Forward Stochastic Reachability Analysis for Uncontrolled Linear Systems using Fourier Transforms

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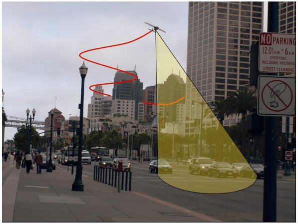
Motivation



- Stochastic non-adversarial target capture
  - Applications: rescue operations, UAV monitoring
  - Scalable, non-conservative, and exact solution
  - Globally optimal probabilistic guarantees via convex optimization
- Forward stochastic reachability
  - Forward stochastic reach probability measure/density
  - Forward stochastic reach set
- Convexity properties

Main contributions

- Exact and scalable techniques for forward stochastic reachability
  - Convolution-based recursion
  - Fourier transform-based approach
- Convexity properties of the forward stochastic reach set and probability measure/density
- Convex optimization for the target capture

Related work

Backward stochastic reachability
Abate, Amin, Prandini, Lygeros, & Sastry (2007); Abate, Prandini, Lygeros, & Sastry (2008); Summers, & Lygeros (2011); Kariotoglou, Summers, Summers, Kamgarpour, & Lygeros (2013); Lesser, Oishi, & Erwin (2013); Manganini, Pirotta, Restelli, Piroddi, & Prandini (2015)

Non-deterministic system reachability (continuous time)
Tomlin, Mitchell, Bayen, & Oishi (2003); Mitchell, Bayen, & Tomlin (2005); Bokanowski, Forcadel, & Zidani (2010); Huang, Ding, Zhang, & Tomlin (2015); Chen, Herbert, & Tomlin (2017)

Non-deterministic system reachability (discrete time)
Kvasnica, Grieder, Baotic, & Morari (2004); Girard (2005); Kurzhanskiy & Varaiya (2006); Kvasnica, Takacs, Holaza, & Ingole (2015)

Forward stochastic reachability
Lasota & Mackey (1985); Althoff, Stursberg, & Buss (2009); Asselborn & Stursberg (2015); HomChaudhuri, Vinod, & Oishi (2017)

Non-adversarial target capture
Kumar, Rus, & Singh (2004); Geyer (2008); Hollinger, Singh, Djugash, & Kehagias (2009)

Outline

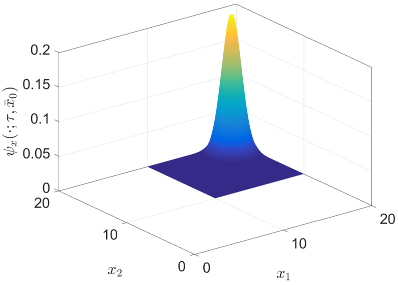
Problem setup

$$\mathbf{x}[k+1] = A\mathbf{x}[k] + B\mathbf{w}[k], \qquad \mathbf{x}[k] \in \mathbb{R}^n$$

- Known initial state $\mathbf{x}[0] = \bar{x}_0 \in \mathcal{X}$
- IID $\mathbf{w}[k]$ in $(\mathcal{W}, \mathscr{B}(\mathcal{W}), \mathbb{P}_{\mathbf{w}}), \mathcal{W} \subseteq \mathbb{R}^p$ with pdf $\psi_{\mathbf{w}}(\cdot)$
- $\mathbf{x}[k]$ in $(\mathcal{X}, \mathscr{B}(\mathcal{X}), \mathbb{P}_{\mathbf{x}}^{k,\bar{x}_0})$

Forward stochastic reachability

- $\mathbb{P}_{\mathbf{x}}^{k,\bar{x}_0}$ defines probability density $\psi_{\mathbf{x}}(\cdot; k, \bar{x}_0)$ (FSRPD)
- $\mathrm{FSReach}(k, \bar{x}_0) = \mathrm{supp}(\mathbf{x}[k])$



HomChaudhuri, Vinod, and Oishi, ACC 2017

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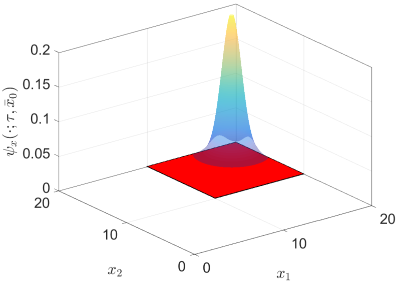
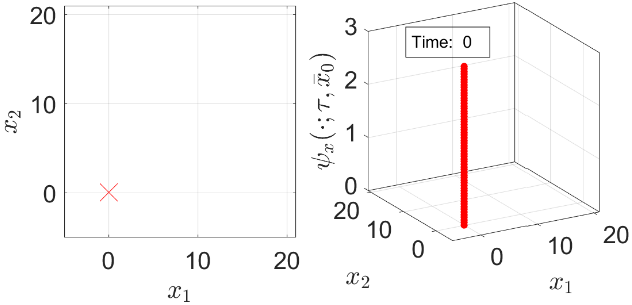


HomChaudhuri, Vinod, and Oishi, ACC 2017

Forward stochastic reachability

- Probability of the state $\mathbf{x}[\tau]$ belonging to a set $\mathcal{S} \in \mathscr{B}(\mathcal{X})$:

$$\mathbb{P}_{\mathbf{x}}^{\tau,\bar{x}_0}\left\{\mathbf{x}[\tau] \in \mathcal{S}\right\} = \int_{\mathcal{S}} \psi_{\mathbf{x}}(\bar{z};\tau,\bar{x}_0)d\bar{z}$$



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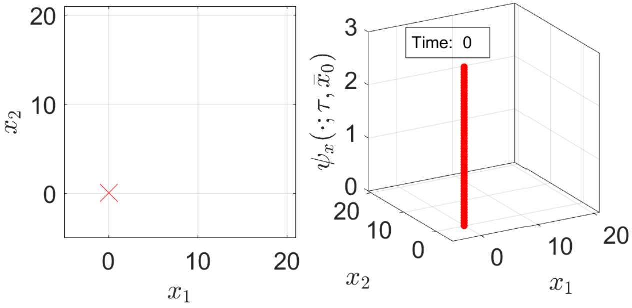


Probability theory and Fourier transforms

- Characteristic function (CF) of p-dimensional $\mathbf{w}$ with pdf $\psi_{\mathbf{w}}$

$$\begin{aligned}\Psi_{\mathbf{w}}(\bar{\alpha}) &\triangleq \mathbb{E}_{\mathbf{w}}\left[\exp\left(j\bar{\alpha}^\top \mathbf{w}\right)\right] \\ &= \int_{\mathbb{R}^p} e^{j\bar{\alpha}^\top \bar{z}}\psi_{\mathbf{w}}(\bar{z})d\bar{z} \\ &= \mathscr{F}\left\{\psi_{\mathbf{w}}(\cdot)\right\}(-\bar{\alpha})\end{aligned}$$

- Given a CF $\Psi_{\mathbf{w}}(\bar{\alpha})$, the pdf $\psi_{\mathbf{w}}$

$$\begin{aligned}\psi_{\mathbf{w}}(\bar{z}) &= \left(\frac{1}{2\pi}\right)^p \int_{\mathbb{R}^p} e^{-j\bar{\alpha}^\top \bar{z}}\Psi_{\mathbf{w}}(\bar{\alpha})d\bar{\alpha} \\ &= \mathscr{F}^{-1}\left\{\Psi_{\mathbf{w}}(\cdot)\right\}(-\bar{z})\end{aligned}$$

- Useful properties:
  - addition of random vectors $\rightarrow$ convolution in pdf
  - affine transformation $\rightarrow$ analytical expression for CF
  - concatenate independent random vectors $\rightarrow$ product in pdf and CF

Cramer, Princeton Univ. Press 1961
Stein and Weiss, Princeton Univ. Press 1971

Problem statement 1 — forward stochastic reachability

Construct analytical expressions for

1. $\psi_{\mathbf{x}}(\cdot;\tau,\bar{x}_0)$, the forward stochastic reach probability density, and
2. $\overline{\mathrm{FSReach}}(\tau,\bar{x}_0)$, an overapproximation of the forward stochastic reach set

via an iterative and a non-iterative approach

$\mathrm{FSReach}(\tau,\bar{x}_0) \rightarrow$ *support of the density*
Iterative approach $\rightarrow$ *Using convolution*
Non-iterative approach $\rightarrow$ *Using Fourier transform (characteristic functions)*

Problem statement 2 — stochastic non-adversarial target capture

For non-adversarial target capture,

1. formulate the maximized probability of capture as a convex optimization problem, and
2. construct the corresponding optimal controller for the pursuer

using log-concavity of $\mathbb{P}_x^{\tau,\bar{x}_0}$

Outline

Convolution-based recursion

- Addition of two random vectors

$$\mathbf{x}[k+1] = A\mathbf{x}[k] + B\mathbf{w}[k]$$

- Convolution in pdfs

$$\psi_{\mathbf{x}}(\bar{y}; k+1, \bar{x}_0) = \big(\psi_{A\mathbf{x}}(\cdot; k, \bar{x}_0) * \psi_{B\mathbf{w}}\big)(\bar{y})$$

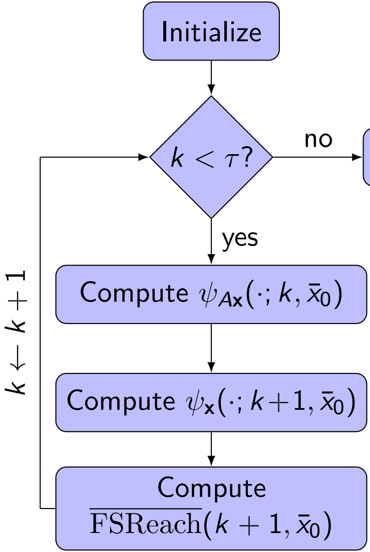
- Overapproximation of $\mathrm{FSReach}(k+1, \bar{x}_0)$

$$\overline{\mathrm{FSReach}}(k+1, \bar{x}_0) = A(\overline{\mathrm{FSReach}}(k, \bar{x}_0)) \oplus B(\mathrm{supp}(\mathbf{w}))$$

- Requirements: τ n-dimensional integrations, a grid, and $\det A \neq 0$
- Extension to nonlinear systems with discrete-valued disturbance

HomChaudhuri, Vinod, and Oishi, ACC 2017
Lasota and Mackey, Springer 1994

Convolution-based recursion



$$k = 0$$

$$\psi_{\mathbf{x}}(\bar{y}; 0, \bar{x}_0) = \delta(y - \bar{x}_0)$$

$$\overline{\mathrm{FSReach}}(0, \bar{x}_0) = \{\bar{x}_0\}$$

$$\psi_{A\mathbf{x}}(\bar{z}; k, \bar{x}_0) = (\det A)^{-1} \psi_{\mathbf{x}}(A^{-1}\bar{z}; k, \bar{x}_0) \quad \text{if } \det A \neq 0$$

$$\psi_{\mathbf{x}}(\bar{y}; k+1, \bar{x}_0) = \left(\psi_{A\mathbf{x}}(\cdot; k, \bar{x}_0) * \psi_{B\mathbf{w}}\right)(\bar{y})$$

$$\overline{\mathrm{FSReach}}(k+1, \bar{x}_0) = A(\overline{\mathrm{FSReach}}(k, \bar{x}_0)) \oplus B(\mathrm{supp}(\mathbf{w}))$$

HomChaudhuri, Vinod, and Oishi, ACC 2017

Fourier transform-based approach

- Trajectory of the linear system

$$\mathbf{x}[\tau] = A^\tau \bar{x}_0 + \mathscr{C}_{n\times(\tau p)}\mathbf{W}$$

  - $\mathbf{W} = \left[\mathbf{w}[N-1]^\top \ \ldots \ \mathbf{w}[0]^\top\right]^\top$
  - $\mathscr{C}_{n\times(\tau p)} = [B\ AB\ A^2B\ \ldots\ A^{\tau-1}B] \in \mathbb{R}^{n\times(\tau p)}$

- CF of $\mathbf{x}[\tau]$ and $\overline{\mathrm{FSReach}}(\tau, \bar{x}_0)$

$$\Psi_{\mathbf{x}}(\bar{\alpha}; \tau, \bar{x}_0) = \exp\left(j\bar{\alpha}^\top (A^\tau \bar{x}_0)\right) \Psi_{\mathbf{W}}(\mathscr{C}_{n\times(\tau p)}^\top \bar{\alpha})$$
$$\overline{\mathrm{FSReach}}(\tau, \bar{x}_0) = A^\tau\{\bar{x}_0\} \oplus \mathscr{C}_{n\times(\tau p)}(\mathrm{supp}(\mathbf{w}))^\tau$$

- Pdf via inverse Fourier transform (IFT)

Fourier transform-based approach

Compute $\Psi_{\mathbf{W}}(\cdot)$ — IID $\mathbf{w}[k] \Rightarrow \Psi_{\mathbf{W}} \leftarrow \prod_{k=0}^{\tau} \Psi_{\mathbf{w}}$

$\downarrow$

Compute $\Psi_{\mathbf{x}}(\cdot;\tau,\bar{x}_0)$ — $\Psi_{\mathbf{x}}(\bar{\alpha};\tau,\bar{x}_0) \leftarrow \exp\left(j\bar{\alpha}^\top(A^\tau\bar{x}_0)\right)\Psi_{\mathbf{W}}(\mathscr{C}_{n\times(\tau p)}^\top\bar{\alpha})$

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Compute $\psi_{\mathbf{x}}(\cdot;\tau,\bar{x}_0)$ — $\psi_{\mathbf{x}}(\bar{y};\tau,\bar{x}_0) \leftarrow \mathscr{F}^{-1}\left\{\Psi_{\mathbf{x}}(\bar{\alpha};\tau,\bar{x}_0)\right\}(-\bar{y})$

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Compute $\overline{\mathrm{FSReach}}(\tau,\bar{x}_0)$ — $\overline{\mathrm{FSReach}}(\tau,\bar{x}_0) \leftarrow A^\tau\{\bar{x}_0\} \oplus \mathscr{C}_{n\times(\tau p)}(\mathrm{supp}(\mathbf{w}))^\tau$

At most one n-dimensional integration, no grid, and A can be singular

Applicable only to linear systems

Fourier transform-based approach

Compute $\Psi_{\mathbf{W}}(\cdot)$	IID $\mathbf{w}[k] \Rightarrow \Psi_{\mathbf{W}} \leftarrow \prod_{k=0}^{\tau} \Psi_{\mathbf{w}}$
↓	
Compute $\Psi_{\mathbf{x}}(\cdot;\tau,\bar{x}_0)$	$\Psi_{\mathbf{x}}(\bar{\alpha};\tau,\bar{x}_0) \leftarrow \exp\left(j\bar{\alpha}^\top(A^\tau\bar{x}_0)\right)\Psi_{\mathbf{W}}(\mathscr{C}_{n\times(\tau p)}^\top\bar{\alpha})$
↓	
Compute $\psi_{\mathbf{x}}(\cdot;\tau,\bar{x}_0)$	$\psi_{\mathbf{x}}(\bar{y};\tau,\bar{x}_0) \leftarrow \mathscr{F}^{-1}\left\{\Psi_{\mathbf{x}}(\bar{\alpha};\tau,\bar{x}_0)\right\}(-\bar{y})$ Closed-form available in certain cases
↓	
Compute $\overline{\mathrm{FSReach}}(\tau,\bar{x}_0)$	$\overline{\mathrm{FSReach}}(\tau,\bar{x}_0) \leftarrow A^\tau\{\bar{x}_0\} \oplus \mathscr{C}_{n\times(\tau p)}(\mathrm{supp}(\mathbf{w}))^\tau$

At most one n-dimensional integration, no grid, and A can be singular
Applicable only to linear systems

Convexity results

- Log-concave $\mathbb{P}_{\mathbf{w}}$ iff for all non-empty $\mathcal{S}_1, \mathcal{S}_2 \in \mathscr{B}(\mathcal{W})$, $\forall \theta \in (0,1)$

$$\mathbb{P}_{\mathbf{w}}\{\mathbf{w} \in \theta\mathcal{S}_1 \oplus (1-\theta)\mathcal{S}_2\} \geq \mathbb{P}_{\mathbf{w}}\{\mathbf{w} \in \mathcal{S}_1\}^{\theta}\mathbb{P}_{\mathbf{w}}\{\mathbf{w} \in \mathcal{S}_2\}^{1-\theta}$$

- Examples of log-concave $\mathbb{P}_{\mathbf{w}}$: Gaussian, exponential, and others
- Log-concave $\mathbb{P}_{\mathbf{w}}$ implies
  - $\Rightarrow$ convex $\mathrm{FSReach}(\tau, \bar{x}_0)$
  - $\Rightarrow$ log-concave $\mathbb{P}_{\mathbf{x}}^{\tau,\bar{x}_0}$
  - $\Rightarrow$ log-concave $\psi_{\mathbf{x}}(\cdot;\tau,\bar{x}_0)$ if affine hull of $\mathrm{FSReach}(\tau, \bar{x}_0)$ is $\mathbb{R}^n$

Dharmadhikari and Joag-Dev, Elsevier 1988

Outline

Maximizing capture probability

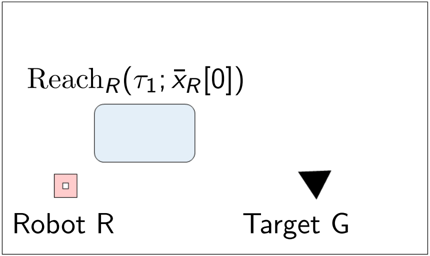
$$\text{Prob1}: \quad \underset{\tau, \bar{x}_R[\tau]}{\text{maximize}} \quad \mathrm{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R[\tau]; \bar{x}_G[0])$$

$$\text{subject to} \quad \begin{cases} \tau \in \mathbb{Z}_{[1,N]} \\ \bar{x}_R[\tau] \in \mathrm{Reach}_R(\tau; \bar{x}_R[0]) \end{cases}$$

$$\mathrm{Reach}_R(\tau; \bar{x}_R[0]) = \left\{ \bar{y} \in \mathcal{X} \middle| \exists \bar{\pi}_\tau \in \overline{\mathcal{M}}_\tau \text{ s.t. } \bar{x}_R[\tau] = \bar{y} \right\}$$

$$\begin{aligned} \mathrm{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R[\tau]; \bar{x}_G[0]) &= \mathbb{P}^{\tau, \bar{x}_G[0]}_{\mathbf{x}_G}\{\mathbf{x}_G[\tau] \in \mathrm{CaptureSet}(\bar{x}_R[\tau])\} \\ &= \int_{\mathrm{CaptureSet}(\bar{x}_R[\tau])} \psi_{\mathbf{x}_G}(\bar{y}; \tau, \bar{x}_G[0]) d\bar{y} \end{aligned}$$

Maximizing capture probability



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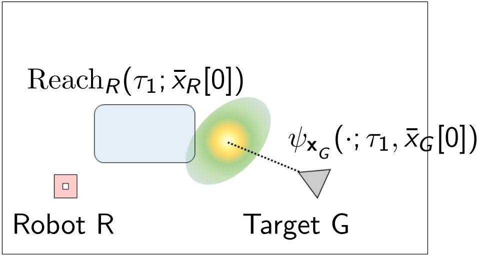

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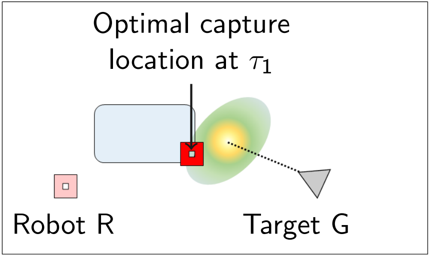
Maximizing capture probability



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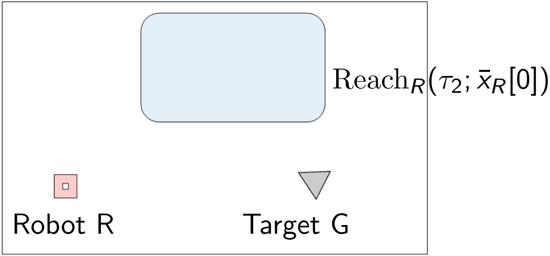

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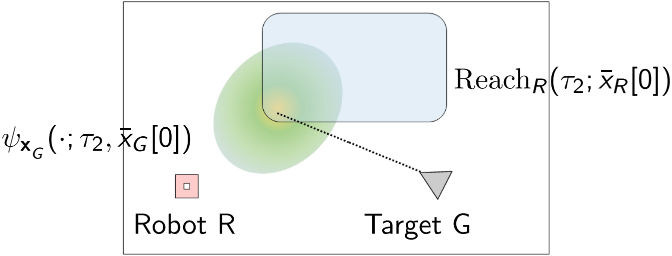

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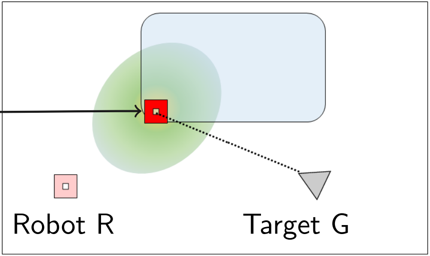
Maximizing capture probability

Optimal capture location at τ_2



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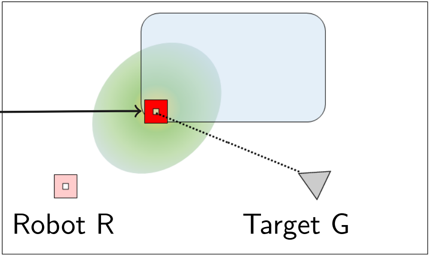
Maximizing capture probability

Optimal capture location at τ_2



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$$\text{subject to} \quad \begin{cases} \tau \in \mathbb{Z}_{[1,N]} \\ \bar{x}_R[\tau] \in \mathrm{Reach}_R(\tau; \bar{x}_R[0]) \end{cases}$$

$$\mathrm{Reach}_R(\tau; \bar{x}_R[0]) = \left\{ \bar{y} \in \mathcal{X} \middle| \exists \bar{\pi}_\tau \in \overline{\mathcal{M}}_\tau \text{ s.t. } \bar{x}_R[\tau] = \bar{y} \right\}$$

$$\begin{aligned} \mathrm{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R[\tau]; \bar{x}_G[0]) &= \int_{\mathbb{R}^n} \psi_{\mathbf{x}_G}(\bar{x}; \tau, \bar{x}_G[0]) 1_{\mathrm{CaptureSet}(\bar{x}_R[\tau])}(\bar{x}) d\bar{x} \\ &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \Psi_{\mathbf{x}_G}(\bar{\gamma}; \tau, \bar{x}_G[0]) \mathscr{F}\left(1_{\mathrm{CaptureSet}(\bar{x}_R[\tau])}(\bar{x})\right) d\bar{\gamma} \end{aligned}$$

Convex formulation

- Log-concave $\mathbb{P}_{\mathbf{x}_G}^{\tau,\bar{x}_G[0]} \Rightarrow$ log-concave $\mathrm{CapturePr}$ in $\bar{x}_R[\tau]$ for each τ
- Solve for each $\tau \in \mathbb{Z}_{[1,N]}$

$$\mathrm{Prob2}: \quad \begin{array}{rl} \underset{\bar{x}_R[\tau]}{\text{maximize}} & \log(\mathrm{CapturePr}_{\bar{x}_R}(\tau, \bar{x}_R[\tau]; \bar{x}_G[0])) \\ \text{subject to} & \bar{x}_R[\tau] \in \mathrm{Reach}_R(\tau; \bar{x}_R[0]) \end{array}$$

  - Prob2 is concave
  - compute τ^* by finite max to solve Prob1

Optimal controller synthesis

- Prob1 returns $(\tau^*, \bar{x}_R^*[\tau^*])$
- Optimal controller synthesis via

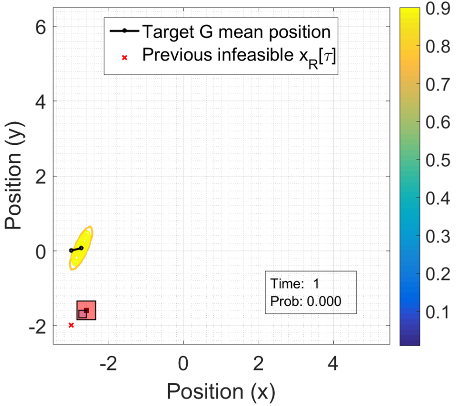
$$\text{Prob3}: \quad \begin{array}{rl} \underset{\bar{\pi}_{\tau^*}}{\text{minimize}} & J_\pi(\bar{\pi}_{\tau^*}) \\ \text{subject to} & \begin{cases} \bar{\pi}_{\tau^*} \in \overline{\mathcal{M}}_{\tau^*} \\ \mathscr{C}_R \bar{\pi}_{\tau^*} = \bar{x}_R^*[\tau^*] - \bar{x}_R[0] \end{cases} \end{array}$$

$$J_\pi(\bar{\pi}_{\tau^*}) = \begin{cases} 0 & \text{feasible controller} \\ \bar{\pi}_{\tau^*}^\top \bar{R} \bar{\pi}_{\tau^*}, \bar{R} \in \mathbb{R}^{(m\tau^*)\times(m\tau^*)} & \text{minimal control effort} \end{cases}$$

Outline

MATLAB codes available at
`http://hscl.unm.edu/software.html`

Gaussian noise perturbed target G



Overall computation time $\sim$ 5.32 s

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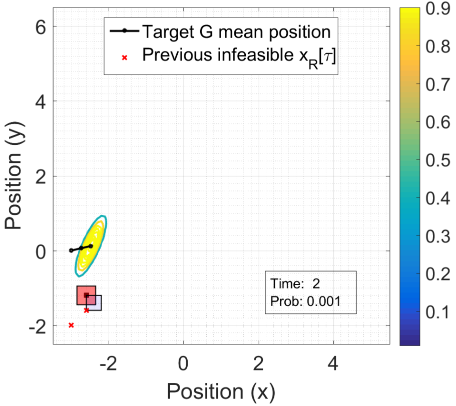

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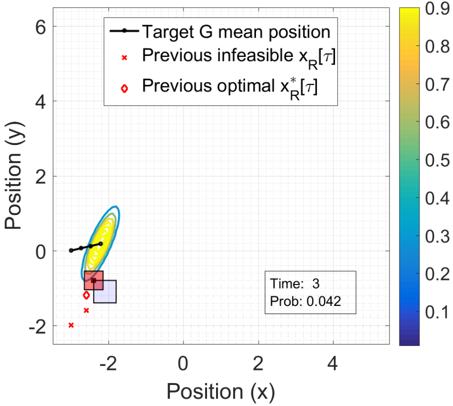

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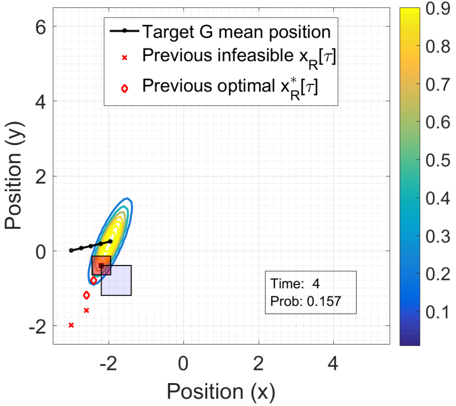

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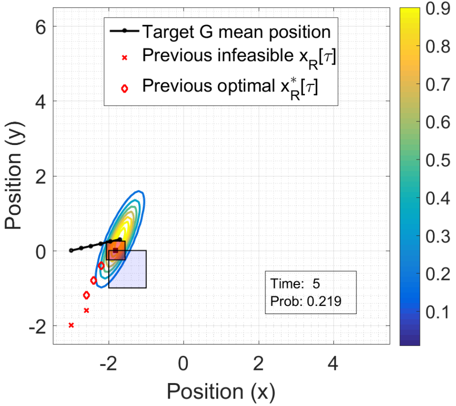

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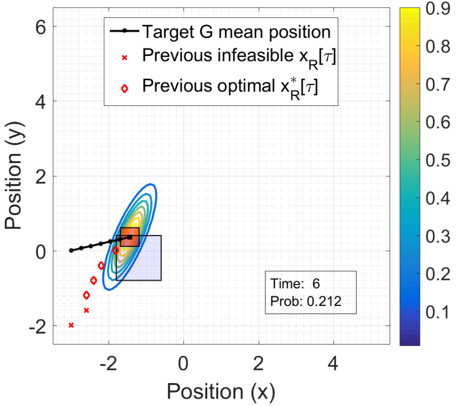

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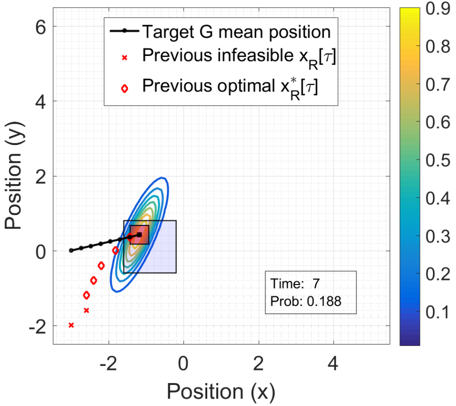

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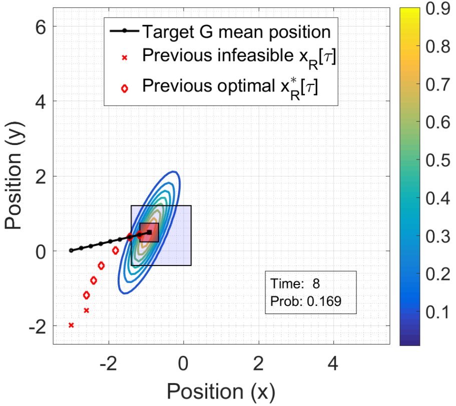

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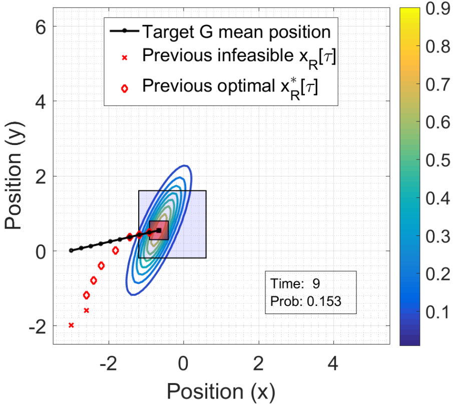

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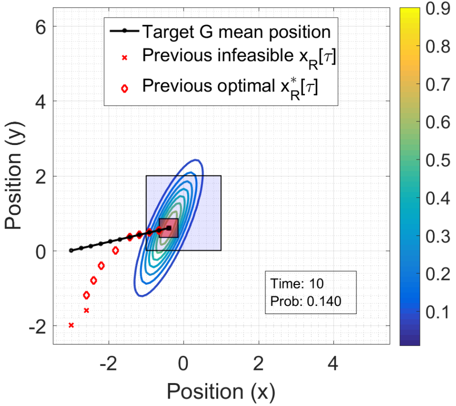

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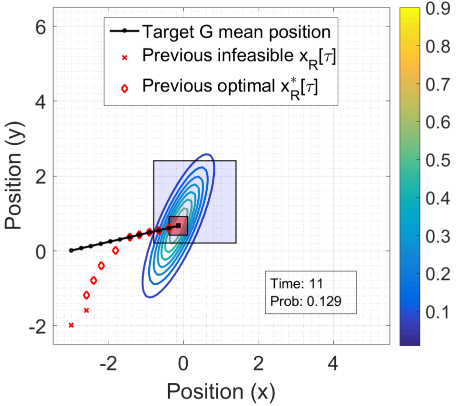

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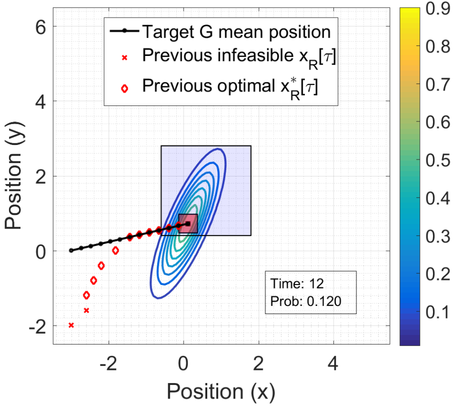

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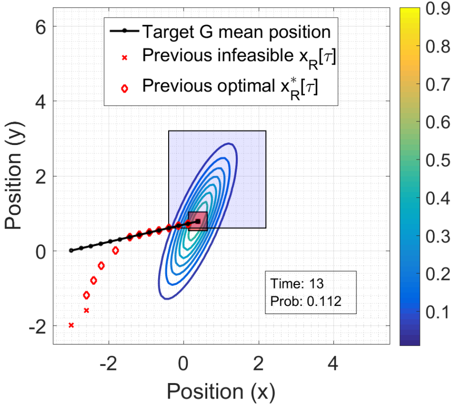

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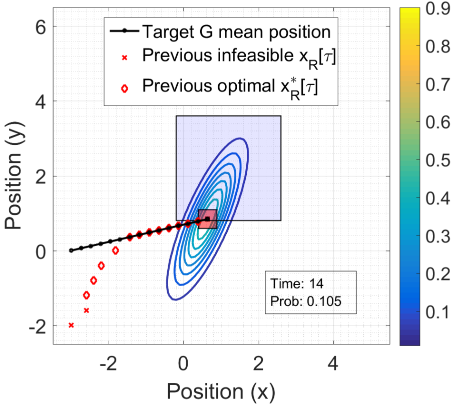

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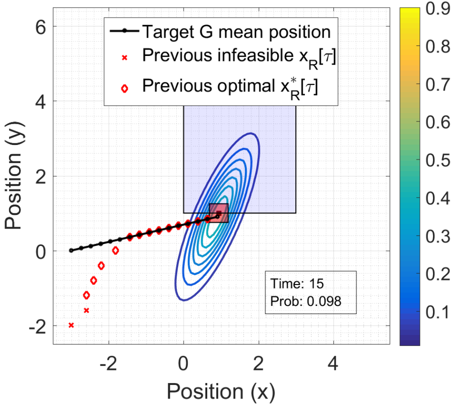

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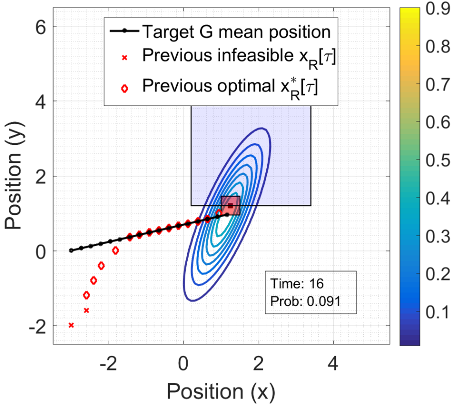

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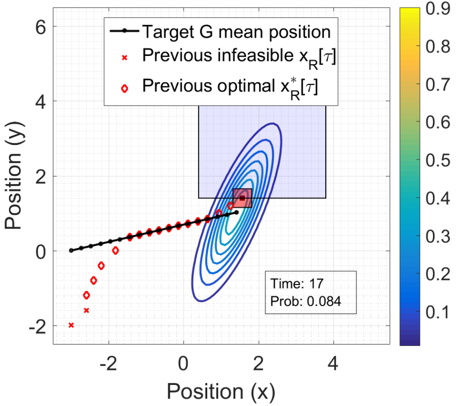

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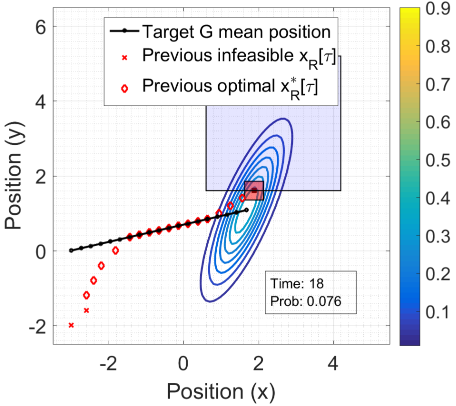

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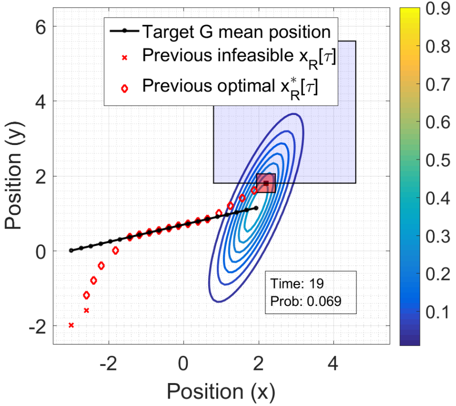

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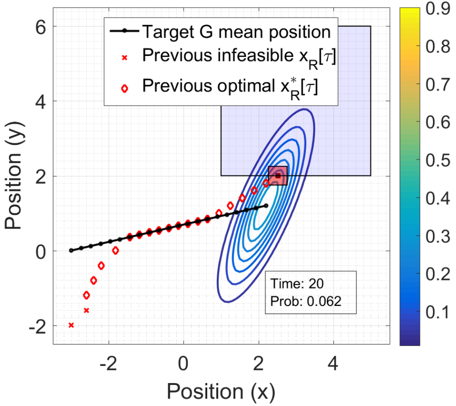

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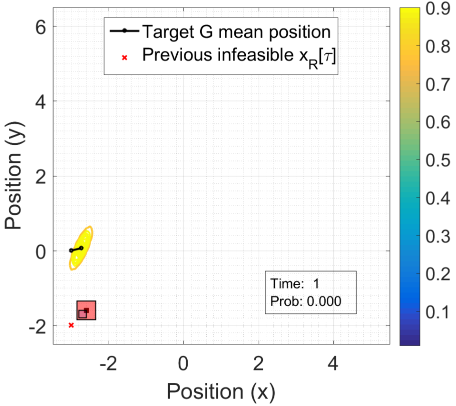

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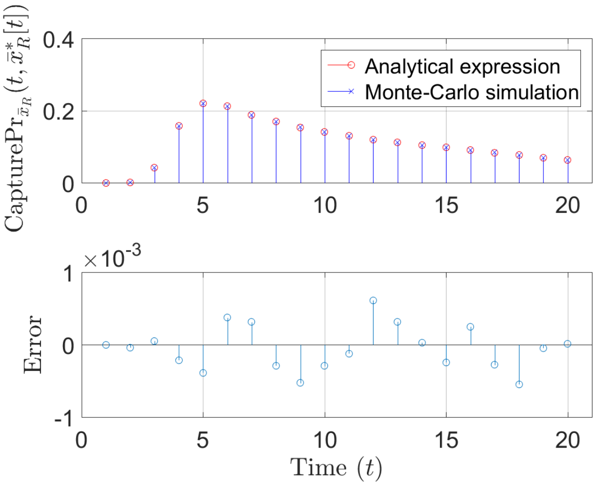
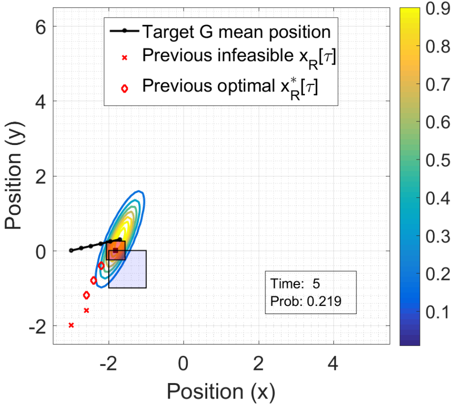


- Validation via Monte-Carlo simulation — $500,000$ particles

Gaussian noise perturbed target G



$\tau^* = 5$

Robot R reaches $\mu_G[\tau]$ only at $\tau = 6$

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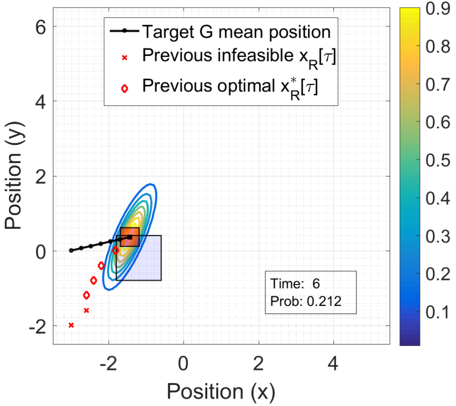

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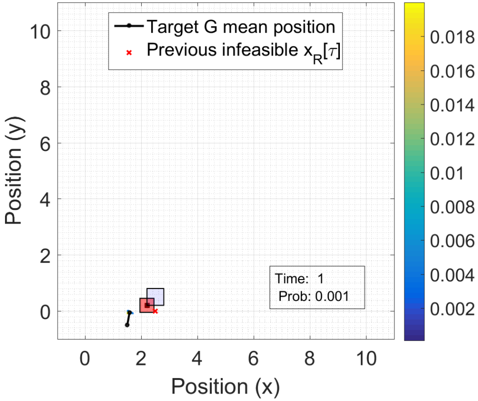

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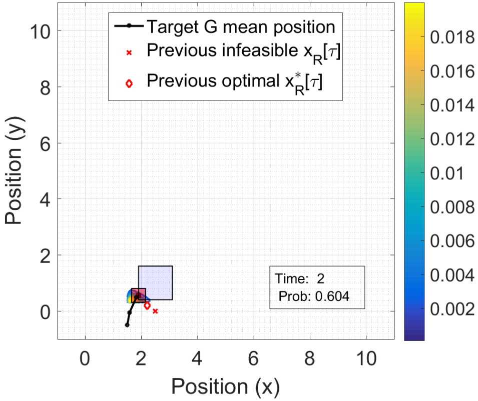

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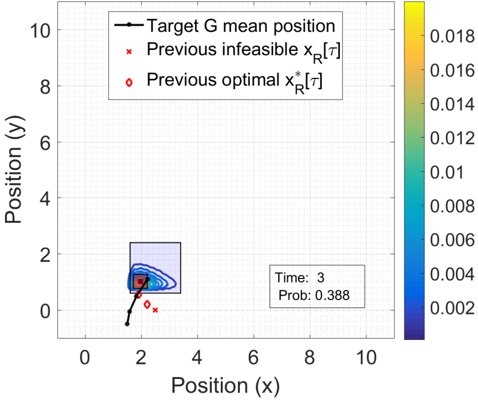

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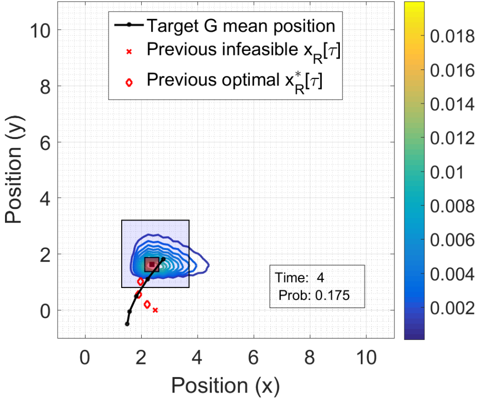

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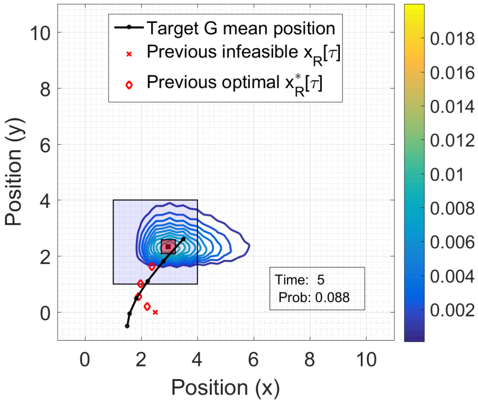

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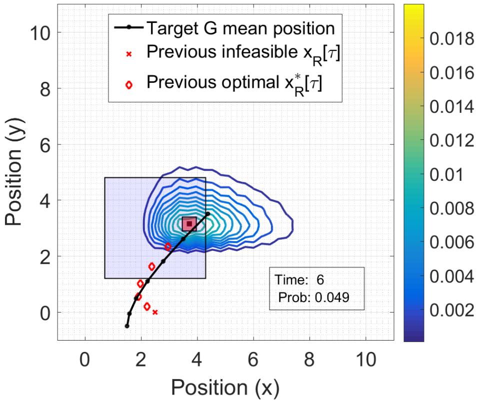

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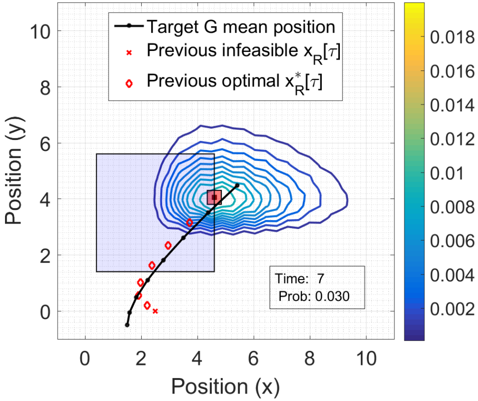

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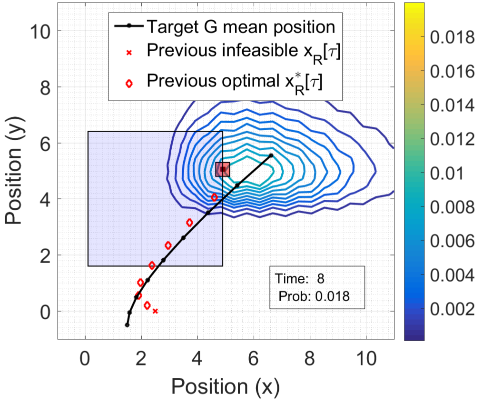

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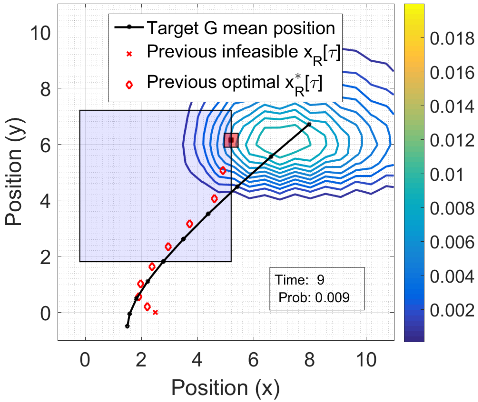

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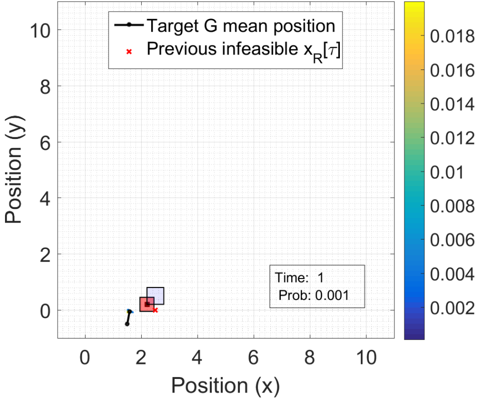

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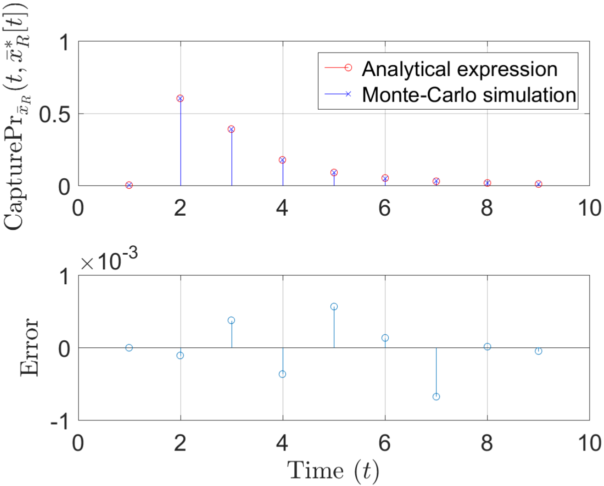

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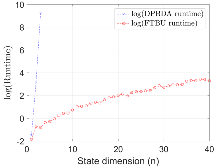
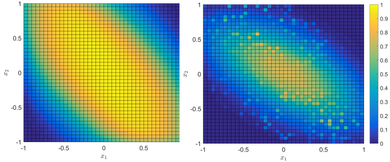
Summary

- Exact and scalable techniques for forward stochastic reachability
  1. Convolution-based recursion
  2. Fourier transform-based approach
- Convexity properties of $\mathrm{FSReach}(\tau, \bar{x}_0), \mathbb{P}_{\mathbf{x}}^{k,\bar{x}_0}$, and $\psi_{\mathbf{x}}(\cdot; k, \bar{x}_0)$
- Convex formulation for the stochastic target capture problem
  - Global optimum for capture location and time
  - Non-conservative solution
  - Prob1 accommodates delayed vs early capture tradeoff
  - Tracking mean not necessarily optimal

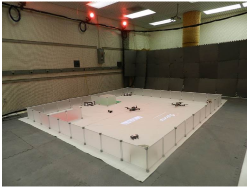
Current and Future Work

- Scalable underapproximation of stochastic reach-avoid probability
- Underapproximation of reach-avoid sets via Lagrangian methods
- Stochastic non-adversarial target capture with multiple pursuers (with Sandia National Laboratories & UNM MARHES Lab, R. Fierro)
- Stochastic switched linear systems

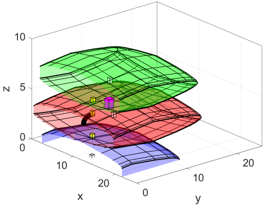


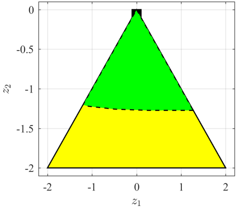


Vinod and Oishi, L-CSS 2017 (submitted)



MARHES Lab (http://marhes.unm.edu)





Gleason, Vinod, and Oishi, CDC 2017 (submitted)

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MATLAB codes available at
`http://hscl.unm.edu/software.html`