Underapproximation of Reach-Avoid Sets for Discrete-Time Stochastic Systems via Lagrangian Methods

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Abstract— We examine Lagrangian techniques for computing underapproximations of finite-time horizon, stochastic reach-avoid level sets for discrete-time, nonlinear systems. We use the concept of reachability of a target tube to define robust reach-avoid sets which are parameterized by the target set, safe set, and the set which the disturbance is drawn from. We unify two existing Lagrangian approaches to compute these sets, and establish that there exists an optimal control policy for the robust reach-avoid sets which is a Markov policy. Based on these results, we characterize the subset of the disturbance space whose corresponding robust reach-avoid set for a given target and safe set is a guaranteed underapproximation of the stochastic reach-avoid level set of interest. The proposed approach dramatically improves the computational efficiency for obtaining an underapproximation of stochastic reach-avoid level sets when compared to the traditional approaches based on gridding. Our method, while conservative, does not rely on a grid, implying scalability as permitted by constraints due to computational geometry. We demonstrate the method on two examples: a simple two-dimensional integrator, and a space vehicle rendezvous-docking problem.

I. Introduction

Reach-avoid analysis is an established verification tool that provides formal guarantees of both safety (by avoiding unsafe regions) and performance (by reaching a target set). It has been used in systems that are safety-critical or expensive, such as space systems [1], avionics [2], [3], biomedical systems [4], and other applications [5], [6], [7]. The reach-avoid set is the set of states for which there exists a control that enables the state to reach a target within some finite time horizon, while remaining within a safe set (avoiding an unsafe set) for all instants in the time horizon. In a probabilistic system, satisfaction of the reach-avoid objective is accomplished stochastically. The *stochastic reach-avoid level set* for a given likelihood is the set of states for which probabilistic success of the reach-avoid objective is assured with at least the desired likelihood.

The theoretical framework for the probabilistic reach-avoid calculation is based on dynamic programming [7], [8], [9], and, hence, is computationally infeasible for even

moderate-sized systems due to the gridding of not only the state-space, but also of the input and disturbance spaces [10]. Recent work has focused on alternatives to dynamic programming, including approximate dynamic programming [6], [11], [12], Gaussian mixtures [12], particle filters [1], [6], and convex chance-constrained optimization [1], [5]. These methods have been applied to systems that are at most 10-dimensional, at high memory and computational costs [6]. Further, since an analytical expression of the value function is not accessible, stochastic reach-avoid level sets can be computed only up to the accuracy of the gridding.

We propose a method to compute an underapproximation of the probabilistic reach-avoid set via the *robust reach-avoid set*, the set of states assured to reach the target set and remain in the safe region despite *any* disturbance input. Robust reach-avoid sets can be theoretically posed as the solution to the *reachability of a target tube* problem [13], [14], [15], originally framed to compute reachable sets of discrete-time controlled systems with bounded disturbance sets. Motivated by the scalability of the Lagrangian method proposed in [4], [16] for viability analysis in deterministic systems (that is, systems without a disturbance input but with a control input), we seek a similar approach to compute the robust reach-avoid sets via tractable set theoretic operations. Lagrangian methods rely on computational geometry, whose scalability depends on the representation and the operation used [17], including polyhedrons (implementable using Model Parametric Toolbox (MPT) [18]), support functions [19], and ellipsoids (implementable via the Ellipsoidal Toolbox [20]).

In this paper, we unify these two approaches to create an efficient algorithm for underapproximation of the stochastic reach-avoid set, and demonstrate our approach on practical examples. Our main contributions are: a) synthesis of the approaches presented in [4], [16] and [13], [14], [15] to compute the robust reach-avoid sets, b) sufficient conditions under which an optimal control policy for a given robust reach-avoid set is a Markov policy, and c) an algorithm to compute an underapproximation of the stochastic reach-avoid level sets using the robust reach-avoid sets. Specifically, we establish sufficient conditions under which an optimal control policy is comprised of universally measurable state-feedback laws, and then characterize the subset of the disturbance space whose corresponding robust reach-avoid set is a guaranteed underapproximation of the desired stochastic reach-avoid level set. Leveraging established Lagrangian methods, we demonstrate that our approach dramatically reduces the computation time required for computing a conservative underapproximation of the desired stochastic reach-avoid

level set. Further, since the Lagrangian approach does not rely on grids, the underapproximated sets are free from numerical artifacts due to discretization.

The remainder of the paper is as follows: Section II describes the problem formulation. In Section III, we describe the relationship between the the recursion established in [13] for the robust reach-avoid set and the Lagrangian approach in [4], and establish the desired measurability properties of the optimal controller. We present an algorithm for underapproximation of stochastic reach-avoid level sets in Section IV. We demonstrate our algorithm on a simple two-dimensional integrator and a space vehicle rendezvous-docking problem in Section V, and provide conclusions and directions of future work in Section VI.

II. PROBLEM STATEMENT

The following notation is used throughout the paper: we denote discrete-time time intervals by $\mathbb{Z}_{[a,b]} = \mathbb{Z} \cap \{a, a+1, \ldots, b-1, b\}$ for $a, b \in \mathbb{Z}$, $a \geq b$; the set of natural numbers (including zero) as $\mathbb{N}$; the Minkowski sum of two sets $\mathcal{S}_1, \mathcal{S}_2$ as $\mathcal{S}_1 \oplus \mathcal{S}_2 = \{s_1 + s_2 : s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2\}$; the Minkowski difference (or Pontryagin difference) of two sets $\mathcal{S}_1, \mathcal{S}_2$ as $\mathcal{S}_2 \ominus \mathcal{S}_1 = \{s : s + s_1 \in \mathcal{S}_2 \ \forall s_1 \in \mathcal{S}_1\}$; the indicator function corresponding to a set $\mathcal{S}$ as $\mathbf{1}_{\mathcal{S}} : \mathcal{X} \to \{0, 1\}$ where $\mathbf{1}_{\mathcal{S}}(x) = 1$ if $x \in \mathcal{S}$ and is zero otherwise; and, the Cartesian product of the set $\mathcal{S}$ with itself $k \in \mathbb{N}$ times is $\mathcal{S}^k$.

A. System formulation

Consider a discrete-time, nonlinear, time-invariant dynamical system with an affine disturbance,

$$x_{k+1} = f(x_k, u_k) + w_k \tag{1}$$

with state $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$, input $u_k \in \mathcal{U} \subseteq \mathbb{R}^m$, disturbance $w_k \in \mathcal{W} \subseteq \mathbb{R}^n$, and a function $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$. We denote the origin of $\mathbb{R}^n$ as 0_n and assume $0_n \in \mathcal{W}$ without loss of generality. We also consider the discrete, LTI system

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{2}$$

with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. We assume A is non-singular, which holds true for discrete-time systems that arise from the discretization of continuous-time systems.

B. Robust reach-avoid sets

Let $t \in \mathbb{N}$ and $\mathcal{F}$ denote the set of admissible state-feedback laws, $\nu : \mathcal{X} \to \mathcal{U}$. We define a control policy as a sequence of state-feedback laws, $\rho_t = [\nu_0(\cdot), \ldots, \nu_{t-1}(\cdot)]$ with $\nu_k \in \mathcal{F}$ for $k \in \mathbb{Z}_{[0,t-1]}$. We denote the corresponding set of admissible control policies as $\mathcal{P}_t$.

Let $\mathcal{E} \subseteq \mathcal{W}$ be a subset of the disturbance set. We define the t-time *robust reach-avoid set* corresponding to $\mathcal{E}$ as the set of initial states $x_0 \in \mathcal{X}$ such that there exists an admissible control policy $\rho_t \in \mathcal{P}_t$ that ensures x_k remains in a safe set $\mathcal{K} \subseteq \mathcal{X}$ for $k \in \mathbb{Z}_{[0,t-1]}$, and x_t lies in a target set $\mathcal{T} \subseteq \mathcal{X}$ (reach-avoid objective), despite presence of the disturbance $w_k \in \mathcal{E}$ at each instant. Denoting

$\bar{w}_t = [w_0^\top, \ldots, w_{t-1}^\top]^\top \in \mathcal{W}^t$, the t-time robust reach-avoid set is

$$\mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E}) = \big\{ x_0 \in \mathcal{X} : \exists \rho_t \in \mathcal{P}_t, \forall \bar{w}_t \in \mathcal{E}^t,$$
$$\forall k \in \mathbb{Z}_{[0,t-1]}, x_k \in \mathcal{K}, x_t \in \mathcal{T} \big\}. \tag{3}$$

Note that for $\mathcal{E} = \{0_n\}$, the system (1) is equivalent to a deterministic, discrete-time, nonlinear system

$$x_{k+1} = f(x_k, u_k) \tag{4}$$

when $w_k \in \mathcal{E}$. The t-time *viable set* of the system (4) is the set of initial states $x_0 \in \mathcal{X}$ such that there exists an admissible control policy $\rho_t \in \mathcal{P}_t$ such that x_k remains in a safe set $\mathcal{K}$ for $k \in \mathbb{Z}_{[0,t]}$. That is,

$$\mathcal{V}_t(\mathcal{K}) = \big\{ x_0 \in \mathcal{X} : \exists \rho_t \in \mathcal{P}_t, \forall k \in \mathbb{Z}_{[0,t]}, x_k \in \mathcal{K} \big\} \tag{5}$$
$$= \mathcal{D}_t(\mathcal{K}, \mathcal{K}, \{0_n\}). \tag{6}$$

The authors in [4] presented a Lagrangian formulation to compute $\mathcal{V}_t(\mathcal{K})$, and discussed the scalability of the viability analysis using MPT, ET, and support functions.

C. Stochastic reach-avoid level sets

In this subsection, we further assume the disturbance w_k in (1) is an n-dimensional random vector defined in the probability space $(\mathcal{W}, \sigma(\mathcal{W}), \mathbb{P}_w)$. Here, $\sigma(\mathcal{W})$ denotes the minimal σ-algebra associated with the random vector w_k. We assume the disturbance w_k is absolutely continuous with a probability density function ψ_w, the disturbance process $\{w_k\}_{k=0}^{N-1}$ is an independent and identically distributed (i.i.d.) random process, and $N \in \mathbb{N}$ is a finite time horizon. The assumption that w_k is i.i.d. is commonplace, and absolute continuity is necessary for the disturbance to have a density function. We assume that f is Borel-measurable, $\mathcal{U}$ is compact, the sets $\mathcal{K}, \mathcal{T}$ are Borel, and ψ_w is continuous.

We denote the set of universally measurable state-feedback laws $\mu(\cdot) : \mathcal{X} \to \mathcal{U}$ as $\mathcal{F}_u$. We define the Markov control policy as $\pi = [\mu_0(\cdot), \ldots, \mu_{N-1}(\cdot)]$ where $\mu_k \in \mathcal{F}_u \ \forall k \in \mathbb{Z}_{[0,N-1]}$, and $\mathcal{M}$ is the set of admissible Markov policies. Since no measurability restrictions were imposed on the feedback laws in Section II-B, $\mathcal{F}_u \subseteq \mathcal{F}$ and $\mathcal{M} \subseteq \mathcal{P}_N$.

Given a Markov policy π and initial state $x_0 \in \mathcal{X}$, the concatenated state vector $\bar{x} = [x_1, \ldots, x_N]$ for the system (1) is a random vector defined in the probability space $(\mathcal{X}^N, \sigma(\mathcal{X}^N), \mathbb{P}_{\bar{x}}^{N,\pi})$. The probability measure $\mathbb{P}_{\bar{x}}^{N,\pi}$ is induced from the probability measure $\mathbb{P}_w$ via (1) [7]. We will denote the probability space associated with the random vector $\bar{x}_k = [x_{k+1}, \ldots, x_N]$ as $(\mathcal{X}^{N-k}, \sigma(\mathcal{X}^{N-k}), \mathbb{P}_{\bar{x}_k}^{N-k,\pi})$ for $k \in \mathbb{Z}_{[0,N-1]}$.

For stochastic reachability analysis, we are interested in the maximum likelihood that the system (1) starting at an initial state $x_0 \in \mathcal{X}$ will achieve the reach-avoid objective using a Markov policy. The maximum likelihood and the optimal Markov policy can be determined as the solution to the optimization problem, [7]

$$\sup_{\pi \in \mathcal{M}} \mathbb{E}_{\bar{x}}^{N,\pi} \left[\left(\prod_{i=0}^{N-1} \mathbf{1}_{\mathcal{K}}(x_i) \right) \mathbf{1}_{\mathcal{T}}(x_N) \right]. \tag{7}$$

A dynamic programming approach was presented in [7] to solve problem (7). Let the optimal solution to problem (7) be $\pi^* = [\mu_0^*(\cdot) \ \ldots \ \mu_{N-1}^*(\cdot)]$, the *maximal Markov policy in the terminal sense* [7, Def. 10]. The existence of a Markov policy is guaranteed for a continuous ψ_w, compact $\mathcal{U}$, Borel $\mathcal{K}, \mathcal{T}$, and Borel-measurable f [21, Thm. 1]. The approach in [7] generates value functions $V_k^* : \mathcal{X} \to [0, 1]$ for $k \in [0, N]$,

$$V_k^*(x) = \mathbf{1}_{\mathcal{K}}(x) \int_{\mathcal{X}} V_{k+1}^*(y) \psi_w(y - f(x, \mu_k^*(x))) dy$$
$$= \mathbf{1}_{\mathcal{K}}(x) \mathbb{P}_{\bar{x}_k}^{N-k, \pi^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \ldots,$$
$$x_{k+1} \in \mathcal{K} | x) \quad (8)$$

initialized with

$$V_N^*(x) = \mathbf{1}_{\mathcal{T}}(x). \quad (9)$$

By definition, the optimal value function $V_0^*(x_0)$ provides the maximum likelihood, optimal value of problem (7), of achieving the reach-avoid objective by the system (1) for the time horizon N and the initial state $x_0 \in \mathcal{X}$.

For $\beta \in [0, 1]$ and $k \in \mathbb{Z}_{[0,N]}$, the *stochastic reach-avoid β-level set*,

$$\mathcal{L}_k(\beta) = \left\{ x \in \mathcal{X} : V_{N-k}^*(x) \geq \beta \right\}, \quad (10)$$

is the set of states x that achieve the reach-avoid objective by the time horizon with a minimum probability β, in the time interval $\mathbb{Z}_{[0,k]}$.

D. Problem statement

The following problems are addressed in this paper:

Problem 1. *Construct a recursion for exact computation of the robust reach-avoid set (3) for the system (1).*

Problem 2. *Given a set $\mathcal{E} \subseteq \mathcal{W}$ and the corresponding robust reach-avoid set (3), characterize the sufficient conditions under which there exists an optimal control policy that is a Markov control policy for the system (1).*

Problem 3. *Given a value $\beta \in [0, 1]$, characterize $\mathcal{E} \subseteq \mathcal{W}$ whose corresponding robust reach-avoid set (3) underapproximates the stochastic reach-avoid β-level set (10).*

Problem 3a. *Construct an algorithm for scalable computation of an underapproximation to the robust reach-avoid set for a nonlinear system (1) with a Gaussian disturbance.*

III. Robust Reach-Avoid Set Computation

In this section, we characterize the robust reach-avoid set for the system described in (1). To solve Problem 1, we first extend the approach presented in [4], [16] to reproduce the results presented in [13]. The authors in [4] demonstrated the scalability of Lagrangian methods for viability analysis in deterministic systems. By unifying these approaches, we aim for a tractable and efficient Lagrangian computation of the robust reach-avoid set with established scalability properties. We also demonstrate that the recursion for the viable set computation in a deterministic system [4] is a special case of the proposed Lagrangian approach. Finally,

we solve Problem 2 and establish that there is an optimal control policy for the robust reach-avoid set that is also a Markov policy.

A. Iterative computation for robust reach-avoid sets

Similarly to [4], for the system (1), we define the unperturbed, one-step forward reach set from a point $x \in \mathcal{X}$ as $\mathscr{F}_1(x)$, and the unperturbed, one-step backward reach set from a set $\mathcal{S} \subseteq \mathcal{X}$ as $\mathscr{R}_1(\mathcal{S})$. Formally, for the system (1),

$$\mathscr{F}_1(x) \triangleq \{x^+ \in \mathcal{X} : \exists u \in \mathcal{U}, \ x^+ = f(x, u)\} \quad (11)$$
$$\mathscr{R}_1(\mathcal{S}) \triangleq \{x^- \in \mathcal{X} : \exists u \in \mathcal{U}, \exists y \in \mathcal{S}, \ y = f(x^-, u)\}$$
$$= \{x^- \in \mathcal{X} : \mathscr{F}_1(x^-) \cap \mathcal{S} \neq \emptyset\} \quad (12)$$

For the LTI system (2), we can write

$$\mathscr{F}_1(x) = A\{x\} \oplus B\mathcal{U}, \quad (13)$$
$$\mathscr{R}_1(\mathcal{S}) = A^{-1}(\mathcal{S} \oplus (-B\mathcal{U})). \quad (14)$$

Proposition 1. *Given a set $\mathcal{E} \subseteq \mathcal{W}$, the finite horizon robust reach-avoid sets for the system (1) can be computed recursively as follows for $k \geq 1, k \in \mathbb{N}$:*

$$\mathcal{D}_0(\mathcal{T}, \mathcal{K}, \mathcal{E}) = \mathcal{T} \quad (15)$$
$$\mathcal{D}_k(\mathcal{T}, \mathcal{K}, \mathcal{E}) = \{x_0 \in \mathcal{K} :$$
$$\mathscr{F}_1(x_0) \cap (\mathcal{D}_{k-1}(\mathcal{T}, \mathcal{K}) \ominus \mathcal{E}) \neq \emptyset\}. \quad (16)$$

Proof: We first show the case in which $k = 1$. From (1) and (3),

$$\mathcal{D}_1(\mathcal{T}, \mathcal{K}, \mathcal{E}) = \{x_0 \in \mathcal{X} : \exists \nu_0(\cdot) \in \mathcal{F}, x_0 \in \mathcal{K}, \forall w_0 \in \mathcal{E},$$
$$\exists x^+ \in \mathcal{T}, x^+ = f(x_0, \nu_0(x_0)) + w_0\}$$
$$= \{x_0 \in \mathcal{X} : x_0 \in \mathcal{K}, \exists u \in \mathcal{U}, \exists \nu_0(\cdot) \in \mathcal{F},$$
$$\exists y \in (\mathcal{T} \ominus \mathcal{E}), y = f(x_0, \nu_0(x_0)), u = \nu_0(x_0)\}$$
$$= \{x_0 \in \mathcal{K} : \exists y, y \in \mathscr{F}_1(x_0) \wedge y \in (\mathcal{T} \ominus \mathcal{E})\}$$
$$= \{x_0 \in \mathcal{K} : \mathscr{F}_1(x_0) \cap (\mathcal{D}_0(\mathcal{T}, \mathcal{K}, \mathcal{E}) \ominus \mathcal{E}) \neq \emptyset\}$$

For any $t \in \mathbb{N}, t > 1$, from (3),

$$\mathcal{D}_{t-1}(\mathcal{T}, \mathcal{K}, \mathcal{E}) = \{x_0 \in \mathcal{X} : \exists \rho_{t-1} \in \mathcal{P}_{t-1}, \forall \bar{w}_{t-1} \in \mathcal{E}^{t-1},$$
$$\forall k \in \mathbb{Z}_{[0,t-2]}, x_k \in \mathcal{K}, x_{t-1} \in \mathcal{T}\}. \quad (17)$$

Using (17), we construct $\mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E})$ in the form of (16).

$$\mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E})$$
$$= \{x_0 \in \mathcal{X} : \exists \rho_t \in \mathcal{P}_t, \forall \bar{w}_t \in \mathcal{E}^t, \forall k \in \mathbb{Z}_{[0,t-1]},$$
$$x_k \in \mathcal{K}, x_t \in \mathcal{T}\}$$
$$= \{x_0 \in \mathcal{X} : x_0 \in \mathcal{K}, \exists \nu_0(\cdot) \in \mathcal{F}, \exists \rho_{t-1} \in \mathcal{P}_{t-1},$$
$$\rho_t = [\nu_0, \rho_{t-1}], \forall w_0 \in \mathcal{E}, \forall \bar{w}_{t-1} \in \mathcal{E}^{t-1},$$
$$\forall k \in \mathbb{Z}_{[1,t-1]}, x_k \in \mathcal{K}, x_t \in \mathcal{T}\} \quad (18)$$
$$= \{x_0 \in \mathcal{K} : \exists \nu_0(\cdot) \in \mathcal{F}, \ \forall w_0 \in \mathcal{E}, \exists x_0^+ \in \mathcal{D}_{t-1}(\mathcal{T}, \mathcal{K}, \mathcal{E})$$
$$x_0^+ = f(x_0, \nu_0(x_0)) + w_0\} \quad (19)$$
$$= \{x_0 \in \mathcal{K} : \exists \nu_0(\cdot) \in \mathcal{F}, \exists y \in \mathcal{D}_{t-1}(\mathcal{T}, \mathcal{K}) \ominus \mathcal{E},$$
$$y = f(x_0, \nu_0(x_0))\}$$
$$= \{x_0 \in \mathcal{K} : \mathscr{F}_1(x_0) \cap (\mathcal{D}_{t-1}(\mathcal{T}, \mathcal{K}, \mathcal{E}) \ominus \mathcal{E}) \neq \emptyset\}.$$
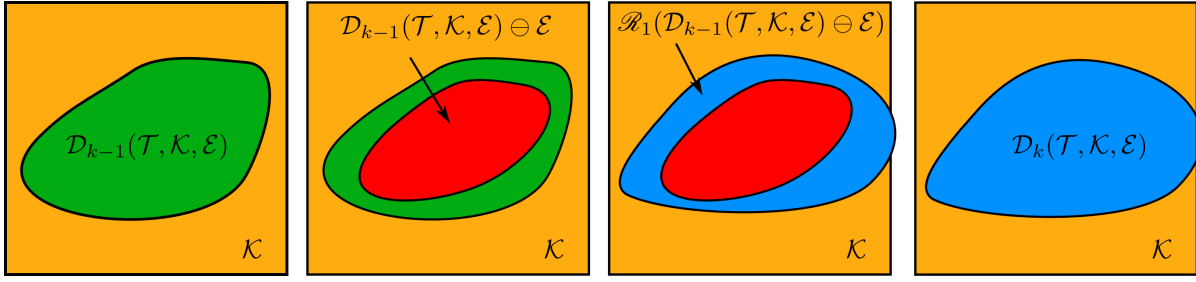
Fig. 1: Graphical representation of Lagrangian methods for computing $\mathcal{D}_k(\mathcal{T}, \mathcal{K}, \mathcal{E})$ from $\mathcal{D}_{k-1}(\mathcal{T}, \mathcal{K}, \mathcal{E})$ via (21).

Since the choice of w_0 depends only $(x_0, \nu(x_0))$ and ρ_{t-1} is independent of w_0, we can exchange the terms $\exists \rho_{t-1}$ and $\forall w_0$ in (18). We obtain (19) after applying (17) to (18). ∎

Theorem 1. *For the system given in (1), the finite-time robust reach-avoid sets $\mathcal{D}_k$ can be computed using the recursion for $k \geq 1, k \in \mathbb{N}$:*

$$\mathcal{D}_0(\mathcal{T}, \mathcal{K}, \mathcal{E}) = \mathcal{T} \tag{20}$$
$$\mathcal{D}_k(\mathcal{T}, \mathcal{K}, \mathcal{E}) = \mathcal{K} \cap \mathscr{R}_1(\mathcal{D}_{k-1}(\mathcal{T}, \mathcal{K}, \mathcal{E}) \ominus \mathcal{E}) \tag{21}$$

Proof: Follows from Proposition 1 and (12). ∎

Figure 1 depicts the recursion in Theorem 1 (21) graphically. From (21), we have the following corollary.

Corollary 1. $\mathcal{D}_k(\mathcal{T}, \mathcal{K}, \mathcal{E}) \subseteq \mathcal{K} \; \forall k \geq 1, k \in \mathbb{N}$.

For completeness, we establish that the viability analysis presented in [4] is a special case of Theorem 1. From [22, Theorem 2.1], for any $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{X}$, $\mathcal{S}_2 \ominus \mathcal{S}_1 = \cap_{s \in \mathcal{S}_1}(\mathcal{S}_2 \oplus \{-s\})$. Hence,

$$\mathcal{S}_2 \ominus \{0_n\} = \mathcal{S}_2. \tag{22}$$

Corollary 2. *[4, Theorem 1] The finite horizon viable sets for (4) can be computed recursively as follows:*

$$\begin{aligned} \mathcal{V}_0(\mathcal{K}) &= \mathcal{K} \\ \mathcal{V}_k(\mathcal{K}) &= \mathcal{K} \cap \mathscr{R}_1(\mathcal{V}_{k-1}(\mathcal{K})) \end{aligned} \tag{23}$$

Proof: Follows from Theorem 1, (4), and (22). ∎

A similar recursion can be constructed to compute the reach-avoid set for a deterministic system.

Lemma 1. *[13, Proposition 3] For the dynamics (2), if $\mathcal{U}, \mathcal{K}, \mathcal{T}$ are convex and compact sets, $\mathcal{E}$ is a compact set, and state matrix A in (2) is non-singular, then $\mathcal{D}_k(\mathcal{T}, \mathcal{K}, \mathcal{E})$ is convex and compact, $\forall k \in \mathbb{N}$.*

Note that convexity of $\mathcal{D}_k(\mathcal{T}, \mathcal{K}, \mathcal{E})$ does not require convexity of $\mathcal{E}$. Further, for polyhedral $\mathcal{U}, \mathcal{K}, \mathcal{T}$, the robust reach-avoid set $\mathcal{D}_k(\mathcal{T}, \mathcal{K}, \mathcal{E})$ is polyhedral for $k \in \mathbb{N}$. Note that the same can not be said be for ellipsoids [13, Sec. 4]. A detailed discussion for the implementation of Theorem 1 for polyhedral sets using support functions is given in [13, App. A].

B. Minmax problem for robust reach-avoid set computation

We now address Problem 2. A minmax optimization problem was presented in [13, Sec. 1], [23, Sec. 4.6.2] to compute the robust reach-avoid sets (3) for the system (1). The optimization problem is:

$$\underset{\rho_t}{\text{minimize}} \; \underset{\bar{w}_t}{\text{maximize}} \; J(\rho_{t-1}, \bar{w}_{t-1}; x_0, t) = \sum_{k=0}^{t} g_k(x_k)$$

$$\text{subject to} \; \begin{cases} x_{k+1} = f(x_k, \nu_k(x_k)) + w_k \\ w_k \in \mathcal{E} \\ \nu_k(\cdot) \in \mathcal{F} \end{cases}, \quad k \in \mathbb{Z}_{[0,t-1]} \tag{24}$$

where the decision variables are ρ_t and $\bar{w}_t$. Here, $g_k(\cdot) = 1 - \mathbf{1}_{\mathcal{K}}(\cdot)$ for $k \in \mathbb{Z}_{[0,t-1]}$ and $g_t(\cdot) = 1 - \mathbf{1}_{\mathcal{T}}(\cdot)$. The objective function $J(\cdot)$ is parameterized by the initial state $x_0 \in \mathcal{X}$ and the time horizon $t \in \mathbb{N}$. Problem (24) can be solved using dynamic programming [23, Sec 1.6] to generate the value functions $J_k^*(x; t) : \mathcal{X} \to \mathbb{Z}_{[0,t-k+1]}$ for $k \in \mathbb{Z}_{[0,t]}$

$$H_k^*(u, x; t) = \sup_{w \in \mathcal{E}} \left[J_{k+1}^*(f(x, u) + w; t) + g_k(x) \right] \tag{25}$$
$$J_k^*(x; t) = \inf_{u \in \mathcal{U}} H_k^*(u, x; t) \tag{26}$$

initialized with $J_t^*(x; t) = g_t(x)$. The optimal value of problem (24) when starting at x_0 is $J_0^*(x_0; t)$. Further,

$$\mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E}) = \{x \in \mathcal{X} : J_t^*(x_0; t) = 0\}. \tag{27}$$

Recall that lower semicontinuous functions are functions whose sublevel sets are closed and upper semicontinuous functions are functions whose negative is a lower semicontinuous function [24, Definition 7.13]. Also, the supremum of a lower semicontinuous function is the negative of the infimum of an upper semicontinuous function. Let an optimal control policy for problem (24) be $\rho_t^* = [\nu_0^*(\cdot) \; \cdots \; \nu_{t-1}^*(\cdot)]$. Note that ρ_t^* need not be unique.

Theorem 2. *For closed sets $\mathcal{K}, \mathcal{T}$ and compact set $\mathcal{U}$, there exists an optimal policy ρ_t^* for problem (24) which is also a Markov policy.*

Proof: We first show by induction that the optimal value functions J_k^* of (24) are lower semicontinuous, and that there exists a Borel-measurable state-feedback law $\nu_k^*(\cdot)$ for every $k \in \mathbb{Z}_{[0,N-1]}$. Then, since Borel measurability implies universal measurability [24, Definition 7.20], the proof of Theorem 2 follows directly, exploiting the definition of a Markov policy.

$$V_{N-t}^*(x) = \mathbb{P}_{\bar{x}_t}^{t,\pi^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K}|x)\mathbf{1}_{\mathcal{K}}(x)$$

$$= \mathbb{P}_{\bar{x}_t}^{t,\pi^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K}|x, \bar{w}_t \in \mathcal{E}^t)\mathbb{P}_{\bar{w}_t}^t(\bar{w}_t \in \mathcal{E}^t)$$

$$+ \mathbb{P}_{\bar{x}_t}^{t,\pi^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K}|x, \bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t))\mathbb{P}_{\bar{w}_t}^t(\bar{w}_t \in (\mathcal{W}^t \setminus \mathcal{E}^t)) \quad (28)$$

$$\geq \mathbb{P}_{\bar{x}_t}^{t,\pi^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots, x_{N-t+1} \in \mathcal{K}|x, \bar{w}_t \in \mathcal{E}^t)\mathbb{P}_{\bar{w}_t}^t(\bar{w}_t \in \mathcal{E}^t). \quad (29)$$

The closedness property of $\mathcal{K}, \mathcal{T}$ implies that $g_k(\cdot)$ is lower semicontinuous for $k \in \mathbb{Z}_{[0,t]}$. Hence, $J_t^*(x;t)$ is lower semicontinuous.

Consider the base case $k = t - 1$. From [24, Prop. 7.32 (b)], we see that $H_{t-1}^*(u, x; t)$ is lower semi-continuous. From [24, Prop. 7.33], we conclude that $J_{t-1}^*(x; t)$ is lower semi-continuous and an optimal state-feedback policy $\nu_{t-1}^*(\cdot)$ exists, and is also Borel-measurable.

Let $\tau \in \mathbb{Z}_{[1,t-2]}$. Assume, for induction, the case $k = \tau$ is true, i.e, J_τ^* is lower semicontinuous. That $J_{\tau-1}^*$ is lower semicontinuous and the existence of a Borel-measurable $\nu_{\tau-1}^*(\cdot)$ follows from [24, Prop. 7.32(b) and 7.33]. This completes the induction. ∎

IV. CONSERVATIVE APPROXIMATION OF STOCHASTIC REACH-AVOID LEVEL SET

We now use the theory developed in Section III to solve Problems 3 and 3a.

Theorem 3. *Given closed sets $\mathcal{K}, \mathcal{T}$, compact set $\mathcal{U}$, and a set $\mathcal{E} \subseteq \mathcal{W}$, for every $x \in \mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E})$, with $t \in \mathbb{Z}_{[1,N]}$,*

$$\mathbb{P}_{\bar{x}_t}^{t,\rho_N^*}(x_N \in \mathcal{T}, x_{N-1} \in \mathcal{K}, \dots,$$
$$x_{N-t+1} \in \mathcal{K}|x, \bar{w}_t \in \mathcal{E}^t) = 1. \quad (30)$$

Proof: Follows from Theorem 2 and the definition of $\mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E})$ (3). ∎

Theorem 4. *Given $\beta \in [0,1]$, closed sets $\mathcal{K}, \mathcal{T}$, and a compact set $\mathcal{U}$, if for any $t \in \mathbb{Z}_{[0,N]}$, $\mathcal{E} \subseteq \mathcal{W}$ such that $\mathbb{P}_w(w_k \in \mathcal{E}) = \beta^{\frac{1}{t}}$ for all $k \in \mathbb{Z}_{[0,t-1]}$, then $\mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E}) \subseteq \mathcal{L}_t(\beta)$.*

Proof: The case for $t = 0$ follows trivially from (9), (10), and (15). Let $t > 0$ and $x \in \mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E})$. We are interested in underapproximating $\mathcal{L}_t(\beta) = \{x : V_{N-t}^*(x) \geq \beta\}$ as defined in (10). Equation (28) follows from (8), the law of total probability, and Corollary 1, which implies $\mathbf{1}_{\mathcal{K}}(x) = 1$ Equation (29) follows from (28) after ignoring the second term (which is non-negative). Simplifying (29) using Theorem 3 and the i.i.d. assumption of the disturbance process, we obtain

$$V_{N-t}^*(x) \geq \mathbb{P}_{\bar{w}_t}^k(\bar{w}_t \in \mathcal{E}^t) = (\mathbb{P}_w(w_k \in \mathcal{E}))^t = \beta. \quad (31)$$

Thus, $\mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E}) \subseteq \mathcal{L}_t(\beta)$ by (10). ∎

Theorem 4 solves Problem 3 for an arbitrary density ψ_w. Computation of $\mathcal{D}_t(\mathcal{T}, \mathcal{K}, \mathcal{E})$ can be done via Theorem 1. Note that $\mathcal{E}$ characterized by Theorem 4 is not unique. We

prescribe that $\mathcal{E}$ has the least Lebesgue measure to reduce the the degree of conservativeness in Theorem 4. We also recommend the set $\mathcal{E}$ be convex, compact, and contains $\{0_n\}$ for computational ease.

Next, we provide a method to compute $\mathcal{E} \subseteq \mathcal{W}$ for any $t \in \mathbb{Z}_{[0,N-1]}$ such that $\mathbb{P}_w(w_k \in \mathcal{E}) = \beta^{\frac{1}{t}}$ for all $k \in \mathbb{Z}_{[0,t-1]}$ when the disturbance in the system (1) is Gaussian.

A. Computation of $\mathcal{E}$ for a Gaussian disturbance

Let the disturbance in (1) be $w_k = v$, an n-dimensional Gaussian random variable with mean vector μ and covariance matrix Σ. The probability density of a multivariate Gaussian random vector is [25, Ch. 29]

$$\psi_v(s) = (2\pi)^{\frac{-n}{2}}|\Sigma|^{\frac{-1}{2}} \exp\left(-\frac{(s-\mu)^\top \Sigma^{-1}(s-\mu)}{2}\right).$$

Consider $\mathcal{E}$ that is a n-dimensional ellipsoid, parameterized by $R^2 \in [0, \infty)$,

$$\mathcal{E}_{R^2} = \left\{s \in \mathbb{R}^n : (s-\mu)^\top \Sigma^{-1}(s-\mu) \leq R^2\right\}. \quad (32)$$

For $\mu = 0, \Sigma = r^2 I_n$, we have $\mathcal{E}_{R^2} = \{w : w^\top w \leq r^2 R^2\}$, a n-dimensional hypersphere of radius rR. We aim to compute the parameter R^2 such that $\mathbb{P}_v\{v \in \mathcal{E}_{R^2}\} = \beta^{\frac{1}{t}}$, for application of Theorem 4.

Given a normally distributed n-dimensional random vector $\eta \sim \mathcal{N}(0, I_n)$, we have $v = \Sigma^{\frac{1}{2}}\eta + \mu$ [25, Ch. 29]. Also, $\mathcal{E}_{R^2} = \Sigma^{\frac{1}{2}}\mathcal{E}_{\eta,R^2} \oplus \{\mu\}$ with $\mathcal{E}_{\eta,R^2} = \{s \in \mathbb{R}^n : s^\top s \leq R^2\}$. Since the affine transformation of η to v is deterministic, $\mathbb{P}_v\{v \in \mathcal{E}_{R^2}\} = \mathbb{P}_\eta\{\eta \in \mathcal{E}_{\eta,R^2}\} = \beta^{\frac{1}{t}}$. From [25, Ex. 20.16], we have

$$F_{\chi^2(n)}(R^2) = \mathbb{P}\left\{\chi^2(n) \leq R^2\right\} = \mathbb{P}\{\eta \in \mathcal{E}_{\eta,R^2}\} = \beta^{\frac{1}{t}}$$

where $\chi^2(n)$ is a chi-squared random variable with n degrees of freedom and $F_{\chi^2(n)}(\cdot)$ denotes its cumulative distribution function. Consequently, we have

$$R^2 = F_{\chi^2(n)}^{-1}\left(\beta^{\frac{1}{t}}\right). \quad (33)$$

By solving (33) and then using the result in (32), we can obtain a feasible $\mathcal{E}$ for any Gaussian disturbance.

B. Computing the stochastic level set underapproximation

We solve Problem 3a using Algorithm 1. This algorithm computes the underapproximation of the N-time stochastic reach-avoid β-level set via robust reach-avoid sets. Note that while nonlinear dynamics (1) are permitted, the computation of $\mathcal{R}_1(S)$ is accessible only for the linear system (2) as defined in (14). Further, for linear dynamics, Lemma 1

guarantees convexity and compactness of the robust reach-avoid set, allowing for easy representation.

Algorithm 1: Underapproximation of the N-time stochastic reach-avoid β-level set for system (1) with a Gaussian disturbance.

Input : Safe set, $\mathcal{K}$; target set, $\mathcal{T}$; system dynamics (1), desired probability level $\beta \in [0, 1]$, Gaussian covariance matrix and mean, Σ, μ; and time horizon, N

Output: N-time stochastic reach-avoid β-level set underapproximation, $\mathcal{D}_N(\mathcal{K}, \mathcal{T}, \mathcal{E})$

$R^2 \leftarrow F^{-1}_{\chi^2(n)}\left(\beta^{\frac{1}{t}}\right)$ // from (33)
$\mathcal{E} \leftarrow \mathcal{E}_{R^2}$ // from (32)
$\mathcal{D}_0(\mathcal{K}, \mathcal{T}, \mathcal{E}) \leftarrow \mathcal{T}$
for $i = 1, 2, \ldots, N$ **do**
    $S \leftarrow \mathcal{D}_{i-1}(\mathcal{K}, \mathcal{T}, \mathcal{E}) \ominus \mathcal{E}$ // from (21)
    $E \leftarrow \mathscr{R}_1(S)$ // from (12)
    $\mathcal{D}_i(\mathcal{K}, \mathcal{T}, \mathcal{E}) \leftarrow \mathcal{K} \cap E$ // from (21)
end

Since the robust reach-avoid set is computed via Lagrangian techniques, Algorithm 1 is scalable and computationally more efficient than the dynamic programming based discretization approach. Algorithm 1 requires a number of basic geometric operations. We focus on the implementation of Algorithm 1 in a polyhedral representation, using MPT [18]. The robust reach-avoid set computation requires a Minkowski difference operation as well as an intersection operation in (21). Since for polytopes, both facet (for efficient intersection) and vertex representations (for efficient Minkowski difference) are required, numerical implementations will be limited by the well known vertex-facet enumeration problem. Additional problems, such as redundancy in vertices and facets, also commonly arise using polytopic representations.

From Lemma 1, support function methods could also be used [4], [13] to implement Algorithm 1. Support functions would not be subject to the computational issues affecting the polytopic approach, but require analytic solutions to support vector calculations, and will yield an overapproximation of the stochastic reach-avoid level set.

The conservativeness of the underapproximations obtained using Algorithm 1 is highly problem dependent. The system dynamics, the strength of the disturbance process, and the size of the target and safe sets all have non-trivial effects on the resulting conservativeness.

V. EXAMPLES

All results were obtained using MPT with MATLAB R2016a running on Windows 7 computer with and Intel Core i7-2600 CPU, 3.6 GHz, and 8 GB RAM. All simulation code can be found at `http://hscl.unm.edu/software.html`.

Grid Size	Dynamic Programming	Algorithm 1	Ratio
41×41	8.16	0.98	8.3
82×82	59.76	0.98	60.9

TABLE I: Computation times, in seconds, for the double integrator problem, solved via dynamic programming and via Algorithm 1. The ratio of computation times is the comp. time via dynamic programming divided by comp. time via Algorithm 1; note that the main benefit of Algorithm 1 is that it does not require a grid.

A. Two-dimensional double integrator

We first consider a simple example which allows direct comparison of the proposed underapproximation as well as the approximative result via dynamic programming, for conservatism and computational speed.

The discretized double integrator dynamics are

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u_k + w_k \qquad (34)$$

with state $x_k \in \mathcal{X} \subseteq \mathbb{R}^2$, input $u_k \in \mathcal{U} \subseteq \mathbb{R}$, $T = 0.25$, and Gaussian disturbance $w_k \sim \mathcal{N}(0, 0.005 \cdot I_2)$.

Figure 2 compares the underapproximation via Algorithm 1 to the level sets computed via dynamic programming, as in [7]. The underapproximation is tighter initially and becomes progressively more conservative as the time horizon N increases. For time horizons $N_1, N_2 \in \mathbb{N}$ with $N_2 > N_1$, we know that $\beta^{\frac{1}{N_2}} \geq \beta^{\frac{1}{N_1}}$ for all $\beta \in [0, 1]$. Hence, from Section IV-A, $R^2_{N_2} \geq R^2_{N_1}$, indicating that $\mathcal{E}_{N_1} \subseteq \mathcal{E}_{N_2}$, implying increased conservativeness by (3). For the example shown in Figure 2 with $N = 1, 2, 3, 4, 5$, $R^2 = 3.22, 4.50, 5.27, 5.83, 6.26$, respectively.

A comparison between the total computation time for both approaches is provided in Table I. The accuracy of dynamic programming relies on its grid size, resulting in a trade-off between accuracy and computation speed, from which Algorithm 1 does not suffer.

For systems with Gaussian disturbance processes that have a low variance, the underapproximation obtained through the Lagrangian methods tightly underapproximates the stochastic level set, and is computed significantly faster—over 7 times faster than the dynamic programming approach using a 41×41 grid. Figure 3 shows a comparison of the stochastic level set and the Lagrangian underapproximation when the Gaussian disturbance is of the form $w_k \sim \mathcal{N}(0, 10^{-5} I_2)$. The irregularities on the exterior of the stochastic level set are a numerical artifact from the state-space gridding.

B. Application to space-vehicle dynamics

We now consider a more realistic problem, motivated by the rendezvous and docking problem for a pair of space vehicles. The goal is for one spacecraft, referred to as the deputy, to approach and dock to an orbiting satellite, referred to as the chief, while remaining in a predefined line-of-sight cone, in which accurate sensing of the other vehicle
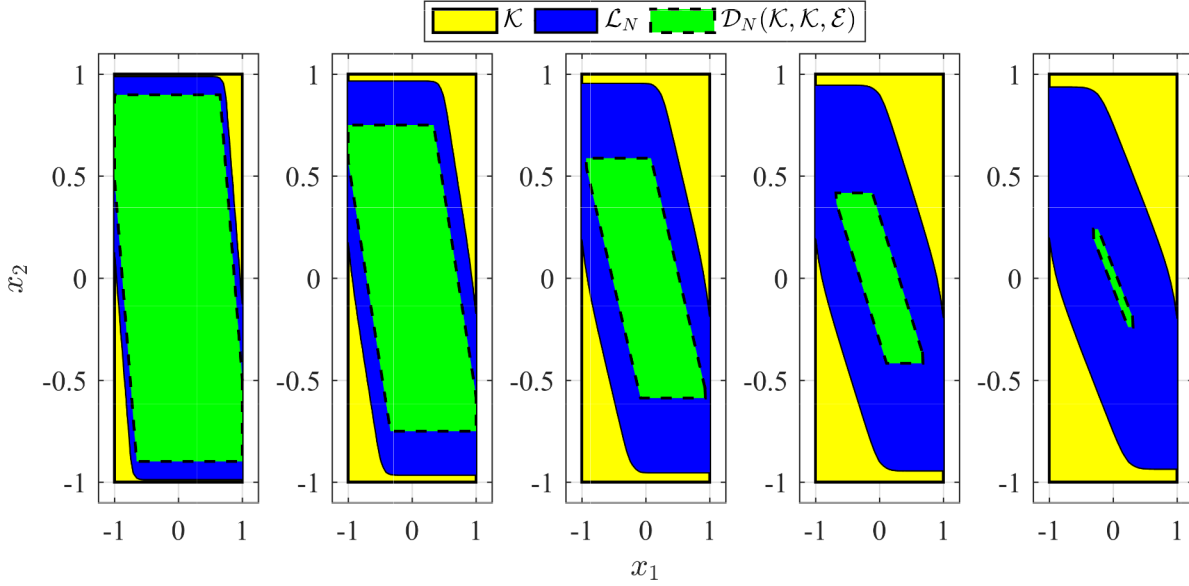
Fig. 2: Comparison between stochastic reach-avoid β-level sets (10), with $\beta = 0.8$, computed via dynamic programming, $\mathcal{L}_N$, and via the Lagrangian underapproximation, $\mathcal{D}_N(\mathcal{K}, \mathcal{K}, \mathcal{E})$, at times $N \in \{1, 2, 3, 4, 5\}$ (from left), for the double integrator system (34) with a Gaussian disturbance $w_k \sim \mathcal{N}(0, 0.005 I_2)$. From Section IV-A, $\mathcal{E} = \{w : w^\top w \leq 0.005 R_N^2\}$, with $R_N^2 = 3.22, 4.50, 5.27, 5.83, 6.26$, respectively.
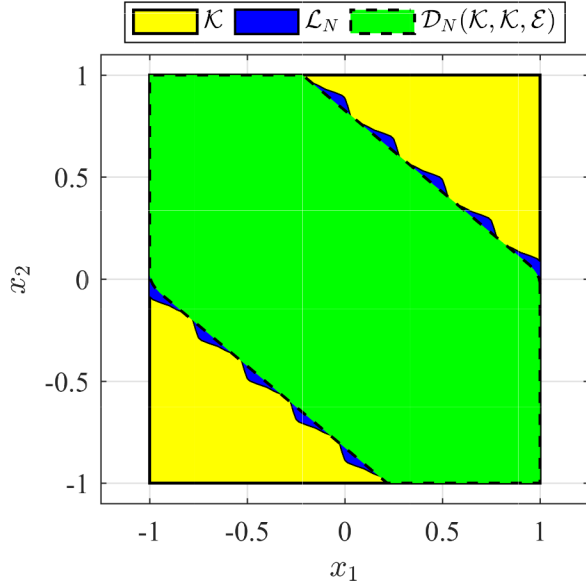

Fig. 3: Comparison between the $N = 5$ stochastic reach-avoid β-level set (10), with $\beta = 0.8$, computed via dynamic programming, $\mathcal{L}_N$, and via the Lagrangian underapproximation, $\mathcal{D}_N(\mathcal{K}, \mathcal{K}, \mathcal{E})$, for the double integrator system (34) with a Gaussian disturbance $w_k \sim \mathcal{N}(0, 10^{-5} \cdot I_2)$. From Section IV-A, $\mathcal{E} = \{w : w^\top w \leq 10^{-5} R^2\}$, with $R^2 = 6.26$.

is possible. The dynamics are described by the Clohessy-Wiltshire-Hill (CWH) equations [26]

$$\ddot{x} - 3\omega x - 2\omega \dot{y} = \frac{F_x}{m_d}$$
$$\ddot{y} + 2\omega \dot{x} = \frac{F_y}{m_d} \tag{35}$$

The chief is located at the origin, the position of the deputy is $x, y \in \mathbb{R}$, $\omega = \sqrt{\mu/R_0^3}$ is the orbital frequency, μ is the gravitational constant, and R_0 is the orbital radius of the spacecraft.

We define the state as $z = [x, y, \dot{x}, \dot{y}] \in \mathbb{R}^4$ and input as $u = [F_x, F_y] \in \mathcal{U} \subseteq \mathbb{R}^2$. We discretize the dynamics (35) in time to obtain the discrete-time LTI system,

$$z_{k+1} = A z_k + B u_k + w_k \tag{36}$$

with $w_k \in \mathbb{R}^4$ a Gaussian i.i.d. disturbance, with $\mathbb{E}[w_k] = 0$, $\Sigma = \mathbb{E}[w_k w_k^\top] = 10^{-4} \times \text{diag}(1, 1, 5 \times 10^{-4}, 5 \times 10^{-4})$.

We define the target set and the constraint set as in [1]

$$\mathcal{T} = \{z \in \mathbb{R}^4 : |z_1| \leq 0.1, -0.1 \leq z_2 \leq 0,$$
$$|z_3| \leq 0.01, |z_4| \leq 0.01\} \tag{37}$$
$$\mathcal{K} = \{z \in \mathbb{R}^4 : |z_1| \leq z_2, |z_3| \leq 0.05, |z_4| \leq 0.05\} \tag{38}$$
$$\mathcal{U} = [-0.1, 0.1] \times [-0.1, 0.1]. \tag{39}$$

For a horizon, $N = 5$, and a level set, $\beta = 0.8$, $\mathcal{E} = \{s : s^\top \Sigma^{-1} s \leq 6.26\}$, from (32). Figure 4 shows a cross-section at $\dot{x} = \dot{y} = 0$ of the resulting underapproximation of the $N = 5$ stochastic reach-avoid level set. The computation time for this level set was 14.5 seconds. Direct comparison of results via dynamic programming is not possible due
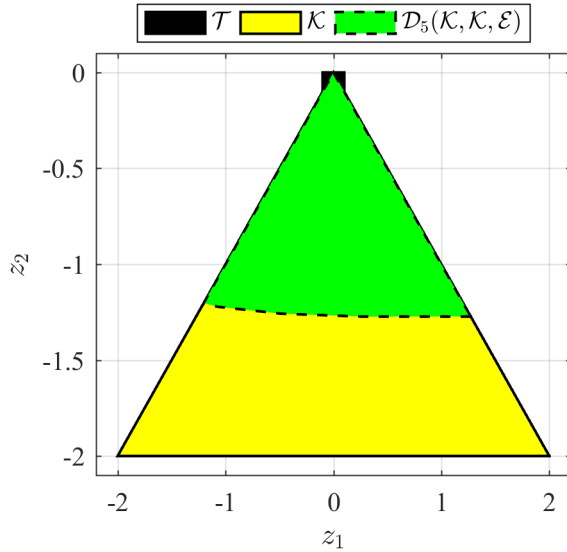
Fig. 4: Cross-section of the underapproximation of the $N = 5$ stochastic reach-avoid β-level set (10), $\beta = 0.8$, for the spacecraft rendezvous docking problem (36) at $\dot{x} = \dot{y} = 0$.

to dimensionality of the state. However, in [1, Figure 2], a cross-section of $\dot{x} = \dot{y} = 0.9$ of the stochastic reach-avoid set was approximated via convex chance-constrained optimization and particle approximation methods. Although these approaches require gridding, they are computationally feasible, unlike dynamic programming. The computation time reported in [1] is approximately 20 minutes (about 82 times slower) for just a subset of the state space.

VI. CONCLUSION

In this paper, we describe a Lagrangian method to compute an underapproximation of the stochastic reach-avoid level set using robust reach-avoid sets. We synthesize approaches in [4], [16] and [13], [14], [15], and characterize sufficient conditions under which an optimal control policy for the robust reach-avoid set is also a Markov policy. We demonstrate that our proposed Lagrangian approach to compute the underapproximation is significantly faster than dynamic programming and other gridding-based methods. The utility of this method is problem dependent, as the conservativeness of the underapproximations.

Future work includes examination of methods to reduce the conservativeness of the underapproximation, and extension of the computation of the disturbance set $\mathcal{E}$ in Theorem 4 for non-Gaussian disturbances.

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