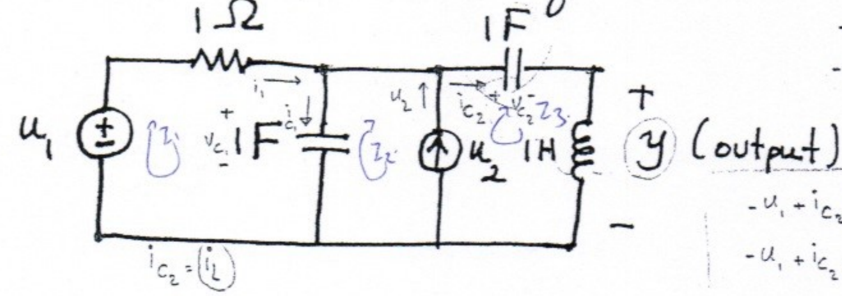
Sample exams with solutions

Linear Systems (Concordia University)

Linear System

ELEC 481/ENGR6131 Exam # 1 Oct. 21, 05

1. For the circuit shown below, write the state and output equations. Also, express the equations in a compact form.



$-u_1 + i_1 + v_{c_1} = 0$

$-u_1 + i_1 + v_{c_2} + v_L = 0$

$i_1 + u_2 = i_{c_2} + i_{c_1}$

$-u_1 + i_{c_2} + i_{c_1} - u_2 + v_{c_1} = 0$

$-u_1 + i_{c_2} + i_{c_1} - u_2 + v_{c_2} + v_L = 0$

2. Find the fundamental matrix $M(t)$ associated with the system $\dot{x} = A(t)x$, where $A(t) = \begin{bmatrix} -1 & 0 \\ 2e^{t} & -2 \end{bmatrix}$.

3. Verify whether the following map is (i) linear or nonlinear (ii) time-invariant or time-varying, and (iii) causal or noncausal (show ALL work).

$$y(t) = \int_{-\infty}^{t} \cos(t - \tau + 1)\, e^{u(\tau)+1}\, d\tau .$$

$2\tau + 2$

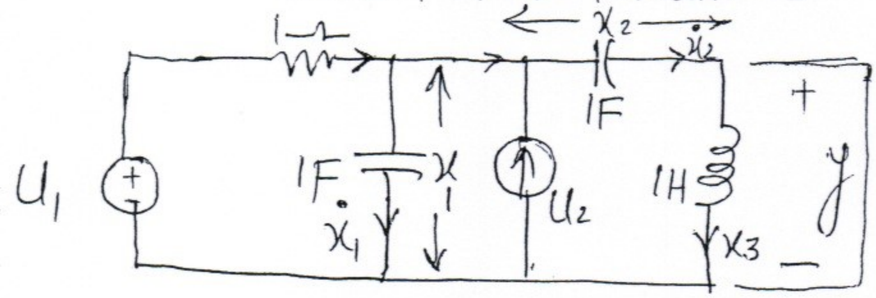
mid2

4. Given $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, find A^{99} by using Caley-Hamilton technique.

All Problems Are Equally Weighted.

MidTerm Exam-1 2005

Q 1



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix}-1 & 0 & -1\\ 0 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}}_{n\times n}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}1 & +1\\ 0 & 0\\ 0 & 0\end{bmatrix}}_{n\times q}\underbrace{\begin{bmatrix}U_1\\ U_2\end{bmatrix}}_{q\times 1}$$

$$\frac{U_1 - x_1}{1} = \dot{x}_1 \mp U_2 + \dot{x}_2$$

$$U_1 - x_1 = \dot{x}_1 \mp U_2 + x_3$$

$$\boxed{\dot{x}_1 = (U_1 \mp U_2) - x_1 - x_3}$$

$$\boxed{\dot{x}_2 = x_3}$$

$$x_1 = x_2 + x_3$$

$$\boxed{\dot{x}_3 = x_1 - x_2}$$

$$y = \cancel{x_3\, x_1 = x_3}$$
$$= \dot{x}_3 = x_1 - x_2$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & 0 & -1\\ 0 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} =$$

$$y = \underbrace{\begin{bmatrix}1 & -1 & 0\end{bmatrix}}_{m\times n}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}$$

Q 2

$M(t) = ?$

$\dot{x} = A(t)\, x \qquad A(t) = \begin{bmatrix}-1 & 0\\ 2e^{t} & -2\end{bmatrix}$

Let $M(t) = [m_1(t) \quad m_2(t)]$

Also $m_1(t) = [x_{11} \quad x_{12}]^T$

$\dot{x}_{11} = -x_{11} \Rightarrow x_{11}(t) = e^{-t},\ 0$

$\dot{x}_{12} = 2e^{t}x_{11} - 2x_{12}$

if $x_{11}(t) = 0$

$\dot{x}_{12} = -2x_{12} \Rightarrow x_{12}(t) = e^{-2t}$ ✓

if $x_{11}(t) = e^{-t}$

$\dot{x}_{12} = 2e^{0} - 2x_{12} = 2 - 2x_{12}$

$x_{12}(t) = 1$ or $1 + e^{-2t}$

To see Linear Combinations

$$\alpha_1 \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\alpha_1 e^{-t} = 0 \Rightarrow \alpha_1 = 0.$$
$$\alpha_1 + \alpha_2 e^{-2t} \Rightarrow \alpha_2 = 0$$

So

$$M(t) = \begin{bmatrix} e^{-t} & 0 \\ 1 & e^{-2t} \end{bmatrix}$$

To check $M(t)$ satisfies the equation

$$\dot{M}(t) = A(t) \cdot M(t).$$

$$\begin{bmatrix} -e^{-t} & 0 \\ 0 & -2e^{-2t} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2e^{t} & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 1 & e^{-2t} \end{bmatrix} = \begin{bmatrix} -e^{-t} & 0 \\ 0 & -2e^{-2t} \end{bmatrix}$$

Q3 Verify the following map for

$$y(t) = \int_{-\infty}^{t} \cos(t-\tau+1) \cdot e^{u(\tau)+1} \cdot d\tau.$$

(i) Check Linearity ?

$$y_1(t) = \int_{-\infty}^{t} \cos(t-\tau+1) \cdot e^{u_1(\tau)+1} \cdot d\tau.$$

$$y_2(t) = \int_{-\infty}^{t} \cos(t-\tau+1) \cdot e^{u_2(\tau)+1} \cdot d\tau.$$

$$y(t) = a_1 u_1 + a_2 u_2$$

$a_1 u_1(\tau) + a_2 u_2(\tau) + 1$

$$y(t) = \int_{-\infty}^{t} \cos(t-\tau+1) \cdot e^{a_1 u_1(\tau)} \cdot e^{a_2 u_2(\tau)} \cdot e^{1} \cdot d\tau.$$

$$a_1 y_1 + a_2 y_2 = a_1 \int_{-\infty}^{t} \cos(t-\tau+1) e^{u_1(\tau)} \cdot e^{1} \, d\tau.$$
$$+ a_2 \int_{-\infty}^{t} \cos(t-\tau+1) \cdot e^{u_2(\tau)} \cdot e^{1} \cdot d\tau$$
$$\neq y(t)$$

So System is Non Linear.

(ii) Time invariant / Time Varying.

$$y(t) = \int_{-\infty}^{t} \cos(t-\tau+1) \cdot e^{u(\tau)+1} \cdot d\tau.$$

Shift the input by T sec

$$y_1(t) = \int_{-\infty}^{t} \cos(t-\tau+1) \cdot e^{u(\tau-T)+1} \cdot d\tau$$

$$y(t-T) = \int_{-\infty}^{t-T} \cos(t-T-\tau+1) \cdot e^{u(\tau)+1} \Longrightarrow$$

Let $\tau' = \tau - T$

$$\tau = \tau' + T, \quad d\tau = d\tau'$$

$$y_1(t-T) = \int_{-\infty}^{t} \cos(t-\tau'-T+1) \cdot e^{u(\tau')+1} \cdot d\tau'$$

$$= \int_{-\infty}^{t-T} \cos(t-\tau'-T+1) \cdot e^{u(\tau')+1} \cdot d\tau' = (A)$$

Since $y_1(t-T) = y(t)$ So System is Time invar

(iii) <u>TO check Causality</u>

The output at time t should not depend on input after time t

There should be No output when input = 0.

$$y(t) = \int_{-\infty}^{t} \cos(t-\tau+1) \cdot e^{u(\tau)+1} \cdot d\tau$$

in this case our system doesnot depend for future inputs so system is Causal.

<u>Q4</u> $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, A^{99} by using Cayley Hamilton.

$P(A) = A^{99}$

$P(\lambda) = \lambda^{99}$

$R(\lambda) = \alpha\lambda + \beta$

$\Pi_A(\lambda) = \det(A - \lambda I) = 0 = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} =$

so $\lambda_{1,2} = 2, 2$, $m = 2$

$R(\lambda) = P(\lambda)$

$\lambda^{99} = \alpha\lambda + \beta \Longrightarrow$ A

$2^{99} = 2\alpha + \beta$ —(i)

$99\lambda^{98} = \alpha$

$99 \cdot (2)^{98} = \alpha$

(i) $\Rightarrow$ $\beta = (2)^{99} - 198(2)^{98}$

(A) $\Rightarrow$

$$\lambda^{99} = 99(2)^{98}\lambda + 2^{99} - 198(2)^{98}$$

$$P(A) = R(A).$$

$$A^{99} = 99(2)^{98}A + [2^{99} - 198(2)^{98}]\dot{I}$$

$$= 99\,2^{98}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} + [2^{99} - 198(2)^{98}]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 198(2)^{98} & 99(2)^{98} \\ 0 & 198(2)^{98} \end{bmatrix} + \begin{bmatrix} 2^{99} - 198(2)^{98} & 0 \\ 0 & 2^{99} - 198(2)^{98} \end{bmatrix}$$

$$= \begin{bmatrix} 2^{99} & 99(2)^{98} \\ 0 & 2^{99} \end{bmatrix} \quad \text{Ans} //$$

$$= \begin{bmatrix} 6.338\times10^{29} & 3.137\times10^{31} \\ 0 & 6.338\times10^{29} \end{bmatrix}$$

Midterm Exam # 1 Oct. 2001

Q 1 Same as Oct. 2005.

Q 2 Response of a LTI system

$u_1(t)$ is $\longrightarrow y_1(t) = e^{-t}\,u_s(t)$

$u_2(t) \longrightarrow y_2(t) = \cos 2t\; u_s(t)$

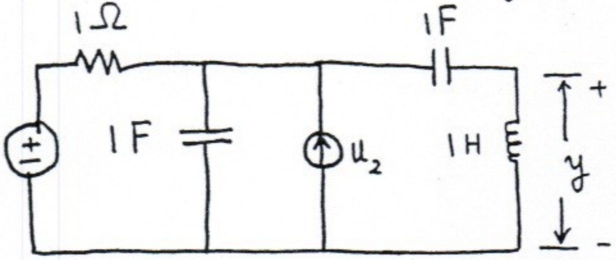
$u_3(t) = 2u_1(t-1) + \frac{d}{dt}u_2(t+1) \longrightarrow y_3(t) = ?$

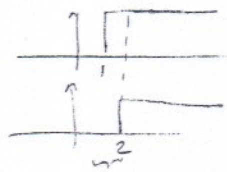
Since system is Linear and Time invariant

ENGR 471/6131 First Hour Exam Oct. 2001

1. Write the state & output equations for the network shown below:

35%





$u(t-2)$

$u(t-1)$

$u(t-2) \cdot u(t-1)$

2. The response of a LTI causal system to the input

30%

$u_1(t)$ is $y_1(t) = e^{-t}\, u_s(t)$

$-2\sin 2t\, u_s(t) +$ $\delta(t)$ $t=0$

$\cos 2t$

The response to a second input $u_2(t)$ is $y_2(t) = \cos 2t\ u_s(t)$,

where $u_s(t)$ is the unit step signal. Compute the system response

$-2\sin 2t\, u_s(t) + \delta(t)$

to $u_3(t) = 2\, u_1(t-1) + \dfrac{du_2}{dt}(t+1)$.

$\delta(t+1)$ Cos

LTI

3. Verify whether the following map is

(i) linear or nonlinear

35%

(ii) time-invariant or time-varying

(iii) causal or non-causal

$$y(t) = \int_{-\infty}^{t} \sin(t-\tau)\, e^{u(\tau)}\, d\tau.$$

Show **ALL** work.

So

$u_3(t) =$ Sum of two Responses

$$u_3(t) = 2\,u_1(t-1) + \frac{d}{dt} u_2(t+1)$$

$$= 2\left[e^{-(t-1)} \cdot u_s(t-1) \right] + \frac{d}{dt}\left[\cos 2(t+1)\, u_s(t+1) \right]$$

$$= 2\, e^{-(t-1)} u_s(t-1) \cdot \frac{d}{dt}\left[\cos 2t \cdot u_s(t) \right]$$

$$\frac{d}{dt}\left[\cos 2t \cdot u_s(t)\right] = -2\sin 2t\, u_s(t) + \cos 2t_0\, \delta(t) \quad t=0$$

$$= -2\sin 2(t+1)\, u_s(t+1) + \delta(t+1)$$

So

$$y_3(t) = 2\, e^{-(t-1)}\, u_s(t-1) - 2\sin 2(t+1)\, u_s(t+1) + \delta(t+$$

Q. 3 Verify the following

$$y(t) = \int_{-\infty}^{t} \sin(t-\tau) \cdot e^{u(\tau)} \cdot d\tau$$

(i) Linearity = ?

$$y_1(t) = \int_{-\infty}^{t} \sin(t-\tau) \cdot e^{u_1(\tau)}\, d\tau$$

$$y_2(t) = \int_{-\infty}^{t} \sin(t-\tau) \cdot e^{u_2(\tau)} \cdot d\tau$$

$$y(t) = \int_{-\infty}^{t} \sin(t-\tau) \cdot e^{a_1 u_1(\tau) + a_2 u_2(\tau)} \cdot d\tau$$

∴ $y(t) \neq a_1 y_1 + a_2 y_2$ ... So System is Non-Linear

ENGR 471 EXAM 1 Nov. 9, 199

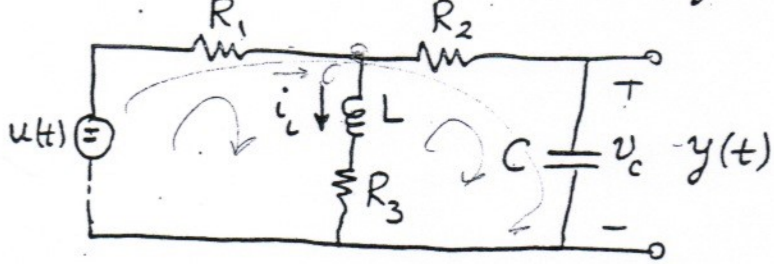
(1) Consider a linear system with the following response to $\delta(t$

$$h(t, \tau) = u(t-\tau) - u(t-2\tau)$$

(a) Is this system time-invariant? (<u>SHOW</u> your work)

(b) Is it causal? (<u>SHOW</u> your work)

(2) Write the state and output equations for the following circ



using i_L and v_c as state variables. Write the state and outp equations when the inductor and capacitor are time-varying, i. $L = L(t)$ and $C = C(t)$. In both problems determine A, B, C and D elements of the equations. $u_s(t)$ $t-1$

$t=1$

(3) A causal linear time-invariant system has step response $y(t) = (1 - e^{-t})\, u_s(t)$. Compute the response to the input signal

$e_i = u_s(t)$

$$u(t) = u_s(t-1)\, u_s(2-t) \quad \text{for } t \le 2.$$

$t>0$

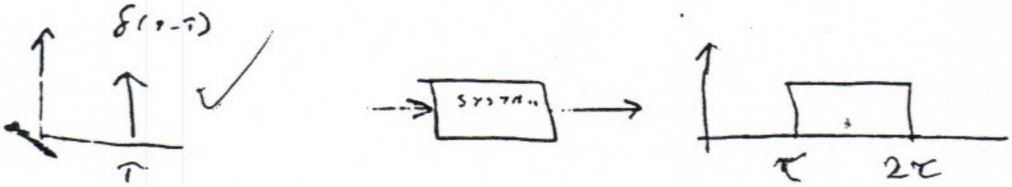
* All PROBLEMS ARE EQUALLY WEIGHTED

RESULTS
HI 84
LO 20
AVE 51

11/10/91 (1)

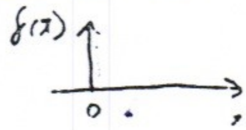
EXAM REVIEW:

$$h(t,\tau) = u(t-\tau) - u(t-2\tau)$$



∴ time VARIANT

let $\tau = 0$



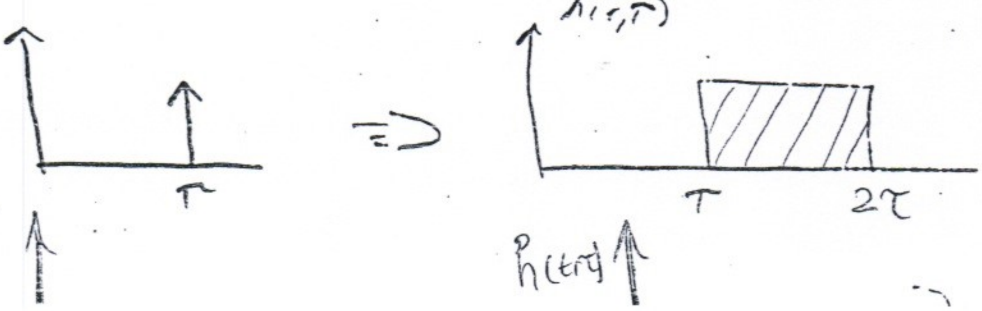
$$h(t,0) = u(t) - u(t) = 0$$

Causality: "system should have no output when input is zero"

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)\,u(\tau)\,d\tau = \int_{-\infty}^{t} h(t,\tau)u(\tau)d\tau + \int_{t}^{+\infty} h(t,\tau)u(\tau)\,d\tau$$

If this is causal system then $h(t,\tau) = 0$, $\tau > t$

$$\therefore\ y(t) = \int_{-\infty}^{t} h(t,\tau)\,u(\tau)\,d\tau$$



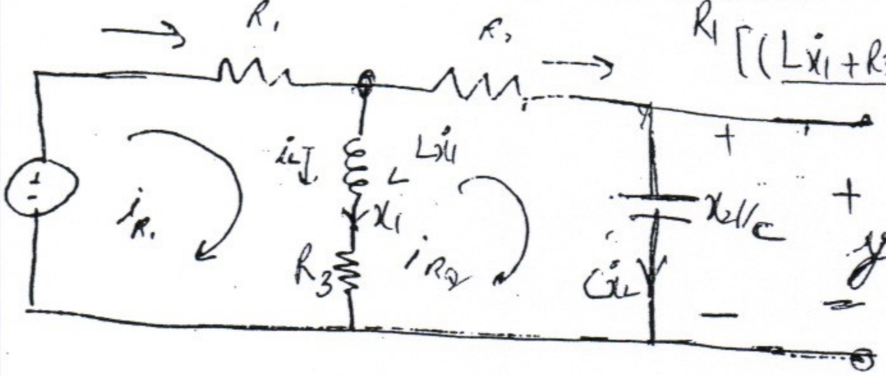
appears causal because $y/t = 0$ for all $t < \tau$

BUT

#2 (A)

$$\frac{U(t)-(L\dot{x}_1+R_3x_1)}{R_1} = x_1 + \left[\frac{(L\dot{x}_1+R_3x_1)-x_2}{R_2}\right] \quad \text{— (i)}$$

$$\left[\frac{(L\dot{x}_1+R_3x_1)-x_2}{R_2}\right] = C\dot{x}_2 \quad \text{— (ii)}$$



$$\begin{cases} i_{R_1} = i_L + i_{R_2} \\ u = R_1 i_{R_1} + L\dfrac{di_L}{dt} + R_3 i_L \\ R_2 i_2 + v_c - R_3 i_L - L\dfrac{di_L}{dt} = 0 \end{cases}$$

$$i_{R_2} = C\frac{dv_c}{dt} \quad \text{assuming } C = \text{constant}$$

(i) — $U = R_1(x_1 - C\dot{x}_2) + (L\dot{x}_1 + R_3x_1)$

(ii) — (KCL) $(L\dot{x}_1 + R_3x_1) = C\dot{x}_2R_2 + x_2$

$$L\dot{x}_1 = -R_3x_1 + C\dot{x}_2R_2 + x_2$$

(KVL) (i) $\Rightarrow U = R_1x_1 - R_1C\dot{x}_2 + L\dot{x}_1 + R_3x_1$

$$U = R_1x_1 - R_1C\dot{x}_2 + (-R_3x_1 + C\dot{x}_2R_2 + x_2) + R_3x_1$$

$$U = R_1x_1 + x_2 - (R_1C + R_2C)\dot{x}_2$$

(KVL)

$$\dot{x}_2 = \frac{1}{C(R_1-R_2)} + \frac{R_1}{C(R_1+R_2)}x_1$$

$$\begin{bmatrix} \dfrac{dV_c}{dt} \\ \dfrac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{C} & -\dfrac{R_1}{C} \\ \dfrac{R_1}{L} & \dfrac{-(R_1R_3+R_2R_3+R_1R_2)}{L} \end{bmatrix} \frac{1}{(R_1+R_2)} \begin{bmatrix} V_c \\ i_L \end{bmatrix}$$

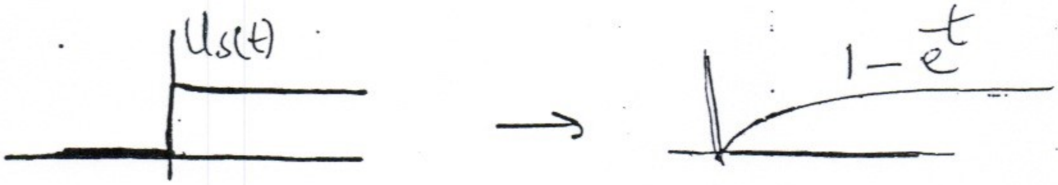
$$\frac{1}{R_1+R_2}\begin{bmatrix} Y_c \\ R_2/L \end{bmatrix}$$

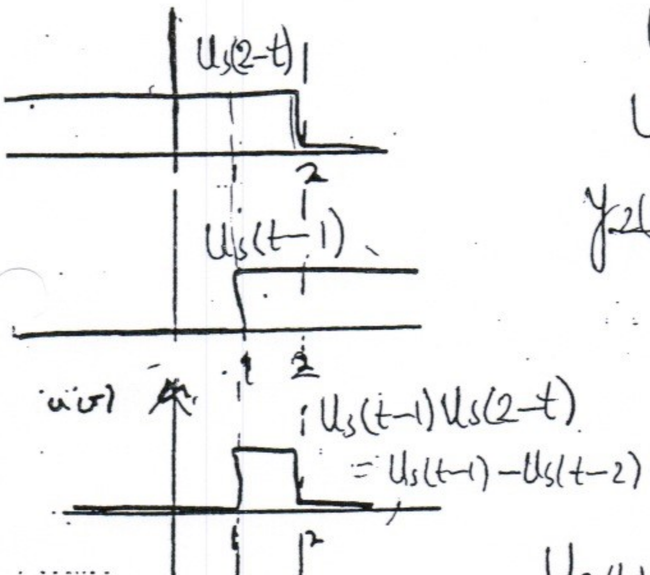
Problem 1.14 De Carlo

$$u = u_s(t) \longrightarrow y(t) = \left(1 - e^{-t}\right) u_s(t)$$

$$u = u_s(t-1)\, u_s(2-t) \rightarrow y(t) = ? \qquad t \le 2$$

since it is causal, we need ~~not~~ concern $t < 0$





$$U = U_s(t) \longrightarrow y(t) = (1 - e^{-t})\, U_s(t)$$

$$U = U_s(t-1)\, U_s(2-t) \longrightarrow y_2(t) = ?$$

$$\cancel{y_2(t) = \left(1 - e^{-(t-1)}\right)\left(1 - e^{-(2-t)}\right) U_s(t)}$$

$$\cancel{= \left(1 - e^{-(2-t)} - e^{-(t-1)} + e^{-t+1-2+2t}\right)} U_s(t)$$

$$\cancel{\left(1 - e^{-2+t} - e^{-t+1} + e^{t-1}\right)} U_s(t)$$

$$U_2(t) = U_s(t-1) \cdot U_s(2-t) = U_s(t-1) - U_s(t-2)$$

$$U_s(t-1) = \left[1 - e^{-(t-1)}\right] U_s(t-1)$$

$$U_s(t-2) = \left[1 - e^{-(t-2)}\right] U_s(t-2)$$

$$U_s(t-1) - U_s(t-2) = \left[1 - e^{-(t-1)}\right] U_s(t-1) - \left[1 - e^{-(t-2)}\right] U_s(t-2)$$

$$y(t) = \left[1 - e^{-(t-1)}\right] U_s(t-1) \cdot \left[1 - e^{-(2-t)}\right] U_s(2-t)$$

$$= \left[1 - e^{-(2-t)} - e^{-(t-1)} + e^{-t+1-2+t}\right] U_s(t-1)\, U_s(2-t)$$

$$= \left[1 - e^{-(2-t)} - e^{-(t-1)} + e^{-1}\right] U_s(t-1)\, U_s(2-t) \qquad \text{Ans}$$

ENGR 471/613 1^{st} Hour Exam Oct. 13, 200c

1. Given the following system where $u(t)$ is the input and $y(t)$ is the outp

(20%)

$$\begin{cases} \dot{x}_1(t) = x_1(t-1) + |u(t)| \\ \dot{x}_2(t) = x_1(t) \end{cases}$$

$$y(t) = x_1(t+1) + x_2(t) - u(t)$$

Find the appropriate description of the system among:

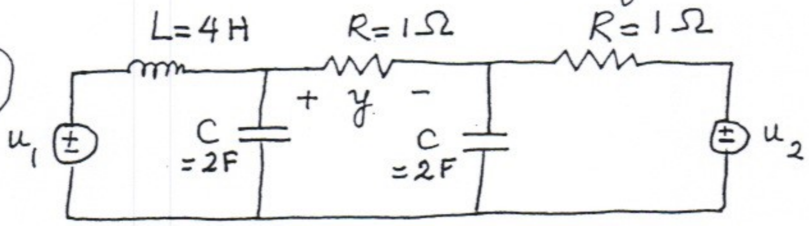
Linear	Time invariant	Causal	Lumped	Continuous-tim
Nonlinear	Time variant	Noncausal	Distributed	Discrete-time

impulse RESPONSE

Justify your answer briefly.

Write the state and output equations for the following network

(30%)



where u_1 and u_2 are input signals and y is the output signal.

. Consider a linear system with the following response to $\delta(t-\tau)$

(20%)

$$h(t,\tau) = 2u(t-\tau) - u(t-3\tau) - u(t-4\tau)$$

a) Is this system time invariant? (show all work).

b) Is it causal? (show all work).

Given the input-output differential equation

(%)

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = \dot{u} + 5u$$

Obtain the controllable and Jordan canonical forms (show all work).

ENGR 471 FIRST HOUR EXAM Oct. 12, 19

CLOSED BOOK AND CLOSED NOTES

1. Given the following system of equations where $u(t)$ is the input and $y(t)$ is the output:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t-1) + |u(t)| \\ x_1(t) \end{bmatrix}$$

$$y(t) = x_1(t+1) + x_2(t) - u(t)$$

Find the appropriate description of the system among:

Linear	Time invariant	Causal	Lumped	Continuous-t
Nonlinear	Time Variant	Non causal	Distributed	Discrete-ti

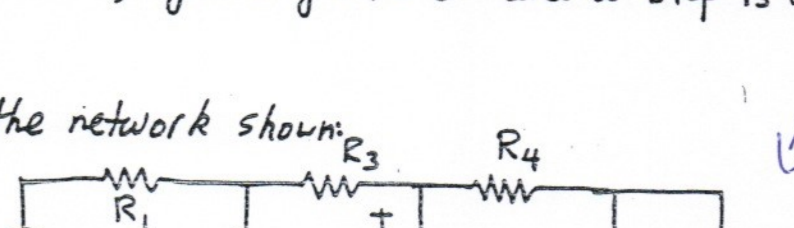
Justify your answer briefly.

2. A system has the differential equation

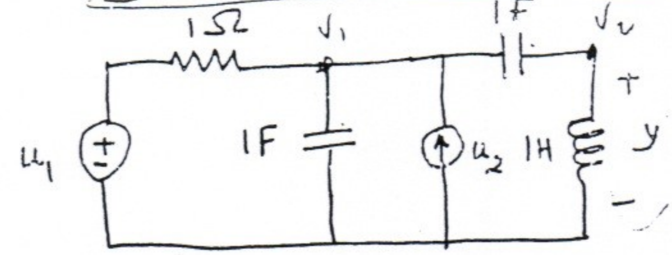
$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = \dot{u} + 5u$$

a) Obtain its transfer function and its impulse response. (b) Find the step response of the relaxed system. (c) If at $t=1$ the conditions are $y(1) = 1$, $\dot{y}(1) = \ddot{y}(1) = 0$ and a step is applied, find $y(t)$ for $t \geq$

3. Consider the network shown:



Consider the circuit shown below. Write the state and output equations using the voltage across the inductor as the output.



2. The response of a LTI causal system to the input

$u_1(t)$ is $e^{-t} u_s(t) = y_1(t)$

The response to a second input u_2 is $y_2(t) = \cos 2t \; u_s(t)$

where $u_s(t)$ is the unit step. Compute the response to the signal

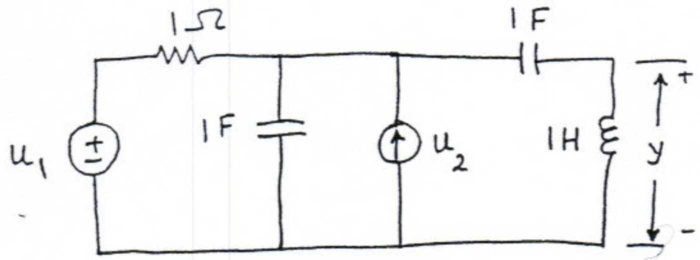
$$u_3(t) = 2u_1(t) + \frac{du_2}{dt}(t+1).$$

3. Verify whether the following map is (i) linear or nonlinear, (ii) time-invariant or time-varying and (iii) causal or non-causal (show all work)

$$y(t) = \int_{-\infty}^{t} \sin(t-\tau)\, e^{u(\tau)}\, d\tau \longrightarrow Q$$

471/613 First Hour Exam Oct. 24, 97

rite the state and output equations for the network shown below.



e response of a LTI causal system to the input

$u_1(t)$ is $y_1(t) = e^{-t} u_s(t)$

response to a second input $u_2(t)$ is $y_2(t) = \cos 2t \; u_s(t)$,

ere $u_s(t)$ is the unit step input. Compute the response to the signal

$$u_3(t) = 2\,u_1(t-1) + \frac{du_2}{dt}(t+1) .$$

erify whether the following map is

(i) linear or nonlinear
(ii) time-invariant or time varying
(iii) Causal or non-causal

$$(t) = \int_{-\infty}^{t} \sin(t-\tau)\, e^{u(\tau)}\, d\tau$$

w all work and provide reasoned arguments.

ELEC 481 / ENGR 6131

Midterm Exam #1

(1) Verify whether the input-output map

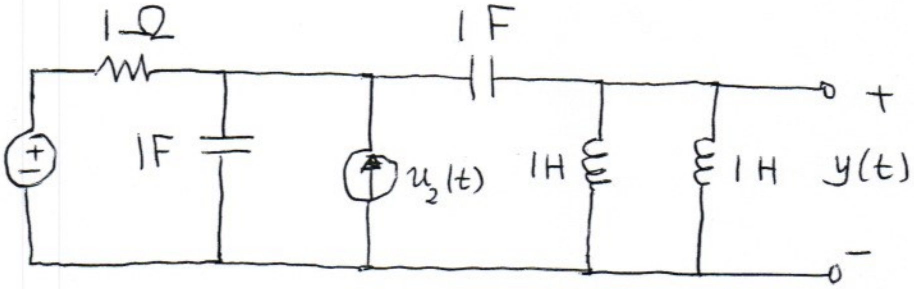
25 b

$$y(t) = \int_{-\infty}^{t} \sin(t-\tau)\, e^{u(\tau-1)}\, d\tau$$

is (i) linear or nonlinear, (ii) time-invariant or time-varying, (iii) causal or noncausal. Show all work and provide reasoned arguments.

(2) Write the state and output equations for the network shown below. Also, obtain the A, B, C, and D matrices.

25 b



(3) The response of a LTI causal system to the input $u_1(t)$ is $y_1(t) = e^{-2t}\sin(t-1)\,u_s(t)$, where $u_s(t)$ is the unit step input. The response to a second input $u_2(t)$ is $y_2(t) = \cos 2t\, u_s(t)$.

25 b

Compute the response to $u_3(t) = 2u_1(t-1) + u_2(t-2) + \dfrac{du_2(t+1)}{dt}$.

(4) Consider the LTI system
$$\dot{x} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

25 b

Let $u(t) = \begin{bmatrix} 4t^3 \\ -t^3 \end{bmatrix}$ for $t \geq 0$, and suppose $y(t) = \cos t - 3\sin t + t^3$.

Find $x(\pi/2)$?

ENGR 1[illegible]3 Midterm Exam I Oct. 23, 96

1) Co[illegible] linear system with the following response to $\delta(t-\tau)$

(35%) $h(t,\tau) = 2u(t-\tau) - u(t-3\tau) - u(t-4\tau)$

a) Is this system time invariant? (Show all work)

b) Is it causal? (show all work)

$x_2(t) = \Phi(t, t_0) \cdot x_1$

2) Suppose $\dot{x}(t) = A(t)\, x(t)$ where

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{pmatrix}$$

a) Find $x_1(t)$ and $x_2(t)$ in terms of the state transition matrix

(40%)

$$\Phi(t,t_0) = \begin{pmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ \Phi_{21}(t,t_0) & \Phi_{22}(t,t_0) \end{pmatrix}$$

b) Show that the associate state transition matrix has the form

$$\Phi(t,t_0) = \begin{pmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ 0 & \Phi_{22}(t,t_0) \end{pmatrix}$$

where $\dot{\Phi}_{ii}(t,t_0) = A_{ii}(t)\, \Phi_{ii}(t,t_0)$ for $i = 1,2$.

c) Find an expression for $\Phi_{12}(t,t_0)$.

3) Let the state transition matrix of a particular system have the form

(25%)

$$\Phi(t,0) = e^{-t}\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Find $\Phi(t,t_0)$ and $A(t)$.

a

$$x_1(t) = \Phi_{11}(t,t_0)\, x_1(t_0) + \Phi_{12}(t,t_0)\, x_2(t_0)$$

$$x_2(t) = \Phi_{21}(t,t_0)\, x_1(t_0) + \Phi_{22}(t,t_0)\, x_2(t_0)$$

b

$$\dot{\Phi}_{11}(t,t_0) = a_{11}\, \Phi_{11}(t,t_0)$$

$$\dot{x}_1(t) = a_{11}(t)\, x_1(t) + a_{12}(t)\, x_2(t)$$

$$\dot{x}_2(t) = a_{22}(t)\, x_2(t)$$

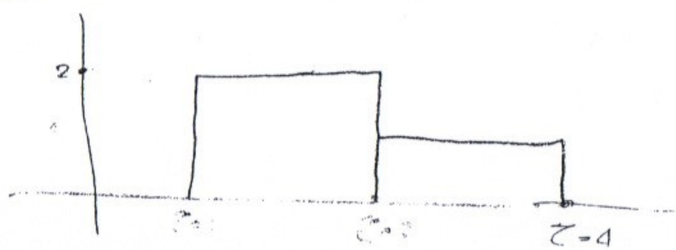
c

$$\Phi_{12}(t,t_0)$$

$$\dot{x}_1 = a_{11} x_1 + a_{12}\, \cancel{\Phi_{22}(t,t_0)}\, x_2(t_0)$$

$$* \quad x_1(t) = \Phi_{11}(t,t_0)\, x_1(t_0) + \int_{t_0}^{t} \Phi_{11}(t,\tau)\, a_{22}$$

$$\Phi_{12}(t,t_0) = \int_{t_0}^{t} \Phi_{11}(t,\tau)\, a_{12}(\tau)\, \Phi_{22}(\tau,t_0)\, d\tau$$



$[A_{11}$

when $\tau = 0$ $\quad h = 0 = h(t,\tau)$

$$\dot{\Phi} = A\, \Phi,$$

t

ENGR 471

Oct. 23, 96

Midterm Exam I

1.) Given the input-output differential equation of a dynamical system

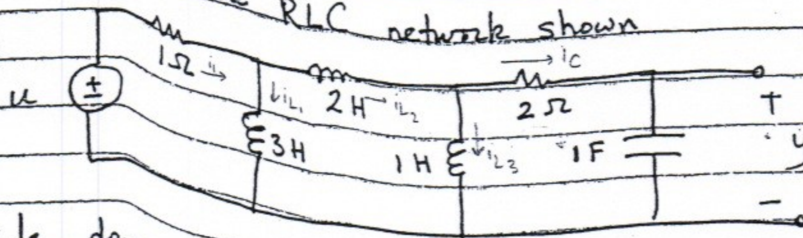
(45 pts) $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = \dot{u} + 5u$

a) Find the transfer function $\frac{Y(s)}{U(s)}$ and its impulse response.

b) Obtain the controllable and Jordan canonical forms

2.) Consider the RLC network shown

(35 pts)



Write down the state and output equations in the compact form.

3.) Find the fundamental matrix $M(t)$ and the state transition matrix $\Phi(t, o$ for the following system

(20 pts) $$\dot{x} = \begin{bmatrix} -3 & e^{2t} \\ 0 & -2 \end{bmatrix} x .$$

$$-\frac{di_{L_3}}{dt} + 2i_c + V_c = 0$$

$$i_{L_2} = i_{L_3} + i_C$$

$$-3\frac{di_{L_1}}{dt} + 2\frac{di_{L_2}}{dt} + \frac{di_{L_3}}{dt} = 0$$

$$i_{L_2} - i_{L_3}$$

$$-u + i_1 + 3\frac{di_L}{dt} = 0 \Rightarrow \boxed{-u + i_{L_1} + i_{L_2} = -3\frac{di_{L_1}}{dt}}$$

$$i_1 = i_{L_1} + i_{L_2}$$

$$\boxed{2i_{L_2} - 2i_{L_3}}$$

$$-u + i_{L_1} + i_{L_2} + 2\frac{di_{L_2}}{dt} + 2i_c + V_c = 0$$

ECR 471

First Exam

The response of a LTI causal system to the input $u_1(t) = u_s(t)u_s(2-t)$

s $y(t) = (1-e^{-t})u_s(t)u_s(2-t) + e^{-(t-2)} u_s(t-2)$. Find the response of the system to the input $u_2(t) = (t-2)u_s(t-2)u_s(4-t)$ for $t \le 4$.

Note $u_s(t)$ is the unit step input.

Consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u$$

$2X(0) - 1$

b) $y = \begin{bmatrix} 1 & -1 \end{bmatrix} x$

t $u(t) = \begin{bmatrix} -6t \\ 6t \end{bmatrix}$ for $t \ge 0$ and suppose that $y(t) = 3 - 3t^2 + t^3$

$x(0)$ $\quad SX(S) - X(0)$

Find $x(1)$?

$X = (SI$

Given

b) $$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t-1) + |u(t)| \\ x_1(t) \end{bmatrix}$$

$$y(t) = x_1(t+1) + x_2(t) - u(t)$$

ind the appropriate description of the system among:

✗ Linear	Time-Invariant ✓	Causal	Lumped	✓ Continuous-time
Nonlinear ✓	Time Variant	✓ Non causal	Distributed ✓	Discrete-time

Suppose $\Phi(t,0) = \begin{bmatrix} e^{-2t}(\sin t + 1) & e^{-2t}(\cos t - 1) \\ -e^{-2t}(\cos t - 1) & e^{-2t}(\sin t - 1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \Phi(t,t) \ne I$

) mod 2

Is the above matrix a valid state transition matrix? Verify your answer.

ELEC 481 / ENGR 6131/2

Midterm Exam #1

ALL PROBLEMS ARE EQUALLY WEIGHTED

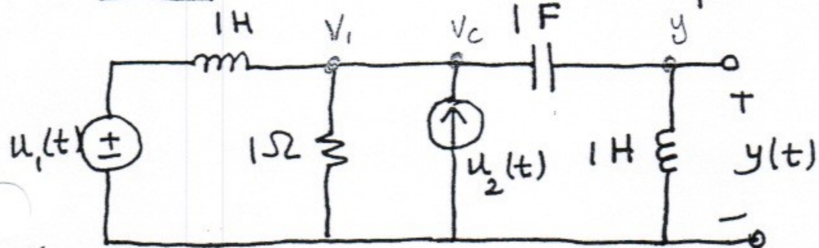
(1) Verify whether the input-output map

$$y(t) = \int_{+t_0}^{t} e^{\sin(t-\tau+1)} \cos(u(\tau)-1)\, d\tau \qquad t_0 \le t$$

(i) linear or nonlinear, (ii) time-invariant or time varying, and (iii) causal or non-causal.

Show all work and provide reasoned arguments.

(2) Write the state and output equations for the network shown bel



Also, obtain the A, B, C, and D matrices.

(3) The response of a LTI causal system to the input $u_1(t)$ is $y_1(t) = e^{-2t}\sin(t-1)\,u_s(t)$, where $u_s(t)$ is the unit step input. The response to a second input $u_2(t)$ is $y_2(t) = \cos 2t\; u_s(t)$.

Compute the response to $u_3(t) = 2u_1(t-1) + \dfrac{d u_2(t+1)}{dt}$.

$\delta(t+)$

(4) Consider the state equation $\dot{x} = Ax$. It is known that the fundamental matrix at $t=3$ is $M(3) = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}$

and further $M(0) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Find the state transition matrix $\Phi(6,0)$.

ENGR 471/613 1^{st} Hour Exam Oct. 13, 2000

Given the following system where $u(t)$ is the input and $y(t)$ is the output:

$$\begin{cases} \dot{x}_1(t) = x_1(t-1) + |u(t)| \\ \dot{x}_2(t) = x_1(t) \end{cases}$$

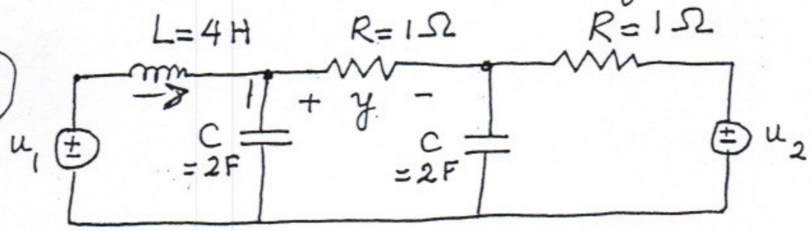
$$y(t) = x_1(t+1) + x_2(t) - u(t)$$

Find the appropriate description of the system among: 20

Linear	Time invariant	Causal	Lumped	Continuous-time
Nonlinear	Time variant	Noncausal	Distributed	Discrete-time

Justify your answer briefly.

Write the state and output equations for the following network



30

here u_1 and u_2 are input signals and y is the output signal.

Consider a linear system with the following response to $\delta(t-\tau)$

$$h(t,\tau) = 2u(t-\tau) - u(t-3\tau) - u(t-4\tau)$$

.) Is this system time invariant? (show all work). 10

) Is it causal? (show all work).

Given the input-output differential equation

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = \dot{u} + 5u$$

$$\frac{s+5}{s^3+6s^2+11s+6}$$ 20

Obtain the controllable and Jordan canonical forms (show all work).

Problem 1:

linear vs. nonlinear: non linear because | | is non linear

$$|u_1 + u_2| \neq |u_1| + |u_2|$$

TV vs TI

if we have $t x_1(t-1) \Rightarrow$ TV

but we have ~~time~~ const coeffecient $\Rightarrow$ Time Invariant

nonCausal because the output pres depends on future input

The delay in time causes infinite number of storage

So in the problem $x(t-1)$ causes the delay

if we have $\dot{x} = u(t-1)$ $\qquad$ $x(t-\tau) \rightarrow e^{-\tau s}$

$$\frac{X(s)}{U(s)} = \frac{e^{-s}}{s} \qquad e^{-s} = \qquad e^{s} = 1 + s + \frac{s^2}{2} + \frac{s^3}{3!} + \cdots$$

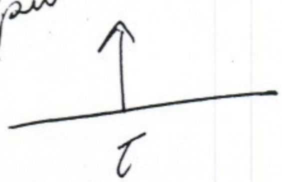
$\dot{x} = u(t)$ lumped

Problem 3:

$$h(t,\tau) = 2u(t-\tau) - u(t-3\tau) - u(t-4\tau)$$

Time variant because it is function of $t - 3\tau$ not just $t - \tau$

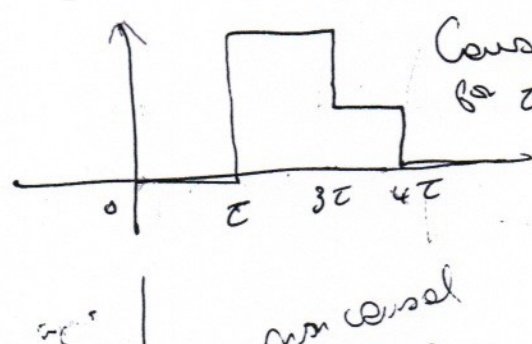
input



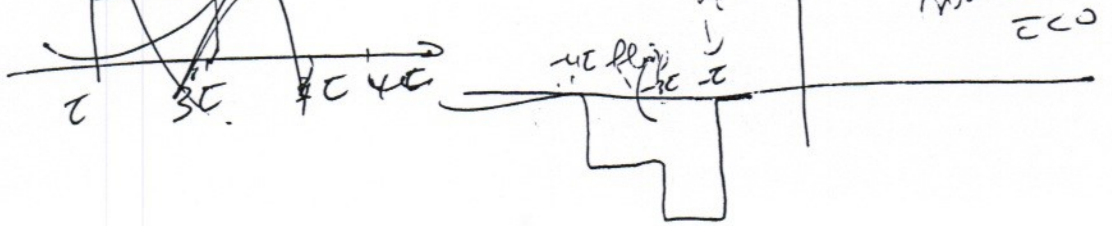
$h(t,\tau)$

set $\tau = 0$ $\qquad$ $h(t,0) = 0 \;\; \forall t$

$\tau \neq 0$ $\qquad$ $\neq 0$



Causal for $\tau > 0$



non causal $\tau < 0$

Apron

linear | multiplicati

ELEC 481 / ENGR 6131 / 2
Mid term Exam #1
ALL PROBLEMS ARE EQUALLY WEIGHTED

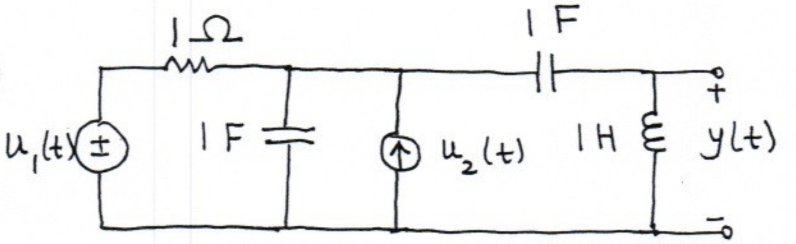
(1) Verify whether the input-output map

$$y(t) = \int_{-\infty}^{t} \sin(t-\tau)\, e^{u(\tau)}\, d\tau$$

(i) linear or nonlinear
(ii) time-invariant or time-varying
(iii) Causal or non-causal

Show all work and proide reasoned arguments.

(2) Write the state and output equations for the network shown below:



Also, obtain the A, B, C, and D matrices.

(3) The response of a LTI causal system to the input $u_1(t)$ is $y_1(t) = e^{-2t}\sin(t-1)\,u_s(t)$, where $u_s(t)$ is the unit step input. The response to a second input $u_2(t)$ is $y_2(t) = \cos 2t\; u_s(t)$. Compute the response to $u_3(t) = 2u_1(t-1) + \dfrac{d\,u_2(t+1)}{dt}$

(4) Consider the state equation $\dot{x} = Ax$. It is known that the fundamental matrix at $t=3$ is $M(3) = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}$ and further $M(0) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Find the state transition matrix $\longrightarrow \Phi(6,0)$.

(ii). <u>Check time invariance</u>.

Shift the input by T sec.

$$y_1(t-T) = \int_{-\infty}^{t} \sin(t-\tau) \cdot e^{u(\tau-T)} \cdot d\tau .$$

Let $\tau' = \tau - T$

$$\tau = \tau' + T \Rightarrow d\tau' = d\tau .$$

$$y_1(t-T) = \int_{-\infty}^{t-T} \sin(t-\tau'-T) \cdot e^{u(\tau')} \cdot d\tau'$$

(iii) Check Causality

Since system is not dependent to future inputs so system is Causal.

ELEC 481/ ENGR 6131

Midterm Exam

SAMZ

ALL PROBLEMS ARE EQUALLY WEIGHTED.

(1) The response of a LTI causal system to the input $u_1(t) = u_s(t)\, u_s(2-t)$ is $y_1(t) = (1-e^{-t})\, u_s(t)\, u_s(2-t) + e^{-(t-2)}\, u_s(t-2)$. Find the response of the system to a new input $u_2(t) = (t-2)\, u_s(t-2)\, u_s(4-t)$ for $t \le 4$. Note that $u_s(t)$ designates a unit step function.

(2) Consider a LTI system
$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} x$$

Let $u(t) = \begin{bmatrix} -3e^{-t} + 5e^{-2t} \\ 3e^{-2t} \end{bmatrix}$ for $t \ge 0$, and suppose that $y(t) = -2e^{-t} + 6e^{-2t}$. If $\lim_{t\to\infty} x(t) = 0$, find $x(0)$.

(3) Given $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t-1) + |u(t)| \\ x_1(t) \end{bmatrix}$; $y(t) = x_1(t+1) + x_2(t) - u(t)$

Find the appropriate description of the system among:

Linear	Time-Invariant	Causal	Lumped	Continuous-time
Nonlinear	Time-Variant	Noncausal	Distributed	Discrete-time

(4) Find a state space representation (A, B, C, D) for the system described by

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} =$ (2×3) 3×1

2×1

$H(s) = \dfrac{Y(s)}{U(s)}$

$$\ddot{y}_1 + 2\dot{y}_1 + 3(y_1 - y_2) = 2u_1 + \dot{u}_2$$

$Y(s) = H(s)\, \underset{\sim}{U}(s)$

$$\dot{y}_2 - 4(y_1 - y_2) = 2u_2 + 4\dot{u}_1$$

$e^{-2t} = -2e^{-7t}$

3×1 :

(3×2) 2×1

ENGR 471 / 613 Mid term Exam # 2 Nov. 17, 2000

1. A characteristic polynomial of a system is given by

(25%)

$$(\lambda - \lambda_1)^4 (\lambda - \lambda_2)^2 = 0$$

List all the possible Jordan Canonical forms of the system.

2. Given

(20%)

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \quad \text{and} \quad x(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Compute $x(2)$ when $u(t)$ = unit step.

3. Suppose the state trajectory for $\dot{x} = Ax + Bu$ is given by

(25%) solved Mid term 2

$$x(t) = \begin{bmatrix} e^{-t} & e^{-t} - e^{t} \\ 0 & e^{t} \end{bmatrix} x(0) + \begin{bmatrix} e^{t} - 1 \\ -e^{t} + 1 \end{bmatrix}$$

with $u(t)$ = unit step. Find A and B matrices of the system.

4. Let the state transition ~~matrix~~ matrix of a particular system have the form

(30%)

$$\Phi(t, 0) = e^{-t} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Find $\Phi(t, t_0)$ and $A(t)$.

Midterm Exam #2

NOV. 17, 20[illegible]

① A char. poly. of a system is given by

$$(\lambda-\lambda_1)^4(\lambda-\lambda_2)^2=0$$

List all possible Jordan Canonical forms of the system

In $(\lambda-\lambda_1)^4(\lambda-\lambda_2)^2=0$ we have total 6 eigenvalue

$\lambda=\lambda_1,\lambda_1,\lambda_1,\lambda_1$ & also $\lambda=\lambda_2,\lambda_2$

Case 1)

$$A=\begin{bmatrix}\lambda_1&&&&&\\&\lambda_1&&&&\\&&\lambda_1&&&\\&&&\lambda_1&&\\&&&&\lambda_2&\\&&&&&\lambda_2\end{bmatrix}$$

Nullity of $\lambda_1=4$
Nullity of $\lambda_2=2$

Case 2)

$$A=\begin{bmatrix}\lambda_1&&&&&\\&\lambda_1&&&&\\&&\lambda_1&&&\\&&&\lambda_1&&\\&&&&\lambda_2&1\\&&&&&\lambda_2\end{bmatrix}$$

Nullity $\lambda_1=4$
$\lambda_2=1$

$$A=\begin{bmatrix}\lambda_1&1&&&&\\&\lambda_1&&&&\\&&\lambda_1&&&\\&&&\lambda_1&&\\&&&&\lambda_2&\\&&&&&\lambda_2\end{bmatrix}$$

Nullity $\lambda_1=3$
Nullity $\lambda_2=2$

$$A=\begin{bmatrix}\lambda_1&1&&&&\\&\lambda_1&&&&\\&&\lambda_1&&&\\&&&\lambda_1&&\\&&&&\lambda_2&1\\&&&&&\lambda_2\end{bmatrix}$$

Nullity $\lambda_1=3$
Nullity $\lambda_2=1$

$$A=\begin{bmatrix}\lambda_1&1&&&&\\&\lambda_1&&&&\\&&\lambda_1&1&&\\&&&\lambda_1&&\\&&&&\lambda_2&\\&&&&&\lambda_2\end{bmatrix}$$

Nullity $\lambda_1=2$, $\lambda_2=2$

$$A=\begin{bmatrix}\lambda_1&1&&&&\\&\lambda_1&&&&\\&&\lambda_1&1&&\\&&&\lambda_1&&\\&&&&\lambda_2&1\\&&&&&\lambda_2\end{bmatrix}$$

Nullity $\lambda_1=2$, $\lambda_2=1$

$$A=\begin{bmatrix}\lambda_1&1&&&&\\&\lambda_1&1&&&\\&&\lambda_1&1&&\\&&&\lambda_1&&\\&&&&\lambda_2&0\\&&&&&\lambda_2\end{bmatrix}$$

Nullity $\lambda_1=1$ [illegible]

$$A=\begin{bmatrix}\lambda_1&1&&&&\\&\lambda_1&1&&&\\&&\lambda_1&1&&\\&&&\lambda_1&&\\&&&&\lambda_2&1\\&&&&&\lambda_2\end{bmatrix}$$

[illegible]

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\Phi(t,t_0) = e^{t_0 - t}\begin{bmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix}$$

$e^{-(t-t_0)}$

For $A(t) = \dot{\Phi}(t,t_0)\Big|_{t_0=t}$ than found At

$$A(t) = \frac{d}{dt}\left[e^{t_0 - t}\begin{pmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{pmatrix}\right]_{t_0=t}$$

$$= e^{t_0-t}\begin{bmatrix} -\sin(t-t_0) & \cos(t-t_0) \\ -\cos(t-t_0) & -\sin(t-t_0) \end{bmatrix}$$

$$- \begin{bmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix} e^{t_0-t}\Big|_{t_0=t}$$

$$= 1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boxed{A(t) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}}$$

(c) $X(2)$ when $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\Phi(t,t_0) = e^{t_0-t}\begin{bmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix}$$

$$x(2) = \Phi(t,t_0)\, x(0)$$

$$x(2) = \Phi(2,0)\, x(0)$$

$$\Phi(2,0) = e^{-2}\begin{bmatrix} \cos 2 & \sin 2 \\ -\sin 2 & \cos 2 \end{bmatrix}$$

$$x(2) = e^{-2}\begin{bmatrix} \cos 2 & \sin 2 \\ -\sin 2 & \cos 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} x_1(2) \\ x_2(2) \end{pmatrix} x(2) = e^{-2}\begin{bmatrix} \cos 2 + \sin 2 \\ \cos 2 - \sin 2 \end{bmatrix}$$

ELEC 481 / ENGR 6131/2

Midterm Exam #1

ALL PROBLEMS ARE EQUALLY WEIGHTED

Oct, 2006

(1) Verify whether the input-output map

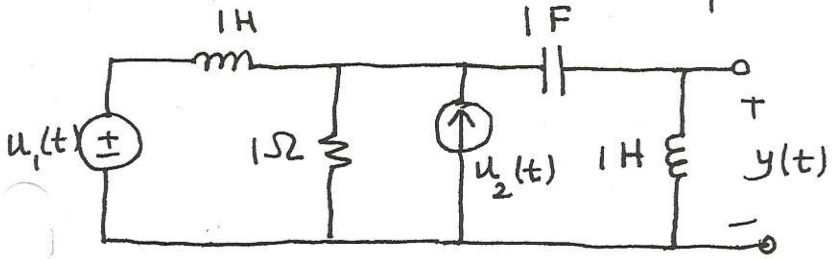
$$y(t) = \int_{t_0}^{t} e^{\sin(t-\tau+1)} \cos(u(\tau)-1)\, d\tau \qquad t_0 \le t$$

(i) linear or nonlinear (ii) time-invariant or time varying, and (iii) causal or non-causal.

because $t_0 - T$

Show all work and provide reasoned arguments.

(2) Write the state and output equations for the network shown below



Also, obtain the A, B, C, and D matrices.

$$\begin{pmatrix} -1 & 0 & +1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \end{pmatrix} \quad [0 \;\; 1]$$

(3) The response of a LTI causal system to the input $u_1(t)$ is $y_1(t) = e^{-2t}\sin(t-1)\,u_s(t)$, where $u_s(t)$ is the unit step input. The response to a second input $u_2(t)$ is $y_2(t) = \cos 2t\; u_s(t)$. Compute the response to $u_3(t) = 2u_1(t-1) + \boxed{\dfrac{du_2(t+1)}{dt}}$ = 0 because system causal

(4) Consider the state equation $\dot{x} = Ax$. It is known that the fundamental matrix at $t=3$ is $M(3) = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}$ and further $M(0) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Find the state transition matrix $\Phi(6,0)$.

$\Phi(6,0) = \Phi(3,0)\Phi(3,0)$ $\begin{pmatrix} 29 & -12 \\ -12 & 5 \end{pmatrix}$

find $\Phi(3,0) = M(3)\, M^{-1}(0)$

ENGR 471/2 Hour Exam #2 Nov. 2

Given $\Phi(t,0) = \begin{bmatrix} e^{-2t}\cos 3t & -e^{-2t}\sin 3t \\ e^{-2t}\sin 3t & e^{-2t}\cos 3t \end{bmatrix}$

35%

Find $\Phi(t,t_0)$ and $A(t)$.

For a LTV system

35%

$$\dot{x} = \overbrace{\begin{bmatrix} -2 & e^{4t} \\ 0 & -4 \end{bmatrix}}^{A(t)} x$$

1- $M(t)$
$\Phi(t,t_0) = M(t)\, M^{-1}(t_0)$
$\Phi(t,0) = M(t)\, M^{-1}(0)$

[Fin]d the Fundamental matrix $M(t)$ and $\Phi(t,0)$.

Given

30%

$$\dot{x} = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

the design objectives is to have the closed-loop poles at $\lambda_1 = -5$ and $\lambda_2 = -6$. Show to what extend one can mee[t] the design objectives. Show all work. [Carry out the design a[s] far as it can be taken].

9

71 L, 13 2nd Exam Nov. 20, 96

$$\dot{x} = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

design objective is to have the closed-loop poles at $\lambda = -5$ and $\lambda = -6$.

to what extend one can meet the desig. objectives. Show all work.

Carry out the design as far as it can be taken). Use $u = -Kx$ and discuss your results.

feedback

en $\dot{x} = A(t)x$ find the state transition matrix $\Phi(t, t_0)$ when

$$A(t) = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix}$$. ~~Hint:~~ First Compute $\Phi(t, 0)$ by defining

$$(t) = \int_0^t \omega(\tau)\, d\tau$$

characteristic polynomial is $(s - \lambda_1)^4 (s - \lambda_2)^2 = 0$. List all the

ible Jordan Canonical forms of the matrix.

$= n - \text{Rank}(\lambda I - A)$

5

ven

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \quad \text{and} \quad x(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

pute $x(2)$ when $u(t) =$ units step.

5 1 4 2 3 3 2 2 3 2

$$Ac = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

C_X Ac $(\lambda I - A)$

$$J_c = (\lambda+1)(\lambda+2) \begin{bmatrix} \lambda+1 & -3 \\ 0 & \lambda+2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

ELEC 481 / ENGR 6131 Exam 2 Nov. 23, 05

1. Given
$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$
(20%)
$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} x$$

det controllability & observability matrix $\neq 0$

$b_2 \neq 0$

$\exists\, b_1 \neq \frac{b_2}{2}$

(a) Under what conditions on b_1 & b_2 the system is controllable

(b) Under what conditions on c_1 & c_2 the system is observable

2. Let the state transition matrix of a system be given by

(30%)
$$\Phi(t,0) = \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

(a) Find $\Phi(t, t_0)$; (b) Find the A matrix; (c) Find $x(2)$ when $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. Let the state trajectory of $\dot{x} = Ax + Bu$, $y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$ be given by

(30%)
$$x(t) = \begin{bmatrix} e^{-t} & e^{-t} - e^{t} \\ 0 & e^{t} \end{bmatrix} x(0) + \begin{bmatrix} e^{t} - 1 \\ -e^{t} + 1 \end{bmatrix}$$

with $u(t)$ = unit step. (a) Find the transfer function of the system; (b) Find the A & B elements of the system.

4. Consider the matrix

(20%)
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$
Find A^s, where s is a real number.

ELEC 481 / ENGR 6131 Exam #2 Abrems Nov. 28, 07

① A characteristic polynomial is $(\lambda - \lambda_1)^5 = 0$. List all the possible Jordan Canonical forms of the matrix and for each matrix find the number of linearly independent eigenvectors.

② Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -2 & -2 \\ -1 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.5 & 1 \end{bmatrix} x$$

Find if the system is controllable and observable. Show all work.

③ Let the state transition matrix of a system be given by

$$\Phi(t,0) = \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

(a) Compute $\Phi(t, t_0)$. (b) Find A. (c) Find $x(2)$ when $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

④ Find the Fundamental and the State Transition matrices $M(t)$ and $\Phi(t,0)$ for the following system

$$\dot{x}(t) = \begin{bmatrix} -3 & e^{-2t} \\ 0 & -2 \end{bmatrix} x(t).$$

ALL PROBLEMS ARE EQUALLY WEIGHTED.

ELEC 481/ENGR 6131 EXAM #2 NOV. 26, 08

1. Given $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ find A^{99} by using the Caley-Hamilton technique.

2. Given

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} x$$

(a) What conditions on b_1, b_2, b_3 make the system controllable?

(b) What conditions on c_1, c_2, c_3 make the system observable?

3. Find the fundamental matrix $M(t)$ and the state transition matrix $\Phi(t,0)$ for the following system

$$\dot{x} = \begin{bmatrix} -3 & e^{2t} \\ 0 & -2 \end{bmatrix} x$$

4. Let the state transition matrix of a system be given by

$$\Phi(t,0) = \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

(a) Compute $\Phi(t, t_0)$

(b) Find A.

Note: ALL PROBLEMS ARE EQUALLY WEIGHTED.

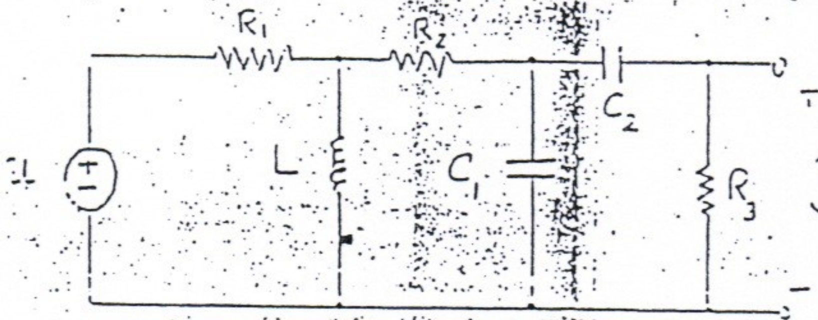
$x_{11} = \frac{1}{3}$. ⟹ $\dot{x}_{11} = 0$

$\dot{x}_{11} = -1 + 1 = 0$

ENGR 471
Linear Systems
Fall 1990 (Final)

2 Final

1. Consider the network shown in Figure 1. Write the state and output equations. Find the transfer function of the network. Let $R_1 = R_2 = R_3 = 1$, $L = 2$ and $C_1 = C_2 = 1$.



mid 2

2. A characteristic equation is $(S - \lambda_1)^5 = 0$. List all the possible Jordan Canonical Forms of the matrix and with each matrix find the number of linearly independent eigenvectors.

(14) possibility

3. Consider the open loop transfer function of a linear system as:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s^2\,(s-1)}$$

where the input is generated by state feedback

$$u(t) = r(t) - K\,x(t),$$

find the gain vector K such that the following design specifications are satisfied simultaneously: (1) zero steady-state-error with r(t) = unit step, (2) two of the dominant closed loop poles at $-1 \pm j$. Find the location of the third pole under the above design conditions.

mid 2

4. Given the transfer function,

cont & obs

$$G(s) = \frac{2s + 5}{s^3 + 11s^2 + 10s + 15}$$

express the state equation in controllable and observable canonical forms and check to see if the system is controllable and observable.

5. Given the system,

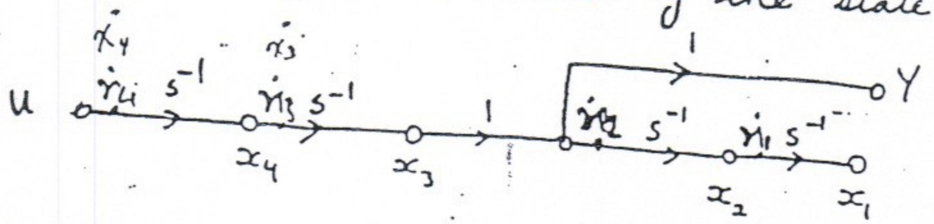
$$\dot{x} = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = x_1$$

design a full order observer having eigenvalues at -5 and -7. Draw a detailed all-integrator block diagram for the full order observer (not in matrix form).

reyeated ... EXAM # 3 Dec. 4,

1) Consider the system described by the state diagram

25 pts



a) Write the system state equations.
b) Find the transfer function $Y(s)/U(s)$.
c) Determine if the system is controllable and observable.
(d) Compare part (b) and (c)

2) Consider

50 pts

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$(s+1)(s+5)$

a) Find a state feedback $u = -Kx + r$ such that the desired closed-loop poles are at -1 and -8.
b) Assume that the state vector x is not accessible. Design an observer with a settling time of at most 4 sec. (Full order or Reduced order).

$\zeta = 1$ $\quad \tau = 4\,sec \quad \omega_n =$ $\quad s^2 + 2\zeta\omega_n s + \omega_n^2 \quad G = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$

$\zeta\omega_n = \frac{1}{\tau} \quad 0.25 \quad s^2 + 2s + 1$

3) Given

25 pts

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \quad ; \quad x(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Compute $x(2)$ when $u(t) = u_s(t)$.

$(Is - A + BK)$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \\ 2k_1 & 2k_2 \end{bmatrix}$$

$y = Cx + Du$

ENGR 471 FINAL EXAM Dec. 12,

1/2

1. Two pendulums coupled by a spring are to be controlled by two equal and opposite forces u, which are applied to the pendulum bobs as shown in figure 1. The equations of motion are

35 pts

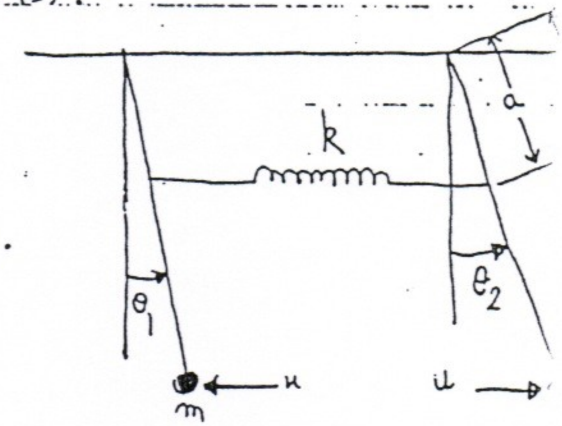
$$m l^2 \ddot{\theta}_1 = -k a^2 (\theta_1 - \theta_2) - m g l \theta_1 - u$$

$$m l^2 \ddot{\theta}_2 = -k a^2 (\theta_2 - \theta_1) - m g l \theta_2 + u$$

states θ_1, $\dot{\theta}_1$, $\dot{\theta}_2$

a) Draw the simulation (state) diagram of the above set of e
b) Find the state equation and output equation using θ_1, θ_2, $\dot{\theta}_1$, $\dot{\theta}_2$ as states, u as input and θ_1 and θ_2 as outputs. Obtain A, B, C and D matrices.
c) Is this system controllable and observable (show all work)?
d) Find the transfer function $\theta_1(s)/U(s)$.

Note: For simplicity take $m=1$, $l=1$
$k=1$, $a=1$ & $g=1$ for parts c) & d).



2. Consider the system $A = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and assume that we are using state feedback of the form $u = -K$
where r is the reference input signal.

20 pts

a) Show that (A, C) is observable.
b) Show that there exists a K such that (A−BK, C) is unobservable
c) Compute a K as in (b) such that $K = [1 \ \ k_2]$, that is, find k_2
d) Compute the open loop and closed-loop transfer functions. What is the unobservability due to?

OVER

ENGR 471 FINAL EXAM Dec

mod 2

3. Suppose $\dot{x}(t) = A(t)\,x(t)$ where

(25 b)

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix}$$

ϕ_{11} ϕ_{21} ϕ_{21} ϕ_{21} ϕ_{ii}

a) Find $x_1(t)$ and $x_2(t)$ in terms of the state transition m

$$\Phi(t,t_0) = \begin{bmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ \Phi_{21}(t,t_0) & \Phi_{22}(t,t_0) \end{bmatrix}$$

b) Show that the associate state transition matrix has the form

$$\Phi(t,t_0) = \begin{bmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ 0 & \Phi_{22}(t,t_0) \end{bmatrix}$$ odr 23, 96

where $\dot{\Phi}_{ii}(t,t_0) = A_{ii}(t)\,\Phi_{ii}(t,t_0)$ for $i = 1, 2$.

mod 2 c) Find an expression for $\Phi_{12}(t,t_0)$.

16
16
20 10
18

4. For the system e^{At}

(20 b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2t & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad ; \quad x(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Compute $\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix}$ given that $u(t) = u_s(t)$ $[sI-A]^{-1}$

$\begin{bmatrix} 1 \\ -4 \end{bmatrix}$

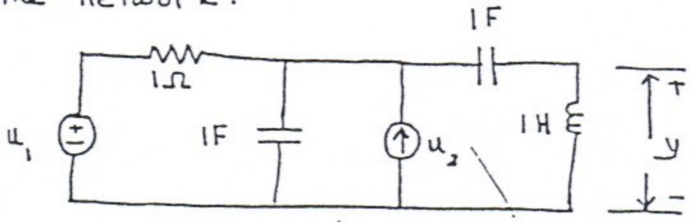
$x(t) = e^{\int_{t_0}^{t} a(\tau)\,d\tau}\, x(t_0)$ for $t_0 = 1$

$x(t) = \Phi(t,t_0)\,x(t_0)$ $\Phi(2,1)$

$x(2) = \Phi(2,1)\;x(1)$

JGR 471 Final Exam Dec. 21, 89

1. Consider the network shown in figure 1. Write the state and output equations. Find the input-output description of the network, i.e. the transfer function matrix of the network.



mid2

2. Consider the dynamical equation

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

Which modes are controllable and observable? cont & unob

3. Consider the transfer function mid2

$$G(s) = \frac{4s^3 + 25s^2 + 45s + 34}{2s^3 + 12s^2 + 20s + 16} = 2 + \frac{s^2 + 5s + 2}{2s^3 + 12s^2 + 20s + 16}$$

D_c

Obtain the controllable and observable canonical forms.

$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -16 & -20 & -12 \end{bmatrix}$

$B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $C_c = [2 \ 5 \ 1]$

$D_c = 2$

4. Consider the state equation

$$\dot{x} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u$$

Is it possible to find a gain vector K such that with state feedback $u = Kx$ closed loop system has eigenvalues −1, −1, −2, −2 and −2? Is it possible have eigenvalues −2, −1, −1, −1, and −1? Explain your answers.

5. Consider the system

$$\dot{x} = \begin{bmatrix} -3 & 10 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Design a state feedback law such that for the closed loop system, $\omega_n = 29.7$ and $\zeta = .7$. Find the steady state error ($e = y - r$) to a unit step input $r = 1$. Note: $u = Kx + r$.

$E_{ss} = \lim e(t) = \lim s \left[\frac{R(s)}{1 + G(s)H(s)} \right]$

A ttempt ALL 4 problems

ELEC 481 / ENGR 6131
Hour Exam #2

1. A Characteristics equation is given by $(s-\lambda_1)^5 = 0$. List all the possible Jordan Canonical Forms of the matrix associated with this equation and with each matrix find the number of linearly independent eigenvectors.

20%

2. Is the following system controllable and observable? Specify which modes are controllable and observable and which modes are not.

20%

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ & & 1 & 1 & 0 & & \\ & & 0 & 1 & 0 & & \\ & & 0 & 0 & 1 & 0 & 0 \\ & & & & & 2 & 0 \\ & & & & & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 & 3 \\ -1 & 0 & 1 \\ 2 & 3 & 9 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} u$$

$\lambda_1 = 1$ NC, NO

$\lambda_2 = 2$ NC, O

$$y = \begin{bmatrix} 5 & 2 & 1 & 1 & 1 & 1 & 0 \\ -2 & 4 & -1 & -1 & 0 & 0 & 1 \\ 3 & 3 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} x$$

3. The following system is to be analyzed for a delayed ramp input

$$t^2 \ddot{y} + t\dot{y} - y = u$$

with $u(t) = \begin{cases} t-1 & \text{for } t \geq 1 \\ 0 & \text{for } t < 1 \end{cases}$ and $y(1) = \dot{y}(1) = 1$. Find (a) the fundamental matrix, (b) state transition matrix, and (c) the general expression for the solution to output $y(t)$ [Do NOT solve the equation].

20%

Note: A general solution to $t^2\ddot{y} + t\dot{y} - y = 0$ is of the form $y = t^k$ for real constant k.

OVER

4. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ $A^{)} = \begin{bmatrix} -S & 0 & 0 \\ 0 & -S & (2)\cdot S \\ 0 & 0 & (2^{)}-2)-2 \end{bmatrix}$

30 $\frac{9}{b}$

Find A^{s}, where s is a real number.

Linear Systems

Mid Term-2 (19th NOV-04)

1. A characteristics equation is given by $(s-\lambda_1)^5 = 0$. List all the possible Jordan canonical forms of the matrix associated with this equation and with each matrix find the number of linearly independent eigen vectors.

Sol^n The chara. eqn is $(s-\lambda_1)^5 = 0$

$\therefore$ The eigen values are $\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_1$ ie repeated 5 times $\therefore$ Multiplicity $= k = 5$.

Case-1: Nullity $= 5$ i.e No. of Lin. inde. ei. ve. $= 5$

$$J_1 = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_1 & & 0 & \\ & & \lambda_1 & & \\ & 0 & & \lambda_1 & \\ & & & & \lambda_1 \end{bmatrix}$$

Case-2: Nullity $= 4$ ie No of lin. inde. e.v. $= 4$

$$J_2 = \left[\begin{array}{cc|ccc} \lambda_1 & 1 & & 0 & \\ 0 & \lambda_1 & & & \\ \hline & & \lambda_1 & & \\ & 0 & & \lambda_1 & 0 \\ & & 0 & & \lambda_1 \end{array}\right]$$

Case-3. Nullity $= 3$ ie No. of l.i.e.v. $= 3$

$$J_3 = \left[\begin{array}{ccc|cc} \lambda_1 & 1 & & & \\ 0 & \lambda_1 & 1 & & \\ 0 & 0 & \lambda_1 & & \\ \hline & & & \lambda_1 & 0 \\ & & & 0 & \lambda_1 \end{array}\right]$$

Case-4:- Nullity $= 3$ $\therefore$ No. of l.i.e.v. $= 3$

$$J_4 = \left[\begin{array}{cc|cc|c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_1 \end{array}\right]$$

Case-5:- Nullity $= 2$ $\therefore$ No of l.i.e.v. $= 2$

$$J_5 = \left[\begin{array}{cccc|c} \lambda_1 & 1 & & & 0 \\ & \lambda_1 & 1 & & 0 \\ & & \lambda_1 & 1 & 0 \\ & & & \lambda_1 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_1 \end{array}\right]$$

Case-6:- Nullity $= 2$ $\therefore$ No of l.i.e.v. $= 2$

$$J_6 = \left[\begin{array}{ccc|cc} \lambda_1 & 1 & 0 & & \\ 0 & \lambda_1 & 1 & & 0 \\ 0 & 0 & \lambda_1 & & \\ \hline & 0 & & \lambda_1 & 1 \\ & & & 0 & \lambda_1 \end{array}\right]$$

Case-7:- Nullity $= 1$ $\therefore$ No of l.i.e.v. $= 1$

$$J_7 = \begin{bmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & 1 & & \\ & & \lambda_1 & 1 & \\ & & & \lambda_1 & 1 \\ & & & & \lambda_1 \end{bmatrix}$$

(2) Is the following system controllable and observable? Specify which modes are controllable and observable and which modes are not?

$$\dot{x} = \begin{array}{c} \\ \end{array}\left[\begin{array}{ccc|cc|c|cc}
\underbrace{}_{A_1} \\
1 & 1 & 0 & & & & & \\
0 & 1 & 0 & & & & & \\
 & & 1 & 1 & 0 & & & \\
 & & 0 & 1 & 0 & & & \\
 & 0 & 0 & 0 & 1 & & 0 & 0 \\
 & & & & & & 2 & 0 \\
 & & & & & & 0 & 2
\end{array}\right]$$

$$\dot{x} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix} x + \begin{array}{r} B_{11}^l \rightarrow \\ \\ B_{12}^l \rightarrow \\ B_{13}^l \rightarrow \\ B_{21}^l \rightarrow \\ B_{22}^l \rightarrow \end{array}\begin{bmatrix}
1 & 1 & 3 \\
-1 & 0 & 1 \\
2 & 3 & 9 \\
1 & 1 & 0 \\
1 & 2 & 1 \\
0 & 0 & 0 \\
1 & -1 & 0
\end{bmatrix} u$$

A_{11}, A_{12}, A_{13} (in A_1); A_{21}, A_{22} (in A_2); B_{11}, B_{12}, B_{13}, B_{21}, B_{22}

$$y = \begin{bmatrix}
5 & 2 & 1 & 1 & 1 & 1 & 0 \\
-2 & 4 & -1 & -1 & 0 & 0 & 1 \\
3 & 3 & 0 & 0 & 1 & 1 & 0
\end{bmatrix} x$$

$\leftarrow C_1 \rightarrow$ | $\leftarrow C_2 \rightarrow$; C_{11}, C_{12}, C_{13}, C_{21}, C_{22}; C_{11}^1, C_{12}^1, C_{13}^1, C_{21}^1, C_{22}^1

Sol^n Matrix - A has two eigen values $\lambda_1 = 1$ (Repeated 5 times) and $\lambda_2 = 2$ (Repeated 2 times)

$\therefore$ A has two main blocks A_1 and A_2 associated with $\lambda_1 = 1$ and $\lambda_2 = 2$ resp.

Also, A_1 has 3 Jordan blocks A_{11}, A_{12} and A_{13} ie Nullity (for $\lambda_1 = 1$) $= 3$ ie. 3 l.i.e.v. (for $\lambda_1 = 1$)

and A_2 has 2 Jordan blocks A_{21}, and A_{22} ie Nullity (for $\lambda_2 = 2$) $= 2$ ie 2 l.i.e.v. (for $\lambda_2 = 2$)

For $\lambda_1 = 1$

$$B_1^l = \begin{bmatrix} B_{11}^l \\ B_{12}^l \\ B_{13}^l \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$\therefore$ Det $B_1^l = 0$ and Rank $B_1^l = 2 \neq$ Full rank

$\therefore$ Mode $\lambda_1 = 1$ is not controllable

$$C_1^1 = [C_{11}^1 \quad C_{12}^1 \quad C_{13}^1]$$

$$= \begin{bmatrix} 5 & 1 & 1 \\ -2 & -1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$\therefore$ Det $C_1^1 = 0$ and Rank $C_1^1 = 2 \neq$ Full rank (ie 3)

$\therefore$ Mode $\lambda_1 = 1$ NOT OBSERVABLE

For $\lambda_2 = 2$

$$B_2^l = \begin{bmatrix} B_{21}^l \\ B_{22}^l \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Rank $B_2^l = 1 \neq$ Full rank (ie 2)

$\therefore$ Mode $\lambda_2 = 2$ is NOT CONTROLLABLE.

$$C_2^1 = [C_{21}^1 \quad C_{22}^1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

As the column of C_2^1 are lin. inde.

ie

$$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \text{ and } \alpha_1 = \alpha_2 = 0$$

$\therefore$ Rank $C_2^1 = 2$ = Full Rank

$\therefore$ Mode $\lambda_2 = 2$ is OBSERVABLE.

(3) The following system is to be analyzed for a delayed ramp i/p. $t^2\ddot{y} + t\dot{y} - y = u$ with

$$u(t) = \begin{cases} t-1 & \text{for } t \geq 1 \\ 0 & \text{for } t < 1 \end{cases}$$ and

$y(1) = \dot{y}(1) = 1$. Find (a) the ~~general~~ fundamental matrix (b) State transition matrix and (c) the general expression for the soln to o/p $y(t)$ [DO NOT solve the eqn]

<u>Note</u>: A general sol^n to $t^2\ddot{y}+t\dot{y}-y=0$ is of the form $y=t^K$ for real constant K.

<u>Sol^n</u>

$$t^2\ddot{y}+t\dot{y}-y=u$$

Let's 1st represent this equ^n in state space.

i.e. $\ddot{y}=\frac{u}{t^2}+\frac{y}{t^2}-\frac{\dot{y}}{t}$ --- (a)

Integrating on both the sides,

$$\Rightarrow \quad y=\int\left[\int\left(\frac{u}{t^2}+\frac{y}{t^2}\right)-\frac{y}{t}\right]$$

Let $x_1=y$ and $x_2=\dot{y}$

$$\Rightarrow \quad \dot{x}_1=\dot{y}=\int\left(\frac{u}{t^2}+\frac{y}{t^2}\right)-\frac{y}{t}=\int\left(\frac{u}{t^2}+\frac{x_1}{t^2}\right)-\frac{x_1}{t} \quad \text{--- ①}$$

$$\dot{x}_2=\ddot{y}=\frac{u}{t^2}+\frac{y}{t^2}-\frac{\dot{y}}{t}=\frac{u}{t^2}+\frac{x_1}{t^2}-\frac{x_2}{t} \quad \text{--- ②}$$

put $\frac{u}{t^2}+\frac{x_1}{t^2}=\ddot{y}+\frac{\dot{y}}{t}$ in ①

$$\Rightarrow \dot{x}_1=\int\left(\ddot{y}+\frac{\dot{y}}{t}\right)-\frac{x_1}{t}=\dot{y}+\frac{y}{t}-\frac{x_1}{t}$$

$\Rightarrow$ put $\dot{y}=x_2$, $y=x_1$

$$=x_2+\cancel{\frac{x_1}{t}}-\cancel{\frac{x_1}{t}}$$

$$\left.\begin{aligned}\Rightarrow \dot{x}_1&=x_2 &&\text{--- ③}\\ \dot{x}_2&=\frac{x_1}{t^2}-\frac{x_2}{t}+\frac{u}{t^2} &&\text{--- ②}\end{aligned}\right\}\text{ state equ}^n$$

$y=x_1 \longrightarrow$ o/p equ^n

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/t^2 & -1/t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/t^2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(a) $$A = \begin{bmatrix} 0 & 1 \\ 1/t^2 & -1/t \end{bmatrix}$$

imp → As the gen. solⁿ of diff. eqⁿ, $t^2 \ddot{y} + t\dot{y} - y = 0$ is $y = t^K$,

$y = t^K$ has to satisfy the above diff. eqⁿ,

ie put $y = t^K$ in above diff. eqⁿ

$$\Rightarrow t^2 \frac{d}{dt^2}(t^K) + t \frac{d}{dt}(t^K) - (t^K) = 0$$

$$\Rightarrow t^2 [K(K-1)t^{K-2}] + t[Kt^{K-1}] - t^K = 0$$

$$\Rightarrow (K^2 - K)t^K + Kt^K - t^K = 0$$

$$\Rightarrow (K^2 - K + K - 1)t^K = 0 \Rightarrow \boxed{(K^2-1)t^K = 0}$$

As $t^K \neq 0$, $K^2 - 1 = 0$

$\Rightarrow K = \pm 1$

For $K = 1 \Rightarrow y = t$

For $K = -1 \Rightarrow y = 1/t$

For $y = t$, $x_1 = y = t$ and $x_2 = \dot{y} = 1$

For $y = 1/t$, $x_1 = y = 1/t$ and $x_2 = \dot{y} = -1/t^2$

Let $M(t)$ be the fundamental matrix which is made up of two l.i. vectors $M_1(t)$ and $M_2(t)$

$M(t) = [\,M_1(t) \;\; M_2(t)\,]$

where $M_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ at $y = t$

ie $M_1(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$

and $M_2(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ at $y = 1/t$

ie $M_2(t) = \begin{bmatrix} 1/t \\ -1/t^2 \end{bmatrix}$

$\Rightarrow \boxed{M(t) = \begin{bmatrix} t & 1/t \\ 1 & -1/t^2 \end{bmatrix}}$ Now as $t_0 = 1$

$\therefore M(t_0) = M(1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\Rightarrow M^{-1}(1) = \dfrac{\text{Adj } M(1)}{\det M(1)} = -\dfrac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

(b) $\Phi(t, t_0) = M(t)\, M^{-1}(t_0)$ Here $t_0 = 1$

$\Rightarrow \Phi(t,1) = M(t)\, M^{-1}(1)$

$= \begin{bmatrix} t & 1/t \\ 1 & -1/t^2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$\Rightarrow \Phi(t,1) = \begin{bmatrix} (t/2 + 1/2t) & (t/2 - 1/2t) \\ (\frac{1}{2} - 1/2t^2) & (1/2 + 1/2t^2) \end{bmatrix}$

(c) The general solⁿ of $y(t)$ is given by,

$$y(t) = C(t)\,\Phi(t,t_0)\,x(t_0) + \int_{t_0}^{t} C(t)\,\Phi(t,\tau)\,B(\tau)\,U(\tau)\,d\tau$$

$$C(t) = [1 \quad 0] \quad , \quad t_0 = 1$$

$$\Phi(t,t_0) = \Phi(t,1) = \begin{bmatrix} (t/2 + 1/2t) & (t/2 - 1/2t) \\ (1/2 - 1/2t^2) & (1/2 + 1/2t^2) \end{bmatrix}$$

$$B(\tau) = \begin{bmatrix} 0 \\ 1/\tau^2 \end{bmatrix}$$

$$U(\tau) = \tau - 1$$

$$\Phi(t,\tau) =$$

$$x(t_0) = x(1) = \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where
$x_1(1) = y(1) = 1$
$x_2(1) = \dot{y}(1) = 1$

$$\Rightarrow y(t) = [1 \quad 0] \begin{bmatrix} (t/2 + 1/2t) & (t/2 - 1/2t) \\ (1/2 - 1/2t^2) & (1/2 + 1/2t^2) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} +$$

$$\int_{1}^{t} [1 \quad 0] \begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} 0 \\ 1/\tau^2 \end{bmatrix} (\tau - 1)\,d\tau$$

$$y(t) = \frac{t}{4} - \frac{1}{4t} + \frac{t}{2}\ln t + 1$$

(7) Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ Find A^S where S is a real number.

Sol^n

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\pi_A(\lambda) = \det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 0 & \lambda-2 \end{vmatrix} = 0$

$\Rightarrow (\lambda-1)(\lambda-1)(\lambda-2) = 0$

$\Rightarrow \lambda_1 = 1, 1$ and $\lambda_2 = 2$

Now by C.H. tech. let

$P(A) = A^S$ and $R(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$

($\because n = 3$ for A)

$P(\lambda) = \lambda^S$ and $R(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2$

For $\lambda_1 = 1$, $P(\lambda_1) = R(\lambda_1)$

$\Rightarrow \lambda_1^S = \alpha_0 + \alpha_1\lambda_1 + \alpha_2\lambda_1^2$

$\Rightarrow \boxed{1^S = 1 = \alpha_0 + \alpha_1 + \alpha_2}$ --- (1)

$\lambda_1 = 1$ $\quad \left.\dfrac{dP(\lambda)}{d\lambda}\right|_{\lambda=\lambda_1} = \left.\dfrac{dR(\lambda)}{d\lambda}\right|_{\lambda=\lambda_1}$

$\Rightarrow \left. S\lambda^{S-1}\right|_{\lambda=\lambda_1=1} = \left. 0 + \alpha_1 + 2\lambda\alpha_2 \right|_{\lambda=\lambda_1=1}$

$\Rightarrow \boxed{S = \alpha_1 + 2\alpha_2}$ --- (2)

For $\lambda_2 = 2$ $\quad P(\lambda_2) = R(\lambda_2)$

$$\lambda_2^5 = \alpha_0 + \lambda_2\alpha_1 + \lambda_2^2\alpha_2$$

$$\Rightarrow 2^5 = \alpha_0 + 2\alpha_1 + 4\alpha_2 \quad --- (3)$$

$$\Rightarrow 2^5 = \alpha_0 + 2(\alpha_1 + 2\alpha_2)$$

put $\alpha_1 + 2\alpha_2 = 5$ from (2)

$$\Rightarrow 2^5 = \alpha_0 + 2 5$$

$$\Rightarrow \boxed{\alpha_0 = 2^5 - 25}$$

put α_0 in (1)

$$\Rightarrow 1 = (2^5 - 25) + \alpha_1 + \alpha_2$$

$$\Rightarrow 1 - 2^5 + 25 = \alpha_1 + \alpha_2 \quad -- (4)$$

$$- \quad 5 = _-\alpha_1 +_- 2\alpha_2 \quad -- (2)$$

$$\Rightarrow -\alpha_2 = 1 - 2^5 + 5$$

$$\Rightarrow \boxed{\alpha_2 = 2^5 \cancel{+} 5 - 1}$$

$$\Rightarrow \alpha_1 + 2\alpha_2 = 5$$

$$\Rightarrow \alpha_1 + 2(2^5 - 5 - 1) = 5$$

$$\Rightarrow \boxed{\alpha_1 = -2^{5+1} + 35 + 2}$$

Now by C.H. tech.

$$P(A) = R(A)$$

$$\Rightarrow A^5 = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$= \alpha_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow A^s = \begin{bmatrix} (\alpha_0+\alpha_1+\alpha_2) & 0 & 0 \\ 0 & (\alpha_0+\alpha_1+\alpha_2) & (\alpha_1+3\alpha_2) \\ 0 & 0 & (\alpha_0+2\alpha_1+4\alpha_2) \end{bmatrix}$$

From eq. (1) and (3)

$$\begin{cases} \alpha_0+\alpha_1+\alpha_2 = s \\ \alpha_0+2\alpha_1+4\alpha_2 = 2^s \end{cases}$$

and $\alpha_1+3\alpha_2 = -2^{s+1}+3s+2+3(2^s)-3s-3$

$= 2^s-1$

$$\Rightarrow A^s = \begin{bmatrix} s & 0 & 0 \\ 0 & s & (2^s-1) \\ 0 & 0 & 2^s \end{bmatrix}$$

Right

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2^s-1 \\ 0 & 0 & 2^s \end{bmatrix}$$

$1 + 2\times 2^s - 2^{s+1}$

Final

Q) Given System:

$$\dot{x} = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

design objective is to have the closed loop poles at $\lambda_1 = -5$ and $\lambda_2 = -6$. Show to what extent one can meet the design objective.

Step 1 Check Controllability & Stability.

$\det(sI - A) = 0$ $\begin{vmatrix} s+1 & -3 \\ 0 & s+2 \end{vmatrix} = 0 \Rightarrow (s+1)(s+2) = 0.$
$\lambda = -1, -2.$

So system is stable i.e open Loop

$C_x = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow$ Rank deficient $\neq 2$.
$[B \ AB]$

So System is Uncontrollable
Now we Canot design the system unless we know if any mode is Controllable. ~~for~~ For that we need to transform the system into Jordan C.F

$$\dot{x}_J = M^{-1}AM x_J + M^{-1}B u.$$
$$y = CM x_J$$

Step 2 Finding Eigen Values of A (system).

$\det(sI - A) = 0. \Rightarrow \lambda_1 = -1$ & $\lambda_2 = -2$
(Distinct Eigen Values).

Step 3 Find Eigen Vectors

$$(A - \lambda I)x_i = 0 = \begin{bmatrix} -1-\lambda & 3 \\ 0 & -2-\lambda \end{bmatrix} x_i = 0$$

Now at $\lambda_1 = -1$

$$\begin{bmatrix} 0 & 3 \\ 0 & -1 \end{bmatrix} x_1 = 0 \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

at $\lambda_2 = -2$.

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} x_2 = 0 \Rightarrow x_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

<u>So modal matrix will be</u>

$$M = \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$$

$$\hat{A} = M^{-1} A M = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\hat{B} = M^{-1} B = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So Jordan C.F will be.

$$\dot{X}_J = \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & -2 \end{array}\right] x_J + \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right] u.$$

Now it's clear that $\lambda = -1$ is Controllable while $\lambda = -2$ is UnControllable Mode
So $\lambda = -1$ Can be placed at either -5 or
while $\lambda = -2$ is fixed so it should incl in X-tic polynomial $\alpha_c(s)$ always.

Now we can start designing the system.

Step 4 desired x-tics poly is.

$$\alpha c(s) = (s+2)(s+5) \quad \text{or} \quad (s+2)(s+6).$$

$$\alpha c(s) = s^2+7s+10 \Rightarrow (A) \quad \text{or} \quad s^2+8s+12.$$

Step 5 Calculate closed Loop x-tics poly.

$$\det(sI-(A-BK)) = 0.$$

$$A-BK = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [k_1 \quad k_2].$$

$$= \begin{bmatrix} -1-k_1 & 3-k_2 \\ 0 & -2 \end{bmatrix}$$

$$\det(sI-(A-BK)) = \begin{vmatrix} s+1+k_1 & -3+k_1 \\ 0 & s+2 \end{vmatrix} = 0.$$

$$= s^2+s(3+k_1)+(2+2k_1) = 0 \Rightarrow (B)$$

Equate both eqs

$$3+k_1 = 7 \Rightarrow k_1 = 4 \quad \text{or}$$
$$2+2k_1 = 10 \Rightarrow k_1 = 4$$

So k_1 is consistent, It means all o.k

Now selection of K2

We will use any arbitrarily value, but not infi[nite]

$[k_1 \quad k_2] = [4 \quad 0]$

Q// Given the System

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

design a full order observer having eigen Val
at -5 & -7. Draw a detailed all integrato
diagram

Sol

Step 1 Check observability first

$$O_x = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$, full Rank so System is
Can proceed for design

Step 2 ∴ Observer Eigen Values given by desired x-tic equation

$$\alpha_e(s) = (s+5)(s+7) = s^2+12s+35 \Longrightarrow$$

Step 3 Finding closed Loop x-tic poly assuming Contr
$U = -Kx$

$$\det(sI-(A-GC)) \quad , \quad GC = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} a-g_1 & 0 \\ -g_2 & 2 \end{bmatrix}} = \cancel{\begin{bmatrix} s-a+g_1 & 0 \\ g_2 & s-2 \end{bmatrix}}$$

$$= \begin{bmatrix} s & 0 \end{bmatrix} \begin{bmatrix} -g_1 & 2 \end{bmatrix} \begin{bmatrix} s+g_1 & -2 \end{bmatrix}$$

$$\det(sI-(A-GC)) = s^2+g_1s-s-g_1+2g_2-6.$$

$$= s^2+(g_1-1)s+(-g_1+2g_2-6) = 0. \Rightarrow (B)$$

by Comparing with eq (A) get,

$$g_1-1 = 12 \Rightarrow g_1 = 13$$

$$(-g_1+2g_2-6) = 35 \Rightarrow g_2 = 27$$

So observer gain

$$G = [13 \quad 27]^T \;//$$

(b) Observer eqs is

$$\dot{\hat{x}} = (A-GC)\hat{x} + Bu + Gy$$

$$\hat{y} = C\hat{x}$$

$$A-GC = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 13 & 0 \\ 27 & 0 \end{bmatrix} = \begin{bmatrix} -13 \\ -24 \end{bmatrix}$$

So

$$\begin{cases} \dot{\hat{x}} = \begin{bmatrix} -13 & 2 \\ -24 & 1 \end{bmatrix}\hat{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u + \begin{bmatrix} 13 \\ 27 \end{bmatrix}y \\ \hat{y} = [1 \quad 0]\hat{x} \end{cases}$$

Plant eqs

$$\begin{cases} \dot{x}_1 = 2x_2 \\ \dot{x}_2 = 3x_1 + x_2 + u \\ y = x_1 \end{cases}$$

Estimator equati

$$\dot{\hat{x}}_1 = -13\hat{x}_1 + 2\hat{x}_2 + 1$$

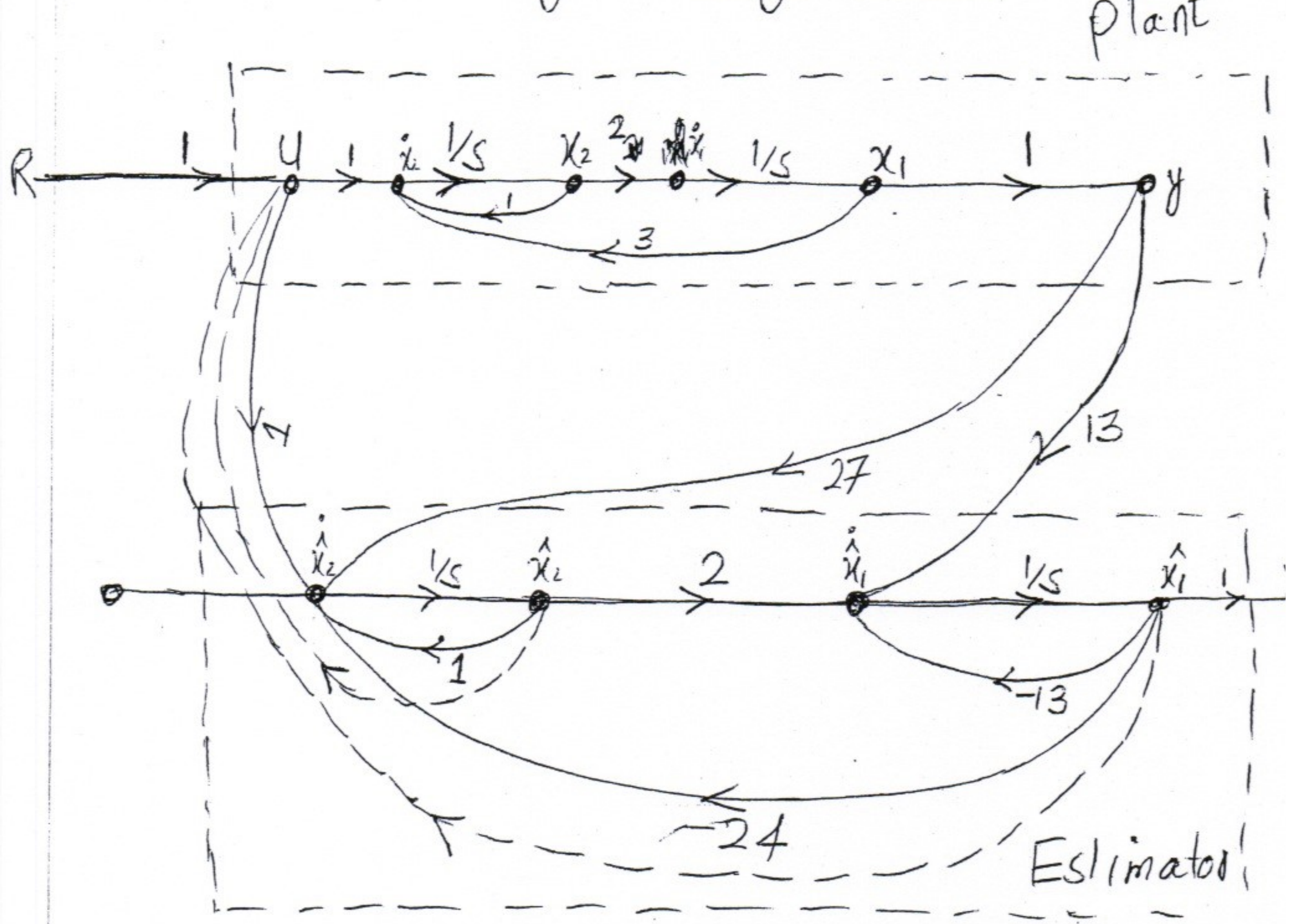
$$\dot{\hat{x}}_2 = -24\hat{x}_1 + \hat{x}_2 + 1$$

27

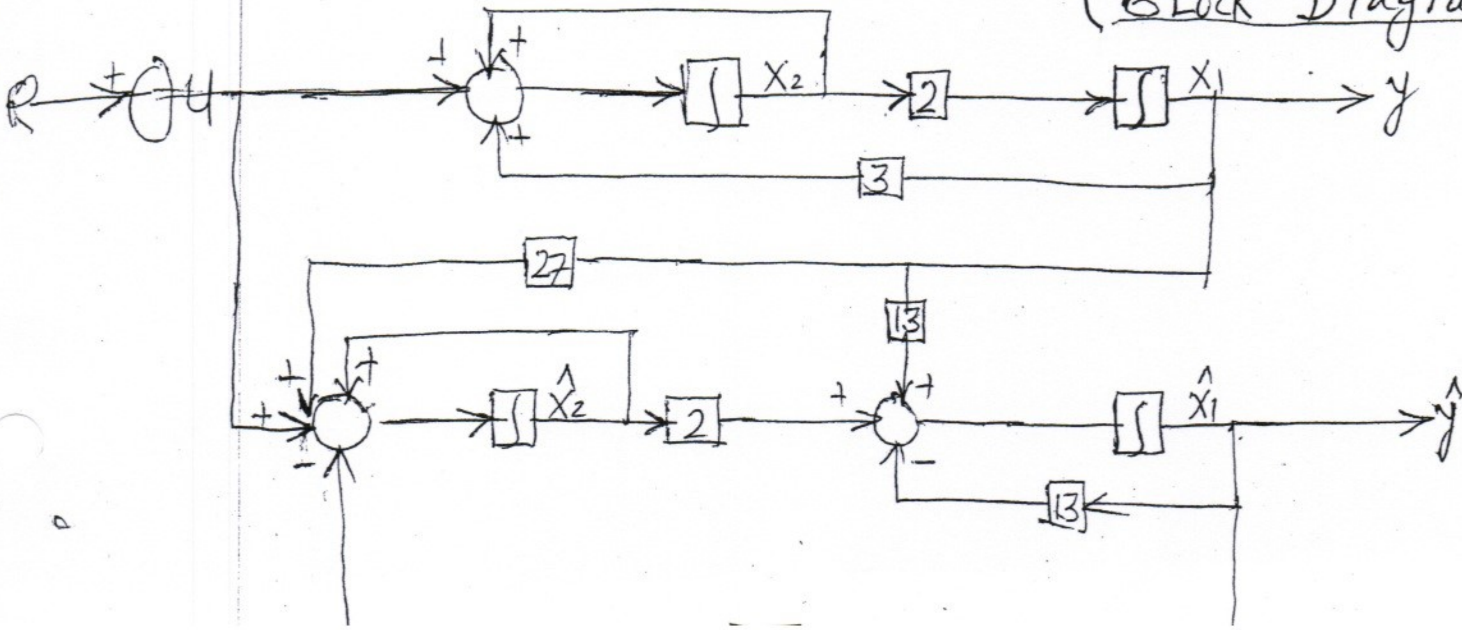
$$\hat{y} = \hat{x}_1$$

4.

Now state diagram (Signal Flow Graph)



(Block Diagram



Effect of feed back on system properties

(i) Controllability

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Control law $\Rightarrow$ $u = -Kx + r$.

* Open loop Controllability Matrix

$$C_x = [B \ \ AB \ \ A^2B \ldots\ldots A^{n-1}B]$$

* Closed Loop Controllability Matrix

$$\dot{x} = Ax + B(-Kx + r) \Rightarrow (A - BK)x + Br$$
$$y = Cx + D(-Kx + r) \Rightarrow (C - DK)x + Dr.$$

$$C_x^* = [B \ \ (A-BK)B \ \ (A-BK)^2B \ldots\ldots]$$

Its clear that Rank $\{C_x\}$ = Rank $\{C_x^*\}$ always so, thus state feed back doesnot alter Controllability

(ii) Observability

Open Loop $O_x = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Closed Loop

$$O_x^* = [(C-DK) \ \ (C-DK)(A-BK) \ldots\ldots]^T$$

It is obvious that when state feedback is used observability is lost if $C = DK$

7

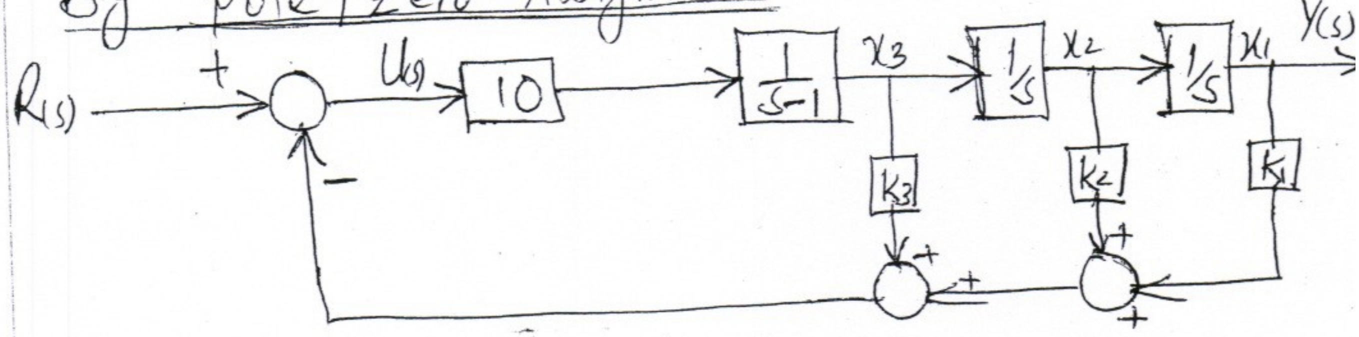
Q// Consider the open Loop T/f of ~~LTI~~ Linear System

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s^2(s-1)}$$

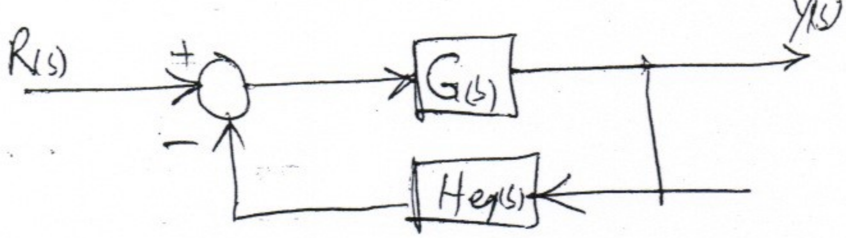
Feed back law is $u = r(t) - Kx(t) = -Kx + r$. Find the gain Vector K such that following design specif are meet

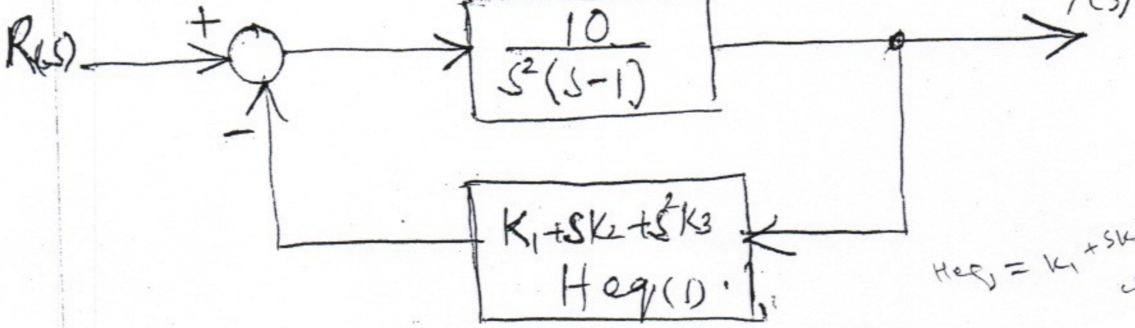
(i) $e_{ss} = 0$ (ii) Two of dominant Closed Loop poles at $-1 \pm j1$. Find location of 3rd pole under above.

Sol By Pole/Zero Assignment



(i) Reduced block diagram





$Heq_s = k_1 + sk_2 + k_3(s^2 - s)$ check.

(ii) Find Closed Loop T/F

$\therefore$ Two dominant poles i.e $s_{1,2} = -1 \pm j1$ given $\Rightarrow \xi = 0.7$ & $\omega_n = 1.4$

Now to ensure the dominance condition, the third closed loop pole should lie at $s \leq -10\xi\omega_n \Rightarrow s_3 = -10\xi\omega_n = -10$

$$(s+1+j1)(s+1-j1) = s^2+s+s-j1+j1+1+sj1-j1\cdot j1 \ldots$$
$$= s^2+2s+2$$
$$= s^2+2\xi\omega_n s+\omega_n^2$$

$$\frac{Y(s)}{R(s)} = \frac{10/s^2(s-1)}{1+\frac{10}{s^2(s-1)} \times (s^2k_3+sk_2+k_1)}$$

$$= \frac{10}{s^3-s^2+10s^2k_3+10sk_2+10k_1}$$

$$\frac{Y(s)}{R(s)} = \frac{10}{s^3+(10k_3-1)s^2+s10k_2+10k_1} \Longrightarrow (A)$$

(II) Find desired Closed Loop T/F

Two dominant poles are given i.e $s_{1,2} = -1 \pm j$

I place third pole (Non dominant) far away at $s_3 = -$

So desired T/F will be

$$\frac{Y(s)}{R(s)} = \frac{10}{(s+50)(s+1+j1)(s+1-j1)}$$ = same as above

$$\frac{Y(s)}{R(s)} = \frac{10}{s^3+52s^2+102s+100} \Longrightarrow (B)$$

$\therefore e_{ss} \rightarrow 0$ so (A) $\Rightarrow$

$$\frac{10}{10k_1} = 1 \Rightarrow k_1 = 1$$

Compare (A) & (B) for remaining paramete

$$10k_3 - 1 = 52 \Rightarrow k_3 = 5.3$$
$$10k_2 = 102 \Rightarrow k_2 = 10.2.$$

So Gain Vector

$$K = [k_1 \;\; k_2 \;\; k_3] = [1 \;\; 10.2 \;\; 5.3]$$

Q. Consider the system

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

and assume that we are using state feedback of the form $u = -Kx + r$

(a) Show that (A, C) is observable. [~~A-BK~~]

$$O_x = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \text{ full rank so its observable.}$$

(b) Show that there exists a K such that $(A-BK, C)$ is unobservable.

$$\cancel{A} \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix} = A - \underline{\underline{BK}} \begin{bmatrix} -2-k_1 & 1-k_2 \\ 1 & 0 \end{bmatrix}$$

$$\overset{*}{O}_x = [C \quad C(A-BK)]^T = \cancel{\begin{bmatrix} 1 & 2 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -2-k_1 & 1-k_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -k_1 & 1-k_2 \end{bmatrix}$$

$$\overset{*}{O}_x = \begin{bmatrix} 1 & 2 \\ -k_1 & 1-k_2 \end{bmatrix} \checkmark \quad \text{So } K = [1 \quad 3]$$

$\det O_x^* = 0$ so unobserv.

~~So if $k_2 = 1$ and k_1 any value then~~
~~$\det (C_x^*) = 0 \Rightarrow$ System is unobservable for~~

$\cancel{K = [k_1 \quad 1]}$ $\quad \det O_x = 1 - k_2 + 2k_1$

So for $k_1 = 1$ & $k_2 = 3$, System is unobservable

(c) Compute a K in (b) such that $K = [1 \quad k_2]$, i.e. fi[nd] k_2

Now if $k_1 = 1$ then value of k_2 should be any value which should cause the system unobservable

$$\det(\overset{*}{O}_x) = \begin{vmatrix} 1 & 2 \\ -1 & 1-k_2 \end{vmatrix} = 0 \Rightarrow \cancel{1 - k_2 + 2 = 0} \Rightarrow k_2 = 3$$

So Now for

$K = [k_1 \quad k_2] \Rightarrow [1 \quad 3]$ again $\overset{*}{O}_x = $ singular so unobserva[ble]

(d) Compute open Loop and closed loop T/f.

* ~~open Loop T/f~~

$$\frac{Y(s)}{U(s)} = C(sI-A)^{-1}B = [1 \quad 2]\begin{bmatrix} s+2 & -1 \\ -1 & s \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= [1 \quad 2]\frac{\begin{bmatrix} s & 1 \\ 1 & s+2 \end{bmatrix}}{s^2+2s-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[1 \quad 2]\frac{\begin{bmatrix} s \\ 1 \end{bmatrix}}{s^2+2s-1} = \frac{s+2}{s^2+2s-1.}$$

* Closed Loop T/f will be.

$$G(s) = C(sI-(A-BK))^{-1}B$$

$$(A-BK) = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1 \quad 3] = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

$$sI-(A-BK) = \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix} \Rightarrow \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix}}{s^2+3s+2}$$

$$[1 \quad 2]\frac{\begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix}}{s^2+3s+2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{[1 \quad 2]\begin{bmatrix} s \\ 1 \end{bmatrix}}{s^2+3s+2}$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{s+2}{s^2+3s+2}} \;/// \quad = \frac{1}{s+1.} \;///$$

1

Exam # 2 NOV. 6, 1991.

1

$$\Phi(t,0) = \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

a

$$\Phi(t,t_0) = \Phi(t,0)\cdot\Phi(0,t_0) \quad \text{transition.}$$

$$= \Phi(t,0)\cdot\Phi^{-1}(t_0,0) \longrightarrow (i)$$

$$\Phi(t_0,0) = \begin{bmatrix} e^{-t_0}\cos t_0 & e^{-t_0}\sin t_0 \\ -e^{-t_0}\sin t_0 & e^{-t_0}\cos t_0 \end{bmatrix}$$

$$det = e^{-2t_0}\cos^2 t_0 + e^{-2t_0}\sin^2 t_0 = e^{-2t_0} \to \star$$

$$C_{factor} = \begin{bmatrix} e^{-t_0}\cos t_0 & +e^{-t_0}\sin t_0 \\ -e^{-t_0}\sin t_0 & e^{-t_0}\cos t_0 \end{bmatrix}$$

$$C^T = \begin{bmatrix} e^{-t_0}\cos t_0 & -e^{-t_0}\sin t_0 \\ +e^{-t_0}\sin t_0 & e^{-t_0}\cos t_0 \end{bmatrix}$$

$$\Phi^{-1}(t_0,0) - \frac{C^T}{det} = \begin{bmatrix} e^{t_0}\cos t_0 & -e^{t_0}\sin t_0 \\ e^{t_0}\sin t_0 & e^{t_0}\cos t_0 \end{bmatrix} \longrightarrow \star$$

from (i)

$$\Phi(t,t_0) = \Phi(t,0)\cdot\Phi^{-1}(t_0,0)$$

$$= \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}\begin{bmatrix} e^{t_0}\cos t_0 & -e^{t_0}\sin t_0 \\ e^{t_0}\sin t_0 & e^{t_0}\cos t_0 \end{bmatrix}$$

$$\Phi(t,t_0) = \begin{bmatrix} e^{-t}e^{t_0}\cos t\cdot\cos t_0 + e^{-t}e^{t_0}\sin t\cdot\sin t_0 & -e^{-t}e^{t_0}\cos t\sin t_0 + e^{-t}e^{t_0}\sin t\cos t \\ -e^{-t}e^{t_0}\sin t\cos t_0 + e^{-t}e^{t_0}\cos t\sin t_0 & +e^{-t}e^{t_0}\sin t\sin t_0 + e^{-t}e^{t_0}\cos t\cos \end{bmatrix}$$

$$= \begin{bmatrix} e^{(t_0-t)}\cdot\cos(t-t_0) & e^{(t_0-t)}\sin(t-t_0) \\ -e^{(t_0-t)}\sin(t-t_0) & +e^{(t_0-t)}\cdot\cos(t-t_0) \end{bmatrix}$$

$t_0 = t$

b. $$\frac{d}{dt}\Phi(t,t_0)\Big| = A(t_0)$$

$$\frac{d\Phi(t,t_0)}{dt}\Big|_{t_0=t} = A(t)$$

$$\Phi(t,t_0) = e^{(t_0-t)}\begin{bmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix}$$

$U \cdot V$

$$\Phi(t,t_0) = e^{(t_0-t)}\begin{bmatrix} -\sin(t-t_0) & \cos(t-t_0) \\ -\cos(t-t_0) & -\sin(t-t_0) \end{bmatrix} - e^{(t_0-t)}\begin{bmatrix} \cos(t-t_0) & \sin(t- \\ -\sin(t-t_0) & \cos(t \end{bmatrix}$$

$$\Phi(t,t_0)\Big|_{t_0=t} = 1\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A(t)\ \cancel{\Phi(t,t_0)} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$ #

c. $$x(t) = \Phi(t,t_0)\ x(t_0)$$

$$x(2) = \begin{bmatrix} e^{-2}\cos 2 & e^{-2}\sin 2 \\ -e^{-2}\sin 2 & e^{-2}\cos 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X(2) = \begin{bmatrix} e^{-2}(\cos 2 + \sin 2) \\ -e^{-2}(\sin 2 - \cos 2) \end{bmatrix}$$ #

(2)

$e^{a x} \ e^{a-x} \ e^{a+x}$
$e^{-a x} \ e^{a x}$

Q2 $\dot{x} = \begin{bmatrix} -3 & e^{2t} \\ 0 & -2 \end{bmatrix} x$

let $M(t) = [M_1(t) \quad M_2(t)]$

$\dot{x}_1 = -3x_1 + e^{2t} x_2 \longrightarrow (1)$

$\dot{x}_2 = -2x_2 \rightarrow x_2(t) = e^{-2t}, 0 \longrightarrow (2)$

from (2)

$\Rightarrow$ when $x_2 = e^{-2t} \Rightarrow \dot{x}_1 = -3x_1 + 1$

$\Rightarrow$

$\Rightarrow x_2(t) = e^{-\int_0^t 3\,dt} + \int_0^t e^{-\int_\tau^t 3\,dt} \cdot 1 \, d\tau$

$= e^{-3t} + \int_0^t e^{-(3t-3\tau)} \cdot d\tau$

$= e^{-3t} + e^{-3t} \int_0^t e^{3\tau}\, d\tau$

$= e^{-3t} + \frac{e^{-3t}}{3}\left[e^{3t} - 1\right]$

$= e^{-3t} + \frac{1}{3} - \frac{e^{-3t}}{3} = \boxed{\frac{1}{3} + \frac{2}{3}e^{-3t}}$ ☆

when $x_2 = 0$

$\dot{x}_1 = -3x_1 \Rightarrow \boxed{x_1 = e^{-3t}}$ ☆

$M(t) = \begin{bmatrix} 1/3 + \frac{2}{3}e^{-3t} & e^{-3t} \\ e^{-2t} & 0 \end{bmatrix}$ #

$M(0) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$M^{-1}(0) \Rightarrow \det = -1 \Rightarrow C_{fact} = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \Rightarrow C^T = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$

$$M^{-1}(0) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Phi(t,0) = M(t) \cdot M^{-1}(0)$$

$$= \begin{bmatrix} 1/3 + 2/3 e^{-3t} & e^{-3t} \\ e^{-2t} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Phi(t,0) = \begin{bmatrix} -e^{-3t} & 1/3 + 1/3 e^{-3t} \\ 0 & e^{-2t} \end{bmatrix}$$

Q3

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} u$$

$$y = [1 \quad -1]\, x$$

$$u(t) = \begin{bmatrix} -6t \\ 6t \end{bmatrix} \qquad t \geq 0$$

$$y(t) = 3 - 3t^2 + t^3 \qquad x(1) = ?$$

$$\dot{x}_1 = x_2 + u_1 + u_2 \longrightarrow (1)$$
$$\dot{x}_2 = \cancel{x_2 +} u_2 \longrightarrow (2)$$

$$y = x_1 - x_2 \longrightarrow (3)$$

given $y(t) = 3 - 3t^2 + t^3 \longrightarrow (4)$

from (3) $\dot{y} = \dot{x}_1 - \dot{x}_2$

$$= x_2 + u_1 + \cancel{u_2} - \cancel{u_2}$$

$$\dot{y} = x_2 + u_1 \longrightarrow (4)$$

(3)

from (3) & ☆

$$x_1 - x_2 = 3 - 3t^2 + t^3 \longrightarrow (5)$$

& $y(t) = -6t + 3t^2$; so from (4)

$$x_2 + u_1 = -6t + 3t^2 \longrightarrow (6)$$

$$x_2 = -6t + 3t^2 - u_1 \longrightarrow ☆$$

from (5) $x_1 = 3 - \cancel{3t^2} + t^3 - 6t + \cancel{3t^2} - u_1$

$$x_1 = 3 - 6t + t^3 - u_1 \longrightarrow ☆$$

$$x(t) = \begin{bmatrix} 3 + t^3 - 6t - u_1 \\ 3t^2 - 6t - u_1 \end{bmatrix} ; \quad u(t) = \begin{bmatrix} -6t \\ 6t \end{bmatrix}$$

$$X(1) = \begin{bmatrix} 3 + 1 - 6 - (-6) \\ 3 - 6 - (-6) \end{bmatrix}$$

$$\Rightarrow x(1) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

24

ENGR 471 — Midterm Exam I — Oct. 23, 96

1) Given the input-output differential equation of a dynamical system

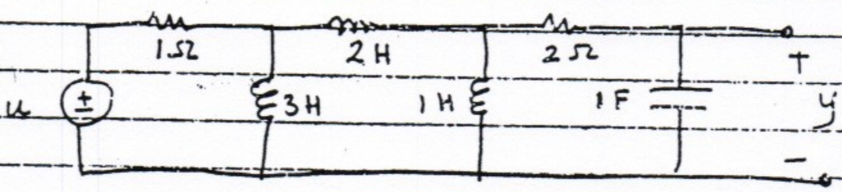
(45%) $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = \dot{u} + 5u$

a) Find the transfer function $\frac{Y(s)}{U(s)}$ and its impulse response.

b) Obtain the controllable and Jordan canonical forms

2) Consider the RLC network shown

(35%)



Write down the state and output equations in the compact form.

3) Find the fundamental matrix $M(t)$ and the state transition matrix $\Phi(t,$

for the following system

(20%) $\dot{x} = \begin{bmatrix} -3 & e^{2t} \\ 0 & -2 \end{bmatrix} x$

Answer available back

october 96

Q.1 $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = \dot{u} + 5u$

a. $(s^3 + 6s^2 + 11s + 6)\, Y(s) = (s+5)\, U(s)$

$$\frac{Y(s)}{U(s)} = \frac{s+5}{(s^3+6s^2+11s+6)}$$

$$= \frac{s+5}{(s+1)(s+2)(s+3)}$$

$$a = \left.\frac{s+5}{(s+2)(s+3)}\right|_{s=-1} \Rightarrow a = 2$$

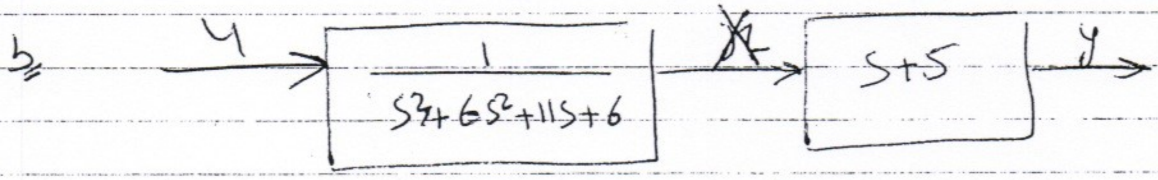
$$b = \left.\frac{s+5}{(s+1)(s+3)}\right|_{s=-2} \Rightarrow b = -3$$

$$c = \left.\frac{s+5}{(s+1)(s+2)}\right|_{s=-3} = 1$$

$$\boxed{H(s) = \frac{2}{(s+1)} - \frac{3}{(s+2)} + \frac{1}{(s+3)}} \rightarrow *$$

impulse response $h(t) = \left(2e^{-t} - 3e^{-2t} + e^{-3t}\right) u_s(t)$. $\boxed{t \geq 0}$ &

b.



$$\frac{X}{U} = \frac{1}{s^3+6s^2+11s+6} \Rightarrow \dddot{x} + 6\ddot{x} + 11\dot{x} + 6x = u$$

Let $\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ x_3 = \ddot{x} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 & \rightarrow (1) \\ \dot{x}_2 = x_3 & \rightarrow (2) \\ \dot{x}_3 = u - 6x_3 - 11x_2 - 6x_1 & \rightarrow (3) \end{cases}$

$$\frac{y}{x} = s+5 \Rightarrow y = \dot{x} + 5x$$

$$\boxed{y = x_2 + 5x_1} \rightarrow \star$$

$$\underset{3\times 1}{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}} = \underset{A_c \quad 3\times 3}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} + \underset{\substack{3\times 1 \\ B_c}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} u \end{bmatrix}_{1\times 1}$$

$$\begin{bmatrix} y \end{bmatrix} = \underset{C_c}{\begin{bmatrix} 5 & 1 & 0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underset{D_c}{\begin{bmatrix} 0 \end{bmatrix}} \begin{bmatrix} u \end{bmatrix}$$

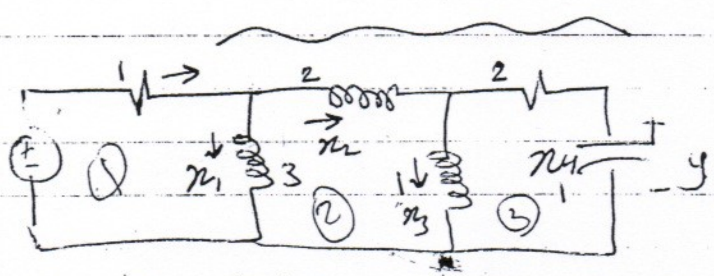
Jordan form

$$H(s) = \frac{2}{(s+1)} - \frac{3}{(s+2)} + \frac{1}{s+3}$$

$$A = \begin{bmatrix} -1 & & 0 \\ & -2 & \\ 0 & & -3 \end{bmatrix} \; ; \; B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \; ; \; D = \begin{bmatrix} 0 \end{bmatrix}$$

Q2



assume the states are: x_1, x_2, x_3 are the ~~state~~ current through the inductors.

& x_4 is the voltage across the capacitor

Loop (1) $-u + (x_1 + x_2)\cdot 1 + 3\dot{x}_1 = 0$

$$\boxed{\dot{x}_1 = \frac{u}{3} - \frac{x_1}{3} - \frac{x_2}{3}} \longrightarrow (1)$$

loop (2)

$$-3\dot{x}_1 + 2\dot{x}_2 + \dot{x}_3 = 0 \longrightarrow (2)^*$$

Loop (3) $-\dot{x}_3 + 2(x_2 - x_3) + x_4 = 0$

$$\boxed{\dot{x}_3 = 2x_2 - 2x_3 + x_4} \longrightarrow (3)$$

$$x_4 = \frac{1}{C}\int (x_2 - x_3)\, dt$$

$$\boxed{\dot{x}_4 = x_2 - x_3} \longrightarrow (4)$$

subsitute (3) & (1) in (2)*

$$-\cancel{3}\left(\frac{u}{\cancel{3}} - \frac{x_1}{\cancel{3}} - \frac{x_2}{\cancel{3}}\right) + 2\dot{x}_2 + 2x_2 - 2x_3 + x_4$$

$$\boxed{\dot{x}_2 = \frac{1}{2}u - \frac{1}{2}x_1 - \frac{3}{2}x_2 + x_3 - \frac{x_4}{2}} \longrightarrow (2)$$

$$\boxed{y = x_4} \longrightarrow \star$$

$$\underset{4\times1}{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}} = \underset{A \quad 4\times4}{\begin{bmatrix} -1/3 & -1/3 & 0 & 0 \\ -1/2 & -3/2 & 1 & -1/2 \\ 0 & 2 & -2 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} + \underset{B \quad 4\times1}{\begin{bmatrix} 1/3 \\ 3/2 \\ 0 \\ 0 \end{bmatrix}} \underset{1\times1}{[u]}$$

$$\underset{1\times1}{[y]} = \underset{4\times4}{\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0] \underset{1\times1}{[u]}$$

Q3

$$M(t) = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$$

$$\dot{x}_{11} = -3x_{11} + e^{2t} x_{12} \longrightarrow (1)$$

$$\dot{x}_{12} = -2x_{12} \longrightarrow x_{12} = e^{-2t} x_0 \longrightarrow (2)$$

Subsitute (2) in (1)

$\Rightarrow$ when $x_{12} = e^{-2t}$

$$\dot{x}_{11} = -3\cancel{e^{-2t}} + -3x_{11} + 1$$

$$x_{11} = e^{\int_0^t 3\,dt} + \int_0^t e^{\int_\tau^t 3\,d\tau} \cdot 1\, d\tau.$$

$$= e^{-3t} + \int_0^t e^{-3t} \cdot e^{3\tau}\, d\tau$$

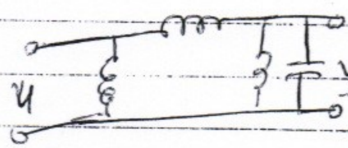
$$= e^{-3t} + e^{-3t} \int_0^t e^{3\tau}\, d\tau$$

$$= e^{-3t} + e^{-3t} \frac{1}{3} \left[e^{3\tau} \right]_0^t$$

$$= e^{-3t} + e^{-3t} \frac{1}{3} \left[e^{3t} - 1 \right]$$

$$= e^{-3t} + \frac{1}{3} - \frac{1}{3} e^{-3t}$$

$$\boxed{x_{11} = \frac{1}{3} + \frac{2}{3} e^{-3t}} \quad ☆$$



when $x_{12} = 0$

$$\dot{x}_{11} = -3x_{11} \Rightarrow x_{11} = e^{-3t}$$

$$\rightarrow M(t) = \cancel{\begin{bmatrix} e^{-2t} & 0 \\ 1/3 + 2/3\, e^{-3t} & e^{-3t} \end{bmatrix}} = \begin{bmatrix} 1/3 + 2/3\, e^{-3t} & e^{-3t} \\ e^{-2t} & 0 \end{bmatrix}$$

to check $\dot{M}(t) = \begin{bmatrix} -2e^{-3t} & -3e^{-3t} \\ -2e^{-2t} & 0 \end{bmatrix} = \begin{bmatrix} -3 & e^{2t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1/3 + 2/3\, e^{-3t} & e^{-3t} \\ e^{-2t} & 0 \end{bmatrix}$

e 471/613 2nd Exam Nov. 20, 96

Given $\dot{x} = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

The design objective is to have the closed-loop poles at $\lambda = -5$ and $\lambda = -6$.

Show to what extend one can meet the desig. objectives. Show all work. (Carry out the design as far as it can be taken). Use $u = -Kx$ and discuss your results.

2. Given $\dot{x} = A(t)x$ find the state transition matrix $\Phi(t, t_0)$ when

(25) $A(t) = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix}$. ~~Hint:~~ First Compute $\Phi(t,0)$ by defining

$\beta(t) = \int_0^t \omega(\tau)\, d\tau$.

3. A characteristic polynomial is $(s-\lambda_1)^4 (s-\lambda_2)^2 = 0$. List all the possible Jordan Canonical forms of the matrix.

(25)

$= n - \text{Rank}(\lambda_1 - A)$

5

5 1 2 1 3 2 3 2

4. Given

(25) $\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$ and $x(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Compute $x(2)$ when $u(t)$ = units step.

$A_c = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$C_x \quad A_c \quad (\lambda I - A)$

$\mathcal{C} = (\lambda+1)(\lambda+2) \begin{bmatrix} \lambda+1 & -3 \\ 0 & \lambda+2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Exam # 2 nov. 20.96

Q1 $\dot{x} = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

~~Same as exam # 2 Q3 1992~~

Q2 $\dot{x} = A(t)\, x$

$$A(t) = \begin{bmatrix} 0 & \omega(t) \\ -\omega(t) & 0 \end{bmatrix}$$

$\therefore$ $A(t)$ & $A(t_0)$ commute, so we can use Method # 3

$$\Phi(t,0) = e^{\int_0^t A(\tau)\, d\tau}$$

$$\int_0^t A(\tau)\, d\tau = \int_0^t \begin{bmatrix} 0 & \omega(\tau) \\ -\omega(\tau) & 0 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 0 & B(t) \\ -B(t) & 0 \end{bmatrix} \Rightarrow B$$

$$e^{\begin{bmatrix} 0 & B(t) \\ -B(t) & 0 \end{bmatrix}}$$

$p(B) = e^{B}$; $p(\lambda) = e^{\lambda}$

$p(\lambda) = \alpha_0 + \alpha_1 \lambda$

$$\det(\lambda I - B) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & B(t) \\ -B(t) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & -B(t) \\ B(t) & \lambda \end{bmatrix} \Rightarrow \lambda^2 + B^2(t) \Rightarrow \lambda_{1,2} = \pm B(t).$$

$P(\lambda) = R(\lambda)$

$\Rightarrow e^{B(t)} = \alpha_0 + \alpha_1 B(t) \longrightarrow (1)$

$\Rightarrow e^{-B(t)} = \alpha_0 - \alpha_1 B(t) \longrightarrow (2)$

add (1) + (2) $\quad e^{B(t)} + e^{-B(t)} = 2\alpha_0$

$$\boxed{\alpha_0 = \frac{e^{B(t)} + e^{-B(t)}}{2}} \longrightarrow (1)^*$$

from (1) $e^{B(t)} = \frac{e^{B(t)} + e^{-B(t)}}{2} + \alpha_1 B(t)$

$$\Rightarrow \cancel{B(t)}\alpha_1 = \frac{e^{B(t)} - e^{-B(t)}}{2}$$

$$\boxed{\alpha_1 = \frac{e^{B(t)} - e^{-B(t)}}{2B(t)}} \longrightarrow (2)^*$$

$$e^{B} = \alpha_0 I + \alpha_1 B$$

$$= \begin{bmatrix} \frac{1}{2}(e^{B(t)} + e^{-B(t)}) & 0 \\ 0 & \frac{1}{2}(e^{B(t)} + e^{-B(t)}) \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}(e^{B(t)} - e^{-B(t)}) \\ -\frac{1}{2}(e^{B(t)} - e^{-B(t)}) & 0 \end{bmatrix}$$

$$e^{B} = \begin{bmatrix} \frac{1}{2}(e^{B(t)} + e^{-B(t)}) & \frac{1}{2}(e^{B(t)} - e^{-B(t)}) \\ -\frac{1}{2}(e^{B(t)} - e^{-B(t)}) & \frac{1}{2}(e^{B(t)} + e^{-B(t)}) \end{bmatrix}$$

$$\Phi(t, 0) = e^{B}$$

$$\Phi(t, t_0) = \Phi(t, 0) \cdot \Phi(0, t_0)$$
$$= \Phi(t, 0) \cdot \Phi^{-1}(t_0, 0)$$

$$\Phi^{-1}(t_0, 0) \Rightarrow \begin{bmatrix} \frac{1}{2}(e^{B(t_0)} + e^{-B(t_0)}) & \frac{1}{2}(e^{B(t_0)} - e^{-B(t_0)}) \\ -\frac{1}{2}(e^{B(t_0)} - e^{-B(t_0)}) & \frac{1}{2}(e^{B(t_0)} - e^{-B(t_0)}) \end{bmatrix}$$

$$\cancel{\det = \frac{1}{4}(e^{B(t_0)} + e^{-B(t_0)})^2 + \frac{1}{4}(e^{B(t_0)}}$$

$$det = \frac{1}{4}\left(e^{B(t_0)} + e^{-B(t_0)}\right)\left(e^{B(t_0)} - e^{-B(t_0)}\right) + \frac{1}{4}\left(e^{B(t_0)} - e^{-B(t_0)}\right)\left(e^{B(t_0)} + e^{-B(t_0)}\right)$$

$$= \frac{1}{2}\left(e^{B(t_0)} + e^{-B(t_0)}\right)\left(e^{B(t_0)} - e^{-B(t_0)}\right)$$

$$= \frac{1}{2}\left[e^{2B(t_0)} - e^{-2B(t_0)}\right] \longrightarrow \star$$

$$C_{factor} = \begin{bmatrix} \frac{1}{2}\left(e^{B(t_0)} + e^{-B(t_0)}\right) & -\frac{1}{2}\left(e^{B(t_0)} - e^{-B(t_0)}\right) \\ \frac{1}{2}\left(e^{B(t_0)} - e^{-B(t_0)}\right) & \frac{1}{2}\left(e^{B(t_0)} + e^{-B(t_0)}\right) \end{bmatrix}$$

$$C_{factor}^T = \begin{bmatrix} \frac{1}{2}\left(e^{B(t_0)} + e^{-B(t_0)}\right) & \frac{1}{2}\left(e^{B(t_0)} - e^{-B(t_0)}\right) \\ -\frac{1}{2}\left(e^{B(t_0)} - e^{-B(t_0)}\right) & \frac{1}{2}\left(e^{B(t_0)} + e^{-B(t_0)}\right) \end{bmatrix}$$

$$\Phi^{-1}(t_0, 0) = \frac{C_{factor}^T}{det} = \begin{bmatrix} \frac{1}{\left(e^{B(t_0)} - e^{-B(t_0)}\right)} & \frac{1}{\left(e^{B(t_0)} + e^{-B(t_0)}\right)} \\ -\frac{1}{\left(e^{B(t_0)} + e^{-B(t_0)}\right)} & \frac{1}{\left(e^{B(t_0)} - e^{-B(t_0)}\right)} \end{bmatrix}$$

$$\Phi(t, t_0) = \Phi(t, 0) \cdot \Phi^{-1}(t_0, 0)$$

$$= \begin{bmatrix} \frac{1}{2}\left(e^{B(t)} + e^{-B(t)}\right) & \frac{1}{2}\left(e^{B(t)} - e^{-B(t)}\right) \\ -\frac{1}{2}\left(e^{B(t)} - e^{-B(t)}\right) & \frac{1}{2}\left(e^{B(t)} + e^{-B(t)}\right) \end{bmatrix} \begin{bmatrix} \frac{1}{e^{B(t_0)} - e^{-B(t_0)}} & \frac{1}{\left(e^{B(t_0)} + e^{-B(t_0)}\right)} \\ -\frac{1}{\left(e^{B(t_0)} + e^{-B(t_0)}\right)} & \frac{1}{\left(e^{B(t_0)} - e^{-B(t_0)}\right)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\left[\frac{e^{B(t)} + e^{-B(t)}}{e^{B(t_0)} - e^{-B(t_0)}} - \frac{e^{B(t)} - e^{-B(t)}}{\left(e^{B(t_0)} + e^{-B(t_0)}\right)}\right] & \frac{1}{2}\left[\frac{e^{B(t)} + e^{-B(t)}}{e^{B(t_0)} + e^{-B(t_0)}} + \frac{e^{B(t)} - e^{-B(t)}}{e^{B(t_0)} - e^{-B(t_0)}}\right] \\ -\frac{1}{2}\left[\frac{e^{B(t)} - e^{-B(t)}}{e^{B(t_0)} - e^{-B(t_0)}} + \frac{\left(e^{B(t)} + e^{-B(t)}\right)}{\left(e^{B(t_0)} + e^{-B(t_0)}\right)}\right] & -\frac{1}{2}\left[\frac{e^{B(t)} + e^{-B(t)}}{e^{B(t_0)} + e^{-B(t_0)}} - \frac{\left(e^{B(t)} + e^{-B(t)}\right)}{\left(e^{B(t_0)} - e^{-B(t_0)}\right)}\right] \end{bmatrix}$$

Q.4

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\dot{x}_1 = 0 \longrightarrow (1)$

$\dot{x}_2 = x_1 + u \longrightarrow (2) \Rightarrow \dot{x}_2 = x_1 + 1 \Rightarrow x_1 = \dot{x}_2 - 1 \longrightarrow (2)^*$

$\dot{x}_3 = x_1 + 2x_2 \longrightarrow (3) \Rightarrow x_2 = \frac{x_1 - \dot{x}_3}{2} \Rightarrow \frac{\dot{x}_2 - \dot{x}_3 - 1}{2} = x_2 \rightarrow (3)^*$

~~x_1~~ = from $(2)^*$ & $(3)^*$

~~assume~~ $\dot{x}_2 = 2$; $\dot{x}_3 = 1$; $\dot{x}_1 = 0$

$x(2) = \begin{bmatrix} 0 \\ \\ \end{bmatrix}$

Mishaal Alkaabi

ELEC 481/ ENGR 6131

Midterm Exam

ALL PROBLEMS ARE EQUALLY WEIGHTED.

(1) The response of a LTI causal system to the input $u_1(t) = u_s(t)\, u_s(2-t)$ is $y_1(t) = (1-e^{-t})\, u_s(t)\, u_s(2-t) + e^{-(t-2)} u_s(t-2)$. Find the response of the system to a new input $u_2(t) = (t-2)\, u_s(t-2)\, u_s(4-t)$ for $t \le 4$. Note that $u_s(t)$ designates a unit step function.

(2) Consider a LTI system

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} x$$

Let $u(t) = \begin{bmatrix} -3e^{-t} + 5e^{-2t} \\ 3e^{-2t} \end{bmatrix}$ for $t \ge 0$, and suppose that $y(t) = -2e^{-t} + 6e^{-2t}$. If $\lim_{t\to\infty} x(t) = 0$, find $x(0)$.

(3) Given $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t-1) + |u(t)| \\ x_1(t) \end{bmatrix}$; $y(t) = x_1(t+1) + x_2(t) - u(t)$

Find the appropriate description of the system among:

Linear	Time-Invariant	Causal	Lumped	Continuous-time
Nonlinear	Time-Variant	Noncausal	Distributed	Discrete-time

(4) Find a state space representation (A, B, C, D) for the system described by

$$\ddot{y}_1 + 2\dot{y}_1 + 3(y_1 - y_2) = 2u_1 + \dot{u}_2$$

$$\dot{y}_2 - 4(y_1 - y_2) = 2u_2 + 4\dot{u}_1$$

Hint: Use the state diagram approach.

ELEC 481 / ENGR 6131 Exam # 2 Nov. 2012

1. Is the following system controllable and observable? Specify which modes are controllable and observable and which ones are not.

$$\dot{x} = \left[\begin{array}{ccc|ccc|cc} 1 & 1 & 0 & & & & & \\ 0 & 1 & 0 & & & 0 & & \\ \hline & & 1 & 1 & 0 & & & \\ & & 0 & 1 & 0 & & & \\ & & 0 & 0 & 1 & 0 & 0 & \\ \hline & 0 & & & & 2 & 0 & \\ & & & & & 0 & 2 & \end{array}\right] x + \left[\begin{array}{ccc} 1 & 1 & 3 \\ -1 & 0 & 1 \\ \hline 2 & 3 & 9 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ \hline 0 & 0 & 0 \\ 1 & -1 & 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cc|ccc|cc} 5 & 2 & 1 & 1 & 1 & 1 & 0 \\ -2 & 4 & -1 & -1 & 0 & 0 & 1 \\ 3 & 3 & 0 & 0 & 1 & 1 & 0 \end{array}\right] x$$

2. A characteristics polynomial of a system is given by $(\lambda - \lambda_1)^4 (\lambda - \lambda_2)^2 = 0$. List all possible Jordan Canonical form of the A matrix associated with the above polynomial.

3. Find the fundamental matrix $M(t)$ and the state transition matrix $\Phi(t,0)$ for the following system

$$\dot{x} = \begin{bmatrix} -5 & e^{3t} \\ 0 & -3 \end{bmatrix} x$$

4. Let the state transition matrix of a system be given by

$$\Phi(t,0) = e^{-t} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

(a) Compute $\Phi(t, t_0)$, (b) Find A.

NOTE: ALL PROBLEMS ARE EQUALLY WEIGHTED