Scalable Underapproximation for the Stochastic Reach-Avoid Problem for High-Dimensional LTI Systems Using Fourier Transform

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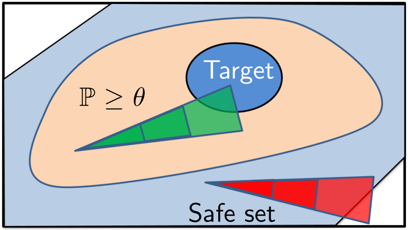
Motivation

- Need for probabilistic guarantees of safety and performance:
  - Motion planning in stochastic environments
  - Human-automation collaboration systems
- Stochastic reach-avoid problem — viability and reachability
- High computation costs and lacks scalability

Motivation



- Need for probabilistic guarantees of safety and performance:
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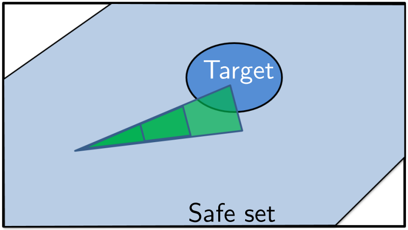


- Need for probabilistic guarantees of safety and performance:
  - Motion planning in stochastic environments
  - Human-automation collaboration systems
- Stochastic reach-avoid problem — viability and reachability
- High computation costs and lacks scalability

Objective: Scalable underapproximation of $\max \mathbb{P}$ from an initial state.

Related work

Backward stochastic reachability
Abate, Amin, Prandini, Lygeros, & Sastry (2007); Abate, Prandini, Lygeros, & Sastry (2008); Summers, & Lygeros (2010)

Approximation techniques
Lesser, Oishi, & Erwin (2013); Manganini, Pirotta, Restelli, Piroddi, & Prandini (2015); Kariotoglou, Kamgarpour, Summers, & Lygeros (2017); Gleason, Vinod, & Oishi (2017)

Forward stochastic reachability
Lasota & Mackey (1985); Althoff, Stursberg, & Buss (2009); Vinod, HomChaudhuri, & Oishi (2017); HomChaudhuri, Vinod, & Oishi (2017)

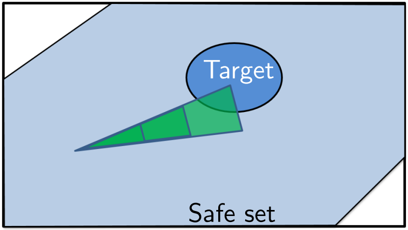
Uncertain system reachability (discrete time)
Bertsekas and Rhodes (1971); Girard (2005); Kurzhanskiy & Varaiya (2006); Kvasnica, Takács, Holaza, & Ingole (2015); Althoff (2015); Bak & Duggirala (2017)

Uncertain system reachability (continuous time)
Tomlin, Mitchell, Bayen, & Oishi (2003); Bokanowski, Forcadel, & Zidani (2010); Huang, Ding, Zhang, & Tomlin (2015); Chen, Herbert, & Tomlin (2017)

Main contributions

Focus: Terminal stochastic reach-avoid problem

- Fourier transform-based underapproximation
  - Grid-free and recursion-free
  - Solution for a known initial condition
  - Open-loop policies
- Transformation to a convex optimization



Outline

1. Introduction
2. Problem statements
3. Fourier transform-based underapproximation
4. Examples
5. Conclusion

System formulation

- Discrete-time LTI system (time horizon N and initial state $\overline{x}_0 \in \mathcal{X}$)

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\overline{u}_k + \boldsymbol{w}_k \qquad \begin{array}{l} \boldsymbol{x}_k \in \mathcal{X} \subseteq \mathbb{R}^n, \overline{u}_k \in \mathcal{U} \subseteq \mathbb{R}^m \\ \boldsymbol{w}_k \in \mathcal{W} \subseteq \mathbb{R}^p, \boldsymbol{w}_k \sim \psi_{\boldsymbol{w}} \end{array}$$

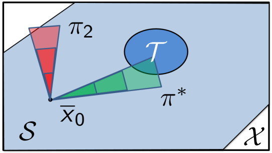
- ($\equiv$) Continuous state Markov decision process (when IID noise)

$$Q(d\overline{y}|\overline{x}, \overline{u}) = \psi_{\boldsymbol{w}}(\overline{y} - A\overline{x} - B\overline{u})d\overline{y}$$

- Markov policy $\pi : \mathbb{N}_{[0,N-1]} \times \mathcal{X} \to \mathcal{U}$, $\pi \in \mathcal{M}$
- $\boldsymbol{X} = [\boldsymbol{x}_1^\top \ \boldsymbol{x}_2^\top \ \ldots \ \boldsymbol{x}_{N-1}^\top]^\top$ random vector (rv) in $(\mathcal{X}^N, \mathscr{B}(\mathcal{X}^N), \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0})$

Summers and Lygeros, Automatica 2010
Abate, Prandini, Lygeros, and Sastry, Automatica 2008

Stochastic reach-avoid problem and its solution



- Borel $\mathcal{S}, \mathcal{T} \subseteq \mathcal{X}$
- Terminal stochastic reach-avoid problem

$$\hat{V}_0^*(\overline{x}_0) = \underset{\pi}{\max.} \ \mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\left\{\boldsymbol{X} \in \left(\mathcal{S}^{N-1} \times \mathcal{T}\right)\right\}$$
$$\text{s. t.} \quad \pi \in \mathcal{M}$$

  - Solution via dynamic programming

$$\hat{V}_N^*(\overline{x}) = 1_{\mathcal{T}}(\overline{x})$$
$$\hat{V}_k^*(\overline{x}) = \sup_{\overline{u} \in \mathcal{U}} 1_{\mathcal{S}}(\overline{x}) \int_{\mathcal{X}} \hat{V}_{k+1}^*(\overline{y}) Q(d\overline{y}|\overline{x}, \overline{u})$$

$\overline{x}_0 \in \mathcal{S}$
$\boldsymbol{x}_1 \in \mathcal{S}$ $\boldsymbol{x}_2 \in \mathcal{S}$ $\vdots$ $\boldsymbol{x}_{N-1} \in \mathcal{S}$ $\boldsymbol{x}_N \in \mathcal{T}$
$\boldsymbol{X} \in \mathcal{S}^{N-1} \times \mathcal{T}$

- Discretization approach
  - For compact $\mathcal{U}, \mathcal{S}, \mathcal{T}$ and Lipschitz Q
  - Curse of dimensionality — $n \leq 3$

Abate, Amin, Prandini, Lygeros, and Sastry, HSCC 2007
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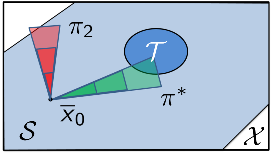
Stochastic reach-avoid problem and its solution



- Borel $\mathcal{S}, \mathcal{T} \subseteq \mathcal{X}$
- Terminal stochastic reach-avoid problem

$$\begin{aligned} \hat{V}_0^*(\overline{x}_0) = \underset{\pi}{\max.} \quad & \mathbb{E}_{x_0}^{\pi}\left[\left(\prod_{k=0}^{N-1} 1_{\mathcal{S}}(\boldsymbol{x}_k)\right) 1_{\mathcal{T}}(\boldsymbol{x}_N)\right] \\ \text{s. t.} \quad & \pi \in \mathcal{M} \end{aligned}$$

  - Solution via dynamic programming

$$\begin{aligned} \hat{V}_N^*(\overline{x}) &= 1_{\mathcal{T}}(\overline{x}) \\ \hat{V}_k^*(\overline{x}) &= \sup_{\overline{u} \in \mathcal{U}} 1_{\mathcal{S}}(\overline{x}) \int_{\mathcal{X}} \hat{V}_{k+1}^*(\overline{y}) Q(d\overline{y} | \overline{x}, \overline{u}) \end{aligned}$$

- Discretization approach
  - For compact $\mathcal{U}, \mathcal{S}, \mathcal{T}$ and Lipschitz Q
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$$\begin{array}{|c|} \hline \overline{x}_0 \in \mathcal{S} \\ \hline \boldsymbol{x}_1 \in \mathcal{S} \\ \boldsymbol{x}_2 \in \mathcal{S} \\ \vdots \\ \boldsymbol{x}_{N-1} \in \mathcal{S} \\ \boldsymbol{x}_N \in \mathcal{T} \\ \hline \boldsymbol{X} \in \mathcal{S}^{N-1} \times \mathcal{T} \\ \hline \end{array}$$

Abate, Amin, Prandini, Lygeros, and Sastry, HSCC 2007
Summers and Lygeros, Automatica 2010

Stochastic reach-avoid problem and its solution

- Borel $\mathcal{S}, \mathcal{T} \subseteq \mathcal{X}$
- Terminal stochastic reach-avoid problem

$$\hat{V}_0^\ast(\overline{x}_0) = \underset{\pi}{\max.} \ \ \mathbb{E}_{x_0}^{\pi}\left[\left(\prod_{k=0}^{N-1} 1_{\mathcal{S}}(\boldsymbol{x}_k)\right) 1_{\mathcal{T}}(\boldsymbol{x}_N)\right]$$

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- Discretization approach
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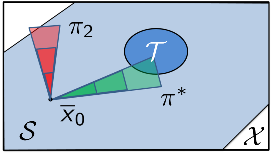
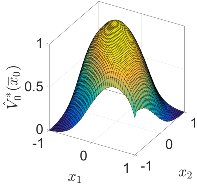
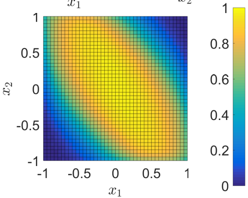






Abate, Amin, Prandini, Lygeros, and Sastry, HSCC 2007
Summers and Lygeros, Automatica 2010

Approximation via open-loop policies

- Open-loop policy $\rho : \mathcal{X} \to \mathcal{U}^N$ ($\rho(\overline{x}_0) \notin \mathcal{M}$)

$$\boldsymbol{W} = [\boldsymbol{w}_0^\top \ \boldsymbol{w}_1^\top \ \ldots \ \boldsymbol{w}_{N-1}^\top]^\top$$

$$\boldsymbol{X} = \overline{A}\overline{x}_0 + \overline{H}\ \overline{U} + \overline{G}\boldsymbol{W} \Longrightarrow \boldsymbol{X} \text{ rv in } (\mathcal{X}^N, \mathscr{B}(\mathcal{X}^N), \mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0})$$

Stochastic reach-avoid problem	Approximation
max. $\mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\}$ s.t. $\pi \in \mathcal{M}$	max. $\mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\}$ s.t. $\rho(\overline{x}_0) \in \mathcal{U}^N$
Optimal value function $\hat{V}_0^\ast(\overline{x}_0)$	Optimal value function $\hat{W}_0^\ast(\overline{x}_0)$
Search over Markov policies	Search over open-loop policies

Lesser, Oishi, and Erwin, CDC 2013

Approximation via open-loop policies

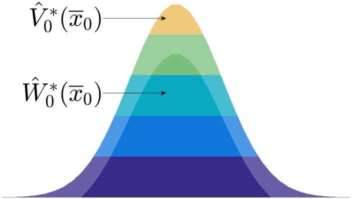
- Open-loop policy $\rho : \mathcal{X} \to \mathcal{U}^N$ $(\rho(\overline{x}_0) \notin \mathcal{M})$

$$\boldsymbol{W} = [\boldsymbol{w}_0^\top \ \boldsymbol{w}_1^\top \ \ldots \ \boldsymbol{w}_{N-1}^\top]^\top$$

$$\boldsymbol{X} = \overline{A}\overline{x}_0 + \overline{H}\ \overline{U} + \overline{G}\boldsymbol{W} \implies \boldsymbol{X} \text{ rv in } (\mathcal{X}^N, \mathscr{B}(\mathcal{X}^N), \mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0})$$

Stochastic reach-avoid problem	Approximation
max. $\mathbb{P}_{\boldsymbol{X}}^{\pi,\overline{x}_0}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\}$ s.t. $\pi \in \mathcal{M}$	max. $\mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\}$ s.t. $\rho(\overline{x}_0) \in \mathcal{U}^N$
Optimal value function $\hat{V}_0^\ast(\overline{x}_0)$	Optimal value function $\hat{W}_0^\ast(\overline{x}_0)$
Search over Markov policies	Search over open-loop policies

- Does $\hat{W}_0^\ast(\overline{x}_0)$ underapproximate $\hat{V}_0^\ast(\overline{x}_0)$?
  - Conservative in the correct direction
  - Useful information for verification
  - Not trivial since $\rho(\overline{x}_0) \notin \mathcal{M}$



Problem statements

Stochastic reach-avoid problem	Underapproximation (?)
$\begin{array}{rl} \text{max.} & \mathbb{P}^{\pi,\overline{x}_0}_{\boldsymbol{X}}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\} \\ \text{s.t.} & \pi \in \mathcal{M} \end{array}$	$\begin{array}{rl} \text{max.} & \mathbb{P}^{\rho,\overline{x}_0}_{\boldsymbol{X}}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\} \\ \text{s.t.} & \rho(\overline{x}_0) \in \mathcal{U}^N \end{array}$
Optimal value function $\hat{V}_0^\ast(\overline{x}_0)$	Optimal value function $\hat{W}_0^\ast(\overline{x}_0)$
Search over Markov policies	Search over open-loop policies

Q1 Compute $\mathbb{P}^{\rho,\overline{x}_0}_{\boldsymbol{X}}$ using Fourier transform

Q2 Establish $\hat{W}_0^\ast(\overline{x}_0) \leq \hat{V}_0^\ast(\overline{x}_0)$ a.s.

- Q2a Sufficient conditions for Borel-measurability of $\hat{V}_k^\ast(\cdot)$

Q3 Convex formulation of the underapproximation for LTI system with Gaussian $\boldsymbol{w}_k$

Compute $\mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0}$ using Fourier transform

$$\begin{array}{rl}\text{max.} & \mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0}\left\{\boldsymbol{X}\in(\mathcal{S}^{N-1}\times\mathcal{T})\right\}\\ \text{s.t.} & \rho(\overline{x}_0)\in\mathcal{U}^N\end{array} \quad\Leftrightarrow\quad \boxed{\begin{array}{rl}\text{max.} & \int_{\mathcal{S}^{N-1}\times\mathcal{T}}\psi_{\boldsymbol{X}}(\overline{X};\overline{x}_0,\overline{U})d\overline{X}\\ \text{s.t.} & \overline{U}\in\mathcal{U}^N\end{array}}$$

$$\begin{tikzcd} & \psi_{\boldsymbol{W}} \arrow[rr, "\text{Fourier transform}"] & & \Psi_{\boldsymbol{W}} \arrow[d] \\ \psi_{\boldsymbol{X}}(\cdot;\rho,\overline{x}_0) & & & \Psi_{\boldsymbol{X}}(\cdot;\rho,\overline{x}_0) \arrow[lll, "\text{Inverse Fourier transform}"'] \end{tikzcd}$$

$$\boldsymbol{X}=\overline{A}\overline{x}_0+\overline{H}\ \overline{U}+\overline{G}\boldsymbol{W}$$

$$\Psi_{\boldsymbol{w}}(\overline{\alpha})=\mathbb{E}_{\boldsymbol{w}}\left[\exp\left(j\overline{\alpha}^\top\boldsymbol{w}\right)\right]=\int_{\mathbb{R}^p}e^{j\overline{\alpha}^\top\overline{z}}\psi_{\boldsymbol{w}}(\overline{z})d\overline{z}$$

$$\psi_{\boldsymbol{w}}(\overline{z})=\left(\frac{1}{2\pi}\right)^p\int_{\mathbb{R}^p}e^{-j\overline{\alpha}^\top\overline{z}}\Psi_{\boldsymbol{w}}(\overline{\alpha})d\overline{\alpha}$$

Vinod, HomChaudhuri, and Oishi, HSCC 2017

Compute $\mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0}$ using Fourier transform

$$\begin{array}{rl}\text{max.} & \mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\} \\ \text{s.t.} & \rho(\overline{x}_0) \in \mathcal{U}^N\end{array} \quad \Leftrightarrow \quad \boxed{\begin{array}{rl}\text{max.} & \int_{\mathcal{S}^{N-1}\times\mathcal{T}} \psi_{\boldsymbol{X}}(\overline{X}; \overline{x}_0, \overline{U})d\overline{X} \\ \text{s.t.} & \overline{U} \in \mathcal{U}^N\end{array}}$$

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$$\boldsymbol{X} = \overline{A}\overline{x}_0 + \overline{H}\ \overline{U} + \overline{G}\boldsymbol{W}$$

$$\boldsymbol{x}_\tau = A^\tau \overline{x}_0 + \mathscr{C}_U^\tau \rho(\overline{x}_0) + \mathscr{C}_W^\tau \boldsymbol{W}$$

$$\Psi_{\boldsymbol{v}}(\overline{\alpha}) = \exp(j\overline{q}^\top \alpha)\Psi_{\boldsymbol{w}}(P^\top \overline{\alpha}) \qquad \text{for } \boldsymbol{v} = P\boldsymbol{w} + \overline{q}$$

$$\Psi_{\boldsymbol{v}}(\overline{\alpha}) = \Psi_{\boldsymbol{w}_1}(\overline{\alpha}_1)\Psi_{\boldsymbol{w}_2}(\overline{\alpha}_2) \qquad \text{for } \boldsymbol{v} = \begin{bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{bmatrix}$$

Vinod, HomChaudhuri, and Oishi, HSCC 2017

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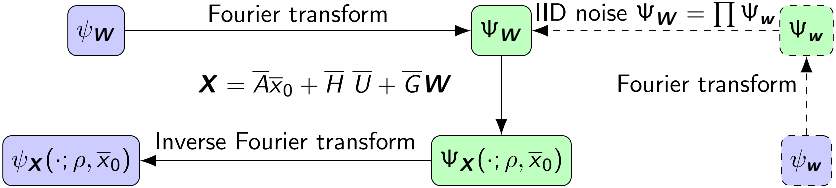

$$\boldsymbol{x}_\tau = A^\tau \overline{x}_0 + \mathscr{C}_U^\tau \rho(\overline{x}_0) + \mathscr{C}_W^\tau \boldsymbol{W}$$

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Vinod, HomChaudhuri, and Oishi, HSCC 2017

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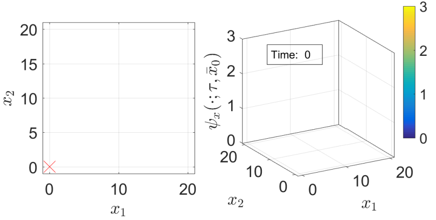
$$\begin{tikzcd}
\psi_{\boldsymbol{W}} \arrow[r, "\text{Fourier transform}"] & \Psi_{\boldsymbol{W}} \arrow[d, "\boldsymbol{X} = \overline{A}\overline{x}_0 + \overline{H}\,\overline{U} + \overline{G}\boldsymbol{W}"'] & \Psi_{\boldsymbol{w}} \arrow[l, dashed, "\text{IID noise } \Psi_{\boldsymbol{W}} = \prod \Psi_{\boldsymbol{w}}"'] \\
\psi_{\boldsymbol{X}}(\cdot; \rho, \overline{x}_0) & \Psi_{\boldsymbol{X}}(\cdot; \rho, \overline{x}_0) \arrow[l, "\text{Inverse Fourier transform}"'] & \psi_{\boldsymbol{w}} \arrow[u, dashed, "\text{Fourier transform}"']
\end{tikzcd}$$

- Grid-free and recursion-free



Vinod, HomChaudhuri, and Oishi, HSCC 2017

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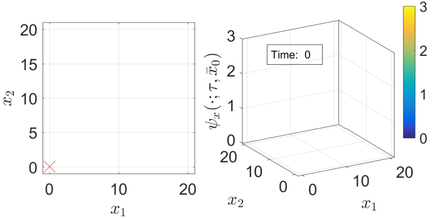
\end{tikzcd}$$

- Grid-free and recursion-free



Vinod, HomChaudhuri, and Oishi, HSCC 2017

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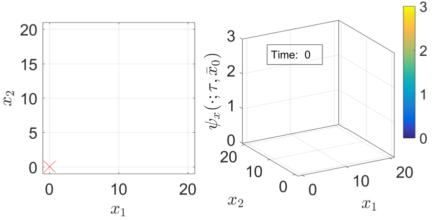
$\psi_{\boldsymbol{W}} \xrightarrow{\text{Fourier transform}} \Psi_{\boldsymbol{W}}$

$\psi_{\boldsymbol{w}} \xrightarrow{\text{Fourier transform}} \Psi_{\boldsymbol{w}} \xrightarrow{\text{IID noise } \Psi_{\boldsymbol{W}} = \prod \Psi_{\boldsymbol{w}}} \Psi_{\boldsymbol{W}}$

$\Psi_{\boldsymbol{W}} \xrightarrow{\boldsymbol{X} = \overline{A}\overline{x}_0 + \overline{H}\,\overline{U} + \overline{G}\boldsymbol{W}} \Psi_{\boldsymbol{X}}(\cdot;\rho,\overline{x}_0)$

$\Psi_{\boldsymbol{X}}(\cdot;\rho,\overline{x}_0) \xrightarrow{\text{Inverse Fourier transform}} \psi_{\boldsymbol{X}}(\cdot;\rho,\overline{x}_0)$

- Grid-free and recursion-free



Vinod, HomChaudhuri, and Oishi, HSCC 2017

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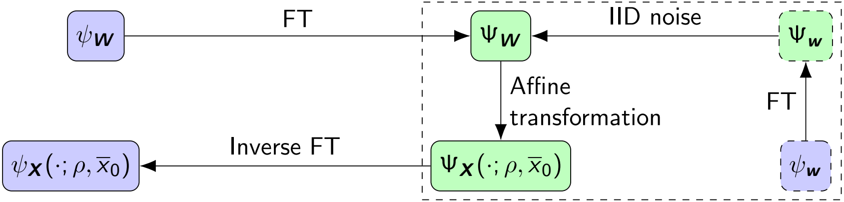
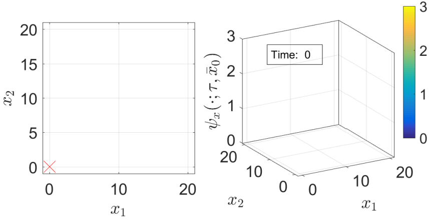


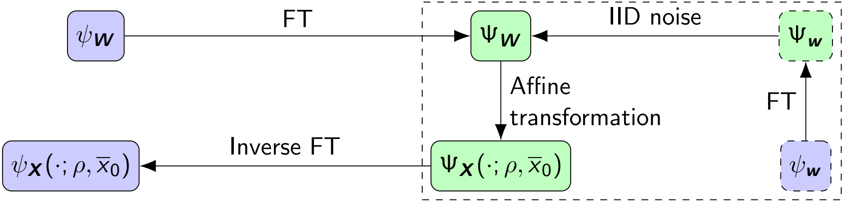
- Grid-free and recursion-free
- Closed-form expressions (in some cases)



Vinod, HomChaudhuri, and Oishi, HSCC 2017

Compute $\mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0}$ using Fourier transform

$$\begin{array}{rl} \text{max.} & \mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0}\left\{\boldsymbol{X} \in (\mathcal{S}^{N-1} \times \mathcal{T})\right\} \\ \text{s.t.} & \rho(\overline{x}_0) \in \mathcal{U}^N \end{array} \quad \Leftrightarrow \quad \boxed{\begin{array}{rl} \text{max.} & \int_{\mathcal{S}^{N-1}\times\mathcal{T}} \psi_{\boldsymbol{X}}(\overline{X};\overline{x}_0,\overline{U})d\overline{X} \\ \text{s.t.} & \overline{U} \in \mathcal{U}^N \end{array}}$$



- Grid-free and recursion-free
- Closed-form expressions (in some cases)

$\boxed{\text{Curse of dim.} \rightarrow nN\text{-dim. quad.}}$

Vinod, HomChaudhuri, and Oishi, HSCC 2017

$\hat{W}_0^*(\cdot)$ underapproximates $\hat{V}_0^*(\cdot)$

- Require:

  1. Compact $\mathcal{U}$ 2. Borel $\mathcal{S}$ and $\mathcal{T}$ 3. Continuous $Q(\cdot|\overline{x},\overline{u})$

  $$Q(d\overline{y}|(\overline{x}_i,\overline{u}_i)) \xrightarrow{i\to\infty} Q(d\overline{y}|(\overline{x},\overline{u}))\ \forall\ (x_i,u_i) \xrightarrow{i\to\infty} (x,u)$$

Thm. 1 $\hat{W}_0^*(\overline{x}_0) \leq \hat{V}_0^*(\overline{x}_0)$ a.s.

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$$\sup_{\overline{U}\in\mathcal{U}^N} \mathbb{E}_{x_0}^{\rho}\left[\left(\prod_{k=0}^{N-1} 1_{\mathcal{S}}(\boldsymbol{x}_k)\right) 1_{\mathcal{T}}(\boldsymbol{x}_N)\right]$$

$$\leq \sup_{\overline{u}_0\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}\left[1_{\mathcal{S}}(\boldsymbol{x}_1) \sup_{\overline{u}_1\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}\left[1_{\mathcal{S}}(\boldsymbol{x}_2) \sup_{\overline{u}_2\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}\left[\right.\right.\right.$$

$$\left.\left.\left.\ldots \mathbb{E}_{x_0}^{\pi}\left[1_{\mathcal{S}}(\boldsymbol{x}_{N-1}) \sup_{\overline{u}_{N-1}\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}[1_{\mathcal{T}}(\boldsymbol{x}_N)|\boldsymbol{x}_{N-1}]\middle|\boldsymbol{x}_{N-2}\right]\ldots\middle|\boldsymbol{x}_1\right]\middle|\boldsymbol{x}_0\right] \text{ a.s.}$$

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$$\overbrace{\sup_{\overline{U}\in\mathcal{U}^N} \mathbb{E}_{x_0}^{\rho}\left[\left(\prod_{k=0}^{N-1} 1_{\mathcal{S}}(\boldsymbol{x}_k)\right) 1_{\mathcal{T}}(\boldsymbol{x}_N)\right]}^{\hat{W}_0^*(\overline{x}_0)}$$

$$\boxed{\begin{aligned}&\mathbb{P}_{\boldsymbol{X}}^{\rho,\overline{x}_0}\left\{\boldsymbol{X}\in(\mathcal{S}^{N-1}\times\mathcal{T})\right\}\\ &\qquad\qquad= \mathbb{E}_{x_0}^{\rho}\left[\left(\textstyle\prod_{k=0}^{N-1} 1_{\mathcal{S}}(\boldsymbol{x}_k)\right) 1_{\mathcal{T}}(\boldsymbol{x}_N)\right]\end{aligned}}$$

$$\leq \sup_{\overline{u}_0\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}\Bigg[1_{\mathcal{S}}(\boldsymbol{x}_1) \sup_{\overline{u}_1\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}\Bigg[1_{\mathcal{S}}(\boldsymbol{x}_2) \sup_{\overline{u}_2\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}\Bigg[$$

$$\ldots \mathbb{E}_{x_0}^{\pi}\Bigg[1_{\mathcal{S}}(\boldsymbol{x}_{N-1}) \sup_{\overline{u}_{N-1}\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}[1_{\mathcal{T}}(\boldsymbol{x}_N)|\boldsymbol{x}_{N-1}]\Bigg|\boldsymbol{x}_{N-2}\Bigg]\ldots\Bigg|\boldsymbol{x}_1\Bigg]\Bigg|\boldsymbol{x}_0\Bigg] \text{ a.s.}$$

$\hat{W}_0^*(\cdot)$ underapproximates $\hat{V}_0^*(\cdot)$

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Thm. 1 $\hat{W}_0^*(\overline{x}_0) \leq \hat{V}_0^*(\overline{x}_0)$ a.s.

$$\begin{aligned}
&\sup_{\overline{U}\in\mathcal{U}^N} \mathbb{E}_{x_0}^{\rho}\left[\left(\prod_{k=0}^{N-1} 1_{\mathcal{S}}(\boldsymbol{x}_k)\right) 1_{\mathcal{T}}(\boldsymbol{x}_N)\right] \\
&\leq \underbrace{\sup_{\overline{u}_0\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}\left[1_{\mathcal{S}}(\boldsymbol{x}_1) \sup_{\overline{u}_1\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}\left[1_{\mathcal{S}}(\boldsymbol{x}_2) \sup_{\overline{u}_2\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}\left[\right.\right.\right. \ldots \mathbb{E}_{x_0}^{\pi}\left[1_{\mathcal{S}}(\boldsymbol{x}_{N-1}) \sup_{\overline{u}_{N-1}\in\mathcal{U}} \mathbb{E}_{x_0}^{\pi}[1_{\mathcal{T}}(\boldsymbol{x}_N)|\boldsymbol{x}_{N-1}]\middle|\boldsymbol{x}_{N-2}\right] \ldots \left.\left.\middle|\boldsymbol{x}_1\right]\middle|\boldsymbol{x}_0\right]}_{\hat{V}_0^*(\overline{x}_0)} \text{ a.s.}
\end{aligned}$$

Via conditional expectation properties provided $\hat{V}_k^*(\cdot)$ is Borel-measurable

$\hat{W}_0^*(\cdot)$ underapproximates $\hat{V}_0^*(\cdot)$

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  1. Compact $\mathcal{U}$ 2. Borel $\mathcal{S}$ and $\mathcal{T}$ 3. Continuous $Q(\cdot|\overline{x},\overline{u})$

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Thm. 1 $\hat{W}_0^*(\overline{x}_0) \leq \hat{V}_0^*(\overline{x}_0)$ a.s.

Thm. 2 $\hat{V}_k^*(\cdot)$ are Borel-measurable for $k \in \mathbb{N}_{[0,N]}$

- Dynamic programming recursion via conditional expectations

$$\begin{aligned}\hat{V}_k^*(\overline{x}) &= \sup_{\overline{u}\in\mathcal{U}} 1_{\mathcal{S}}(\overline{x}) \int_{\mathcal{X}} \hat{V}_{k+1}^*(\overline{y}) Q(d\overline{y}|\overline{x},\overline{u}) \qquad k \in \mathbb{N}_{[0,N-1]}\\ &= \sup_{\overline{u}\in\mathcal{U}} 1_{\mathcal{S}}(\overline{x}) \mathbb{E}_{\boldsymbol{x}}^{\overline{u}}\left[\hat{V}_{k+1}^*(\boldsymbol{x}_{k+1})\middle|\boldsymbol{x}_k=\overline{x}\right]\\ \hat{V}_k^*(\boldsymbol{x}_k) &= \sup_{\overline{u}\in\mathcal{U}} 1_{\mathcal{S}}(\overline{x}) \mathbb{E}_{\boldsymbol{x}}^{\overline{u}}\left[\hat{V}_{k+1}^*(\boldsymbol{x}_{k+1})\middle|\boldsymbol{x}_k\right] \text{ a.s.} \qquad k \in \mathbb{N}_{[0,N-1]}\end{aligned}$$

LTI systems with Gaussian disturbance

- Log-concave optimization
  - Convex compact $\mathcal{U}$
  - Borel convex $\mathcal{S}, \mathcal{T}$

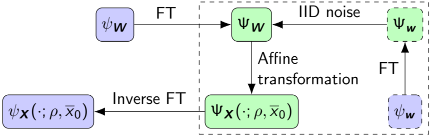
$$\begin{array}{ll} \text{max.} & \int_{\mathcal{S}^{N-1}\times\mathcal{T}} \psi_{\boldsymbol{X}}(\overline{X}; \overline{x}_0, \overline{U}) d\overline{X} \\ \text{s.t.} & \overline{U} \in \mathcal{U}^N \end{array}$$

$$\Psi_{\boldsymbol{w}}(\overline{\alpha}) = \exp\left(j\overline{\alpha}^\top \overline{m} - \frac{\overline{\alpha}^\top \Sigma \overline{\alpha}}{2}\right)$$

$$\boldsymbol{X} \sim \mathcal{N}(\overline{m}_{\boldsymbol{X}}, \Sigma_{\boldsymbol{X}})$$

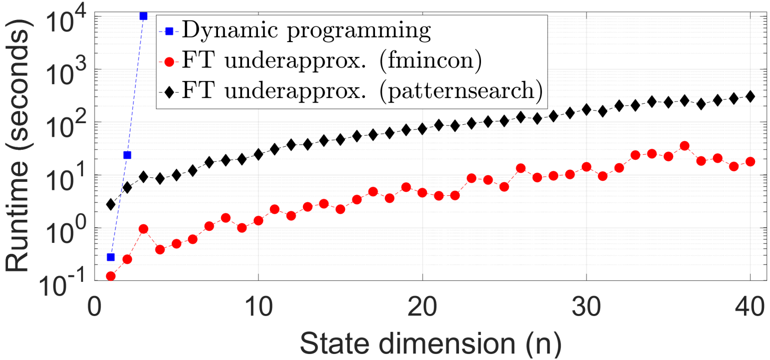
$$\overline{m}_{\boldsymbol{X}} = \overline{G}(\overline{1}_{N\times 1} \otimes \overline{m}) + \overline{A}\overline{x}_0 + \overline{H}\ \overline{U}$$

$$\Sigma_{\boldsymbol{X}} = \overline{G}(I_N \otimes \Sigma)\overline{G}^\top$$



- Integrate Gaussian $\psi_{\boldsymbol{X}}$ over polytopic $\mathcal{S}$ and $\mathcal{T}$
  - Genz's algorithm: Choose particles for desired accuracy $\epsilon > 0$
  - Noisy objective evaluation $\to$ Choice of solver crucial

Scalability



- Dynamics — chain of integrators
- Scalability due to convex optimization and Fourier transform

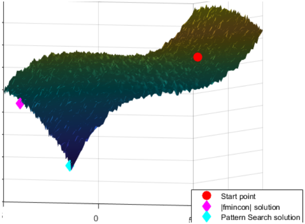
Solver comparison — *patternsearch* vs *fmincon*

$n = 40$

$$\boxed{\hat{W}_0^*(\overline{x}_0) \leq \hat{V}_0^*(\overline{x}_0) \leq 1}$$

Initial state of interest $\overline{x}_0 \in \mathbb{R}^{40}$	$\hat{W}_0^*(\overline{x}_0)$		$\hat{V}_0^*(\overline{x}_0)$	Runtime (s)	
	fm	*ps*		*fm*	*ps*
[0 0 0 ... 0]	0.999	0.999	[0.999, 1]	12	302
[2.5 2.5 2.5 ...]	0.983	0.985	[0.985, 1]	798	1196
[-8.5 8 -8.5 8 ...]	0.500	0.998	[0.998, 1]	12	441

- Non-trivial lower bounds for 40D systems
- *patternsearch* — tighter lower bounds at higher computation costs



Mathworks, Optim. of Stoch. Obj. Func.

Solver comparison — *patternsearch* vs *fmincon*

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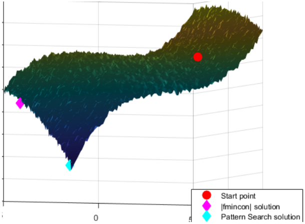
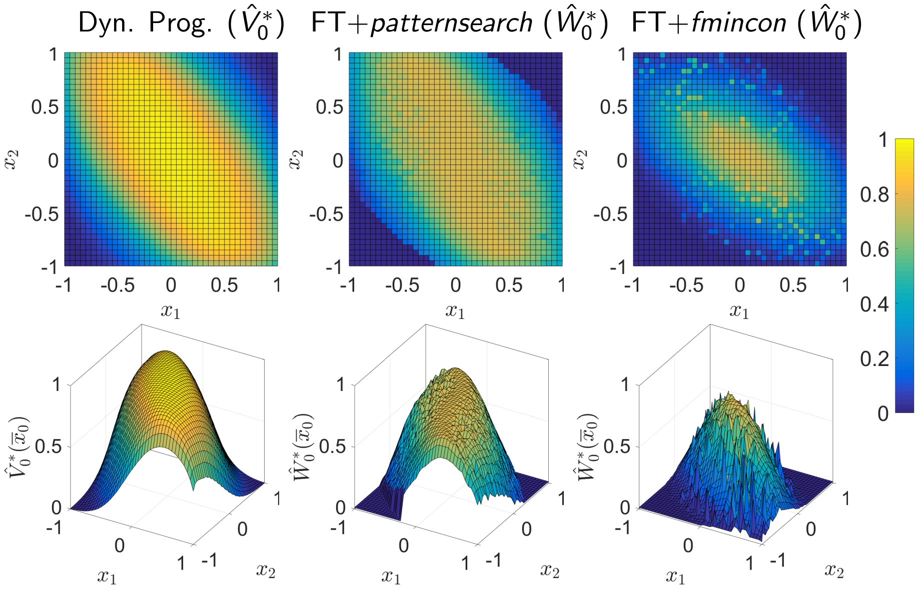
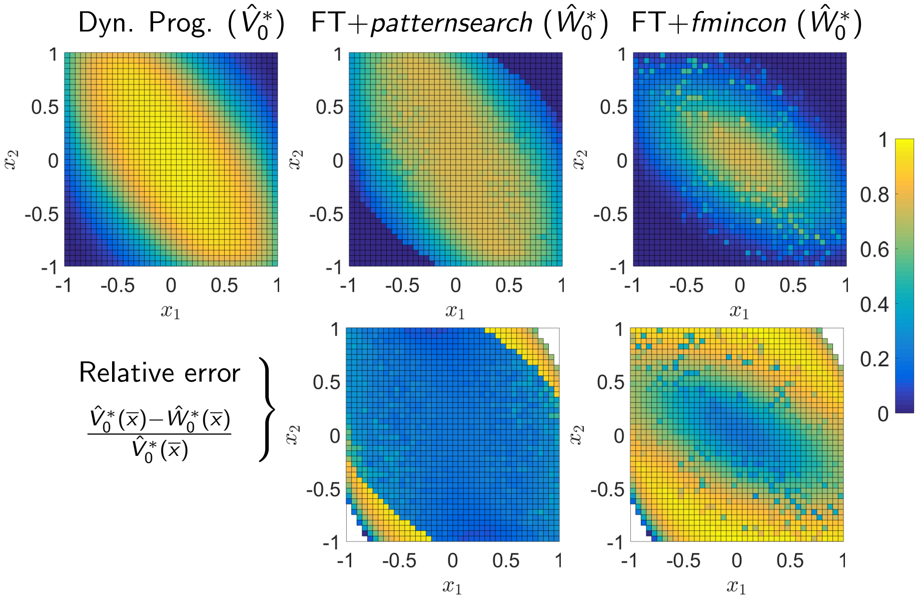


Mathworks, Optim. of Stoch. Obj. Func.

Underapproximation quality



Underapproximation quality

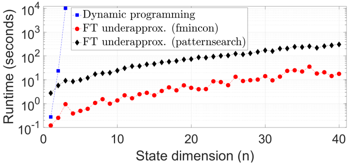


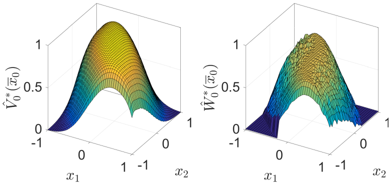
Relative error

$$\frac{\hat{V}_0^*(\overline{x}) - \hat{W}_0^*(\overline{x})}{\hat{V}_0^*(\overline{x})}$$

Conclusion

- Summary
  - Underapproximation for known $\overline{x}_0$
    - Fourier transform-based
    - Grid-free and recursion-free
    - Open-loop policies
  - Transformation to convex optimization for scalability
- Future work
  - Scalable underapproximation of stochastic reach-avoid set
  - Mitigate noisy optimization
  - Investigate non-Gaussian





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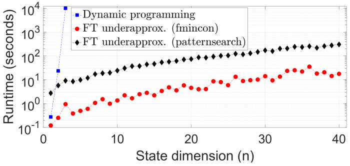

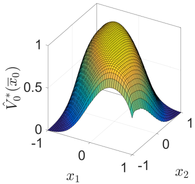

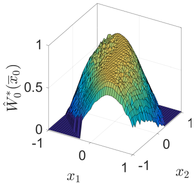

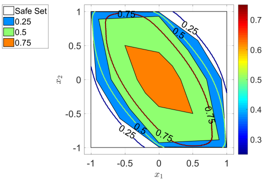

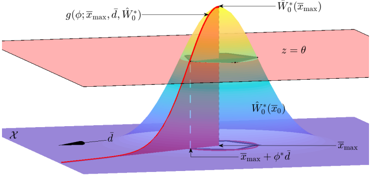

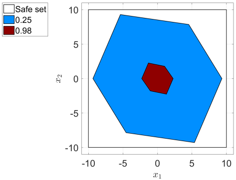

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- IIS-1528047

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