DETAILED NUMERICAL EXPERIMENT FOR "NONCONVEX QUASI-VARIATIONAL INEQUALITIES: STABILITY ANALYSIS AND APPLICATION TO NUMERICAL OPTIMIZATION"

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1. INTRODUCTION

This manuscript is a supplement material for the paper titled "*Nonconvex quasi-variational inequalities: stability analysis and application to numerical optimization*" [54]. It contains a number of testing problems and corresponding solved solutions. The setting for the experiments in the next section is described in [54, Section 5]. More details about the problems solved, including their dimensions, can be found in the BOLIB Library of Bilevel Optimization Test Problems and Solvers [48]. Before testing each example, let us describe some indicators that will be recorded in each example.

- λ: parameter with choices $\{3^2, 3^1, 3^0, 3^{-1}, 3^{-2}\}$.
- $(\hat{x}, \hat{y})$: solution obtained by our method.
- F and f: the upper and lower level objective function values respectively, i.e,
$$F = F(\hat{x}, \hat{y}), \ f = f(\hat{x}, \hat{y}).$$
- Iter: number of iterations used by our method, with being less than 2000.
- Error: $\|\Phi(\zeta^k)\|$ with $\{\zeta^k\}$ is the sequence generated by our method.
- Time: CPU time consumed by our method.
- $\mathrm{Err}_{y\xi}$: relative error between y^k and ξ^k in the last step, defined by
$$\mathrm{Err}_{y\xi} := \frac{\|y^k - \xi^k\|}{\max\{1, \|y^k\|, \|\xi^k\|\}}.$$
- Err_{vw}: relative error between v^k and w^k in the last step, defined by
$$\mathrm{Err}_{vw} := \frac{\|v^k - w^k\|}{\max\{1, \|v^k\|, \|w^k\|\}}.$$

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2. NUMERICAL EXPERIMENTS ON THE BOLIB NONLINEAR TEST SET

Problem name: `AiyoshiShimizu1984Ex2`

Source: [1]

description: Aiyoshi and Shimizu 1984 defined Example 2 as follows

$$\begin{aligned}
F(x,y) &:= 2x_1+2x_2-3y_1-3y_2-60\\
G(x,y) &:= \begin{bmatrix} x_1+x_2+y_1-2y_2-40\\ x_1-50\\ x_2-50\\ -x_1\\ -x_2 \end{bmatrix}\\
f(x,y) &:= (y_1-x_1+20)^2+(y_2-x_2+20)^2\\
g(x,y) &:= \begin{bmatrix} 2y_1-x_1+10\\ 2y_2-x_2+10\\ -y_1-10\\ y_1-20\\ -y_2-10\\ y_2-20 \end{bmatrix}
\end{aligned}$$

Comment: The glaobal optimal soluion provided in [1] was $x^* = (25\ 30)^T, y^* = (5\ 10)^T$ with $F(x^*,y^*) = 5$ and $f(x^*,y^*) = 0$. Notice that Ishizuka and Aiyoshi in [27] stated an (local) optimal solution was $x^* = (0,0)^T, y^* = (-10,-10)^T$ with $F(x^*,y^*) = 0$ and $f(x^*,y^*) = 200$ which also derived by Colson, Marcotte and Savard in [8]. Moreover, in [47] another (local) optimal solution was $x^* = (0,30)^T, y^* = (-10,10)^T$ with $F(x^*,y^*) = 0$ and $f(x^*,y^*) = 100$. Results were listed in following table, where two starting points were $x^0 = (10.00,30.00)^T, y^0 = (5.00,5.00)^T$ and $x^0 = (20.00,20.00)^T, y^0 = (0.00,20.00)^T$. Our method achieved the optimal one when $\lambda = 3^{-1}$ and $\lambda = 3^{-2}$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	11.57	117.57	131	0.30	7.39E-02	6.92E-03	7.86E+00	5.86 0.00	-9.95 -10.00
3^1	0.00	200.00	31	0.04	6.35E-12	0.00E+00	1.85E+00	0.00 0.00	-10.00 -10.00
3^0	10.00	100.00	35	0.05	1.21E-08	1.24E-01	2.00E-01	20.00 0.00	0.00 -10.00
3^{-1}	5.00	0.00	10	0.01	6.60E-12	3.61E-01	1.50E+00	25.00 30.00	5.00 10.00
3^{-2}	5.00	0.00	8	0.02	2.08E-07	1.08E+00	1.50E+00	25.00 30.00	5.00 10.00
3^2	15.00	100.00	34	0.05	7.39E-10	6.74E-03	8.25E+00	25.00 50.00	5.00 20.00
3^1	10.00	100.00	19	0.03	2.69E-08	4.16E-02	1.80E+00	20.00 0.00	0.00 -10.00
3^0	31.25	0.56	10	0.01	4.22E-07	1.57E-01	1.00E+00	16.00 10.50	-4.00 -8.75
3^{-1}	5.00	0.00	11	0.02	2.06E-10	3.61E-01	1.50E+00	25.00 30.00	5.00 10.00
3^{-2}	5.00	0.00	13	0.01	6.09E-08	1.08E+00	1.50E+00	25.00 30.00	5.00 10.00

Problem name: `AllendeStill2013`

source: [2]

description: Allende and Still 2013 defined one example as follows

$$\begin{aligned}
F(x,y) &:= -x_1^2 - 2x_1 + x_2^2 - 2x_2 + y_1^2 + y_2^2 \\
G(x,y) &:= \begin{bmatrix} -x_1 \\ -x_2 \\ -y_1 \\ -y_2 \\ x_1 - 2 \end{bmatrix} \\
f(x,y) &:= y_1^2 - 2x_1y_1 + y_2^2 - 2x_2y_2 \\
g(x,y) &:= \begin{bmatrix} (y_1-1)^2 - 0.25 \\ (y_2-1)^2 - 0.25 \end{bmatrix}
\end{aligned}$$

Comment: Results were listed in following table. Allende and Still in [2] stated that the minimizer was $x^* = (0.5\ 0.5)^T$ and $y^* = (0.5\ 0.5)^T$ with $F(x^*, y^*) = -1.5$ and $f(x^*, y^*) = -0.5$. Two starting points were $x^0 = (0.00, 0.00)^T, y^0 = (0.50, 0.50)^T$ and $x^0 = (0.00, 0.00)^T, y^0 = (1.00, 1.00)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-6.25	-4.00	356	0.67	1.38E-07	3.47E-02	5.00E+00	2.00 0.50	1.50 0.50
3^1	-6.25	-4.00	379	0.52	9.64E-07	1.02E-01	1.00E+00	2.00 0.50	1.50 0.50
3^0	-7.14	-3.56	22	0.03	7.30E-07	3.33E-01	1.00E+00	2.00 0.50	1.17 0.50
3^{-1}	-8.15	-2.73	76	0.09	6.84E-08	6.04E-01	9.93E-01	2.00 0.67	0.70 0.50
3^{-2}	-8.49	-2.39	10	0.02	2.41E-08	6.67E-01	8.25E-01	2.00 0.89	0.50 0.50
3^2	-2.50	-1.25	192	0.21	1.41E+00	1.00E-01	3.53E-05	1.00 0.50	1.00 0.50
3^1	-6.25	-4.00	31	0.03	7.15E-07	1.02E-01	1.00E+00	2.00 0.50	1.50 0.50
3^0	-7.14	-3.56	15	0.01	4.37E-07	3.33E-01	1.00E+00	2.00 0.50	1.17 0.50
3^{-1}	-8.15	-2.73	75	0.12	1.41E-11	6.04E-01	9.93E-01	2.00 0.67	0.70 0.50
3^{-2}	-8.49	-2.39	280	0.35	1.89E-12	6.67E-01	8.25E-01	2.00 0.89	0.50 0.50

Problem name: `AnEtal2009`

Source: [3]

Description: An et al. 2009 defined one example as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}z^\top Hz + c^\top z \\ G(x,y) & := & \begin{bmatrix} -x \\ -y \\ Ax+By+d \end{bmatrix} \\ f(x,y) & := & y^\top Px + \frac{1}{2}y^\top Qy + q^\top y \\ g(x,y) & := & Dx+Ey+b \end{array}$$

where $z = (x^\top, y^\top)^\top$ and H, c, A, B, d, P, Q, q, D, E, b respectively are defined as follows

$$H := \begin{bmatrix} -3.8 & 4.4 & 1.2 & -2.2 \\ 4.4 & -2.2 & 0.6 & 1.8 \\ 1.2 & 0.6 & 0.0 & 0.4 \\ -2.2 & 1.8 & 0.4 & 0.0 \end{bmatrix}, \quad c := \begin{bmatrix} 935.74474 \\ 87.53654 \\ 121.96196 \\ 299.24825 \end{bmatrix},$$

$$A := \begin{bmatrix} 0.00000 & 3.88889 \\ -2.00000 & 8.77778 \end{bmatrix}, \quad B := \begin{bmatrix} 4.88889 & 7.44444 \\ -5.11111 & 0.88889 \end{bmatrix}, \quad d := \begin{bmatrix} -61.57778 \\ -0.80000 \end{bmatrix}$$

$$P := \begin{bmatrix} -17.85000 & 6.57500 \\ 30.32500 & 30.32500 \end{bmatrix}, \quad Q := \begin{bmatrix} 21.10204 & 11.81633 \\ 11.81633 & -14.44898 \end{bmatrix}, \quad q := \begin{bmatrix} -18.21053 \\ 13.05263 \end{bmatrix}$$

$$D := \begin{bmatrix} 5.00000 & 7.44444 \\ -8.33333 & 3.00000 \\ -8.66667 & -8.55556 \\ 6.44444 & -5.11111 \end{bmatrix}, \quad E := \begin{bmatrix} 3.88889 & 1.77778 \\ 6.88889 & 6.11111 \\ -5.33333 & -7.00000 \\ 1.44444 & 4.44444 \end{bmatrix}, \quad b := \begin{bmatrix} -39.62222 \\ -60.00000 \\ 72.37778 \\ -17.28889 \end{bmatrix}$$

Comments: Results were listed in following table. For different starting points and different λ, solutions were the same as one obtained by An et al. in [3]. Two starting points were $x^0 = (1.00, 1.00)^\top, y^0 = (1.00, 1.00)^\top$ and $x^0 = (0.00, 0.00)^\top, y^0 = (0.00, 0.00)^\top$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	2251.55	565.78	267	0.88	1.05E-05	3.26E-16	8.17E+00	0.20 2.00	4.00 4.60
3^1	2251.55	565.78	210	0.52	1.55E-05	1.63E-16	2.42E+00	0.20 2.00	4.00 4.60
3^0	2251.55	565.78	115	0.21	1.98E-05	1.31E-06	1.85E+00	0.20 2.00	4.00 4.60
3^{-1}	2251.55	565.78	26	0.05	9.25E-07	1.22E-06	1.50E+00	0.20 2.00	4.00 4.60
3^{-2}	2251.55	565.78	45	0.08	6.37E-07	1.22E-06	1.69E+00	0.20 2.00	4.00 4.60
3^2	2251.55	565.78	475	0.86	1.05E-05	4.61E-16	8.17E+00	0.20 2.00	4.00 4.60
3^1	2251.55	565.78	143	0.25	1.55E-05	7.29E-16	2.42E+00	0.20 2.00	4.00 4.60
3^0	2251.55	565.78	138	0.23	4.19E-06	7.67E-07	1.43E+00	0.20 2.00	4.00 4.60
3^{-1}	2251.55	565.78	116	0.19	2.38E-05	1.22E-06	1.50E+00	0.20 2.00	4.00 4.60
3^{-2}	2251.55	565.78	117	0.18	6.36E-06	1.22E-06	1.69E+00	0.20 2.00	4.00 4.60

Problem name: `Bard1988Ex1`

Source: [4]

Description: Bard 1988 defined Example 1 as follows

$$\begin{array}{rcl} F(x,y) & := & (x-5)^2+(2y+1)^2 \\ G(x,y) & := & -x \\ f(x,y) & := & (y-1)^2-1.5xy \\ g(x,y) & := & \begin{bmatrix} -3x+y+3 \\ x-0.5y-4 \\ x+y-7 \\ -y \end{bmatrix} \end{array}$$

Comment: Bard in [4] stated the global optimal solution was $x^*=1, y^*=0$ and one local optimal solution was $x^*=5, y^*=2$. Results were listed in following table, where two starting points were $x^0=5.00, y^0=0.00$ and $x^0=5.00, y^0=1.00$. Our method achieved the global optimal solution when $\lambda=1$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	48.42	-10.16	13	0.12	6.50E-09	2.53E-01	1.00E+00	3.17	2.86
3^1	30.59	-6.93	10	0.01	5.81E-07	5.41E-01	1.00E+00	2.64	2.00
3^0	17.00	1.00	10	0.01	1.14E-09	7.25E-18	8.52E-01	1.00	0.00
3^{-1}	7.68	0.44	12	0.01	1.97E-09	9.78E-01	1.00E+00	2.50	0.10
3^{-2}	2.00	1.00	19	0.03	2.53E-08	1.00E+00	1.06E+00	4.00	0.00
3^2	48.42	-10.16	13	0.01	6.50E-09	2.53E-01	1.00E+00	3.17	2.86
3^1	30.59	-6.93	10	0.01	5.81E-07	5.41E-01	1.00E+00	2.64	2.00
3^0	17.00	1.00	10	0.01	1.01E-07	5.81E-19	7.84E-01	1.00	0.00
3^{-1}	7.68	0.44	14	0.01	1.28E-10	9.78E-01	1.00E+00	2.50	0.10
3^{-2}	2.00	1.00	19	0.02	2.53E-08	1.00E+00	1.06E+00	4.00	0.00

Problem name: `Bard1988Ex2`

Source: [4]

Description: Bard 1988 defined Example 2 as follows

$$\begin{aligned} F(x,y) &:= -(200-y_1-y_3)(y_1+y_3)-(160-y_2-y_4)(y_2+y_4)\\ f(x,y) &:= (y_1-4)^2+(y_2-13)^2+(y_3-35)^2+(y_4-2)^2\\ G(x,y) &:= \begin{bmatrix} x_1+x_2+x_3+x_4-40\\ x_1-10\\ x_2-5\\ x_3-15\\ x_4-20\\ -x \end{bmatrix},\ g(x,y):=\begin{bmatrix} 0.4y_1+0.7y_2-x_1\\ 0.6y_1+0.3y_2-x_2\\ 0.4y_3+0.7y_4-x_3\\ 0.6y_3+0.3y_4-x_4\\ y_1-20\\ y_2-20\\ y_3-40\\ y_4-40\\ -y \end{bmatrix}\end{aligned}$$

Comment: This version of the problem is taken from [7]. The original one in [4] has two lower-level problems. However, Colson etal. in [8] combined the two lower-level problems together and got a solution satisfying $F(x^*,y^*)=-6600$, $f(x^*,y^*)=57.48$. Results were listed in following table, where our method achieved a better one since $F(\hat{x},\hat{y})=-6600$, $f(\hat{x},\hat{y})=54$. Two starting points were $x^0=(5.00,5.00,5.00,5.00)^T, y^0=(0.00,0.00,0.00,0.00)^T$ and $x^0=(0.00,0.00,0.00,0.00)^T, y^0=(0.00,0.00,0.00,0.00)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-5772.97	193.40	34	0.19	4.58E-11	8.07E-02	8.94E+00	0.00 0.00 14.54 20.00	0.00 0.00 32.13 2.41
3^1	-5852.78	197.89	28	0.06	1.69E-07	1.12E-01	1.95E+00	0.00 0.00 15.00 20.00	0.00 0.00 31.67 3.33
3^0	-6600.00	69.00	26	0.05	2.02E-10	2.34E-12	6.49E+00	7.91 4.37 11.09 16.63	2.28 10.00 27.72 0.00
3^{-1}	-6600.00	54.00	27	0.06	3.77E-09	1.12E-10	1.33E+01	7.00 3.00 12.00 18.00	0.00 10.00 30.00 0.00
3^{-2}	-6600.00	54.00	97	0.26	2.79E-11	1.06E-14	1.32E+01	7.00 3.00 12.00 18.00	0.00 10.00 30.00 0.00
3^2	-5772.97	193.40	28	0.05	1.22E-08	8.07E-02	8.94E+00	0.00 0.00 15.00 20.00	0.00 0.00 32.13 2.41
3^1	-6600.00	69.01	37	0.07	8.73E-07	4.65E-09	1.22E+01	7.91 4.37 11.09 16.63	2.28 10.00 27.72 0.00
3^0	-6600.00	66.49	49	0.09	1.06E-09	3.42E-11	6.72E+00	7.82 4.23 11.18 16.77	2.05 10.00 27.95 0.00
3^{-1}	-6600.00	68.93	18	0.03	2.11E-07	6.96E-09	1.06E+01	7.91 4.37 11.09 16.63	2.28 10.00 27.72 0.00
3^{-2}	-6600.00	54.00	31	0.06	6.48E-12	1.95E-13	1.33E+01	7.00 3.00 12.00 18.00	0.00 10.00 30.00 0.00

Problem name: `Bard1988Ex3`

Source: [4]

Description: Bard 1988 defined Example 3 as follows

$$
\begin{aligned}
F(x,y) &:= -x_1^2 - 3x_2 - 4y_1 + y_2^2 \\
G(x,y) &:= \begin{bmatrix} x_1^2 + 2x_2 - 4 \\ -x_1 \\ -x_2 \end{bmatrix} \\
f(x,y) &:= 2x_1^2 + y_1^2 - 5y_2 \\
g(x,y) &:= \begin{bmatrix} -x_1^2 + 2x_1 - x_2^2 + 2y_1 - y_2 - 3 \\ -x_2 - 3y_1 + 4y_2 + 4 \\ -y_1 \\ -y_2 \end{bmatrix}
\end{aligned}
$$

Comment: Results were listed in following table. Colson, Marcotte and Savard in [8] got one solution x^*, y^* with $F(x^*, y^*) = -12.68, f(x^*, y^*) = -1.02$. This was same as the solution obtained by our method when $\lambda = 3$. Two starting points were $x^0 = (2.00, 3.00)^T, y^0 = (2.00, 0.00)^T$ and $x^0 = (1.00, 1.00)^T, y^0 = (1.00, 1.00)^T$

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-10.36	9.56	15	0.13	4.44E-07	1.43E-11	9.25E+00	2.00 0.00	1.60 0.20
3^1	-12.68	-1.02	16	0.03	1.45E-07	2.64E-01	1.64E+00	0.00 2.00	1.88 0.91
3^0	-12.79	-1.01	19	0.03	4.05E-07	7.33E-01	3.99E-01	0.00 2.00	1.92 0.94
3^{-1}	-10.36	9.56	18	0.03	8.15E-11	1.16E-13	1.87E+00	2.00 0.00	1.60 0.20
3^{-2}	-10.36	9.56	24	0.05	4.44E-09	6.40E-11	1.84E+00	2.00 0.00	1.60 0.20
3^2	-12.68	-1.02	13	0.02	1.19E-07	8.81E-02	7.64E+00	0.00 2.00	1.88 0.91
3^1	-12.68	-1.02	16	0.02	1.59E-07	2.64E-01	1.64E+00	0.00 2.00	1.88 0.91
3^0	-12.79	-1.01	13	0.02	2.52E-07	7.33E-01	3.99E-01	0.00 2.00	1.92 0.94
3^{-1}	-18.64	6.94	13	0.02	4.42E-07	8.13E-01	1.00E+00	0.00 2.00	3.33 0.83
3^{-2}	-20.98	11.13	14	0.02	1.22E-09	8.40E-01	1.02E+00	0.00 2.00	3.94 0.88

Problem name: `Bard1991Ex1`

Source: [5]

description: Bard 1991 defined Example 2.1 as follows

$$
\begin{aligned}
F(x,y) &:= x + y_2\\
G(x,y) &:= \begin{bmatrix} -x+2 \\ x-4 \end{bmatrix}\\
f(x,y) &:= 2y_1 + xy_2\\
g(x,y) &:= \begin{bmatrix} x - y_1 - y_2 + 4 \\ -y_1 \\ -y_2 \end{bmatrix}
\end{aligned}
$$

Optimization operators: Results were listed in following table. $x^* = 2, y^* = (6\ 0)^\top$ is the unique optimal solution stated by Bard in [5]. Our method got its unique optimal solution for all cases. Two starting points were $x^0 = 0.00, y^0 = (4.00, 4.00)^\top$ and $x^0 = 2.00, y^0 = (4.00, 4.00)^\top$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	2.00	12.00	13	0.13	3.71E-07	4.90E-05	8.02E+00	2.00	6.00 0.00
3^1	2.00	12.00	10	0.02	5.77E-12	7.85E-02	2.06E+00	2.00	6.00 0.00
3^0	2.00	12.00	13	0.02	6.76E-10	2.11E-04	5.00E-01	2.00	6.00 0.00
3^{-1}	2.00	12.00	12	0.02	1.72E-09	7.07E-01	8.33E-01	2.00	6.00 0.00
3^{-2}	2.00	12.00	17	0.03	2.06E-08	1.41E+00	1.02E+00	2.00	6.00 0.00
3^2	2.00	12.00	442	0.80	5.36E-10	1.11E-03	8.02E+00	2.00	6.00 0.00
3^1	2.00	12.00	345	0.48	1.75E-10	7.85E-02	2.06E+00	2.00	6.00 0.00
3^0	2.00	12.00	13	0.01	1.00E-10	2.35E-01	5.00E-01	2.00	6.00 0.00
3^{-1}	2.00	12.00	11	0.01	4.08E-10	7.07E-01	8.33E-01	2.00	6.00 0.00
3^{-2}	2.00	12.00	33	0.03	5.03E-09	1.41E+00	1.02E+00	2.00	6.00 0.00

Problem name: `BardBook1998`

Source: [6]

description: Bard 1998 defined one example as follows

$$\begin{aligned} F(x,y) &:= (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\ g(x,y) &:= \begin{bmatrix} x_1 - 50 \\ x_2 - 50 \\ -x_1 \\ -x_2 \end{bmatrix} \\ f(x,y) &:= 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60 \\ g(x,y) &:= \begin{bmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ 2y_1 - x_1 + 10 \\ 2y_2 - x_2 + 10 \\ y_1 - 20 \\ y_2 - 20 \\ -y_1 - 10 \\ -y_2 - 10 \end{bmatrix} \end{aligned}$$

Comments: Results were listed in following table. For three following different starting points, solutions were the same when $\lambda \leq 1$. Two starting points were $x^0 = (50.00, 50.00)^T, y^0 = (-10.00, -10.00)^T$ and $x^0 = (30.00, 30.00)^T, y^0 = (20.00, 20.00)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	182.25	-8.84	178	0.39	4.08E+00	1.16E-01	1.00E+00	-1.66 30.00	-8.16 10.00
3^1	11.11	-3.33	12	0.02	1.72E-07	6.40E-09	2.79E+00	23.33 30.00	6.67 10.00
3^0	0.00	5.00	20	0.03	1.66E-12	2.38E-16	1.98E-13	25.00 30.00	5.00 10.00
3^{-1}	0.00	5.00	12	0.02	4.98E-09	3.11E-10	6.67E-01	25.00 30.00	5.00 10.00
3^{-2}	0.00	5.00	13	0.03	3.06E-12	0.00E+00	8.89E-01	25.00 30.00	5.00 10.00
3^2	11.11	-3.33	157	0.44	1.64E-11	8.87E-15	7.66E+00	23.33 30.00	6.67 10.00
3^1	147.47	-2.60	126	0.29	4.08E+00	3.80E-01	9.99E-01	19.15 -1.67	2.52 -10.00
3^0	0.00	5.00	9	0.01	3.48E-09	1.59E-16	2.61E-10	25.00 30.00	5.00 10.00
3^{-1}	0.00	5.00	14	0.02	2.15E-07	2.15E-08	6.67E-01	25.00 30.00	5.00 10.00
3^{-2}	0.00	5.00	16	0.02	1.71E-10	1.71E-11	8.89E-01	25.00 30.00	5.00 10.00

Problem name: `CalamaiVicente1994a`

Source: [10]

Description: Calamai and Vicente 1994 defined one example, a quadratic bilevel program, as follows:

$$\begin{aligned} F(x,y) &:= \tfrac{1}{2}(x-1)^2 + \tfrac{1}{2}y^2 \\ f(x,y) &:= \tfrac{1}{2}y - xy \\ g(x,y) &:= \begin{bmatrix} x-y-1 \\ -x-y+1 \\ x+y-\rho \end{bmatrix} \end{aligned}$$

Comment: It is assumed in [10] that the parameter $\rho \geq 1$. We consider the following scenarios studied in the latter reference:

(i) For $\rho = 1$, the point $(1,0)$ is global optimum of the problem.

(ii) For $1 < \rho < 2$, the point $\frac{1}{2}(1+\rho, -1+\rho)$ is a global optimal solution, while $\frac{1}{2}(1,1)$ is a local optimal solution of the problem.

(iii) For $\rho = 2$, the points $\frac{1}{2}(1,1)$ and $\frac{1}{2}(3,1)$ are global optimal solution.

(iv) For $\rho > 2$, the point $\frac{1}{2}(1,1)$ is global optimum of the problem.

Comments: Results were listed in following table, in which the starting point was $x^o = 0, y^o = 1$ for all cases. One can notice that for each ρ, our method was able to render the global optimal solution under one case.

ρ	λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
	3^2	0.00	0.00	106	0.30	7.13E-06	5.69E-06	4.00E+00	1.00	0.00
	3^1	0.00	0.00	15	0.03	2.76E-08	1.93E-08	1.00E+00	1.00	0.00
1	3^0	0.00	0.00	17	0.02	1.74E-11	1.06E-12	6.69E-04	1.00	0.00
	3^{-1}	0.00	0.00	18	0.02	1.40E-11	3.73E-12	6.67E-01	1.00	0.00
	3^{-2}	0.00	0.00	13	0.02	6.21E-07	2.26E-07	4.52E-01	1.00	0.00
	3^2	0.25	0.00	18	0.12	2.05E-07	1.11E-01	5.00E-01	0.50	0.50
	3^1	0.25	0.00	11	0.02	1.50E-10	3.33E-01	5.00E-01	0.50	0.50
1.5	3^0	0.06	-0.06	21	0.03	1.93E-13	5.00E-01	2.50E-01	0.75	0.25
	3^{-1}	0.06	-0.18	1000	1.15	2.19E-03	6.95E-03	7.43E-01	1.25	0.25
	3^{-2}	0.00	-0.03	11	0.02	8.22E-07	4.35E-01	5.08E-01	1.01	0.06
	3^2	0.25	0.00	14	0.11	3.41E-09	1.11E-01	5.00E-01	0.50	0.50
	3^1	0.25	0.00	10	0.02	2.82E-09	3.33E-01	5.00E-01	0.50	0.50
2	3^0	0.25	0.00	14	0.02	7.99E-07	6.67E-01	5.00E-01	0.50	0.50
	3^{-1}	0.03	-0.06	11	0.02	4.17E-07	8.57E-01	3.38E-01	0.83	0.17
	3^{-2}	0.00	-0.02	26	0.04	6.65E-07	9.47E-01	4.44E-01	0.94	0.06
	3^2	0.25	0.00	12	0.11	6.83E-08	1.11E-01	5.00E-01	0.50	0.50
	3^1	1.00	-1.50	46	0.06	2.89E-07	1.22E-07	1.33E+00	2.00	1.00
3	3^0	0.25	0.00	10	0.02	6.44E-08	6.67E-01	5.00E-01	0.50	0.50
	3^{-1}	0.11	-0.06	13	0.02	3.83E-07	8.57E-01	3.24E-01	0.67	0.33
	3^{-2}	0.01	-0.04	15	0.03	7.27E-08	9.47E-01	3.95E-01	0.89	0.11

Problem name: `CalamaiVicente1994b`

source: [10]

description: Calamai and Vicente 1994 defined one example, a quadratic bilevel program, as follows:

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}\sum_{i=1}^{4}(x_i-1)^2+\frac{1}{2}\sum_{i=1}^{2}y_i^2 \\ f(x,y) & := & \sum_{i=1}^{2}(\frac{1}{2}y_i^2-x_iy_i) \\ g(x,y) & := & \begin{bmatrix} x_1-y_1-1 \\ x_2-y_2-1 \\ x_1+y_1-1.5 \\ x_2+y_2-3 \\ -x_1-y_1+1 \\ -x_2-y_2+1 \end{bmatrix} \end{array}$$

Comment: Results were listed in following table, where two starting points were $x^0=(0.5,1,0,0)^T$, $y^0=(1,2)^T$ and $x^0=(1,1,0,0)^T, y^0=(0.5,2)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.44	-1.43	13	0.12	9.05E-12	4.43E-02	1.11E-01	0.74 1.53 1.00 1.00	0.71 1.42
3^1	1.25	-1.74	10	0.02	9.74E-07	1.23E-01	1.00E+00	0.70 2.00 1.00 1.00	0.64 1.00
3^0	0.50	-0.25	11	0.02	6.38E-09	5.00E-01	3.43E-09	0.50 0.50 1.00 1.00	0.50 0.50
3^{-1}	0.31	-0.41	8	0.02	9.97E-08	7.44E-01	7.76E-01	1.25 0.50 1.00 1.00	0.25 0.50
3^{-2}	0.06	-0.33	10	0.03	9.77E-07	8.66E-01	1.00E+00	1.07 0.86 1.00 1.00	0.14 0.27
3^2	1.44	-1.43	13	0.02	9.05E-12	4.43E-02	1.11E-01	0.74 1.53 1.00 1.00	0.71 1.42
3^1	1.25	-1.74	10	0.02	9.74E-07	1.23E-01	1.00E+00	0.70 2.00 1.00 1.00	0.64 1.00
3^0	0.50	-0.25	17	0.03	3.12E-07	5.00E-01	1.59E-07	0.50 0.50 1.00 1.00	0.50 0.50
3^{-1}	0.31	-0.41	33	0.06	9.98E-07	7.44E-01	9.14E-01	1.25 0.50 1.00 1.00	0.25 0.50
3^{-2}	0.06	-0.33	10	0.01	9.77E-07	8.66E-01	1.00E+00	1.07 0.86 1.00 1.00	0.14 0.27

Problem name: `CalamaiVicente1994c`

source: [10]

description: Calamai and Vicente 1994 defined one example, a quadratic bilevel program, as follows:

$$\begin{array}{lll} F(x,y) & := & \frac{1}{2}x^TAx + \frac{1}{2}y^TBy + a^Tx + 2 \\ f(x,y) & := & \frac{1}{2}y^TBy + x^TCy \\ g(x,y) & := & Dx + Ey + d \end{array}$$

where

$$A := \begin{bmatrix} 197.2 & 32.4 & -129.6 & -43.2 \\ 32.4 & 110.8 & -43.2 & -14.4 \\ -129.6 & -43.2 & 302.8 & -32.4 \\ -43.2 & -14.4 & -32.4 & 389.2 \end{bmatrix}, B = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, a = \begin{bmatrix} -8.56 \\ -9.52 \\ -9.92 \\ -16.64 \end{bmatrix}$$

$$C := \begin{bmatrix} -132.4 & -10.8 \\ -10.8 & -103.6 \\ 43.2 & 14.4 \\ 14.4 & 4.8 \end{bmatrix}, D = \begin{bmatrix} 13.24 & 1.08 & -4.32 & -1.44 \\ 1.08 & 10.36 & -1.44 & -0.48 \\ 13.24 & 1.08 & -4.32 & -1.44 \\ 1.08 & 10.36 & -1.44 & -0.48 \\ -13.24 & -1.08 & 4.32 & 1.44 \\ -1.08 & -10.36 & 1.44 & 0.48 \end{bmatrix}$$

$$E = \begin{bmatrix} -10 & 0 \\ 0 & -10 \\ 10 & 0 \\ 0 & 10 \\ -10 & 0 \\ 0 & -10 \end{bmatrix}, d = \begin{bmatrix} -1 \\ -1 \\ -1.5 \\ -3 \\ 1 \\ 1 \end{bmatrix}$$

Comment: Results were listed in following table, in which solutions were different with the optimal solution with the corresponding upper-level objective function value being 0.3125 report in [10]. Results were listed in following table, where two starting points were $x^0 = (0,0,0,0)^T, y^0 = (0.1,0.1)^T$ and $x^0 = (0,0,0,0)^T, y^0 = (1.00,1.00)^T$. Our method achieved the same one under $\lambda = 3^{-1}$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.3998	-1.2887	10	0.13	2.48E-07	7.44E-03	1.08E-01	0.06 0.16 0.09 0.06	0.05 0.14
3^1	0.5000	-0.2500	8	0.02	6.36E-07	2.36E-02	8.28E-07	0.07 0.06 0.09 0.06	0.05 0.05
3^0	0.5000	-0.2500	12	0.02	4.62E-07	7.07E-02	6.04E-07	0.07 0.06 0.09 0.06	0.05 0.05
3^{-1}	0.3125	-0.4063	7	0.01	7.30E-07	1.50E-01	9.11E-01	0.13 0.05 0.10 0.07	0.02 0.05
3^{-2}	0.0592	-0.3334	10	0.02	5.22E-08	1.89E-01	1.00E+00	0.11 0.09 0.10 0.07	0.01 0.03
3^2	1.2123	-1.4449	66	0.13	7.98E-14	4.95E-03	7.71E+00	0.13 0.15 0.11 0.07	0.02 0.14
3^1	0.5000	-0.2500	10	0.02	2.66E-07	2.36E-02	3.48E-07	0.07 0.06 0.09 0.06	0.05 0.05
3^0	0.5000	-0.2500	37	0.06	7.80E-07	7.07E-02	1.02E-06	0.07 0.06 0.09 0.06	0.05 0.05
3^{-1}	0.3125	-0.4063	11	0.02	3.89E-07	1.50E-01	9.14E-01	0.13 0.05 0.10 0.07	0.03 0.05
3^{-2}	0.0592	-0.3334	10	0.01	5.30E-10	1.89E-01	1.00E+00	0.11 0.09 0.10 0.07	0.01 0.03

Problem name: `CalveteGale1999P1`

Source: [9]

Description: Calvete Galé 1999 defined P1 as follows

$$
\begin{aligned}
F(x,y) &:= -8x_1 - 4x_2 + y_1 - 40y_2 - 4y_3 \\
G(x,y) &:= \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} \\
f(x,y) &:= \frac{1 + x_1 + x_2 + 2y_1 - y_2 + y_3}{6 + 2x_1 + y_1 + y_2 - 3y_3} \\
g(x,y) &:= \begin{bmatrix} -y_1 \\ -y_2 \\ -y_3 \\ -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - \frac{1}{2}y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - \frac{1}{2}y_3 - 1 \end{bmatrix}
\end{aligned}
$$

Comment: The optimal solution is $x^* = (0.0, 0.9)^T, y^* = (0.0, 0.6, 0.4)^T$ with values of the upper and lower-level objective function are -29.2 and 0.31 [9]. Results were listed in following table, where two starting points were $x^0 = (0.00, 0.00)^T, y^0 = (0.00, 0.20, 0.80)^T$ and $x^0 = (0.00, 0.00)^T, y^0 = (0.00, 1.00, 1.00)^T$. When $\lambda = 3^2$, we got the same solution.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-29.20	0.31	19	1.84	6.38E-13	1.75E-16	1.08E+01	0.00 0.90	0.00 0.60 0.40
3^1	-13.00	0.78	7	0.63	1.41E-12	2.62E-13	2.11E+00	0.75 0.75	0.00 0.00 1.00
3^0	-58.00	0.58	15	1.41	1.04E-08	8.79E-01	3.33E+01	0.00 0.00	1.50 1.50 1.00
3^{-1}	-58.00	0.58	15	1.36	1.44E-10	8.79E-01	3.35E+01	0.00 0.00	1.50 1.50 1.00
3^{-2}	-58.00	0.58	16	1.50	8.34E-09	8.79E-01	3.35E+01	0.00 0.00	1.50 1.50 1.00
3^2	-29.20	0.31	64	5.82	5.56E-08	3.41E-16	1.08E+01	0.00 0.90	0.00 0.60 0.40
3^1	-16.00	1.63	18	1.61	3.07E-09	6.50E-18	2.17E+00	1.50 0.00	1.00 0.00 2.00
3^0	-58.00	0.58	19	1.76	3.22E-08	8.79E-01	3.33E+01	0.00 0.00	1.50 1.50 1.00
3^{-1}	-58.00	0.58	15	1.33	5.24E-08	8.79E-01	3.35E+01	0.00 0.00	1.50 1.50 1.00
3^{-2}	-58.00	0.58	20	1.85	5.55E-08	8.79E-01	3.35E+01	0.00 0.00	1.50 1.50 1.00

Problem name: `ClarkWesterberg1990a`

Source: [11]

description: Clark and Westerberg 1990 defined one example as follows

$$\begin{aligned} F(x,y) &:= (x-3)^2+(y-2)^2 \\ G(x,y) &:= \begin{bmatrix} x-8 \\ -x \end{bmatrix} \\ f(x,y) &:= (y-5)^2 \\ g(x,y) &:= \begin{bmatrix} -2x+y-1 \\ x-2y-2 \\ x+2y-14 \end{bmatrix} \end{aligned}$$

Comments: The best known optimal solution of the problem is $x^* = 1, y^* = 3$ [43]. Results were listed in following table where our method achieved the same one nuder three cases. Two starting points were $x^0 = -1.00, y^0 = 1.00$ and $x^0 = 3.00, y^0 = 2.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	5.00	4.00	20	0.13	1.28E-11	4.25E-12	7.50E+00	1.00	3.00
3^1	7.43	0.10	16	0.03	1.01E-11	1.10E-01	3.16E-01	3.47	4.68
3^0	5.00	4.00	10	0.02	1.22E-08	4.06E-09	5.00E-01	1.00	3.00
3^{-1}	1.70	2.99	11	0.01	3.35E-10	3.89E-01	1.00E+00	3.29	3.27
3^{-2}	0.36	5.83	130	0.22	5.32E-11	5.24E-01	1.00E+00	3.13	2.58
3^2	9.49	0.00	16	0.02	2.97E-08	5.33E-02	9.84E-02	2.11	4.95
3^1	5.00	4.00	11	0.01	3.00E-07	0.00E+00	1.50E+00	1.00	3.00
3^0	4.58	0.83	9	0.01	1.23E-10	2.24E-01	9.09E-01	3.45	4.09
3^{-1}	1.70	2.99	8	0.01	5.20E-08	3.89E-01	1.00E+00	3.29	3.27
3^{-2}	0.36	5.83	9	0.01	2.88E-13	5.24E-01	1.00E+00	3.13	2.58

Problem name: `Colson2002BIPA1`

Source: [7]

description: BIPA1 is defined as follows

$$
\begin{aligned}
F(x,y) &:= (10-x)^3 + (10-y)^3 \\
G(x,y) &:= \begin{bmatrix} x-5 \\ -x+y \\ -x \end{bmatrix} \\
f(x,y) &:= (x+2y-15)^4 \\
g(x,y) &:= \begin{bmatrix} x+y-20 \\ y-20 \\ -y \end{bmatrix}
\end{aligned}
$$

Comments: Results were listed in following table. Two starting points were $x^0 = 5, y^0 = 10$ and $x^0 = 10, y^0 = 5$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	250.00	0.00	29	0.15	5.03E-09	1.00E+00	4.28E-09	5.00	5.00
3^1	250.00	0.00	32	0.05	4.28E-13	6.67E-01	4.28E-13	5.00	5.00
3^0	250.00	0.00	27	0.04	1.83E-11	1.00E+00	1.83E-11	5.00	5.00
3^{-1}	250.00	0.00	10	0.01	4.51E-11	1.00E+00	4.51E-11	5.00	5.00
3^{-2}	250.00	0.00	15	0.03	1.34E-08	9.97E-01	3.62E-09	5.00	5.00
3^2	303.44	1.36	169	0.19	2.99E-01	6.99E-01	1.00E+00	4.75	4.59
3^1	250.00	0.00	42	0.05	8.44E-12	6.66E-01	8.44E-12	5.00	5.00
3^0	250.00	0.00	13	0.01	3.40E-08	9.99E-01	1.12E-08	5.00	5.00
3^{-1}	250.00	0.00	15	0.01	1.29E-08	1.00E+00	1.29E-08	5.00	5.00
3^{-2}	250.00	0.00	20	0.02	8.43E-07	9.99E-01	8.43E-07	5.00	5.00

Problem name: `Colson2002BIPA2`

Source: [7]

description: BIPA2 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^2 + (2y+1)^2 \\ G(x,y) &:= -x \\ f(x,y) &:= (y-1)^2 - 1.5xy + x^3 \\ g(x,y) &:= \begin{bmatrix} -3x+y+3 \\ x-0.5y-4 \\ x+y-7 \\ -y \end{bmatrix} \end{aligned}$$

Comments: Solution obtained by Colson, Marcotte and Savard in [8] is $x^* = 1, y^* = 0$. Results were listed in following table, in which method produced the same solution when $\lambda = 1$. Two starting points were $x^0 = 5.00, y^0 = 2.00$ and $x^0 = 7.00, y^0 = 0.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	25.00	111.00	18	0.13	1.26E-08	2.22E-16	6.08E+00	5.00	2.00
3^1	30.59	11.53	11	0.02	1.20E-11	5.41E-01	1.00E+00	2.64	2.00
3^0	17.00	2.00	11	0.02	5.47E-13	3.52E-17	8.50E-01	1.00	0.00
3^{-1}	17.00	2.00	12	0.02	6.62E-08	4.67E-08	8.90E-01	1.00	0.00
3^{-2}	2.00	65.00	17	0.03	3.07E-12	1.00E+00	1.06E+00	4.00	0.00
3^2	48.42	21.83	19	0.02	1.07E-08	2.53E-01	1.00E+00	3.17	2.86
3^1	30.59	11.53	16	0.02	4.10E-11	5.41E-01	1.00E+00	2.64	2.00
3^0	17.00	2.00	8	0.01	5.16E-09	2.90E-09	8.52E-01	1.00	0.00
3^{-1}	25.00	111.00	12	0.01	8.75E-12	4.00E-15	6.48E-01	5.00	2.00
3^{-2}	2.00	65.00	15	0.02	4.42E-11	1.00E+00	1.06E+00	4.00	0.00

Problem name: `Colson2002BIPA3`

Source: [7]

description: BIPA3 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^4 + (2y+1)^4 \\ G(x,y) &:= \begin{bmatrix} x+y-4 \\ -x \end{bmatrix} \\ f(x,y) &:= e^{-x+y} + x^2 + 2xy + y^2 + 2x + 6y \\ g(x,y) &:= \begin{bmatrix} -x+y-2 \\ -y \end{bmatrix} \end{aligned}$$

Comments: Results were listed in following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = 1.00, y^0 = 1.00$. For different starting points and different λ, solutions were the same as the one derived by Colson, Marcotte and Savard in [8].

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	2.00	24.02	9	0.11	1.42E-10	0.00E+00	8.86E+00	4.00	0.00
3^1	2.00	24.02	8	0.01	2.39E-13	0.00E+00	2.86E+00	4.00	0.00
3^0	2.00	24.02	10	0.01	3.12E-07	3.04E-21	8.56E-01	4.00	0.00
3^{-1}	2.00	24.02	10	0.01	8.68E-08	5.50E-23	1.89E-01	4.00	0.00
3^{-2}	2.00	24.02	12	0.02	3.79E-07	1.13E-22	3.29E-02	4.00	0.00
3^2	2.00	24.02	17	0.02	3.38E-10	4.25E-13	8.86E+00	4.00	0.00
3^1	2.00	24.02	13	0.02	5.23E-12	5.79E-24	2.86E+00	4.00	0.00
3^0	2.00	24.02	17	0.02	4.88E-13	4.62E-23	8.56E-01	4.00	0.00
3^{-1}	2.00	24.02	11	0.01	2.70E-12	1.78E-18	1.89E-01	4.00	0.00
3^{-2}	2.00	24.02	11	0.01	1.40E-11	8.04E-24	3.29E-02	4.00	0.00

Problem name: `Colson2002BIPA4`

Source: [7]

description: BIPA4 is defined as follows

$$\begin{aligned} F(x,y) &:= x^2+(y-10)^2 \\ G(x,y) &:= \begin{bmatrix} x+2y-6 \\ -x \end{bmatrix} \\ f(x,y) &:= x^3+2y^3+x-2y-x^2 \\ g(x,y) &:= \begin{bmatrix} -x+2y-3 \\ -y \end{bmatrix} \end{aligned}$$

Comments: Colson, Marcotte and Savard in [8] got solution such that $F(x^*,y^*) = 88.79$, $f(x^*,y^*) = -0.77$. Results were listed in following table , where two starting points were $x^0 = 2.00, y^0 = 2.00$ and $x^0 = 0.00, y^0 = 0.00$. Similar solutions were generated by our algorithm when $\lambda = 3^2$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	88.79	-0.77	12	0.13	4.32E-07	3.02E-01	1.24E-23	0.00	0.58
3^1	86.94	-0.73	11	0.02	7.32E-07	6.76E-01	7.39E-01	0.00	0.68
3^0	80.08	0.22	14	0.02	2.82E-07	1.00E+00	1.00E+00	0.00	1.05
3^{-1}	69.90	5.83	10	0.01	1.00E-09	1.00E+00	1.00E+00	0.29	1.64
3^{-2}	62.52	20.11	10	0.03	2.56E-10	1.00E+00	1.01E+00	1.46	2.23
3^2	88.79	-0.77	13	0.02	3.58E-07	3.02E-01	0.00E+00	0.00	0.58
3^1	86.94	-0.73	12	0.02	4.48E-07	6.76E-01	7.39E-01	0.00	0.68
3^0	80.08	0.22	9	0.01	1.48E-08	1.00E+00	1.00E+00	0.00	1.05
3^{-1}	69.90	5.83	12	0.02	1.95E-10	1.00E+00	1.00E+00	0.29	1.64
3^{-2}	62.52	20.11	12	0.01	5.90E-09	1.00E+00	1.01E+00	1.46	2.23

Problem name: `Colson2002BIPA5`

Source: [7]

description: BIPA5 is defined as follows

$$
\begin{aligned}
F(x,y) &:= (x-y_2)^4+(y_1-1)^2+(y_1-y_2)^2\\
G(x,y) &:= -x\\
f(x,y) &:= 2x+e^{y_1}+y_1^2+4y_1+2y_2^2-6y_2\\
g(x,y) &:= \begin{bmatrix} 6x+y_1^2+e^{y_2}-15\\ 5x+y_1^4-y_2-25\\ y_1-4\\ y_2-2\\ -y_1\\ -y_2 \end{bmatrix}
\end{aligned}
$$

Comments: Colson, Marcotte and Savard in [8] got solutions x^*, y^* such that $F(x^*,y^*)=2.75$, $f(x^*,y^*)=0.57$. Results were listed in following table, in which our method got one solution $\hat{x}=1.94, \hat{y}=(0,1.21)^T$ such that $F(\hat{x},\hat{y})\approx F(x^*,y^*)$, $f(\hat{x},\hat{y})\approx f(x^*,y^*)$. Two starting points were $x^0=1.00, y^0=(0.00,0.00)^T$ and $x^0=0.00, y^0=(0.00,0.00)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	2.75	0.55	14	0.14	3.00E-07	2.81E-09	7.12E+00	1.94	0.00 1.21
3^1	2.75	0.55	16	0.02	7.08E-09	2.11E-10	1.12E+00	1.94	0.00 1.21
3^0	2.96	-0.94	1000	1.61	1.61E-02	3.00E-01	9.60E-01	1.27	0.00 1.40
3^{-1}	0.89	1.52	21	0.03	1.08E-08	4.61E-01	1.00E+00	1.18	0.43 1.18
3^{-2}	0.20	4.39	14	0.03	2.18E-07	5.76E-01	1.00E+00	1.44	0.75 1.09
3^2	2.75	0.55	20	0.03	1.37E-13	1.47E-15	7.12E+00	1.94	0.00 1.21
3^1	3.25	-0.42	113	0.17	1.46E-05	1.43E-01	1.00E+00	1.54	0.00 1.50
3^0	2.96	-0.94	1000	1.24	2.42E-02	3.01E-01	9.60E-01	1.27	0.00 1.40
3^{-1}	0.95	2.52	16	0.02	4.31E-10	3.73E-01	1.00E+00	1.69	0.43 1.18
3^{-2}	0.20	4.39	19	0.02	2.02E-09	5.76E-01	1.00E+00	1.44	0.75 1.09

Problem name: `Dempe1992a`

Source: [12]

Description: Dempe 1992 defined one example as follows

$$\begin{aligned} F(x,y) &:= y_2 \\ G(x,y) &:= x_1^2 + (x_2+1)^2 - 1 \\ f(x,y) &:= \tfrac{1}{2}(y_1-1)^2 + \tfrac{1}{2}y_2^2 \\ g(x,y) &:= \begin{bmatrix} y_1 + y_2 x_1 + x_2 \\ y_1 \end{bmatrix} \end{aligned}$$

Comments: Solutions were unknown. Results were listed in following table. Two starting points were $x^0 = (1.00, -1.00)^\top, y^0 = (0.00, 0.00)^\top$ and $x^0 = (1.00, 1.00)^\top, y^0 = (1.00, 1.00)^\top$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.50	12	0.13	8.02E-09	1.11E-01	8.00E+00	0.25 -0.33	0.00 0.00
3^1	0.00	0.50	10	0.02	2.52E-11	3.33E-01	2.00E+00	-0.26 -1.09	0.00 0.00
3^0	-0.33	0.55	11	0.02	3.33E-08	6.76E-01	1.37E+00	0.33 -0.05	0.00 -0.33
3^{-1}	-0.81	0.83	16	0.02	9.26E-12	1.48E+00	1.00E+00	0.81 -0.41	0.00 -0.81
3^{-2}	-2.25	5.19	23	0.05	4.04E-09	1.24E+00	1.00E+00	0.98 -0.78	-1.30 -2.25
3^2	0.00	0.50	43	0.06	9.10E-07	1.11E-01	8.00E+00	0.00 0.00	0.00 0.00
3^1	0.00	0.50	38	0.05	9.81E-07	3.33E-01	2.00E+00	0.00 0.00	0.00 0.00
3^0	0.00	0.50	28	0.03	8.26E-07	1.00E+00	1.04E-06	0.00 0.00	0.00 0.00
3^{-1}	-0.81	0.83	15	0.02	5.35E-10	1.48E+00	1.00E+00	0.81 -0.41	0.00 -0.81
3^{-2}	0.00	0.50	20	0.03	1.32E-08	1.00E+00	8.89E-01	-0.04 0.00	0.00 0.00

Problem name: `Dempe1992b`

Source: [12]

Description: Dempe 1992 defined one example as follows

$$\begin{aligned} F(x,y) &:= (x-3.5)^2+(y+4)^2 \\ f(x,y) &:= (y-3)^2 \\ g(x,y) &:= y^2-x \end{aligned}$$

Comments: Results were listed in following table, in which solutions generated by our method when $\lambda \in \{3^2, 3^1\}$ were the same as the one derived by Colson, Marcotte and Savard in [8]. Two starting points were $x^0 = 1.00, y^0 = 0.00$ and $x^0 = 4.00, y^0 = 2.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	31.25	4.00	17	0.10	1.67E-07	4.62E-14	5.50E+00	1.00	1.00
3^1	31.25	4.00	12	0.02	6.72E-07	1.07E-10	5.00E-01	1.00	1.00
3^0	18.45	7.79	10	0.01	6.07E-07	8.72E-01	1.00E+00	2.64	0.21
3^{-1}	6.69	19.79	10	0.01	3.40E-07	1.83E+00	1.00E+00	3.08	-1.45
3^{-2}	4.39	24.11	12	0.02	1.31E-12	2.00E+00	7.72E-01	3.65	-1.91
3^2	31.25	4.00	23	0.02	2.37E-08	8.62E-14	5.50E+00	1.00	1.00
3^1	31.25	4.00	13	0.01	3.28E-07	6.02E-13	5.00E-01	1.00	1.00
3^0	18.45	7.79	8	0.01	1.58E-07	8.72E-01	1.00E+00	2.64	0.21
3^{-1}	6.69	19.79	17	0.02	8.73E-08	1.83E+00	1.00E+00	3.08	-1.45
3^{-2}	4.39	24.11	116	0.14	4.55E-10	2.00E+00	7.72E-01	3.65	-1.91

Problem name: `DempeDutta2012Ex24`

Source: [13]

Description: Dempe and Dutta 2012 defined Example 2.4 as follows

$$\begin{aligned} F(x,y) &:= (x-1)^2 + y^2 \\ f(x,y) &:= x^2 y \\ g(x,y) &:= y^2 \end{aligned}$$

Comment: The global optimal solution of the problem is $(1,0)$, seen [13]. Results were listed in following table, where two starting points were $x^0 = 2.00, y^0 = 0.00$ and $x^0 = 1.00, y^0 = 1.00$. Under some λ, solutions were similar to the global optimal solution.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	14.05	-1.15	190	0.27	2.33E-02	1.52E-01	9.33E-01	-2.75	-0.15
3^1	172.12	-61.91	1000	0.56	1.80E-01	3.61E-01	5.66E-01	-12.11	-0.42
3^0	251.19	-241.28	483	0.26	1.22E+00	9.68E-01	9.68E-01	-14.81	-1.10
3^{-1}	0.00	0.00	1000	0.50	6.93E-02	2.42E-03	8.95E-01	1.00	0.00
3^{-2}	0.00	-0.04	1000	0.46	5.11E-03	2.96E-02	9.94E-01	1.00	-0.04
3^2	1.89	-0.03	1000	0.57	2.23E+01	4.33E-01	1.50E+01	-0.20	-0.68
3^1	295.73	-124.23	1000	0.52	2.25E-01	3.54E-01	2.41E-01	-16.19	-0.47
3^0	0.00	0.00	1000	0.53	9.43E-02	5.26E-04	2.02E-01	1.00	0.00
3^{-1}	0.00	-0.01	1000	0.48	6.83E-01	3.50E-02	7.09E-01	0.99	-0.01
3^{-2}	0.00	-0.06	1000	0.46	2.84E-02	4.37E-02	1.00E+00	1.00	-0.06

Problem name: `DempeDutta2012Ex31`

Source: [13]

Description: Dempe and Dutta 2012 defined Example 3.1 as follows

$$
\begin{aligned}
F(x,y) &:= -y_2 \\
G(x,y) &:= \begin{bmatrix} -x_1 \\ -x_2 \\ y_1y_2 \\ -y_1y_2 \end{bmatrix} \\
f(x,y) &:= y_1^2 + (y_2+1)^2 \\
g(x,y) &:= \begin{bmatrix} (y_1-x_1)^2 + (y_2-x_1-1)^2 - 1 \\ (y_1+x_2)^2 + (y_2-x_2-1)^2 - 1 \end{bmatrix}
\end{aligned}
$$

Comment: The point $x^* = (\sqrt{2}/2, \sqrt{2}/2)^T, y^* = (0,1)^T$ is the best known solution of the problem provided in [13, 38] with $F(x^*,y^*) = -1$ and $f(x^*,y^*) = 4$. Results were listed in following table. Two starting points were $x^0 = (0.80, 0.80)^T, y^0 = (0.80, 0.80)^T$ and $x^0 = (0.50, 0.50)^T, y^0 = (0.50, 0.50)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	1.00	127	0.43	1.43E-03	2.77E-03	7.97E+00	0.00 0.00	0.00 0.00
3^1	-0.16	1.35	19	0.03	1.52E-10	1.52E-01	1.63E+00	0.15 0.01	0.00 0.16
3^0	-0.46	2.12	18	0.02	8.57E-07	3.88E-01	2.34E-01	0.38 0.00	0.00 0.46
3^{-1}	-1.12	4.48	17	0.03	3.33E-10	7.53E-01	7.63E-01	0.76 0.01	0.00 1.12
3^{-2}	0.00	1.00	152	0.20	1.22E-03	5.07E-02	6.17E+00	0.00 0.00	-0.03 0.00
3^2	-0.06	1.11	29	0.05	5.69E-09	5.40E-02	7.55E+00	0.05 0.05	0.00 0.06
3^1	-0.16	1.35	345	0.54	5.70E-07	1.52E-01	1.63E+00	0.11 0.15	0.00 0.16
3^0	-0.46	2.12	12	0.01	3.31E-09	3.88E-01	2.34E-01	0.38 0.01	0.00 0.46
3^{-1}	-1.12	4.48	27	0.04	1.69E-11	7.53E-01	7.63E-01	0.76 0.00	0.00 1.12
3^{-2}	0.00	1.00	117	0.22	1.44E-04	1.71E-02	8.42E-01	0.00 0.00	0.01 0.00

Problem name: `DempeEtal2012`

Source: [17]

Description: Dempe et al. 2012 defined one example as follows

$$\begin{aligned} F(x,y) &:= x \\ G(x,y) &:= \begin{bmatrix} -1-x \\ -1+x \end{bmatrix} \\ f(x,y) &:= xy \\ g(x,y) &:= \begin{bmatrix} -y \\ -1+y \end{bmatrix} \end{aligned}$$

Comment: The best known optimal solution is $x^* = -1, y^* = 1$, seen [17]. Results were listed in following table where our method achieved the same one under many cases. Two starting points were $x^0 = -1.00, y^0 = 0.00$ and $x^0 = 1.00, y^0 = 1.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.00	11	0.12	3.03E-14	1.11E-01	1.35E-14	0.00	0.36
3^1	0.00	0.00	158	0.24	4.75E-10	3.33E-01	3.01E-10	0.00	0.59
3^0	-1.00	-1.00	12	0.02	8.60E-08	0.00E+00	2.12E-22	-1.00	1.00
3^{-1}	-1.00	-1.00	12	0.01	5.31E-12	0.00E+00	6.67E-01	-1.00	1.00
3^{-2}	-1.00	-1.00	11	0.02	2.08E-10	4.18E-11	8.89E-01	-1.00	1.00
3^2	-1.00	-1.00	9	0.01	1.03E-09	0.00E+00	8.00E+00	-1.00	1.00
3^1	0.00	0.00	10	0.01	4.76E-13	3.33E-01	2.14E-13	0.00	0.66
3^0	-1.00	-1.00	10	0.01	4.19E-11	0.00E+00	0.00E+00	-1.00	1.00
3^{-1}	-1.00	-1.00	21	0.02	2.23E-07	1.88E-07	6.67E-01	-1.00	1.00
3^{-2}	-1.00	-1.00	10	0.01	1.32E-07	4.27E-08	8.89E-01	-1.00	1.00

Problem name: `DempeFranke2011Ex41`

Source: [14]

Description: Dempe and Franke 2011 defined Example 4.1 as follows

$$\begin{array}{rcl} F(x,y) & := & x_1 + y_1^2 + y_2^2 \\ G(x,y) & := & \begin{bmatrix} -1 - x_1 \\ -1 + x_1 \\ -1 - x_2 \\ 1 + x_2 \end{bmatrix} \\ f(x,y) & := & x^\top y \\ g(x,y) & := & \begin{bmatrix} -2y_1 + y_2 \\ y_1 - 2 \\ y_2 - 2 \\ -y_2 \end{bmatrix} \end{array}$$

Comment: The best known optimal solution is $x^* = (0,-1)^\top, y^* = (1,2)^\top$, seen [14]. Results were listed in following table where our method achieved the same one when $\lambda = 3^2$. Two starting points were $x^0 = (-1.00,-1.00)^\top, y^0 = (1.00,2.00)^\top$ and $x^0 = (0.00,-1.00)^\top, y^0 = (1.00,1.00)^\top$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	5.00	-2.00	58	0.20	8.49E-07	4.86E-02	4.00E+00	0.00 -1.00	1.00 2.00
3^1	4.52	-1.87	22	0.03	2.68E-09	1.48E-01	1.00E+00	0.00 -1.00	1.00 1.87
3^0	2.56	-1.25	45	0.05	1.49E-09	4.42E-01	1.00E+00	0.00 -1.00	1.00 1.25
3^{-1}	-0.22	-1.25	25	0.03	1.15E-07	6.87E-01	1.00E+00	-1.00 -1.00	0.62 0.63
3^{-2}	-0.88	-0.50	31	0.04	9.69E-07	8.75E-01	1.00E+00	-1.00 -1.00	0.25 0.25
3^2	5.00	-2.00	38	0.04	7.81E-07	4.86E-02	4.00E+00	0.00 -1.00	1.00 2.00
3^1	4.52	-1.87	20	0.03	6.29E-08	1.48E-01	1.00E+00	0.00 -1.00	1.00 1.87
3^0	2.56	-1.25	36	0.07	2.64E-07	4.42E-01	1.00E+00	0.00 -1.00	1.00 1.25
3^{-1}	-0.22	-1.25	24	0.02	9.12E-07	6.87E-01	1.00E+00	-1.00 -1.00	0.63 0.63
3^{-2}	-0.88	-0.50	63	0.06	1.61E-10	8.75E-01	1.00E+00	-1.00 -1.00	0.25 0.25

Problem name: `DempeFranke2011Ex42`

Source: [14]

Description: Dempe and Franke 2011 defined Example 4.2 as follows

$$\begin{aligned}
F(x,y) &:= x_1 + (y_1-1)^2 + y_2^2 \\
G(x,y) &:= \begin{bmatrix} -1-x_1 \\ -1+x_1 \\ -1-x_2 \\ 1+x_2 \end{bmatrix} \\
f(x,y) &:= x^\top y \\
g(x,y) &:= \begin{bmatrix} -y_1+y_2-1 \\ y_1+y_2-3.5 \\ y_2-2 \end{bmatrix}
\end{aligned}$$

Comment: Dempe and Franke reported in [14], there were three local strictly optimal solutions (i) $x^* = (-1,-1)^\top, y^* = (2.25, 1.25)^\top$, (ii) $x^* = (0,-1)^\top, y^* = (2,2)^\top$ and (iii) $x^* = (1,-1)^\top, y^* = (0,1)^\top$. Results were listed in following table where our method closely achieved the first one when $\lambda = 3$ under the first starting point and $\lambda = 3^2$ under the second starting point. Two starting points were $x^0 = (0.00,-1.00)^\top, y^0 = (2.50, 1.00)^\top$ and $x^0 = (0.00,-1.00)^\top, y^0 = (2.00, 1.50)^\top$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	3.13	-3.50	54	0.18	4.61E-12	3.62E-02	5.50E+00	-1.00 -1.00	2.96 0.54
3^1	2.13	-3.50	12	0.02	3.48E-07	9.56E-03	5.00E-01	-1.00 -1.00	2.30 1.20
3^0	0.61	-2.75	18	0.03	5.46E-07	2.16E-01	1.00E+00	-1.00 -1.00	2.08 0.67
3^{-1}	-0.62	-1.87	21	0.02	6.56E-07	4.51E-01	1.00E+00	-1.00 -1.00	1.42 0.45
3^{-2}	-0.93	-1.35	15	0.02	7.12E-07	7.18E-01	1.00E+00	-1.00 -1.00	1.10 0.25
3^2	2.13	-3.50	17	0.01	7.79E-10	3.68E-03	5.50E+00	-1.00 -1.00	2.19 1.31
3^1	2.23	-3.50	12	0.01	1.01E-09	4.33E-02	5.00E-01	-1.00 -1.00	2.02 1.48
3^0	0.68	-2.75	16	0.02	7.37E-07	2.65E-01	1.00E+00	-1.00 -1.00	1.60 1.15
3^{-1}	-0.61	-1.88	23	0.03	2.81E-07	4.73E-01	1.00E+00	-1.00 -1.00	1.37 0.50
3^{-2}	-0.93	-1.35	17	0.01	4.42E-07	7.11E-01	1.00E+00	-1.00 -1.00	1.10 0.25

Problem name: `DempeFranke2014Ex38`

Source: [15]

Description: Dempe and Franke 2014 defined Example 3.8 as follows

$$\begin{aligned}
F(x,y) &:= 2x_1 + x_2 + 2y_1 - y_2 \\
G(x,y) &:= \begin{bmatrix} -1 - x_1 \\ -1 + x_1 \\ -1 - x_2 \\ 0.75 + x_2 \end{bmatrix} \\
f(x,y) &:= x^\top y \\
g(x,y) &:= \begin{bmatrix} -2y_1 + y_2 \\ y_1 - 2 \\ y_2 - 2 \\ -y_2 \end{bmatrix}
\end{aligned}$$

Comment: The best known optimal solution is $x^* = (-1,-1)^\top, y^* = (2,2)^\top$, seen [15]. Results were listed in following table, where two starting points were $x^0 = (-1.00,-1.00)^\top, y^0 = (0.00,0.00)^\top$ and $x^0 = (-1.00,-1.00)^\top, y^0 = (1.00,1.00)^\top$. Our method achieved the same one under four cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-1.00	-4.00	49	0.17	2.49E-09	0.00E+00	7.65E+00	-1.00 -1.00	2.00 2.00
3^1	-1.00	-4.00	124	0.18	6.45E-03	1.41E-07	2.12E+00	-1.00 -1.00	2.00 2.00
3^0	-2.00	-3.50	87	0.10	2.93E-09	1.77E-01	1.00E+00	-1.00 -1.00	1.50 2.00
3^{-1}	-3.00	-3.00	20	0.02	1.01E-11	3.54E-01	7.99E-01	-1.00 -1.00	1.00 2.00
3^{-2}	-3.00	-3.00	13	0.03	7.55E-08	3.54E-01	1.05E+00	-1.00 -1.00	1.00 2.00
3^2	-1.00	-4.00	684	0.74	1.37E-10	0.00E+00	7.65E+00	-1.00 -1.00	2.00 2.00
3^1	-1.00	-4.00	21	0.03	7.35E-09	0.00E+00	2.12E+00	-1.00 -1.00	2.00 2.00
3^0	-2.00	-3.50	80	0.08	5.08E-08	1.77E-01	1.00E+00	-1.00 -1.00	1.50 2.00
3^{-1}	-3.00	-3.00	110	0.11	1.22E-04	3.54E-01	7.99E-01	-1.00 -1.00	1.00 2.00
3^{-2}	-3.00	-3.00	30	0.03	1.43E-07	3.54E-01	1.05E+00	-1.00 -1.00	1.00 2.00

Problem name: `DempeLohse2011Ex31a`

Source: [16]

description: Dempe and Lohse 2011 defined Example 3.1a as follows

$$\begin{array}{rcl} F(x,y) & := & (x_1-0.5)^2+(x_2-0.5)^2-3y_1-3y_2 \\ f(x,y) & := & x_1y_1+x_2y_2 \\ g(x,y) & := & \begin{bmatrix} y_1+y_2-2 \\ -y_1+y_2 \\ -y_1 \\ -y_2 \end{bmatrix} \end{array}$$

Comment: Results were listed in following table, where two starting points were $x^0=(-1,-1)^\top$, $y^0=(1,1)^\top$ and $x^0=(1,1)^\top, y^0=(1,1)^\top$. When $\lambda=3^{-2}$, solutions were close to the unique optimal solution $x^*=(0.5,0.5)^\top, y^*=(1,1)^\top$ reported in [16].

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.00	7	0.11	4.35E-09	2.77E-09	1.99E+00	0.50 0.50	0.00 0.00
3^1	-5.50	0.00	9	0.01	9.15E-08	3.33E-01	3.00E+00	0.00 0.00	1.00 1.00
3^0	-5.50	0.00	9	0.02	9.77E-07	1.00E+00	3.00E+00	0.00 0.00	1.00 1.00
3^{-1}	-5.94	0.67	7	0.01	2.32E-10	1.00E+00	2.93E+00	0.33 0.33	1.00 1.00
3^{-2}	-5.99	0.89	9	0.02	8.72E-07	1.00E+00	3.02E+00	0.44 0.44	1.00 1.00
3^2	0.00	0.00	7	0.01	4.35E-09	2.77E-09	1.99E+00	0.50 0.50	0.00 0.00
3^1	-5.50	0.00	7	0.01	3.77E-09	3.33E-01	3.00E+00	0.00 0.00	1.00 1.00
3^0	-5.50	0.00	9	0.01	9.77E-07	1.00E+00	3.00E+00	0.00 0.00	1.00 1.00
3^{-1}	-5.94	0.67	7	0.01	2.32E-10	1.00E+00	2.93E+00	0.33 0.33	1.00 1.00
3^{-2}	-5.99	0.89	9	0.01	8.72E-07	1.00E+00	3.02E+00	0.44 0.44	1.00 1.00

Problem name: `DempeLohse2011Ex31b`

Source: [16]

description: Dempe and Lohse 2011 defined Example 3.1b as follows

$$\begin{array}{rcl} F(x,y) & := & (x_1-0.5)^2+(x_2-0.5)^2+x_3^2-3y_1-3y_2-6y_3 \\ f(x,y) & := & x_1y_1+x_2y_2+x_3y_3 \\ g(x,y) & := & \begin{bmatrix} y_1+y_2+y_3-2 \\ -y_1+y_2 \\ -y_1 \\ -y_2 \\ -y_3 \end{bmatrix} \end{array}$$

Comment: Results were listed in following table, where solutions generated by our method with $\lambda \leq 3$ were the optimal ones. Two starting points were $x^0 = (1.00, 1.00, 0.00)^T, y^0 = (0.00, 0.00, 0.00)^T$ and $x^0 = (0.00, 0.00, 0.00)^T, y^0 = (0.00, 0.00, 1.00)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	4.04	-0.47	1000	1.52	7.63E-01	1.92E-01	1.71E+00	0.50 0.50 0.86	0.00 0.00 -0.55
3^1	-12.00	0.00	236	0.25	2.14E-07	1.54E-07	9.70E+00	0.50 0.50 0.00	0.00 0.00 2.00
3^0	-12.00	0.00	28	0.03	7.77E-07	9.31E-07	7.35E+00	0.50 0.50 0.00	0.00 0.00 2.00
3^{-1}	-12.00	0.00	39	0.04	8.87E-07	2.90E-06	7.09E+00	0.50 0.50 0.00	0.00 0.00 2.00
3^{-2}	-12.00	0.00	89	0.11	4.34E-07	4.01E-06	7.00E+00	0.50 0.50 0.00	0.00 0.00 2.00
3^2	1.81	-0.15	1000	1.09	2.07E-01	2.05E-01	1.16E+00	0.33 0.33 0.89	0.04 0.04 -0.20
3^1	-12.00	0.00	289	0.29	1.28E-09	9.16E-10	9.83E+00	0.50 0.50 0.00	0.00 0.00 2.00
3^0	-12.00	0.00	19	0.02	8.43E-07	7.77E-07	7.35E+00	0.50 0.50 0.00	0.00 0.00 2.00
3^{-1}	-12.00	0.00	45	0.04	8.72E-07	2.83E-06	7.09E+00	0.50 0.50 0.00	0.00 0.00 2.00
3^{-2}	-12.00	0.00	49	0.05	9.46E-07	8.75E-06	7.00E+00	0.50 0.50 0.00	0.00 0.00 2.00

Problem name: `DeSilva1978`

source: [18]

description: De Silva 1978 defined one example as follows

$$\begin{array}{rcl} F(x,y) & := & x_1^2 - 2x_1 + x_2^2 - 2x_2 + y_1^2 + y_2^2 \\ f(x,y) & := & (y_1 - x_1)^2 + (y_2 - x_2)^2 \\ g(x,y) & := & \begin{bmatrix} -y_1 + 0.5 \\ -y_2 + 0.5 \\ y_1 - 1.5 \\ y_2 - 1.5 \end{bmatrix} \end{array}$$

Comment: Results were listed in following table, where two starting points were $x^0 = (0.5, 0.5)^\top$, $y^0 = (0,0)^\top$ and $x^0 = (0,0)^\top, y^0 = (0,0)^\top$. Solutions generated by our method with $\lambda \geq 3^0$ were similar to $x^* = y^* = (0.5, 0.5)^\top$ obtained by Colson, Marcotte and Savard in [8].

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-1.00	0.00	25	0.14	7.17E-07	7.86E-02	3.27E-06	0.50 0.50	0.50 0.50
3^1	-1.00	0.00	7	0.01	2.25E-09	2.36E-01	4.40E-09	0.50 0.50	0.50 0.50
3^0	-1.00	0.00	15	0.02	6.53E-07	5.00E-01	2.98E-06	0.50 0.50	0.50 0.50
3^{-1}	-1.28	0.06	8	0.01	1.34E-09	6.67E-01	5.67E-01	0.67 0.67	0.50 0.50
3^{-2}	-1.48	0.30	10	0.02	4.09E-09	6.67E-01	1.34E+00	0.89 0.89	0.50 0.50
3^2	-1.00	0.00	25	0.03	7.17E-07	7.86E-02	3.27E-06	0.50 0.50	0.50 0.50
3^1	-1.00	0.00	10	0.01	9.55E-10	2.36E-01	1.97E-10	0.50 0.50	0.50 0.50
3^0	-1.00	0.00	15	0.02	6.53E-07	5.00E-01	2.98E-06	0.50 0.50	0.50 0.50
3^{-1}	-1.28	0.06	8	0.01	1.34E-09	6.67E-01	5.67E-01	0.67 0.67	0.50 0.50
3^{-2}	-1.48	0.30	10	0.01	4.09E-09	6.67E-01	1.34E+00	0.89 0.89	0.50 0.50

Problem name: `EdmundsBard1991`

Source: [19]

Description: Edmunds and Bard 1991 defined one example as follows

$$\begin{aligned}
F(x,y) &:= 2x_1 + 2x_2 - 3y_1 - 3y_2 - 60\\
G(x,y) &:= \begin{bmatrix} -x_1\\ -x_2\\ x_1 - 50\\ x_2 - 50 \end{bmatrix}\\
f(x,y) &:= (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2\\
g(x,y) &:= \begin{bmatrix} y_1 - 20\\ y_2 - 20\\ -y_1 - 10\\ -y_2 - 10\\ 2y_1 - x_1 + 10\\ 2y_2 - x_2 + 10\\ x_1 + x_2 + y_1 - 2y_2 - 40 \end{bmatrix}
\end{aligned}$$

Comment: The best know solution of the problem is $x^* = (0,0)^T$, $y^* = (-10,-10)^T$, seen [49]. Results were listed in following table, where two starting points were $x^0 = (-20.00,-20.00)^T$, $y^0 = (-20.00,-20.00)^T$ and $x^0 = (-10.00,-10.00)^T, y^0 = (-20.00,-20.00)^T$. Under $\lambda = 3^0$, solutions were the same as the above mentioned one.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	90.00	1000.00	24	0.15	2.89E-08	2.64E-10	7.99E+00	40.00 50.00	-10.00 20.00
3^1	10.00	200.00	38	0.05	6.71E-07	0.00E+00	1.90E+00	0.00 50.00	-10.00 20.00
3^0	0.00	200.00	21	0.02	5.76E-11	0.00E+00	1.50E-01	0.00 0.00	-10.00 -10.00
3^{-1}	5.00	0.00	40	0.07	4.22E-11	3.61E-01	1.58E+00	25.00 30.00	5.00 10.00
3^{-2}	5.00	0.00	15	0.04	6.20E-08	1.08E+00	1.58E+00	25.00 30.00	5.00 10.00
3^2	90.00	1000.00	24	0.04	2.89E-08	2.64E-10	7.99E+00	40.00 50.00	-10.00 20.00
3^1	10.00	200.00	38	0.06	6.71E-07	0.00E+00	1.90E+00	0.00 50.00	-10.00 20.00
3^0	0.00	200.00	21	0.03	5.76E-11	0.00E+00	1.50E-01	0.00 0.00	-10.00 -10.00
3^{-1}	14.00	102.25	12	0.02	4.45E-10	2.12E-01	8.21E-01	11.50 0.00	-7.00 -10.00
3^{-2}	-10.50	276.13	17	0.02	3.05E-10	1.75E-01	1.00E+00	0.00 0.00	-8.25 -8.25

Problem name: `FalkLiu1995`

source: [20]

description: Falk and Liu 1995 defined one example as follows

$$\begin{aligned} F(x,y) &:= x_1^2 - 3x_1 + x_2^2 - 3x_2 + y_1^2 + y_2^2 \\ f(x,y) &:= (y_1 - x_1)^2 + (y_2 - x_2)^2 \\ g(x,y) &:= \begin{bmatrix} -y_1 + 0.5 \\ -y_2 + 0.5 \\ y_1 - 1.5 \\ y_2 - 1.5 \end{bmatrix} \end{aligned}$$

Comment: The best known optimal upper and lower level objective function values got in Colson, Marcotte and Savard [8] are -2.25 and 0. Results were same when $\lambda \geq 3^0$. Two starting points were $x^0 = (1.50, 1.50)^T, y^0 = (1.00, 1.00)^T$ and $x^0 = (0.50, 0.50)^T, y^0 = (0.50, 0.50)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-2.25	0.00	10	0.12	8.92E-09	1.00E-01	3.15E-09	0.75 0.75	0.75 0.75
3^1	-2.25	0.00	10	0.01	5.05E-12	2.50E-01	5.05E-12	0.75 0.75	0.75 0.75
3^0	-2.25	0.00	12	0.02	4.14E-07	5.00E-01	1.08E-06	0.75 0.75	0.75 0.75
3^{-1}	-3.72	0.83	12	0.02	1.24E-11	6.43E-01	1.00E+00	1.18 1.18	0.54 0.54
3^{-2}	-3.98	1.58	7	0.02	1.18E-11	6.67E-01	1.05E+00	1.39 1.39	0.50 0.50
3^2	-2.25	0.00	10	0.01	8.92E-09	1.00E-01	3.15E-09	0.75 0.75	0.75 0.75
3^1	-2.25	0.00	10	0.01	5.05E-12	2.50E-01	5.05E-12	0.75 0.75	0.75 0.75
3^0	-2.25	0.00	13	0.01	9.08E-07	5.00E-01	2.38E-06	0.75 0.75	0.75 0.75
3^{-1}	-3.72	0.83	9	0.01	2.88E-07	6.43E-01	1.00E+00	1.18 1.18	0.54 0.54
3^{-2}	-3.98	1.58	9	0.01	3.85E-07	6.67E-01	1.05E+00	1.39 1.39	0.50 0.50

Problem name: `FloudasZlobec1998`

Source: [21]

Description: Floudas and Zlobec 1998 defined one example as follows

$$
\begin{aligned}
F(x,y) &:= x^3 y_1 + y_2 \\
G(x,y) &:= \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\
f(x,y) &:= -y_2 \\
g(x,y) &:= \begin{bmatrix} -y_1 - 1 \\ y_1 - 1 \\ -y_2 \\ y_2 - 100 \\ xy_1 - 10 \\ y_1^2 + xy_2 - 1 \end{bmatrix}
\end{aligned}
$$

Comment: Notice that explicit bounds on the variable y were added. This is same as [35]. The global optimal solution is $x^* = 1, y^* = (0,1)^T$ according to [22, 21, 35]. In following table, where two starting points were $x^0 = 1.00, y^0 = (0.00, 0.00)^T$ and $x^0 = 0.00, y^0 = (0.00, 1.00)^T$, results were similar to the unique one when $\lambda = 3^2$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.93	-1.00	9	0.13	9.58E-11	6.26E-02	7.00E+00	1.00	-0.06 1.00
3^1	100.00	-100.00	89	0.19	6.49E-07	1.99E-02	1.00E+00	0.00	0.99 100.00
3^0	0.00	0.00	142	0.29	1.54E-07	1.00E+00	1.00E+00	0.00	1.00 0.00
3^{-1}	0.00	0.00	30	0.05	4.64E-07	1.00E+00	1.20E+00	0.00	-1.00 0.00
3^{-2}	-1.00	0.00	627	0.99	1.62E-07	1.41E+00	1.67E+00	1.00	-1.00 0.00
3^2	100.00	-100.00	35	0.04	9.37E-07	7.32E-07	7.00E+00	0.01	0.00 100.00
3^1	100.00	-100.00	33	0.04	9.85E-07	5.13E-07	1.00E+00	0.00	-1.00 100.00
3^0	100.00	-100.00	34	0.04	6.01E-10	6.19E-04	1.00E+00	0.00	0.06 100.00
3^{-1}	0.00	0.00	121	0.24	9.71E-05	1.00E+00	1.20E+00	0.00	-1.00 0.00
3^{-2}	-0.13	0.00	570	0.89	7.06E-01	1.12E+00	1.08E+00	0.50	-1.00 0.00

Problem name: `GumusFloudas2001Ex1`

Source: [22]

description: Gumus and Floudas 2001 defined Example 1 as follows

$$\begin{aligned} F(x,y) &:= 16x^2+9y^2 \\ G(x,y) &:= \begin{bmatrix} -x \\ x-12.5 \\ -4x+y \end{bmatrix} \\ f(x,y) &:= (x+y-20)^4 \\ g(x,y) &:= \begin{bmatrix} -y \\ y-50 \\ 4x+y-50 \end{bmatrix} \end{aligned}$$

Comment: Notice that explicit upper bounds on the variables were added. This is same as [35], in which Mitsos and Barton got solution $x^* = 11.25, y^* = 5$ with $F(x^*, y^*) = 2250$, $f(x^*, y^*) = 197.75$. Our method achieved the same one under three cases. Two starting points were $x^0 = 12.50, y^0 = 0.00$ and $x^0 = 5.00, y^0 = 5.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	2250.00	197.75	34	0.14	1.47E-09	1.07E-15	7.57E+00	11.25	5.00
3^1	2250.00	197.75	13	0.03	9.21E-13	3.55E-16	1.57E+00	11.25	5.00
3^0	2018.86	2.97	13	0.02	1.65E-08	4.54E-01	1.00E+00	7.27	11.42
3^{-1}	1863.78	18.48	11	0.01	2.45E-13	4.96E-01	1.00E+00	7.17	10.76
3^{-2}	1652.20	98.89	13	0.03	3.65E-13	5.56E-01	1.00E+00	6.90	9.95
3^2	2196.03	0.05	538	0.65	6.70E-07	4.15E-01	4.44E-01	7.27	12.25
3^1	2250.00	197.75	14	0.01	1.04E-11	5.33E-16	1.57E+00	11.25	5.00
3^0	2230.69	49.16	16	0.01	1.57E-12	7.51E-02	1.00E+00	10.70	6.65
3^{-1}	1863.78	18.48	11	0.01	2.10E-09	4.96E-01	1.00E+00	7.17	10.76
3^{-2}	1652.20	98.89	11	0.01	6.65E-07	5.56E-01	1.00E+00	6.90	9.95

Problem name: `GumusFloudas2001Ex3`

Source: [22]

description: Gumus and Floudas 2001 defined Example 3 as follows

$$
\begin{aligned}
F(x,y) &:= -8x_1 - 4x_2 + y_1 - 40y_2 - 4y_3 \\
G(x,y) &:= \begin{bmatrix} -x_1 \\ -x_2 \\ x_1 - 2 \\ x_2 - 2 \end{bmatrix} \\
f(x,y) &:= \frac{1 + x_1 + x_2 + 2y_1 - y_2 + y_3}{6 + 2x_1 + y_1 + y_2 - 3y_3} \\
g(x,y) &:= \begin{bmatrix} -y_1 \\ -y_2 \\ -y_3 \\ y_1 - 2 \\ y_2 - 2 \\ y_3 - 2 \\ -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - \frac{1}{2}y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - \frac{1}{2}y_3 - 1 \end{bmatrix}
\end{aligned}
$$

Comment: Notice that explicit upper bounds on the variables were added. This is same as [35], in which best known solution is $x^* = (0, 0.9)^T, y^* = (0, 0.6, 0.4)^T$. In following table, where two starting points were $x^0 = (0.00, 1.00)^T, y^0 = (0.00, 0.00, 1.00)^T$ and $x^0 = (0.00, 1.00)^T, y^0 = (0.00, 0.20, 0.80)^T$, results were same as the best known one under two cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-58.00	0.58	90	8.49	8.48E-10	8.79E-01	3.14E+01	0.00 0.00	1.50 1.50 1.00
3^1	-58.00	0.58	50	4.47	2.82E-09	8.79E-01	3.27E+01	0.00 0.00	1.50 1.50 1.00
3^0	-58.00	0.58	22	1.98	1.07E-08	8.79E-01	3.33E+01	0.00 0.00	1.50 1.50 1.00
3^{-1}	-29.20	0.31	54	4.76	3.87E-11	9.33E-17	8.15E-01	0.00 0.90	0.00 0.60 0.40
3^{-2}	-58.00	0.58	20	1.78	5.21E-07	8.79E-01	3.35E+01	0.00 0.00	1.50 1.50 1.00
3^2	-29.20	0.31	82	7.88	5.11E-10	5.42E-11	1.08E+01	0.00 0.90	0.00 0.60 0.40
3^1	-58.00	0.58	34	3.07	3.87E-07	8.79E-01	3.27E+01	0.00 0.00	1.50 1.50 1.00
3^0	-58.00	0.58	19	1.71	7.92E-09	8.79E-01	3.33E+01	0.00 0.00	1.50 1.50 1.00
3^{-1}	-58.00	0.58	19	1.72	4.36E-11	8.79E-01	3.35E+01	0.00 0.00	1.50 1.50 1.00
3^{-2}	-58.00	0.58	25	2.27	2.45E-10	8.79E-01	3.35E+01	0.00 0.00	1.50 1.50 1.00

Problem name: `GumusFloudas2001Ex4`

Source: [22]

description: Gumus and Floudas 2001 defined Example 4 as follows

$$\begin{aligned} F(x,y) &:= (x-3)^2+(y-2)^2 \\ G(x,y) &:= \begin{bmatrix} -x \\ x-8 \\ -2x+y-1 \\ x-2y+2 \\ x+2y-14 \end{bmatrix} \\ f(x,y) &:= (y-5)^2 \\ g(x,y) &:= \begin{bmatrix} -y \\ y-10 \end{bmatrix} \end{aligned}$$

Comment: Notice that explicit upper bounds on the variable y were added. This is same as [35]. The unique optimal solution is $x^* = 3, y^* = 5$, seen [22, 35]. In following table, where two starting points were $x^0 = 8.00, y^0 = 10.00$ and $x^0 = 0.00, y^0 = 0.00$, results were same to the unique one when $\lambda \geq 3^0$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	9.00	0.00	13	0.12	9.22E-12	6.25E-02	9.59E-17	3.00	5.00
3^1	9.00	0.00	13	0.02	8.18E-07	1.67E-01	4.84E-18	3.00	5.00
3^0	9.00	0.00	10	0.02	2.22E-08	3.75E-01	2.53E-09	3.00	5.00
3^{-1}	4.84	0.64	11	0.02	2.98E-09	5.80E-01	1.00E+00	3.00	4.20
3^{-2}	1.00	4.00	11	0.03	3.58E-08	7.00E-01	1.00E+00	3.00	3.00
3^2	9.00	0.00	10	0.01	4.71E-07	6.25E-02	8.49E-10	3.00	5.00
3^1	9.00	0.00	9	0.01	6.67E-11	1.67E-01	8.22E-18	3.00	5.00
3^0	9.00	0.00	10	0.01	1.63E-07	3.75E-01	9.08E-15	3.00	5.00
3^{-1}	4.84	0.64	10	0.01	3.48E-10	5.80E-01	1.00E+00	3.00	4.20
3^{-2}	1.00	4.00	11	0.01	3.17E-10	7.00E-01	1.00E+00	3.00	3.00

Problem name: `GumusFloudas2001Ex5`

Source: [22]

description: Gumus and Floudas 2001 defined Example 5 as follows

$$\begin{aligned} F(x,y) &:= x \\ G(x,y) &:= \begin{bmatrix} x-10 \\ -x+0.1 \end{bmatrix} \\ f(x,y) &:= -y_1 + 0.5864 y_1^{0.67} \\ g(x,y) &:= \begin{bmatrix} y_1 - 10 \\ y_2 - 10 \\ -y_1 + 0.1 \\ -y_2 + 0.1 \\ 0.0332333/y_2 + 0.1y_1 - 1 \\ (4x + 2x^{-0.71})/y_2 + 0.0332333x^{-1.3} - 1 \end{bmatrix} \end{aligned}$$

Comment: Mitsos and Barton in [35] stated the unique optimal solution was $x^* = 0.194, y^* = (9.967, 10)^T$. In following table, where two starting points were $x^0 = 0.10, y^0 = (1.00, 10.00)^T$ and $x^0 = 0.10, y^0 = (1.00, 1.00)^T$, there were many cases were same as the unique one.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$	
3^2	0.19	-7.23	23	0.41	4.96E-08	3.50E-09	8.00E+00	0.194	9.967	10.00
3^1	0.19	-7.23	34	0.06	2.01E-07	5.28E-10	2.00E+00	0.194	9.967	10.00
3^0	0.19	0.00	211	0.51	1.41E-01	9.19E-05	6.07E-03	0.194	0.000	10.00
3^{-1}	0.19	-7.23	13	0.02	3.92E-07	6.48E-13	6.66E-01	0.194	9.967	10.00
3^{-2}	0.19	-0.72	14	0.03	3.96E-07	6.01E-01	9.96E-01	0.194	1.478	10.00
3^2	0.19	-7.23	224	0.56	9.21E-02	2.24E-09	1.81E+00	0.191	9.967	10.00
3^1	2.18	-7.23	34	0.05	5.37E-07	3.78E-08	1.99E+00	2.183	9.967	10.00
3^0	0.19	-7.23	67	0.12	5.28E-07	3.72E-08	3.20E-02	0.194	9.967	10.00
3^{-1}	2.18	-7.26	124	0.16	3.33E-03	2.34E-03	6.69E-01	2.183	10.000	10.00
3^{-2}	0.19	-7.23	61	0.09	2.39E-07	1.69E-08	8.89E-01	0.194	9.967	10.00

Problem name: `HatzEtal2013`

Source: [23]

description: Hatz et al. defined one example as follows

$$\begin{array}{rcl} F(x,y) & := & -x + 2y_1 + y_2 \\ f(x,y) & := & (x-y_1)^2 + y_2^2 \\ g(x,y) & := & \begin{bmatrix} -y_1 \\ -y_2 \end{bmatrix} \end{array}$$

Comment: Hatz et al. in [23] stated the unique optimal solution was $x^* = 0, y^* = (0,0)^T$. In following table, where two starting points were $x^0 = 1.00, y^0 = (1.00, 1.00)^T$ and $x^0 = 1.00, y^0 = (1.00, 0.00)^T$, there were same results as the unique one when $\lambda \leq 3^0$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	359.99	0.00	106	0.25	1.08E-01	2.19E-04	1.18E-01	360.10	360.05 0.00
3^1	7414.39	0.02	108	0.14	2.77E-01	3.29E-05	3.08E-01	7414.65	7414.52 0.00
3^0	0.00	0.00	7	0.01	6.83E-11	6.30E-01	1.03E+00	0.00	0.00 0.00
3^{-1}	0.00	0.00	8	0.01	9.97E-11	1.00E+00	1.09E+00	0.00	0.00 0.00
3^{-2}	0.00	0.00	8	0.02	6.47E-13	1.00E+00	1.00E+00	0.00	0.00 0.00
3^2	359.99	0.00	106	0.12	1.08E-01	2.19E-04	1.18E-01	360.10	360.05 0.00
3^1	7414.39	0.02	108	0.11	2.77E-01	3.29E-05	3.08E-01	7414.65	7414.52 0.00
3^0	0.00	0.00	7	0.01	6.90E-11	6.41E-01	1.02E+00	0.00	0.00 0.00
3^{-1}	0.00	0.00	8	0.01	9.97E-11	1.00E+00	1.09E+00	0.00	0.00 0.00
3^{-2}	0.00	0.00	9	0.01	8.32E-12	1.00E+00	1.22E+00	0.00	0.00 0.00

Problem name: `HendersonQuandt1958`

source: [24] or MacMPEC (Facchinei, Jiang and Qi [26])

description: Henderson and Quandt 1958 defined one example as follows

$$\begin{array}{rcl} F(x,y) & := & x^2/2 + xy/2 - 95x \\ G(x,y) & := & \begin{bmatrix} x - 200 \\ -x \end{bmatrix} \\ f(x,y) & := & y^2 + (x/2 - 100)y \\ g(x,y) & := & -y \end{array}$$

Comment: Results were listed in following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = 100.00, y^0 = 100.00$. All cases derived results being similar to the one $x^* = 93.33, y^* = 26.67$ with $F(x^*, x^*) = -3266.67$ and $f(x^*, x^*) = -711.11$ obtained by Facchinei, Jiang and Qi in [26].

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-3266.67	-711.11	10	0.11	4.43E-11	8.86E-02	2.04E-21	93.33	26.67
3^1	-3266.67	-711.11	10	0.02	5.03E-11	2.26E-01	5.94E-25	93.33	26.67
3^0	-3266.67	-711.11	10	0.01	4.82E-11	4.67E-01	3.16E-29	93.33	26.67
3^{-1}	-3266.67	-711.11	9	0.01	4.06E-13	7.24E-01	8.27E-16	93.33	26.67
3^{-2}	-3266.67	-711.11	8	0.02	4.93E-07	8.87E-01	1.59E-07	93.33	26.67
3^2	-3266.67	-711.11	10	0.01	4.43E-11	8.86E-02	2.04E-21	93.33	26.67
3^1	-3266.67	-711.11	10	0.01	5.03E-11	2.26E-01	5.94E-25	93.33	26.67
3^0	-3266.67	-711.11	9	0.01	8.74E-14	4.67E-01	8.61E-14	93.33	26.67
3^{-1}	-3266.67	-711.11	9	0.01	1.87E-10	7.24E-01	1.87E-10	93.33	26.67
3^{-2}	-3266.67	-711.11	6	0.01	2.42E-09	8.87E-01	2.42E-09	93.33	26.67

Problem name: `HenrionSurowiec2011`

Source: [25]

Description: Henrion and Surowiec 2011 defined one example as follows

$$\begin{aligned} F(x,y) &:= x^2 + cy \\ f(x,y) &:= 0.5y^2 - xy \end{aligned}$$

Comment: Here, c is a real-valued parameter. The optimal solution of the problem is $-0.5c(1,1)$, seen [25]. Results were listed in following table, where the starting point was $x^0 = 1.00, y^0 = 1.00$. One can notice that for each c, our method was able to render the global optimal solution for all cases.

c	λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
	3^2	0.00	0.00	1	0.05	0.00E+00	0.00E+00	0.00E+00	0.00	0.00
	3^1	0.00	0.00	1	0.01	0.00E+00	0.00E+00	0.00E+00	0.00	0.00
0	3^0	0.00	0.00	1	0.01	0.00E+00	0.00E+00	0.00E+00	0.00	0.00
	3^{-1}	0.00	0.00	1	0.00	5.23E-17	1.11E-16	0.00E+00	0.00	0.00
	3^{-2}	0.00	0.00	1	0.02	0.00E+00	0.00E+00	0.00E+00	0.00	0.00
	3^2	-0.25	-0.13	1	0.05	6.28E-16	1.11E-01	0.00E+00	-0.50	-0.50
	3^1	-0.25	-0.13	1	0.01	7.02E-16	3.33E-01	0.00E+00	-0.50	-0.50
1	3^0	-0.25	-0.13	1	0.01	0.00E+00	1.00E+00	0.00E+00	-0.50	-0.50
	3^{-1}	-0.25	-0.13	1	0.01	4.27E-16	1.20E+00	0.00E+00	-0.50	-0.50
	3^{-2}	-0.25	-0.13	1	0.02	0.00E+00	1.06E+00	0.00E+00	-0.50	-0.50
	3^2	-0.25	-0.13	1	0.05	6.28E-16	1.11E-01	0.00E+00	0.50	0.50
	3^1	-0.25	-0.13	1	0.02	3.14E-16	3.33E-01	0.00E+00	0.50	0.50
-1	3^0	-0.25	-0.13	1	0.01	0.00E+00	1.00E+00	0.00E+00	0.50	0.50
	3^{-1}	-0.25	-0.13	1	0.01	0.00E+00	1.20E+00	0.00E+00	0.50	0.50
	3^{-2}	-0.25	-0.13	1	0.02	2.38E-16	1.06E+00	0.00E+00	0.50	0.50

Problem name: `IshizukaAiyoshi1992a`

Source: [27]

description: Ishizuka and Aiyoshi 1992 defined Example 1 as follows

$$\begin{aligned} F(x,y) &:= xy_2^2 \\ G(x,y) &:= -x-M \\ f(x,y) &:= y_1 \\ g(x,y) &:= \begin{bmatrix} -x \\ -x-y_1 \\ -x+y_1 \\ -M-y_1-y_2 \\ -M+y_1+y_2 \end{bmatrix} \end{aligned}$$

where $M > 1$.

Comment: Ishizuka and Aiyoshi in[27] stated the unique optimal solution was $x^* = M, y^* = (-M, 0)^T$. We fixed $M = 4$ as results of other M were similar. Our method got the unique optimal solution when $\lambda = 3$ under the first starting point. Two starting points were $x^0 = 1.00, y^0 = (-1.00, 1.00)^T$ and $x^0 = 5.00, y^0 = (-5.00, 0.00)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.00	100	0.28	1.48E-09	2.41E-02	8.00E+00	0.00	0.00 0.01
3^1	0.00	-4.00	12	0.03	9.25E-07	1.12E-03	2.00E+00	4.00	-4.00 0.00
3^0	0.00	0.00	109	0.20	1.69E-05	2.00E-01	2.69E-04	0.00	0.00 0.00
3^{-1}	0.00	0.00	41	0.07	8.96E-07	3.76E-01	6.37E-01	0.00	0.00 0.37
3^{-2}	0.00	0.00	112	0.19	2.89E-05	2.67E-03	8.89E-01	0.00	0.00 0.00
3^2	0.00	0.00	109	0.21	3.71E-05	1.00E+00	8.00E+00	0.00	0.00 0.00
3^1	0.00	0.00	42	0.06	4.28E-09	1.43E+00	3.75E+00	0.00	0.00 1.71
3^0	0.00	0.00	168	0.32	3.58E-04	1.00E+00	3.02E-02	0.00	0.00 0.00
3^{-1}	0.00	0.00	20	0.03	6.50E-07	9.95E-01	6.67E-01	0.00	0.00 -0.02
3^{-2}	0.00	-4.00	124	0.13	4.50E-02	9.96E-01	8.17E-01	559.59	-4.00 0.00

Problem name: `KleniatiAdjiman2014Ex3`

Source: [28]

Description: Kleniati and Adjiman 2014 tested one example as follows

$$\begin{aligned} F(x,y) &:= x-y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= xy^2/2-xy^3 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: This is a variant of Example 3.13 in [35]. Kleniati and Adjiman in [28] stated that the unique optimal solution is $x^*=0, y^*=1$. Results were listed in following table, where two starting points were $x^0=1.00, y^0=1.00$ and $x^0=1.00, y^0=0.00$. Same solutions were achieved under three cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-0.45	0.00	61	0.19	9.28E-07	1.40E+00	4.33E-03	-0.05	0.40
3^1	-1.00	0.00	11	0.02	2.30E-08	1.67E-01	1.00E+00	0.00	1.00
3^0	-1.00	0.00	18	0.02	1.08E-07	5.00E-01	1.00E+00	0.00	1.00
3^{-1}	-1.00	0.00	13	0.02	1.14E-12	1.50E+00	1.00E+00	0.00	1.00
3^{-2}	-1.86	0.26	24	0.04	2.78E-09	1.86E+00	1.00E+00	-1.00	0.86
3^2	-1.00	0.00	12	0.02	7.93E-13	1.11E-01	1.93E-20	-1.00	0.00
3^1	-1.33	-0.02	11	0.01	7.98E-11	3.33E-01	8.53E-17	-1.00	0.33
3^0	-1.33	-0.02	10	0.01	5.78E-11	1.00E+00	1.56E-12	-1.00	0.33
3^{-1}	-1.48	0.00	13	0.01	5.72E-08	1.48E+00	2.12E-01	-1.00	0.48
3^{-2}	-1.86	0.26	11	0.01	5.83E-12	1.86E+00	1.00E+00	-1.00	0.86

Problem name: `KleniatiAdjiman2014Ex4`

Source: [28]

Description: Kleniati and Adjiman 2014 tested one example as follows

$$\begin{aligned} F(x,y) &:= -\sum_{j=1}^{5}(x_j^2+y_j^2) \\ f(x,y) &:= y_1^3+y_2^2x_1+y_2^2x_2+0.1y_3+(y_4^2+y_5^2)x_3x_4x_5 \\ G(x,y) &:= \begin{bmatrix} -x_1-1 \\ \vdots \\ -x_5-1 \\ x_1-1 \\ \vdots \\ x_5-1 \\ y_1y_2-x_1 \\ x_2y_1^2 \\ x_1-e^{x_2}+y_3 \end{bmatrix} \quad g(x,y) := \begin{bmatrix} -y_1-1 \\ \vdots \\ -y_5-1 \\ y_1-1 \\ \vdots \\ y_5-1 \\ x_1-0.2-y_3^2 \end{bmatrix} \end{aligned}$$

Comment: This is a variant of Example 3.28 in [35]. Kleniati and Adjiman in [28] stated that one best known solution x^*, y^* satisfied $F(x^*,y^*) = -10$ and $f(x^*,y^*) = -3.1$. Results were listed in following table, where two starting points were $x^0 = (2.00,2.00,2.00,2.00,2.00)^T, y^0 = (1.00,1.00,1.00,1.00,1.00)^T$ and $x^0 = y^0 = (2.00,2.00,2.00,2.00,2.00)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}
3^2	-3.18	0.00	205	0.84	4.55E-1	1.00E+0	4.82E+0
3^1	-7.00	-0.10	19	0.04	1.65E-8	5.76E-1	2.97E+0
3^0	-9.00	-2.10	143	0.41	1.41E+0	2.17E-3	1.00E+0
3^{-1}	-3.73	0.84	134	0.28	2.10E-3	1.58E+0	1.73E+0
3^{-2}	-5.04	1.04	45	0.10	7.75E-12	1.38E+0	1.01E+0
3^2	-6.00	-1.90	143	0.46	8.18E-4	1.12E+0	9.00E+0
3^1	-5.00	-0.10	47	0.08	9.43E-7	7.07E-1	2.20E+0
3^0	-7.00	-0.10	21	0.03	8.87E-10	1.28E+0	2.90E+0
3^{-1}	-6.69	1.94	214	0.56	1.41E+0	1.84E+0	2.12E+3
3^{-2}	-7.00	2.10	232	0.47	8.69E-5	1.61E+0	1.56E+0

λ	$\hat{x}$	$\hat{y}$
3^2	0.43 0.00 -1.00 -1.00 -1.00	0.00 0.00 0.00 0.00 0.00
3^1	1.00 -1.00 1.00 1.00 1.00	0.00 1.00 -1.00 0.00 0.00
3^0	1.00 -1.00 -1.00 -1.00 -1.00	0.00 -1.00 -1.00 1.00 1.00
3^{-1}	0.20 0.00 0.75 0.75 0.75	0.00 0.00 0.01 -1.00 -1.00
3^{-2}	0.00 -1.00 1.00 1.00 1.00	1.00 0.00 0.01 0.14 0.14
3^2	0.00 0.00 -1.00 -1.00 -1.00	0.00 0.00 1.00 -1.00 -1.00
3^1	0.00 1.00 1.00 1.00 1.00	0.00 0.00 -1.00 0.00 0.00
3^0	1.00 -1.00 1.00 1.00 1.00	0.00 -1.00 -1.00 0.00 0.00
3^{-1}	0.00 1.00 0.75 0.75 0.75	0.00 -1.00 1.00 -1.00 -1.00
3^{-2}	-0.01 0.01 1.00 1.00 1.00	-0.01 1.00 1.00 1.00 1.00

Problem name: `LampariellloSagratella2017Ex23`

Source: [30]

Description: Lampariello and Sagratella 2017 tested Example 2.3 as follows

$$\begin{aligned} F(x,y) &:= x \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= (x-y_1)^2+(y_2+1)^2 \\ g(x,y) &:= \begin{bmatrix} y_1^3-y_2 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: The best known optimal solution is $x^* = -1, y^* = (-1,0)^\top$, seen [30] . Results were listed in following table, where two starting points were $x^0 = -1.00, y^0 = (-1.00,-1.00)^\top$ and $x^0 = 1.00, y^0 = (1.00, 1.00)^\top$. Solutions were the same as the global optimal solution for all cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-1.00	1.00	9	0.13	3.80E-10	8.03E-12	8.00E+00	-1.00	-1.00 0.00
3^1	-1.00	1.00	10	0.02	6.04E-11	3.37E-29	2.00E+00	-1.00	-1.00 0.00
3^0	-1.00	1.00	10	0.01	2.73E-10	0.00E+00	0.00E+00	-1.00	-1.00 0.00
3^{-1}	-1.00	1.00	11	0.02	7.56E-09	1.06E-08	6.67E-01	-1.00	-1.00 0.00
3^{-2}	-1.00	1.00	11	0.03	4.64E-09	2.10E-08	8.89E-01	-1.00	-1.00 0.00
3^2	-1.00	1.00	119	0.20	5.56E-09	1.23E-10	8.00E+00	-1.00	-1.00 0.00
3^1	-1.00	1.00	145	0.22	1.36E-12	2.27E-13	2.00E+00	-1.00	-1.00 0.00
3^0	-1.00	1.00	10	0.01	1.88E-09	2.12E-22	0.00E+00	-1.00	-1.00 0.00
3^{-1}	-1.00	1.00	14	0.02	4.54E-08	1.79E-15	6.67E-01	-1.00	-1.00 0.00
3^{-2}	-1.00	1.00	17	0.02	8.48E-13	0.00E+00	8.89E-01	-1.00	-1.00 0.00

Problem name: `LampariellloSagratella2017Ex31`

Source: [29]

Description: Lampariello and Sagratella 2017 tested Example 3.1 as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & -x+1 \\ f(x,y) & := & y \\ g(x,y) & := & -x-y+1 \end{array}$$

Comment: The best known optimal solution is $x^* = 1, y^* = 0$, seen [29]. Results were listed in following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = 1.00, y^0 = 1.00$. For different starting points and λ, solutions were the same as the global optimal solution.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.00	0.00	8	0.09	6.20E-07	6.20E-07	8.00E+00	1.00	0.00
3^1	1.00	0.00	8	0.01	6.20E-07	6.20E-07	2.00E+00	1.00	0.00
3^0	1.00	0.00	8	0.01	6.20E-07	6.19E-07	1.77E-09	1.00	0.00
3^{-1}	1.00	0.00	9	0.01	8.38E-12	8.18E-12	6.67E-01	1.00	0.00
3^{-2}	1.00	0.00	8	0.02	7.04E-07	2.88E-07	8.89E-01	1.00	0.00
3^2	1.00	0.00	9	0.01	6.20E-07	6.20E-07	8.00E+00	1.00	0.00
3^1	1.00	0.00	9	0.01	6.20E-07	6.20E-07	2.00E+00	1.00	0.00
3^0	1.00	0.00	9	0.01	6.20E-07	6.20E-07	9.98E-13	1.00	0.00
3^{-1}	1.00	0.00	9	0.01	6.20E-07	6.20E-07	6.67E-01	1.00	0.00
3^{-2}	1.00	0.00	9	0.01	6.20E-07	6.20E-07	8.89E-01	1.00	0.00

Problem name: `LamparielloSagratella2017Ex32`

Source: [29]

Description: Lampariello and Sagratella 2017 tested Example 3.2 as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ f(x,y) & := & (x+y-1)^2 \end{array}$$

Comment: The best known optimal solution is $x^* = 0.5, y^* = 0.5$, seen [29]. Results were listed in following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = 1.00, y^0 = 0.00$. Our method rendered the same one for all cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.50	0.00	1	0.05	1.69E-15	5.56E-02	0.00E+00	0.50	0.50
3^1	0.50	0.00	1	0.01	5.55E-16	1.67E-01	0.00E+00	0.50	0.50
3^0	0.50	0.00	1	0.01	0.00E+00	5.00E-01	0.00E+00	0.50	0.50
3^{-1}	0.50	0.00	1	0.01	0.00E+00	7.50E-01	0.00E+00	0.50	0.50
3^{-2}	0.50	0.00	1	0.02	1.11E-16	9.00E-01	0.00E+00	0.50	0.50
3^2	0.50	0.00	1	0.00	2.37E-15	5.56E-02	0.00E+00	0.50	0.50
3^1	0.50	0.00	1	0.00	5.55E-16	1.67E-01	0.00E+00	0.50	0.50
3^0	0.50	0.00	1	0.00	0.00E+00	5.00E-01	0.00E+00	0.50	0.50
3^{-1}	0.50	0.00	1	0.00	0.00E+00	7.50E-01	0.00E+00	0.50	0.50
3^{-2}	0.50	0.00	1	0.00	1.11E-16	9.00E-01	0.00E+00	0.50	0.50

Problem name: `LamparielloSagratella2017Ex33`

Source: [29]

Description: Lampariello and Sagratella 2017 tested Example 3.3 as follows

$$\begin{aligned} F(x,y) &:= x^2 + (y_1 + y_2)^2 \\ G(x,y) &:= -x + 0.5 \\ f(x,y) &:= y_1 \\ g(x,y) &:= \begin{bmatrix} -x - y_1 - y_2 + 1 \\ -y_1 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: The best known optimal solution is $x^* = 0.5, y^* = (0, 0.5)^T$, seen [29]. Results were listed in following table, where two starting points were $x^0 = 1.00, y^0 = (1.00, 1.00)^T$ and $x^0 = 0.50, y^0 = (1.00, 1.00)^T$. All solutions were the same as the global optimal solution.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.50	0.00	20	0.15	7.39E-08	5.29E-03	8.06E+00	0.50	0.00 0.50
3^1	0.50	0.00	16	0.03	5.94E-08	4.10E-03	2.24E+00	0.50	0.00 0.50
3^0	0.50	0.00	15	0.03	4.90E-07	7.35E-03	1.00E+00	0.50	0.00 0.50
3^{-1}	0.50	0.00	14	0.02	1.13E-07	6.67E-03	1.20E+00	0.50	0.00 0.50
3^{-2}	0.50	0.00	14	0.03	9.13E-10	5.97E-03	1.34E+00	0.50	0.00 0.50
3^2	0.50	0.00	20	0.03	7.39E-08	5.29E-03	8.06E+00	0.50	0.00 0.50
3^1	0.50	0.00	16	0.03	5.94E-08	4.10E-03	2.24E+00	0.50	0.00 0.50
3^0	0.50	0.00	15	0.02	4.90E-07	7.35E-03	1.00E+00	0.50	0.00 0.50
3^{-1}	0.50	0.00	14	0.02	1.13E-07	6.67E-03	1.20E+00	0.50	0.00 0.50
3^{-2}	0.50	0.00	14	0.02	9.13E-10	5.97E-03	1.34E+00	0.50	0.00 0.50

Problem name: `LampariielloSagratella2017Ex35`

Source: [29]

Description: Lampariello and Sagratella 2017 tested Example 3.5 as follows

$$\begin{aligned} F(x,y) &:= x^2 + y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= -y \\ g(x,y) &:= \begin{bmatrix} 2x+y-2 \\ y-1 \\ -y \end{bmatrix} \end{aligned}$$

Comment: The best known optimal solution is $x^* = 0.8, y^* = 0.4$, seen [29]. Results were listed in following table, where two starting points were $x^0 = 1.00, y^0 = 0.50$ and $x^0 = 1.00, y^0 = 1.00$. Same optimal solution was generated such as $\lambda = 3^0$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.25	-1.00	9	0.12	3.79E-09	0.00E+00	6.17E+00	0.50	1.00
3^1	0.80	-0.40	21	0.03	2.68E-07	1.87E-11	1.20E+00	0.80	0.40
3^0	0.80	-0.40	9	0.01	2.78E-09	4.59E-11	8.00E-01	0.80	0.40
3^{-1}	0.03	-0.17	42	0.06	1.89E-07	8.33E-01	1.00E+00	0.00	0.17
3^{-2}	0.00	-0.06	17	0.04	6.15E-10	9.44E-01	1.00E+00	0.00	0.06
3^2	1.00	0.00	13	0.04	1.16E-10	7.72E-11	6.32E+00	1.00	0.00
3^1	1.25	-1.00	10	0.01	2.73E-08	0.00E+00	1.78E-03	0.50	1.00
3^0	0.80	-0.40	11	0.01	4.99E-09	4.99E-09	8.00E-01	0.80	0.40
3^{-1}	0.03	-0.17	21	0.02	3.39E-12	8.33E-01	1.00E+00	0.00	0.17
3^{-2}	0.00	-0.06	14	0.04	3.02E-10	9.44E-01	1.00E+00	0.00	0.06

Problem name: `LucchettiEtal1987`

Source: [32]

Description: Lucchetti et al. 1987 tested one example as follows

$$\begin{aligned} F(x,y) &:= 0.5(1-x)+xy \\ G(x,y) &:= \begin{bmatrix} -x \\ -1+x \end{bmatrix} \\ f(x,y) &:= (x-1)y \\ g(x,y) &:= \begin{bmatrix} -y \\ -1+y \end{bmatrix} \end{aligned}$$

Comment: The best known optimal solution is $x^* = 1, y^* = 0$ with $F(x^*,y^*) = 0, f(x^*,y^*) = 0$, seen [32]. Results were listed in following table, where two starting points were $x^0 = 0.50, y^0 = 0.50$ and $x^0 = 0.00, y^0 = 0.00$, and our method rendered same solution under many cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.00	21	0.13	2.20E-08	1.00E+00	1.00E+00	1.00	0.00
3^1	0.78	-0.22	10	0.02	3.94E-09	1.25E-01	2.50E-01	0.75	0.88
3^0	0.00	0.00	7	0.01	2.37E-07	4.13E-04	1.00E+00	1.00	0.00
3^{-1}	0.00	0.00	11	0.02	6.93E-10	9.99E-01	1.00E+00	1.00	0.00
3^{-2}	0.50	-1.00	23	0.04	2.44E-11	2.44E-11	8.89E-01	0.00	1.00
3^2	0.00	0.00	21	0.02	2.20E-08	1.00E+00	1.00E+00	1.00	0.00
3^1	0.78	-0.22	10	0.01	3.94E-09	1.25E-01	2.50E-01	0.75	0.88
3^0	0.00	0.00	6	0.01	3.55E-07	9.93E-01	1.00E+00	1.00	0.00
3^{-1}	0.53	-0.47	10	0.01	6.77E-08	3.75E-01	7.50E-01	0.25	0.62
3^{-2}	0.50	-1.00	11	0.01	5.65E-07	5.65E-07	8.89E-01	0.00	1.00

Problem name: `LuDebSinha2016a`

Source: [31]

Description: Lu, Deb and Sinha 2016 defined one example as follows

$$
\begin{aligned}
F(x,y) &:= 2-\exp\left[-\left(\frac{0.2y-x+0.6}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{0.15y-0.4+x}{0.3}\right)^{2}\right]\\
G(x,y) &:= \begin{bmatrix} -x\\ -1+x\\ -y\\ -2+y \end{bmatrix}\\
f(x,y) &:= 2-\exp\left[-\left(\frac{1.5y-x}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{2y-3+x}{0.5}\right)^{2}\right]
\end{aligned}
$$

Comment: $x^* = 1.4, y^* = 0.2$ with $F(x^*,y^*) = 1.943, f(x^*,y^*) = 1.961$ is a possible solution for this problem, seen [31]. Clearly, the $x^* = 1.4$ violates the constraints. Results were listed in following table, where two starting points were $x^0 = 0.46, y^0 = 0.69$ and $x^0 = 0.50, y^0 = 1.00$, and our method provided better solutions for all cases. It seems like $\hat{x} = 0.29$ and $\hat{y} = 0.79$ is the best optimal solution.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.11	1.95	8	0.45	1.90E-08	4.27E-03	0.00E+00	0.29	0.79
3^1	1.11	1.95	8	0.35	1.90E-08	1.28E-02	0.00E+00	0.29	0.79
3^0	1.11	1.95	8	0.32	1.90E-08	3.84E-02	0.00E+00	0.29	0.79
3^{-1}	1.11	1.95	8	0.33	1.90E-08	1.15E-01	0.00E+00	0.29	0.79
3^{-2}	1.11	1.95	8	0.34	1.90E-08	3.46E-01	0.00E+00	0.29	0.79
3^2	1.14	1.18	15	0.79	3.30E-12	1.22E-04	0.00E+00	0.20	1.40
3^1	1.11	1.95	109	4.99	2.60E-12	1.28E-02	0.00E+00	0.29	0.79
3^0	1.51	1.87	11	0.49	8.43E-07	8.45E-02	0.00E+00	0.65	0.71
3^{-1}	1.51	1.87	7	0.28	2.35E-10	2.54E-01	0.00E+00	0.65	0.71
3^{-2}	1.51	1.87	7	0.27	2.35E-10	7.61E-01	0.00E+00	0.65	0.71

Problem name: `LuDebSinha2016b`

Source: [31]

Description: Lu, Deb and Sinha 2016 defined one example as follows

$$\begin{aligned} F(x,y) &:= (x-0.5)^2+(y-1)^2 \\ G(x,y) &:= \begin{bmatrix} -x \\ -1+x \\ -y \\ -2+y \end{bmatrix} \\ f(x,y) &:= 2-\exp\left[-\left(\frac{1.5y-x}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{2y-3+x}{0.5}\right)^{2}\right] \end{aligned}$$

Comment: $x^*=0.5, y^*=1$ with $F(x^*,y^*)=0$ and $f(x^*,y^*)=1.6645$ is a possible solution for this problem, seen [31]. Results were listed in following table, where two starting points were $x^0=0.46, y^0=0.69$ and $x^0=0.50, y^0=1.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.07	1.93	5	0.35	1.82E-07	1.95E-02	0.00E+00	0.43	0.75
3^1	0.07	1.93	6	0.28	1.46E-07	5.84E-02	0.00E+00	0.43	0.75
3^0	0.07	1.93	6	0.27	1.45E-07	1.75E-01	0.00E+00	0.43	0.75
3^{-1}	0.07	1.93	6	0.28	1.47E-07	4.11E-01	0.00E+00	0.43	0.75
3^{-2}	0.07	1.93	6	0.26	1.47E-07	6.77E-01	0.00E+00	0.43	0.75
3^2	0.07	1.93	11	0.47	1.14E-12	1.95E-02	0.00E+00	0.43	0.75
3^1	0.05	1.17	13	0.56	1.61E-12	4.31E-03	0.00E+00	0.60	1.20
3^0	0.07	1.93	10	0.43	2.68E-11	1.75E-01	0.00E+00	0.43	0.75
3^{-1}	0.07	1.93	9	0.40	2.28E-09	4.11E-01	0.00E+00	0.43	0.75
3^{-2}	0.07	1.93	7	0.36	3.11E-08	6.77E-01	0.00E+00	0.43	0.75

Problem name: `LuDebSinha2016c`

Source: [31]

Description: Lu, Deb and Sinha 2016 defined one example as follows

$$
\begin{aligned}
F(x,y) &:= 2-\exp\left[-\left(\frac{0.2y-x+0.6}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{0.15y-0.4+x}{0.3}\right)^{2}\right]\\
G(x,y) &:= \begin{bmatrix} -x\\ -1+x\\ -y\\ -2+y \end{bmatrix}\\
f(x,y) &:= (x-0.5)^2+(y-1)^2
\end{aligned}
$$

Comment: $x^* = 0.26, y^* = 1$ with $F(x^*,y^*) = 1.12, f(x^*,y^*) = 0.06$ is a possible solution for this problem, seen [31]. Results were listed in following table, where two starting points were $x^0 = 0.50, y^0 = 0.50$ and $x^0 = 1.00, y^0 = 1.00$, and our method reached the same one under several cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.12	0.06	272	0.52	1.42E-10	2.95E-03	0.00E+00	0.26	1.00
3^1	1.97	1.08	1000	0.95	1.13E+00	2.18E-02	0.00E+00	-0.54	1.00
3^0	1.12	0.06	9	0.01	3.05E-09	2.59E-02	0.00E+00	0.26	1.00
3^{-1}	1.12	0.06	9	0.01	3.05E-09	7.39E-02	0.00E+00	0.26	1.00
3^{-2}	1.12	0.06	9	0.02	3.05E-09	1.93E-01	0.00E+00	0.26	1.00
3^2	1.64	0.03	326	0.39	3.59E-12	2.01E-02	0.00E+00	0.67	1.00
3^1	1.12	0.06	117	0.14	1.88E-08	8.79E-03	0.00E+00	0.26	1.00
3^0	1.64	0.03	171	0.17	2.08E-11	1.56E-01	0.00E+00	0.67	1.00
3^{-1}	1.64	0.03	55	0.05	3.10E-08	3.56E-01	0.00E+00	0.67	1.00
3^{-2}	1.12	0.06	143	0.13	1.21E-11	1.93E-01	0.00E+00	0.26	1.00

Problem name: `LuDebSinha2016d`

Source: [31]

Description: Lu, Deb and Sinha 2016 defined one example as follows

$$
\begin{aligned}
F(x,y) &:= -x_2\\
G(x,y) &:= \begin{bmatrix} -\left(\frac{y_1}{14}+\frac{16}{7}\right)(x_1-2)^2+x_2\\ 12.5\left(\frac{y_1}{14}+\frac{16}{7}\right)(x_1-5)-x_2\\ -5\left[x_1+4-\left(\frac{y_1}{14}+\frac{16}{7}\right)\right]\left[x_1+8-\left(\frac{y_1}{14}+\frac{16}{7}\right)\right]+x_2\\ -4-x_1\\ -10+x_1\\ -100-x_2\\ -200+x_2\\ -4-y_1\\ -10+y_1\\ -100-y_2\\ -200+y_2 \end{bmatrix}\\
f(x,y) &:= -y_2\\
g(x,y) &:= \begin{bmatrix} -\left(\frac{x_1}{14}+\frac{16}{7}\right)(y_1-2)^2+y_2\\ 12.5\left(\frac{x_1}{14}+\frac{16}{7}\right)(y_1-5)-y_2\\ -5\left[y_1+4-\left(\frac{x_1}{14}+\frac{16}{7}\right)\right]\left[y_1+8-\left(\frac{x_1}{14}+\frac{16}{7}\right)\right]+y_2 \end{bmatrix}
\end{aligned}
$$

Comment: Results were listed in following table, where two starting points were $x^0=(0,200)^T$, $y^0=(0,200)^T$ and $x^0=(10,200)^T, y^0=(10,200)^T$. It seems that solution $x^*=y^*=(10,192)^T$ was the best one among all solutions.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-155.04	-0.02	23	0.17	1.02E-09	1.01E+00	8.35E+00	10.00 155.04	1.92 0.02
3^1	-19.56	-19.56	61	0.12	2.32E-08	1.01E+00	1.42E+00	-0.97 19.56	-0.97 19.56
3^0	19.91	0.00	163	0.30	5.00E-01	1.00E+00	8.29E-01	-3.71 -19.91	1.98 0.00
3^{-1}	-16.27	-127.30	127	0.25	3.24E-01	1.17E+00	3.17E-01	-0.34 16.27	9.50 127.30
3^{-2}	-192.00	-192.00	37	0.08	6.30E-08	9.17E-01	6.45E-01	10.00 192.00	10.00 192.00
3^2	-132.36	19.99	138	0.22	9.68E-01	9.31E-01	9.03E+00	10.00 132.36	-3.05 -19.99
3^1	-178.29	-75.00	113	0.19	5.67E-01	1.37E+00	2.00E+01	10.00 178.29	7.00 75.00
3^0	-192.01	-192.00	157	0.27	7.02E-01	1.02E+00	1.12E+00	10.00 192.01	10.00 192.00
3^{-1}	-144.98	-15.67	189	0.27	2.89E-01	1.06E+00	2.76E-01	10.00 144.98	-0.29 15.67
3^{-2}	19.97	95.28	142	0.19	1.19E-01	9.64E-01	2.50E-02	-3.92 -19.97	-4.00 -95.28

Problem name: `LuDebSinha2016e`

Source: [31]

Description: Lu, Deb, Sinha 2016 defined one example as follows

$$
\begin{aligned}
F(x,y) &:= \left(\frac{y_2-50}{30}\right)^2+\left(\frac{x-2.5}{0.2}\right)^2\\
G(x,y) &:= \begin{bmatrix} 2-x\\ -3+x\\ -4-y_1\\ -10+y_1\\ -100-y_2\\ -200+y_2 \end{bmatrix}\\
f(x,y) &:= -y_2\\
g(x,y) &:= \begin{bmatrix} -x(y_1-2)^2+y_2\\ 12.5x(y_1-5)-y_2\\ -5(y_1+4-x)(y_1+8-x)+y_2 \end{bmatrix}
\end{aligned}
$$

Comment: Results were listed in following table, wherestarting points were $x^0 = 2.00, y^0 = (5.00, 10.00)^T$ and $x^0 = 2.00, y^0 = (5.00, 100.00)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	9.75	0.00	154	0.40	1.08E+00	1.00E+00	8.70E+00	1.97	2.00 0.00
3^1	7.85	-16.00	121	0.18	8.74E-01	1.01E+00	2.38E+00	3.01	-0.30 16.00
3^0	28.65	-192.00	140	0.16	4.99E-01	1.27E+00	5.60E-01	3.00	10.00 192.00
3^{-1}	5.44	20.00	48	0.07	4.11E-07	1.02E+00	1.11E+00	2.50	-3.50 -20.00
3^{-2}	8.35	-135.71	159	0.20	1.10E-01	9.36E-01	6.27E-01	2.41	9.50 135.71
3^2	143.57	19.91	1000	1.65	2.61E+00	1.04E+00	1.67E+03	0.15	-5.69 -19.91
3^1	11.79	20.00	169	0.22	9.26E-01	9.98E-01	3.29E+00	3.00	-3.00 -20.00
3^0	28.65	-192.00	30	0.04	3.93E-07	1.00E+00	3.16E-01	3.00	10.00 192.00
3^{-1}	18.29	-174.03	222	0.25	3.13E-01	1.06E+00	1.17E-01	2.72	10.00 174.03
3^{-2}	5.26	18.78	130	0.11	1.10E-01	9.92E-01	2.66E-01	2.49	-4.00 -18.78

Problem name: `LuDebSinha2016f`

Source: [31]

Description: Lu, Deb and Sinha 2016 defined one example as follows

$$\begin{aligned} F(x,y) &:= -x_2 \\ G(x,y) &:= \begin{bmatrix} 2-y \\ -4+y \\ -80-x_1 \\ -200+x_1 \\ -100-x_2 \\ -200+x_2 \\ -y\left(\frac{x_1}{20}-2\right)^2+x_2 \\ 12.5y\left(\frac{x_1}{20}-5\right)-x_2 \\ -5\left(\frac{x_1}{20}+4-y\right)\left(\frac{x_1}{20}+8-y\right)+x_2 \end{bmatrix} \\ f(x,y) &:= \left(\frac{x_1-50}{28}\right)^2+\left(\frac{y-2.5}{0.2}\right)^2 \end{aligned}$$

Comment: The optimal solution should have the form $y^* = 2.5$ since the lower level model is unconstrained. Results were listed in following table, where two starting points were $x^0 = (200.00, 200.00)^\top, y^0 = 2.50$ and $x^0 = (50.00, 50.00)^\top, y^0 = 2.50$. It seems that $\hat{x} = (200, 160)^\top, \hat{y} = 2.5$ is the best optimal solution.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-160.00	28.70	21	0.13	3.17E-13	5.69E-02	0.00E+00	200.00 160.00	2.50
3^1	-160.00	28.70	21	0.03	3.17E-13	1.71E-01	0.00E+00	200.00 160.00	2.50
3^0	-160.00	28.70	27	0.04	1.51E-09	5.12E-01	0.00E+00	200.00 160.00	2.50
3^{-1}	-160.00	28.70	23	0.03	6.09E-07	1.54E+00	0.00E+00	200.00 160.00	2.50
3^{-2}	-18.54	5.30	27	0.03	4.20E-08	2.30E-01	0.00E+00	-14.47 18.54	2.50
3^2	-18.54	5.30	22	0.03	2.23E-08	3.68E-03	0.00E+00	-14.47 18.54	2.50
3^1	-18.54	5.30	22	0.03	2.25E-08	1.10E-02	0.00E+00	-14.47 18.54	2.50
3^0	-18.54	5.30	22	0.02	2.36E-08	3.22E-02	0.00E+00	-14.47 18.54	2.50
3^{-1}	-18.54	5.30	22	0.02	3.70E-08	9.08E-02	0.00E+00	-14.47 18.54	2.50
3^{-2}	-18.54	5.30	23	0.02	6.14E-12	2.30E-01	0.00E+00	-14.47 18.54	2.50

Problem name: `MacalHurter1997`

Source: [33]

description: Macal and Hurter defined one example as follows

$$\begin{aligned} F(x,y) &:= (x-1)^2 + (y-1)^2 \\ f(x,y) &:= 0.5y^2 + 500y - 50xy \end{aligned}$$

Comment: The unique optimal solution reported in [33] was $x^* = 10.02, y^* = 0.820$. In following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = 1.00, y^0 = 1.00$, we achieved the unique one for every case.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	81.33	-0.34	1	0.05	1.04E-12	4.01E-02	0.00E+00	10.02	0.82
3^1	81.33	-0.34	1	0.01	1.86E-13	1.20E-01	0.00E+00	10.02	0.82
3^0	81.33	-0.34	1	0.01	1.25E-13	3.61E-01	0.00E+00	10.02	0.82
3^{-1}	81.33	-0.34	1	0.00	2.03E-14	1.08E+00	0.00E+00	10.02	0.82
3^{-2}	81.33	-0.34	1	0.02	2.09E-14	1.34E+00	0.00E+00	10.02	0.82
3^2	81.33	-0.34	1	0.00	1.08E-12	4.01E-02	0.00E+00	10.02	0.82
3^1	81.33	-0.34	1	0.00	1.28E-13	1.20E-01	0.00E+00	10.02	0.82
3^0	81.33	-0.34	1	0.00	1.06E-13	3.61E-01	0.00E+00	10.02	0.82
3^{-1}	81.33	-0.34	1	0.00	2.16E-15	1.08E+00	0.00E+00	10.02	0.82
3^{-2}	81.33	-0.34	1	0.00	9.48E-15	1.34E+00	0.00E+00	10.02	0.82

Problem name: `Mirrlees1999`

Source: [34]

Description: Mirrlees 1999 defined one example as follows

$$\begin{aligned} F(x,y) &:= (x-2)^2 + (y-1)^2 \\ f(x,y) &:= -xe^{-(y+1)^2} - e^{-(y-1)^2} \end{aligned}$$

Comment: This problem is known in the literature as Mirrlees problem. It is usually used to illustrate how the KKT reformulation of the bilevel optimization problem is not appropriate for problems with nonconvex lower-level problems. The best known optimal solution reported in [34] for the problem is $x^* = 1, y^* = 0.958$ with $F(x^*, y^*) = 1.002$ and $f(x^*, y^*) = -1.020$. Another local optimal solution reported in [34] is $x^* = 1.99, y^* = 0.895$. In following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = 1.00, y^0 = 1.00$, our method produced the local one for all cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.01	-1.04	8	0.08	8.65E-14	1.86E-02	0.00E+00	1.99	0.89
3^1	0.01	-1.04	8	0.01	8.88E-14	5.59E-02	0.00E+00	1.99	0.89
3^0	0.01	-1.04	6	0.00	7.31E-08	1.68E-01	0.00E+00	1.99	0.89
3^{-1}	0.01	-1.04	6	0.00	7.42E-08	5.03E-01	0.00E+00	1.99	0.89
3^{-2}	0.01	-1.04	8	0.01	7.91E-14	1.51E+00	0.00E+00	1.99	0.89
3^2	0.01	-1.04	7	0.00	3.58E-10	1.86E-02	0.00E+00	1.99	0.89
3^1	0.01	-1.04	7	0.00	3.93E-10	5.59E-02	0.00E+00	1.99	0.89
3^0	0.01	-1.04	6	0.00	1.13E-12	1.68E-01	0.00E+00	1.99	0.89
3^{-1}	0.01	-1.04	6	0.00	1.13E-12	5.03E-01	0.00E+00	1.99	0.89
3^{-2}	0.01	-1.04	6	0.00	1.13E-12	1.51E+00	0.00E+00	1.99	0.89

Problem name: `MitsosBarton2006Ex38`

Source: [35]

description: Mitsos and Barton defined Example 3.8 as follows

$$\begin{array}{rcl} F(x,y) & := & y^2 \\ G(x,y) & := & \begin{bmatrix} -x+1 \\ x-1 \\ -y+0.1 \\ y-0.1 \end{bmatrix} \\ f(x,y) & := & yx + ye^x \\ g(x,y) & := & \begin{bmatrix} -y+1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: The unique optimal solution reported in [35] was $x^* = -0.5671, y^* = 0$. In following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = 1.00, y^0 = 1.00$, we achieved the unique one for different λ and different x^o, y^o.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.00	72	0.20	2.78E-10	0.00E+00	0.00E+00	-0.5671	0.00
3^1	0.00	0.00	16	0.02	3.50E-11	2.12E-22	2.63E-24	-0.5671	0.00
3^0	0.00	0.00	9	0.01	2.79E-07	1.78E-07	2.36E-17	-0.5671	0.00
3^{-1}	0.00	0.00	6	0.01	1.54E-07	4.60E-09	1.37E-07	-0.5671	0.00
3^{-2}	0.00	0.00	6	0.02	6.83E-07	1.01E-07	6.83E-07	-0.5671	0.00
3^2	0.00	0.00	13	0.02	4.11E-09	0.00E+00	4.62E-22	-0.5671	0.00
3^1	0.00	0.00	15	0.02	3.54E-10	1.28E-13	1.91E-15	-0.5671	0.00
3^0	0.00	0.00	12	0.01	1.23E-09	2.94E-12	3.24E-11	-0.5671	0.00
3^{-1}	0.00	0.00	11	0.01	3.57E-08	6.82E-08	1.41E-16	-0.5671	0.00
3^{-2}	0.00	0.00	21	0.02	1.24E-07	7.10E-07	2.02E-13	-0.5671	0.00

Problem name: `MitsosBarton2006Ex39`

Source: [35]

description: Mitsos and Barton defined Example 3.9 as follows

$$\begin{aligned} F(x,y) &:= x \\ G(x,y) &:= \begin{bmatrix} -x+y \\ -x-10 \\ x-10 \end{bmatrix} \\ f(x,y) &:= y^3 \\ g(x,y) &:= \begin{bmatrix} -y+1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The unique optimal solution reported in [35] was $x^* = -1, y^* = -1$. In following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = -1.00, y^0 = 0.00$, we achieved the unique one when $\lambda \leq 3^{-1}$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-0.02	0.00	28	0.14	8.63E-07	9.81E-01	1.09E-03	-0.02	-0.02
3^1	-0.06	0.00	7	0.02	6.71E-07	9.39E-01	1.12E-02	-0.06	-0.06
3^0	-0.33	-0.04	12	0.02	1.42E-07	6.67E-01	3.33E-01	-0.33	-0.33
3^{-1}	-1.00	-1.00	42	0.06	1.62E-11	1.62E-11	3.33E-01	-1.00	-1.00
3^{-2}	-1.00	-1.00	68	0.10	2.83E-09	0.00E+00	5.56E-01	-1.00	-1.00
3^2	-0.02	0.00	59	0.07	8.38E-07	9.81E-01	1.09E-03	-0.02	-0.02
3^1	-0.06	0.00	12	0.01	8.07E-07	9.39E-01	1.12E-02	-0.06	-0.06
3^0	-0.33	-0.04	16	0.02	4.46E-08	6.67E-01	3.33E-01	-0.33	-0.33
3^{-1}	-1.00	-1.00	138	0.24	1.08E-10	8.53E-12	3.33E-01	-1.00	-1.00
3^{-2}	-1.00	-1.00	11	0.01	5.28E-10	5.28E-10	5.56E-01	-1.00	-1.00

Problem name: `MitsosBarton2006Ex310`

Source: [35]

description: Mitsos and Barton defined Example 3.10 as follows

$$\begin{aligned} F(x,y) &:= y \\ G(x,y) &:= \begin{bmatrix} -x+0.1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= x(16y^4+2y^3-8y^2-1.5y+0.5) \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has infinitely many optimal solution points $x^* \in [0.1,1], y^* = 0.5$ with $F(x^*,y^*) = 0.5$, $f(x^*,y^*) = -x^*$. Results were listed in following table, where two starting points were $x^0 = 1.00, y^0 = 1.00$ and $x^0 = 0.80, y^0 = 1.00$, and solutions under 8 cases were optimal ones.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.50	-1.00	10	0.12	8.70E-12	2.93E-03	2.04E-27	1.00	0.50
3^1	-0.09	0.06	50	0.07	4.60E-09	2.09E-01	4.42E-17	0.10	-0.09
3^0	0.50	-0.10	10	0.01	1.78E-10	2.63E-01	6.16E-33	0.10	0.50
3^{-1}	0.50	-0.18	18	0.02	1.17E-11	4.33E-01	8.20E-19	0.18	0.50
3^{-2}	0.50	-1.00	12	0.02	7.22E-09	2.37E-01	7.22E-09	1.00	0.50
3^2	-0.75	0.00	125	0.17	9.94E-02	1.75E+00	9.15E-03	0.00	-0.75
3^1	0.50	-0.12	75	0.10	3.58E-11	7.59E-02	1.29E-27	0.12	0.50
3^0	0.50	-0.10	10	0.01	1.09E-09	2.63E-01	6.16E-33	0.10	0.50
3^{-1}	0.50	-0.18	13	0.01	4.67E-07	4.36E-01	6.42E-10	0.18	0.50
3^{-2}	0.50	-1.00	9	0.03	7.81E-09	2.37E-01	5.07E-09	1.00	0.50

Problem name: `MitsosBarton2006Ex311`

Source: [35]

description: Mitsos and Barton defined Example 3.11 as follows

$$\begin{aligned} F(x,y) &:= y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= x(16y^4+2y^3-8y^2-1.5y+0.5) \\ g(x,y) &:= \begin{bmatrix} -y-0.8 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has unique optimal solution point $x^* = 0, y^* = -0.8$. Results were listed in following table, where starting points were $x^0 = -1.00, y^0 = -1.00$ and $x^0 = 1.00, y^0 = -1.00$, and solution under three cases was the unique optimal solution.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-0.50	0.00	14	0.12	3.85E-07	3.00E-01	1.33E-08	-0.01	-0.50
3^1	0.50	-0.99	13	0.02	1.79E-09	8.82E-03	1.67E-17	0.99	0.50
3^0	0.50	0.89	13	0.02	8.35E-08	2.97E-02	1.16E-22	-0.89	0.50
3^{-1}	-0.80	0.00	10	0.01	1.76E-09	2.39E-10	9.95E-01	0.00	-0.80
3^{-2}	-0.80	0.00	14	0.03	9.21E-10	3.36E-10	9.94E-01	0.00	-0.80
3^2	-0.50	0.00	11	0.01	1.04E-10	4.28E-03	5.27E-11	1.00	-0.50
3^1	-0.50	0.00	13	0.02	1.94E-11	1.29E-02	5.47E-13	0.99	-0.50
3^0	0.10	0.27	175	0.20	1.80E+00	2.41E-05	1.32E-01	0.99	0.10
3^{-1}	-0.50	0.00	17	0.02	5.42E-11	2.91E-01	1.32E-11	-0.40	-0.50
3^{-2}	-0.80	0.00	9	0.01	3.84E-11	6.69E-12	9.99E-01	0.00	-0.80

Problem name: `MitsosBarton2006Ex312`

Source: [35]

description: Mitsos and Barton defined Example 3.12 as follows

$$\begin{array}{rcl} F(x,y) & := & -x + xy + 10y^2 \\ G(x,y) & := & \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) & := & -xy^2 + y^4/2 \\ g(x,y) & := & \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{array}$$

Comment: Mitsos and Barton in [35] stated that this problem has unique optimal solution $x^* = 0, y^* = 0$. Results were listed in following table, where two starting points were $x^0 = -1.00, y^0 = 0.00$ and $x^0 = 1.00, y^0 = 0.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-1.00	0.00	11	0.11	1.55E-08	5.56E-02	1.44E-26	1.00	0.00
3^1	-1.00	0.00	10	0.02	3.13E-08	1.67E-01	0.00E+00	1.00	0.00
3^0	-1.00	0.00	10	0.01	3.00E-08	5.00E-01	0.00E+00	1.00	0.00
3^{-1}	-1.01	0.00	10	0.01	1.93E-10	9.82E-01	3.58E-02	1.00	-0.02
3^{-2}	-1.02	0.00	12	0.03	7.25E-10	9.60E-01	7.95E-02	1.00	-0.04
3^2	-1.00	0.00	10	0.01	1.56E-07	5.56E-02	4.46E-14	1.00	0.00
3^1	-1.00	0.00	232	0.28	4.01E-11	1.67E-01	7.93E-12	1.00	0.00
3^0	-1.00	0.00	57	0.06	1.23E-09	5.00E-01	1.01E-23	1.00	0.00
3^{-1}	-1.01	0.00	23	0.02	5.80E-07	9.82E-01	3.58E-02	1.00	-0.02
3^{-2}	-1.02	0.00	14	0.01	2.33E-08	9.60E-01	7.95E-02	1.00	-0.04

Problem name: `MitsosBarton2006Ex313`

Source: [35]

description: Mitsos and Barton defined Example 3.13 as follows

$$\begin{aligned} F(x,y) &:= x-y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= xy^2/2-x^3y \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has unique optimal solution point $x^*=0, y^*=1$. Results were listed in following table, where two starting points were $x^0=-1.00, y^0=-1.00$ and $x^0=1.00, y^0=1.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	-0.50	17	0.12	3.84E-12	5.57E-02	4.99E-01	1.00	1.00
3^1	0.30	-0.20	11	0.01	7.96E-12	2.40E-01	4.43E-01	-0.46	-0.76
3^0	-2.00	0.50	12	0.02	1.21E-11	7.26E-04	1.00E+00	-1.00	1.00
3^{-1}	-2.00	0.50	11	0.01	1.31E-09	3.65E-03	1.00E+00	-1.00	1.00
3^{-2}	-2.00	0.50	9	0.02	1.41E-12	2.00E+00	1.22E+00	-1.00	1.00
3^2	0.00	-0.50	10	0.01	1.60E-10	1.11E-01	4.98E-03	1.00	1.00
3^1	0.00	-0.50	11	0.02	1.24E-07	1.68E-01	4.97E-01	1.00	1.00
3^0	0.13	-0.04	18	0.02	1.29E-10	1.50E+00	6.75E-02	0.62	0.50
3^{-1}	0.00	-0.50	13	0.01	5.37E-13	1.50E+00	5.00E-01	1.00	1.00
3^{-2}	-2.00	0.50	50	0.06	1.44E-07	2.00E+00	1.22E+00	-1.00	1.00

Problem name: `MitsosBarton2006Ex314`

Source: [35]

description: Mitsos and Barton defined Example 3.14 as follows

$$
\begin{aligned}
F(x,y) &:= (x-1/4)^2 + y^2\\
G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix}\\
f(x,y) &:= y^3/3 - xy\\
g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix}
\end{aligned}
$$

Comment: Mitsos and Barton in [35] stated that this problem has unique optimal solution point $x^* = 1/4, y^* = 1/2$ with $F(x^*, y^*) = 1/4$ and $f(x^*, y^*) = -1/12$. Results were listed in following table, where starting points were $x^0 = 1.00, y^0 = 1.00$ and $x^0 = -1.00, y^0 = -1.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.06	0.00	16	0.12	3.56E-13	5.56E-02	0.00E+00	0.00	0.00
3^1	0.06	0.00	10	0.02	2.47E-08	1.67E-01	1.15E-41	0.00	0.00
3^0	0.06	0.00	9	0.01	2.79E-10	5.00E-01	3.16E-12	0.00	0.00
3^{-1}	0.03	0.00	10	0.01	3.08E-11	9.79E-01	8.64E-02	0.09	0.02
3^{-2}	0.00	0.00	9	0.02	3.69E-10	9.88E-01	1.95E-01	0.20	0.01
3^2	0.06	0.00	15	0.02	7.23E-08	5.56E-02	1.82E-16	0.00	0.00
3^1	1.00	-0.08	9	0.01	6.87E-11	2.22E-16	5.00E-01	0.25	-1.00
3^0	0.06	0.00	11	0.01	6.87E-13	5.00E-01	8.69E-26	0.00	0.00
3^{-1}	0.03	0.00	12	0.01	1.23E-07	9.79E-01	8.64E-02	0.09	0.02
3^{-2}	0.00	0.00	8	0.01	2.40E-09	9.88E-01	1.95E-01	0.20	0.01

Problem name: `MitsosBarton2006Ex315`

Source: [35]

description: Mitsos and Barton defined Example 3.15 as follows

$$\begin{aligned} F(x,y) &:= x+y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= xy^2/2 - y^3/3 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has unique optimal solution point $x^* = -1, y^* = 1$. Results were listed in following table, where two starting points were $x^0 = 1.00, y^0 = 1.00$ and $x^0 = 2.00, y^0 = 2.00$, and our method achieved the unique optimal solution when $\lambda = 3^2$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	-0.83	12	0.12	1.10E-11	0.00E+00	7.50E+00	-1.00	1.00
3^1	-2.00	-0.17	13	0.02	4.47E-11	3.33E-01	3.58E-05	-1.00	-1.00
3^0	-2.00	-0.17	11	0.02	5.22E-08	4.87E-03	9.95E-01	-1.00	-1.00
3^{-1}	-2.00	-0.17	9	0.01	6.57E-13	7.30E-03	9.98E-01	-1.00	-1.00
3^{-2}	-2.00	-0.17	9	0.02	8.54E-12	4.34E-03	1.00E+00	-1.00	-1.00
3^2	0.00	-0.83	13	0.02	3.84E-11	0.00E+00	7.50E+00	-1.00	1.00
3^1	-0.33	-0.32	15	0.02	8.67E-07	3.33E-01	1.00E+00	-1.00	0.67
3^0	-0.18	-0.51	20	0.02	8.94E-08	1.84E-01	1.00E+00	-1.00	0.82
3^{-1}	-2.00	-0.17	12	0.01	7.88E-07	2.00E+00	3.34E-01	-1.00	-1.00
3^{-2}	-2.00	-0.17	11	0.01	1.25E-08	2.00E+00	7.78E-01	-1.00	-1.00

Problem name: `MitsosBarton2006Ex316`

Source: [35]

description: Mitsos and Barton defined Example 3.16 as follows

$$\begin{aligned} F(x,y) &:= 2x+y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= -xy^2/2 - y^4/4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has two optimal solutions $x^* = -1, y^* = 0$ and $x^* = -1/2, y^* = -1$. Results were listed in following table, where starting points were $x^0 = -1.00, y^0 = -1.00$ and $x^0 = -1.00, y^0 = 1.00$, and our method achieved the first optimal one under two cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-0.54	0.00	18	0.12	2.70E-11	1.19E+00	2.65E-02	-0.18	-0.19
3^1	-0.25	0.03	11	0.02	8.06E-08	1.46E+00	6.61E-02	-0.35	0.46
3^0	-2.00	0.00	16	0.02	7.32E-07	1.00E+00	8.87E-07	-1.00	0.00
3^{-1}	-3.00	0.25	64	0.10	3.47E-15	2.00E+00	2.33E+00	-1.00	-1.00
3^{-2}	-3.00	0.25	73	0.10	7.91E-09	1.58E-03	1.00E+00	-1.00	-1.00
3^2	0.12	0.00	21	0.02	1.46E-12	8.89E-01	1.02E-24	-0.06	0.25
3^1	-2.00	0.00	91	0.09	3.69E-10	3.33E-01	1.33E-17	-1.00	0.00
3^0	-3.00	0.25	10	0.01	1.20E-10	1.22E-04	1.00E+00	-1.00	-1.00
3^{-1}	-3.00	0.25	12	0.01	4.98E-10	2.00E+00	2.33E+00	-1.00	-1.00
3^{-2}	-3.00	0.25	12	0.01	7.43E-07	1.23E-01	1.03E+00	-1.00	-1.00

Problem name: `MitsosBarton2006Ex317`

Source: [35]

description: Mitsos and Barton defined Example 3.17 as follows

$$\begin{aligned} F(x,y) &:= (x+1/2)^2 + y^2/2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= xy^2/2 + y^4/4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has two optimal solutions $x^* = -1/4, y^* = \pm 1/2$. Results were listed in following table, where two starting points were $x^0 = -1.00, y^0 = 1.00$ and $x^0 = 1.00, y^0 = 1.00$. When $\lambda = 3^2, 3^1$, our method got the optimal solution.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.19	-0.02	12	0.11	2.60E-08	1.11E-01	5.08E-13	-0.25	0.50
3^1	0.19	-0.02	13	0.02	1.09E-07	3.33E-01	3.98E-11	-0.25	0.50
3^0	0.00	0.00	113	0.16	3.30E-07	7.86E-07	1.01E-14	-0.50	0.00
3^{-1}	0.00	0.00	9	0.01	3.26E-07	2.91E-07	5.33E-10	-0.50	0.00
3^{-2}	0.00	0.00	9	0.02	1.38E-09	7.84E-09	2.19E-11	-0.50	0.00
3^2	0.19	-0.02	14	0.02	6.67E-12	1.11E-01	3.48E-24	-0.25	0.50
3^1	0.19	-0.02	16	0.02	2.53E-07	3.33E-01	2.31E-10	-0.25	0.50
3^0	0.00	0.00	20	0.02	4.39E-09	9.32E-09	0.00E+00	-0.50	0.00
3^{-1}	0.00	0.00	11	0.01	3.53E-07	1.91E-06	3.53E-09	-0.50	0.00
3^{-2}	0.00	0.00	12	0.01	1.76E-11	2.53E-10	9.50E-15	-0.50	0.00

Problem name: `MitsosBarton2006Ex318`

Source: [35]

description: Mitsos and Barton defined Example 3.18 as follows

$$\begin{aligned} F(x,y) &:= -x^2+y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= xy^2-y^4/2 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has a unique optimal solution $x^* = 1/2, y^* = 0$. Results were listed in following table, where two starting points were $x^0 = -1.00, y^0 = -1.00$ and $x^0 = 1.00, y^0 = 1.00$, and our method achieved solutions such that $-1 = F(\hat{x},\hat{y}) < F(x^*,y^*) = -1/4, f(\hat{x},\hat{y}) = f(x^*,y^*) = 0$. The solution $\hat{x} = 1, \hat{y} = 0$ is same as that obtained in [43].

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-1.00	0.00	9	0.11	1.02E-08	1.13E-21	0.00E+00	1.00	0.00
3^1	-1.00	0.00	9	0.02	1.02E-08	7.97E-21	1.36E-20	1.00	0.00
3^0	-1.00	0.00	9	0.01	1.02E-08	3.02E-21	0.00E+00	1.00	0.00
3^{-1}	-1.00	0.00	8	0.01	1.45E-09	2.14E-09	1.26E-09	-1.00	0.00
3^{-2}	-1.00	0.00	8	0.02	6.95E-10	2.90E-09	6.45E-10	-1.00	0.00
3^2	0.25	0.13	11	0.01	3.75E-10	7.86E-02	2.63E-13	0.50	0.71
3^1	0.25	0.12	9	0.01	2.62E-12	2.36E-01	7.44E-19	0.50	0.71
3^0	0.00	0.50	16	0.02	2.66E-12	9.97E-01	1.99E+00	1.00	1.00
3^{-1}	0.00	0.00	10	0.01	1.53E-11	1.00E+00	1.72E-12	0.00	0.00
3^{-2}	0.00	0.00	11	0.01	1.61E-13	1.00E+00	1.57E-13	0.00	0.00

Problem name: `MitsosBarton2006Ex319`

Source: [35]

description: Mitsos and Barton defined Example 3.19 as follows

$$\begin{aligned} F(x,y) &:= xy - y + y^2/2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\ f(x,y) &:= -xy^2 + y^4/2 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has a unique optimal solution $x^* = 0.189, y^* = 0.434$ with $F(x^*,y^*) = -0.258, f(x^*,y^*) = -0.018$. Results were listed in following table, where two starting points were $x^0 = 0.00, y^0 = 0.50$ and $x^0 = 0.50, y^0 = 1.00$, and our method approached the unique solution under four cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-0.258	-0.018	12	0.12	2.24E-08	5.56E-02	2.00E-15	0.189	0.434
3^1	-0.258	-0.018	6	0.01	4.21E-08	1.67E-01	1.09E-08	0.189	0.434
3^0	-0.258	-0.018	8	0.01	2.23E-10	5.00E-01	1.63E-11	0.189	0.434
3^{-1}	-0.692	0.157	17	0.02	1.83E-10	1.38E+00	8.76E-01	-1.000	0.382
3^{-2}	-1.201	0.688	18	0.03	1.98E-08	1.74E+00	1.00E+00	-1.000	0.736
3^2	0.000	0.000	10	0.02	2.96E-12	1.11E-01	1.34E-17	-0.999	0.000
3^1	0.000	0.000	10	0.01	9.95E-07	3.49E-01	5.29E-08	-0.914	0.000
3^0	-0.258	-0.018	9	0.01	1.21E-10	5.00E-01	2.81E-26	0.189	0.434
3^{-1}	0.000	0.000	35	0.04	1.70E-07	3.58E-04	1.30E-07	1.000	0.000
3^{-2}	-1.201	0.688	11	0.01	1.58E-10	1.74E+00	1.00E+00	-1.000	0.736

Problem name: `MitsosBarton2006Ex320`

Source: [35]

description: Mitsos and Barton defined Example 3.20 as follows

$$\begin{aligned}
F(x,y) &:= (x-1/4)^2+y^2\\
G(x,y) &:= \begin{bmatrix} -x-1\\ x-1\end{bmatrix}\\
f(x,y) &:= y^3/3-x^2y\\
g(x,y) &:= \begin{bmatrix} -y-1\\ y-1\end{bmatrix}
\end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has a unique optimal solution $x^*=1/2, y^*=1/2$ with $F(x^*,y^*)=0.3125, f(x^*,y^*)=-0.0833$. Results were listed in following table, where two starting points were $x^0=0.25, y^0=0.00$ and $x^0=1.00, y^0=1.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.05	0.00	22	0.13	5.83E-07	1.00E+00	6.24E-04	0.02	0.00
3^1	0.04	0.00	16	0.02	9.82E-08	1.00E+00	3.88E-03	0.06	0.00
3^0	0.02	0.00	14	0.02	1.98E-09	1.07E+00	9.73E-03	0.12	-0.07
3^{-1}	0.00	0.00	12	0.02	2.39E-11	9.91E-01	3.52E-02	0.19	0.01
3^{-2}	0.00	0.00	12	0.03	8.28E-08	9.97E-01	5.06E-02	0.23	0.00
3^2	0.03	0.00	10	0.01	1.34E-08	1.11E-01	9.32E-14	0.12	0.12
3^1	0.03	0.00	15	0.01	2.81E-07	3.33E-01	2.93E-14	0.13	0.13
3^0	0.02	0.00	10	0.01	1.82E-07	9.25E-01	1.12E-02	0.13	0.07
3^{-1}	0.00	0.00	13	0.01	3.19E-11	9.91E-01	3.52E-02	0.19	0.01
3^{-2}	0.00	0.00	14	0.01	2.88E-07	9.97E-01	5.06E-02	0.23	0.00

Problem name: `MitsosBarton2006Ex321`

Source: [35]

description: Mitsos and Barton defined Example 3.21 as follows

$$\begin{aligned}
F(x,y) &:= (x+0.6)^2+y^2\\
G(x,y) &:= \begin{bmatrix} -x-1\\ x-1 \end{bmatrix}\\
f(x,y) &:= y^4+\tfrac{2}{15}(-x+1)y^3+(-0.02x^2+0.16x-0.4)y^2\\
&\quad +(0.004x^3-0.036x^2+0.08x)y\\
g(x,y) &:= \begin{bmatrix} -y-1\\ y-1 \end{bmatrix}
\end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has a unique optimal solution $x^*=-0.5545, y^*=0.4554$ with $F(x^*,y^*)=0.2095, f(x^*,y^*)=-0.0656$. Results were listed in following table, where two starting points were $x^0=1.00, y^0=1.00$ and $x^0=0.00, y^0=1.00$, and our method achieved same one when $\lambda\geq 3^0$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.2095	-0.0656	11	0.12	2.32E-11	4.90E-02	2.48E-12	-0.5545	0.4554
3^1	0.2095	-0.0656	13	0.02	1.70E-09	1.47E-01	3.05E-11	-0.5545	0.4554
3^0	0.2095	-0.0656	10	0.01	1.02E-09	4.41E-01	1.79E-11	-0.5545	0.4554
3^{-1}	0.0036	0.0018	12	0.01	3.56E-11	3.44E-01	2.56E-13	-0.5941	-0.0594
3^{-2}	0.0036	0.0018	26	0.04	8.07E-10	1.03E+00	8.03E-10	-0.5941	-0.0594
3^2	0.2095	-0.0656	23	0.03	2.47E-07	4.90E-02	2.46E-07	-0.5545	0.4554
3^1	0.2095	-0.0656	10	0.01	6.15E-13	1.47E-01	3.96E-18	-0.5545	0.4554
3^0	0.2095	-0.0656	11	0.01	3.12E-08	4.41E-01	1.24E-09	-0.5545	0.4554
3^{-1}	0.0036	0.0018	17	0.02	7.64E-11	3.44E-01	3.46E-13	-0.5941	-0.0594
3^{-2}	0.0036	0.0018	14	0.01	7.95E-10	1.03E+00	7.91E-10	-0.5941	-0.0594

Problem name: `MitsosBarton2006Ex322`

Source: [35]

description: Mitsos and Barton defined Example 3.22 as follows

$$
\begin{aligned}
F(x,y) &:= (x+0.6)^2 + y^2\\
G(x,y) &:= \begin{bmatrix} -x-1\\ x-1 \end{bmatrix}\\
f(x,y) &:= y^4 + \tfrac{2}{15}(-x+1)y^3 + (-0.02x^2 + 0.16x - 0.4)y^2\\
&\quad +(0.004x^3 - 0.036x^2 + 0.08x)y\\
g(x,y) &:= \begin{bmatrix} -y-1\\ y-1\\ 0.01(1+x)^2 - y^2 \end{bmatrix}
\end{aligned}
$$

Comment: Mitsos and Barton in [35] stated that this problem has a unique optimal solution $x^* = -0.5545, y^* = 0.4554$ with $F(x^*, y^*) = 0.2095, f(x^*, y^*) = -0.0656$. Results were listed in following table, where starting points were $x^0 = 1.00, y^0 = 1.00$ and $x^0 = 0.00, y^0 = 1.00$, and our method achieved same one when $\lambda \geq 3^0$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.2095	-0.0656	12	0.12	1.20E-07	4.90E-02	5.67E-10	-0.5545	0.4554
3^1	0.2095	-0.0656	11	0.02	3.62E-07	1.47E-01	8.94E-09	-0.5545	0.4554
3^0	0.2095	-0.0656	13	0.02	4.98E-07	4.41E-01	1.62E-07	-0.5545	0.4554
3^{-1}	0.0136	0.0002	14	0.02	1.54E-09	8.83E-01	2.17E+00	-0.5987	-0.1165
3^{-2}	0.0136	-0.0131	9	0.02	3.13E-10	8.84E-01	1.23E+00	-0.5849	0.1158
3^2	0.2095	-0.0656	12	0.01	9.49E-07	4.90E-02	3.17E-14	-0.5545	0.4554
3^1	0.2095	-0.0656	12	0.01	1.48E-12	1.47E-01	9.49E-16	-0.5545	0.4554
3^0	0.2095	-0.0656	13	0.01	1.71E-11	4.41E-01	1.71E-11	-0.5545	0.4554
3^{-1}	0.0136	0.0002	20	0.02	6.19E-09	8.83E-01	2.17E+00	-0.5987	-0.1165
3^{-2}	0.0136	-0.0131	23	0.03	3.07E-07	8.84E-01	1.23E+00	-0.5849	0.1158

Problem name: `MitsosBarton2006Ex323`

Source: [35]

description: Mitsos and Barton defined Example 3.23 as follows

$$\begin{aligned}
F(x,y) &:= x^2 \\
G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \\ 1+x-9x^2-y \end{bmatrix} \\
f(x,y) &:= y \\
g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \\ y^2(x-0.5) \end{bmatrix}
\end{aligned}$$

Comment: Mitsos and Barton in [35] stated that this problem has a unique optimal solution $x^* = -0.4191, y^* = -1$. Results were listed in following table, where two starting points were $x^0 = -0.50, y^0 = 0.00$ and $x^0 = -1.00, y^0 = -1.00$, and our method achieved the optimal solution under many scenarios.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.1757	-1.0000	10	0.11	1.64E-07	2.69E-11	7.90E+00	-0.4191	-1.0000
3^1	0.1757	-1.0000	11	0.02	2.55E-12	0.00E+00	1.90E+00	-0.4191	-1.0000
3^0	0.0464	0.8493	143	0.18	2.59E-01	1.85E+00	1.01E+02	0.2154	0.8493
3^{-1}	0.1757	-1.0000	95	0.10	9.70E-09	0.00E+00	7.65E-01	-0.4191	-1.0000
3^{-2}	0.1757	-1.0000	33	0.05	1.69E-07	7.11E-09	9.87E-01	-0.4191	-1.0000
3^2	0.0031	1.0139	109	0.17	1.96E-02	1.99E+00	1.54E+01	0.0558	1.0139
3^1	0.1757	-1.0000	13	0.01	5.98E-11	2.15E-14	1.90E+00	-0.4191	-1.0000
3^0	0.1757	-1.0000	45	0.05	2.96E-09	0.00E+00	9.81E-02	-0.4191	-1.0000
3^{-1}	0.2500	-0.7500	118	0.13	1.26E-08	2.50E-01	3.12E-01	0.5000	-0.7500
3^{-2}	0.1757	-1.0000	91	0.12	6.14E-08	6.14E-08	9.87E-01	-0.4191	-1.0000

Problem name: `MitsosBarton2006Ex324`

Source: [35]

description: Mitsos and Barton defined Example 3.24 as follows

$$\begin{array}{rcl} F(x,y) & := & x^2 - y \\ G(x,y) & := & \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\ f(x,y) & := & \left((y-1-0.1x)^2 - 0.5 - 0.5x\right)^2 \\ g(x,y) & := & \begin{bmatrix} -y \\ y-3 \end{bmatrix} \end{array}$$

Comment: Mitsos and Barton in [35] stated that this problem has a unique optimal solution $x^* = 0.2106, y^* = 1.7993$ with $F(x^*, y^*) - 1.7549, f(x^*, y^*) = 0$. Results were listed in following table, where two starting points were $x^0 = 0.00, y^0 = 2.00$ and $x^0 = 1.00, y^0 = 3.00$, and similar solutions were obtained.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-1.7547	0.0000	15	0.12	2.01E-08	1.28E-02	4.26E-24	0.2107	1.7991
3^1	-1.7547	0.0000	11	0.02	7.47E-07	3.83E-02	7.26E-13	0.2107	1.7991
3^0	-1.7547	0.0000	9	0.01	3.11E-10	1.15E-01	4.89E-22	0.2107	1.7991
3^{-1}	-1.7547	0.0000	8	0.01	3.21E-11	3.44E-01	1.38E-17	0.2107	1.7991
3^{-2}	-1.7621	0.0001	10	0.02	8.98E-09	1.00E+00	3.61E-02	0.2071	1.8050
3^2	-1.7547	0.0000	15	0.02	1.06E-07	1.28E-02	1.02E-20	0.2107	1.7991
3^1	-1.7547	0.0000	9	0.01	2.19E-10	3.83E-02	1.70E-13	0.2107	1.7991
3^0	-1.7547	0.0000	11	0.01	8.19E-09	1.15E-01	1.48E-11	0.2107	1.7991
3^{-1}	-1.7547	0.0000	10	0.01	1.83E-10	3.44E-01	1.46E-13	0.2107	1.7991
3^{-2}	-1.7621	0.0001	13	0.01	2.24E-10	1.00E+00	3.61E-02	0.2071	1.8050

Problem name: `MitsosBarton2006Ex325`

Source: [35]

description: Mitsos and Barton defined Example 3.25 as follows

$$\begin{aligned}
F(x,y) &:= x_1y_1 + x_2y_1^2 - x_1x_2y_3 \\
G(x,y) &:= \begin{bmatrix} -x_1-1 \\ -x_2-1 \\ x_1-1 \\ x_2-1 \\ 0.1y_1y_2-x_1^2 \\ x_2y_1^2 \end{bmatrix} \\
f(x,y) &:= x_1y_1^2 + x_2y_2y_3 \\
g(x,y) &:= \begin{bmatrix} -y_1-1 \\ -y_2-1 \\ -y_3-1 \\ y_1-1 \\ y_2-1 \\ y_3-1 \\ y_1^2-y_2y_3 \\ y_2^2y_3-y_1x_1 \\ -y_3^2+0.1 \end{bmatrix}
\end{aligned}$$

Comment: Mitsos and Barton in [35] stated that the best objective function values $F(x^*,y^*) = -1$ and $f(x^*,y^*) = -2$. One of probable solutions is $x^* = (-1,-1)^T$, $y^* = (-1,1,1)^T$. Results were listed in following table, where two starting points were $x^0 = (0.00, 0.00)^T, y^0 = (-1.00, 1.00, 1.00)^T$ and $x^0 = (-1.00, -1.00)^T, y^0 = (-1.00, 0.00, 0.00)^T$. We achieved the same one when $\lambda = 3^0$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.00	178	0.78	3.66E-05	1.31E+00	7.91E-01	0.00 0.01	0.00 0.00 -0.32
3^1	0.00	0.00	289	0.74	1.00E-01	3.22E-01	9.97E-01	0.00 -0.13	0.01 0.01 0.32
3^0	-1.00	-2.00	60	0.10	8.73E-07	1.58E-07	1.29E+00	-1.00 -1.00	1.00 -1.00 -1.00
3^{-1}	0.00	0.00	175	0.48	1.00E-01	3.18E-01	1.06E+02	0.00 1.00	0.00 0.00 0.00
3^{-2}	0.00	0.00	145	0.36	1.00E-01	7.55E-01	2.66E+02	0.00 1.00	0.00 -0.04 0.00
3^2	0.00	0.00	392	1.00	5.27E-06	1.57E-03	2.67E-02	0.00 0.00	0.00 0.00 -0.32
3^1	-0.43	-0.55	315	0.89	6.14E-01	2.51E-01	5.53E+02	0.00 -0.76	-0.75 -0.85 -0.86
3^0	-1.00	-2.00	17	0.02	3.37E-12	8.98E-15	5.62E-01	-1.00 -1.00	-1.00 1.00 1.00
3^{-1}	0.00	0.00	191	0.30	5.35E-05	1.32E+00	1.26E+00	0.00 0.00	0.00 0.00 -0.32
3^{-2}	0.00	0.00	140	0.34	4.85E-01	6.29E-01	9.90E-01	0.00 0.00	0.00 0.00 0.00

Problem name: `MitsosBarton2006Ex326`

Source: [35]

description: Mitsos and Barton defined Example 3.26 as follows

$$
\begin{aligned}
F(x,y) &:= x_1y_1 + x_2y_2^2 + x_1x_2y_3^3\\
G(x,y) &:= \begin{bmatrix} 0.1 - x_1^2\\ 1.5 - y_1^2 - y_2^2 - y_3^2\\ -2.5 + y_1^2 + y_2^2 + y_3^2\\ -x_1 - 1\\ -x_2 - 1\\ x_1 - 1\\ x_2 - 1 \end{bmatrix}\\
f(x,y) &:= x_1y_1^2 + x_2y_2^2 + (x_1 - x_2)y_3^2\\
g(x,y) &:= \begin{bmatrix} -y_1 - 1\\ -y_2 - 1\\ -y_3 - 1\\ y_1 - 1\\ y_2 - 1\\ y_3 - 1 \end{bmatrix}
\end{aligned}
$$

Comment: Mitsos and Barton in [35] stated that this problem has a unique optimal solution $x^* = (-1,-1)^\top, y^* = (1,1,-0.707)^\top$. Results were listed in following table, where two starting points were $x^0 = (0.00, 0.00)^\top, y^0 = (1.00, 1.00, 0.71)^\top$ and $x^0 = (-1.00,-1.00)^\top, y^0 = (0.80, 0.80, -0.80)^\top$. Our method achieved the unique optimal solution when $\lambda = 3^1$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-2.00	-2.00	16	0.14	8.01E-07	5.74E-01	8.75E+00	-1.000 -1.000	1.000 1.000 0.000
3^1	-2.35	-2.00	15	0.04	5.55E-08	2.01E-01	1.71E+00	-1.000 -1.000	1.000 1.000 -0.707
3^0	0.17	-0.36	125	0.26	7.92E-10	4.72E-01	4.99E-01	-0.316 -0.100	-0.658 -0.258 -1.000
3^{-1}	-1.19	-1.42	177	0.29	2.08E-01	8.18E-01	2.32E+01	-1.000 -0.494	1.191 0.007 -0.007
3^{-2}	-2.00	-2.00	25	0.06	1.91E-08	5.40E-01	2.86E-01	-1.000 -1.000	1.000 1.000 0.000
3^2	0.15	0.24	43	0.07	6.08E-12	1.86E+00	3.87E-01	0.316 0.156	0.000 -0.856 0.876
3^1	-2.35	-2.00	11	0.02	1.15E-14	2.02E-01	1.71E+00	-1.000 -1.000	1.000 1.000 -0.707
3^0	0.00	0.00	125	0.17	1.00E-01	4.02E-01	5.54E-07	0.000 0.000	-0.954 -0.002 -0.800
3^{-1}	-1.00	-2.00	59	0.10	6.39E-08	6.81E-01	4.86E-01	-1.000 0.000	1.000 0.704 1.000
3^{-2}	-1.00	-1.79	103	0.17	1.60E-08	1.11E-01	7.28E-01	-1.000 -0.039	1.000 0.843 0.889

Problem name: `MitsosBarton2006Ex327`

Source: [35]

description: Mitsos and Barton defined Example 3.27 as follows

$$F(x,y) := \sum_{j=1}^{5}(x_j^2+y_j^2)$$

$$f(x,y) := y_1^3+y_2^2x_1+y_2^2x_2+0.1y_3+(y_4^2+y_5^2)x_3x_4x_5$$

$$G(x,y) := \begin{bmatrix} -x_1-1 \\ \vdots \\ -x_5-1 \\ x_1-1 \\ \vdots \\ x_5-1 \\ y_1y_2-x_1 \\ x_2y_1^2 \\ x_1-e^{x_2}+y_3 \end{bmatrix} \quad g(x,y) := \begin{bmatrix} -y_1-1 \\ \vdots \\ -y_5-1 \\ y_1-1 \\ \vdots \\ y_5-1 \\ y_1y_2-0.3 \\ x_1-0.2-y_3^2 \\ -e^{y_3}+y_4y_5-0.1 \end{bmatrix}$$

Comment: Mitsos and Barton in [35] stated that the best objective function value $F(x^*,y^*)=2$ and $f(x^*,y^*)=-1.1$. One of probable solutions is $x^*=(0,0,0,0,0)^T$, $y^*=(-1,0,-1,0,0)^T$. Two starting points were $x^0=(0,0,0,0,0)^T$, $y^0=(0,0,0,0,0)^T$ and $x^0=(0,0,0,0,0)^T$, $y^0=(-1,-1,-1,-1,-1)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}
2^4	0.20	-0.05	13	0.15	4.02E-7	5.50E-1	1.00E-1
2^2	0.02	-0.01	11	0.04	1.44E-7	8.50E-1	1.00E-1
2^0	0.00	-0.01	12	0.04	7.42E-7	9.50E-1	1.00E-1
2^{-2}	0.00	0.00	13	0.05	1.93E-7	9.83E-1	1.00E-1
2^{-4}	0.00	0.00	11	0.05	9.77E-7	9.94E-1	1.00E-1
2^4	0.20	-0.05	69	0.26	8.97E-7	5.95E-1	1.00E-1
2^2	0.02	-0.02	131	0.43	3.00E-1	9.26E-1	1.00E+0
2^0	0.00	-0.01	126	0.32	1.33E-7	9.77E-1	1.00E-1
2^{-2}	0.04	0.00	24	0.07	8.08E-7	1.00E+0	8.58E-1
2^{-4}	0.04	0.00	113	0.26	2.79E-3	9.99E-1	9.99E-1

λ	$\hat{x}$	$\hat{y}$
3^2	0.00 0.00 0.00 0.00 0.00	0.00 0.00 -0.45 0.00 0.00
3^1	0.00 0.00 0.00 0.00 0.00	0.00 0.00 -0.15 0.00 0.00
3^0	0.00 0.00 0.00 0.00 0.00	0.00 0.00 -0.05 0.00 0.00
3^{-1}	0.00 0.00 0.00 0.00 0.00	0.00 0.00 -0.02 0.00 0.00
3^{-2}	0.00 0.00 0.00 0.00 0.00	0.00 0.00 -0.01 0.00 0.00
3^2	0.00 0.00 0.00 0.00 0.00	0.00 0.00 -0.45 0.00 0.00
3^1	0.00 -0.01 0.00 0.00 0.00	-0.02 0.00 -0.15 0.00 0.00
3^0	0.00 0.00 0.00 0.00 0.00	0.00 0.00 -0.05 0.00 0.00
3^{-1}	0.20 0.00 0.00 0.00 0.00	0.00 0.00 -0.02 0.00 0.00
3^{-2}	0.20 0.00 0.00 0.00 0.00	0.00 0.00 -0.01 0.00 0.00

Problem name: `MitsosBarton2006Ex328`

Source: [35]

description: Mitsos and Barton defined Example 3.28 as follows

$$
\begin{aligned}
F(x,y) &:= -\sum_{j=1}^{5}(x_j^2+y_j^2)\\
f(x,y) &:= y_1^3+y_2^2x_1+y_2^2x_2+0.1y_3+(y_4^2+y_5^2)x_3x_4x_5\\
G(x,y) &:= \begin{bmatrix} -x_1-1\\ \vdots\\ -x_5-1\\ x_1-1\\ \vdots\\ x_5-1\\ y_1y_2-x_1\\ x_2y_1^2\\ x_1-e^{x_2}+y_3 \end{bmatrix} \quad g(x,y) := \begin{bmatrix} -y_1-1\\ \vdots\\ -y_5-1\\ y_1-1\\ \vdots\\ y_5-1\\ y_1y_2-0.3\\ x_1-0.2-y_3^2\\ -e^{y_3}+y_4y_5-0.1 \end{bmatrix}
\end{aligned}
$$

Comment: Mitsos and Barton in [35] stated that the best objective function value $F(x^*,y^*)=-10$ and $f(x^*,y^*)=-3.1$. One of probable solutions is $x^*=(1,-1,-1,-1,-1)^T, y^*=(-1,1,-1,-1,1)^T$. Results were listed in following table, where two starting points were $x^0=(-0.25,-0.25,-0.25,-0.25,-0.25)^T, y^0=(-0.25,-0.25,-0.25,-0.25,-0.25)^T$ and $x^0=(0.50,0.50,0.50,0.50,0.50)^T, y^0=(0.50,0.50,0.50,0.50,0.50)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}
3^2	-2.61	-0.22	1000	2.62	4.15E-3	1.05E+0	1.00E+0
3^1	-4.24	-0.12	1000	1.63	6.91E-1	9.37E-1	1.61E+0
3^0	-8.68	0.11	1000	2.85	1.87E+1	1.68E+0	2.91E+3
3^{-1}	-7.94	-2.04	25	0.05	1.13E-8	9.99E-1	9.15E-1
3^{-2}	-10.00	-2.90	23	0.05	6.48E-9	1.07E+0	1.09E+0
3^1	-0.53	0.05	1000	3.03	2.52E-4	1.38E+0	1.01E-1
3^1	-0.02	0.01	48	0.09	2.64E-7	1.08E+0	1.00E-1
3^0	-5.00	-0.10	170	0.44	1.40E-2	8.23E-1	2.00E+0
3^{-1}	-2.94	-0.10	185	0.39	9.02E-4	1.15E+0	2.33E+0
3^{-2}	-8.16	-1.58	12	0.02	3.14E-8	7.23E-1	7.92E-1

λ	$\hat{x}$	$\hat{y}$
3^2	0.00 0.00 -1.00 1.00 0.00	-0.64 0.00 0.45 0.00 0.00
3^1	0.22 -1.00 -1.00 -1.00 1.00	0.01 -0.41 0.15 0.00 0.00
3^0	0.27 2.09 -1.00 -1.00 1.00	0.08 0.00 1.12 0.00 0.00
3^{-1}	1.00 -1.00 -1.00 -1.00 -1.00	-1.00 0.00 -1.00 -0.68 -0.68
3^{-2}	-1.00 -1.00 -1.00 -1.00 -1.00	1.00 -1.00 1.00 -1.00 -1.00
3^2	0.00 0.00 0.28 0.28 0.28	0.00 0.00 0.45 0.00 0.31
3^1	0.00 0.00 0.00 0.00 0.00	0.00 0.00 0.15 0.00 0.00
3^0	1.00 -1.00 0.00 1.00 1.00	0.00 0.00 -1.00 0.00 0.00
3^{-1}	0.00 0.00 0.00 0.00 -1.00	0.00 0.00 -1.00 0.68 0.68
3^{-2}	1.00 1.00 -1.00 -1.00 -1.00	0.00 0.40 1.00 1.00 1.00

Problem name: `MorganPatrone2006a`

Source: [36]

Description: Morgan and Patrone 2006 tested one example as follows

$$\begin{aligned} F(x,y) &:= -(x+y) \\ G(x,y) &:= \begin{bmatrix} -x-0.5 \\ x-0.5 \end{bmatrix} \\ f(x,y) &:= xy \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The best known optimal solution reported in [36] is $x^* = 0, y^* = 1$; In following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = 1.00, y^0 = 1.00$, we achieved the unique many cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-1.00	0.00	32	0.14	3.33E-07	1.11E-01	1.00E+00	0.00	1.00
3^1	-1.00	0.00	12	0.02	5.72E-11	3.33E-01	1.00E+00	0.00	1.00
3^0	-1.00	0.00	13	0.02	7.08E-12	1.00E+00	1.00E+00	0.00	1.00
3^{-1}	-1.50	0.50	26	0.04	5.25E-11	2.00E+00	9.72E-01	0.50	1.00
3^{-2}	-1.50	0.50	10	0.02	4.28E-09	2.00E+00	1.07E+00	0.50	1.00
3^2	0.78	-0.10	17	0.02	1.54E-08	1.11E-01	1.11E-01	0.11	-0.89
3^1	-1.00	0.00	13	0.02	6.63E-09	3.33E-01	1.00E+00	0.00	1.00
3^0	-1.00	0.00	10	0.01	1.12E-07	1.00E+00	1.00E+00	0.00	1.00
3^{-1}	-1.50	0.50	10	0.01	1.04E-12	2.00E+00	9.72E-01	0.50	1.00
3^{-2}	-1.50	0.50	10	0.01	9.41E-11	2.00E+00	1.07E+00	0.50	1.00

Problem name: `MorganPatrone2006b`

Source: [36]

Description: Morgan and Patrone 2006 tested one example as follows

$$\begin{aligned} F(x,y) &:= -(x+y) \\ f(x,y) &:= \begin{cases} (x+0.25)y & \text{if } x \in [-0.5,-0.25] \\ 0 & \text{if } x \in [-0.25,0.25] \\ (x-0.25)y & \text{if } x \in [0.25,0.5] \end{cases} \\ g(x,y) &:= \begin{bmatrix} -x-0.5 \\ x-0.5 \\ -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The best known optimal solution reported in [36] is $x^* = 0.25, y^* = 1$; Results were listed in following table, where two starting points were $x^0 = 1.00, y^0 = 1.00$ and $x^0 = 0.00, y^0 = 0.00$, and our method reached the same one when $\lambda = 3^0$ under the first starting point.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.53	-0.10	14	0.12	2.90E-09	1.11E-01	1.11E-01	0.36	-0.89
3^1	-0.75	0.00	49	0.07	5.07E-07	3.33E-01	1.00E+00	-0.25	1.00
3^0	-1.25	0.00	10	0.02	2.91E-07	1.00E+00	1.00E+00	0.25	1.00
3^{-1}	-1.50	0.25	11	0.02	7.34E-10	2.00E+00	9.50E-01	0.50	1.00
3^{-2}	-1.50	0.25	11	0.03	8.47E-10	2.00E+00	1.00E+00	0.50	1.00
3^2	-0.50	-0.25	16	0.02	7.15E-10	6.95E-10	3.00E+00	-0.50	1.00
3^1	-0.50	-0.25	18	0.03	5.82E-08	5.82E-08	1.54E+00	-0.50	1.00
3^0	-1.50	0.25	55	0.06	4.18E-11	2.00E+00	1.62E-01	0.50	1.00
3^{-1}	-1.50	0.25	23	0.03	1.53E-07	2.00E+00	9.56E-01	0.50	1.00
3^{-2}	-1.50	0.25	17	0.02	4.61E-11	2.00E+00	1.00E+00	0.50	1.00

Problem name: `MorganPatrone2006c`

Source: [36]

Description: Morgan and Patrone 2006 tested one example as follows

$$\begin{aligned} F(x,y) &:= -(x+y) \\ f(x,y) &:= \begin{cases} \left(x+\frac{7}{4}\right)y & \text{if } x \in [-2,-7/4] \\ 0 & \text{if } x \in [-7/4,7/4] \\ \left(x-\frac{7}{4}\right)y & \text{if } x \in [7/4,2] \end{cases} \\ g(x,y) &:= \begin{bmatrix} -x-2 \\ x-2 \\ -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: The best known optimal solution reported in [36] is $x^* = 2, y^* = -1$; Results were listed in following table, where two starting points were $x^0 = 2.00, y^0 = -4.00$ and $x^0 = 0.00, y^0 = -4.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.00	-0.25	16	0.13	6.99E-07	6.99E-07	3.00E+00	-2.00	1.00
3^1	1.00	-0.25	31	0.05	2.81E-10	2.69E-10	1.52E+00	-2.00	1.00
3^0	1.00	-0.25	584	0.75	4.14E-12	4.14E-12	1.37E+00	-2.00	1.00
3^{-1}	-3.00	0.25	28	0.04	4.32E-11	2.00E+00	9.89E-01	2.00	1.00
3^{-2}	-3.00	0.25	17	0.03	4.13E-11	2.00E+00	1.00E+00	2.00	1.00
3^2	1.00	-0.25	36	0.04	7.28E-10	7.28E-10	3.00E+00	-2.00	1.00
3^1	1.00	-0.25	35	0.04	2.95E-10	3.05E-10	1.52E+00	-2.00	1.00
3^0	-3.00	0.25	40	0.04	4.80E-11	2.00E+00	1.16E+00	2.00	1.00
3^{-1}	-3.00	0.25	30	0.02	3.72E-11	2.00E+00	9.88E-01	2.00	1.00
3^{-2}	-3.00	0.25	27	0.02	3.25E-11	2.00E+00	1.00E+00	2.00	1.00

Problem name: `MuuQuy2003Ex1`

Source: [37]

Description: Muu and Quy 2003 tested Example 1 as follows

$$\begin{aligned} F(x,y) &:= x^2-4x+y_1^2+y_2^2 \\ G(x,y) &:= \begin{bmatrix} -x \\ -2+x \end{bmatrix} \\ f(x,y) &:= y_1^2+\tfrac{1}{2}y_2^2+y_1y_2+(1-3x)y_1+(1+x)y_2 \\ g(x,y) &:= \begin{bmatrix} 2y_1+y_2-2x-1 \\ -y_1 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: $x^* = 0.844$, $y^* = (0.766, 0)^T$ is the best known solution according to [37]. Results were listed in following table, where two starting points were $x^0 = 1.00, y^0 = (1.00, 1.00)^T$ and $x^0 = 1.00, y^0 = (2.00, 0.00)^T$, and our method closely reached x^*, y^* when $\lambda = 3^2$. Notice that solutions under $\lambda \le 3^0$ lead to $F(\hat{x},\hat{y}) < F(x^*,y^*) = -2.077$ and $f(\hat{x},\hat{y}) < f(x^*,y^*) = -0.586$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-2.08	-0.59	22	0.15	2.55E-08	8.55E-02	7.71E+00	0.85	0.77 0.00
3^1	1.30	0.34	109	0.15	3.80E-01	2.84E-02	1.09E+00	-0.30	0.14 0.06
3^0	-2.24	-0.74	9	0.02	7.76E-11	4.62E-01	2.76E-01	0.91	0.76 0.00
3^{-1}	-3.01	-1.23	8	0.01	4.57E-09	6.46E-01	8.45E-01	1.21	0.61 0.00
3^{-2}	-3.68	-1.17	8	0.02	1.05E-09	8.28E-01	9.73E-01	1.55	0.35 0.00
3^2	-2.08	-0.59	22	0.06	2.55E-08	8.55E-02	7.71E+00	0.85	0.77 0.00
3^1	1.30	0.34	109	0.12	3.80E-01	2.84E-02	1.09E+00	-0.30	0.14 0.06
3^0	-2.24	-0.74	9	0.01	7.76E-11	4.62E-01	2.76E-01	0.91	0.76 0.00
3^{-1}	-3.01	-1.23	10	0.02	8.50E-12	6.46E-01	8.45E-01	1.21	0.61 0.00
3^{-2}	-3.68	-1.17	10	0.01	5.68E-12	8.28E-01	9.73E-01	1.55	0.35 0.00

Problem name: `MuuQuy2003Ex2`

Source: [37]

Description: Muu and Quy 2003 tested Example 2 as follows

$$\begin{aligned}
F(x,y) &:= -7x_1 + 4x_2 + y_1^2 + y_3^2 - y_1y_3 - 4y_2 \\
G(x,y) &:= \begin{bmatrix} -x_1 \\ -x_2 \\ x_1 + x_2 - 1 \end{bmatrix} \\
f(x,y) &:= y_1^2 + \tfrac{1}{2}y_2^2 + \tfrac{1}{2}y_3^2 + y_1y_2 + (1-3x_1)y_1 + (1+x_2)y_2 \\
g(x,y) &:= \begin{bmatrix} 2y_1 + y_2 - y_3 + x_1 - 2x_2 + 2 \\ -y_1 \\ -y_2 \\ -y_3 \end{bmatrix}
\end{aligned}$$

Comment: The best known solution reported in [37] is $x^* = (0.609, 0.391)^T$, $y^* = (0, 0, 1.828)^T$ with $F(x^*, y^*) = 0.6426$ and $f(x^*, y^*) = 1.6708$. Results were listed in following table, where two starting points were $x^0 = (1.00, 1.00)^T, y^0 = (1.00, 1.00, 1.00)^T$ and $x^0 = (0.00, 0.00)^T, y^0 = (0.00, 0.00, 0.00)^T$, and our method reached x^*, y^* for all cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.64	1.68	30	0.15	2.79E-11	1.21E-16	9.03E+00	0.61 0.39	0.00 0.00 1.83
3^1	0.64	1.68	59	0.12	2.86E-09	3.31E-19	3.15E+00	0.61 0.39	0.00 0.00 1.83
3^0	0.64	1.68	10	0.02	5.86E-07	3.19E-07	1.42E+00	0.61 0.39	0.00 0.00 1.83
3^{-1}	0.64	1.68	11	0.02	1.70E-10	1.31E-10	1.08E+00	0.61 0.39	0.00 0.00 1.83
3^{-2}	0.64	1.68	24	0.05	4.00E-08	3.09E-08	1.03E+00	0.61 0.39	0.00 0.00 1.83
3^2	0.64	1.68	28	0.04	8.85E-08	3.46E-14	9.03E+00	0.61 0.39	0.00 0.00 1.83
3^1	0.64	1.68	19	0.03	7.94E-09	2.74E-20	3.15E+00	0.61 0.39	0.00 0.00 1.83
3^0	0.64	1.68	10	0.01	3.16E-08	1.53E-15	1.42E+00	0.61 0.39	0.00 0.00 1.83
3^{-1}	0.64	1.68	12	0.01	9.55E-10	5.32E-15	1.08E+00	0.61 0.39	0.00 0.00 1.83
3^{-2}	0.64	1.68	17	0.02	1.77E-08	1.36E-08	1.03E+00	0.61 0.39	0.00 0.00 1.83

Problem name: `NieWangYe2017Ex34`

Source: [38]

Description: Nie, Wang and Ye 2017 defined Example 3.4 as follows

$$\begin{array}{rcl} F(x,y) & := & x + y_1 + y_2 \\ G(x,y) & := & \begin{bmatrix} -x+2 \\ -3+x \end{bmatrix} \\ f(x,y) & := & x(y_1+y_2) \\ g(x,y) & := & \begin{bmatrix} -y_1^2 + y_2^2 + (y_1^2+y_2^2)^2 \\ -y_1 \end{bmatrix} \end{array}$$

Comment: Results were listed in following table, where two starting points were $x^0 = 2.50, y^0 = (0.75, 0.25)^\top$ and $x^0 = 2.50, y^0 = (0.00, 0.00)^\top$. Many cases rendered solutions same as the global optimal solution $x^* = 2.00, y^* = (0.00, 0.00)^\top$ of this problem provided by Nie, Wang and Ye in [38].

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.98	-0.05	467	0.90	6.11E-04	2.47E-02	9.82E-01	2.00	0.00 -0.02
3^1	2.00	0.00	573	0.82	2.04E-03	4.49E-02	8.27E+02	2.00	0.00 0.00
3^0	2.00	0.00	240	0.35	1.78E-03	4.01E-02	2.02E+03	2.00	0.00 0.00
3^{-1}	2.00	0.00	160	0.23	1.62E-03	3.99E-02	4.27E+02	2.00	0.00 0.00
3^{-2}	2.00	0.00	138	0.21	1.73E-03	3.99E-02	1.10E+03	2.00	0.00 0.00
3^2	3.03	0.08	1000	1.17	3.11E+00	8.98E-01	1.00E+00	3.00	0.16 -0.13
3^1	2.01	0.02	197	0.22	1.30E-03	1.43E-01	9.98E-01	2.00	0.10 -0.10
3^0	1.95	-0.10	413	0.41	2.61E-03	5.10E-02	9.99E-01	2.00	0.00 -0.05
3^{-1}	2.00	0.00	538	0.53	2.01E-03	4.48E-02	1.08E+03	2.00	0.00 0.00
3^{-2}	1.96	-0.08	328	0.24	4.90E-02	6.40E-02	9.95E-01	2.00	0.00 -0.04

Problem name: `NieWangYe2017Ex52`

Source: [38]

Description: Nie, Wang and Ye 2017 defined Example 5.2 as follows

$$\begin{aligned}
F(x,y) &:= x_1y_1 + x_2y_2 + x_1x_2y_1y_2y_3\\
G(x,y) &:= \begin{bmatrix} x_1-1\\ x_2-1\\ -x_1-1\\ -x_2-1\\ y_1y_2-x_1^2 \end{bmatrix}\\
f(x,y) &:= x_1y_1^2 + x_2^2y_2y_3 - y_1y_3^2\\
g(x,y) &:= \begin{bmatrix} 1-y_1^2-y_2^2-y_3^2\\ y_1^2+y_2^2+y_3^2-2 \end{bmatrix}
\end{aligned}$$

Comment: The best known solution provided in [38] is $x^* = (-1,-1)^\top, y^* = (1.110, 0.314, -0.818)^\top$ with $F(x^*,y^*) = -1.710$ and $f(x^*,y^*) = -2.232$. Results were listed in following table, where two starting points were $x^0 = (-1.00,-1.00)^\top, y^0 = (1.00, 0.71, -0.71)^\top$ and $x^0 = (-1.00,-1.00)^\top, y^0 = (1.00, 0.00, 0.00)^\top$, and our method achieved a better solution in terms of smaller F when $\lambda = 3^0$ under the first starting point.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-0.44	0.04	44	0.19	1.32E-08	2.63E-01	2.17E-02	0.05 0.43	0.56 -1.08 -0.18
3^1	0.35	-0.24	127	0.26	3.78E-09	2.59E-01	2.78E-01	1.00 0.00	0.35 0.00 1.02
3^0	-2.14	-2.08	11	0.02	5.94E-07	2.54E-01	5.25E-01	-1.00 -1.00	1.01 0.64 -0.75
3^{-1}	-0.07	0.43	51	0.08	2.34E-08	1.70E+00	1.22E+00	0.13 0.00	-0.57 0.00 -0.82
3^{-2}	-1.41	-2.00	12	0.03	4.85E-08	5.19E-09	5.35E-01	-1.00 0.00	1.41 0.00 0.00
3^2	0.00	0.00	7	0.01	3.68E-07	4.08E-02	9.78E-09	0.00 -0.44	1.36 0.00 0.00
3^1	-1.41	-2.00	12	0.02	3.98E-07	1.32E-08	2.35E+00	-1.00 0.00	1.41 0.00 0.00
3^0	-1.41	-2.00	10	0.01	8.35E-07	1.89E-07	3.54E-01	-1.00 0.00	1.41 0.00 0.00
3^{-1}	0.00	0.00	151	0.19	1.41E+00	1.36E+00	8.52E-07	0.00 0.00	-0.10 1.00 0.00
3^{-2}	0.00	0.00	39	0.06	4.45E-11	1.22E+00	1.01E+00	0.55 0.00	0.00 0.00 1.41

Problem name: `NieWangYe2017Ex54`

Source: [38]

Description: Nie, Wang and Ye 2017 defined Example 5.4 as follows

$$\begin{array}{rcl} F(x,y) & := & x_1^2y_1 + x_2y_2 + x_3y_3^2 + x_4y_4^2 \\ G(x,y) & := & \begin{bmatrix} \|x\|^2-1 \\ y_1y_2 - x_1 \\ y_3y_4 - x_3^2 \end{bmatrix} \\ f(x,y) & := & y_1^2 - y_2(x_1+x_2) - (y_3+y_4)(x_3+x_4) \\ g(x,y) & := & \begin{bmatrix} \|y\|^2-1 \\ y_2^2+y_3^2+y_4^2-y_1 \end{bmatrix} \end{array}$$

Comment: The best known solution obtained in [38] is

$$x^* = (0,0,-0.707,-0.707)^\top, y^* = (0.618, 0, -0.556, -0.556)^\top$$

with $F(x^*,y^*) = -0.437$ and $f(x^*,y^*) = -1.190$. Results were listed in following table, where two starting points were $x^0 = (-1,-1,-1,-1)^\top, y^0 = (1,-1,-1,-1)^\top$ and $x^0 = y^0 = (-1,-1,-1,-1)^\top$. One can notice that our method achieved the same solution when $\lambda = 3^2$ under the first starting point.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-0.44	-1.19	21	0.14	1.35E-07	4.72E-09	8.32E+00	0.00 0.00 -0.71 -0.71	0.62 0.00 -0.56 -0.56
3^1	0.00	0.00	180	0.37	5.16E-06	2.74E-03	4.22E-04	0.00 0.00 0.00 0.00	0.00 -0.01 0.00 0.00
3^0	0.00	0.00	28	0.05	8.32E-07	5.48E-03	2.89E-05	0.00 0.00 0.00 0.00	0.00 -0.01 0.00 0.00
3^{-1}	0.00	0.00	112	0.15	4.55E-06	1.82E-02	2.04E-04	0.00 0.00 0.00 0.00	0.00 -0.01 0.00 0.00
3^{-2}	0.00	0.00	112	0.18	2.40E-06	1.18E-02	1.19E-04	0.00 0.00 0.00 0.00	0.00 0.00 0.00 0.00
3^2	0.00	0.00	120	0.21	2.65E-04	3.35E-03	8.58E-03	0.00 0.00 -0.10 0.10	0.00 -0.03 0.00 0.00
3^1	0.00	0.00	35	0.04	9.80E-07	6.43E-04	3.34E-05	0.00 0.00 0.71 -0.71	0.00 0.00 0.00 0.00
3^0	-0.13	-0.03	22	0.03	1.12E-09	3.33E-01	3.49E-01	0.27 -0.25 0.73 -0.58	0.21 0.18 0.05 0.41
3^{-1}	0.00	0.00	164	0.21	1.86E-06	1.35E-02	1.12E-04	0.00 0.00 0.00 0.00	0.00 0.00 0.00 0.00
3^{-2}	0.00	0.00	32	0.04	8.77E-07	1.78E-03	5.96E-06	0.00 0.00 -0.70 0.70	0.00 0.00 0.00 0.00

Problem name: `NieWangYe2017Ex57`

Source: [38]

Description: Nie, Wang and Ye 2017 defined Example 5.7 as follows

$$
\begin{aligned}
F(x,y) &:= \tfrac{1}{2}x_1^2y_1 + x_2y_2^2 - (x_1 + x_2^2)y_3 \\
G(x,y) &:= \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \\ -x_1 - 1 \\ -x_2 - 1 \\ -x_1 - x_2 + x_1^2 + y_1^2 + y_2^2 \end{bmatrix} \\
f(x,y) &:= x_2(y_1y_2y_3 + y_2^2 - y_3^3) \\
g(x,y) &:= \begin{bmatrix} -x_1 + y_1^2 + y_2^2 + y_3^2 \\ -1 + 2y_2y_3 \end{bmatrix}
\end{aligned}
$$

Comment: The point $x^* = (1,1)^\top, y^* = (0,0,1)^\top$ with $F(x^*,y^*) = -2$ and $f(x^*,y^*) = -1$ is the best known solution of the problem provided in [38]. Results were listed in following table, where two starting points were $x^0 = (1.00, 1.00)^\top, y^0 = (0.00, 0.50, 1.00)^\top$ and $x^0 = (0.00, 0.00)^\top, y^0 = (1.00, 1.00, 1.00)^\top$, and our method achieved one which was close to above mentioned solution under three cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.00	119	0.38	1.28E-03	1.30E-02	3.52E-03	0.00 0.00	0.00 0.02 0.00
3^1	0.00	0.00	132	0.18	8.17E-01	6.85E-03	8.71E-01	-0.33 1.00	0.00 0.00 0.01
3^0	-2.04	-0.98	11	0.02	8.89E-12	9.82E-02	6.73E-01	1.00 1.00	-0.11 0.02 0.99
3^{-1}	0.00	0.00	138	0.33	5.75E-05	3.22E-03	1.44E-04	0.00 0.00	0.00 0.00 0.00
3^{-2}	-2.06	-0.94	90	0.14	2.05E-09	2.13E-01	1.80E-01	1.00 1.00	-0.21 0.01 0.98
3^2	0.00	0.00	120	0.15	1.41E+00	1.85E-04	1.32E+00	0.00 0.02	0.00 0.00 0.00
3^1	0.00	0.00	126	0.20	8.24E-05	3.76E-03	3.80E-03	0.00 0.01	0.00 0.00 0.00
3^0	0.00	0.00	153	0.19	2.16E-05	6.29E-04	6.27E-04	0.00 0.00	0.00 0.00 0.00
3^{-1}	0.00	0.00	190	0.30	2.93E-04	8.40E-03	5.34E-03	0.00 0.00	0.00 0.00 -0.01
3^{-2}	-2.06	-0.96	579	0.58	9.99E-01	1.05E+00	1.00E+00	1.00 1.00	-0.17 0.00 0.99

Problem name: `NieWangYe2017Ex58`

Source: [38]

Description: Nie, Wang and Ye 2017 defined Example 5.8 as follows

$$\begin{array}{rcl} F(x,y) & := & (x_1+x_2+x_3+x_4)(y_1+y_2+y_3+y_4) \\ G(x,y) & := & \begin{bmatrix} \|x\|^2-1 \\ y_3^2-x_4 \\ y_2y_4-x_1 \end{bmatrix} \\ f(x,y) & := & x_1y_1+x_2y_2+0.1y_3+0.5y_4-y_3y_4 \\ g(x,y) & := & \begin{bmatrix} y_1^2+2y_2^2+3y_3^2+4y_4^2-x_1^2-x_3^2-x_2-x_4 \\ y_2y_3-y_1y_4 \end{bmatrix} \end{array}$$

Comment: There is an error of the constraint $g(x,y) \le 0$. The original one in [38] was $-y_2y_3 + y_1y_4 \le 0$ which clearly can not be satisifed by the best known solution $x^* = (0.514, 0.505, 0.488, 0.493)^T, y^* = (-0.835, -0.410, -0.211, -0.289)^T$ with $F(x^*,y^*) = -3.488$, $f(x^*,y^*) = -0.862$ provided in [38]. Therefore we replace it by $y_2y_3 - y_1y_4 \le 0$. Results were listed in following table, where two starting points were $x^0 = (0.5,0.5,0.5,0.5)^T, y^0 = -(0.5,0.5,0.5,0.5)^T$ and $x^0 = (1,1,1,1)^T, y^0 = -(1,1,1,1)^T$, and our method approached better ones $F(\hat{x},\hat{y}) < F(x^*,y^*)$ under four cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.05	194	0.54	3.25E-07	1.05E-08	5.81E-03	0.00 0.00 -0.71 0.71	-0.50 -0.10 0.50 0.10
3^1	0.00	0.05	72	0.11	8.46E-09	1.14E-10	4.83E-03	0.00 0.00 -0.71 0.71	-0.50 -0.10 0.50 0.10
3^0	0.00	0.04	47	0.08	1.46E-08	3.82E-13	8.29E-14	0.01 -0.01 -0.71 0.70	-0.52 -0.13 0.52 0.13
3^{-1}	-3.53	-0.84	13	0.02	4.78E-08	1.34E-01	9.72E-01	0.50 0.50 0.50 0.50	-0.85 -0.42 -0.27 -0.22
3^{-2}	-3.54	-0.83	12	0.02	2.74E-09	1.49E-01	9.06E-01	0.50 0.50 0.50 0.50	-0.85 -0.42 -0.28 -0.22
3^2	0.00	0.05	153	0.26	5.37E-07	6.53E-13	4.74E-07	0.00 0.00 -0.66 0.66	-0.27 -0.33 0.50 0.10
3^1	0.00	-0.05	147	0.27	3.61E-07	3.87E-01	3.14E-01	-0.02 0.22 -0.35 0.13	0.08 -0.37 0.25 0.04
3^0	-3.53	-0.84	120	0.15	8.82E-09	1.02E-01	1.17E+00	0.50 0.50 0.50 0.50	-0.85 -0.42 -0.26 -0.24
3^{-1}	-3.53	-0.84	948	1.24	5.24E-08	1.34E-01	9.72E-01	0.50 0.50 0.50 0.50	-0.85 -0.42 -0.27 -0.22
3^{-2}	0.00	-0.32	150	0.18	7.13E-11	8.30E-11	3.01E-01	-0.07 -0.42 -0.34 0.84	0.11 0.31 -0.17 -0.25

Problem name: `NieWangYe2017Ex61`

Source: [38]

Description: Nie, Wang and Ye 2017 defined Example 6.1 as follows

$$
\begin{aligned}
F(x,y) &:= y_1^3(x_1^2 - 3x_1x_2) - y_1^2y_2 + y_2x_2^3 \\
G(x,y) &:= \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \\ -x_1 - 1 \\ -x_2 - 1 \\ -y_2 - y_1(1 - x_1^2) \end{bmatrix} \\
f(x,y) &:= y_1y_2^2 - y_2^3 - y_1^2(x_2 - x_1^2) \\
g(x,y) &:= y_1^2 + y_2^2 - 1
\end{aligned}
$$

Comment: The point $x^* = (0.571, -1)^T, y^* = (-0.164, 0.987)^T$ with $F(x^*, y^*) = -1.022$ and $f(x^*, y^*) = -1.084$ is the best known solution of the problem provided in [38]. Results were listed in following table, where two starting points were $x^0 = (1.00, -2.00)^T, y^0 = (1.00, 1.00)^T$ and $x^0 = (1.00, -1.00)^T, y^0 = (0.00, 2.00)^T$, and our method closely achieved the above mentioned solution when $\lambda = 3^0$ under the first starting point and when $\lambda = 3^1$ under the second starting point.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.00	18	0.15	3.36E-07	5.38E-06	0.00E+00	0.05 0.00	0.00 0.00
3^1	-0.11	-1.15	110	0.16	1.48E-05	5.87E-02	2.08E+00	-0.19 0.22	-0.35 0.94
3^0	-1.03	-1.08	477	0.66	3.65E-06	6.48E-02	3.50E-01	0.62 -1.00	-0.19 0.98
3^{-1}	0.00	0.00	36	0.06	2.92E-07	6.63E-01	1.35E-21	-0.98 0.23	0.00 0.00
3^{-2}	-4.16	1.88	48	0.07	2.01E-10	1.97E+00	1.77E+00	1.00 -1.00	-0.98 0.20
3^2	0.00	0.00	145	0.25	2.50E-05	1.00E+00	9.20E-10	-1.00 0.00	0.00 0.00
3^1	-1.02	-1.08	585	0.77	3.69E-07	1.78E-02	2.34E+00	0.59 -1.00	-0.17 0.99
3^0	-0.13	-1.10	15	0.02	9.38E-08	1.49E-01	8.68E-02	0.20 -0.16	-0.36 0.93
3^{-1}	-0.36	-0.70	82	0.08	1.82E-09	5.84E-01	4.11E-01	0.22 0.05	-0.71 0.71
3^{-2}	-0.01	0.01	19	0.02	4.80E-08	1.14E+00	7.76E-02	-0.05 -0.28	0.19 0.21

Problem name: `Outrata1990Ex1a`

Source: [39]

Description: Outrata 1990 tested one example as follows

$$
\begin{aligned}
F(x,y) &:= 0.1(x_1^2+x_2^2)+\frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2-12.5\\
f(x,y) &:= \frac{1}{2}y^T\begin{bmatrix} 1 & -2\\ -2 & 5\end{bmatrix}y-x^Ty\\
g(x,y) &:= \begin{bmatrix} -0.333y_1+y_2-2\\ y_1-0.333y_2-2\\ -y_1\\ -y_2\end{bmatrix}
\end{aligned}
$$

Comment: One solution of this problem reported in [39] is $x^*=(0.97,3.14)^T, y^*=(2.6,1.8)^T$ with $F(x^*,y^*)=-8.92, f(x^*,y^*)=-6.05$. Results were listed in following table, where two starting points were $x^0=(0.00,0.00)^T, y^0=(3.00,3.00)^T$ and $x^0=(3.00,3.00)^T, y^0=(3.00,3.00)^T$. Similar solutions were produced under many cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-8.92	-6.14	69	0.20	8.62E-08	2.30E-02	8.71E+00	1.03 3.10	2.60 1.79
3^1	-8.92	-6.14	46	0.07	4.39E-08	6.90E-02	2.71E+00	1.03 3.10	2.60 1.79
3^0	-8.92	-6.14	11	0.02	5.04E-09	2.07E-01	7.11E-01	1.03 3.10	2.60 1.79
3^{-1}	-9.01	-5.91	11	0.01	8.33E-07	6.01E-01	7.92E-02	1.00 3.01	2.60 1.81
3^{-2}	-11.15	0.69	10	0.02	4.60E-08	6.97E-01	9.25E-01	0.46 1.39	2.83 2.50
3^2	9.24	-13.24	1000	1.23	2.05E+01	1.23E-01	1.02E+01	-5.06 11.51	0.02 2.28
3^1	-8.92	-6.14	54	0.07	3.17E-11	6.90E-02	2.71E+00	1.03 3.10	2.60 1.79
3^0	-8.92	-6.14	11	0.01	5.04E-09	2.07E-01	7.11E-01	1.03 3.10	2.60 1.79
3^{-1}	-9.01	-5.91	9	0.01	1.57E-11	6.01E-01	7.92E-02	1.00 3.01	2.60 1.81
3^{-2}	-11.15	0.69	10	0.01	4.60E-08	6.97E-01	9.25E-01	0.46 1.39	2.83 2.50

Problem name: `Outrata1990Ex1b`

Source: [39]

Description: Outrata 1990 tested one example as follows

$$\begin{aligned} F(x,y) &:= x_1^2 + x_2^2 + \frac{1}{2}(y_1-3)^2 + \frac{1}{2}(y_2-4)^2 - 12.5 \\ f(x,y) &:= \frac{1}{2}y^T \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} y - x^T y \\ g(x,y) &:= \begin{bmatrix} -0.333y_1 + y_2 - 2 \\ y_1 - 0.333y_2 - 2 \\ -y_1 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: One of solutions reported in [39] is $x^* = (0.28, 0.48)^T$, $y^* = (2.34, 1.03)^T$ with $F(x^*, y^*) = -7.56, f(x^*, y^*) = -0.58$. Results were listed in following table, where two starting points were $x^0 = (3.00, 3.00)^T, y^0 = (3.00, 3.00)^T$ and $x^0 = (0.00, 0.00)^T, y^0 = (3.00, 3.00)^T$, and similar results ware obtained when $\lambda \geq 3^0$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-7.58	-0.57	13	0.12	3.65E-07	4.78E-02	2.00E+00	0.28 0.47	2.34 1.03
3^1	-7.58	-0.57	13	0.02	6.00E-07	1.43E-01	2.00E+00	0.28 0.47	2.34 1.03
3^0	-7.58	-0.57	13	0.02	7.66E-09	4.30E-01	2.00E+00	0.28 0.47	2.34 1.03
3^{-1}	-9.25	0.71	8	0.01	9.43E-10	5.48E-01	1.04E+00	0.08 0.25	2.51 1.53
3^{-2}	-11.21	4.54	11	0.02	2.30E-07	6.88E-01	9.60E-01	0.04 0.13	2.81 2.42
3^2	-7.58	-0.57	13	0.01	3.65E-07	4.78E-02	2.00E+00	0.28 0.47	2.34 1.03
3^1	-7.58	-0.57	13	0.01	6.00E-07	1.43E-01	2.00E+00	0.28 0.47	2.34 1.03
3^0	-7.58	-0.57	12	0.01	3.86E-07	4.30E-01	2.00E+00	0.28 0.47	2.34 1.03
3^{-1}	-9.25	0.71	10	0.01	1.13E-09	5.48E-01	1.04E+00	0.08 0.25	2.51 1.53
3^{-2}	-11.21	4.54	11	0.02	3.45E-08	6.88E-01	9.60E-01	0.04 0.13	2.81 2.42

Problem name: `Outrata1990Ex1c`

Source: [39]

Description: Outrata 1990 tested one example as follows

$$\begin{aligned} F(x,y) &:= \frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2-12.5 \\ f(x,y) &:= \frac{1}{2}y^T\begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}y - x^Ty \\ g(x,y) &:= \begin{bmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y_1 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: One of solutions of this problem reported in [39] is $x^*=(20.26,42.81)^T$, $y^*=(3,3)^T$ with $F(x^*,y^*)=-12$, $f(x^*,y^*)=-112.71$. Results were listed in following table, where two starting points were $x^0=(3.00,3.00)^T, y^0=(3.00,3.00)^T$ and $x^0=(0.00,0.00)^T, y^0=(3.00,3.00)^T$. Our method rendered $\hat{y}=y^*$, and thus leaded to similar $F(\hat{x},\hat{y})$. But one can discern that for the lower level problem, it is unbounded from blow since x is unbounded.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-6.00	-20.00	513	0.88	2.84E-07	1.36E-20	2.00E+00	5.54 20.00	0.00 2.00
3^1	0.00	0.00	292	0.40	3.39E-09	1.66E-21	6.64E-01	-1.50 -1.33	0.00 0.00
3^0	-12.00	-76.87	16	0.02	8.87E-10	0.00E+00	1.19E+00	12.21 38.92	3.00 3.00
3^{-1}	-12.00	-80.93	9	0.01	7.10E-07	0.00E+00	3.36E-06	12.00 40.48	3.00 3.00
3^{-2}	-12.00	-79.81	12	0.02	9.11E-13	0.00E+00	2.34E-07	12.00 40.11	3.00 3.00
3^2	0.00	0.00	51	0.05	7.75E-09	1.44E-17	5.04E-01	-0.33 -0.55	0.00 0.00
3^1	0.00	0.00	143	0.15	5.26E-07	7.92E-20	8.08E-04	-1.50 -2.00	0.00 0.00
3^0	-12.00	-76.87	16	0.02	8.87E-10	0.00E+00	1.19E+00	12.21 38.92	3.00 3.00
3^{-1}	-12.00	-80.93	9	0.04	7.10E-07	0.00E+00	3.36E-06	12.00 40.48	3.00 3.00
3^{-2}	-12.00	-79.81	12	0.01	9.11E-13	0.00E+00	3.66E-08	12.00 40.11	3.00 3.00

Problem name: `Outrata1990Ex1d`

Source: [39]

Description: Outrata 1990 tested one example as follows

$$\begin{aligned} F(x,y) &:= 0.1(x_1^2+x_2^2)+\frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2-12.5 \\ f(x,y) &:= \frac{1}{2}y^T\begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix} y - x^T y \\ g(x,y) &:= \begin{bmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y_1 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: One solution of this problem reported in [39] is $x^*=(2,0.06)^T, y^*=(2,0)^T$ with $F(x^*,y^*)=-3.60, f(x^*,y^*)=-2$. Results were listed in following table, where two starting points were $x^0=(0.00,0.00)^T, y^0=(3.00,3.00)^T$ and $x^0=(3.00,3.00)^T, y^0=(3.00,3.00)^T$. Our method rendered a similar one when $\lambda=3^0$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-0.38	-0.18	33	0.15	1.44E-08	4.23E-02	2.71E+00	0.00 1.90	0.00 0.19
3^1	-0.37	-0.19	180	0.32	4.08E-03	1.28E-01	6.17E-01	-0.05 1.92	0.00 0.19
3^0	-3.60	-2.00	12	0.02	3.94E-08	2.00E-01	5.10E-01	2.00 0.00	2.00 0.00
3^{-1}	-3.88	-2.03	12	0.02	3.99E-11	6.88E-01	1.00E+00	2.33 0.16	2.03 0.10
3^{-2}	-6.89	7.94	11	0.02	1.57E-08	1.00E+00	1.00E+00	1.17 0.46	2.11 0.82
3^2	25.67	-8.97	115	0.11	1.38E+00	3.72E-01	3.00E+01	-14.40 8.53	-0.31 0.95
3^1	-0.37	-0.19	151	0.21	3.97E-03	1.28E-01	5.99E-01	-0.06 1.92	0.00 0.19
3^0	-3.60	-2.00	10	0.01	3.84E-08	2.00E-01	5.10E-01	2.00 0.00	2.00 0.00
3^{-1}	-3.88	-2.03	12	0.01	3.99E-11	6.88E-01	1.00E+00	2.33 0.16	2.03 0.10
3^{-2}	-6.89	7.94	11	0.01	1.57E-08	1.00E+00	1.00E+00	1.17 0.46	2.11 0.82

Problem name: `Outrata1990Ex1e`

Source: [39]

Description: Outrata 1990 tested one example as follows

$$\begin{aligned}
F(x,y) &:= 0.1(x_1^2+x_2^2)+\frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2-12.5\\
f(x,y) &:= \frac{1}{2}y^T\begin{bmatrix}1 & 3\\ 3 & 10\end{bmatrix}y-y^T\begin{bmatrix}-1 & 2\\ 3 & -3\end{bmatrix}x\\
g(x,y) &:= \begin{bmatrix}-0.333y_1+y_2-2\\ y_1-0.333y_2-2\\ -y_1\\ -y_2\end{bmatrix}
\end{aligned}$$

Comment: One solution of this problem reported in [39] is $x^*=(2.42,-3.65)^T, y^*=(0,1.58)^T$ with $F(x^*,y^*)=-3.15, f(x^*,y^*)=-16.29$. This problem is the same as TP5 tested by Sinha, Malo and Deb in [47], with best know upper and lower level objective function values $\overline{F}=-3.6, \overline{f}=-2.0$. Results were listed in following table, where two starting points were $x^0=(3.00,3.00)^T, y^0=(3.00,3.00)^T$ and $x^0=(0.00,0.00)^T, y^0=(3.00,3.00)^T$. Our method rendered a solution $\hat{x}=(3.16,-3.16)^T, \hat{y}=(0.00,1.89)^T$ similar to x^*, y^* but with better upper and lower level objective function values.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-3.92	-2.00	70	0.21	6.24E-07	4.44E-03	7.58E+00	-0.40 0.80	2.00 0.00
3^1	-3.79	-17.95	78	0.09	1.09E-08	3.70E-02	1.84E+00	3.16 -3.16	0.00 1.89
3^0	-3.92	-2.00	14	0.02	8.22E-08	4.00E-02	4.34E-01	-0.40 0.80	2.00 0.00
3^{-1}	-4.21	-1.74	14	0.02	5.66E-07	2.01E-01	1.00E+00	-0.30 0.97	2.02 0.07
3^{-2}	-6.36	5.28	13	0.03	5.32E-10	1.00E+00	1.00E+00	-0.16 1.39	2.21 0.64
3^2	-0.47	-0.13	66	0.09	5.37E-08	5.78E-02	1.94E-01	1.42 0.87	0.05 0.15
3^1	-3.79	-17.95	78	0.09	1.09E-08	3.70E-02	1.84E+00	3.16 -3.16	0.00 1.89
3^0	-3.92	-2.00	14	0.02	8.22E-08	4.00E-02	4.34E-01	-0.40 0.80	2.00 0.00
3^{-1}	-4.21	-1.74	14	0.01	5.66E-07	2.01E-01	1.00E+00	-0.30 0.97	2.02 0.07
3^{-2}	-6.36	5.28	13	0.01	5.32E-10	1.00E+00	1.00E+00	-0.16 1.39	2.21 0.64

Problem name: `Outrata1990Ex2a`

Source: [39]

Description: Outrata 1990 tested one example as follows

$$\begin{aligned} F(x,y) &:= \tfrac{1}{2}\left[(y_1-3)^2+(y_2-4)^2\right] \\ G(x,y) &:= -x \\ f(x,y) &:= \tfrac{1}{2}y^\top y-(3+1.333x)y_1-xy_2 \\ g(x,y) &:= \begin{bmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y_1 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: One solution of this problem reported in [39] is $x^* = 2.07, y^* = (3,3)^\top$. Results were listed in following table, where two starting points were $x^0 = 3.00, y^0 = (3.00, 4.00)^\top$ and $x^0 = 3.00, y^0 = (3.00, 3.00)^\top$. Our method rendered similar ones for 8 cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	7.77	-3.12	1000	1.70	6.12E-01	1.06E-01	6.55E+00	-0.32	2.06 0.17
3^1	0.50	-14.53	156	0.23	1.39E-09	0.00E+00	2.17E+00	2.08	3.00 3.00
3^0	0.50	-14.59	11	0.01	9.41E-07	3.22E-09	4.27E-01	2.08	3.00 3.00
3^{-1}	0.50	-14.54	13	0.02	6.31E-08	1.08E-08	6.68E-01	2.08	3.00 3.00
3^{-2}	0.50	-14.57	17	0.03	7.64E-10	1.35E-10	8.54E-01	2.08	3.00 3.00
3^2	0.75	-12.67	1000	1.25	3.42E-01	1.56E-16	8.20E+00	1.80	2.93 2.78
3^1	0.50	-14.53	14	0.01	8.40E-08	4.70E-09	2.17E+00	2.08	3.00 3.00
3^0	0.50	-14.53	9	0.01	9.22E-10	2.48E-13	4.29E-01	2.08	3.00 3.00
3^{-1}	0.50	-14.53	10	0.01	7.65E-10	8.47E-14	6.69E-01	2.08	3.00 3.00
3^{-2}	0.50	-14.56	8	0.01	8.80E-10	5.07E-13	8.54E-01	2.08	3.00 3.00

Problem name: `Outrata1990Ex2b`

Source: [39]

Description: Outrata 1990 tested one example as follows

$$\begin{aligned}
F(x,y) &:= \tfrac{1}{2}(y_1-3)^2+\tfrac{1}{2}(y_2-4)^2\\
G(x,y) &:= -x\\
f(x,y) &:= \tfrac{1}{2}(1+x)y_1^2-(3+1.333x)y_1-xy_2\\
g(x,y) &:= \begin{bmatrix} -0.333y_1+y_2-2\\ y_1-0.333y_2-2\\ -y_1\\ -y_2 \end{bmatrix}
\end{aligned}$$

Comment: One solution of this problem reported in [39] is $x^*=0, y^*=(3,3)^\top$. Results were listed in following table, where two starting points were $x^0=3.00, y^0=(3.00,4.00)^\top$ and $x^0=3.00, y^0=(3.00,3.00)^\top$. For different starting points and different λ, solutions were the same as the above one.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.50	-4.50	44	0.19	7.46E-07	1.86E-07	1.20E+00	0.00	3.00 3.00
3^1	0.50	-4.50	94	0.19	9.62E-07	4.43E-08	1.19E+00	0.00	3.00 3.00
3^0	0.50	-4.50	26	0.04	3.84E-07	6.15E-08	1.19E+00	0.00	3.00 3.00
3^{-1}	0.50	-4.50	116	0.17	2.34E-04	2.02E-01	1.09E+00	0.00	3.00 3.00
3^{-2}	0.50	-4.50	190	0.33	8.48E-05	5.25E-01	1.10E+00	0.00	3.00 3.00
3^2	0.50	-4.50	45	0.05	7.72E-07	2.14E-07	1.20E+00	0.00	3.00 3.00
3^1	0.50	-4.50	87	0.11	9.46E-07	4.36E-08	1.19E+00	0.00	3.00 3.00
3^0	0.50	-4.50	22	0.03	2.45E-07	3.92E-08	1.19E+00	0.00	3.00 3.00
3^{-1}	0.50	-4.50	117	0.16	2.34E-04	3.39E-01	1.04E+00	0.00	3.00 3.00
3^{-2}	0.50	-4.50	43	0.07	6.03E-07	1.46E-07	1.19E+00	0.00	3.00 3.00

Problem name: `Outrata1990Ex2c`

Source: [39]

Description: Outrata 1990 tested one example as follows

$$\begin{aligned} F(x,y) &:= \tfrac{1}{2}(y_1-3)^2+\tfrac{1}{2}(y_2-4)^2 \\ G(x,y) &:= -x \\ f(x,y) &:= \tfrac{1}{2}(1+x)y_1^2+\tfrac{1}{2}(1+0.1x)y_2^2-(3+1.333x)y_1-xy_2 \\ g(x,y) &:= \begin{bmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y_1 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: One solution of this problem reported in [39] is $x^* = 3.456$, $y^* = (1.707, 2.569)^T$ with $F(x^*, y^*) = 1.860$, $f(x^*, y^*) = -10.931$. Results were listed in following table, where two starting points were $x^0 = 3.00, y^0 = (3.00, 4.00)^T$ and $x^0 = 3.00, y^0 = (3.00, 3.00)^T$. Our method reached x^*, y^* under four cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	4.34	-5.67	1000	1.71	1.93E+00	2.44E-01	1.42E+03	1.34	1.95 1.25
3^1	1.86	-10.93	112	0.17	7.17E-07	4.57E-02	1.18E+00	3.46	1.71 2.57
3^0	1.86	-10.93	12	0.02	3.20E-08	1.37E-01	1.18E+00	3.46	1.71 2.57
3^{-1}	1.86	-10.93	696	1.02	5.48E-10	4.12E-01	1.18E+00	3.46	1.71 2.57
3^{-2}	0.51	-0.07	16	0.03	5.35E-08	7.72E-01	1.01E+00	0.02	2.97 2.99
3^2	4.65	-5.62	1000	1.36	2.75E+01	4.20E-01	2.10E+00	1.47	2.46 1.00
3^1	6.15	-4.19	1000	1.37	1.19E+00	2.59E-01	8.66E-01	-0.21	2.64 0.51
3^0	2.98	-3.68	151	0.13	2.35E+00	5.12E-01	1.00E+00	0.55	1.99 1.78
3^{-1}	1.86	-10.93	546	0.62	2.17E-08	4.12E-01	1.18E+00	3.46	1.71 2.57
3^{-2}	0.51	-0.07	15	0.01	2.91E-07	7.72E-01	1.01E+00	0.02	2.97 2.99

Problem name: `Outrata1990Ex2d`

Source: [39]

Description: Outrata 1990 tested one example as follows

$$\begin{aligned} F(x,y) &:= \tfrac{1}{2}(y_1-3)^2+\tfrac{1}{2}(y_2-4)^2 \\ G(x,y) &:= -x \\ f(x,y) &:= \tfrac{1}{2}y^Ty-(3+1.333x)y_1-xy_2 \\ g(x,y) &:= \begin{bmatrix} (-0.333+0.1x)y_1+y_2-x \\ y_1+(-0.333-0.1x)y_2-2 \\ -y_1 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: One solution of this problem reported in [39] is $x^*=2.498$, $y^*=(3.632,2.8)^T$ with $F(x^*,y^*)=0.92$, $f(x^*,y^*)=-19.47$. Results were listed in following table, where two starting points were $x^0=3.00, y^0=(3.00,4.00)^T$ and $x^0=3.00, y^0=(3.00,3.00)^T$. Despite our method could not render the same solution under those cases, but it offered solutions 9 of which are such that $F(\hat{x},\hat{y})<F(x^*,y^*)$ and $f(\hat{x},\hat{y})<f(x^*,y^*)$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.85	-22.95	21	0.16	2.22E-07	9.69E-10	7.85E+00	2.86	3.88 3.04
3^1	0.85	-22.95	10	0.02	2.66E-12	8.57E-14	1.86E+00	2.86	3.88 3.04
3^0	0.85	-22.95	10	0.01	1.40E-10	1.44E-11	2.79E-01	2.86	3.88 3.04
3^{-1}	0.85	-22.95	12	0.01	1.47E-10	3.06E-11	8.56E-01	2.86	3.88 3.04
3^{-2}	0.35	-25.75	9	0.02	4.72E-08	1.33E-01	9.31E-01	3.36	3.52 3.35
3^2	7.21	-3.39	195	0.34	4.16E-01	6.66E-02	9.87E-01	-0.21	2.10 0.31
3^1	0.85	-22.95	13	0.02	2.56E-08	4.55E-10	1.86E+00	2.86	3.88 3.04
3^0	0.85	-22.95	11	0.02	1.50E-08	7.16E-10	2.79E-01	2.86	3.88 3.04
3^{-1}	0.85	-22.95	11	0.02	5.71E-11	1.19E-11	8.56E-01	2.86	3.88 3.04
3^{-2}	0.35	-25.75	12	0.02	1.45E-07	1.33E-01	9.31E-01	3.36	3.52 3.35

Problem name: `Outrata1990Ex2e`

Source: [39]

Description: Outrata 1990 tested one example as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2 \\ G(x,y) & := & -x \\ f(x,y) & := & \frac{1}{2}(1+x)y_1^2+\frac{1}{2}y_2^2-(3+1.333x)y_1-xy_2 \\ g(x,y) & := & \left[\begin{array}{r} (-0.333+0.1x)y_1+y_2-x \\ y_1+(-0.333-0.1x)y_2-2 \\ -y_1 \\ -y_2 \end{array}\right] \end{array}$$

Comment: One solution of this problem reported in [39] is $x^*=3.999$, $y^*=(1.665,3.887)^\top$ with $F(x^*,y^*)=0.90$, $f(x^*,y^*)=-14.94$. Our method reached x^*,y^* when $\lambda=3^0$ and $\lambda=3^{-1}$. Two starting points were $x^0=3.00$, $y^0=(3.00,4.00)^\top$ and $x^0=3.00$, $y^0=(3.00,3.00)^\top$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	6.96	-3.84	206	0.44	4.16E-01	9.20E-02	9.77E-01	-0.209	2.117 0.374
3^1	6.26	-3.77	117	0.15	4.16E-01	1.70E-01	9.77E-01	-0.209	2.175 0.561
3^0	0.90	-14.93	11	0.02	2.02E-09	6.29E-02	1.31E-01	3.999	1.665 3.887
3^{-1}	0.90	-14.93	11	0.02	1.70E-07	1.89E-01	5.63E-02	3.999	1.665 3.887
3^{-2}	0.66	-14.76	16	0.03	5.82E-10	4.34E-01	9.51E-01	3.982	1.857 3.861
3^2	6.96	-3.84	139	0.16	4.16E-01	9.20E-02	9.86E-01	-0.210	2.117 0.374
3^1	6.26	-3.77	170	0.18	4.16E-01	1.70E-01	9.92E-01	-0.210	2.175 0.560
3^0	0.90	-14.93	13	0.02	1.18E-10	6.29E-02	1.31E-01	3.999	1.665 3.887
3^{-1}	0.90	-14.93	14	0.02	3.86E-08	1.89E-01	5.63E-02	3.999	1.665 3.887
3^{-2}	0.66	-14.76	16	0.02	2.54E-11	4.34E-01	9.51E-01	3.982	1.857 3.861

Problem name: `Outrata1993Ex31`

source: [40]

description: Outrata 1993 defined Example 3.1 as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2 \\ G(x,y) & := & -x \\ f(x,y) & := & \frac{1}{2}(1+0.2x)y_1^2+\frac{1}{2}(1+0.1x)y_2^2-(3+1.33x)y_1-xy_2 \\ g(x,y) & := & \begin{bmatrix} (-0.333+0.1x)y_1+y_2+0.1x-2 \\ y_1+(-0.333-0.1x)y_2+0.1x-2 \\ -y_1 \\ -y_2 \end{bmatrix} \end{array}$$

Comment: Outrata in obtained [40] solution as $x^*=1.909, y^*=(2.978,2.232)^T$ with $F(x^*,y^*)=1.56$, $f(x^*,y^*)=-11.68$. Results were listed in following table, where two starting points were $x^0=3.00, y^0=(3.00,4.00)^T$ and $x^0=3.00, y^0=(3.00,3.00)^T$, and our method closely achieved the same solution wunder six cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.56	-11.68	402	0.64	5.93E-07	1.32E-02	7.99E+00	1.910	2.979 2.232
3^1	11.96	-3.67	99	0.18	1.03E-10	7.97E-02	2.13E+00	17.976	0.093 0.066
3^0	1.56	-11.68	13	0.02	4.13E-10	1.19E-01	8.82E-01	1.910	2.979 2.232
3^{-1}	1.56	-11.68	10	0.03	1.10E-11	3.56E-01	1.14E+00	1.910	2.979 2.232
3^{-2}	0.88	-5.49	13	0.03	4.57E-09	7.16E-01	1.08E+00	0.838	3.030 2.671
3^2	1.56	-11.68	11	0.02	1.14E-08	1.32E-02	7.99E+00	1.910	2.979 2.232
3^1	11.96	-3.67	459	0.72	1.81E-07	7.97E-02	2.13E+00	17.976	0.093 0.066
3^0	1.56	-11.68	14	0.04	4.69E-08	1.19E-01	8.82E-01	1.910	2.979 2.232
3^{-1}	1.56	-11.68	14	0.02	1.91E-11	3.56E-01	1.14E+00	1.910	2.979 2.232
3^{-2}	0.88	-5.49	11	0.01	5.40E-12	7.16E-01	1.08E+00	0.838	3.030 2.671

Problem name: `Outrata1993Ex32`

source: [40]

description: Outrata 1993 defined Example 3.2 as follows

$$\begin{aligned}
F(x,y) &:= \tfrac{1}{2}(y_1-3)^2+\tfrac{1}{2}(y_2-4)^2\\
G(x,y) &:= -x\\
f(x,y) &:= \tfrac{1}{2}(1+0.2x)y_1^2+\tfrac{1}{2}(1+0.1x)y_2^2-(3+1.33x)y_1-xy_2\\
g(x,y) &:= \begin{bmatrix} -0.333y_1+y_2+0.1x-1\\ y_1^2+y_2^2-0.1x-9\\ -y_1\\ -y_2 \end{bmatrix}
\end{aligned}$$

Comment: Outrata in obtained [40] solution as $x^*=4.061, y^*=(2.682,1.487)^T$ with $F(x^*,y^*)=3.208$, $f(x^*,y^*)=-20.531$. Results were listed in following table, where two sstarting points were $x^0=2.00, y^0=(2.00,2.00)^T$ and $x^0=3.00, y^0=(3.00,4.00)^T$, and our method achieved the similar solutions when $\lambda=3^2$ under the first starting point and when $\lambda=3^1$ under the second starting point.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	3.20	-20.26	204	0.62	5.28E-10	1.01E-02	5.56E+00	3.998	2.678 1.492
3^1	8.06	-74.75	53	0.12	7.48E-08	6.60E-10	1.99E+00	21.099	3.333 0.000
3^0	3.12	-18.37	17	0.03	2.07E-07	8.76E-02	1.25E+00	3.561	2.651 1.527
3^{-1}	3.01	-15.45	14	0.02	1.10E-07	2.42E-01	1.28E+00	2.889	2.607 1.579
3^{-2}	2.82	-9.97	15	0.03	1.54E-07	5.78E-01	1.32E+00	1.640	2.521 1.676
3^2	8.06	-74.75	282	0.46	2.25E-12	4.50E-15	8.00E+00	21.099	3.333 0.000
3^1	3.18	-19.74	39	0.05	9.01E-12	2.99E-02	1.91E+00	3.878	2.671 1.502
3^0	3.12	-18.37	16	0.02	3.34E-07	8.76E-02	1.25E+00	3.561	2.651 1.527
3^{-1}	3.01	-15.45	19	0.02	5.36E-08	2.42E-01	1.28E+00	2.889	2.607 1.579
3^{-2}	2.82	-9.97	18	0.02	2.46E-07	5.78E-01	1.32E+00	1.640	2.521 1.676

Problem name: `Outrata1994Ex31`

source: [41]

description: Outrata 1994 defined Example 3.1 as follows

$$
\begin{aligned}
F(x,y) &:= \tfrac{1}{2}(y_1-3)^2+\tfrac{1}{2}(y_2-4)^2\\
G(x,y) &:= \begin{bmatrix} -x\\ -10+x \end{bmatrix}\\
f(x,y) &:= \tfrac{1}{2}(1+0.2x)y_1^2+\tfrac{1}{2}(1+0.1x)y_2^2-(3+1.333x)y_1-xy_2\\
g(x,y) &:= \begin{bmatrix} -0.333y_1+y_2+0.1x-1\\ y_1^2+y_2^2-0.1x-9\\ -y_1\\ -y_2 \end{bmatrix}
\end{aligned}
$$

Comment: Outrata in obtained [40] solution as $x^*=4.060, y^*=(2.682,1.487)^\top$ with $F(x^*,y^*)=3.208$, $f(x^*,y^*)=-20.531$. Results were listed in following table, where two starting points were $x^0=3.00, y^0=(3.00,4.00)^\top$ and $x^0=2.00, y^0=(2.00,2.00)^\top$, and our method achieved the similar solutions when $\lambda=3^2$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	3.20	-20.26	189	0.46	9.29E-09	1.01E-02	5.56E+00	3.998	2.678 1.492
3^1	3.18	-19.74	62	0.10	2.83E-09	2.99E-02	1.91E+00	3.878	2.671 1.502
3^0	3.12	-18.37	21	0.03	2.59E-07	8.76E-02	1.25E+00	3.561	2.651 1.527
3^{-1}	3.01	-15.45	20	0.03	7.24E-12	2.42E-01	1.28E+00	2.889	2.607 1.579
3^{-2}	2.82	-9.97	17	0.07	6.23E-09	5.78E-01	1.32E+00	1.640	2.521 1.676
3^2	3.20	-20.26	31	0.05	1.03E-08	1.01E-02	5.56E+00	3.998	2.678 1.492
3^1	3.18	-19.74	19	0.02	1.43E-09	2.99E-02	1.91E+00	3.878	2.671 1.502
3^0	3.12	-18.37	15	0.02	2.89E-09	8.76E-02	1.25E+00	3.561	2.651 1.527
3^{-1}	3.01	-15.45	10	0.01	1.37E-08	2.42E-01	1.28E+00	2.889	2.607 1.579
3^{-2}	2.82	-9.97	14	0.02	4.88E-11	5.78E-01	1.32E+00	1.640	2.521 1.676

Problem name: `OutrataCervinka2009`

Source: [42]

Description: Outrata and Cervinka 2009 tested one example as follows

$$\begin{array}{rcl} F(x,y) & := & -2x_1 - 0.5x_2 - y_2 \\ G(x,y) & := & x_1 \\ f(x,y) & := & y_1 - y_2 + x^\top y + \frac{1}{2} y^\top y \\ g(x,y) & := & \begin{bmatrix} y_2 \\ y_2 - y_1 \\ y_2 + y_1 \end{bmatrix} \end{array}$$

Comment: According to [42], $x^* = y^* = (0,0)^\top$ is a solution of the problem. Results were listed in following table, where two starting points were $x^0 = (-1.00, 1.00)^\top, y^0 = (0.00, -1.00)^\top$ and $x^0 = (-1.00, 0.00)^\top, y^0 = (0.00, -1.00)^\top$, and our method closely achieved the same solution under six cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.00	23	0.15	5.94E-07	7.86E-02	8.47E+00	0.00 0.01	-0.00 0.00
3^1	102.79	-14.24	145	0.21	5.47E-01	6.00E-01	1.84E+00	-40.74 -42.54	0.40 -0.04
3^0	0.00	0.00	15	0.03	6.35E-07	7.07E-01	5.00E-01	0.00 0.00	-0.00 0.00
3^{-1}	0.00	-1.00	80	0.13	9.98E-07	6.00E-01	5.00E-01	0.00 2.00	-1.00 -1.00
3^{-2}	0.00	0.00	16	0.04	6.63E-07	1.00E+00	3.89E-01	0.00 0.00	0.00 0.00
3^2	0.00	0.00	23	0.03	5.94E-07	7.86E-02	8.47E+00	0.000 0.01	0.00 0.00
3^1	54.62	-4.22	115	0.14	1.39E-01	1.92E-01	1.92E+00	0.000 -109.32	0.00 0.04
3^0	0.00	0.00	15	0.02	7.04E-07	7.07E-01	5.00E-01	0.00 0.00	0.00 0.00
3^{-1}	0.00	-1.00	80	0.16	9.98E-07	6.00E-01	5.00E-01	0.00 2.00	1.00 -1.00
3^{-2}	0.00	0.00	16	0.02	6.63E-07	1.00E+00	3.89E-01	0.00 0.00	0.00 0.00

Problem name: `PaulaviciusEtal2017a`

Source: [43]

Description: Paulavicius et. al 2017 tested one problem as follows

$$\begin{aligned} F(x,y) &:= x^2+y^2 \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \\ -y-1 \\ y-1 \end{bmatrix} \\ f(x,y) &:= xy^2-\tfrac{1}{2}y^4 \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: This problem a slight modification of `MitsosBarton06Ex318`, just with the upper-level objective function there replaced by x^2+y^2. Doing so, the point $x^*=0.5, y^*=0$ remains optimal for the new problem [43]. Results were listed in following table, where two starting points were $x^0=0.00, y^0=0.00$ and $x^0=0.00, y^0=1.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	0.00	1	0.06	6.66E-16	2.30E-17	1.57E-16	0.000	0.000
3^1	0.00	0.00	1	0.02	7.85E-16	8.80E-17	2.48E-16	0.000	0.000
3^0	0.00	0.00	1	0.01	6.66E-16	2.30E-17	1.57E-16	0.000	0.000
3^{-1}	0.00	0.00	1	0.01	6.66E-16	2.30E-17	1.57E-16	0.000	0.000
3^{-2}	0.00	0.00	1	0.03	6.66E-16	2.30E-17	1.57E-16	0.000	0.000
3^2	1.00	-0.50	10	0.02	2.61E-07	6.97E-13	3.00E+00	0.000	-1.000
3^1	1.00	-0.50	8	0.01	6.75E-07	4.19E-09	1.97E-07	0.000	1.000
3^0	0.00	0.00	107	0.23	1.27E-06	9.98E-01	1.27E-06	-0.001	-0.001
3^{-1}	0.00	0.00	17	0.02	1.50E-07	9.92E-01	5.00E-08	0.000	0.000
3^{-2}	0.00	0.00	7	0.01	1.48E-10	9.99E-01	8.45E-11	0.000	0.000

Problem name: `PaulaviciusEtal2017b`

Source: [43]

Description: Paulavicius et. al 2017 tested one problem as follows

$$\begin{aligned} F(x,y) &:= x+y \\ G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \\ -y-1 \\ y-1 \end{bmatrix} \\ f(x,y) &:= 0.5xy^2 - x^3y \\ g(x,y) &:= \begin{bmatrix} -y-1 \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: This problem a slight modification of `MitsosBarton06Ex313`, just with the *minus* in upper-level objective function replaced by a *plus*. Doing so, the optimal solution the new problem above is $x^* = -1, y^* = -1$ according to [43]. Results were listed in following table, where two starting points were $x^0 = -1.00, y^0 = 0.00$ and $x^0 = -1.00, y^0 = 1.00$. Our method rendered the same optimal solution under many cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-0.20	0.00	345	0.57	9.60E-09	9.32E-01	1.11E-02	-0.13	-0.07
3^1	-2.00	-1.50	17	0.03	1.07E-12	1.03E-12	7.50E-01	-1.00	-1.00
3^0	-2.00	-1.50	10	0.01	3.34E-10	2.17E-10	2.50E-01	-1.00	-1.00
3^{-1}	-2.00	-1.50	9	0.01	9.20E-09	3.67E-11	5.83E-01	-1.00	-1.00
3^{-2}	-2.00	-1.50	12	0.03	4.93E-11	2.54E-12	6.94E-01	-1.00	-1.00
3^2	-0.25	0.02	84	0.15	2.57E-07	2.22E-01	1.67E-22	-0.50	0.25
3^1	-0.31	0.00	51	0.09	2.47E-10	1.04E+00	2.76E-02	-0.35	0.04
3^0	-2.00	-1.50	28	0.03	7.05E-11	7.50E-12	4.96E-01	-1.00	-1.00
3^{-1}	-2.00	-1.50	96	0.12	3.56E-11	3.77E-11	5.83E-01	-1.00	-1.00
3^{-2}	-2.00	-1.50	19	0.02	3.83E-10	3.85E-10	6.94E-01	-1.00	-1.00

Problem name: `SahinCiric1998Ex2`

Source: [44]

description: Sahin and Ciric 1998 defined Example 2 as follows

$$
\begin{aligned}
F(x,y) &:= (x-3)^2 + (y-2)^2\\
G(x,y) &:= \begin{bmatrix} -x \\ x-8 \end{bmatrix}\\
f(x,y) &:= (y-5)^2\\
g(x,y) &:= \begin{bmatrix} -2x+y-1 \\ x-2y+2 \\ x+2y-14 \end{bmatrix}
\end{aligned}
$$

Comment: Results were listed in following table, where two starting points were $x^0 = 8.00, y^0 = 2.00$ and $x^0 = 3.00, y^0 = 0.00$, and solution $\hat{x} = 1, \hat{y} = 3$ achieved under two cases was same as that obtained by Sahin and Ciric in [44].

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	13.00	1.00	29	0.14	1.93E-10	0.00E+00	4.65E+00	6.00	4.00
3^1	5.00	4.00	11	0.02	6.92E-08	2.62E-14	1.50E+00	1.00	3.00
3^0	4.58	0.83	16	0.02	3.77E-09	2.24E-01	9.09E-01	3.45	4.09
3^{-1}	1.70	2.99	10	0.01	1.91E-13	3.89E-01	1.00E+00	3.29	3.27
3^{-2}	0.36	5.83	11	0.03	4.55E-12	5.24E-01	1.00E+00	3.13	2.58
3^2	5.00	4.00	21	0.03	3.02E-13	9.99E-14	7.50E+00	1.00	3.00
3^1	13.00	1.00	14	0.02	1.36E-08	2.22E-16	5.25E-01	6.00	4.00
3^0	4.58	0.83	11	0.01	4.95E-07	2.24E-01	9.09E-01	3.45	4.09
3^{-1}	1.70	2.99	14	0.02	2.29E-12	3.89E-01	1.00E+00	3.29	3.27
3^{-2}	0.36	5.83	11	0.01	4.79E-07	5.24E-01	1.00E+00	3.13	2.58

Problem name: `ShimizuAiyoshi1981Ex1`

source: [45]

description: Shimizu and Aiyoshi 1981 defined Example 1 as follows

$$\begin{aligned}
F(x,y) &:= x^2+(y-10)^2\\
G(x,y) &:= \begin{bmatrix} -x+y\\ x-15\\ -x \end{bmatrix}\\
f(x,y) &:= (x+2y-30)^2\\
g(x,y) &:= \begin{bmatrix} x+y-20\\ y-20\\ -y \end{bmatrix}
\end{aligned}$$

Comment: The unique optimal solution reported in [45] was $x^*=10, y^*=10$. In following table, where two starting points were $x^0=0.00, y^0=0.00$ and $x^0=1.00, y^0=1.00$, we closely achieved the unique one when $\lambda=3^2$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	99.09	0.02	11	0.11	1.25E-09	9.13E-03	5.50E-01	9.95	9.95
3^1	97.33	0.16	10	0.02	3.75E-10	2.67E-02	1.00E+00	9.86	9.86
3^0	92.60	1.33	10	0.01	4.71E-11	7.41E-02	1.00E+00	9.62	9.62
3^{-1}	82.00	9.00	8	0.01	6.34E-08	1.82E-01	1.00E+00	9.00	9.00
3^{-2}	66.33	41.33	10	0.02	1.52E-10	3.53E-01	1.00E+00	7.86	7.86
3^2	99.09	0.02	11	0.01	1.25E-09	9.13E-03	5.50E-01	9.95	9.95
3^1	97.33	0.16	10	0.01	3.75E-10	2.67E-02	1.00E+00	9.86	9.86
3^0	92.60	1.33	10	0.01	4.71E-11	7.41E-02	1.00E+00	9.62	9.62
3^{-1}	82.00	9.00	8	0.01	6.34E-08	1.82E-01	1.00E+00	9.00	9.00
3^{-2}	66.33	41.33	10	0.01	1.52E-10	3.53E-01	1.00E+00	7.86	7.86

Problem name: `ShimizuAiyoshi1981Ex2`

source: [45]

description: Shimizu and Aiyoshi 1981 defined Example 2 as follows

$$\begin{aligned} F(x,y) &:= (x_1-30)^2+(x_2-20)^2-20y_1+20y_2 \\ G(x,y) &:= \begin{bmatrix} -x_1-2x_2+30 \\ x_1+x_2-25 \\ x_2-15 \end{bmatrix} \\ f(x,y) &:= (x_1-y_1)^2+(x_2-y_2)^2 \\ g(x,y) &:= \begin{bmatrix} y_1-10 \\ y_2-10 \\ -y_1 \\ -y_2 \end{bmatrix} \end{aligned}$$

Comment: Shimi and Aiyo in [45] obtained solution $x^*=(20,5)^\top, y^*=(10,5)^\top$ with $F(x^*,y^*)=225$ and $f(x^*,y^*)=100$. In following table, where two starting points were $x^0=(10.00,10.00)^\top$, $y^0=(0.00,0.00)^\top$ and $x^0=(0.00,15.00)^\top, y^0=(0.00,0.00)^\top$, we achieved the same one when $\lambda \geq 3^1$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	225.00	100.00	97	0.36	6.46E-12	9.48E-02	9.00E+00	20.00 5.00	10.00 5.00
3^1	225.00	100.00	13	0.03	6.80E-14	2.56E-01	3.00E+00	20.00 5.00	10.00 5.00
3^0	175.00	106.25	11	0.02	3.72E-10	5.30E-01	1.00E+00	20.00 5.00	10.00 2.50
3^{-1}	118.06	118.06	10	0.02	4.84E-08	7.07E-01	7.78E-01	19.17 5.83	10.00 0.00
3^{-2}	113.12	113.12	13	0.03	6.68E-12	7.07E-01	1.04E+00	18.06 6.94	10.00 0.00
3^2	225.00	100.00	787	1.34	3.48E-10	9.48E-02	9.00E+00	20.00 5.00	10.00 5.00
3^1	225.00	100.00	14	0.01	5.09E-08	2.56E-01	3.00E+00	20.00 5.00	10.00 5.00
3^0	175.00	106.25	11	0.01	4.47E-11	5.30E-01	1.00E+00	20.00 5.00	10.00 2.50
3^{-1}	118.06	118.06	12	0.01	7.97E-09	7.07E-01	7.78E-01	19.17 5.83	10.00 0.00
3^{-2}	113.12	113.12	13	0.01	8.50E-11	7.07E-01	1.04E+00	18.06 6.94	10.00 0.00

Problem name: `ShimizuEtal1997a`

Source: [46]

description: Shimizu et al. 1997 defined one example as follows

$$
\begin{aligned}
F(x,y) &:= (x-5)^2 + (2y+1)^2 \\
f(x,y) &:= (y-1)^2 - 1.5xy \\
g(x,y) &:= \begin{bmatrix} -3x+y+3 \\ x-0.5y-4 \\ x+y-7 \end{bmatrix}
\end{aligned}
$$

Comment: Results were listed in following table, where two starting points were $x^0 = 4.00, y^0 = 3.00$ and $x^0 = 5.00, y^0 = 1.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	48.42	-10.16	9	0.11	2.64E-09	2.53E-01	1.00E+00	3.17	2.86
3^1	30.59	-6.93	8	0.01	7.40E-07	5.41E-01	1.00E+00	2.64	2.00
3^0	14.04	-3.35	9	0.01	1.60E-07	8.01E-01	1.00E+00	2.50	0.90
3^{-1}	25.00	-14.00	15	0.02	2.95E-10	8.88E-16	6.48E-01	5.00	2.00
3^{-2}	25.00	-14.00	7	0.02	6.32E-14	2.80E-14	8.70E-01	5.00	2.00
3^2	48.42	-10.16	14	0.02	8.72E-11	2.53E-01	1.00E+00	3.17	2.86
3^1	25.00	-14.00	14	0.01	3.76E-12	1.68E-12	2.46E+00	5.00	2.00
3^0	16.89	1.58	10	0.01	4.10E-07	5.27E-16	7.22E-01	0.95	-0.16
3^{-1}	2.92	-0.83	13	0.01	2.04E-09	9.20E-01	1.07E+00	4.12	0.23
3^{-2}	1.66	2.13	15	0.02	2.34E-13	1.05E+00	1.05E+00	3.93	-0.14

Problem name: `ShimizuEtal1997b`

Source: [46]

description: Shimizu et al. 1997 defined one example as follows

$$\begin{aligned} F(x,y) &:= 16x^2 + 9y^2 \\ G(x,y) &:= \begin{bmatrix} y-4x \\ -x \end{bmatrix} \\ f(x,y) &:= (x+y-20)^4 \\ g(x,y) &:= \begin{bmatrix} 4x+y-50 \\ -y \end{bmatrix} \end{aligned}$$

Comment: According to [46], $x^* = 11.25, y^* = 5$ is the global optimal solution of the problem and $x^* = 7.2, y^* = 12.8$ is a local optimal solution. Results were listed in following table, where two starting points were $x^0 = 15.00, y^0 = 0.00$ and $x^0 = 10.00, y^0 = 10.00$. Our method obtained the global one when $\lambda = 3^1$ under the first starting point and closely achieved the local one under many cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	2329.21	1.81	10	0.11	3.71E-13	7.34E-02	1.00E+00	10.16	8.68
3^1	2250.00	197.75	12	0.02	4.23E-09	1.07E-15	1.57E+00	11.25	5.00
3^0	2018.86	2.97	11	0.02	4.56E-11	4.54E-01	1.00E+00	7.27	11.42
3^{-1}	1863.78	18.48	12	0.02	4.98E-12	4.96E-01	1.00E+00	7.17	10.76
3^{-2}	1652.20	98.89	11	0.02	4.60E-07	5.56E-01	1.00E+00	6.90	9.95
3^2	2196.03	0.05	266	0.42	1.86E-12	4.15E-01	4.44E-01	7.27	12.25
3^1	2126.02	0.42	11	0.01	2.15E-09	4.29E-01	1.00E+00	7.28	11.91
3^0	2018.86	2.97	11	0.01	9.26E-13	4.54E-01	1.00E+00	7.27	11.42
3^{-1}	1863.78	18.48	12	0.01	2.70E-07	4.96E-01	1.00E+00	7.17	10.76
3^{-2}	1652.20	98.89	11	0.01	6.94E-07	5.56E-01	1.00E+00	6.90	9.95

Problem name: `SinhaMaloDeb2014TP3`

Source: [47]

description: Sinha, Malo and Deb 2014 defined TP3 as follows

$$
\begin{aligned}
F(x,y) &:= -x_1^2 - 3x_2^2 - 4y_1 + y_2^2 \\
G(x,y) &:= \begin{bmatrix} -x_1 \\ -x_2 \\ x_1^2 + 2x_2 - 4 \end{bmatrix} \\
f(x,y) &:= 2x_1^2 + y_1^2 - 5y_2 \\
g(x,y) &:= \begin{bmatrix} -y_1 \\ -y_2 \\ -x_2 - 3y_1 + 4y_2 + 4 \\ -x_1^2 + 2x_1 - x_2^2 + 2y_1 - y_2 - 3 \end{bmatrix}
\end{aligned}
$$

Comment: The best known solution x^*, y^* reported in [47] satisfied $F(x^*, y^*) = -18.679$ and $f(x^*, y^*) = -1.016$. In following table, where two starting points were $x^0 = (0.00, 2.00)^T, y^0 = (2.00, 2.00)^T$ and $x^0 = (0.00, 2.00)^T, y^0 = (2.00, 0.00)^T$, we achieved solutions with same upper and lower objective function value when $\lambda = 3^1$ under the first starting point.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-5.27	1.78	14	0.14	2.73E-08	4.56E-09	7.48E+00	0.27 0.16	1.28 0.00
3^1	-18.68	-1.02	20	0.04	4.17E-07	2.64E-01	1.64E+00	0.00 2.00	1.88 0.91
3^0	-18.79	-1.01	16	0.02	7.25E-07	7.33E-01	3.99E-01	0.00 2.00	1.92 0.94
3^{-1}	-24.64	6.94	18	0.03	3.08E-07	8.13E-01	1.00E+00	0.00 2.00	3.33 0.83
3^{-2}	-26.98	11.13	18	0.03	5.09E-09	8.40E-01	1.02E+00	0.00 2.00	3.94 0.88
3^2	-10.36	9.56	17	0.02	8.42E-12	5.23E-12	9.25E+00	2.00 0.00	1.60 0.20
3^1	-8.64	8.38	1000	1.49	2.96E-02	8.36E-04	9.88E-01	1.82 0.00	1.33 0.00
3^0	-18.79	-1.01	18	0.01	1.95E-08	7.33E-01	3.99E-01	0.00 2.00	1.92 0.94
3^{-1}	-24.64	6.94	16	0.01	6.39E-07	8.13E-01	1.00E+00	0.00 2.00	3.33 0.83
3^{-2}	-26.98	11.13	11	0.01	4.38E-08	8.40E-01	1.02E+00	0.00 2.00	3.94 0.88

Problem name: `SinhaMaloDeb2014TP6`

Source: [47]

description: Sinha, Malo and Deb 2014 defined TP6 as follows

$$
\begin{aligned}
F(x,y) &:= (x-1)^2 - 2x + 2y_1 \\
G(x,y) &:= -x \\
f(x,y) &:= (2y_1-4)^2 + (2y_2-1)^2 + xy_1 \\
g(x,y) &:= \begin{bmatrix} -y_1 \\ -y_2 \\ 4x+5y_1+4y_2-12 \\ -4x-5y_1+4y_2+4 \\ 4x-4y_1+5y_2-4 \\ -4x+4y_1+5y_2-4 \end{bmatrix}
\end{aligned}
$$

Comment: The best known solution x^*, y^* reported in [47] satisfied $F(x^*,y^*) = -1.209$ and $f(x^*,y^*) = 7.615$. In following table, where two starting points were $x^0 = 1.00, y^0 = (1.00, 0.00)^\top$ and $x^0 = 0.50, y^0 = (0.50, 0.00)^\top$, we achieved same solution for all cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-1.21	7.62	15	0.14	1.50E-11	1.11E-16	8.11E+00	1.89	0.89 0.00
3^1	-1.21	7.62	221	0.43	7.07E-12	1.11E-16	2.12E+00	1.89	0.89 0.00
3^0	-1.21	7.62	7	0.01	8.26E-07	0.00E+00	2.69E-01	1.89	0.89 0.00
3^{-1}	-1.21	7.62	15	0.02	2.55E-09	3.19E-14	6.10E-01	1.89	0.89 0.00
3^{-2}	-1.21	7.62	15	0.03	5.77E-10	4.48E-11	8.23E-01	1.89	0.89 0.00
3^2	-1.21	7.62	15	0.02	1.50E-11	1.11E-16	8.11E+00	1.89	0.89 0.00
3^1	-1.21	7.62	755	1.18	8.04E-11	2.90E-21	2.12E+00	1.89	0.89 0.00
3^0	-1.21	7.62	7	0.01	8.26E-07	3.33E-16	2.69E-01	1.89	0.89 0.00
3^{-1}	-1.21	7.62	15	0.02	2.55E-09	3.20E-14	6.10E-01	1.89	0.89 0.00
3^{-2}	-1.21	7.62	15	0.02	5.77E-10	4.48E-11	8.23E-01	1.89	0.89 0.00

Problem name: `SinhaMaloDeb2014TP7`

Source: [47]

description: Sinha, Malo and Deb 2014 defined TP7 as follows

$$\begin{aligned}
F(x,y) &:= -\frac{(x_1+y_1)(x_2+y_2)}{1+x_1y_1+x_2y_2}\\
G(x,y) &:= \begin{bmatrix} -x_1\\ -x_2\\ x_1-x_2\\ x_1^2+x_2^2-100 \end{bmatrix}\\
f(x,y) &:= \frac{(x_1+y_1)(x_2+y_2)}{1+x_1y_1+x_2y_2}\\
g(x,y) &:= \begin{bmatrix} -y_1\\ -y_2\\ y_1-x_1\\ y_2-x_2 \end{bmatrix}
\end{aligned}$$

Comment: The best known solution x^*, y^* reported in [47] satisfied $F(x^*,y^*) = -1.96$ and $f(x^*,y^*) = 1.96$. In following table, where two starting points were $x^0 = (7.00, 7.00)^\top, y^0 = (7.00, 7.00)^\top$ and $x^0 = (1.00, 1.00)^\top, y^0 = (1.00, 1.00)^\top$, we achieved solutions with better F under 9 cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-1.98	1.98	36	2.40	2.44E-07	7.87E-10	8.75E-13	7.07 7.07	6.93 6.93
3^1	-1.98	1.98	12	0.76	5.20E-11	8.98E-11	5.88E-12	7.07 7.07	6.93 6.93
3^0	-1.98	1.98	18	1.15	1.45E-09	2.27E-09	2.22E-10	7.07 7.07	6.93 6.93
3^{-1}	-1.98	1.98	36	2.33	2.34E-10	8.80E-09	2.35E-17	7.07 7.07	6.93 6.93
3^{-2}	-1.98	1.98	62	3.87	5.13E-07	2.29E-02	2.91E-04	7.07 7.07	6.91 6.91
3^2	0.00	0.00	31	1.89	6.56E-07	1.05E-14	1.96E-04	0.00 0.00	0.00 0.00
3^1	-1.98	1.98	60	3.64	5.09E-07	0.00E+00	0.00E+00	7.07 7.07	6.93 6.93
3^0	-1.98	1.98	21	1.26	3.46E-09	3.17E-08	3.14E-09	7.07 7.07	6.93 6.93
3^{-1}	-1.98	1.98	23	1.39	2.88E-07	8.72E-06	9.31E-09	7.07 7.07	6.93 6.93
3^{-2}	-1.98	1.98	64	3.92	5.78E-07	0.00E+00	0.00E+00	7.07 7.07	6.93 6.93

Problem name: `SinhaMaloDeb2014TP8`

Source: [47]

description: Sinha, Malo and Deb 2014 defined TP8 as follows

$$
\begin{aligned}
F(x,y) &:= |2x_1 + 2x_2 - 3y_1 - 3y_2 - 60| \\
g(x,y) &:= \begin{bmatrix} x_1 + x_2 + y_1 - 2y_2 - 40 \\ x_1 - 50 \\ x_2 - 50 \\ -x_1 \\ -x_2 \end{bmatrix} \\
f(x,y) &:= (y_1 - x_1 + 20)^2 + (y_2 - x_2 + 20)^2 \\
g(x,y) &:= \begin{bmatrix} 2y_1 - x_1 + 10 \\ 2y_2 - x_2 + 10 \\ y_1 - 20 \\ y_2 - 20 \\ -y_1 - 10 \\ -y_2 - 10 \end{bmatrix}
\end{aligned}
$$

However we aim at calculating the same one but

$$F(x,y) := (2x_1 + 2x_2 - 3y_1 - 3y_2 - 60)^2$$

Comment: The best known solution x^*, y^* reported in [47] satisfied $F(x^*, y^*) = 0$ and $f(x^*, y^*) = 100$. In following table, where two sstarting points were $x^0 = (-10.00, 10.00)^T, y^0 = (0.00, 0.00)^T$ and $x^0 = (-10.00, 10.00)^T, y^0 = (-10.00, 10.00)^T$, we achieved one solution with same upper and lower objective function values when $\lambda = 3^2$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	100.00	108	0.35	1.62E-04	4.90E-06	8.00E+00	0.00 30.00	-10.00 10.00
3^1	73.47	8.16	10	0.02	5.93E-08	5.20E-01	3.16E+00	12.86 30.00	-4.29 10.00
3^0	25.00	0.00	16	0.02	3.33E-12	1.20E+00	1.50E+01	25.00 30.00	5.00 10.00
3^{-1}	4.63	1.30	13	0.01	2.24E-10	1.47E+00	2.85E+00	24.43 30.00	5.57 10.00
3^{-2}	0.17	153.69	16	0.03	1.53E-09	7.88E-01	1.00E+00	22.40 0.00	2.40 -7.60
3^2	0.00	100.00	108	0.22	1.62E-04	4.90E-06	8.00E+00	0.00 30.00	-10.00 10.00
3^1	73.47	8.16	10	0.02	5.93E-08	5.20E-01	3.16E+00	12.86 30.00	-4.29 10.00
3^0	25.00	0.00	16	0.02	3.33E-12	1.20E+00	1.50E+01	25.00 30.00	5.00 10.00
3^{-1}	4.63	1.30	13	0.02	2.24E-10	1.47E+00	2.85E+00	24.43 30.00	5.57 10.00
3^{-2}	0.17	153.69	16	0.03	1.53E-09	7.88E-01	1.00E+00	22.40 0.00	2.40 -7.60

Problem name: `SinhaMaloDeb2014TP9`

Source: [47]

description: Sinha, Malo and Deb 2014 defined TP9 as follows

$$
\begin{aligned}
F(x,y) &:= \sum_{i=1}^{10}\left[|x_i-1|+|y_i|\right]\\
f(x,y) &:= \exp\left[\left(1+\frac{1}{4000}\sum_{i=1}^{10}y_i^2-\prod_{i=1}^{10}\cos\left(\frac{y_i}{\sqrt{i}}\right)\right)\sum_{i=1}^{10}x_i^2\right]\\
g(x,y) &:= \begin{bmatrix} y_1-\pi\\ \vdots\\ y_{10}-\pi\\ -y_1-\pi\\ \vdots\\ -y_{10}-\pi\end{bmatrix}
\end{aligned}
$$

However, we aim at calculating the following one:

$$
\begin{aligned}
F(x,y) &:= \sum_{i=1}^{10}\left((x_i-1)^2+y_i^2\right)\\
f(x,y) &:= \exp\left[\left(1+\frac{1}{4000}\sum_{i=1}^{10}y_i^2-\prod_{i=1}^{10}\cos\left(\frac{y_i}{\sqrt{i}}\right)\right)\sum_{i=1}^{10}x_i^2\right]\\
g(x,y) &:= \begin{bmatrix} y_1-\pi\\ \vdots\\ y_{10}-\pi\\ -y_1-\pi\\ \vdots\\ -y_{10}-\pi\end{bmatrix}
\end{aligned}
$$

Comment: The best known solution x^*, y^* reported in [47] satisfied $F(x^*,y^*)=0$ and $f(x^*,y^*)=1$. Let $\mathbf{1}:=(1,1,1,1,1,1,1,1,1,1)^T$ and $\mathbf{0}:=(0,0,0,0,0,0,0,0,0,0)^T$. In following table, where the starting point was $x^0=y^0=\mathbf{0}$, our method rendered the same solutions with same upper and lower optimal objective function values.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	1.00	2	7.18	5.97E-15	1.68E-16	0.00E+0	**1**	**0**
3^1	0.00	1.00	2	7.14	1.99E-15	1.68E-16	0.00E+0	**1**	**0**
3^0	0.00	1.00	2	7.11	6.64E-16	1.68E-16	0.00E+0	**1**	**0**
3^{-1}	0.00	1.00	2	7.10	2.21E-16	1.68E-16	0.00E+0	**1**	**0**
3^{-2}	0.00	1.00	2	7.10	7.38E-17	1.68E-16	0.00E+0	**1**	**0**

Problem name: `SinhaMaloDeb2014TP10`

Source: [47]

description: Sinha, Malo and Deb 2014 defined TP10 as follows

$$\begin{aligned} F(x,y) &:= \sum_{i=1}^{10}(|x_i-1|+|y_i|)\\ f(x,y) &:= \exp\left[1+\frac{1}{4000}\sum_{i=1}^{10}x_i^2y_i^2-\prod_{i=1}^{10}\cos\left(\frac{x_iy_i}{\sqrt{i}}\right)\right]\\ g(x,y) &:= \begin{bmatrix} y_1-\pi\\ \vdots\\ y_{10}-\pi\\ -y_1-\pi\\ \vdots\\ -y_{10}-\pi\end{bmatrix}\end{aligned}$$

However, we aim at calculating the following one:

$$\begin{aligned} F(x,y) &:= \sum_{i=1}^{10}\left[(x_i-1)^2+y_i^2\right]\\ f(x,y) &:= \exp\left[1+\frac{1}{4000}\sum_{i=1}^{10}x_i^2y_i^2-\prod_{i=1}^{10}\cos\left(\frac{x_iy_i}{\sqrt{i}}\right)\right]\\ g(x,y) &:= \begin{bmatrix} y_1-\pi\\ \vdots\\ y_{10}-\pi\\ -y_1-\pi\\ \vdots\\ -y_{10}-\pi\end{bmatrix}\end{aligned}$$

Comment: The best known solution x^*, y^* reported in [47] satisfied $F(x^*,y^*)=0$ and $f(x^*,y^*)=1$. Let $\mathbf{1}:=(1,1,1,1,1,1,1,1,1,1)^\top$ and $\mathbf{0}:=(0,0,0,0,0,0,0,0,0,0)^\top$. In following table, the starting point was $x^0=y^0=\mathbf{0}$, our method rendered the same solutions with same upper and lower optimal objective function values.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	1.00	2	7.46	5.97E-16	1.68E-16	0.00E+0	**1**	**0**
3^1	0.00	1.00	2	7.35	1.99E-16	1.68E-16	0.00E+0	**1**	**0**
3^0	0.00	1.00	2	7.37	6.64E-17	1.68E-16	0.00E+0	**1**	**0**
3^{-1}	0.00	1.00	2	7.35	2.21E-17	1.68E-16	0.00E+0	**1**	**0**
3^{-2}	0.00	1.00	2	7.37	7.38E-18	1.68E-16	0.00E+0	**1**	**0**

Problem name: `TuyEtal2007`

Source: [49]

Description: Tuy et al. 2007 tested one example as follows

$$\begin{aligned} F(x,y) &:= x^2+y^2 \\ G(x,y) &:= \begin{bmatrix} -x \\ -y \end{bmatrix} \\ f(x,y) &:= -y \\ g(x,y) &:= \begin{bmatrix} 3x+y-15 \\ x+y-7 \\ x+3y-15 \end{bmatrix} \end{aligned}$$

Comment: The best know solution of the problem is $x^* = 4.492, y^* = 1.523$ with $F(x^*,y^*) = 22.5$ and $f(x^*,y^*) = -1.523$, seen [49]. In following table, where two starting points were $x^0 = 3.75, y^0 = 3.75$ and $x^0 = 0.00, y^0 = 0.00$, we achieved similar ones when $\lambda = 3^1$ under the first starting point.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	24.50	-3.50	7	0.11	5.11E-08	2.24E-10	1.00E+00	3.50	3.50
3^1	22.50	-1.50	27	0.04	9.24E-07	3.47E-06	1.00E+00	4.50	1.50
3^0	0.28	-0.50	14	0.02	1.41E-09	8.99E-01	3.33E-01	0.17	0.50
3^{-1}	0.03	-0.17	15	0.02	1.11E-08	9.67E-01	3.33E-01	0.06	0.17
3^{-2}	0.00	-0.06	14	0.03	4.62E-07	9.89E-01	3.33E-01	0.02	0.06
3^2	25.00	-4.00	18	0.03	5.84E-07	4.62E-08	5.82E-01	3.00	4.00
3^1	18.25	-1.50	14	0.02	1.84E-07	5.00E-01	8.50E-01	4.00	1.50
3^0	0.28	-0.50	19	0.02	1.89E-09	8.99E-01	3.33E-01	0.17	0.50
3^{-1}	0.03	-0.17	13	0.01	1.14E-10	9.67E-01	3.33E-01	0.06	0.17
3^{-2}	0.00	-0.06	16	0.02	3.68E-08	9.89E-01	3.33E-01	0.02	0.06

Problem name: `Vogel2012`

Source: [50]

Description: Vogel 2012 defined one example as follows

$$
\begin{aligned}
F(x,y) &:= (y+1)^2 \\
G(x,y) &:= \begin{bmatrix} -3-x \\ -2+x \end{bmatrix} \\
f(x,y) &:= y^3-3y \\
g(x,y) &:= x-y
\end{aligned}
$$

Comment: Two popssible sloutions reported by Vogel in [50] are $x^* = -2, y^* = -2$ and $x^* = -3, y^* = -1$. In following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = -2.00, y^0 = -1.00$, we achieved the second solution.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.00	2.00	23	0.15	7.76E-11	4.29E-11	5.56E-08	-1.00	-1.00
3^1	0.00	2.00	19	0.03	6.29E-08	4.30E-08	4.64E-06	-1.00	-1.00
3^0	0.00	2.00	9	0.02	9.60E-07	1.71E-07	2.85E-10	-2.99	-1.00
3^{-1}	0.00	2.00	10	0.01	8.32E-14	4.15E-14	8.03E-26	-3.00	-1.00
3^{-2}	0.00	2.00	34	0.07	4.22E-07	6.05E-07	7.59E-08	-1.00	-1.00
3^2	0.00	2.00	24	0.02	1.62E-10	8.94E-11	1.16E-07	-1.00	-1.00
3^1	0.00	2.00	17	0.02	2.38E-08	1.63E-08	1.76E-06	-1.00	-1.00
3^0	0.00	2.00	17	0.02	5.12E-07	2.02E-08	9.43E-17	-1.00	-1.00
3^{-1}	0.00	2.00	12	0.01	3.67E-07	7.22E-08	2.89E-07	-1.00	-1.00
3^{-2}	0.00	2.00	34	0.04	7.95E-07	3.95E-07	7.03E-07	-1.00	-1.00

Problem name: `WanWangLv2011`

Source: [51]

description: Wan, Wang and Lv defined one example as follows

$$\begin{aligned} F(x,y) &:= (1+x_1-x_2+2y_2)(8-x_1-2y_1+y_2+5y_3) \\ f(x,y) &:= 2y_1-y_2+y_3 \\ g(x,y) &:= \begin{bmatrix} -y_1+y_2+y_3-1 \\ 2x_1-y_1+2y_2-0.5y_3-1 \\ 2x_2+2y_1-y_2-0.5y_3-1 \\ -y_1 \\ -y_2 \\ -y_3 \\ -x_1 \\ -x_2 \end{bmatrix} \end{aligned}$$

Comment: Wan, Wang and Lv in [51] obtained the optimal solution $x^* = (0, 0.75)^T$, $y^* = (0, 0.5, 0)^T$ with with $F(x^*, y^*) = 10.62$ and $f(x^*, y^*) = -0.50$. Results were listed in the following table, where two starting points were $x^0 = (0.00, 0.00)^T, y^0 = (0.00, 0.00, 0.00)^T$ and $x^0 = (1.00, 1.00)^T, y^0 = (1.00, 1.00, 1.00)^T$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	11.25	0.00	16	0.13	1.46E-07	2.39E-09	1.10E+01	0.50 0.00	0.00 0.00 0.00
3^1	11.25	0.00	23	0.05	9.81E-07	1.26E-06	3.62E+00	0.50 0.00	0.00 0.00 0.00
3^0	12.28	-0.63	121	0.15	3.51E-01	2.59E-04	1.23E-01	-0.12 0.93	-0.06 0.66 0.15
3^{-1}	7.00	1.00	73	0.11	1.12E-08	7.07E-01	1.36E+00	0.00 0.00	0.50 0.00 0.00
3^{-2}	12.25	1.00	213	0.23	1.77E-10	1.67E-10	8.99E-01	0.75 0.75	0.00 0.00 1.00
3^2	8.00	0.00	14	0.01	6.22E-09	5.00E-01	1.04E+01	0.00 0.00	0.00 0.00 0.00
3^1	7.50	0.00	13	0.01	1.62E-08	9.68E-09	4.40E+00	0.50 0.50	0.00 0.00 0.00
3^0	8.00	0.00	42	0.04	2.65E-09	5.00E-01	2.01E-01	0.00 0.00	0.00 0.00 0.00
3^{-1}	12.25	1.00	133	0.16	8.88E-10	8.40E-10	8.43E-01	0.75 0.75	0.00 0.00 1.00
3^{-2}	12.25	1.00	201	0.31	1.78E-10	1.69E-10	8.99E-01	0.75 0.75	0.00 0.00 1.00

Problem name: `YeZhu2010Ex42`

Source: [52]

Description: Ye and Zhu 2010 defined Example 4.2 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-1)^2 + y^2 \\ G(x,y) &:= \begin{bmatrix} -x-3 \\ x-2 \end{bmatrix} \\ f(x,y) &:= y^3 - 3y \\ g(x,y) &:= x - y \end{aligned}$$

Comment: This is a slightly modified version of `Vogel2012`. The optimal sloution reported by Ye and Zhu in [52] is $x^* = 1, y^* = 1$. Results were listed in the following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = -2.00, y^0 = -2.00$, and similar results were achieved under many cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.00	-2.00	15	0.12	2.26E-07	3.57E-02	7.83E-08	1.00	1.00
3^1	1.00	-2.00	24	0.03	2.96E-07	1.00E-01	7.76E-07	1.00	1.00
3^0	1.00	-2.00	8	0.01	2.46E-07	2.50E-01	6.52E-09	1.00	1.00
3^{-1}	1.00	-2.00	12	0.01	2.37E-07	5.00E-01	6.21E-07	1.00	1.00
3^{-2}	1.00	-2.00	9	0.02	2.98E-07	7.50E-01	5.62E-11	1.00	1.00
3^2	1.00	-2.00	163	0.21	4.46E-07	3.57E-02	1.17E-06	1.00	1.00
3^1	5.00	2.00	51	0.06	2.56E-10	3.33E-01	4.00E+00	-1.00	-1.00
3^0	1.00	-2.00	23	0.03	5.14E-07	2.50E-01	1.34E-06	1.00	1.00
3^{-1}	5.00	2.00	10	0.01	5.43E-10	1.50E+00	4.00E+00	-1.00	-1.00
3^{-2}	5.00	2.00	9	0.01	8.43E-13	1.13E+00	4.00E+00	-1.00	-1.00

Problem name: `YeZhu2010Ex43`

Source: [52]

Description: Ye and Zhu 2010 defined Example 4.3 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-1/2)^2+(y-2)^2 \\ G(x,y) & := & \begin{bmatrix} -x \\ x-4 \end{bmatrix} \\ f(x,y) & := & y^3-3y \\ g(x,y) & := & x-y-3 \end{array}$$

Comment: The optimal sloution reported by Ye and Zhu in [52] is $x^*=1, y^*=1$. Results were listed in the following table, where two starting points were $x^0=1.00, y^0=0.00$ and $x^0=4.00, y^0=1.00$. We got one solution $\hat{x}=0.5, \hat{y}=1$ such that $F(\hat{x},\hat{y})<F(x^*,y^*)=1.25$ and $f(\hat{x},\hat{y})=f(x^*,y^*)=-2$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.00	-2.00	10	0.11	2.12E-10	3.70E-02	1.45E-18	0.50	1.00
3^1	1.00	-2.00	8	0.01	6.59E-09	1.11E-01	4.13E-11	0.50	1.00
3^0	1.00	-2.00	6	0.01	1.95E-07	3.33E-01	1.62E-07	0.50	1.00
3^{-1}	1.00	-2.00	11	0.01	2.89E-08	1.00E+00	1.19E-11	0.50	1.00
3^{-2}	1.00	-2.00	10	0.02	1.63E-09	1.50E+00	1.62E-09	0.50	1.00
3^2	1.00	-2.00	12	0.02	7.30E-12	3.70E-02	6.42E-13	0.50	1.00
3^1	1.00	-2.00	8	0.01	2.15E-09	1.11E-01	3.94E-11	0.50	1.00
3^0	1.00	-2.00	11	0.01	2.63E-09	3.33E-01	2.31E-20	0.50	1.00
3^{-1}	1.00	-2.00	11	0.01	5.93E-07	1.00E+00	5.40E-11	0.50	1.00
3^{-2}	1.00	-2.00	13	0.01	2.32E-09	1.50E+00	6.27E-11	0.50	1.00

Problem name: `Yezza1996Ex31`

Source: [53]

description: Yezza 1996 defined Example 3.1 as follows

$$\begin{aligned} F(x,y) &:= -(4x-3)y+2x+1 \\ G(x,y) &:= \begin{bmatrix} -x \\ x-1 \end{bmatrix} \\ f(x,y) &:= -(1-4x)y-2x-2 \\ g(x,y) &:= \begin{bmatrix} -y \\ y-1 \end{bmatrix} \end{aligned}$$

Comment: Yezza derived the unique optimal solution of this problem is $x^* = 1/4, y^* = 0$. In following table, where two starting points were $x^0 = 0.00, y^0 = 0.00$ and $x^0 = 0.75, y^0 = 0.00$, we achieved this optimal solution when $\lambda \geq 3^0$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	1.50	-2.50	112	0.29	4.76E-08	5.56E-02	2.00E+00	0.25	0.00
3^1	1.50	-2.50	346	0.48	4.26E-11	1.67E-01	2.00E+00	0.25	0.00
3^0	1.50	-2.50	8	0.01	3.56E-08	5.00E-01	2.00E+00	0.25	0.00
3^{-1}	1.00	-2.00	10	0.01	5.86E-10	1.00E+00	2.85E+00	0.00	0.00
3^{-2}	2.00	-1.00	9	0.02	1.46E-12	1.00E+00	1.02E+00	1.00	1.00
3^2	1.50	-2.50	784	0.97	6.19E-07	5.56E-02	2.00E+00	0.25	0.00
3^1	1.50	-2.50	68	0.07	1.84E-09	1.67E-01	2.00E+00	0.25	0.00
3^0	1.50	-2.50	8	0.01	3.56E-08	5.00E-01	2.00E+00	0.25	0.00
3^{-1}	1.00	-2.00	7	0.01	1.83E-08	1.00E+00	2.85E+00	0.00	0.00
3^{-2}	2.48	-2.36	10	0.01	1.40E-07	5.62E-01	1.00E+00	0.81	0.56

Problem name: `Yezza1996Ex41`

Source: [53]

description: Yezza 1996 defined Example 4.1 as follows

$$\begin{aligned} F(x,y) &:= (y-2)^2/2 + (x-y-2)^2/2 \\ f(x,y) &:= y^2/2 + x - y \\ g(x,y) &:= \begin{bmatrix} -y \\ y-x \end{bmatrix} \end{aligned}$$

Comment: Yezza derived the unique optimal solution of this problem is $x^* = 3, y^* = 1$. In following table, where two starting points were $x^0 = 2.00, y^0 = 0.00$ and $x^0 = 2.00, y^0 = 2.00$, we closely achieved this optimal solution under three cases.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	4.00	0.00	26	0.13	8.39E-09	6.74E-12	6.26E+00	0.00	0.00
3^1	0.50	2.50	9	0.01	1.37E-07	3.33E-01	1.37E-07	3.00	1.00
3^0	0.50	2.50	7	0.01	8.14E-10	1.00E+00	7.63E-10	3.00	1.00
3^{-1}	0.18	2.98	9	0.01	1.44E-12	1.00E+00	4.00E-01	3.40	1.40
3^{-2}	0.04	3.49	16	0.03	3.18E-11	1.00E+00	7.27E-01	3.73	1.73
3^2	4.00	0.00	6	0.01	1.32E-10	1.52E-20	6.28E+00	0.00	0.00
3^1	4.00	0.00	8	0.01	2.20E-08	2.71E-20	8.94E-01	0.00	0.00
3^0	0.50	2.50	15	0.01	3.25E-07	1.00E+00	3.88E-07	3.00	1.00
3^{-1}	0.18	2.98	9	0.01	7.64E-07	1.00E+00	4.00E-01	3.40	1.40
3^{-2}	0.04	3.49	9	0.01	3.17E-08	1.00E+00	7.27E-01	3.73	1.73

Problem name: `Zlobec2001a`

Source: [55]

Description: Zlobec 2001 tested one example as follows

$$
\begin{aligned}
F(x,y) &:= -y_1/x \\
f(x,y) &:= -y_1 - y_2 \\
g(x,y) &:= \begin{bmatrix} -1 + y_1 + xy_2 \\ -y_1 \\ -y_2 \end{bmatrix}
\end{aligned}
$$

Comment: This example is used in [55] to illustrate that the objective function of the problem can be discontinuous. As stated in [55], an optimal solution is $x^* = 1, y^* = (1,0)^\top$. Same/similar results were obtained under three cases. Two starting points were $x^0 = 1.00, y^0 = (0.00, 0.00)^\top$ and $x^0 = 1.00, y^0 = (0.00, 1.00)^\top$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	-1.00	-1.00	13	0.15	5.94E-09	1.57E-01	9.06E+00	1.00	1.00 0.00
3^1	0.00	0.00	245	0.39	1.31E-03	2.11E-02	4.15E+00	2608.99	0.00 0.00
3^0	-1.00	-1.00	11	0.02	9.59E-09	1.41E+00	1.41E+00	1.00	1.00 0.00
3^{-1}	0.00	-1.72	544	0.71	1.58E-02	1.00E+00	8.72E-01	4696.25	1.72 0.00
3^{-2}	0.00	-2.99	568	0.52	4.00E-02	5.66E-01	8.86E-01	2018.25	2.99 0.00
3^2	0.00	-1.13	10	0.01	2.84E-08	3.17E-09	7.95E+00	0.89	0.00 1.13
3^1	2.35	-1186.18	158	0.16	2.33E+00	6.92E-01	5.99E-01	-316.82	745.01 441.17
3^0	-1.01	-1.00	10	0.01	2.52E-11	1.41E+00	1.40E+00	0.99	1.00 0.00
3^{-1}	0.38	-83.10	136	0.13	3.94E-01	6.53E-01	6.40E-02	-169.93	63.77 19.34
3^{-2}	0.70	-306.97	205	0.13	1.42E-01	6.95E-01	5.46E-02	-32.85	22.86 284.12

Problem name: `Zlobec2001b`

Source: [55]

Description: Zlobec 2001 tested one example as follows

$$\begin{array}{rcl} F(x,y) & := & x+y \\ G(x,y) & := & \begin{bmatrix} -1+x \\ -x \end{bmatrix} \\ f(x,y) & := & -y \\ g(x,y) & := & \begin{bmatrix} -1+y \\ -y \\ -xy \\ xy \end{bmatrix}. \end{array}$$

Comment: This example is used in [55] to illustrate that the feasible set of a bilevel optimization problem is not necessarily closed. As stated in [55], this problem does not have an optimal solution. Results were listed in following table, where two starting points were $x^0 = 0.00, y^0 = 1.00$ and $x^0 = 1.00, y^0 = 1.00$.

λ	F	f	Iter	Time	Error	Er_{yz}	Er_{vw}	$\hat{x}$	$\hat{y}$
3^2	0.13	-0.13	128	0.35	8.07E-04	8.70E-01	6.84E+01	0.00	0.13
3^1	1.00	-1.00	21	0.03	2.46E-08	0.00E+00	8.88E-01	0.00	1.00
3^0	1.00	-1.00	23	0.04	4.00E-12	1.68E-03	1.08E+00	0.00	1.00
3^{-1}	0.00	0.00	13	0.02	6.91E-08	1.00E+00	6.43E+00	0.00	0.00
3^{-2}	0.08	-0.08	131	0.23	3.19E-03	9.18E-01	3.68E+00	0.00	0.08
3^2	1.00	-1.00	73	0.11	9.21E-08	4.79E-09	1.36E+01	0.00	1.00
3^1	1.00	-1.00	19	0.03	5.57E-10	9.35E-13	9.99E-01	0.00	1.00
3^0	0.01	-0.01	66	0.09	2.41E-08	9.87E-01	1.61E+02	0.00	0.01
3^{-1}	-0.03	0.00	112	0.20	3.29E-02	1.32E-03	2.37E-01	-0.03	0.00
3^{-2}	0.00	0.00	113	0.19	2.26E-06	1.00E+00	1.34E+00	0.00	0.00

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