Voronoi Partition-based Scenario Reduction for Fast Sampling-based Stochastic Reachability Computation of Linear Systems

Hossein Sartipizadeh, Abraham P. Vinod, Behçet Açıkmeşe, and Meeko Oishi

***Abstract*— We address the stochastic reach-avoid problem for linear systems with additive stochastic uncertainty. We seek to compute the maximum probability that the states remain in a safe set over a finite time horizon and reach a target set at the final time. We employ sampling-based methods and provide a lower bound on the number of scenarios required to guarantee that our estimate provides an underapproximation. Due to the probabilistic nature of the sampling-based methods, our underapproximation guarantee is probabilistic, and the proposed lower bound can be used to satisfy a prescribed probabilistic confidence level. To decrease the computational complexity, we propose a Voronoi partition-based to check the reach-avoid constraints at representative scenarios, instead of the original scenarios. The state constraints arising from the safe and target sets are tightened appropriately. We propose a systematic approach for selecting these representative scenarios and provide the flexibility to trade-off the number of cells needed for accuracy with the computational cost.**

I. Introduction

Reach-avoid analysis is an established verification tool for discrete-time stochastic dynamical systems, which provides probabilistic guarantees on the safety and performance [1]–[5]. This paper focuses on the finite time horizon *terminal* hitting time stochastic reach-avoid problem [1] (referred to here as the *terminal time problem*), that is, computation of the maximum probability of hitting a target set at the terminal time, while avoiding an unsafe set during all the preceding time steps using a controller that satisfies the specified control bounds.

The solution to the terminal time problem relies on grid-based dynamic programming [1], [6], [7], and lacks scalability. Researchers have proposed scalable approximations using approximate dynamic programming [8], [9], Gaussian mixtures [8], particle filters [4], [9], convex chance-constrained optimization [4], Fourier transform-based verification [10], [11], Lagrangian approaches [5], and semi-definite programming [12]. Currently, the largest system verified is a 40-dimensional chain of integrators using Fourier transform-based techniques [10], [11]. However, their high computational costs precludes their use in real-time applications.

In this paper, we reconsider the sampling-based approach, proposed in [4]. Similar sampling-based approaches have been used successfully in robotics [13] and in stochastic optimal control [14]–[17]. In the sampling-based stochastic reach-avoid problem, we sample the stochastic disturbance to produce a finite set of *scenarios*, and then formulate a mixed-integer linear program (MILP) to maximize the number of scenarios that satisfy the reach-avoid constraints [4], [13]. The approximated probability will converge to the true terminal time probability as the number of considered scenarios increases. However, the computational complexity of MILP increases exponentially with the number of scenarios, making the approach practically intractable when seeking high accuracy solutions [18, Rem. 1].

The main contributions of this paper are two-fold. We first provide a lower bound on the number of scenarios needed to probabilistically guarantee a user-specified upper bound on the approximation error with a user-specified confidence level using concentration techniques. Using Hoeffding's inequality, we demonstrate that the number of scenarios that need to be considered is inversely proportional to the square of the desired upper bound on the estimate error. Next, we propose a Voronoi partition-based undersampling technique that underapproximates the MILP-based solution in a computationally efficient manner. This approach allows us to partially mitigate the exponential computational complexity, and provides flexibility to the user to select the number of partitions based on the desired online computational complexity. We demonstrate the application of the proposed method in the problem of spacecraft rendezvous and docking.

The organization of the paper is as follows: The problem formulation and preliminary definitions are stated in Section II. A lower bound on the required number of scenarios for the prescribed confidence level is given in Section III. Section IV presents the proposed partition-based method and the approximate solution reconstruction. The performance of the proposed method is investigated on a spacecraft rendezvous maneuvering and docking in Section V.

II. Problem formulation

We presume $\mathbb{R}$ and $\mathbb{N}$ are sets of real and natural numbers, with $\mathbb{R}^n$ a length n vector of real numbers, and $\mathbb{N}_{[a,b]}$ the set

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of natural numbers between (and including) a and b where $a, b \in \mathbb{N}$ and $a \leq b$. For $x \in \mathbb{R}^n$, $x^\top$ denotes the transpose of x. For two vectors $x, y \in \mathbb{R}^n$, we define $x \preceq y$ (elementwise inequality) as the condition that $y - x$ lies in the positive orthant of $\mathbb{R}^n$. We define $\mathbf{1} \in \mathbb{R}^n$ as a n-dimensional vector with all elements 1, and the set $\mathcal{G}^N$ as the Cartesian product of a set $\mathcal{G} \subseteq \mathbb{R}^n$ with itself $N \in \mathbb{N}$ times.

A. System description

Consider a discrete-time stochastic linear system,

$$x_{t+1} = A_t x_t + B_t u_t + w_t \tag{1}$$

with state $x_t \in \mathbb{R}^{n_x}$, input $u_t \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$, disturbance $w_t \in \mathcal{W} \subseteq \mathbb{R}^{n_x}$ at time instant t, and matrices A_t, B_t assumed to be of appropriate dimensions.

The system (1) over a time horizon with length N can be alternatively written in a concatenated form,

$$X = G_x x_0 + G_u U + G_w W, \tag{2}$$

with $X = [x_1^\top, \cdots, x_N^\top]^\top \in \mathbb{R}^{n_x N}$, $U = [u_0^\top, \cdots, u_{N-1}^\top]^\top \in \mathcal{U}^N$, and $W = [w_0^\top, \cdots, w_{N-1}^\top]^\top \in \mathcal{W}^N$ the concatenated state, input, and disturbance vectors over a N-length horizon. The matrices G_x, G_u, and G_w are obtained from the system matrices in (1), see [15], [19].

The concatenated disturbance random vector W is a random vector. As with sampling-based methods, we do not require the explicit knowledge of the probability measure of W, denoted by $\mathbb{P}_W$. We only require the ability to draw samples, according to $\mathbb{P}_W$. Note that W need not be a Gaussian random vector. Due to the stochastic nature of W, the state x_t and the concatenated state vector X are random. We define $\mathbb{P}_X^{x_0,U}$ as the probability measure associated with the random vector X induced from $\mathbb{P}_W$ and (2).

B. Open-loop stochastic reach-avoid problem

We are interested in the terminal time problem [1]. In this problem, we seek a measurable, time-varying, state-feedback control law (also known as Markov policy) that maximizes the probability of reaching a target set at the time horizon while staying within a safe set. However, the computation of these control laws require grid-based dynamic programming [7], which lacks scalability. To assure tractability, we seek open-loop control laws similar to [4], [10], [11]. These control laws are more conservative as compared to Markov policies.

Given an initial state $x_0 \in \mathbb{R}^{n_x}$ and an open-loop control $U \in \mathcal{U}^N$, we define the *open-loop terminal time probability*, $r_{x_0}^U(\mathcal{S}, \mathcal{T})$, as the probability that the state trajectory remains inside the safety set $\mathcal{S} \subseteq \mathbb{R}^{n_x}$ and reaches the target set $\mathcal{T} \subseteq \mathbb{R}^{n_x}$ at time N,

$$\begin{aligned} r_{x_0}^U(\mathcal{S}, \mathcal{T}) &= \mathbb{P}_X^{x_0,U}\left\{x_N \in \mathcal{T} \wedge \forall t \in \mathbb{N}_{[0,N-1]}, x_t \in \mathcal{S}\right\}, \\ &= \mathbb{P}_X^{x_0,U}\left\{X \in \mathcal{R}\right\} 1_{\mathcal{S}}(x_0). \end{aligned} \tag{3}$$

Here, $\mathcal{R} \triangleq \mathcal{S}^{N-1} \times \mathcal{T}$. By construction, $r_{x_0}^U(\mathcal{S}, \mathcal{T})$ underapproximates the maximal terminal time reach probability that may be obtained using a Markov policy [10, Thm. 2].

The open-loop stochastic reach-avoid problem is formulated as:

Problem 1. *Open-loop terminal time problem:*

$$p^*(x_0) = \underset{U \in \mathcal{U}^N}{\text{maximize}} \quad r_{x_0}^U(\mathcal{S}, \mathcal{T}) \tag{4}$$

Problem 1 is equivalent to,

$$p^*(x_0) = \underset{U \in \mathcal{U}^N}{\text{maximize}} \quad 1_{\mathcal{S}}(x_0) \mathbb{E}_z^{x_0,U}[z], \tag{5}$$

where $z = 1_{\mathcal{T}}(x_N) \prod_{t=1}^{N-1} 1_{\mathcal{S}}(x_t) = 1_{\mathcal{R}}(X)$ is a Bernoulli random variable with a discrete probability measure $\mathbb{P}_z^{x_0,U}$ induced from $\mathbb{P}_X^{x_0,U}$ [10, Thm. 2] [1, Sec. 4].

Remark 1. *In Problem 1, $p^*(x_0)$ is trivially zero when $x_0 \notin \mathcal{S}$, irrespective of the choice of the controller.*

In [4], [13], a MILP was formulated as an approximation of Problem 1 when the safe and the target sets are *polytopic*. Note that restriction of the safe and the target sets to polytopes is not severe since convex and compact sets admit tight polytopic underapproximations [20, Ex. 2.25]. We will define the safe set $\mathcal{S}$, the target set $\mathcal{T}$, and the reach-avoid constraint set $\mathcal{R}$ as

$$\mathcal{S} = \{x | f_{\mathcal{S}} x \preceq h_{\mathcal{S}}\}, \tag{6a}$$

$$\mathcal{T} = \{x | f_{\mathcal{T}} x \preceq h_{\mathcal{T}}\}, \tag{6b}$$

$$\mathcal{R} = \{x | FX \preceq h\} \tag{6c}$$

with matrices $f_{\mathcal{S}} \in \mathbb{R}^{l_{\mathcal{S}} \times n_x}$, $f_{\mathcal{T}} \in \mathbb{R}^{l_{\mathcal{T}} \times n_x}$, and $F = [f_{\mathcal{S}}^\top \ \cdots \ f_{\mathcal{S}}^\top \ f_{\mathcal{T}}^\top]^\top \in \mathbb{R}^{L \times n_x N}$, and scalars $l_{\mathcal{S}}, l_{\mathcal{T}}, L \in \mathbb{N}$, $L = (N-1) l_{\mathcal{S}} + l_{\mathcal{T}}$.

Consider a finite set of K samples of the concatenated disturbance vector W,

$$\mathscr{W}_K = \{W^{(1)}, \cdots, W^{(K)}\} \subset \mathbb{R}^{n_x N}. \tag{7}$$

Using the "big-M" approach [4], [13], [18], and a sampling-based empirical mean for $r_{x_0}^U(\mathcal{S}, \mathcal{T})$ using $\mathscr{W}_K$, we obtain the following MILP approximation to Problem 1.

Problem 2. *A MILP approximation to Problem 1 is given by*

$$\begin{aligned} \underset{\substack{U \in \mathcal{U}^N \\ z^{(1)}, \ldots, z^{(K)}}}{\text{maximize}} \quad & \tfrac{1}{K} \textstyle\sum_{i=1}^K z^{(i)} \\ \text{subject to} \quad & X^{(i)} = G_x x_0 + G_u U + G_w W^{(i)}, && i \in \mathbb{N}_{[1,K]} \\ & FX^{(i)} \preceq h + M(1 - z^{(i)})\mathbf{1}, && i \in \mathbb{N}_{[1,K]} \\ & z^{(i)} \in \{0, 1\}, && i \in \mathbb{N}_{[1,K]} \end{aligned}$$

with the optimal value denoted by $p_K^*(x_0)$. Here, $M \in \mathbb{R}$ is some large positive number, $W^{(i)} \in \mathscr{W}_K$ are samples of the concatenated disturbance vector, and $X^{(i)} \in \mathbb{R}^{n_x N}$ are the associated sampled state trajectories (scenarios).

As observed in [4], [13], [18], for every $i \in \mathbb{N}_{[1,K]}$, $z^{(i)} = 1$ implies $FX^{(i)} \preceq h$, i.e., the sampled state trajectory is safe. Conversely, if a sampled state trajectory violates the reach-avoid constraint ($FX^{(i)} \not\preceq h$), then $z^{(i)} = 0$. This concept is illustrated in Figure 1; red crosses indicate the sampled state trajectories which fail to remain in $\mathcal{R}$, and, therefore,

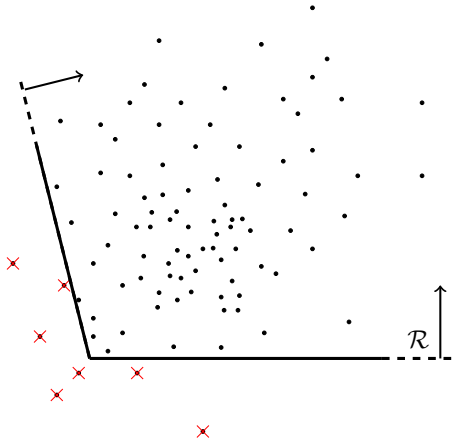

Fig. 1. Illustration of the Problem 2. The reach-avoid set $\mathcal{R}$ is distinguished with lines. Black dots and red crosses represent the sampled state trajectories corresponding to the disturbance samples in $\mathscr{W}_K$ that succeed and fail to remain in $\mathcal{R}$, respectively. Given x_0, Problem 2 seeks $U \in \mathcal{U}^N$ that maximizes the number of sampled trajectories that remain in $\mathcal{R}$.

their corresponding binary variables are zero. We define the random vector $Z = [z^{(1)} \ \dots \ z^{(K)}]^\top$, the concatenation of an i.i.d process consisting of Bernoulli random variables $\{z^{(i)}\}_{i=1}^K$. The probability measure associated with Z is $\mathbb{P}_Z^{x_0,U}$ is induced from $\mathbb{P}_z^{x_0,U}$.

From [4], [13], we have,

$$\lim_{K\to\infty} p_K^*(x_0) = p^*(x_0). \tag{8}$$

However, Problem 2 becomes computationally intractable for large values of K since the worst-case time complexity of MILP problems exponentially increases in the number of binary decision variables [18, Rem. 1].

C. Problem statements

Question 1. *Given a violation parameter $\delta > 0$, risk of failure $\beta \in [0,1]$, initial state $x_0 \in \mathcal{S}$, and the optimal solution $U_K^* \in \mathcal{U}^N$ to Problem 2, characterize a lower bound on the number of scenarios K to guarantee $\mathbb{P}_Z^{x_0,U_K^*}\{p_K^*(x_0) - p^*(x_0) \geq \delta\} \leq \beta$.*

Question 2. *Given $K > 0$, construct an undersampling-based MILP underapproximation to Problem 2 that:*

1) *uses $\hat{K} \ll K$ scenarios, and*
2) *attains an optimal value $p_{\hat{K}}^*(x_0) \leq p_K^*(x_0)$.*

Also, compute $\hat{p}(x_0)$ to tighten the open-loop terminal time probability estimate using the computed optimal input $U_{\hat{K}}^*$ such that $p_{\hat{K}}^*(x_0) \leq \hat{p}(x_0) \leq p_K^*(x_0)$.

We will address Question 1 using Hoeffding's inequality. The solution to Problem 2 can, with the lower bound on K from Question 1, provide an assured underapproximation of the solution to Problem 1. However, solving Problem 2 for large K is hard, which motivates Question 2.

We address Question 2 using Voronoi partitions, and solve an undersampled restriction of Problem 2. The user can control the number of representative samples (Voronoi centers) used, and trade-off accuracy or conservativeness with computational complexity. We tighten the safety constraint $FX^{(\cdot)} \preceq h$ using appropriately defined buffers to ensure that the undersampled problem is indeed a restriction of Problem 2 (and thereby, Problem 1).

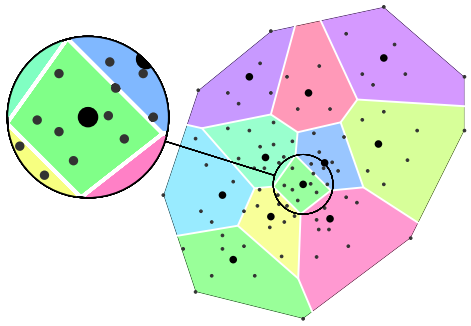

Fig. 2. Voronoi partition of a finite set (smaller dots) using a fewer set of representative points (larger dots).

D. Voronoi partition and data clustering

We revisit concepts from Voronoi partitioning for our solution to Question 2. A Voronoi partition is characterized by a finite set of representative points (centers) for a given set. Given a finite set $\mathcal{Q} = \{q^{(1)}, \cdots, q^{(K)}\}$ with $q^{(\cdot)} \in \mathbb{R}^d$, we can partition $\mathcal{Q}$ into $\hat{K}$ cells using the centers $\mathcal{C} = \{c^{(1)}, \cdots, c^{(\hat{K})}\}$, $c^{(\cdot)} \in \mathbb{R}^d$. Specifically, we obtain the partition $\mathscr{V}_{\mathcal{Q}}(\mathcal{C}) \triangleq \{\mathcal{V}^{(1)}, \dots, \mathcal{V}^{(\hat{K})}\}$, in which each cell $\mathcal{V}^{(j)}$ is defined as

$$\mathcal{V}^{(j)} = \{q \in \mathcal{Q} | d(q, c^{(j)}) < d(q, c^{(\ell)}), \forall \ell \in \mathbb{N}_{[1,\hat{K}]}, \ell \neq j\}. \tag{9}$$

Here, $d(q, c^{(\cdot)})$ is the distance between $q, c^{(\cdot)} \in \mathbb{R}^d$ in any valid metric (Euclidean norm is used in this paper). By construction, every point $q^{(\cdot)} \in \mathcal{Q}$ belongs to only one of the cells in $\mathscr{V}_{\mathcal{Q}}(\mathcal{C})$. We denote the number of elements in $\mathcal{V}^{(j)}$ by $|\mathcal{V}^{(j)}|$.

Figure 2 illustrates the Voronoi partition of a set of points (smaller dots) using the Voronoi centers (larger dots). By choosing $\hat{K}$ to be significantly smaller than K, the centers $c^{(j)}$ can serve as the representative point of all the points $q^{(\cdot)}$ in the cell $\mathcal{V}^{(j)}$ by (9).

While any collection of $\hat{K}$ finite points be used as $\mathcal{C}$, we will compute $\mathcal{C}$ using the k-means method or Lloyd's algorithm [21]. In this clustering approach, we compute the set of centers $\mathcal{C}$ by minimizing the *within-cluster sum of squares*. The within-cluster sum of squares, denoted by $\mathrm{WSS}(\hat{K}, \mathscr{V}_{\mathcal{Q}}(\mathcal{C}))$, simply represents the total sum of squared deviations of points in cells from the centers in $\mathcal{C}$. Formally, given a set of points $\mathcal{Q}$, a set of centers $\mathcal{C}$, and its associated partition $\mathscr{V}_{\mathcal{Q}}(\mathcal{C})$,

$$\mathrm{WSS}(\hat{K}, \mathscr{V}_{\mathcal{Q}}(\mathcal{C})) = \sum_{j=1}^{\hat{K}} \sum_{q^{(i)} \in \mathcal{V}^{(j)}} \|q^{(i)} - c^{(j)}\|^2. \tag{10}$$

Let the set of optimal centers be $\mathcal{C}^*$, given by,

$$\mathcal{C}^* = \underset{\mathcal{C}\subset\mathbb{R}^d}{\arg\min}\ \mathrm{WSS}(\hat{K}, \mathscr{V}_{\mathcal{Q}}(\mathcal{C})), \tag{11}$$

The cells of $\mathscr{V}_{\mathcal{Q}}(\mathcal{C}^*)$ represent the optimal $\hat{K}$ clusters of $\mathcal{Q}$. Solving (11) is known to be a NP-hard problem [22]. However, efficient algorithms exist to compute a local minima. For example, starting from an initial guess of $\mathcal{C}$, a successive algorithm with time complexity $\mathcal{O}(\gamma d K \hat{K})$ for γ iterations can be used to update each center by replacing it with the centroid of its cluster elements [23]. Note that the set of optimal centers $\mathcal{C}^*$ is not necessarily a subset of $\mathcal{Q}$.

III. Scenarios required to meet given failure tolerance

We first recall the Hoeffding's inequality. Given i.i.d. random variables $y^{(1)}, y^{(2)}, \ldots, y^{(K)}$ for $K > 0$ and $y^{(i)} \in [0,1],\ \forall i \in \mathbb{N}_{[1,K]}$ with probability measure $\mathbb{P}_y$, we define the concatenation of these random variables $Y = [y^{(1)}\ y^{(2)}\ \ldots\ y^{(K)}]^\top \in [0,1]^K$. The probability measure associated with Y is $\mathbb{P}_Y^K = \prod_{i=1}^K \mathbb{P}_y$.

Lemma 1. ***(Hoeffding's inequality)*** *[24, Thm. 1] Define $\overline{Y} = \frac{\mathbf{1}^\top Y}{K} = \frac{\sum_{i=1}^K y^{(i)}}{K}$, and $\mu_{\overline{Y}} \triangleq \mathbb{E}\left[\overline{Y}\right]$. For any $\delta > 0$,*

$$\mathbb{P}_Y^K\left\{\overline{Y} - \mu_{\overline{Y}} \geq \delta\right\} \leq e^{-2K\delta^2}. \tag{12}$$

Theorem 1. *Given a violation parameter $\delta > 0$, risk of failure $\beta \in [0,1]$, initial state $x_0 \in \mathcal{S}$, and the optimal solution $U_K^* \in \mathcal{U}^N$ to Problem 2, we have the risk of failure $\mathbb{P}_Z^{x_0,U_K^*}\{p_K^*(x_0) - p^*(x_0) \geq \delta\} \leq \beta$, provided*

$$K \geq \frac{-\ln(\beta)}{2\delta^2}. \tag{13}$$

Proof: Let the optimal solution to Problem 1 be $U^* \in \mathcal{U}^N$. Note that U^* may not be equal to U_K^*, the optimal solution to Problem 2. For $x_0 \in \mathcal{S}$, the optimal values of Problems 1 and 2 are respectively,

$$\begin{aligned} p^*(x_0) &= \mathbb{E}_z^{x_0,U^*}[z] = \tfrac{\mathbb{E}_Z^{x_0,U^*}[\mathbf{1}^\top Z]}{K} && \text{under } U^*, \\ p_K^*(x_0) &= \tfrac{1}{K}\textstyle\sum_{i=1}^K z^{(i)} = \tfrac{\mathbf{1}^\top Z}{K} && \text{under } U_K^*. \end{aligned} \tag{14}$$

Here, $p_K^*(x_0)$ is the empirical mean of z estimated using K samples under the optimal controller U_K^*. By the optimality of U^* in Problem 1,

$$\mathbb{E}_z^{x_0,U^*}[z] \geq \mathbb{E}_z^{x_0,U_K^*}[z]. \tag{15}$$

Using (14),

$$\begin{aligned} &\mathbb{P}_Z^{x_0,U_K^*}\{p_K^*(x_0) - p^*(x_0) \geq \delta\} \\ &\quad = \mathbb{P}_Z^{x_0,U_K^*}\left\{\left(\frac{1}{K}\sum_{i=1}^K z^{(i)} - \mathbb{E}_z^{x_0,U^*}[z]\right) \geq \delta\right\}. \end{aligned} \tag{16}$$

By adding and subtracting $\mathbb{E}_z^{x_0,U_K^*}[z]$, we obtain

$$\begin{aligned} &\left(\frac{1}{K}\sum_{i=1}^K z^{(i)} - \mathbb{E}_z^{x_0,U^*}[z]\right) \\ &= \left(\frac{1}{K}\sum_{i=1}^K z^{(i)} - \mathbb{E}_z^{x_0,U_K^*}[z]\right) + \left(\mathbb{E}_z^{x_0,U_K^*}[z] - \mathbb{E}_z^{x_0,U^*}[z]\right) \\ &\leq \left(\frac{1}{K}\sum_{i=1}^K z^{(i)} - \mathbb{E}_z^{x_0,U_K^*}[z]\right) \end{aligned} \tag{17}$$

where (17) follows from (15). Thus, the set $\left\{\overline{Z} \in \{0,1\}^K : \left(\frac{\mathbf{1}^\top \overline{Z}}{K} - \mathbb{E}_z^{x_0,U^*}[z]\right) \geq \delta\right\}$ is a subset of $\left\{\overline{Z} \in \{0,1\}^K : \left(\frac{\mathbf{1}^\top \overline{Z}}{K} - \mathbb{E}_z^{x_0,U_K^*}[z]\right) \geq \delta\right\}$ by (14) and (17). Recall that $\mathbb{P}\{\mathcal{S}_1\} \leq \mathbb{P}\{\mathcal{S}_2\}$ for any two sets $\mathcal{S}_1 \subseteq \mathcal{S}_2$, and $\mathbb{E}_z^{x_0,U_K^*}[z] = \frac{\mathbb{E}_Z^{x_0,U_K^*}[\mathbf{1}^\top Z]}{K}$ by definition. From these observations, Hoeffding's inequality (Lemma 1), and (16),

$$\begin{aligned} &\mathbb{P}_Z^{x_0,U_K^*}\{p_K^*(x_0) - p^*(x_0) \geq \delta\} \\ &\quad \leq \mathbb{P}_Z^{x_0,U_K^*}\left\{\left(\frac{\mathbf{1}^\top Z}{K} - \frac{\mathbb{E}_Z^{x_0,U_K^*}[\mathbf{1}^\top Z]}{K}\right) \geq \delta\right\} \leq e^{-2K\delta^2}. \end{aligned}$$

To obtain the desired probabilistic guarantee $\mathbb{P}_Z^{x_0,U_K^*}\{p_K^*(x_0) - p^*(x_0) \geq \delta\} \leq \beta$, we require $e^{-2K\delta^2} \leq \beta$. We obtain (13) by solving for K. ∎

Theorem 1 addresses Question 1. Specifically, let K satisfy the lower bound prescribed by Theorem 1 for a user-specified violation parameter δ and risk of failure β. Then, $1 - \mathbb{P}_Z^{x_0,U_K^*}\{p_K^*(x_0) - p^*(x_0) < \delta\} \leq \beta$, or equivalently,

$$\mathbb{P}_Z^{x_0,U_K^*}\{p_K^*(x_0) - \delta < p^*(x_0)\} \geq 1 - \beta. \tag{18}$$

In other words, with a confidence of $(1-\beta)$, we have an underapproximation of the optimal solution to Problem 1. While $p^*(x_0)$ and $p_K^*(x_0)$ are functions of the time horizon N, the lower bound on K is independent of the choice of N. A similar bound is used for the application of aircraft conflict detection in [25].

IV. Partition-based sample reduction

As implied from the concentration probability bounds given in Theorem 1, Problem 2 typically needs a large number of samples to provide a precise approximation for Problem 1 with a small deviation δ and small risk of failure β. Therefore, solving Problem 2 can be computationally expensive or even intractable for real-time applications.

In this section, we address the first part of Question 2 by proposing a *Voronoi partition-based* underapproximation to Problem 2. The key advantage of this formulation is its flexible computational complexity. Specifically, the proposed underapproximation has $\hat{K}$ binary variables, where $\hat{K}$ can be significantly smaller than K and is selected by the user. By (18), an underapproximation of the optimal solution of Problem 2 can be used to underapproximate the optimal open-loop terminal time probability (Problem 1).

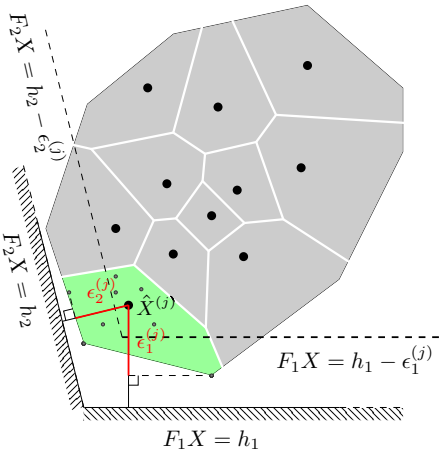

Fig. 3. Voronoi partition of the sampled state trajectories $X^{(\cdot)}$ using the representative state trajectories $\hat{X}^{(\cdot)}$ as in Figure 2. A Voronoi cell $\{G_x x_0 + G_u U\} \oplus \mathcal{V}^{(j)}$ is shown in green for some $j \in \mathbb{N}_{[1,\hat{K}]}$. The satisfaction of the constraint $F\hat{X}^{(j)} \preceq h - \varepsilon^{(j)}$ implies that $X^{(i)} \in \{G_x x_0 + G_u U\} \oplus \mathcal{V}^{(j)}$ will satisfy the constraint $FX \preceq h$.

A. Voronoi partition-based underapproximation

Recall that for a finite set $\mathcal{Q}$ and $\hat{K} < K$, we can compute $\hat{K}$ Voronoi centers by minimizing $\mathrm{WSS}(\cdot)$ (10). Consider the finite set,

$$\mathcal{Q} = G_w \mathscr{W}_K = \{G_w W^{(1)}, \dots, G_w W^{(K)}\} \subset \mathbb{R}^{n_x N}, \quad (19)$$

and let $\mathcal{C} = \{c^{(1)}, \dots, c^{(\hat{K})}\} \subset \mathbb{R}^{n_x N}$ be the $\hat{K}$ representative points (Voronoi centers).

Problem 3. *The Voronoi partition-based underapproximation of Problem 2 is*

$$\begin{aligned} \underset{\substack{U \in \mathcal{U}^N \\ \hat{z}^{(1)}, \dots, \hat{z}^{(\hat{K})}}}{\text{maximize}} \quad & \tfrac{1}{K} \textstyle\sum_{j=1}^{\hat{K}} |\mathcal{V}^{(j)}| \hat{z}^{(j)} \\ \text{subject to} \quad & \hat{X}^{(j)} = G_x x_0 + G_u U + c^{(j)}, && j \in \mathbb{N}_{[1,\hat{K}]} \\ & F\hat{X}^{(j)} \preceq h - \varepsilon^{(j)} + M(1 - \hat{z}^{(j)})\mathbf{1}, && j \in \mathbb{N}_{[1,\hat{K}]} \\ & \hat{z}^{(j)} \in \{0,1\}, && j \in \mathbb{N}_{[1,\hat{K}]} \end{aligned}$$

with the optimal value denoted by $p^*_{\hat{K}}(x_0)$. Here, $M \in \mathbb{R}$ is some large positive number, and $\varepsilon^{(j)} = \left[\epsilon_1^{(j)}, \cdots, \epsilon_L^{(j)}\right]^\top \in \mathbb{R}^L$, $j \in \mathbb{N}_{[1,\hat{K}]}$ are buffers, with

$$\epsilon_\ell^{(j)} \triangleq \max_{q \in \mathcal{V}^{(j)}} \left(F_\ell q - F_\ell c^{(j)}\right), \quad \ell \in \mathbb{N}_{[1,L]} \quad (20)$$

where $F_\ell \in \mathbb{R}^{1 \times n_x N}$ is the ℓ^{th} row of the matrix F.

Figure 3 shows how the buffer $\varepsilon^{(j)}$ tightens the safety constraints for the state trajectories that $\hat{X}^{(j)}$ correspond to the representative disturbance samples $c^{(j)}$. The buffers $\varepsilon^{(j)}$ are defined such that the "furthest" sampled trajectory $X^{(i)}$ in the Voronoi cell $\{G_x x_0 + G_u U\} \oplus \mathcal{V}^{(j)}$ associated with the representative trajectory $\hat{X}^{(j)}$ will satisfy the safety constraint $FX^{(i)} \preceq h$, if $F\hat{X}^{(j)} + \varepsilon^{(j)} \preceq h$. Since Voronoi partitions are translation-invariant, we can compute $\varepsilon^{(j)}$ using $\mathcal{V}^{(j)}$ as in (20), independent of x_0 and U.

Theorem 2. *For every initial state x_0, the optimal value of Problem 3 is a lower bound for the optimal value of Problem 2,*

$$p^*_{\hat{K}}(x_0) \le p^*_K(x_0). \quad (21)$$

Proof: Let $(U, \hat{z}^{(1)}, \dots, \hat{z}^{(\hat{K})})$ be a feasible solution of Problem 3. Next, we construct a solution $(U, z^{(1)}, \dots, z^{(K)})$ for Problem 2 using $(U, \hat{z}^{(1)}, \dots, \hat{z}^{(\hat{K})})$. For every $\hat{z}^{(j)} = 1$, we require the corresponding $z^{(i)} = 1$ where $i \in \mathbb{N}_{[1,K]}$ and $W^{(i)} \in \mathcal{V}^{(j)}$. Consequently,

$$\frac{1}{K} \sum_{i=1}^{K} z^{(i)} \ge \frac{1}{K} \sum_{j=1}^{\hat{K}} |\mathcal{V}^{(j)}| \hat{z}^{(j)} \quad (22)$$

since the unspecified $z^{(i)}$ are non-negative $z^{(i)} \in \{0,1\}$. If this construction satisfies the constraints of Problem 2,

$$\begin{aligned} X^{(i)} &= G_x x_0 + G_u U + G_w W^{(i)}, && i \in \mathbb{N}_{[1,K]} && (23) \\ FX^{(i)} &\preceq h + M(1 - z^{(i)})\mathbf{1}, && i \in \mathbb{N}_{[1,K]} && (24) \\ z^{(i)} &\in \{0,1\}, && i \in \mathbb{N}_{[1,K]} && (25) \end{aligned}$$

then Problem 3 is a restriction of Problem 2. This implies (21), by construction.

Now, we show that the constructed solution $(U, z^{(1)}, \dots, z^{(K)})$ satisfies (23)–(25). Clearly (25) is satisfied. Substituting (23) in (24), we have to show that for any $i \in \mathbb{N}_{[1,K]}$,

$$F(G_x x_0 + G_u U + G_w W^{(i)}) \preceq h + M(1 - z^{(i)})\mathbf{1}. \quad (26)$$

Note that $z^{(i)} = 0$ trivially satisfies (26). Therefore, we have to only show that setting $z^{(i)} = 1$ when $\hat{z}^{(j)} = 1$, does not violate (26).

Consider some $i \in \mathbb{N}_{[1,K]}$. Let the corresponding disturbance realization $W^{(i)} \in \mathcal{V}^{(j)}$ for some $j \in \mathbb{N}_{[1,\hat{K}]}$. Adding and subtracting $FG_w c^{(j)}$ on the LHS of (26),

$$\begin{aligned} & F(G_x x_0 + G_u U + G_w W^{(i)}) \\ & \quad = F(G_x x_0 + G_u U + c^{(j)}) + FG_w(W^{(i)} - c^{(j)}) \\ & \quad \preceq F(G_x x_0 + G_u U + c^{(j)}) + \max_{W \in \mathcal{V}^{(j)}} FG_w(W - c^{(j)}) \\ & \quad = F\hat{X}^{(j)} + \varepsilon^{(j)}. \end{aligned} \quad (27)$$

Here, (27) follows from the definitions and constraints of Problem 3, and the fact that U is a feasible solution for Problem 3. Further,

$$F\hat{X}^{(j)} + \varepsilon^{(j)} \preceq h + M(1 - \hat{z}^{(j)})\mathbf{1}. \quad (28)$$

For $\hat{z}^{(j)} = 1$, the representative state trajectory is safe, $F\hat{X}^{(j)} + \varepsilon^{(j)} \preceq h$. Further, every associated sampled state trajectory is safe, $FX^{(i)} + \varepsilon^{(j)} \preceq h$ for every $i \in \mathbb{N}_{[1,K]}$ with $W^{(i)} \in \mathcal{V}^{(j)}$ by (27). This permits the associated binary variable $z^{(i)}$ to be set to one, while satisfying (26). ■

By (20), the buffers become insignificant when $\mathrm{WSS}(\cdot)$ is small. By (11), $\mathrm{WSS}(\hat{K}, \mathscr{V}_{\mathcal{Q}}(\mathcal{C}^*))$ is non-increasing in $\hat{K}$. Therefore, we can reduce the conservativeness of Problem 3 when compared to Problem 2 by increasing $\hat{K}$. Specifically, for $\hat{K} = K$, Problems 2 and 3 become equivalent. However,

increasing $\hat{K}$ increases the computational cost of solving Problem 3. Thus, we have a trade-off between accuracy and computational complexity in terms of $\hat{K}$.

B. Tightening the underapproximation probability estimate

Let $U^*_{\hat{K}}$ be the optimal open-loop controller for Problem 3. A tighter underapproximation on p^*_K can be recalculated by simply computing the fraction of the original K sampled state trajectories that remain in reach-avoid set $\mathcal{R}$, after applying $U^*_{\hat{K}}$ to the stochastic system (1). In contrast to solving a large MILP (as done in Problem 2) whose computational complexity grows exponentially with K, this improved estimate (see (29)) is obtained by a policy evaluation that has a computational complexity of $\mathcal{O}(K)$. Theorem 3 solves the second part of Question 2.

Theorem 3. *Let $U^*_{\hat{K}}$ denote the optimal solution of Problem 3. Define $\hat{p}(x_0)$ as*

$$\hat{p}(x_0) = \frac{1}{K}\sum_{i=1}^{K} 1_{\mathcal{R}}\left(G_x x_0 + G_u U^*_{\hat{K}} + G_w W^{(i)}\right). \quad (29)$$

Then

$$p^*_{\hat{K}}(x_0) \le \hat{p}(x_0) \le p^*_K(x_0). \quad (30)$$

Proof: Recall that $p^*_K(x_0)$ and $p^*_{\hat{K}}(x_0)$ are the optimal values of Problems 2 and 3.

Proof for $\hat{p}(x_0) \le p^*_K(x_0)$: Follows from the observation that $\hat{p}(x_0)$ is the evaluation of an open-loop controller $U^*_{\hat{K}}$ over K samples, and $p^*_K(x_0)$ is the optimal open-loop terminal time probability with K samples.

Proof for $p^*_{\hat{K}}(x_0) \le \hat{p}(x_0)$: In the proof of Theorem 2, we arrived at (22) by restricting $z^{(i)} = 1$ whenever $\hat{z}^{(j)} = 1$ for the appropriate $j \in \mathbb{N}_{[1,\hat{K}]}$. In (29), we set $z^{(i)} = 1$ whenever $1_{\mathcal{R}}\left(G_x x_0 + G_u U^*_{\hat{K}} + G_w W^{(i)}\right) = 1$. ■

Algorithm 1 summarizes the proposed Voronoi partition-based method to solve the open-loop terminal time problem for linear system (1) and fixed initial state $x_0 \in \mathcal{S}$. It provides a probabilistically guaranteed underapproximation of the open-loop terminal time problem (Problem 1) by (18) and Theorem 3. The computation of the Voronoi centers $\mathcal{C}$ for the set $\mathcal{Q}$ (19) and the buffers $\varepsilon^{(\cdot)}$ are independent of the initial state x_0 and the control vector U. Therefore, these computations can be done offline, allowing for quick evaluation of $\hat{p}(x_0)$ for multiple x_0.

C. Discussion

1) Computation complexity: The computational complexity of the offline phase of Algorithm 1 (steps 3 and 4) is $\mathcal{O}\left(\gamma n_x N K \hat{K} + \sum_{j=1}^{\hat{K}} L|V^{(j)}|\log(|V^{(j)}|)\right)$. This follows from the computational complexity of k-means algorithm [23], and the computational complexity of 1D sorting. The computation of $\epsilon^{(j)}_\ell$ using (20) for $\ell = 1, \cdots, L$ and $j \in \mathbb{N}_{[1,\hat{K}]}$ is a sorting problem in 1D. For the online phase of Algorithm 1, the computational complexity of step 1 is known to have an exponential dependence on $\hat{K}$ [18, Rem. 1], and the computational complexity of step 2 is $\mathcal{O}(K)$ (Section IV-B).

Algorithm 1 Voronoi partition-based underapproximation

Input: Linear system (1), reach-avoid set $\mathcal{R}$, initial state x_0, number of representative state trajectories $\hat{K}$, violation parameter δ, and risk of failure β.

Offline (independent of x_0):

1) Compute the minimum number of samples K required given δ and β using Theorem 1.
2) Generate $\mathscr{W}_K$ by taking K i.i.d. samples from $\mathbb{P}_W$.
3) Synthesize the corresponding Voronoi centers $\mathcal{C}$ and Voronoi partition $\mathscr{V}_{\mathcal{Q}}(\mathcal{C})$ with $\hat{K}$ elements for the finite set $\mathcal{Q} = G_w\mathscr{W}_K$ (see Section II-D).
4) Compute $\varepsilon^{(j)}$ for $j \in \mathbb{N}_{[1,\hat{K}]}$ using (20).

Online (depends on x_0):

1) Solve Problem 3 for $U^*_{\hat{K}}$ and $p^*_{\hat{K}}(x_0)$.
2) Compute $\hat{p}(x_0)$ from (29).

Output: $\max(\hat{p}(x_0) - \delta, 0)$, an underapproximation of optimal open-loop terminal time probability by (18).

2) What is a good choice for $\hat{K}$?: The user can use $\mathrm{WSS}(\cdot)$ to guide his/her selection of $\hat{K}$, as is typically done in clustering approaches [21]. Ideally, we wish the buffers $\varepsilon^{(\cdot)}$ to be small. Since $\mathrm{WSS}(\cdot)$ quantifies the variation of points in a cluster, a smaller $\mathrm{WSS}(\cdot)$ implies more compact clusters. Clearly, increasing $\hat{K}$ will reduce $\mathrm{WSS}(\cdot)$, since $\mathrm{WSS}(\cdot)$ is non-increasing in $\hat{K}$. However, this will also increase the computational complexity of Problem 3, since the number of binary variables ($\hat{K}$) increases. We can compute $\mathrm{WSS}(\cdot)$ as a function of $\hat{K}$, to explore this trade-off. The "knee" of the curve provides an efficient compromise between precision and computational complexity.

3) Extensions: For a stochastic initial state $x_0 \in \mathcal{S}$, Algorithm 1 can be extended by considering samples drawn for the initial state as well as the concatenated disturbance. Defining Y as a $(n_x + n_x N)$-dimensional random vector, the corresponding concatenated state trajectory is,

$$X = G_u U + [G_x\ G_w]\underbrace{\begin{bmatrix} x_0 \\ W \end{bmatrix}}_{Y}. \quad (31)$$

We used Voronoi partition as a convenient method to undersample Problem 2, and construct representative state trajectories. In reality, we can utilize any heuristic to construct the partition and the representative state trajectories, and generate a MILP similar to Problem 3 with $\hat{K}$ binary variables. A distinct advantage of Voronoi partitioning is that it ensures $\varepsilon^{(j)}$ is reasonably small, since it minimizes $\mathrm{WSS}(\cdot)$. This ensures that the feasible solution space of Problem 3 is not severely restricted as compared to that of Problem 2.

V. Illustrative Example: Spacecraft Rendezvous

We consider the spacecraft rendezvous example discussed in [4]. In this example, two spacecraft are in the same

elliptical orbit. One spacecraft, referred to as the deputy, must approach and dock with another spacecraft, referred to as the chief, while remaining in a line-of-sight cone, in which accurate sensing of the other vehicle is possible. The relative dynamics are described by the Clohessy-Wiltshire-Hill (CWH) equations as given in [26],

$$\ddot{x} - 3\omega x - 2\omega\dot{y} = m_d^{-1}F_x, \qquad \ddot{y} + 2\omega\dot{x} = m_d^{-1}F_y. \quad (32)$$

The position of the deputy is denoted by $x, y \in \mathbb{R}$ when the chief, with the mass $m_d = 300$ kg, is located at the origin. For the gravitational constant μ and the orbital radius of the spacecraft R_0, $\omega = \sqrt{\mu/R_0^3}$ represents the orbital frequency. In this example, the spacecraft is in a circular orbit at an altitude of 850 km above the earth.

We define $\zeta = [x, y, \dot{x}, \dot{y}] \in \mathbb{R}^4$ as the system state and $u = [F_x, F_y] \in \mathcal{U} \subseteq \mathbb{R}^2$ as the system input, then discretize the dynamics (32) with a sampling time of 20 s to obtain the discrete-time LTI system,

$$\zeta_{t+1} = A\zeta_t + Bu_t + w_t. \quad (33)$$

The additive stochastic noise, modeled by the Gaussian i.i.d. disturbance $w_t \in \mathbb{R}^4$, with $\mathbb{E}[w_t] = 0$, and $\mathbb{E}[w_t w_t^\top] = 10^{-4} \times \text{diag}(1, 1, 5 \times 10^{-4}, 5 \times 10^{-4})$, accounts for disturbances and model uncertainty.

We define the target set and the safe set as in [4],

$$\mathcal{T} = \{\zeta \in \mathbb{R}^4 : |\zeta_1| \le 0.1, -0.1 \le \zeta_2 \le 0, \\ |\zeta_3| \le 0.01, |\zeta_4| \le 0.01\}, \quad (34)$$

$$\mathcal{S} = \{\zeta \in \mathbb{R}^4 : |\zeta_1| \le \zeta_2, -1 \le \zeta_2 \\ |\zeta_3| \le 0.05, |\zeta_4| \le 0.05\}, \quad (35)$$

with a horizon of $N = 5$. We consider the initial position $x = y = -0.75$ km, the initial velocity $\dot{x} = \dot{y} = 0$ km/s and $\mathcal{U} = [-0.1, 0.1] \times [-0.1, 0.1]$.

All computations were carried out on a 2.8 GHz processor Intel Core i5 with 16 GB RAM. We used SReachTools [27], a MATLAB stochastic reachability toolbox, to implement Algorithm 1, and compare the results with existing methods [4], [10]. SReachTools uses CVX [28] for formulating optimization problems, and MPT3 [29] for computational geometry. We used Gurobi [30] as the backend MILP solver. The open-loop terminal time probability for this problem using Fourier transform method [10] is known to be 0.85, which we assume to be the ground truth. This approach maximizes $r^U_{x_0}(\mathcal{S}, \mathcal{T})$ directly via nonlinear optimization and numerical quadrature.

Figure 4 studies the influence of $\hat{K}$ on WSS$(\cdot)$, $\hat{p}(x_0)$, and the compute time. We choose the violation parameter $\delta = 0.05$ and risk of failure $\beta = 10^{-8}$. By Theorem 1, the minimum number of original samples required is $K = 3685$. We vary $\hat{K} \in \mathbb{N}_{[1,100]}$, and report the mean and the maximum variations of the results of the 50 experiments. Figure 4a shows the WSS$(\cdot)$ curve with $\hat{K} \in \mathbb{N}_{[1,100]}$. Figure 4b compares the open-loop terminal time probability underapproximation $\hat{p}(x_0) - \delta$ (from Algorithm 1) and $p^*_{\hat{K}}(x_0) - \delta$ (from Problem 3). By Theorem 3 and (18), both of these

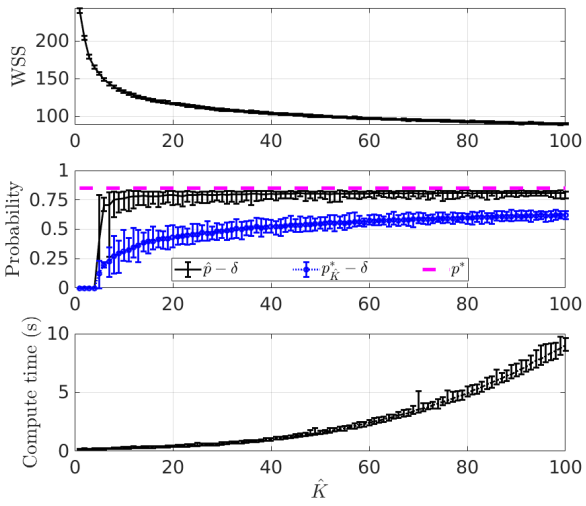

Fig. 4. The mean and maximum deviations of (a) within-cluster sum of squares (used to select $\hat{K}$, offline) (b) underapproximations $\hat{p}(x_0) - \delta$ (Algorithm 1) and $p^*_{\hat{K}}(x_0) - \delta$ (Problem 3) compared to the Fourier transform estimate [10] ($p^*(x_0) = 0.85$), and (c) total run time, with varying $\hat{K}$. The data was obtained from 50 experiments.

underapproximations are strictly below the optimal open-loop terminal time probability $p^*(x_0)$ with a confidence level of $(1 - 10^{-8})$. The underapproximation $\hat{p}(x_0) - \delta$ is a tighter underapproximation, as expected. The total run time (for a fixed $\hat{K}$) of Algorithm 1 is shown in Figure 4c. The run time increases exponentially with $\hat{K}$, since Problem 3 is a MILP.

Table I show that Algorithm 1 significantly outperforms Fourier transform approach [10] in computation time for lower $\hat{K}$, enabling applications to real-time systems. As desired, Algorithm 1 provides a flexible trade-off between the accuracy and computation time via a suitable choice of $\hat{K}$, without compromising on the underapproximation guarantee. The original particle filter (Problem 2) fails for K prescribed by Theorem 1, and it returns a poor approximation (higher than the ground truth) for lower K.

From Figure 4b and Table I, we see that the improvements in the accuracy of Algorithm 1 becomes gradual beyond $\hat{K} = 10$, which is referred to as the "knee" of the trade-off curve. Note that the knees of Figure 4a and 4b coincide. As discussed in Section IV-C.2, the knee of Figure 4a provides a reasonable choice for $\hat{K}$ during the offline phase of Algorithm 1. The computation of Figure 4a took about 10 seconds, and has a computational complexity of $\mathcal{O}(\gamma n_x N K^2)$ (see Section II-D). For a fixed $\hat{K}$, the offline phase took about 0.01 – 0.24 seconds. For $\hat{K} > 10$, we obtain a tighter underapproximation at a higher computational cost.

Figure 5 shows the position trajectory, associated with ζ_1 and ζ_2, obtained by Algorithm 1 and the Fourier method [10], along with the resulting sampled trajectories $X^{(\cdot)}$. We used $\hat{K} = 40$.

VI. Conclusion

In this paper we presented a novel partition-based method for under-approximating the terminal time probability through sample reduction. By using Hoeffding's in-

TABLE I

OPEN-LOOP TERMINAL REACH-AVOID PROBABILITY ESTIMATES AND COMPUTE TIMES (50 EXPERIMENTS, $\delta = 0.05$, AND $\beta = 10^{-8}$ WITH FOURIER TRANSFORM METHOD AS THE GROUND TRUTH)

Method	$r^U_{x_0}(\mathcal{S}, \mathcal{T})$		Time (s)	
	Mean	Std.	Mean	Std.
Algorithm 1 ($K = 3685$)				
$\hat{K} = 7$	0.71	0.05	0.21	0.03
$\hat{K} = 10$	0.77	0.04	0.25	0.03
$\hat{K} = 15$	0.78	0.03	0.32	0.03
$\hat{K} = 20$	0.79	0.02	0.40	0.04
$\hat{K} = 40$	0.80	0.01	1.02	0.06
$\hat{K} = 100$	0.81	0.01	9.17	0.30
Particle filter (Prob. 2) [4], [13]				
$K = 100$	0.89	0.03	7.63	0.30
$K = 3685$	-	-	-	-
Fourier transform [10]	0.85		11.2	

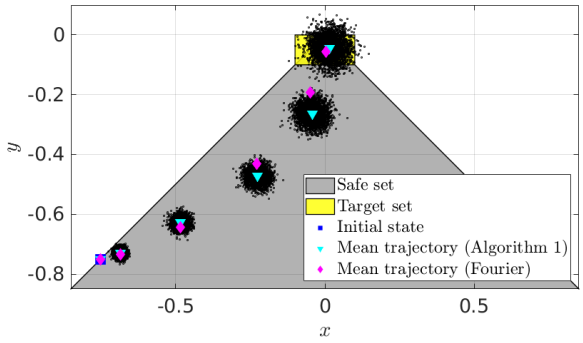

Fig. 5. Position trajectory obtained using the optimal open-loop controllers from Algorithm 1 ($\hat{K} = 40$) and Fourier transform method [10]. The scatter plot shows the sampled trajectories $X^{(i)}$, $i \in \mathbb{N}_{[1,3685]}$ (positions).

equality, we provided a bound on the required number of scenarios to achieve a desired probabilistic bound on the approximation error. Furthermore, we proposed a method which clusters the taken scenarios in few cells, each cell represented by a center, where the number of cells is selected by the user in a systematic manner using the trend of a given curve or based on the desired running time. The proposed method scales easily with dimension since the clustering computational complexity increases linearly with the dimension of data. Numerical experiments demonstrate the computational efficiency of the approach, and its potential for real-time applications.

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