Viable Set Approximation for Linear-Gaussian Systems with Unknown, Bounded Variance

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Motivation

- Current reach-avoid techniques commonly rely on perfectly characterized stochastic processes
  - Assumption often impractical for real-world

Main Contributions

- Underapproximation of viable set for LTI system perturbed by Gaussian noise with unknown bounded variance
- Dynamic programming based solution by applying state transformations using scalable subsets and scalable supersets

Related Work

Stochastic Reach Avoid Sets
Lesser, Oishi, Erwin (2013); Summers & Lygeros (2010); Gao & Lygeros (2007); Abate, Prandini, Lygeros, Sastry (2008); Esfahani, Chaterjee, Lygeros (2011);

Reach Avoid Sets With Bounded Disturbances
Mitchell, Bayen, Tomlin (2005); Ding & Tomlin (2010); Ding, Huang, Tomlin (2011);

Reachability for Partially Observable Systems
Lesser & Oishi (2015); Lesser & Oishi (2014); Ding, Abate, Tomin (2013); Verma and del Vecchio (2012);

Viable Sets

- Probability of achieving viability objective [Summers & Lygeros (2010)], for input policy $\pi = [u_0, \ldots, u_{N-1}]$

$$\begin{aligned} V_0^\pi(x_0) &= \mathbb{E}_{x_0}^\pi \left[\prod_{n=0}^{N} 1_K(x_n) \right] \\ &= \mathbb{P}(x_N \in K, \ldots, x_0 \in K | x_0) \\ &= \mathbb{P}(x_N \in K | x_{N-1}) \times \cdots \times \mathbb{P}(x_0 \in K | x_0) \end{aligned}$$

- Viable sets

$$\mathrm{Viab}(\epsilon) = \{x \mid V_0^\pi(x) \geq \epsilon\}$$

- Optimal control policy and optimal likelihood

$$V_0^{\pi^*}(x_0) = \sup_{\pi \in \Pi} \mathbb{E}_{x_0}^\pi \left[\prod_{n=0}^{N} 1_K(x_n) \right]$$

Problem Statement

- Discrete LTI system, state $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$, input $u_k \in \mathcal{U} \subseteq \mathbb{R}^m$

$$x_{k+1} = Ax_k + Bu_k + w_k$$

- w_k is an i.i.d. Gaussian process with

$$\begin{aligned} \mathbb{E}[w_k] &= \mu \\ \mathbb{E}[w_k w_j^T] - \mu\mu^T &= \mathrm{diag}(\sigma_1^2[k], \dots, \sigma_n^2[k]) \\ \underline{\sigma}_i^2 \leq \sigma_i^2[k] &\leq \overline{\sigma}_i^2, \;\; \forall i \in \{1, \dots, n\} \end{aligned}$$

  - Unknown: $\sigma_i^2[k]$, $i = 1, 2, \dots, n$
  - Known: $\underline{\sigma}_i^2$, $\overline{\sigma}_i^2$, $i = 1, 2, \dots, n$

Compute based on known parameters:

- lower bounding value function $\underline{V}_0^\pi(x) \leq V_0^\pi(x)$
- upper bounding value function $\overline{V}_0^\pi(x) \geq V_0^\pi(x)$

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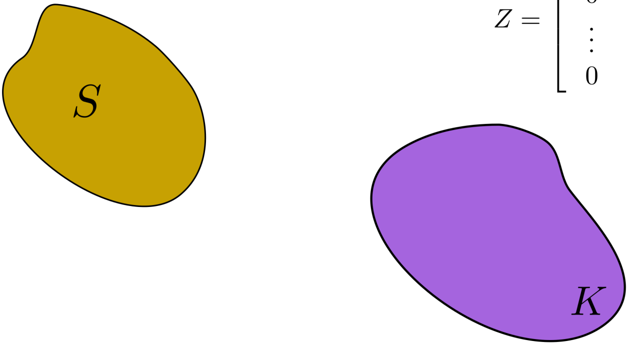
Scalable Subsets and Supersets

- $S \subset \mathbb{R}^n$ is a **Z-scalable subset** of K if for a diagonal matrix $Z = \text{diag}(z_1, \ldots, z_n) \in \mathbb{R}^{n \times n}$, $ZS = \{Zx \in \mathbb{R}^n : x \in S\} \subseteq K$

$$Z = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix}$$



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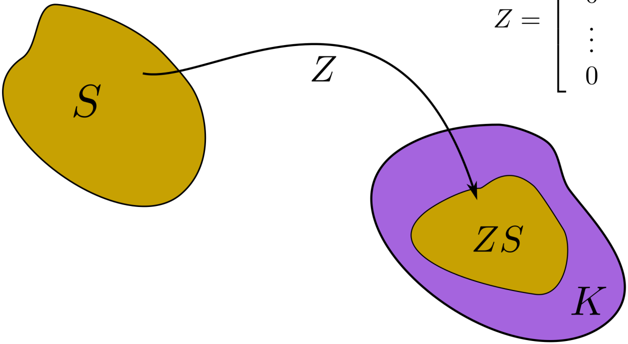

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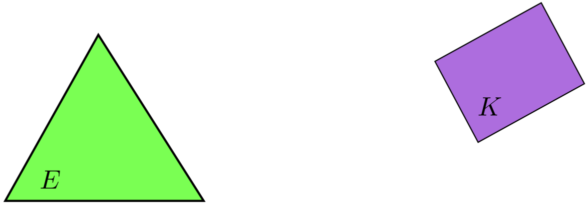
- $E \subset \mathbb{R}^n$ is a ***Z*-scalable superset** of K if for a diagonal matrix $Z = \text{diag}(z_1, \ldots, z_n) \in \mathbb{R}^{n \times n}$, $ZE \supseteq K$



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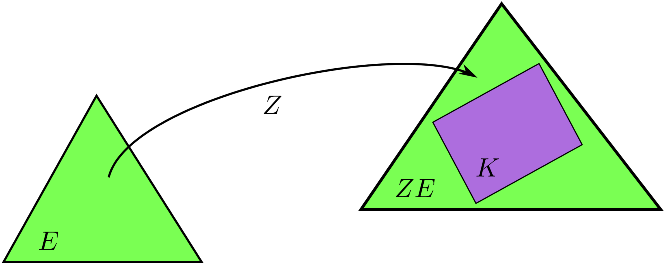

Scalable Subsets and Supersets

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- $S \subset \mathbb{R}^n$ ($E \subset \mathbb{R}^n$) is **$\mathcal{Z}$-scalable subset** (**superset**) of K if for a set of diagonal matrices,

  $$\mathcal{Z} = \left\{ Z \in \mathbb{R}^{n \times n} : Z = \mathrm{diag}(z_1, \ldots, z_n) \right\}$$

  $ZS \subseteq K$ ($ZE \supseteq K$) for all $Z \in \mathcal{Z}$

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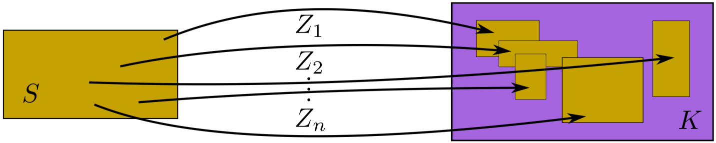


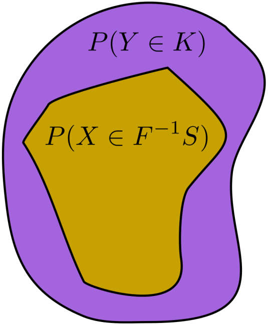
$\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_n\}$

Bounding Probability with Scalable Sets

- X is a Gaussian random variable with unknown, but bounded, variance
- $F = \text{diag}(\overline{\sigma}_1/\sigma_1, \ldots, \overline{\sigma}_n/\sigma_n)$
- $T^{-1}(K) = S$

If S is a F^{-1}-scalable subset of K, then for the random variable defined as $Y = T(FX)$, we have $\mathbb{P}(Y \in K) \leq \mathbb{P}(X \in K)$.

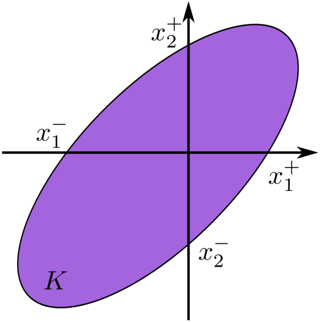
$$\begin{aligned}\mathbb{P}(Y \in K) &= \mathbb{P}(T(FX) \in K) \\ &= \mathbb{P}(X \in F^{-1}S) \\ &\leq \mathbb{P}(X \in K)\end{aligned}$$



- $\text{var}(FX) = \text{diag}(\overline{\sigma}_1^2, \ldots, \overline{\sigma}_n^2)$
- If T is a linear transformation then Y is Gaussian!

Convex Set Scaling

- Can we convert a convex set into an F^{-1} scalable subset?
- $K \subset \mathbb{R}^n$ be a bounded, convex containing the origin



$$x_i^- \triangleq \arg \min_{x \in K} e_i^T x \quad \text{subject to } (I - e_i e_i^T)x = \mathbf{0}$$

$$x_i^+ \triangleq \arg \max_{x \in K} e_i^T x \quad \text{subject to } (I - e_i e_i^T)x = \mathbf{0}$$

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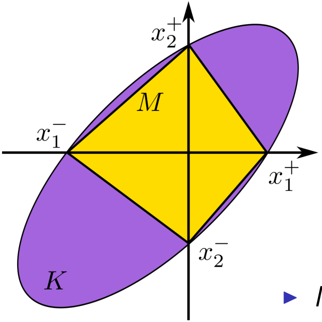

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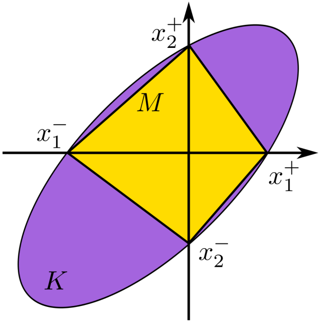
- M is the polytope with vertices x_i^-, x_i^+, $i \in \{1, \cdots, n\}$
- $M \subseteq K$.

Convex Set Scaling...

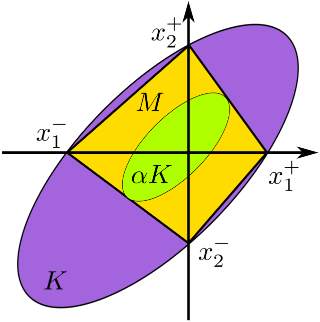


- $K = \cap_i a_i^T x \leq b_i$
- $M = \cap_{j=1}^{2^n} m_j^T x \leq c_j$

$$
\begin{array}{ll}
\text{maximize} & t \\
\text{subject to} & 0 \leq t \leq 1 \\
& a_i^T x \leq b_i \quad \forall i \\
& m_j^T t x \leq c_j \quad \forall j \in \{1, \ldots, 2^n\}
\end{array}
$$

- $\alpha = t^*$

Convex Set Scaling...



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$$

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Convex Set Scaling

Properties of αK:

- convex
- $\mathcal{Z}$-scalable with

$$\mathcal{Z} = \{Z \in \mathbb{R}^{n\times n} : Z = \mathrm{diag}(z_1, \ldots, z_n), 0 \leq z_i \leq 1, \forall i = 1, \ldots, n\}$$

$F = \mathrm{diag}(\overline{\sigma}_1/\sigma_1, \ldots, \overline{\sigma}_n/\sigma_n)$

- $F^{-1} = \mathrm{diag}(\sigma_1/\overline{\sigma}_1, \ldots, \sigma_n/\overline{\sigma}_n) \in \mathcal{Z}$
- Since αK is $\mathcal{Z}$-scalable it is F^{-1}-scalable.

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State Transformation

- Discrete LTI system with Gaussian disturbance, w_k, that has known mean and unknown but bounded variance
- Transformation of state x_k

$$y_k = \alpha_k^{-1} F(x_k - \gamma_k) + \gamma_k$$

  - $F = \text{diag}(\overline{\sigma}_1/\sigma_1, \ldots, \overline{\sigma}_n/\sigma_n)$
  - $\gamma_k \in K$

State Transformation

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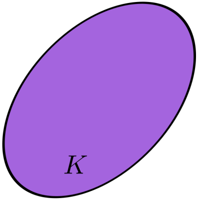

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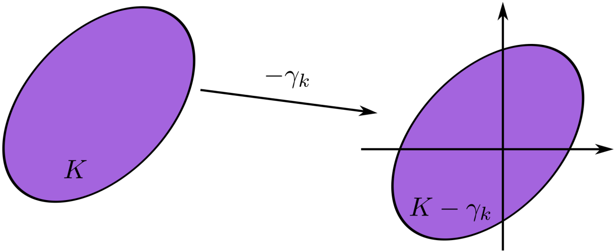

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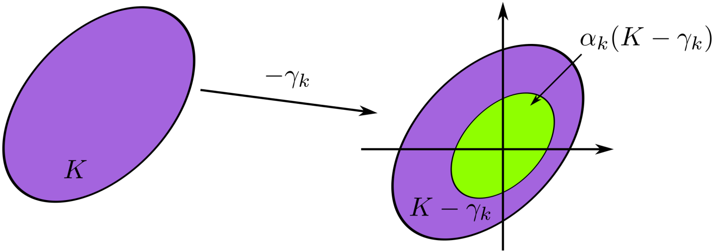

- $\alpha_k(K - \gamma_k)$ is a $\mathcal{Z}$-scalable subset of $(K - \gamma_k)$
  - $F^{-1} \in \mathcal{Z}$

State Transformation

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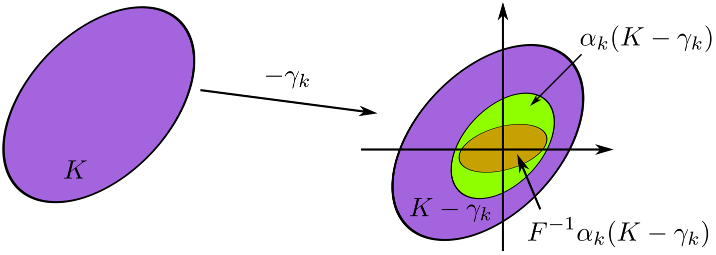

State Transformation

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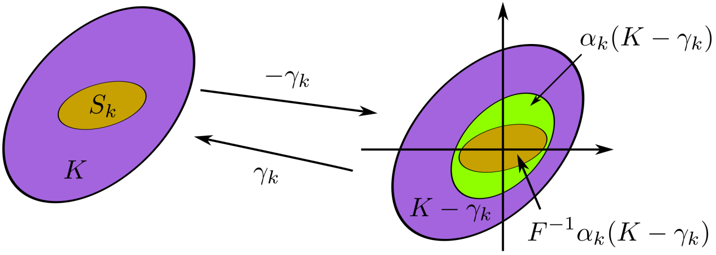
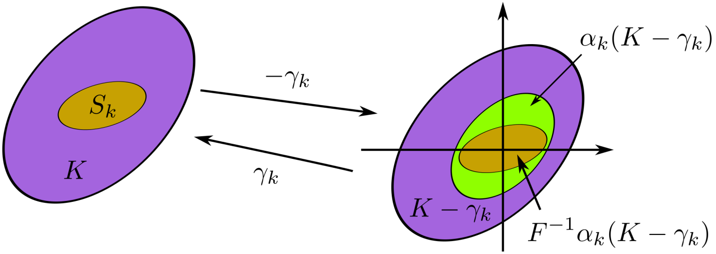


$$S_k = F^{-1}\alpha_k(K - \gamma_k) + \gamma_k \subseteq K$$

State Transformation

$$y_k = \alpha_k^{-1} F(x_k - \gamma_k) + \gamma_k$$



- $\mathbb{P}(y_k \in K) = \mathbb{P}(x_k \in S_k) \leq \mathbb{P}(x_k \in K)$
- y_k is Gaussian

Lower Bounding Viability: Sketch of Proof

$$V_k^\pi = \mathbb{P}(x_N \in K | x_{N-1}) \times \cdots \times \mathbb{P}(x_{k+1} \in K | x_k)$$

$$\begin{aligned}
\underline{V}_k^\pi(x_k) &= \mathbb{P}(y_N \in K, \ldots, y_{k+1} \in K | x_k) \\
&= \mathbb{P}(y_N \in K | y_{N-1}, \ldots, y_{k+1}) \times \cdots \times \mathbb{P}(y_{k+1} \in K | x_k) \\
&= \mathbb{P}(y_N \in K | y_{N-1}) \times \cdots \times \mathbb{P}(y_{k+1} \in K | x_k) \\
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\end{aligned}$$

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&= \mathbb{P}(y_N \in K | x_{N-1}) \times \cdots \times \mathbb{P}(y_{k+1} \in K | x_k)
\end{aligned}$$

Because $\mathbb{P}(y_{k+1} \in K | x_k) \leq \mathbb{P}(x_{k+1} \in K | x_k)$ for all $k = 0, \ldots, N-1$,

$$\underline{V}_k^\pi(x_k) \leq V_k^\pi(x_k)$$

Transformation Properties

$$y_k = \alpha_k^{-1} F(x_k - \gamma_k) + \gamma_k$$

- y_k is Gaussian
- $\mathbb{E}[y_k | x_{k-1}] = \alpha_k^{-1} F(\mathbb{E}[x_k | x_{k-1}] - \gamma_k) + \gamma_k$
- $\text{var}(y_k | x_{k-1}) = \alpha_k^{-1} \, \text{diag}(\overline{\sigma}_1^2, \ldots, \overline{\sigma}_n^2) \alpha_k^{-1}$

Cannot numerically evaluate $\underline{V}_k^\pi(x_k)$ because $\mathbb{E}[y_{k+1} | x_k]$ still depends on F

$$F = \text{diag}(\overline{\sigma}_1 / \sigma_1, \ldots, \overline{\sigma}_n / \sigma_n)$$

Transformation Properties

$$y_k = \alpha_k^{-1} F(x_k - \mathbb{E}[x_k | x_{k-1}]) + \mathbb{E}[x_k | x_{k-1}]$$

- $E[y_k | x_{k-1}] = 0$
- $\mathrm{var}(y_k | x_{k-1}) = \alpha_k^{-1} \operatorname{diag}(\overline{\sigma}_1^2, \ldots, \overline{\sigma}_n^2) \alpha_k^{-1}$
- Define

$$\underline{H}_k^{\pi^*}(x_k) = \begin{cases} \underline{V}_k^{\pi^*}(x_k), & \mathbb{E}[x_n | x_k] \in K, \ k+1 \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

Now have numerically implementable underapproximation

$$\underline{H}_k^{\pi^*}(x_k) \leq \underline{V}_k^{\pi^*}(x_k) \leq V_k^{\pi^*}(x_k)$$

Pseudo-Algorithm

input : Variance bounds: $\underline{\sigma}_i$, $\overline{\sigma}_i$
output: $\underline{H}_0^{\pi}(x)$
for $k = N, N-1, \ldots, 0$ **do**
  for $x_k \in \mathcal{X}$ **do**
    for $\pi \in \Pi$ **do**
      Compute α_k for the set $K - E[x_k|x_{k-1}]$;
      $E[y_k|x_{k-1}] \leftarrow 0$;
      $\mathrm{var}(y_k|x_{k-1}) \leftarrow \alpha_k^{-1}\,\mathrm{diag}(\overline{\sigma}_1^2, \ldots, \overline{\sigma}_n^2)\alpha_k^{-1}$;
      Solve for $H_k^{\pi}(x_k)$ via dynamic programming;
    end
    $H_k^{\pi^*}(x_k) \leftarrow \sup_{\pi} H_k^{\pi}(x_k)$
  end
end

Can similarly find overapproximation of viable sets using scalable supersets

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- Discretized double integrator

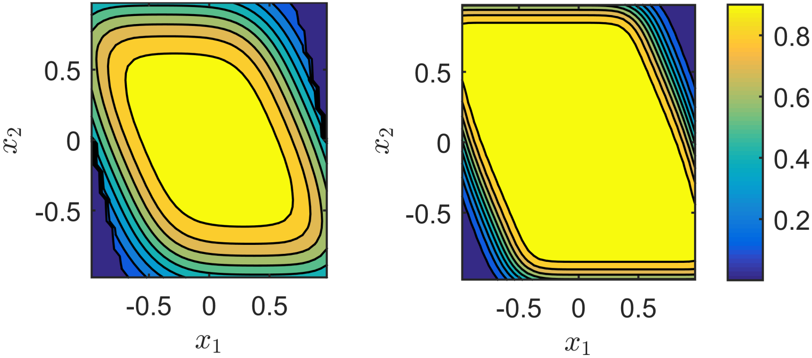
$$x_{k+1} = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{\Delta^2}{2} \\ \Delta \end{bmatrix} u_k + w_k$$

- $x \in \mathbb{R}^2$
- $u \in \{-0.1, 0, 0.1\}$
- w_k a zero-mean, Gaussian random vector

$$\mathbb{E}[w_k w_k^T] = \begin{bmatrix} \sigma_1^2[k] & 0 \\ 0 & \sigma_2^2[k] \end{bmatrix}$$

- $\sigma_i^2[k] \in [0.01, 0.05]$ for all i and k

Example...



Approximate Viable Set

Viable Set with Known Variance

- Viable set boundaries: $[-1, 1] \times [-1, 1]$
  - 40×40 grid

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Summary

- Computed bounding viable sets for LTI system with poorly characterized stochastic process
  - Gaussian with unknown but bounded variance
- State transformation using scalable subsets to provide dynamic programming based solution

Future Work

- Extend to reach and reach-avoid framework
- Improve computation time by applying techniques to convex chance constrained methodology

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Questions?