal{S}$$
$$\boldsymbol{x}_2 \in \mathcal{S}$$
$$\vdots$$
$$\boldsymbol{x}_{N-1} \in \mathcal{S}$$
$$\boldsymbol{x}_N \in \mathcal{T}$$
$$\boldsymbol{X} \in \mathcal{S}^{N-1} \times \mathcal{T}$$

- Discretization approach
  - For compact $\mathcal{U}, \mathcal{S}, \mathcal{T}$ and Lipschitz Q
  - Curse of dimensionality — $n \leq 3$

Abate, Amin, Prandini, Lygeros, and Sastry, HSCC 2007
Summers and Lygeros, Automatica 2010

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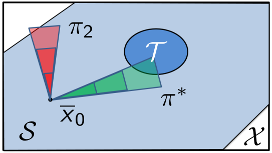

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$$V_0^*(\overline{x}_0) = \underset{\pi}{\max.} \ \mathbb{E}_{\overline{x}_0}^{\pi}\left[\left(\prod_{k=0}^{N-1} 1_{\mathcal{S}}(\boldsymbol{x}_k)\right) 1_{\mathcal{T}}(\boldsymbol{x}_N)\right] \quad \text{s. t.} \quad \pi \in \mathcal{M}$$

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$\overline{x}_0 \in \mathcal{S}$
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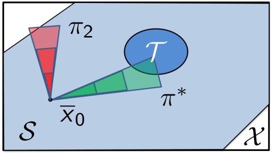

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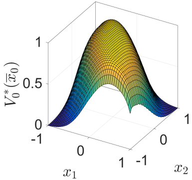

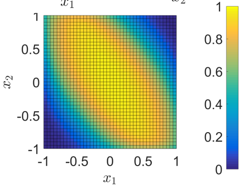

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Underapproximation of the stochastic reach-avoid problem

Stochastic reach-avoid problem	Underapproximation
$\begin{array}{rl} \text{max.} & \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\} \\ \text{s.t.} & \pi \in \mathcal{M} \end{array}$	$\begin{array}{rl} \text{max.} & \mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\} \\ \text{s.t.} & \rho(\overline{x}_0) \in \mathcal{U}^N \end{array}$
Optimal value function $V_0^\ast(\overline{x}_0)$	Optimal value function $W_0^\ast(\overline{x}_0)$
Search over Markov policies	Search over open-loop policies

Vinod and Oishi, LCSS 2017
Vinod, HomChaudhuri, and Oishi, HSCC 2017

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max. $\mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\}$ s.t. $\pi \in \mathcal{M}$	max. $\int_{\mathcal{S}^{N-1}\times\mathcal{T}} \psi_{\boldsymbol{X}}(\overline{X};\rho,\overline{x}_0)d\overline{X}$ s.t. $\rho(\overline{x}_0) \in \mathcal{U}^N$
Optimal value function $V_0^*(\overline{x}_0)$	Optimal value function $W_0^*(\overline{x}_0)$
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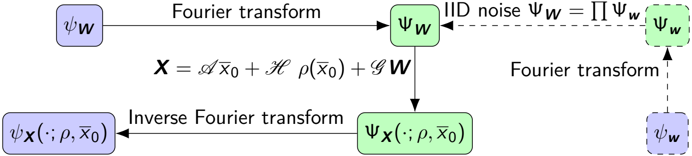

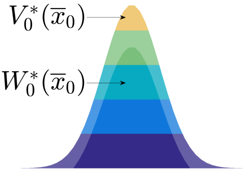

$$\boxed{W_0^*(\overline{x}_0) \leq V_0^*(\overline{x}_0) \text{ a.s.}}$$

Quadrature $\int \psi(\overline{X}; \rho, \overline{x}_0) d\overline{X}$ + `patternsearch`

Vinod and Oishi, LCSS 2017

Vinod, HomChaudhuri, and Oishi, HSCC 2017

Underapproximation of stochastic reach-avoid sets

- Stochastic reach-avoid set and its underapproximation, $\theta \in [0, 1]$

$$\mathcal{L}^{\pi^*}(\theta, \mathcal{S}, \mathcal{T}) = \{\overline{x}_0 \in \mathcal{X} : V_0^*(\overline{x}_0) \geq \theta\}$$
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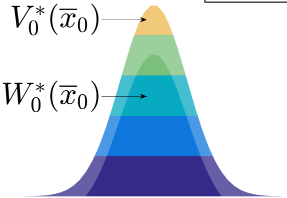

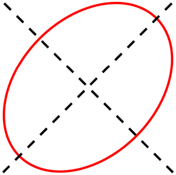

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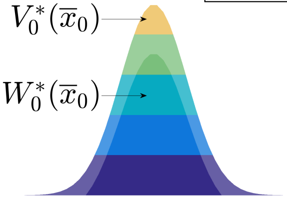

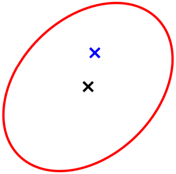

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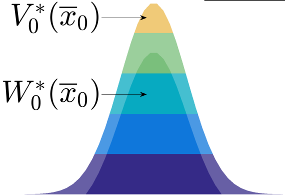

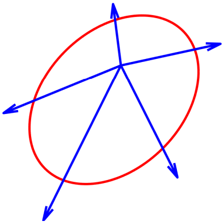

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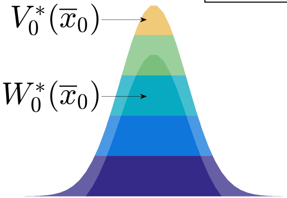

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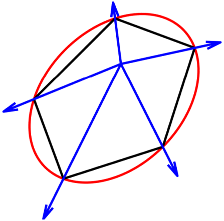

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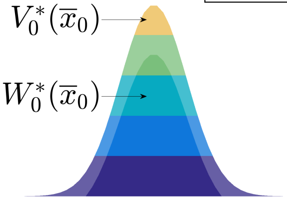

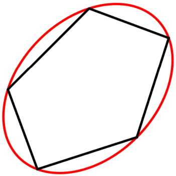

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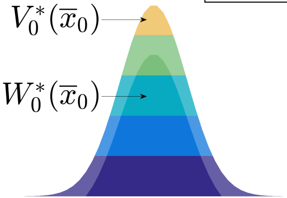

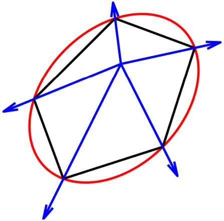

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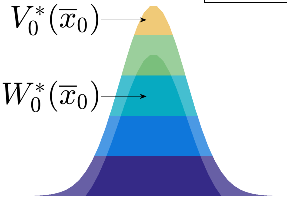

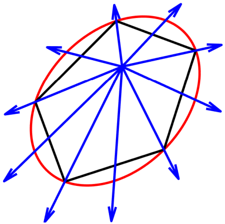

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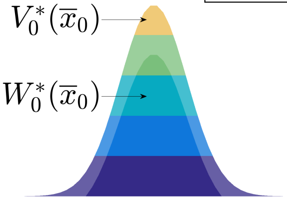

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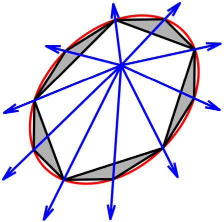

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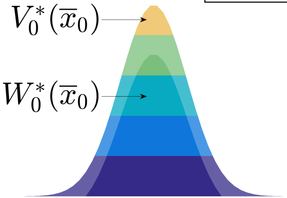
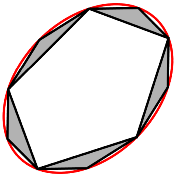
$\boxed{W_0^*(\overline{x}_0) \leq V_0^*(\overline{x}_0) \text{ a.s.}}$





Problem statements

Q1 Compute a tight polytopic underapproximation of $\mathcal{K}^{\rho^*}(\theta, \mathcal{S}, \mathcal{T})$ in a grid-free manner

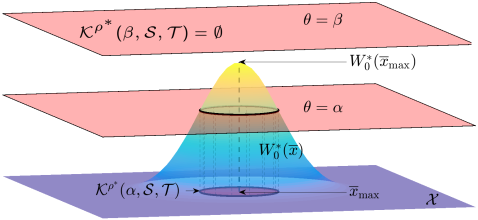
Q2 When are $\mathcal{L}^{\pi^*}(\theta, \mathcal{S}, \mathcal{T})$ and $\mathcal{K}^{\rho^*}(\theta, \mathcal{S}, \mathcal{T})$ convex and compact?

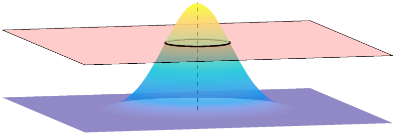
- Log-concavity and upper semicontinuity of $V_0^*(\overline{x}_0)$ and $W_0^*(\overline{x}_0)$
- Admittance of bang-bang optimal Markov policies

Tight polytopic underapproximation of $\mathcal{K}^{\rho^*}(\theta, \mathcal{S}, \mathcal{T})$

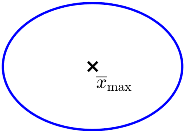
Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x}, \overline{u})$	$\mathcal{U}$
$V_0^*(\overline{x})$ and $W_0^*(\overline{x})$	$\mathcal{L}^{\pi^*}(\theta)$ and $\mathcal{K}^{\rho^*}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex
upper semi-continuity	Closed	Closed	Closed	Continuous	Compact
—	Compact $\theta \in (0, 1]$	Compact			

$$\beta > W_0^*(\overline{x}_{\max}) > \alpha > 0$$





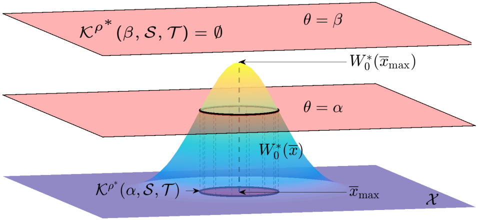
$|D|$ bisections to ϵ accuracy
Runtime: $\mathcal{O}(|D| \log(\lceil \epsilon \rceil))$

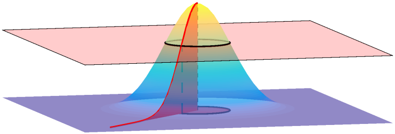


Tight polytopic underapproximation of $\mathcal{K}^{\rho^*}(\theta, \mathcal{S}, \mathcal{T})$

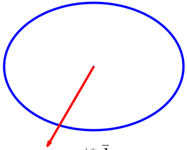
Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^*(\overline{x})$ and $W_0^*(\overline{x})$	$\mathcal{L}^{\pi^*}(\theta)$ and $\mathcal{K}^{\rho^*}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex
upper semi-continuity	Closed	Closed	Closed	Continuous	Compact
—	Compact $\theta \in (0,1]$	Compact			

$$\beta > W_0^*(\overline{x}_{\max}) > \alpha > 0$$





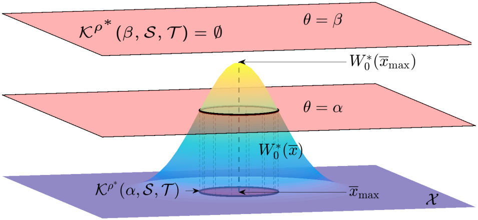
$|D|$ bisections to ϵ accuracy
Runtime: $\mathcal{O}(|D| \log(\lceil \epsilon \rceil))$

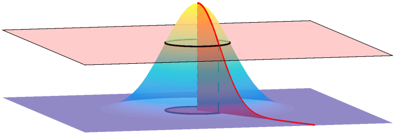


Tight polytopic underapproximation of $\mathcal{K}^{\rho^\ast}(\theta, \mathcal{S}, \mathcal{T})$

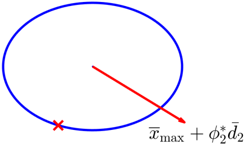
Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^\ast(\overline{x})$ and $W_0^\ast(\overline{x})$	$\mathcal{L}^{\pi^\ast}(\theta)$ and $\mathcal{K}^{\rho^\ast}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex
upper semi-continuity	Closed	Closed	Closed	Continuous	Compact
—	Compact $\theta \in (0, 1]$	Compact			

$\beta > W_0^\ast(\overline{x}_{\max}) > \alpha > 0$





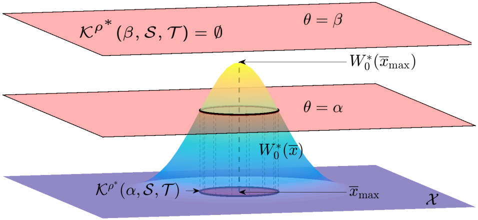
$|D|$ bisections to ϵ accuracy
Runtime: $\mathcal{O}(|D| \log(\lceil \epsilon \rceil))$

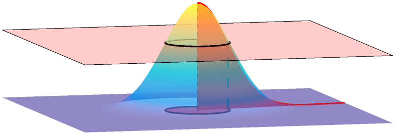


Tight polytopic underapproximation of $\mathcal{K}^{\rho^*}(\theta, \mathcal{S}, \mathcal{T})$

Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^*(\overline{x})$ and $W_0^*(\overline{x})$	$\mathcal{L}^{\pi^*}(\theta)$ and $\mathcal{K}^{\rho^*}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex
upper semi-continuity	Closed	Closed	Closed	Continuous	Compact
—	Compact $\theta \in (0, 1]$	Compact			

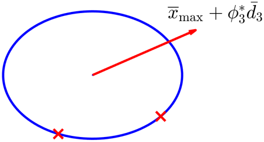
$\beta > W_0^*(\overline{x}_{\max}) > \alpha > 0$





$|D|$ bisections to ϵ accuracy

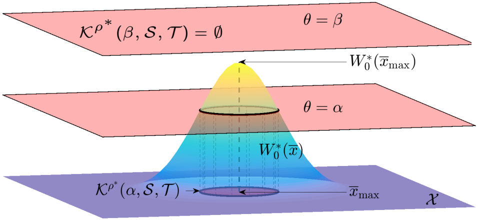
Runtime: $\mathcal{O}(|D| \log(\lceil \epsilon \rceil))$

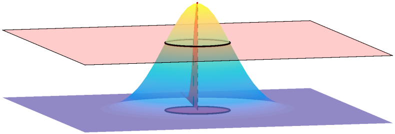


Tight polytopic underapproximation of $\mathcal{K}^{\rho^\ast}(\theta, \mathcal{S}, \mathcal{T})$

Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^\ast(\overline{x})$ and $W_0^\ast(\overline{x})$	$\mathcal{L}^{\pi^\ast}(\theta)$ and $\mathcal{K}^{\rho^\ast}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex
upper semi-continuity	Closed	Closed	Closed	Continuous	Compact
—	Compact $\theta \in (0,1]$	Compact			

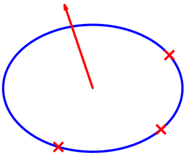
$$\beta > W_0^\ast(\overline{x}_{\max}) > \alpha > 0$$





$|D|$ bisections to ϵ accuracy

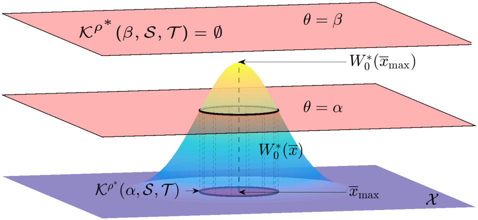
Runtime: $\mathcal{O}(|D| \log(\lceil \epsilon \rceil))$

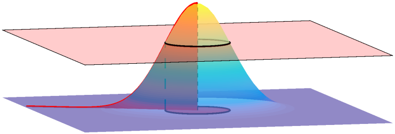


Tight polytopic underapproximation of $\mathcal{K}^{\rho^*}(\theta, \mathcal{S}, \mathcal{T})$

Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\mid\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^*(\overline{x})$ and $W_0^*(\overline{x})$	$\mathcal{L}^{\pi^*}(\theta)$ and $\mathcal{K}^{\rho^*}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex
upper semi-continuity	Closed	Closed	Closed	Continuous	Compact
—	Compact $\theta \in (0, 1]$	Compact			

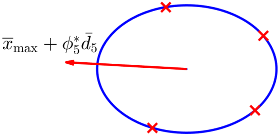
$\beta > W_0^*(\overline{x}_{\max}) > \alpha > 0$





$|D|$ bisections to ϵ accuracy

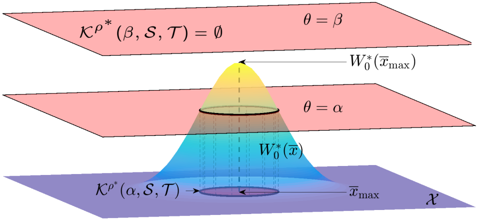
Runtime: $\mathcal{O}(|D| \log(\lceil \epsilon \rceil))$

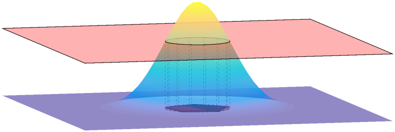


Tight polytopic underapproximation of $\mathcal{K}^{\rho^*}(\theta, \mathcal{S}, \mathcal{T})$

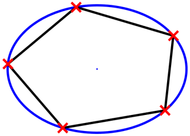
Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot \vert \overline{x}, \overline{u})$	$\mathcal{U}$
$V_0^*(\overline{x})$ and $W_0^*(\overline{x})$	$\mathcal{L}^{\pi^*}(\theta)$ and $\mathcal{K}^{\rho^*}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex
upper semi-continuity	Closed	Closed	Closed	Continuous	Compact
—	Compact $\theta \in (0,1]$	Compact			

$$\beta > W_0^*(\overline{x}_{\max}) > \alpha > 0$$





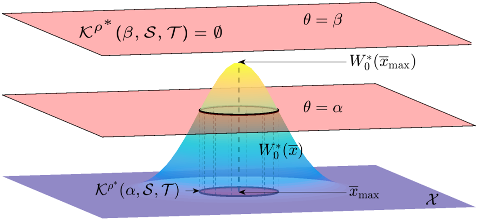
$|D|$ bisections to ϵ accuracy
Runtime: $\mathcal{O}(|D| \log(\lceil \epsilon \rceil))$

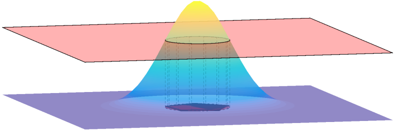


Tight polytopic underapproximation of $\mathcal{K}^{\rho^\ast}(\theta,\mathcal{S},\mathcal{T})$

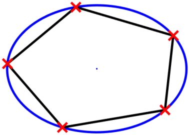
Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^\ast(\overline{x})$ and $W_0^\ast(\overline{x})$	$\mathcal{L}^{\pi^\ast}(\theta)$ and $\mathcal{K}^{\rho^\ast}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex
upper semi-continuity	Closed	Closed	Closed	Continuous	Compact
—	Compact $\theta \in (0,1]$	Compact			

$$\beta > W_0^\ast(\overline{x}_{\max}) > \alpha > 0$$





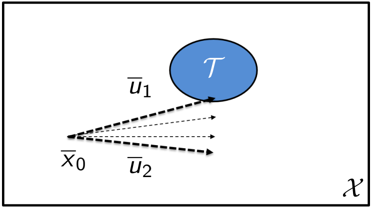
Grid-free, parallelizable, tight polytopic underapproximation!

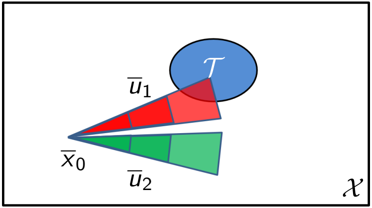


Special case: Bang-bang optimal Markov policies

Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^*(\overline{x})$ and $W_0^*(\overline{x})$	$\mathcal{L}^{\pi^*}(\theta)$ and $\mathcal{K}^{\rho^*}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex

- Avoid $\mathcal{T}$ at N:
  - Reach $\mathcal{X}\setminus\mathcal{T}$ at N,
  - $\mathcal{S}\leftarrow\mathcal{X}$, and
  - $\mathcal{U}=\mathrm{convexHull}(\mathcal{U}_{\mathrm{vertices}})$





Special case: Bang-bang optimal Markov policies

Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^\ast(\overline{x})$ and $W_0^\ast(\overline{x})$	$\mathcal{L}^{\pi^\ast}(\theta)$ and $\mathcal{K}^{\rho^\ast}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex

- Avoid $\mathcal{T}$ at N:
  - Reach $\mathcal{X} \setminus \mathcal{T}$ at N,
  - $\mathcal{S} \leftarrow \mathcal{X}$, and
  - $\mathcal{U} = \mathrm{convexHull}(\mathcal{U}_{\mathrm{vertices}})$

$$
\begin{aligned}
V_0^\ast(\overline{x}_0) &= \sup_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N \in \mathcal{X}\setminus\mathcal{T}\} \\
&= 1 - \inf_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N \in \mathcal{T}\} \\
&\triangleq 1 - V_{0,\mathrm{bang}}^\ast(\overline{x})
\end{aligned}
$$

Special case: Bang-bang optimal Markov policies

Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^\ast(\overline{x})$ and $W_0^\ast(\overline{x})$	$\mathcal{L}^{\pi^\ast}(\theta)$ and $\mathcal{K}^{\rho^\ast}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex

- Avoid $\mathcal{T}$ at N:
  - Reach $\mathcal{X}\setminus\mathcal{T}$ at N,
  - $\mathcal{S}\leftarrow\mathcal{X}$, and
  - $\mathcal{U}=\mathrm{convexHull}(\mathcal{U}_{\mathrm{vertices}})$

$$
\begin{aligned}
V_0^\ast(\overline{x}_0) &= \sup_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N\in\mathcal{X}\setminus\mathcal{T}\}\\
&= 1-\inf_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N\in\mathcal{T}\}\\
&\triangleq 1-V_{0,\mathrm{bang}}^\ast(\overline{x})
\end{aligned}
$$

$$
V_{N,\mathrm{bang}}^\ast(\overline{x}) = 1_{\mathcal{T}}(\overline{x})
$$

$$
V_{k,\mathrm{bang}}^\ast(\overline{x}) = \inf_{\overline{u}\in\mathcal{U}} \int_{\mathcal{X}} V_{k+1}^\ast(\overline{y})Q(d\overline{y}\vert\overline{x},\overline{u})
$$

Special case: Bang-bang optimal Markov policies

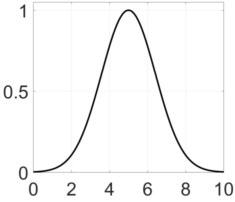
Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^\ast(\overline{x})$ and $W_0^\ast(\overline{x})$	$\mathcal{L}^{\pi^\ast}(\theta)$ and $\mathcal{K}^{\rho^\ast}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex

- Avoid $\mathcal{T}$ at N:
  - Reach $\mathcal{X}\setminus\mathcal{T}$ at N,
  - $\mathcal{S}\leftarrow\mathcal{X}$, and
  - $\mathcal{U}=\mathrm{convexHull}(\mathcal{U}_{\mathrm{vertices}})$

$$
\begin{aligned}
V_0^\ast(\overline{x}_0) &= \sup_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N\in\mathcal{X}\setminus\mathcal{T}\}\\
&= 1-\inf_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N\in\mathcal{T}\}\\
&\triangleq 1-V_{0,\mathrm{bang}}^\ast(\overline{x})
\end{aligned}
$$

$$V_{N,\mathrm{bang}}^\ast(\overline{x}) = 1_{\mathcal{T}}(\overline{x})$$

$$V_{k,\mathrm{bang}}^\ast(\overline{x}) = \inf_{\overline{u}\in\mathcal{U}_{\mathrm{vertices}}} \int_{\mathcal{X}} V_{k+1}^\ast(\overline{y})Q(d\overline{y}\vert\overline{x},\overline{u})$$



$$\exp\left(\frac{-\|x-5\|_2^2}{4}\right)$$

Special case: Bang-bang optimal Markov policies

Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^\ast(\overline{x})$ and $W_0^\ast(\overline{x})$	$\mathcal{L}^{\pi^\ast}(\theta)$ and $\mathcal{K}^{\rho^\ast}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex

- Avoid $\mathcal{T}$ at N:
  - Reach $\mathcal{X}\setminus\mathcal{T}$ at N,
  - $\mathcal{S}\leftarrow\mathcal{X}$, and
  - $\mathcal{U}=\mathrm{convexHull}(\mathcal{U}_{\mathrm{vertices}})$

$$
\begin{aligned}
V_0^\ast(\overline{x}_0) &= \sup_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N\in\mathcal{X}\setminus\mathcal{T}\}\\
&= 1-\inf_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N\in\mathcal{T}\}\\
&\triangleq 1-V_{0,\mathrm{bang}}^\ast(\overline{x})
\end{aligned}
$$

$$V_{N,\mathrm{bang}}^\ast(\overline{x}) = 1_{\mathcal{T}}(\overline{x})$$

$$V_{k,\mathrm{bang}}^\ast(\overline{x}) = \inf_{\overline{u}\in\mathcal{U}_{\mathrm{vertices}}} \int_{\mathcal{X}} V_{k+1}^\ast(\overline{y})Q(d\overline{y}\vert\overline{x},\overline{u})$$

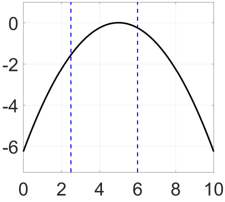
$$\log\left(\exp\left(\frac{-\|x-5\|_2^2}{4}\right)\right)$$

Special case: Bang-bang optimal Markov policies

Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^\ast(\overline{x})$ and $W_0^\ast(\overline{x})$	$\mathcal{L}^{\pi^\ast}(\theta)$ and $\mathcal{K}^{\rho^\ast}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex

- Avoid $\mathcal{T}$ at N:
  - Reach $\mathcal{X} \setminus \mathcal{T}$ at N,
  - $\mathcal{S} \leftarrow \mathcal{X}$, and
  - $\mathcal{U} = \mathrm{convexHull}(\mathcal{U}_{\mathrm{vertices}})$

$$V_0^\ast(\overline{x}_0) = \sup_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N \in \mathcal{X}\setminus\mathcal{T}\}$$
$$= 1 - \inf_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N \in \mathcal{T}\}$$
$$\triangleq 1 - V_{0,\mathrm{bang}}^\ast(\overline{x})$$
$$V_{N,\mathrm{bang}}^\ast(\overline{x}) = 1_{\mathcal{T}}(\overline{x})$$
$$V_{k,\mathrm{bang}}^\ast(\overline{x}) = \inf_{\overline{u}\in\mathcal{U}_{\mathrm{vertices}}} \int_{\mathcal{X}} V_{k+1}^\ast(\overline{y})Q(d\overline{y}\vert\overline{x},\overline{u})$$



$$\log\left(\exp\left(\frac{-\|x-5\|_2^2}{4}\right)\right)$$

$$\mathrm{minimize}\ \exp\left(\frac{-\|x-5\|_2^2}{4}\right)$$
$$\text{s.t } x \in [2.5, 6]$$

Special case: Bang-bang optimal Markov policies

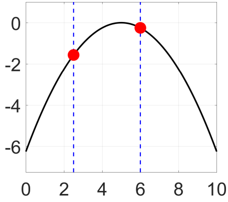
Property		$\mathcal{S}$	$\mathcal{T}$	$Q(\cdot\vert\overline{x},\overline{u})$	$\mathcal{U}$
$V_0^*(\overline{x})$ and $W_0^*(\overline{x})$	$\mathcal{L}^{\pi^*}(\theta)$ and $\mathcal{K}^{\rho^*}(\theta)$				
Log-concave	Convex	Borel Convex	Borel Convex	Continuous Log-concave	Compact Convex

- Avoid $\mathcal{T}$ at N:
  - Reach $\mathcal{X} \setminus \mathcal{T}$ at N,
  - $\mathcal{S} \leftarrow \mathcal{X}$, and
  - $\mathcal{U} = \mathrm{convexHull}(\mathcal{U}_{\mathrm{vertices}})$

$$
\begin{aligned}
V_0^*(\overline{x}_0) &= \sup_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N \in \mathcal{X}\setminus\mathcal{T}\} \\
&= 1 - \inf_{\pi\in\mathcal{M}} \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\{\boldsymbol{x}_N \in \mathcal{T}\} \\
&\triangleq 1 - V_{0,\mathrm{bang}}^*(\overline{x})
\end{aligned}
$$

$$V_{N,\mathrm{bang}}^*(\overline{x}) = 1_{\mathcal{T}}(\overline{x})$$

$$V_{k,\mathrm{bang}}^*(\overline{x}) = \inf_{\overline{u}\in\mathcal{U}_{\mathrm{vertices}}} \int_{\mathcal{X}} V_{k+1}^*(\overline{y}) Q(d\overline{y}\vert\overline{x},\overline{u})$$

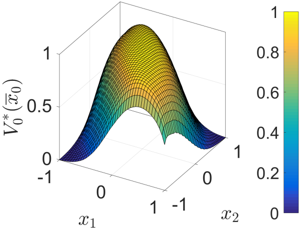


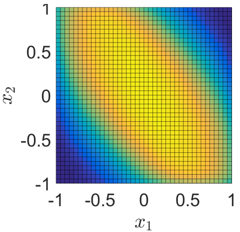
$$\log\left(\exp\left(\frac{-\|x-5\|_2^2}{4}\right)\right)$$

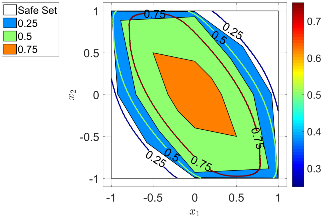
$$\mathrm{minimize}\ \exp\left(\frac{-\|x-5\|_2^2}{4}\right)$$

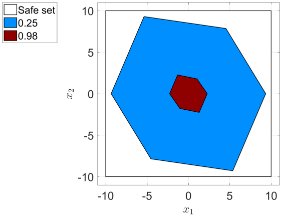
$$\text{s.t } x \in [2.5, 6]$$

Chain of integrators





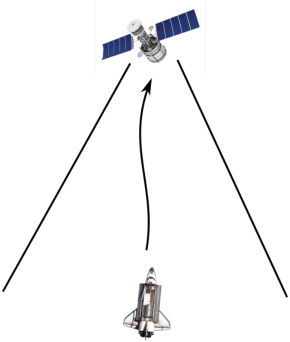




$n = 2$ (4–6 min. vs 34 min.)

$n = 40$ (16–31 min.)

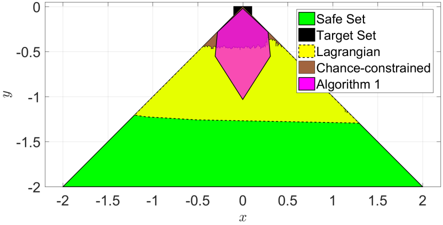
Satellite docking problem



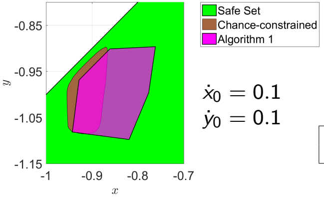
- Clohessy-Wiltshire-Hill dynamics

$$\left.\begin{aligned}\ddot{x} - 3\omega x - 2\omega\dot{y} &= u_1\\ \ddot{y} + 2\omega\dot{x} &= u_2\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}\boldsymbol{x}_{t+1} &= A\boldsymbol{x}_t + B\overline{u}_t + \boldsymbol{w}_t\\ \boldsymbol{x}_t &= [x_t\ y_t\ \dot{x}_t\ \dot{y}_t]^\top\\ \boldsymbol{w}_t &\sim \mathcal{N}(\overline{0}, \Sigma_{\boldsymbol{w}})\end{aligned}\right.$$

- Stay within line-of-sight cone and reach target



$\dot{x}_0 = 0$
$\dot{y}_0 = 0$

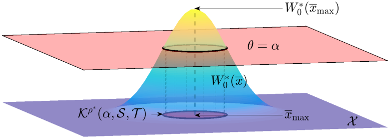


$\dot{x}_0 = 0.1$
$\dot{y}_0 = 0.1$

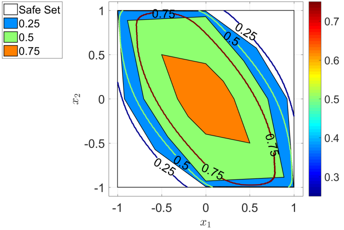
	Proposed method	Chance const. (CDC '13)	Lagrangian (CDC '17)	Dyn. prog. (HSCC '07)	Value of v ($\dot{x}_0 = \dot{y}_0 = v$)
Compute time (min.)	6.52	106.53	0.24	–	0
	9.88	13.12	–	–	0.1

Conclusion

- Summary
  - Polytopic underapproximation for the stochastic reach-avoid set
    - Scalable, grid-free, parallelizable
  - Sufficient conditions for
    - Closed, compact, convex sets
  - Optimal bang-bang policies



- Future work
  - Mitigate noisy optimization
  - Feedback-based underapproximation



Acknowledgement

This work was supported by the following grants:

- NSF CMMI-1254990 (CAREER, Oishi)
- CNS-1329878

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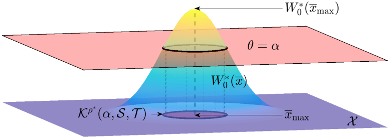

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