UNIVERSITÉ Concordia UNIVERSITY

ELEC 481 / ENGR 6131 Exam #2

1) Given $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ find A^{99} by using the

25% Caley-Hamilton technique.

2) Given $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$

25% $y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} x$

(a) What conditions on b_1, b_2, b_3 will ensure that the system is controllable?

(b) What conditions on c_1, c_2, c_3 will ensure that the system is observable?

3) Find the fundamental matrix $M(t)$ and the state transition matrix $\Phi(t, 0)$ for the following system

25% $\dot{x} = \begin{bmatrix} -3 & e^{2t} \\ 0 & -2 \end{bmatrix} x$

4) Let the state transition matrix of a system be given by

25% $\Phi(t, 0) = \begin{bmatrix} e^{-t} \cos t & e^{-t} \sin t \\ -e^{-t} \sin t & e^{-t} \cos t \end{bmatrix}$

(a) Compute $\Phi(t, t_0)$, $\forall\, t_0 > 0$, and

(b) Find A.

1) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$; $A^{99} = ?$

$P(A) = A^{99}$ $\quad P(\lambda) = \lambda^{99}$; $n = 2$; $R(\lambda) = \alpha + \beta\lambda$

$\det(\lambda I - A) = 0 \longrightarrow \lambda = 2, 2$

$$\begin{cases} P(\lambda) = R(\lambda) \longrightarrow 2^{99} = \alpha + 2\beta \quad (@\ \lambda = 2) \\ \dfrac{dP}{d\lambda} = \dfrac{dR}{d\lambda} \longrightarrow 99\lambda^{98} = \beta \longrightarrow \beta = 99(2)^{98} \mathbin{/\!/} \end{cases}$$

$\therefore \quad \alpha = 2^{99} - 2(99)(2)^{98} = -98(2)^{99} \mathbin{/\!/}$

$P(A) = R(A) \longrightarrow P(A) = \alpha I + \beta A$

$\therefore \quad A^{99} = -98(2)^{99}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 99(2)^{98}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \mathbin{/\!/}$

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2) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$; $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$

(a) $C_x = \begin{bmatrix} B & \underline{AB} & A^2B \end{bmatrix}$

$C_x = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & 3b_3 \\ b_3 & 3b_3 & 9b_3 \end{bmatrix}$; $\det C_x \neq 0$

$\det C_x = b_1\begin{vmatrix} b_3 & 3b_3 \\ 3b_3 & 9b_3 \end{vmatrix} - b_2\begin{vmatrix} b_2 & 3b_3 \\ b_3 & 9b_3 \end{vmatrix} + b_3\begin{vmatrix} b_2 & b_3 \\ b_3 & 3b_3 \end{vmatrix}$

$= b_1\cancelto{0}{(9b_3^2 - 9b_3^2)} - b_2(9b_2b_3 - 3b_3^2) + b_3(3b_2b_3 - b_3^2)$

$$\det \mathcal{C}_x = -9b_2^2 b_3 + 3b_3^2 b_2 + 3b_2 b_3^2 - b_3^3$$

$$= b_3\left(-9b_2^2 + 6b_2 b_3 - b_3^2\right)$$

$$= -b_3\left(3b_2 - b_3\right)^2 \neq 0$$

$$b_3 \neq 0 \ \& \ b_3 \neq 3b_2 \implies \underline{\text{Cont.}}$$

(b) $$\mathcal{O}_x = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

$$\mathcal{O}_x = \begin{bmatrix} c_1 & c_2 & c_3 \\ 0 & c_1 & c_2 + 3c_3 \\ 0 & 0 & c_1 + 3(c_2 + 3c_3) \end{bmatrix}$$

$\det \mathcal{O}_x \neq 0$

$$\det \mathcal{O}_x = c_1 \begin{vmatrix} c_1 & c_2 + 3c_3 \\ 0 & c_1 + 3c_2 + 9c_3 \end{vmatrix}$$

$$= c_1^2\left(c_1 + 3c_2 + 9c_3\right) \neq 0$$

$$c_1 \neq 0 \quad \& \quad c_1 + 3c_2 + 9c_3 \neq 0 \implies \text{Obs.}$$

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3) $$A = \begin{bmatrix} -3 & e^{2t} \\ 0 & -2 \end{bmatrix}$$

$M(t) = ?$; $\Phi(t, 0) = ?$

$$\begin{cases} \dot{x}_1 = -3x_1 + e^{2t} x_2 \leftarrow \\ \dot{x}_2 = -2x_2 \end{cases}$$