Optimal trade-off analysis for efficiency and safety in the spacecraft rendezvous and docking problem

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Abstract: We study the problem of spacecraft rendezvous and docking, specifically the trade-off between safety and efficiency. The safety component in this problem is to stay within a line-of-sight cone and reach the target at an appropriate time in the presence of stochastic disturbance due to sensing and modeling errors. The efficiency component in this problem is our desire to minimize control effort or fuel. We pose this problem as a bi-criterion optimization problem. Using our recent convexity results for the stochastic reach-avoid problem when the control policies are restricted to open-loop policies, we show that this bi-criterion optimization problem is convex. The resulting tractability permits the exploration of the trade-off between safety and efficiency in the stochastic system. We also show that while increasing the control bounds need not always translate to an increase in the attainable safety guarantees, decreasing the control bounds typically translates to a decrease in the attainable safety guarantees.

Keywords: Stochastic reachability, convex optimization, bi-criterion optimization, spacecraft rendezvous and docking problem

1. INTRODUCTION

The safety-critical and expensive nature of space systems makes the need for reliable autonomy paramount. While safety must be assured, despite modeling uncertainties and other disturbance forces, it is generally *only* one of the several factors influencing the design of a controller. For example, in spacecraft rendezvous and docking maneuvers, one may wish to dock safely while minimizing the fuel costs (efficiency). Such maneuvers are crucial in resupplying the space station, collection of data, satellite repair, and other missions. Typically, this class of problems is treated either from the safety perspective or the efficiency perspective, but treatment of both of these important objectives simultaneously is largely lacking in the literature. Model predictive control has provided efficient controllers to perform these maneuvers (Weiss et al. (2012); Park et al. (2011); Gavilan et al. (2012); Hartley et al. (2012); Weiss et al. (2015)). Stochastic reachability has been used to verify the problem, i.e., characterize the initial set of states from which such maneuvers can be performed safely (Lesser et al. (2013); Gleason et al. (2017); Vinod and Oishi (2018)). The safety problem, in this case, is to design controllers that maximize the probability of the docking satellite staying within a predetermined safe region (such as a line-of-sight cone) in the presence of a stochastic disturbance. The goal of this paper is to compute controllers that optimize both the performance and safety of the spacecraft rendezvous and docking problem (a four-dimensional discrete-time stochastic optimal control problem). We achieve this via bi-criterion optimization and exploit our recent results on the convexity of the safety problem (Vinod and Oishi (2017, 2018)).

Reachability analysis is a useful tool for identifying the set of initial states in a dynamical system that can be driven to the desired target set at an appropriate time while avoiding a set of undesired states (the "reach-avoid" problem). For stochastic systems, the stochastic reach-avoid problem was formulated and solved using dynamic programming (Summers and Lygeros (2010) and Abate et al. (2008)). The grid-based dynamic programming implementation of the stochastic reach-avoid problem, proposed in Abate et al. (2007), suffers from the curse of dimensionality and is limited to at most three-dimensional problems. Polynomial optimization-based approximate dynamic programming approaches have been proposed that have shown moderate scalability (Kariotoglou et al. (2013)). However, these approaches typically provide an overapproximation of the stochastic reach-avoid set, resulting in nonconservative solutions. The spacecraft rendezvous and docking problem was approximated using particle filters and convex chance constrained optimization techniques in Lesser et al. (2013). The underlying approximation used in Lesser et al. (2013) was shown to be an underapproximation, and a Fourier transform-based grid-free and recursion-free solution to this underapproximation was proposed (Vinod and Oishi

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(2017)). The Fourier transform-based underapproximation was demonstrated to be faster and less conservative to the particle filter and chance-constrained approaches (Vinod and Oishi (2018)). We rely upon this underapproximation technique in our analysis here. Since the stochastic reach-avoid problem inherently does not optimize for any other cost function apart from safety, the corresponding optimal controller may not be *efficient* with respect to any performance cost other than safety. Therefore, we perform a trade-off analysis between safety and performance in the spacecraft rendezvous and docking problem.

Researchers have recently examined multi-objective optimization with safety as one of the objectives. Lexicographic optimization techniques have been used in performing multi-objective optimization in space applications (Dueri et al. (2016)), and a in more general POMDP framework (Lesser and Abate (2017)). We have analyzed the problem of path-planning (minimizing a trajectory cost) while ensuring probabilistic safety from stochastic obstacles (HomChaudhuri et al. (2017)). In this paper, we pose the problem of guaranteeing both safety (through an underapproximation) and performance as a bi-criterion optimization problem. Building on the convexity results of the Fourier transform-based underapproximation presented in Vinod and Oishi (2018), we show that this problem is convex resulting in a tractable implementation. This analysis quantifies the balance between the desire of lower control effort (performance) and higher probability of constraint satisfaction (safety).

Our main contributions are:

(1) formulate a bi-criterion optimization problem that optimizes simultaneously the performance and the safety of the system (Sec. 3),
(2) provide sufficient conditions for this problem to be convex allowing a tractable implementation (Sec. 3),
(3) construct an optimal trade-off curve between performance and safety (Sec. 4.2), and
(4) investigate the effect of control bounds on the stochastic reach-avoid maneuvers (Sec. 4.3).

We discuss strategies to implement the Fourier transform-based underapproximation in Sec. 3.1. We formulate our problem statement and revisit existing results in the literature in Sec. 2, provide numerical values used in simulations in Sec. 4.1, and provide conclusions and directions for future work in Sec. 5.

2. PROBLEM FORMULATION

We denote a discrete-time time interval by $\mathbb{N}_{[a,b]}$ for $a, b \in \mathbb{N}$ and $a \leq b$, which inclusively enumerates all integers in between (and including) a and b, random variables/vectors with bold case, non-random vectors with an overline, and concatenated random variables/vectors as bold case with overline. We denote the Cartesian product of the set $\mathcal{S}$ with itself $k \in \mathbb{N}$ times as $\mathcal{S}^k$.

2.1 Spacecraft Rendezvous and Docking

We consider two spacecraft in the same circular orbit. One spacecraft, referred to as the deputy, must approach and dock with another spacecraft, referred to as the chief, at a specified time (the control time horizon) while remaining in a line-of-sight cone. The line-of-sight cone is the region where accurate sensing of the deputy is possible. As given in Wiesel (1989), the relative planar dynamics are described by the Clohessy-Wiltshire-Hill (CWH) equations,

$$\ddot{x} - 3\omega x - 2\omega \dot{y} = \frac{F_x}{m_d} \tag{1a}$$

$$\ddot{y} + 2\omega \dot{x} = \frac{F_y}{m_d} \tag{1b}$$

The chief is located at the origin, the position of the deputy is $x, y \in \mathbb{R}$, $\omega = \sqrt{\mu / R_0^3}$ is the orbital frequency, μ is the gravitational constant, and R_0 is the orbital radius of the chief spacecraft. We define the state as $\boldsymbol{x} = [x \ y \ \dot{x} \ \dot{y}]^\top \in \mathbb{R}^4$ and the input as $\overline{u} = [F_x \ F_y]^\top \in \mathcal{U} \subset \mathbb{R}^2$. We discretize the dynamics (1) in time, via zero-order hold, to obtain the discrete-time LTI system and add a Gaussian disturbance to account for the modeling uncertainties and the disturbance forces,

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\overline{u}_k + \boldsymbol{w}_k \tag{2}$$

with $\boldsymbol{w}_k \in \mathbb{R}^4$ as an IID Gaussian zero-mean random process with a known covariance matrix $\Sigma_{\boldsymbol{w}}$. We denote the known initial condition of (2) as $\overline{x}_0 \in \mathbb{R}^4$ and denote the time horizon using $N \in \mathbb{N}$.

2.2 Bi-criterion optimization

A general bi-criterion optimization problem can be stated as

$$\begin{array}{ll} \underset{\pi}{\text{minimize}} \quad (\text{w.r.t. } \mathbb{R}^2_+) & \begin{bmatrix} J_1(\overline{y}) \\ J_2(\overline{y}) \end{bmatrix} \\ \text{subject to} & \overline{y} \in \mathcal{Y}. \end{array} \tag{3}$$

where $\mathcal{Y} \subseteq \mathbb{R}^n$ and $J_i : \mathcal{Y} \rightarrow \mathbb{R}_+$. The objective in (3) is ordered using generalized inequalities defined on $\mathbb{R}^2_+$ (Boyd and Vandenberghe, 2004, Sec. 2.4). As discussed in (Boyd and Vandenberghe, 2004, Sec. 4.7), bi-criterion optimization problems typically do not have an optimal solution since J_i are rarely *non-competing*. In other words, in order to solve (3), compromises have to be made among the objectives. In these cases, the minimal elements of the set of achievable values of the optimization problem are used. Recall that minimal elements (w.r.t $\mathbb{R}^2_+$) of a set means that no other solution of the set lies to the left and below the minimal element of interest (e.g. (2.17) in Boyd and Vandenberghe (2004)). These solutions are known as *Pareto optimal solutions* and the set of Pareto optimal solutions is called the *optimal trade-off curve*. This curve is typically obtained via scalarization of (3). For some $\lambda_1, \lambda_2 \geq 0$, we have

$$\begin{array}{ll} \underset{\pi}{\text{minimize}} & [\lambda_1 \ \lambda_2] \begin{bmatrix} J_1(\overline{y}) \\ J_2(\overline{y}) \end{bmatrix} = \lambda_1 J_1(\overline{y}) + \lambda_2 J_2(\overline{y}) \\ \text{subject to} & \overline{y} \in \mathcal{Y}. \end{array} \tag{4}$$

We obtain the trade-off curve by solving (4) for each pair $\lambda_1, \lambda_2 \geq 0$. Note that λ_1 and λ_2 can not simultaneously be equal to zero. To facilitate analysis, we define a relative weight

$$\lambda = \frac{\lambda_1}{\lambda_2} \geq 0 \tag{5}$$

and simplify (4) as

$$\begin{array}{ll} \underset{\pi}{\text{minimize}} & \lambda J_1(\overline{y}) + J_2(\overline{y}) \\ \text{subject to} & \overline{y} \in \mathcal{Y}. \end{array} \tag{6}$$

We solve (6) for every $\lambda \geq 0$ to obtain the optimal trade-off curve. The optimal trade-off curve consists of the optimal solutions to (3) under varying degree of compromises, modeled via λ, between the objectives.

2.3 Probabilistic safety via stochastic reach-avoid problem

Define a *Markov policy* $\pi = (\mu_0, \mu_1, \ldots, \mu_{N-1}) \in \mathcal{M}$ as a sequence of universally measurable state-feedback laws, $\mu : \mathcal{X} \to \mathcal{U}$. The random vector $\boldsymbol{X} = [\boldsymbol{x}_1^\top\ \boldsymbol{x}_2^\top\ \ldots\ \boldsymbol{x}_N^\top]^\top$ generated from $\overline{x}_0$ under the action of π has a probability measure $\mathbb{P}_{\boldsymbol{X}}^{\overline{x}_0,\pi}$ as described in Summers and Lygeros (2010) and Vinod and Oishi (2017). Further, define the *terminal time probability*, $\hat{r}_{\overline{x}_0}^{\pi}(\mathcal{S}, \mathcal{T})$, for known $\overline{x}_0$ and π, as the probability that the execution with policy π is inside the target set $\mathcal{T}$ at time N and stays within the safe set $\mathcal{S}$ for all time up to N. From Summers and Lygeros (2010),

$$\hat{r}_{\overline{x}_0}^{\pi}(\mathcal{S}, \mathcal{T}) = \mathbb{P}_{\boldsymbol{X}}^{\overline{x}_0,\pi} \left\{\forall k \in \mathbb{N}_{[0,N-1]}\ \boldsymbol{x}_k \in \mathcal{S} \wedge \boldsymbol{x}_N \in \mathcal{T}\right\}. \tag{7}$$

From (Summers and Lygeros, 2010, Def. 10), a Markov policy π^* is a *maximal reach-avoid policy in the terminal sense* if and only if it is the optimal solution of (8),

$$\hat{r}_{\overline{x}_0}^{\pi^*}(\mathcal{S}, \mathcal{T}) = \sup_{\pi \in \mathcal{M}} \hat{r}_{\overline{x}_0}^{\pi}(\mathcal{S}, \mathcal{T}). \tag{8}$$

A dynamic programming-based solution of (8) is given in (Summers and Lygeros, 2010, Thm. 11). However, implementation of a grid-based dynamic programming solution suffers from the curse of dimensionality, preventing its use in this problem.

2.4 Fourier transform-based underapproximation of the stochastic reach-avoid problem

We next propose a grid-free and recursion-free scalable underapproximation to the optimal value function of (8). Define an open-loop policy as $\rho : \mathcal{X} \to \mathcal{U}^N$ that maps an initial condition $\overline{x}_0 \in \mathcal{X}$ to a input vector concatenated over time, such that $\rho(\overline{x}_0) = [\overline{u}_0^\top\ \overline{u}_1^\top\ \ldots\ \overline{u}_{N-1}^\top]^\top \in \mathcal{U}^N$ for each $\overline{x}_0 \in \mathcal{X}$. Given $\overline{x}_0$ and $\boldsymbol{W} = [\boldsymbol{w}_0^\top\ \boldsymbol{w}_1^\top\ \ldots\ \boldsymbol{w}_{N-1}^\top]^\top \in \mathcal{W}^N$, we obtain

$$\boldsymbol{X} = \overline{A}\overline{x}_0 + \overline{H}\rho(\overline{x}_0) + \overline{G}\boldsymbol{W}. \tag{9}$$

The matrices $\overline{A}, \overline{H}, \overline{G}$ are given by specific combinations of the matrices A and B (see (Skaf and Boyd, 2010, Sec. 2)). Here, $\boldsymbol{X}$ is defined using a known ρ and $\overline{x}_0$ has a probability measure $\mathbb{P}_{\boldsymbol{X}}^{\overline{x}_0,\rho}$.

In Vinod and Oishi (2017), an underapproximation for (8) for a given $\overline{x}_0 \in \mathcal{X}$ was proposed as

$$\hat{r}_{\overline{x}_0}^{\rho^*}(\mathcal{S}, \mathcal{T}) = \sup_{\rho(\overline{x}_0) \in \mathcal{U}^N} \hat{r}_{\overline{x}_0}^{\rho}(\mathcal{S}, \mathcal{T}) \tag{10}$$

with decision variable $\rho(\overline{x}_0)$, and

$$\hat{r}_{\overline{x}_0}^{\rho}(\mathcal{S}, \mathcal{T}) = \mathbb{P}_{\boldsymbol{X}}^{\overline{x}_0,\rho} \left\{\forall k \in \mathbb{N}_{[0,N-1]}\ \boldsymbol{x}_k \in \mathcal{S} \wedge \boldsymbol{x}_N \in \mathcal{T}\right\}. \tag{11}$$

The problem (10) can be equivalently written as

$$\begin{array}{ll} \underset{\rho(\overline{x}_0)}{\text{maximize}} & \displaystyle\int_{\mathcal{S}^{N-1} \times \mathcal{T}} \psi_{\boldsymbol{X}}(\overline{Z}; \overline{x}_0, \rho) d\overline{Z} \\ \text{subject to} & \rho(\overline{x}_0) \in \mathcal{U}^N \end{array} \tag{12}$$

where $\psi_{\boldsymbol{X}}$ is the probability density function of $\boldsymbol{X}$. The probability density function $\psi_{\boldsymbol{X}}$ may be obtained using Fourier transforms as described in Vinod et al. (2017). For a Gaussian disturbance $\boldsymbol{w}_k \sim \mathcal{N}(\overline{0}, \Sigma_{\boldsymbol{w}})$,

$$\boldsymbol{X} \sim \mathcal{N}(\overline{m}_{\boldsymbol{X}}, \Sigma_{\boldsymbol{X}}) \tag{13a}$$

$$\overline{m}_{\boldsymbol{X}} = \overline{A}\overline{x}_0 + \overline{H}\rho(\overline{x}_0) \tag{13b}$$

$$\Sigma_{\boldsymbol{X}} = \overline{G}(I_N \otimes \Sigma_{\boldsymbol{w}})\overline{G}^\top. \tag{13c}$$

For polytopic $\mathcal{S}$ and $\mathcal{T}$, the objective of (12) thus simplifies to an integral of a Gaussian distribution over a polytope.

Lemma 1. (Vinod et al., 2017, Thm. 3) Given $\overline{x}_0 \in \mathcal{X}$, if $\psi_{\boldsymbol{w}}$ is a log-concave and continuous density, the sets $\mathcal{X}$, $\mathcal{S}$, and $\mathcal{T}$ are Borel and convex, and the set $\mathcal{U}$ is compact and convex, then (12) is a log-concave optimization problem.

Note that the underapproximation given in (12) is conservative. In other words, given an optimal solution ρ^* of (12), there exists a Markov controller that provides an equal or better probabilistic safety guarantee than ρ^*.

2.5 Problem statement

We apply these techniques to the problem of spacecraft rendezvous and docking. Hence we consider the relative dynamics of two spacecraft as in (2) in the following.

Problem 2. Formulate a tractable bi-criterion convex optimization problem to simultaneously maximize the terminal time probability and minimize the control effort for the spacecraft rendezvous and docking problem.

Problem 3. Compute the optimal trade-off curve for Problem 2.

Problem 4. Characterize the influence of $\overline{u}_{\mathrm{bound}}$ on Problem 2 when $\mathcal{U} = [-\overline{u}_{\mathrm{bound}}, \overline{u}_{\mathrm{bound}}]^2$.

3. BI-CRITERION STOCHASTIC OPTIMAL CONTROL

The bi-criterion stochastic optimal control problem is

$$\begin{array}{lll} \underset{\pi}{\text{minimize}} & (\text{w.r.t. } \mathbb{R}_+^2) & \begin{bmatrix} J_{\overline{x}_0}(\pi) \\ -\log\left(\hat{r}_{\overline{x}_0}^{\pi}(\mathcal{S}, \mathcal{T})\right) \end{bmatrix} \\ \text{subject to} & & \pi \in \mathcal{M} \end{array} \tag{14}$$

where $J_{\overline{x}_0}(\pi) : \mathcal{M} \to \mathbb{R}_+$ is the fuel cost (minimize for efficiency) associated with a Markov policy π. Note that $-\log\left(\hat{r}_{\overline{x}_0}^{\pi}(\mathcal{S}, \mathcal{T})\right)$ ensures that the bi-criterion objective in (14) may be compared using generalized inequalities defined on $\mathbb{R}_+^2$, while still allowing for maximization of the terminal time probability.

Due to the computational challenge in computing $\hat{r}_{\overline{x}_0}^{\pi}(\mathcal{S}, \mathcal{T})$, we only consider open-loop policies, and use a convex function $L_{\overline{x}_0}(\rho) : \mathcal{U}^N \to \mathbb{R}_+$ for the fuel cost,

$$\begin{array}{lll} \underset{\rho(\overline{x}_0)}{\text{minimize}} & (\text{w.r.t. } \mathbb{R}_+^2) & \begin{bmatrix} L_{\overline{x}_0}(\rho) \\ -\log\left(\hat{r}_{\overline{x}_0}^{\rho}(\mathcal{S}, \mathcal{T})\right) \end{bmatrix}. \\ \text{subject to} & & \rho(\overline{x}_0) \in \mathcal{U}^N \end{array} \tag{15}$$

Theorem 5. Given $\overline{x}_0 \in \mathcal{X}$, if $\psi_{\boldsymbol{w}}$ is a log-concave and continuous density, the sets $\mathcal{X}$, $\mathcal{S}$, and $\mathcal{T}$ are Borel and convex, the set $\mathcal{U}$ is compact and convex, and the function $L_{\overline{x}_0}(\rho)$ is convex over $\mathcal{U}^N$, then (15) is a convex optimization problem.

Proof: A bi-criterion optimization problem is convex if each of its objectives are convex and the constraints are convex (see Sec. 4.7.5 of Boyd and Vandenberghe (2004)). Lemma 1 and convexity of $L_{\overline{x}_0}(\rho)$ completes the proof. ■

Theorem 5 solves Problem 2. On scalarization of (15), we obtain using (5)

$$\begin{array}{ll}\underset{\rho(\overline{x}_0)}{\text{minimize}} & \lambda L_{\overline{x}_0}(\rho) - \log(\hat{r}^{\rho}_{\overline{x}_0}(\mathcal{S},\mathcal{T})) \\ \text{subject to} & \rho(\overline{x}_0) \in \mathcal{U}^N\end{array}. \qquad (16)$$

For each $\lambda \geq 0$, (16) is a convex optimization problem. In this paper, we consider the fuel cost to be

$$L_{\overline{x}_0}(\rho) = \|\rho(\overline{x}_0)\|_2 \qquad (17)$$

which is a convex function (Boyd and Vandenberghe, 2004, Sec 3.1.5).

Note that construction of the optimal trade-off curve is highly parallelizable due to the recursion-free property of the Fourier-transform approach and the structure in (16). The optimal trade-off curve can be made arbitrarily accurate by evaluating (16) for more values of λ. The curve provides several insights on the trade-offs present in bi-criterion optimization, as discussed in Sec. 4.7.5 of Boyd and Vandenberghe (2004). We explore some of these insights in Sec. 4.

3.1 Implementation

Despite establishing that (16) is a convex optimization problem, two major challenges arise:

(1) Unavailability of a closed-form expression for $\hat{r}^{\rho}_{\overline{x}_0}(\mathcal{S},\mathcal{T})$, and
(2) Dependence of $\hat{r}^{\rho}_{\overline{x}_0}(\mathcal{S},\mathcal{T})$ on the tail of the distribution $\psi_{\boldsymbol{X}}$ for large regions of $\mathcal{U}^N$.

The first challenge arises from the fact that $\hat{r}^{\rho}_{\overline{x}_0}(\mathcal{S},\mathcal{T})$ is an integral of a Gaussian over a polytope for which no closed-form expression is available. We mitigate this challenge by using Genz's algorithm (see Genz (2017)) to compute an approximation of $\hat{r}^{\rho}_{\overline{x}_0}(\mathcal{S},\mathcal{T})$. Genz's algorithm uses quasi-Monte-Carlo simulations and Cholesky decomposition to evaluate this integral (see Genz (1992)). Since Genz's algorithm provides a noisy evaluation of $\hat{r}^{\rho}_{\overline{x}_0}(\mathcal{S},\mathcal{T})$, we use MATLAB's *patternsearch* solver instead of *fmincon* as suggested in Vinod and Oishi (2017).

The second challenge arises from the fact that if $\rho(\overline{x}_0)$ does not drive the system towards satisfying the reach-avoid constraints, then $\hat{r}^{\rho}_{\overline{x}_0}(\mathcal{S},\mathcal{T})$ will depend heavily on the tail of the distribution $\psi_{\boldsymbol{X}}$. Recall that Monte-Carlo simulations do not model distribution tails well. Further, $\hat{r}^{\rho}_{\overline{x}_0}(\mathcal{S},\mathcal{T})$ will be close to zero for large regions of $\mathcal{U}^N$ causing convergence issues for the solver. We partially mitigate this problem by appropriately initializing the solver when solving (16). We solve the following optimization problem to obtain the initialization for (16),

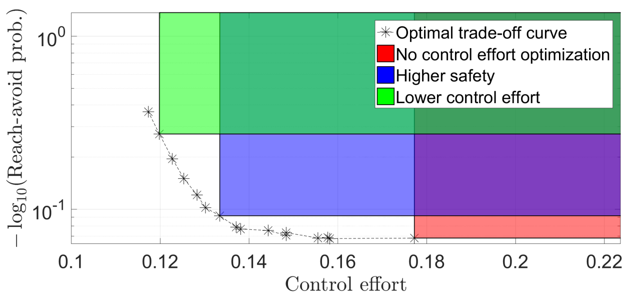

Fig. 1. Pareto optimal curve for $\overline{x}_0 = [0.75\ \ -0.75\ 0\ 0]^\top$ with the corresponding sets of achievable bi-criterion objective values compared to which they are minimal.

$$\begin{array}{ll}\underset{\rho(\overline{x}_0)}{\text{minimize}} & \|\rho(\overline{x}_0)\|_2^2 \\ \text{subject to} & \left\{\begin{array}{l}\overline{m}_{\boldsymbol{X}} \in \mathcal{S}^{N-1}\times\mathcal{T} \\ \rho(\overline{x}_0) \in \mathcal{U}^N\end{array}\right.\end{array}. \qquad (18)$$

using $\overline{m}_{\boldsymbol{X}}$ defined in (13b). Problem (18) optimizes for a safe mean trajectory while minimizing the fuel cost. A safe mean trajectory would typically result in higher values for the safety probability $\hat{r}^{\rho}_{\overline{x}_0}(\mathcal{S},\mathcal{T})$. Empirically, we found this approach to serve as a good initial guess for *patternsearch*. For polytopic $\mathcal{S}$ and $\mathcal{T}$, (18) is a quadratic program. One may consider relaxations of (18) when it is infeasible.

4. TRADE-OFF ANALYSIS

In this section, we compute the Pareto optimal curve for the problem (15). We first discuss the numerical values used in the simulation and then perform trade-off analysis.

4.1 Numerical values

We consider a diagonal disturbance covariance matrix $\Sigma_{\boldsymbol{w}} = 10^{-4}\times\mathrm{diag}(1,1,5\times10^{-4},5\times10^{-4})$, and time horizon of $N=5$. We define the target and safe sets as

$$\begin{aligned}\mathcal{T} = \{z \in \mathbb{R}^4 : &|z_1| \leq 0.1, -0.1 \leq z_2 \leq 0, \\ &|z_3| \leq 0.01, |z_4| \leq 0.01\} \qquad (19)\end{aligned}$$

$$\mathcal{S} = \{z \in \mathbb{R}^4 : |z_1| \leq z_2, |z_3| \leq 0.05, |z_4| \leq 0.05\}. \qquad (20)$$

We analyze the initial condition $\overline{x}_0 = [0.75\ \ -0.75\ 0\ 0]^\top$. Note that $\overline{x}_0$ lies on the boundary of $\mathcal{S}$ implying that the problem of ensuring safety is non-trivial (see Figure 3).

All computations were performed using MATLAB on an Intel core i7-4600 processor with 2.1GHz clock rate and 8 GB RAM. The codes used for this paper are available at `https://github.com/unm-hscl/abyvinod-NAASS2018`.

4.2 Trade-off analysis for a specific $\overline{x}_0 \in \mathcal{S}$

We define the input space as $\mathcal{U} = [-0.1, 0.1]^2$. Figure 1 shows the Pareto optimal curve corresponding to the optimization problem (3), and Figure 2 shows the trade-off between performance and safety. In Figure 2, we see that the maximum terminal time probability that may be attained for the given $\overline{x}_0$ and $\mathcal{U}$ is 0.85. We do not plot the Pareto optimal solution corresponding to $\lambda = 0$, since for $\|\rho^*(\overline{x}_0)\|_2 = 0$, the corresponding $\hat{r}^{\rho^*}_{\overline{x}_0}(\mathcal{S},\mathcal{T})$ is insignificant, as seen in Figure 3.

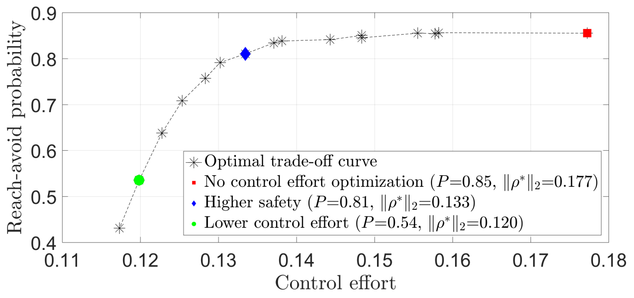

Fig. 2. Trade-off curve between safety and performance for $\overline{x}_0 = [0.75 \ \ -0.75 \ 0 \ 0]^\top$.

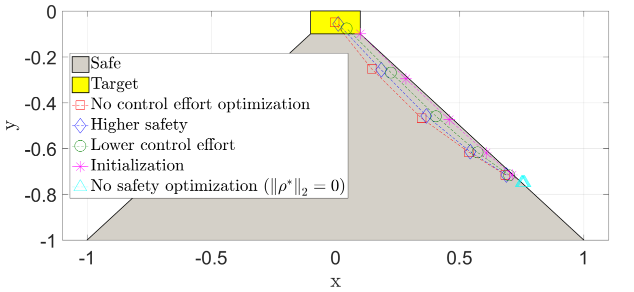

Fig. 3. Pareto optimal trajectories from $\overline{x}_0 = [0.75 \ \ -0.75 \ 0 \ 0]^\top$ for time horizon $N = 5$.

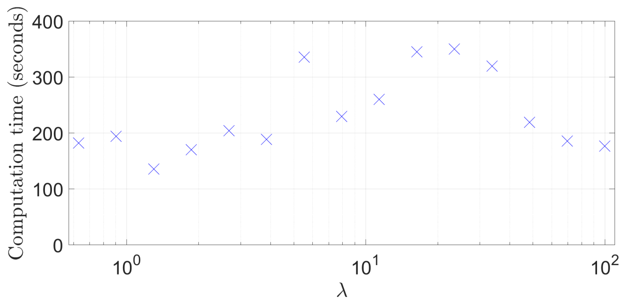

Fig. 4. Computation time for various values of λ (see (5)). The computation time was 44.92 seconds for $\lambda_1 = 0$ ($\lambda = 0$) and 0.03 seconds for $\lambda_2 = 0$ ($\lambda = \infty$).

The shaded regions in Figure 1 are the sets corresponding to the positive cone $\mathbb{R}^2_+$ shifted to the Pareto optimal solutions. These regions cover the achievable bi-criterion objective values that are sub-optimal compared to the respective Pareto optimal solutions. We see that a slight decrease in the desired terminal time probability guarantee provides a compromise that is optimal with respect to a significantly larger portion of the achievable bi-criterion objective values. The fuel cost may be reduced even further if the desired terminal time probability can be further relaxed. Figure 3 shows the trajectories for the optimal solutions corresponding to the compromises, a solution that preferentially weights safety, a solution that preferentially weights efficiency, and the solution to (18). Figure 4 shows the computational time taken for evaluating (16) at different λ values. The total computation time for 17 evaluations of (16) was approximately 59 minutes.

This analysis solves Problem 3 and may be done for any other $\overline{x}_0 \in \mathcal{S}$ as well. Note that the safety probability obtained in this analysis is an underapproximation and may be further improved upon using a Markov policy.

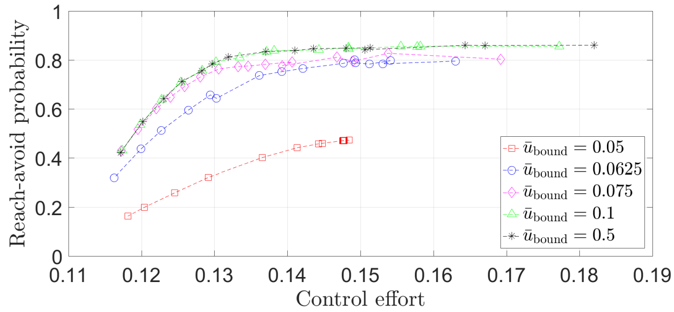

Fig. 5. Influence of bounded control on safety

4.3 Influence of the bounded control on safety

We define the input space as $\mathcal{U} = [-\overline{u}_{\text{bound}}, \overline{u}_{\text{bound}}]^2$ and consider $\overline{u}_{\text{bound}} \in \{0.05, 0.075, 0.1, 0.5\}$. Figure 5 shows the Pareto optimal curve for varying $\overline{u}_{\text{bound}}$ at $\overline{x}_0 = [0.75 \ \ -0.75 \ 0 \ 0]^\top$.

The Pareto optimal curves for $\overline{u}_{\text{bound}} = 0.1$ and $\overline{u}_{\text{bound}} = 0.5$ are very similar, indicating that moderate increases in control bounds from $\overline{u}_{\text{bound}} = 0.1$ will not translate to an increase in safety. On the other hand, even a small decrease in the control bounds from $\overline{u}_{\text{bound}} = 0.1$ show a significant decrease in the achievable terminal time probability. This characterizes the effect of control saturation in the given reach-avoid problem. From Figure 5, we need $\overline{u}_{\text{bound}} \geq 0.75$ to obtain a probabilistic safety guarantee of 0.8. We thus obtain empirical lower bounds on $\overline{u}_{\text{bound}}$ to ensure a desired probabilistic safety guarantee. This solves Problem 4.

5. CONCLUSION

In this paper, we used bi-criterion optimization and stochastic reachability to analyze the trade-off between safety and performance in a stochastic system, namely the spacecraft rendezvous and docking problem. Solving a stochastic reach-avoid problem typically yields controllers that incur large control costs since the control design does not optimize for control effort. We discuss an approach to compute optimal trade-off curves that are conservative with respect to safety, and help minimize control effort while maximizing the terminal time probability. We also show the effect of bounded control authority on the given reach-avoid problem.

REFERENCES

Abate, A., Amin, S., Prandini, M., Lygeros, J., and Sastry, S. (2007). Computational approaches to reachability analysis of stochastic hybrid systems. In *Proc. Hybrid Syst.: Comput. and Ctrl.*, 4–17.

Abate, A., Prandini, M., Lygeros, J., and Sastry, S. (2008). Probabilistic reachability and safety for controlled discrete time stochastic hybrid systems. *Automatica*, 44(11), 2724–2734.

Boyd, S. and Vandenberghe, L. (2004). *Convex optimization*. Cambridge Univ. Press.

Dueri, D., Leve, F.A., and Açikmese, B. (2016). Minimum error dissipative power reduction control allocation via lexicographic convex optimization for momentum control systems. *IEEE Trans. Contr. Sys. Techn.*, 24(2), 678–686.

Gavilan, F., Vazquez, R., and Camacho, E.F. (2012). Chance-constrained model predictive control for spacecraft rendezvous with disturbance estimation. In *Control Engineering Practice*.

Genz, A. (1992). Numerical computation of multivariate normal probabilities. *J. of Comp. and Graph. Stat.*, 1(2), 141–149.

Genz, A. (2017). QSCMVNV. URL `http://www.math.wsu.edu/faculty/genz/software/matlab/qscmvnv.m`.

Gleason, J., Vinod, A., and Oishi, M. (2017). Underapproximation of reach-avoid sets for discrete-time stochastic systems via Lagrangian methods. In *IEEE Conf. Dec. Ctrl.* (accepted). [Online]. Available: `https://arxiv.org/abs/1704.03555`.

Hartley, E.N., Trodden, P.A., Richards, A.G., and Maciejowski, J.M. (2012). Model predictive control system design and implementation for spacecraft rendezvous. In *Control Engineering Practice*.

HomChaudhuri, B., Vinod, A.P., and Oishi, M. (2017). Computation of forward stochastic reach sets: Application to stochastic, dynamic obstacle avoidance. In *Proc. American Ctrl. Conf.* Seattle, WA.

Kariotoglou, N., Summers, S., Summers, T., Kamgarpour, M., and Lygeros, J. (2013). Approximate dynamic programming for stochastic reachability. In *Proc. European Ctrl. Conf.*, 584–589.

Lesser, K. and Abate, A. (2017). Multiobjective optimal control with safety as a priority. *IEEE Trans. Control Syst. Technol.*, PP(99), 1–13.

Lesser, K., Oishi, M., and Erwin, R. (2013). Stochastic reachability for control of spacecraft relative motion. In *IEEE Conf. Dec. Ctrl.*, 4705–4712.

Park, H., Di Cairano, S., and Kolmanovsky, I. (2011). Model predictive control for spacecraft rendezvous and docking with a rotating/tumbling platform and for debris avoidance. In *Proc. American Ctrl. Conf.*

Skaf, J. and Boyd, S. (2010). Design of affine controllers via convex optimization. *IEEE Trans. Auto. Ctr.*, 55(11), 2476–87.

Summers, S. and Lygeros, J. (2010). Verification of discrete time stochastic hybrid systems: A stochastic reach-avoid decision problem. *Automatica*, 46(12), 1951–1961.

Vinod, A.P. and Oishi, M. (2017). Scalable underapproximation for the stochastic reach-avoid problem for high-dimensional LTI systems using fourier transforms. *IEEE Ctrl. Syst. Lett.*, 1(2), 316–321.

Vinod, A.P., HomChaudhuri, B., and Oishi, M. (2017). Forward stochastic reachability analysis for uncontrolled linear systems using Fourier transforms. In *Hybrid Systems: Comp. & Control*, 35–44.

Vinod, A.P. and Oishi, M. (2018). Exploiting compactness and convexity of reach-avoid sets for discrete-time stochastic LTI systems for a scalable polytopic underapproximation. In *Proc. Hybrid Syst.: Comput. and Ctrl.* (submitted).

Weiss, A., Kolmanovsky, I., Baldwin, M., and Erwin, R.S. (2012). Model predictive control of three dimensional relative motion. In *Proc. American Ctrl. Conf.*

Weiss, A., Baldwin, M., Erwin, R.S., and Kolmanovsky, I. (2015). Model Predictive Control for Spacecraft Rendezvous and Docking: Strategies for Handling Constraints and Case Studies. *IEEE Transactions on Control Systems Technology*, 23(4), 1638–1647.

Wiesel, W. (1989). *Spaceflight Dynamics*. McGraw-Hill, New York.