Viable Set Approximation for Linear-Gaussian Systems with Unknown, Bounded Variance

Joseph Gleason, Abraham P. Vinod, Meeko M. K. Oishi, and R. Scott Erwin

Abstract— **Computation of stochastic reachable and viable sets enables assurances of safety and feasibility through the synthesis of optimal control policies. These control policies are typically generated under the assumption of accurate characterization of additive noise processes. We consider the case in which independent noise processes are not fully characterized. Specifically, we consider linear time-invariant dynamics with additive noise, with known mean and bounded (but unknown) variance. We propose a method to compute a conservative underapproximation to the stochastic viable set for problems with convex viable and target sets. We underapproximate expected values by scaling the effect of the variance. We demonstrate this method (via dynamic programming) on simple example systems.**

I. Introduction

Computation of reachable and viable sets is a well established technique for assuring safety and feasibility in safety critical, high risk, or expensive systems. For systems with stochastic disturbances (e.g., process noise), stochastic reachability provides an assurance of a certain likelihood of safety or feasibility. However, in many applications, accurate characterizations of the disturbance process is elusive. This may be due to poor knowledge of the underlying system dynamics (as in many biomedical systems, in which a lack of first principles yields population mean models with "noise" processes that capture unmodeled phenomena), or due to faults or failures that create erratic and unexpected behaviors.

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Consider space vehicle rendezvous and docking, in which the other vehicle may be inoperable or uncontrollable. Latent drift or erratic actuation may generate disturbance forces that are difficult to predict from a history of observations. Or consider biomedical devices, such as infusion pumps for anethesia, insulin, or cancer drugs, that are safety-critical, but reliant upon biological processes which may be highly heterogeneous across subjects as well as temporally variable for a given subject. Because disturbance processes often capture modeling inaccuracies as well as faults and system errors, accurate characterizations may be difficult to obtain. Inaccurate characterizations of noise processes could have an undue effect on the computation of stochastic reachable and viable sets, as well as the resulting optimal controllers, and hence on system safety. For systems with poorly modeled or even non-stationary noise processes, improper assumptions can result in incorrect determination of the stochastic reachable or viable sets.

We consider the case in which a linear, time-invariant (LTI) system that is subject to additive Gaussian process noise with known mean and unknown (but bounded) variance. We pose of the question of calculation of the stochastic viable set despite incomplete characterization of the noise statistics (but with perfect observations, e.g., accurate knowledge of the state). We presume the mean is known because in many systems, it is straightforward to identify a steady-state bias from gathered data, and standard practice to 'detrend' data to remove an observed non-zero mean.

Little work has been done on reachability with unbounded disturbances whose statistics are incompletely characterized. Methods to compute reachable sets with bounded disturbances [1], [2], [3] using a differential game framework rely on worst-case scenarios that would be prohibitively conservative for the systems we consider. Stochastic reachability methods typically rely on *a priori* knowledge of the noise statistics [4], [5], [6]. While work has been done on reachability with incomplete information, in which observations may be corrupted and the true state is inaccessible [7], [8], these methods also rely on accurate knowledge of the noise statistics.

Systems with poorly characterized noise processes have been addressed in other frameworks. Specifically, [9] address the problem of parameter estimation and model forecasting for systems in which no sensor statistical information is present. Researchers in [10] use semi-definite programming techniques to estimate disturbance structures in stochastic systems. In [11], polynomial chaos is used to estimate state trajectories for nonlinear systems with uncertain initial conditions.

The main contribution of this work is a method to create bounded under- and over-approximations of the stochastic reach-avoid set for an LTI system with additive Gaussian noise with unknown but bounded variance, based in dynamic programming. The approximations are designed to be practical, that is, not overly conservative, and applicable to nonstationary systems.

In Section II we formulate the problem. Section III describes the scaling method, and Section IV describes an algorithm to compute bounded under- and over-approximations of the stochastic viable set via scaling. Section V demonstrates this technique on a motivating example. Section VI summarizes the contributions and provides directions for future work.

II. Problem Statement

We consider the discrete-time, linear system

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$

with state $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$, input $u_k \in \mathcal{U} \subseteq \mathbb{R}^n$, and Gaussian noise process w_k with statistics

$$E[w_k] = \mu \tag{2a}$$

$$E[w_k w_j^T] = \begin{cases} \text{diag}(\sigma_1^2[k], \dots, \sigma_n^2[k]) & j = k \\ 0_{n\times n} & j \neq k \end{cases} \tag{2b}$$

$$\underline{\sigma} \le \sigma_i^2[k] \le \overline{\sigma}, \quad \forall i = 1, \dots, n \tag{2c}$$

We presume that while the variance σ is unknown, the upper and lower bounds $\underline{\sigma}$, $\overline{\sigma}$ are known, positive and finite values. The mean μ is known. We also presume that the initial state x_0 is known.

The standard reach-avoid problem is one in which we wish to find the set of states which for which there exists a control which, with given minimum likelihood, will drive the state to reach a target set $L \subseteq \mathbb{R}^n$ at time N while remaining within a constraint set $K \subseteq \mathbb{R}^n$ for all times steps prior to N. We also wish to find the resulting optimal control policy. As in [12], for a given initial condition x_0, we wish to solve

$$p(x_0) = \sup_{\pi \in \Pi} E_{x_0}^{\pi} \left[\left(\prod_{n=0}^{N-1} 1_K(x_n) \right) 1_L(x_N) \right] \tag{3}$$

where $\pi = [u_1, \dots, u_{N-1}]$, $u_i \in \mathcal{U}$, is an input policy, $\pi \in \Pi$, $\Pi = \mathcal{U}^N$, and using standard indicator function notation, $\mathbf{1}_K(x) = 1$ if $x \in K$ and $\mathbf{1}_K(x) = 0$ otherwise. Then $p(x_0)$ is the likelihood of reaching L while avoiding K^c, the complement of K, over the horizon N, and

$$\text{ReachAvoid}(\alpha) = \{x \in \mathbb{R}^n \mid p(x) \ge \alpha\} \tag{4}$$

is the ReachAvoid set associated with minimum likelihood $\alpha \in [0, 1]$.

This optimization can be solved via dynamic programming [4], [5]. We define the final value function

$$V_N^{\pi}(x) = \mathbf{1}_L(x) \tag{5}$$

and iterate backwards in time according to

$$V_k^{\pi}(x) = \mathbf{1}_K(x) \int_X V_{k+1}(y) Q(dy|x, \pi) \tag{6}$$

where $Q : \mathcal{B}(X) \times X \times U \to [0, 1]$ is the transition probability function

$$Q(dy|x_k, u_k) = \frac{1}{(2\pi)^{n/2}\text{det}|\Sigma|} e^{-\frac{1}{2}(y-m_k)^T \Sigma^{-1}(y-m_k)} \tag{7}$$

with $m_k = Ax_k + Bu_k + E[w_k]$ and $\Sigma = \text{var}(w_k)$, and $\mathcal{B}(\Omega)$ is the borel sigma algebra of the set Ω. For a specific control $u \in \mathcal{U}$, we can write (7) as $Q^u(dy|x_k)$.

Then the maximum probability of achieving the reach-avoid objective is

$$V_0^*(x_0) = \sup_{\pi \in \Pi} E_{x_0}^{\pi} \left[\left(\prod_{n=0}^{N-1} 1_K(x_n) \right) 1_L(x_N) \right] \tag{8}$$

Hence the ReachAvoid set is solved via

$$\text{ReachAvoid}(\alpha) = \{x \mid V_0^*(x_0) \ge \alpha\} \tag{9}$$

For special cases of dynamics and target and constraint sets, chance constrained optimization [13], [6], particle filters [6], [14], and approximate dynamic programming methods [15] may be alternately used.

Problem 1: For an LTI system (1) with noise statistics (2) find an under-approximation and over-approximation of the ReachAvoid set (9) by computing approximate value functions $\underline{V}_0(x) \le V_0(x)$ and $\overline{V}_0(x) \ge V_0(x)$ that are lower and upper bounds, respectively, to the true likelihood (8).

Further, we wish to characterize how close $\underline{V}_k(x), \overline{V}_k(x)$ are to the true value $V_k(x)$, as well as the control strategy which corresponds to the under-approximation.

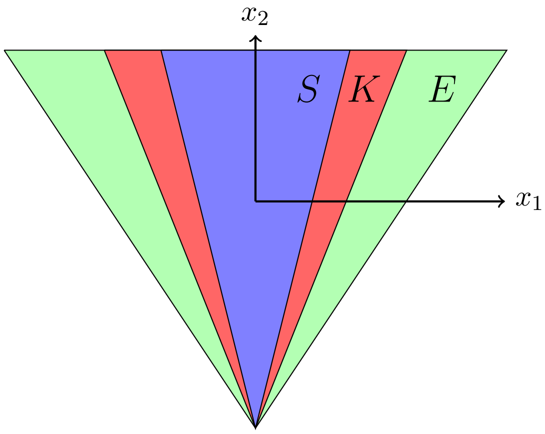

Fig. 1: Scalable subset S and scalable superset E for the convex set K.

III. Scalable Subsets and Supersets

We first introduce two types of sets, *scalable subsets* and *scalable supersets*, that will be used to under- and over-approximate viability probabilities.

Definition 1: The set $S \subseteq K$ is a **scalable subset** if there exists a matrix $Z = \text{diag}(z_1, z_2, \ldots, z_n)$, $z_i \geq 1$ for $1 \leq i \leq n$, such that $Z^{-1}S = \left\{Z^{-1}x : x \in S\right\} = S' \subseteq K$.

Definition 2: The set $E \subseteq K$ is a **scalable superset** if there exists a matrix $Z = \text{diag}(z_1, z_2, \ldots, z_n)$, $z_i \leq 1$ for $1 \leq i \leq n$, such that $Z^{-1}E = \left\{Z^{-1}x : x \in E\right\} = E' \supseteq K$,

Figure 1 shows a *scalable subset* and a scalable superset for a triangular convex set. For an arbitrary bounded, convex set, there are many ways of computing scalable subsets and scalable supersets.

Lemma 1: A bounded, convex set K that contains the origin has at least one scalable subset and one scalable superset.

Proof: Scalable subset: The Euclidian ball $\mathcal{B}(0, r)$ centered at the origin with radius $r = \min_{x \in b(K)} ||x||_2^2$ where $b(K)$ is the boundary of K, is a scalable subset.

Scalable superset: The interval hull [16] $\mathcal{D} = [-a_1, b_1] \times \cdots \times [-a_n, b_n]$ of K is a scalable superset. For any matrix $Z = \text{diag}(z_1, z_2, \ldots, z_n)$, $z_i \leq 1$ for $1 \leq i \leq n$, $Z^{-1}K = [-z_1^{-1}a_1, z_1^{-1}b1] \times \cdots \times [-z_n^{-1}a_n, z_n^{-1}b_n]$ and $[-z_i^{-1}a_i, z_i^{-1}b_i] \supseteq [-a_i, b_i]$ since $z_i^{-1} \geq 1$ for $1 \leq i \leq n$. Hence $Z^{-1}\mathcal{D} \supseteq \mathcal{D} \supseteq K$ ■

Theorem 1: Let X be a random variable with $E[X] = m$ and $E[XX^T] = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ where $\underline{\sigma_i} \leq \sigma_i \leq \overline{\sigma_i}$, $\overline{\sigma_i}, \underline{\sigma_i} \in [0, \infty)$ for $1 \leq i \leq n$. Let K be a bounded, convex set, S be a scalable subset of K, and $T : S \to K$ be an invertible transformation. If we define $Y = T(FX)$ with $F = \text{diag}(\overline{\sigma_1}/\sigma_1, \ldots, \overline{\sigma_n}/\sigma_n)$, then the likelihood that $X \in K$ is at least the likelihood that $Y \in K$: $P(Y \in K) \leq P(X \in K)$ for all σ_i, $1 \leq i \leq n$.

Proof: Note that the diagonal elements of F are greater than or equal to 1.

$$\begin{aligned}
P(Y \in K) &= P(T(FX) \in K) \\
&= P(FX \in T^{-1}(K)) = P(FX \in S) \\
&= P(X \in F^{-1}S) \\
&\leq P(X \in K)
\end{aligned}$$

The inequality results since S is a scalable subset of K, that is, $F^{-1}S \subseteq K$. ■

Corollary 1: For a bounded, convex set K, if E is a scalable superset and an invertible transformation $T : E \to K$, then defining $Y = T(FX)$, with $F = \text{diag}(\underline{\sigma_1}/\sigma_1, \ldots, \underline{\sigma_n}/\sigma_n)$ and X defined as in Theorem 1, $P(Y \in K) \geq P(X \in K)$.

Proof: Similarly as in the proof of Theorem 1, but with the diagonal elements of F less than or equal to 1, such that $F^{-1}E \supseteq K$. ■

We plan to use the matrix F to perform a linear transformation on X, altering its variance from $E[XX^T] = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$ to $E[FXX^TF^T] = \text{diag}(\overline{\sigma_1}^2, \ldots, \overline{\sigma_n}^2)$. This enables under-approximations of the stochastic viable set when we can assure that $F^{-1}M$ is a subset of K. The transformation T guarantees that our approximation is valid by converting K to a scalable subset.

Note that not all scalable subsets or scalable supersets are useful. Consider the scalable superset $\mathcal{B}_\infty = [-\infty, \infty] \times [-\infty, \infty] \times \cdots \times [-\infty, \infty]$ that will provide $P(X \in \mathcal{B}_\infty) = 1$ or the scalable subset, $\emptyset$, that will always give $P(X \in \emptyset) = 0$. For stochastic reachability problems, the following sets would be ideal.

Definition 3: Let X be a random vairable. The set $\overline{S} \subseteq K$ is the **maximal likelihood scalable subset** if $\overline{S}$ is a scalable subset with $P(X \in \overline{S}) \geq P(X \in S)$, where S is any other scalable subset.

Definition 4: Let X be a random vairable. The set $\underline{E} \supseteq K$ is the **minimal likelihood scalable superset** if $\underline{E}$ is an scalable superset with measure, $P(X \in \underline{E}) \leq P(X \in E)$, where E is any other scalable superset.

We aim to find a transformation T which, if it cannot provide the maximal (minimal) likelihood scalable subset (superset), can provide sufficietly large (small) sets to provide under- (over-) approximations of use for systems with bounded but unknown variance. We exploit the fact that linear transformations of Gaussian processes are also Gaussian. Theorem 1 suggests such a linear transformation, and Figure 2 schematically shows how the the values t_i are determined.

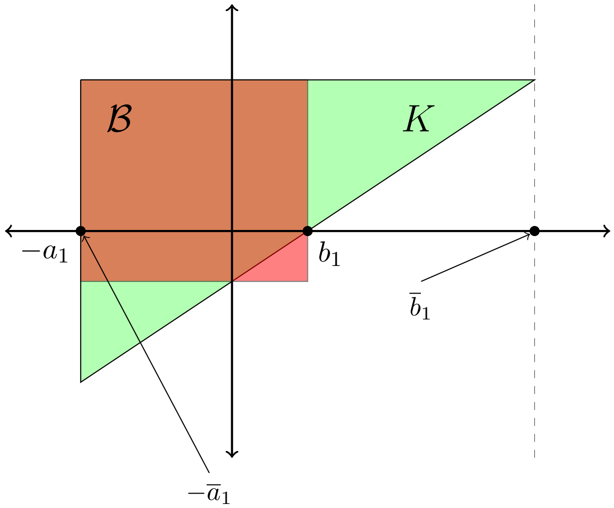

Fig. 2: The scaling matrix element t_1 from (12) is computed for a two dimensional set. First compute the two intercepts $-a_1, b_1$ along the x_1 axis, and the minimum and maximum values of x_1 for any $x \in K$ (e.g., $\bar{a}_1$, $\bar{b}_1$), then choose the maximum ratio, so that $t_1 = \max\left(\frac{\bar{a}_1}{a_1}, \frac{\bar{b}_1}{b_1}\right) = \frac{\bar{b}_1}{b_1}$.

Theorem 2: For a bounded, convex set K that contains the origin, the set $T^{-1}K = \left\{T^{-1}x : x \in K\right\}$ with $T = \mathrm{diag}(t_1, \dots, t_n)$, t_i defined as in (12), is a convex, scalable subset of K. For each axis x_i, we define the intercept values

$$\begin{aligned} a_i &= \max |x_i| \\ &\quad \text{s.t.} \begin{cases} x \in K \\ x_j = 0, j \neq i \\ x_i \leq 0 \end{cases} \\ b_i &= \max |x_i| \\ &\quad \text{s.t.} \begin{cases} x \in K \\ x_j = 0, j \neq i \\ x_i \geq 0 \end{cases} \end{aligned} \tag{10}$$

and the maximum values

$$\begin{aligned} \bar{a}_i &= \max |x_i| \\ &\quad \text{s.t.} \begin{cases} x \in K \\ x_i \leq 0 \end{cases} \\ \bar{b}_i &= \max |x_i| \\ &\quad \text{s.t.} \begin{cases} x \in K \\ x_i \geq 0 \end{cases} \end{aligned} \tag{11}$$

to compute the diagonal element

$$t_i = \max\left(\frac{\bar{a}_i}{a_i}, \frac{\bar{b}_i}{b_i}\right) \tag{12}$$

Proof: Convexity: Since K is convex, $T^{-1}K$ is convex.

Subset: For $y = T^{-1}x$, $x \in K$, we know $y_i \in [-a_i, b_i]$. In the worst case, if $x_i = \bar{b}_i$, then $t_i^{-1}x_i = b_i$ if $\frac{\bar{b}_i}{b_i} \geq \frac{\bar{a}_i}{a_i}$ or $\frac{a_i}{\bar{a}_i}\bar{b}_i \leq b_i$ if $\frac{\bar{b}_i}{b_i} \leq \frac{\bar{a}_i}{a_i}$. Similar reasoning holds for the minimum value $x_i = -a_i$. Hence, if $y \in T^{-1}K$, then $y \in \mathcal{B} = \{[-a_1, b_1] \times [-a_2, b_2] \times \dots \times [-a_n, b_n]\}$, where a_i and b_i are defined in (10). Further, we can claim that $y \notin \mathcal{B} \setminus K$, as seen by contradiction: if $y \in \mathcal{B} \setminus K$ then for all points $z \in K$ with $\mathrm{sign}(z_i) = \mathrm{sign}(y_i)$, $\mathrm{sign}(|z_i| - |y_i|) \ngeq 0$ for all $1 \leq i \leq n$. However, since $y = T^{-1}x = (t_1^{-1}x_1, \dots, t_n^{-1}x_n)$ for some $x \in K$ with $t_i \geq 1$ for all $1 \leq i \leq n$, $\mathrm{sign}(|x_i| - |y_i|) \geq 0$ for all i. This is a contradiction, hence $y \notin \mathcal{B} \setminus K$. Since $\mathcal{B} \cap (\mathcal{B} \setminus K)^c = \mathcal{B} \cap K \subseteq K$, we have $T^{-1}K \subseteq \mathcal{B} \cap K \subseteq K$.

Scalable subset: Since the subset of any scalable subset of K is also a scalable subset of K, and $T^{-1}K \subseteq (\mathcal{B} \cap K)$, we show that $(\mathcal{B} \cap K)$ is a scalable subset of K. For any $y \in \Gamma^{-1}(\mathcal{B} \cap K)$, $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_n)$, $\gamma_i \geq 1$ for $1 \leq i \leq n$, we know $y \in \mathcal{B}$. Since we also know $y \notin \mathcal{B} \setminus K$, we have $y \in (\mathcal{B} \cap K) \subseteq K$. Since $T^{-1}K$ is a subset of the scalable subset $(\mathcal{B} \cap K)$, it is also a scalable subset of K. ■

Corollary 2: For a convex set, K, and T defined in Theorem 2, the set $TK = \{Tx : x \in K\}$ is an scalable superset of K.

Proof: Convexity: Since K is convex, the linear transformation TK is convex.

Superset: Similar approach to Theorem 2 except with $\mathcal{B} = \left\{[-\bar{a}_1, \bar{b}_1] \times \dots \times [-\bar{a}_n, \bar{b}_n]\right\}$.

Scalable superset: Same process as for Theorem 2, except with $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_n)$, $\gamma_i \leq 1$ for $1 \leq i \leq n$.

■

The result of Theorem 2 and Corollary 2 is a linear transformation to create scalable subsets and supersets for bounded, convex sets that contain the origin.

IV. Viable Set Approximation

To solve Problem 1, we under- and over-approximate $V_0^\pi(x)$, the probability of achieving a reach-avoid objective given an input policy π, in a manner that is amenable to dynamic programming and can be computed via a backward recursion, as in [5], [4].

The utility of the previous theorems and corollaries is that for an independent Gaussian random vector X with known mean and unknown but bounded variance, we can provide a new independent Gaussian random vector Y with both a known mean and a known variance which provides either an under- or over-approximation of the probability of being in a convex set K.

Consider the random variable

$$y_k = T_k F(x_k - E[x_k]) + E[x_k] \tag{13}$$

when x_k has dynamics given by (1) with noise statistics (2), T_k is defined by Theorem 2 using the convex set $(K - E[x_k])$ and F is given by Theorem 1. We know that y_k is Gaussian, with $E[y_k] = E[x_k]$ and $\text{var}(y_k) = T_k F \text{var}(x_k) F T_k$. y_k obtains its unique definition because, given a bounded, convex set K, that may or may not contain the origin, $P(y_k \in K) = P(x_k \in S_k)$ where $S_k = F^{-1}T_k^{-1}(K - E[x_k]) + E[x_k]$. If $E[x_k] \in K$ then $S_k \subseteq K$ because T_k is defined using Theorem 2 with the set $(K - E[x_k])$, hence, $F^{-1}T_k^{-1}(K - E[x_k]) \subseteq (K - E[x_k])$, indicating that $S_k \subseteq K$.

We define the following:

$$C^u_{k,K} = \{x \in X : Ax + Bu + E[w_{k-1}] \in K\} \tag{14}$$

$$P(x_{k+1} \in K | x_k \in \mathcal{X}, u_k \in \mathcal{U}) = P^{u_k}(x_{k+1} \in K | x_k) \tag{15}$$

$$E[x_{k+1} | x_k \in \mathcal{X}, u_k \in \mathcal{U}] = E^{u_k}[x_{k+1} | x_k] \tag{16}$$

$Q^u(dy|x)$ is a stochastic kernel such that

$$P^u(x_k \in K | x_{k-1}) = \int_X 1_K(y) Q^u(dy | x_{k-1}) \tag{17}$$

for a system given by (1) with noise statistics (2).

Lemma 2: For an LTI system (1) with noise statistics (2), let y_k be defined by (13), with T_k as determined by Theorem 2 with the convex set $(K - E[x_k])$ and F determined as in Theorem 1. Then $1_{C^{u_k}_{k+1,K}}(x_k) P^{u_k}(y_{k+1} \in K | x_k) \leq P^{u_k}(x_{k+1} \in K | x_k)$ for some convex set $K \in \mathbb{R}^n$.

Proof: Because T_k are defined by Theorem 2 with the set $(K - E[x_k])$ and F is defined by Theorem 1, $P^{u_k}(y_{k+1} \in K | x_k) \leq P^{u_k}(x_{k+1} \in K | x_k)$ if $E^{u_k}[x_{k+1}|x_k] \in K$. Hence, $\mathbf{1}_{C^{u_k}_{k+1,K}}(x_k) P^{u_k}(y_{k+1} \in K | x_k) \leq P^{u_k}(x_{k+1} \in K | x_k)$. ■

Lemma 3: For an LTI system (1) with noise statistics (2), let y_k be defined by (13), T_k as determined by Corollary 2 with the convex set $(K - E[x_k])$ and F determined as in Corollary 1. Then $P^{u_k}(y_{k+1} \in K | x_k) \geq P^{u_k}(x_{k+1} \in K | x_k)$ if $E[x_{k+1}|x_k] \in K$, or, $1 + \mathbf{1}_{C^{u_k}_{k+1,K}}(x_k)(P^{u_k}(y_{k+1} \in K | x_k) - 1) \geq P^{u_k}(x_{k+1} \in K | x_k)$ for any $x_k \in \mathcal{X}$.

Proof: The proof is similar to Lemma 2 but relies on the over-approximation demonstrated by Corollary 1. ■

We now incorporate the under- and over-approximations in the dynamic programming recursion.

Theorem 3: The dynamic program with terminal value function $\underline{V}^\pi_N(x_k) = \mathbf{1}_K(x_k)$ and recursion

$$\underline{V}^\pi_k(x_k) = \mathbf{1}_K(x_k) \int_X \underline{V}_{k+1}(y) \underline{H}^u_{k,K}(dy, x_k) \tag{18}$$

with

$$\underline{H}^u_{k,K}(dy|x) = \begin{cases} 0 & x \notin C^{u_k}_{k,K} \\ \overline{Q}^u_k(dy|x) & \text{otherwise} \end{cases} \tag{19}$$

where $\overline{Q}^u_k(dy|x)$ is the stochastic kernel such that, for y_k defined by (13),

$$P^u(y_k \in K | x_{k-1}) = \int_X \mathbf{1}_K(y) \overline{Q}^u_k(dy | x_{k-1}) \tag{20}$$

yields an under-approximation $\underline{V}^\pi_k \leq V^\pi_k(x_k)$ for any input policy π.

Proof: It can be seen that,

$$\mathbf{1}_{C^u_{k,K}}(x_{k-1}) P^u(y_k \in K | x_{k-1}) = \int_X \mathbf{1}_K(y) H^u_{k,K}(dy | x_{k-1}) \tag{21}$$

From [17, Proposition 7.28] and the fact that the sequences x_k and y_k are first-order Markov chains,

$$\prod_{i=k}^{N-1} P^{u_i}(x_{i+1} \in K | x_i) = \int_{X^{N-k}} \left(\prod_{i=k+1}^{N} \mathbf{1}_K(x_i) \right) \times \prod_{j=k+1}^{N-1} Q^{u_j}(dx_{j+1}|x_j) Q^{u_k}(dx_{k+1}|x_k) \tag{22}$$

and

$$\prod_{i=k}^{N-1} \mathbf{1}_{C^{u_i}_{i+1,K}}(x_i) P^{u_i}(y_{i+1} \in K | x_i) = \int_{X^{N-k}} \left(\prod_{i=k+1}^{N} \mathbf{1}_K(y_i) \right) \prod_{j=k+1}^{N-1} H^{u_j}_{j,K}(dy_{j+1}|y_j) \times H^{u_k}_{k,K}(dy_{k+1}|y_k)$$

Finally we can note that

$$\prod_{i=k}^{N-1} P^{u_i}(x_{i+1} \in K | x_i) \geq \prod_{i=k}^{N-1} \mathbf{1}_{C^{u_i}_{i+1,K}}(x_i) P^{u_i}(y_{i+1} \in K | x_i) \tag{23}$$

since, by Lemma 2, $\mathbf{1}_{C^{u_k}_{k,K}}(x_k) P^{u_k}(y_{k+1} \in K | x_k) \leq P^{u_k}(x_{k+1} \in K | x_k)$ for all $k = 0, \ldots, N-1$.

Since from [5],

$$V_k(x_k) = \mathbf{1}_K(x_k) \prod_{i=k}^{N-1} P^{u_i}(x_{i+1} \in K | x_i) \tag{24}$$

if we define $\underline{V}_k(x_k)$ by the recursion

$$\underline{V}_k(x_k) = \mathbf{1}_K(x_k) \int_X \underline{V}_{k+1}(y) H_{k,K}^{u_k}(dy, x_k) \tag{25}$$

then it can be seen that

$$\underline{V}_k(x_k) = \mathbf{1}_K(x_k) \prod_{i=k}^{N-1} \mathbf{1}_{C_{i+1,K}^{u_i}}(x_i) P(y_{i+1} \in K|x_i) \tag{26}$$

and by (23) $V_k(x_k) \geq \underline{V}_k(x_k)$. ■

Theorem 3 provides a under-approximating recursion to determine viable sets. The only difference between our recursion and those originally shown in [4] and [5] is in the definition of $\underline{H}$, which in our definition is a truncated stochastic kernel for a the stationary Gaussian produced through the transformation of x_k to y_k in (13). The transformation, and the new recursion, converts the problem of the LTI system in (1) with unknown noise statistics (2) into a well understood problem of a discrete LTI system with stationary Gaussian noise. Hence we can find the optimal control policy which maximizes the value function for the approximate recursion [5]

$$\pi_k^*(x_k) = \arg\sup_{u\in\mathcal{U}} \left\{ \mathbf{1}_K(x_k) \int_X \underline{V}_{k+1}(y) \underline{H}_{k,K}^u(dy|x_k) \right\} \tag{27}$$

where π_k^* is the kth element of $\pi^* = [u_0^*, \ldots, u_{N-1}^*]$.

Our methods can be extended to the reach problem or to the reach-avoid problem. For the reach-avoid problem

$$E_{x_0}^{\pi} \left[\left(\prod_{n=0}^{N-1} \mathbf{1}_K(x_n) \right) \mathbf{1}_L(x_N) \right] \tag{28}$$

we use the recursion, $\underline{V}_N^{\pi}(x) = \mathbf{1}_L(x)$,

$$\underline{V}_{N-1}^{\pi}(x) = \mathbf{1}_K(x) \int_X \underline{V}_N^{\pi}(y) \underline{H}_{N-1,L}^u(dy|x) \tag{29}$$

$$\underline{V}_k^{\pi}(x) = \mathbf{1}_K(x) \int_X \underline{V}_{k+1}^{\pi}(y) \underline{H}_{k,K}^u(dy|x) \tag{30}$$

for $k = 0, \ldots, N-2$.

The error in the under-approximation

$$\varepsilon_k^{\pi}(x_k) = \prod_{i=k}^{N-1} P^{u_i}(x_{i+1} \in K|x_i) - \prod_{i=k}^{N-1} \mathbf{1}_{C_{i+1}}(x_i) P^{u_i}(y_{i+1} \in K|x_i) \tag{31}$$

is either

$$\varepsilon_k^{\pi}(x_k) = \prod_{i=k}^{N-1} P^{u_i}(x_{i+1} \in K|x_i) - \prod_{i=k}^{N-1} P^{u_i}(y_{i+1} \in K|x_i) \tag{32}$$

or

$$\varepsilon_k^{\pi}(x_k) = V_k^{\pi}(x_k) \tag{33}$$

depending on evaluation of the indicator function. The value (33) arises from the fact that, for some paths, we set our approximation equivalently to zero. This allows us to provide a consistent under-approximation but provides larger error for certain combinations of states and inputs.

The error given by (32) can be equivalently written as

$$\varepsilon_k^{\pi}(x_k) = \prod_{i=k}^{N-1} P^{u_i}(x_{i+1} \in K|x_i) - \prod_{i=k}^{N-1} P^{u_i}(x_{i+1} \in S_{i+1}|x_i) \tag{34}$$

where $S_{i+1} = F^{-1}T_i^{-1}(K - E[x_{i+1}|x_i]) + E[x_{i+1}|x_i]$, from Theorems 1 and 2. From (22), we obtain

$$\varepsilon_k^{\pi}(x_k) = \prod_{i=k}^{N-1} P^{u_i}(x_{i+1} \in K \setminus S_{i+1}|x_i) \tag{35}$$

In summary,

$$\varepsilon_k^{\pi}(x_k) = \begin{cases} \prod_{i=k}^{N-1} P^{u_i}(x_{i+1} \in K \setminus S_{i+1}) & \text{for } \prod_{i=k}^{N-1} \mathbf{1}_{C_{i+1,K}^{u_i}} = 1 \\ V_k^{\pi}(x_k) & \text{for } \prod_{i=k}^{N-1} \mathbf{1}_{C_{i+1,K}^{u_i}} = 0 \end{cases} \tag{36}$$

Corollary 3: The dynamic program with terminal value function $\overline{V}_N(x_k) = \mathbf{1}_K(x_k)$ and recursion

$$\overline{V}_k^{\pi}(x_k) = \mathbf{1}_K(x_k) \int_X \overline{V}_{k+1}(y) \overline{H}_{k,K}^{u_k}(dy, x_k) \tag{37}$$

with

$$\overline{H}_{k,K}^u(dy|x) = \begin{cases} 0 & x \notin C_{k,K}^u \\ \underline{Q}_k^u(dy|x) & \text{otherwise} \end{cases} \tag{38}$$

where $\underline{Q}_k(dy|x)$ is the stochastic kernel such that, for y_k defined by (13), T_k defined in Corollary 2 with the set $(K - E[x_k])$ and F defined in Corollary 1,

$$P^u(y_k \in K|x_{k-1}) = \int_X \mathbf{1}_K(y) \underline{Q}_k^u(dy|x_{k-1}) \tag{39}$$

yields an under-approximation $\overline{V}_k^\pi(x_k) \geq V_k^\pi(x_k)$ for any input policy π.

Proof: The proof follows similarly to the proof for Theorem 3 except using Lemma 3 to provide over-approximation. ■

Corollary 3 provides a backward recursion technique to determine an over-approximation of the viability value function. Hence we can determine either an over- or under-approximation of viable sets using the Theorem and Corollary presented.

V. Example

To demonstrate our methods we use the intuitive double integrator,

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + w(t) \tag{40}$$

where x is the state vector consisting of a the position and velocity of an object, u is an accelerative force, and w is the additive disturbance input. We discretize (40) via forward Euler with time step Δ to obtain the discrete-time dynamics,

$$x_{k+1} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{\Delta^2}{2} \\ \Delta \end{bmatrix} u_k + w_k \tag{41}$$

The additive input w_k is assumed to be a zero-mean, Gaussian random vector with independent elements, i.e.

$$E[w_k w_k^T] = \begin{bmatrix} \sigma_1^2[k] & 0 \\ 0 & \sigma_2^2[k] \end{bmatrix}$$

The Gaussians are independent with respect to k, or $E[w_k w_j^T] = 0_{2\times 2}$ for $k \neq j$. The variances do not need to be constant over time but must be bounded. We presume $\sigma_i[k] \in [0.01, 0.05]$ for all i and k.

To compute the approximate value function $\underline{V}_k^\pi(x)$ for $x \in X$ (and the known variance value function $V_k^\pi(x)$) we used dynamic programming. With $T = 1$, the constraint set is $K = [-1, 1] \times [-1, 1]$. We consider a uniform grid with 40 elements in each dimension. (Conveniently, since our viable set is rectangular, the linear transformation matrix to convert our convex set to a scalable subset set is $T = I_{2\times 2}$ for any input for which $E[x_{k+1}] \in K$). Our input space was 3 allowable values, $u \in \{-0.1, 0, 0.1\}$. Computation of the approximate viable set for this setup was 9.51 seconds. Figure 3 shows a comparison between the approximate viable set and the viable set with a known variance—assigned to $\sigma_i = 0.03$ for $i = 1, 2$.

A visual inspection of the figure clearly demonstrates that the approximate viable set is conservative. This was made more clear by setting $\sigma_i = 0.03$ for $i = 1, 2$. Since the set was rectangular the approximate set is equivalently computing the viability probability as if $\sigma_i = 0.05$ for $i = 1, 2$, and setting the probability equal to zero whenever the next-step expected value is outside the set.

The over-approximation is computed in a similar manner. Since the viable set is rectangular, the problem is equivalent to a viable set computation as if $\sigma_i = 0.01$ for $i = 1, 2$.

VI. Conclusion

We presented an approach to computing under- and over-approximations of stochastic viable and reachable sets for linear systems with additive Gaussian noise that have an unknown but bounded variance. Our process is applicable to systems perturbed by noise that has a nonstationary variance and a known stationary mean.

Future work will be in investigating the accuracy of the approximation in actual systems, and considering the effect of unknown mean statistics. The work demonstrated in the paper can also be extended to allow for viable set computation when dependent noise is present upon the system.

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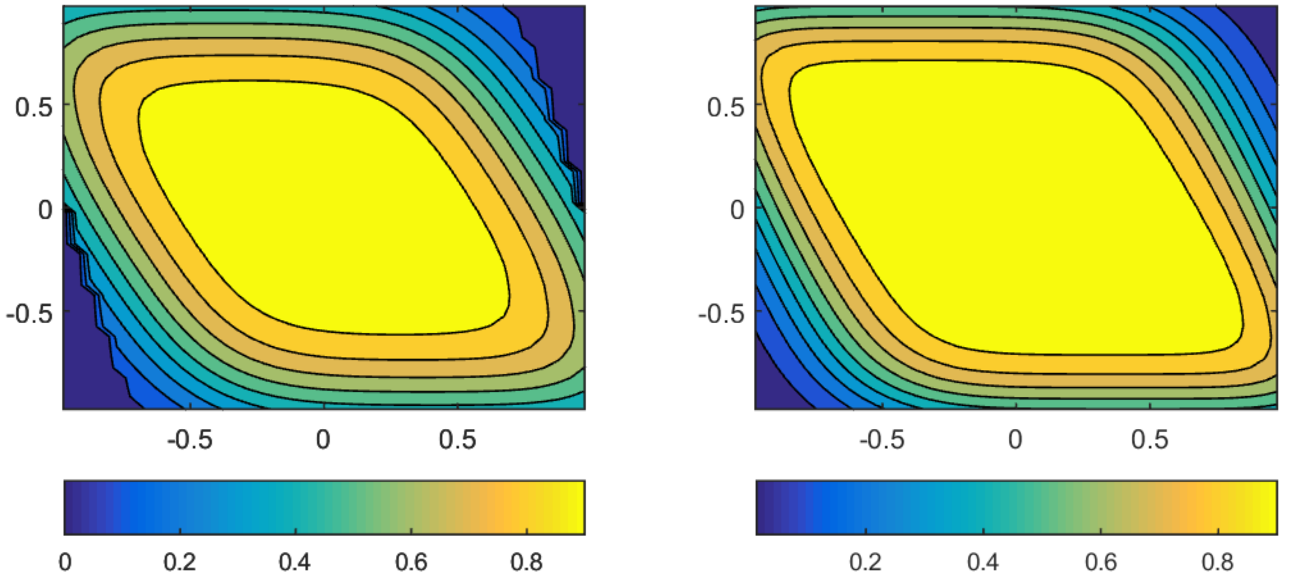

(a) (b)

Fig. 3: Contour plots of viable subsets for the double integrator problem at $kT = 0.5$ with (a) an unknown but bounded variance and (b) and a known variance $\sigma_i = 0.03$ for $i = 1, 2$.

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