5. Plant Dynamic Models

This Chapter provides time and Laplace domain expressions which are useful for linear control implementation and are used in the experiments described later in this manual. Details of the development of these equations as well as their motivation and additional equation forms are provided in Appendix A.

5.1 Two Degree of Freedom Plants

The most general form of the two degree of freedom torsional system is shown in Figure 5-1a where friction is idealized as being viscous. Using the free body diagram of Figure 5-1b and summing torques acting on J_1 we have via the rotational form of Newton's second law:

$$J_1\ddot{\theta}_1+c_1\dot{\theta}_1+k_1\theta_1-k_1\theta_2 = T(t) \tag{5.1-1}$$

Similarly from Figure 5-1c for J_2:

$$J_2\ddot{\theta}_2+c_2\dot{\theta}_2+(k_1+k_2)\theta_2-k_1\theta_1 = 0 \tag{5.1-2}$$

These may be expressed in a state space realization as:

$$\begin{aligned}\dot{x} &= Ax + BT(t)\\ Y &= Cx\end{aligned} \tag{5.1-3}$$

where:

$$X = \begin{bmatrix}\theta_1\\ \dot{\theta}_1\\ \theta_2\\ \dot{\theta}_2\end{bmatrix},\quad A = \begin{bmatrix}0 & 1 & 0 & 0\\ -k_1/J_1 & -c_1/J_1 & k_1/J_1 & 0\\ 0 & 0 & 0 & 1\\ k_1/J_2 & 0 & -(k_1+k_2)/J_2 & -c_2/J_2\end{bmatrix},$$

$$B = \begin{bmatrix}0\\ 1/J_1\\ 0\\ 0\end{bmatrix},\quad C = \begin{bmatrix}C_1 & 0 & 0 & 0\\ 0 & C_2 & 0 & 0\\ 0 & 0 & C_3 & 0\\ 0 & 0 & 0 & C_4\end{bmatrix}$$

and $C_i = 1$ (i=1,2,3,4) when X_i is an output and equals 0 otherwise.

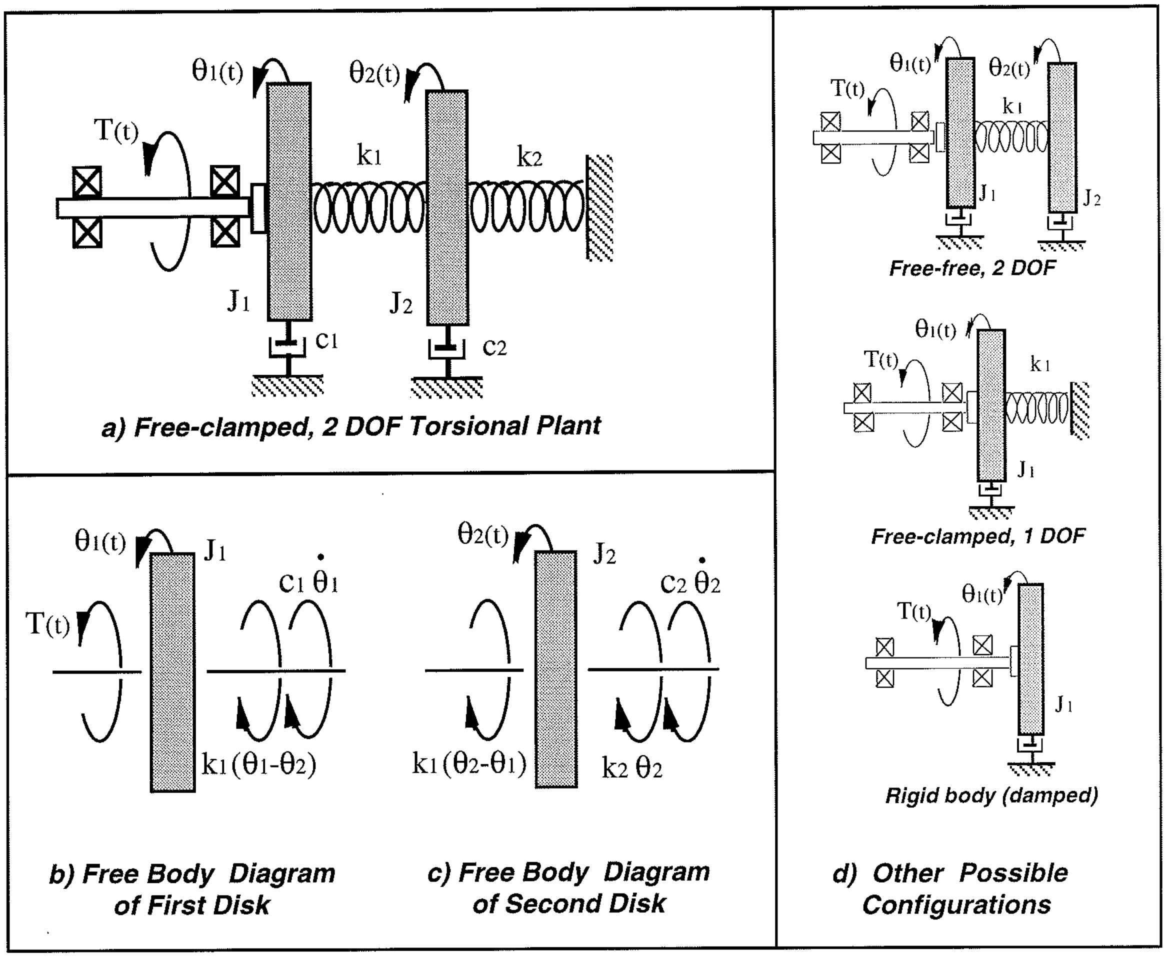

Figure 5-1 Two DOF Plant Models

By Laplace transform of Eq's (5.1-1,-2) and assuming zero valued initial conditions we may solve for the transfer functions:

$$\frac{\theta_1(s)}{T(s)} = \frac{J_2 s^2 + c_2 s + k_1 + k_2}{D(s)} \tag{5.1-4}$$

$$\frac{\theta_2(s)}{T(s)} = \frac{k_1}{D(s)} \tag{5.1-5}$$

where:

$$\begin{aligned} D(s) = J_1 J_2 s^4 + \left(c_1 J_2 + c_2 J_1\right) s^3 + \left(J_1(k_1+k_2) + J_2 k_1 + c_1 c_2\right) s^2 \\ + \left(c_1(k_1+k_2) + c_2 k_1\right) s + k_1 k_2 \end{aligned} \tag{5.1-6}$$

which may also be expressed in the form:

$$\frac{\theta_1(s)}{T(s)} = \frac{K_1\left(s^2 + 2\zeta_z\omega_z s + \omega_z^2\right)}{\left(s^2 + 2\zeta_{p1}\omega_{p1}s + \omega_{p1}^2\right)\left(s^2 + 2\zeta_{p2}\omega_{p2}s + \omega_{p2}^2\right)} \tag{5.1-7}$$

$$\frac{\theta_2(s)}{T(s)} = \frac{K_2}{\left(s^2 + 2\zeta_{p1}\omega_{p1}s + \omega_{p1}^2\right)\left(s^2 + 2\zeta_{p2}\omega_{p2}s + \omega_{p2}^2\right)} \tag{5.1-8}$$

where the ω_i's and ζ_i's are the natural frequencies and damping ratios respectively, and the gains K_1 & K_2, are nominally equal to $1/J_1$ and k_1/J_1J_2 (but often may be measured more directly).

For the case $k_2 = 0$ ($\omega_1 = 0$), a damped rigid body motion exists and Eq's (5.1-7,-8) become:

$$\frac{\theta_1(s)}{T(s)} = \frac{K_1\left(s^2 + 2\zeta_z\omega_z s + \omega_z^2\right)}{s\left(s + c^*\right)\left(s^2 + 2\zeta_{p2}\omega_{p2}s + \omega_{p2}^2\right)} \tag{5.1-9}$$

$$\frac{\theta_2(s)}{T(s)} = \frac{K_2}{s\left(s + c^*\right)\left(s^2 + 2\zeta_{p2}\omega_{p2}s + \omega_{p2}^2\right)} \tag{5.1-10}$$

The equations describing the single DOF plant may be obtained from the above or found in Appendix A.

5.2 Three Degree of Freedom Plants (Model 205a only)

The time domain equations of motion for the three DOF torsional plant are (see Appendix A, Figure A-4):

$$J_1\ddot{\theta}_1 + c_1\dot{\theta}_1 + k_1\theta_1 - k_1\theta_2 = T(t) \tag{5.2-1}$$

$$J_2\ddot{\theta}_2 + c_2\dot{\theta}_2 + (k_1+k_2)\theta_2 - k_1\theta_1 - k_2\theta_3 = 0 \tag{5.2-2}$$

$$J_3\ddot{\theta}_3 + c_3\dot{\theta}_3 + k_2\theta_3 - k_2\theta_2 = 0 \tag{5.2-3}$$

which may be expressed in a state space realization:

$$\begin{aligned} \dot{x} &= Ax + BT(t) \\ Y &= Cx \end{aligned} \tag{5.2-4}$$