Underapproximation of Reach-Avoid Sets for Discrete-Time Stochastic Systems via Lagrangian Methods

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Motivation

- Reach-avoid analysis established tool for safety critical and/or expensive systems
- Determined for either stochastic or bounded disturbances
  - Dynamic programming suffers from the *curse of dimensionality*
  - Bounded uncertainties do not provide information about probabilistic likelihood

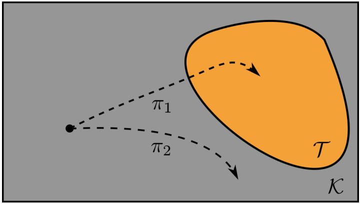
Problem Statement

- Nonlinear, discrete-time system with affine disturbance

$$x_{k+1} = f(x_k, u_k) + w_k$$

- $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^p$, $w_k \sim (\mathcal{W}, \sigma(\mathcal{W}), \mathbb{P}_w)$
- Markov policy $\pi : \mathbb{N}_{[0,N-1]} \times \mathcal{X} \to \mathcal{U} \in \mathcal{M}$
- Terminal time, N, reach-avoid problem

$$V_N^*(x) = \sup_{\pi \in \mathcal{M}} \mathbb{E}_{\bar{x}}^{N,\pi} \left[\left(\prod_{i=0}^{N-1} \mathbf{1}_{\mathcal{K}}(x_i) \right) \mathbf{1}_{\mathcal{T}}(x_N) \right]$$



Abate, Amin, Prandini, Lygeros, Sastry (2007)

Summers, Lygeros (2010)

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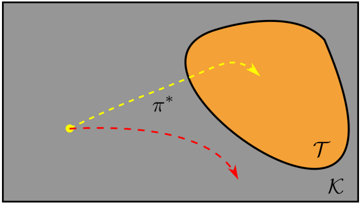

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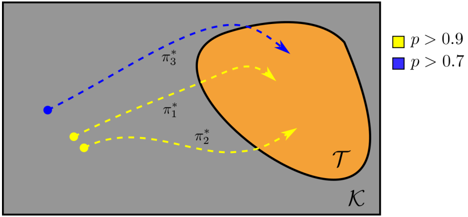
Problem Statement

- Reach-avoid likelihood value function

$$V_t^*(x) = \mathbf{1}_{\mathcal{K}}(x)\mathbb{P}_{\bar{x}_k}^{N-t,\pi^*}\left(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{t+1} \in \mathcal{K} | x\right)$$

- Efficient conservative approximate the stochastic reach-avoid β-level set

$$\mathcal{L}_t(\beta) = \left\{x \in \mathcal{X} : V_{N-t}^*(x) \geq \beta\right\}$$



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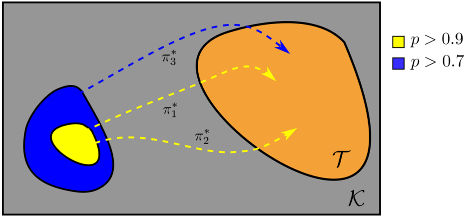


Related Work

Stochastic Reach-Avoid Analysis
Pola, Lygeros, Benedetto (2006); Abate, Amin, Prandini, Lygeros, Sastry (2007); Abate, Prandini, Lygeros, Sastry (2008); Summers, Lygeros (2010); Summers, Kamgarpour, Lygeros, Tomlin (2011); Lesser, Oishi, Erwin (2013); Kariotoglou, Summers, Summers, Kamgarpour, Lygeros (2013); Vinod and Oishi (2017)

Reach-Avoid Sets With Bounded Disturbance
Bertsekas and Rhodes (1971); Kerrigan (2001); Tomlin, Mitchell, Bayen (2003); Raković, Kerrigan, Mayne, Lygeros (2006);

Lagrangian Methods and Computation Tools
Saint-Pierre (1994); Maidens, Kaynama, Mitchell, Oishi, Dumont (2013); Kurzhanskiy, Varaiya (2006); Le Guernic and Girard (2010); Herceg, Kvasnica, Jones, Morari (2013); Bak, Duggirala (2017)

Main Contributions

- Given a value $\beta \in [0, 1]$, characterize bounded $\mathcal{E} \subseteq \mathcal{W}$ which is used to conservatively underapproximete the stochastic reach-avoid β-level set
- An algorithm to compute an underapproximation of the stochastic reach-avoid level sets for a nonlinear system with an affine Gaussian disturbance

Conservative Approximation

$$\mathcal{L}_t(\beta) = \left\{x \in \mathcal{X} : V_{N-t}^*(x) \geq \beta\right\}$$

$$V_{N-t}^*(x) = \mathbb{P}_{\bar{x}_t}^{t,\pi^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x)\mathbf{1}_{\mathcal{K}}(x)$$

Conservative Approximation

$$\mathcal{L}_t(\beta) = \{x \in \mathcal{X} : V^*_{N-t}(x) \geq \beta\}$$

$$V^*_{N-t}(x) = \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \ldots, x_{N-t+1} \in \mathcal{K} | x)\mathbf{1}_{\mathcal{K}}(x)$$
$$= \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \ldots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in \mathcal{E}^t)\mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in \mathcal{E}^t)$$
$$+ \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \ldots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t))\mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t))$$

- $\bar{w}_t = [w_{N-t}^\top, w_{N-t+1}^\top, \ldots, w_{N-1}^\top]^\top$
- $\mathcal{E}^t = \underbrace{\mathcal{E} \times \mathcal{E} \times \cdots \times \mathcal{E}}_{t \text{ times}}$
- From i.i.d. $\mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in \mathcal{E}^t) = \mathbb{P}_{w_{N-t}}(w_{N-t} \in \mathcal{E})^t$

Conservative Approximation

$$\begin{aligned}
\mathcal{L}_t(\beta) &= \left\{x \in \mathcal{X} : V^*_{N-t}(x) \geq \beta\right\} \\
V^*_{N-t}(x) &= \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x) \mathbf{1}_{\mathcal{K}}(x) \\
&= \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in \mathcal{E}^t) \mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in \mathcal{E}^t) \\
&\quad + \underbrace{\mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t)) \mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t))}_{\geq 0}
\end{aligned}$$

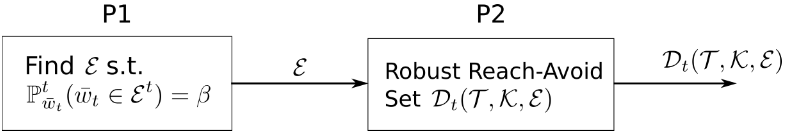
Conservative Approximation

$$
\begin{aligned}
\mathcal{L}_t(\beta) &= \left\{x \in \mathcal{X} : V_{N-t}^*(x) \geq \beta\right\} \\
V_{N-t}^*(x) &= \mathbb{P}_{\bar{x}_t}^{t,\pi^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x)\mathbf{1}_{\mathcal{K}}(x) \\
&= \mathbb{P}_{\bar{x}_t}^{t,\pi^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in \mathcal{E}^t)\mathbb{P}_{\bar{w}_t}^{t}(\bar{w}_t \in \mathcal{E}^t) \\
&\quad + \mathbb{P}_{\bar{x}_t}^{t,\pi^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t))\mathbb{P}_{\bar{w}_t}^{t}(\bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t)) \\
&\geq \mathbb{P}_{\bar{x}_t}^{t,\pi^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in \mathcal{E}^t)\mathbb{P}_{\bar{w}_t}^{t}(\bar{w}_t \in \mathcal{E}^t)
\end{aligned}
$$

Conservative Approximation

$$\mathcal{L}_t(\beta) = \left\{x \in \mathcal{X} : V^*_{N-t}(x) \geq \beta\right\}$$

$$\begin{aligned} V^*_{N-t}(x) &= \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x) \mathbf{1}_{\mathcal{K}}(x) \\ &= \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in \mathcal{E}^t) \mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in \mathcal{E}^t) \\ &\quad + \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t)) \mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t)) \\ &\geq \underbrace{\mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in \mathcal{E}^t)}_{=1} \underbrace{\mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in \mathcal{E}^t)}_{=\beta} = \beta \end{aligned}$$



$$\mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E}) = \left\{x_0 \in \mathcal{X} : \exists \pi \in \mathcal{M}, \forall \bar{w}_t \in \mathcal{E}^t, \forall k \in \mathbb{Z}_{[0,t-1]}, x_k \in \mathcal{K}, x_t \in \mathcal{T}\right\}$$

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$$\mathcal{L}_t(\beta) = \{x \in \mathcal{X} : V^*_{N-t}(x) \geq \beta\}$$

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V^*_{N-t}(x) &= \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x) \mathbf{1}_{\mathcal{K}}(x) \\
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&\quad + \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t)) \mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t)) \\
&\geq \underbrace{\mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in \mathcal{E}^t)}_{=1} \underbrace{\mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in \mathcal{E}^t)}_{=\beta} = \beta
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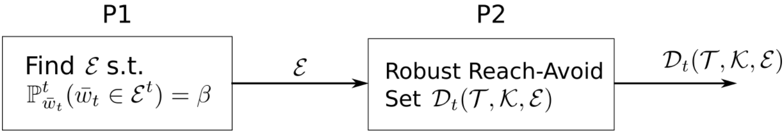

$$x \in \mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E}) \Rightarrow \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in \mathcal{E}^t) = 1$$

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$$\mathcal{L}_t(\beta) = \{x \in \mathcal{X} : V^*_{N-t}(x) \geq \beta\}$$

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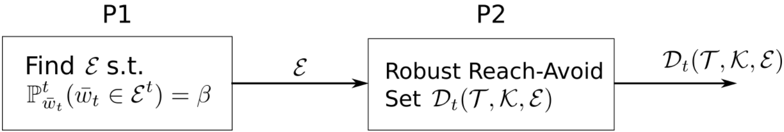


$$\begin{aligned}
x \in \mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E}) &\Rightarrow \mathbb{P}^{t,\pi^*}_{\bar{x}_t}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K} | x, \bar{w}_t \in \mathcal{E}^t) = 1 \\
&\Rightarrow V^*_{N-t}(x) \geq \beta
\end{aligned}$$

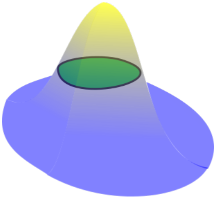
$$\therefore \mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E}) \subseteq \mathcal{L}_t(\beta) = \{x \in \mathcal{X} : V^*_{N-t}(x) \geq \beta\}$$

P1: $\mathcal{E}$ For Gaussian Disturbances

- Find $\mathcal{E}$ such that $\mathbb{P}^t_{\bar{w}_t}(\bar{w}_t \in \mathcal{E}^t) = \beta$ $(\mathbb{P}_{w_{N-t}}(w_{N-t} \in \mathcal{E})^t = \beta)$
- Given $\beta \in [0,1]$ and i.i.d. Gaussian disturbance $w_{N-t} \sim \mathcal{N}(\mu, \Sigma)$
- Let R^2 come from χ^2 distribution

$$R^2 = F^{-1}_{\chi^2(n)}\left(\beta^{\frac{1}{t}}\right)$$

- Ellipsoid $\mathcal{E}_{R^2} = \left\{s \in \mathbb{R}^n : (s-\mu)^\top \Sigma^{-1}(s-\mu) \leq R^2\right\}$ has $\mathbb{P}(w_{N-t} \in \mathcal{E}_{R^2})^t = \beta$



Robust Reach-Avoid Set

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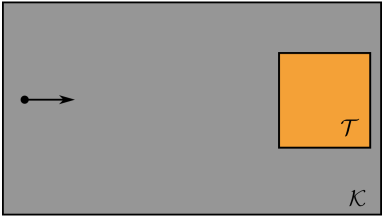

- Also referred to as “reachability of a target tube” [Bertsekas and Rhodes (1971)]

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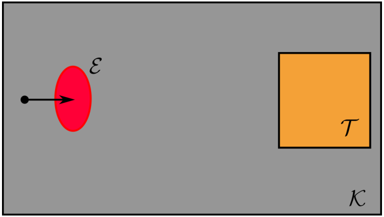

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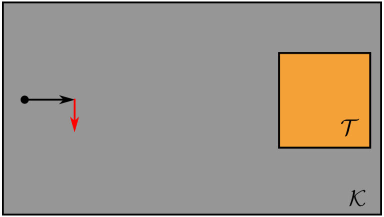

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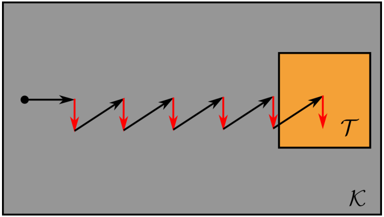

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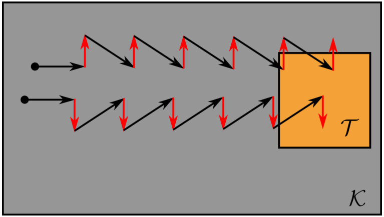

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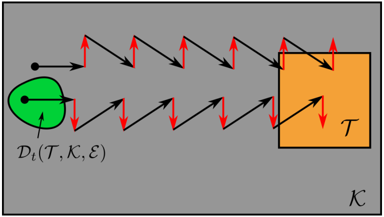

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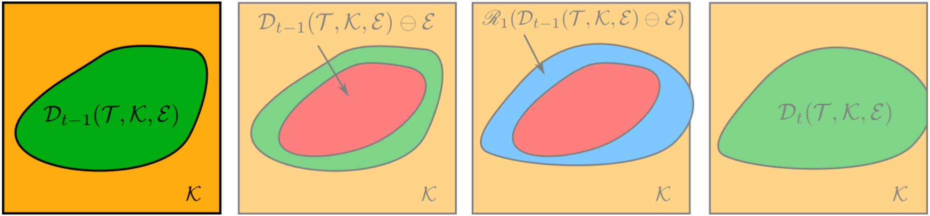
P2: Robust Reach-Avoid Set Recursion

$$\mathcal{D}_0(\mathcal{T},\mathcal{K},\mathcal{E}) = \mathcal{T}$$
$$\mathcal{D}_t(\mathcal{T},\mathcal{K},\mathcal{E}) = \mathcal{K} \cap \mathscr{R}_1(\mathcal{D}_{t-1}(\mathcal{T},\mathcal{K},\mathcal{E}) \ominus \mathcal{E})$$

Sketch of proof:



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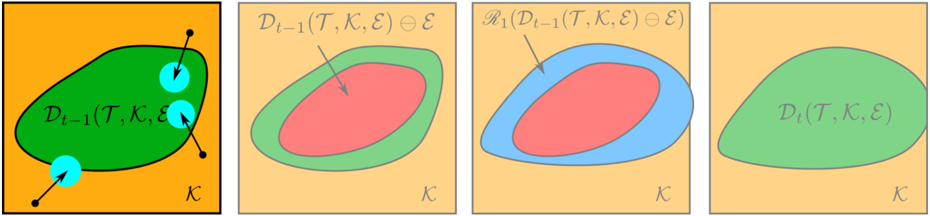

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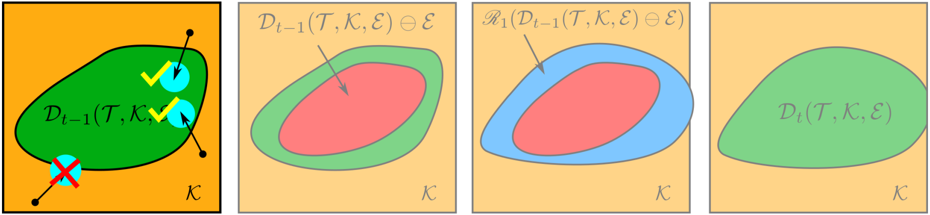

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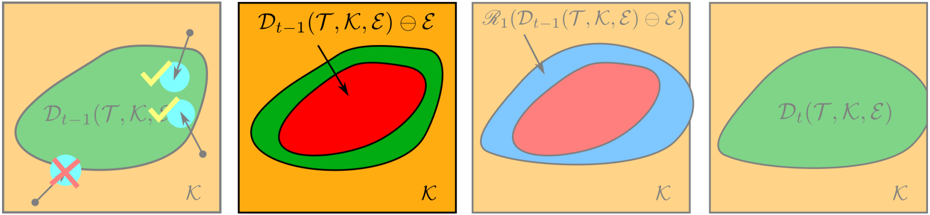
Sketch of proof: $A \ominus B = \{c \in A : c + B \subseteq A\}$



P2: Robust Reach-Avoid Set Recursion

$$\mathcal{D}_0(\mathcal{T},\mathcal{K},\mathcal{E}) = \mathcal{T}$$
$$\mathcal{D}_t(\mathcal{T},\mathcal{K},\mathcal{E}) = \mathcal{K} \cap \mathscr{R}_1(\mathcal{D}_{t-1}(\mathcal{T},\mathcal{K},\mathcal{E}) \ominus \mathcal{E})$$

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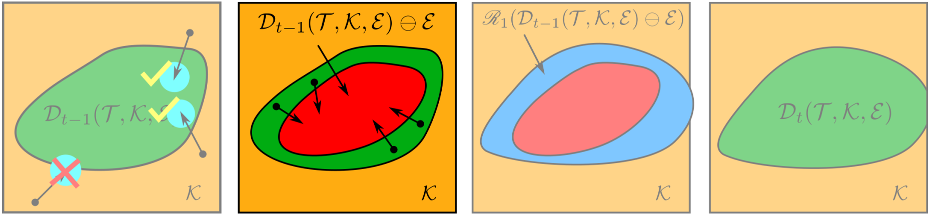

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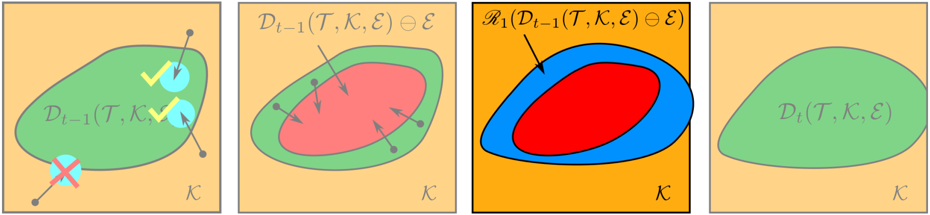
Sketch of proof: $\mathscr{R}_1(\mathcal{S}) \triangleq \left\{x^- \in \mathcal{X} : \exists u \in \mathcal{U}, \exists y \in \mathcal{S},\ y = f(x^-, u)\right\}$



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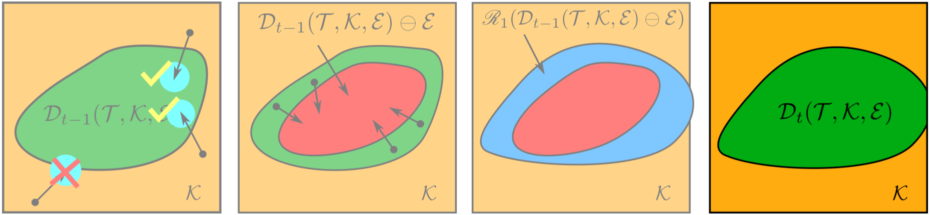
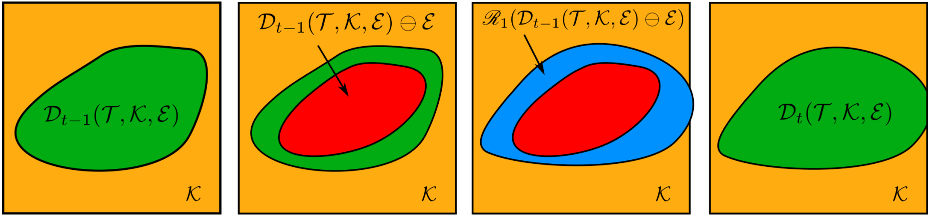


Properties of the Robust Reach-Avoid Set

- $\mathcal{D}_t(\mathcal{T},\mathcal{K},\mathcal{E}) \subseteq \mathcal{L}_t(\beta) = \left\{x \in \mathcal{X} : V^*_{N-t}(x) \geq \beta\right\}$
- For linear dynamics, $\mathcal{U},\mathcal{K},\mathcal{T}$ are convex and compact sets, $\mathcal{E}$ is a compact set, state matrix A is non-singular, then $\mathcal{D}_t(\mathcal{T},\mathcal{K},\mathcal{E})$ is convex and compact, $\forall k \in \mathbb{N}$



Algorithm

Input : Safe set, $\mathcal{K}$; target set, $\mathcal{T}$; system dynamics, desired probability level $\beta \in [0,1]$, Gaussian covariance matrix and mean, Σ, μ; and time horizon, N

Output: N-time stochastic reach-avoid β-level set underapproximation, $\mathcal{D}_N(\mathcal{K},\mathcal{T},\mathcal{E})$

$R^2 \leftarrow F^{-1}_{\chi^2(n)}\left(\beta^{\frac{1}{N}}\right)$

$\mathcal{E} \leftarrow \mathcal{E}_{R^2}$

$\mathcal{D}_0(\mathcal{K},\mathcal{T},\mathcal{E}) \leftarrow \mathcal{T}$

for $i = 1, 2, \ldots, N$ **do**

  $S \leftarrow \mathcal{D}_{i-1}(\mathcal{K},\mathcal{T},\mathcal{E}) \ominus \mathcal{E}$

  $E \leftarrow \mathscr{R}_1(S)$

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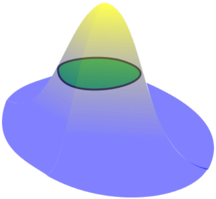

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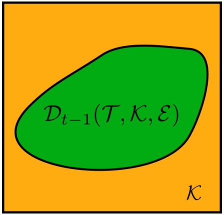

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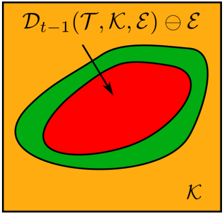

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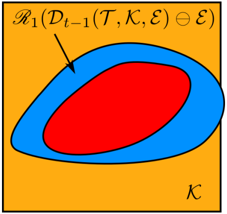

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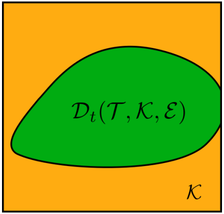
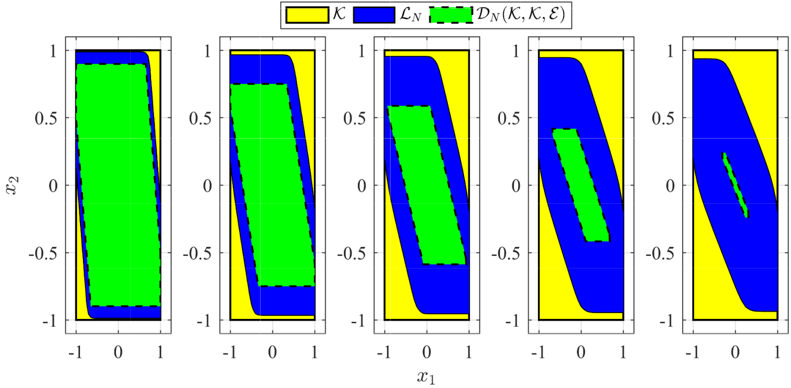


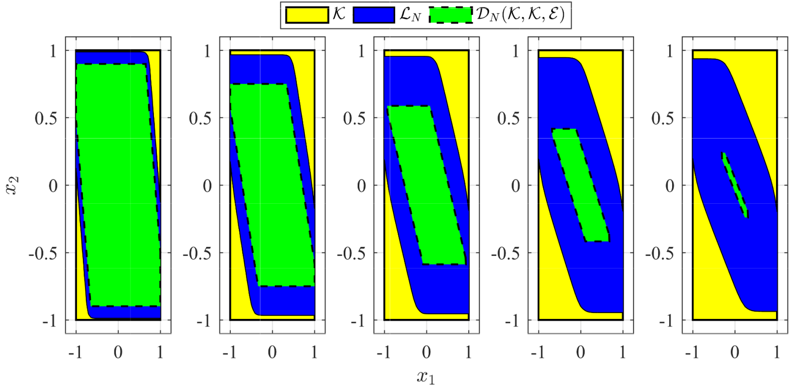
Double Integrator



$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u_k + w_k$$

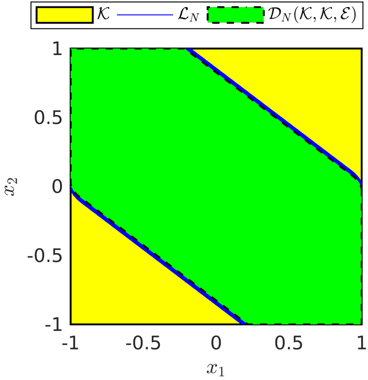
$$w_k \sim \mathcal{N}(0, 0.005 \cdot I_2)$$

Double Integrator



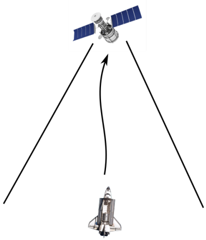
Grid Size	Dynamic Programming [s]	Lagrangian Method [s]	Ratio
41×41	8.16	0.98	8.3
82×82	59.76	0.98	60.9

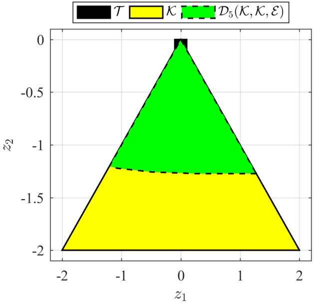
Effect of Variance



- $w_k \sim \mathcal{N}(0, 10^{-5} \cdot I_2)$
- Approximation becomes tight as variance decreases
- Size of $\mathcal{E}$ decreases with variance

Spacecraft Rendevous-Docking





- 4-dimensional Clohessy-Wiltshire-Hill dynamics

$$\ddot{x} - 3\omega x - 2\omega \dot{y} = u_1$$
$$\ddot{y} + 2\omega \dot{x} = u_2$$

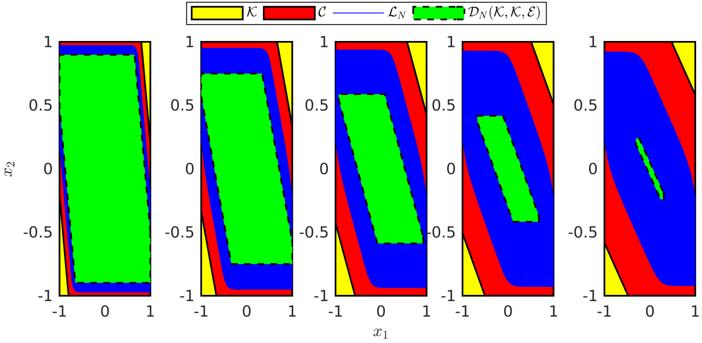
- Underapproximation computation time: 14.5 seconds
- $\Sigma = \mathrm{diag}(10^{-4}, 10^{-4}, 5 \times 10^{-8}, 5 \times 10^{-8})$

Conclusion

- Developed theory for Lagrangian techniques for underapproximating stochastic reach-avoid level sets
- Algorithm for determining $\mathcal{E}$ and the underapproximation for Gaussian noise processes
- Tradeoff between increased computation speed and conservativeness of results

Future Work

- Lagrangian methods for overapproximation the stochastic reach-avoid level-sets



- Develop methods for obtaining $\mathcal{E}$ for other disturbance types, e.g. exponential

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