Concordia University
Electrical and Computer Engineering
Faculty of Engineering and Computer Science

LINEAR SYSTEMS

ELEC 481 / ENGR 6131

DR K. KHORASANI

Contents

Table of Figures

1 Lecture 1

Objectives:

1) ***Introduction to design of physical systems***
2) ***System classification***
3) ***Applications***
4) ***I/O description of systems***
5) ***Linearity***

1.1 Introduction to Design of Physical Systems

a) Performance Specification

Specifications and Requirements

b) Modeling
   - A tool for study
   - A representation of the physical system
   - ODE or difference equations

Time domain vs frequency domain
rise time, %OS
Bandwidth, peak resonance

c) Simulation
   - A tool for studying behavior of ODE's or difference equations

d) Analysis

BIBO bounded input bounded output

   - Qualitative: stability, controllability, observability
   - Quantitative: simulation

PID->Proportional,integral & Derivative

e) Optimization/control

Parsimonious and implicit

   - Optimize parameters of the system

f) Realization

Prototype implementation

PI^nD^M
PI^2D^4

Example 1: Vehicle suspension system

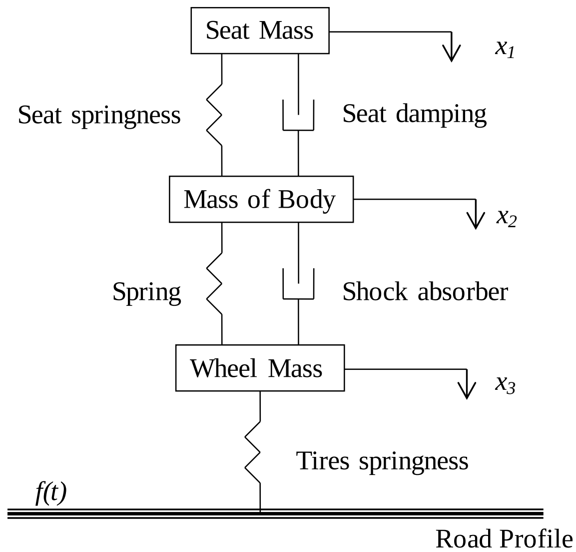


Figure 1-1. Vehicle suspension system.

1.2 System Classification

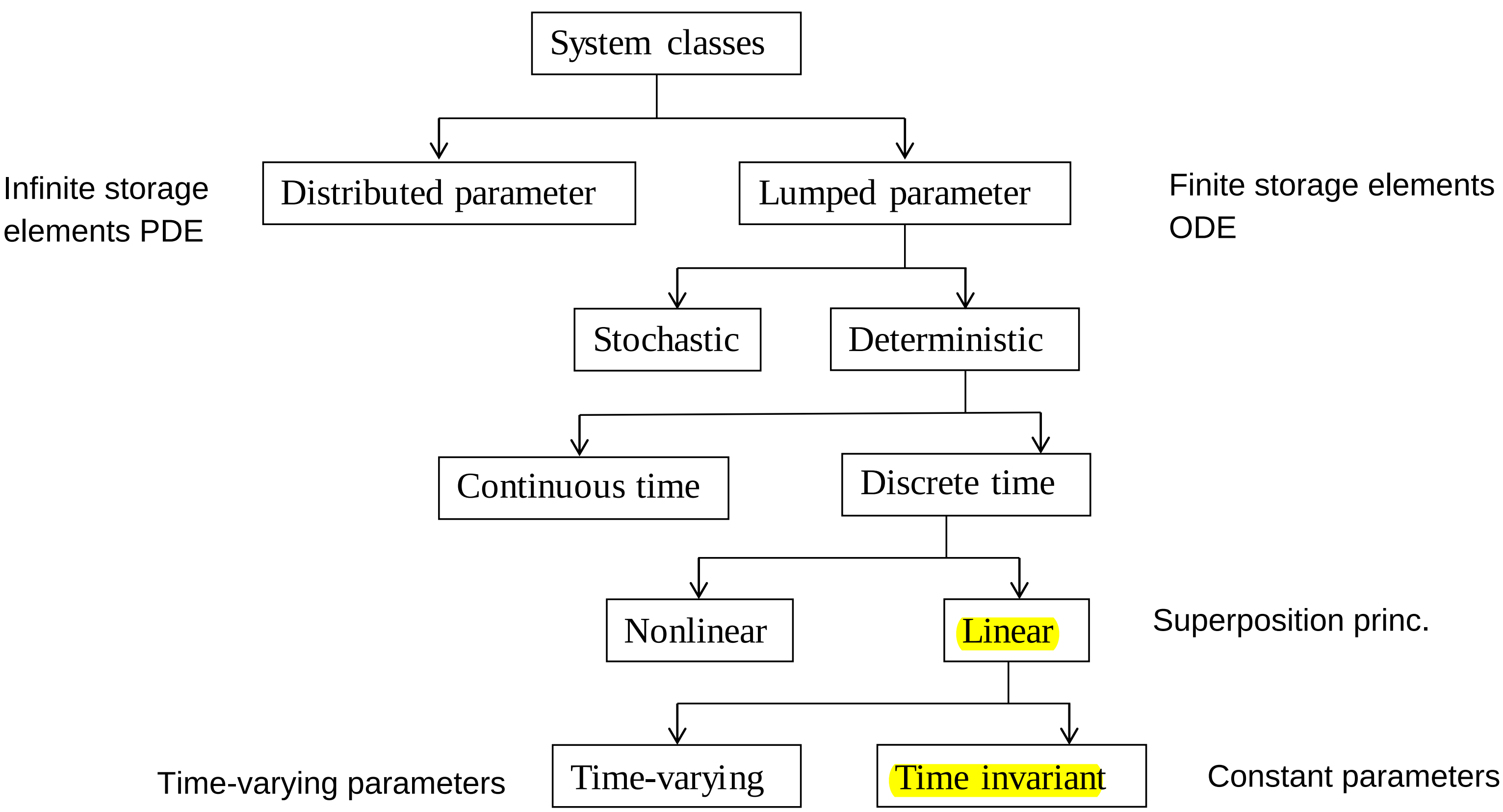


Figure 1-2. System classification.

Linear spring:	Nonlinear spring:
$\begin{aligned} x_1 &\to f_1 = kx_1 \\ x_2 &\to f_2 = kx_2 \\ x_1 + x_2 &\to f_{1,2} = k(x_1 + x_2) \\ &\qquad \equiv f_1 + f_2 \end{aligned}$	$\begin{aligned} x_1 &\to f_1 = kx_1^3 \\ x_2 &\to f_2 = kx_2^3 \\ x_1 + x_2 &\to f_{1,2} = k(x_1 + x_2)^3 \\ &\qquad \neq f_1 + f_2 \end{aligned}$

1.3 Applications

a) Deterministic Control Problem

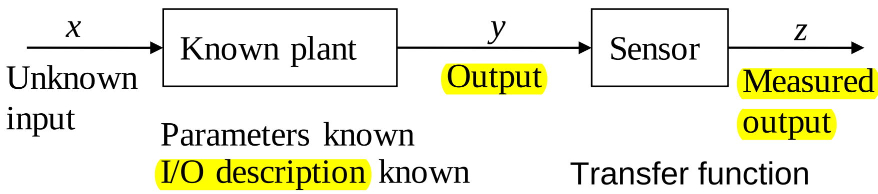


Figure 1-3. Application #1.

b) Estimation Problems

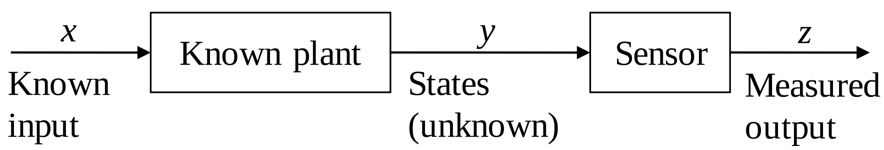


Figure 1-4. Application #2.

Internal representation: State characterization

c) Stochastic Control Problem

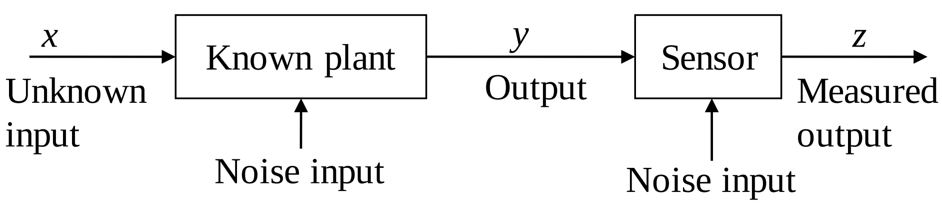
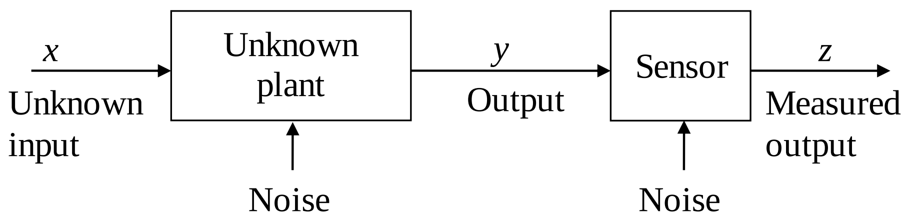


Figure 1-5. Application #3.

d) System Identification



Parameters unknown, however I/O description known

Figure 1-6. Application #4.

e) Adaptive Control Problem

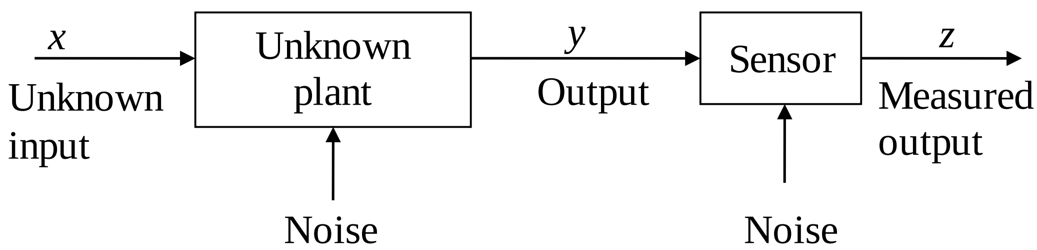
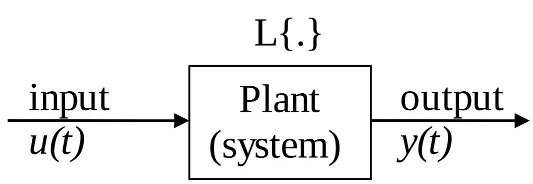


Figure 1-7. Application #5.

1.4 I/O Description of Dynamical Systems

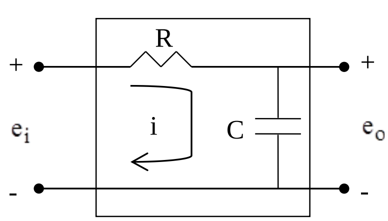
External representation
vs
Internal representation -State space representation



y(t) = L{u(t)} (*) ,

Under what conditions (*) holds.

Figure 1-8. Input/output system representation.



$$\begin{cases} e_i = Ri + \frac{1}{c}\int i dt \\ e_o = \frac{1}{c}\int i dt \end{cases} \Rightarrow \frac{de_o}{dt} = \frac{i}{c} \Rightarrow i = c\frac{de_o}{dt}$$

Figure 1-9. RC network.

$$\therefore \; e_i = RC\frac{de_o}{dt} + e_o$$

R(t)C(t)

1) $e_i = 0$ and $e_o(0) \neq 0$ $\longrightarrow$ zero input response

2) $e_i \neq 0$ and $e_o(0) = 0$ $\longrightarrow$ zero state response

3) $e_i \neq 0$ and $e_o(0) \neq 0$ $\longrightarrow$ general response

Case 1: $e_i = 0 \;\Rightarrow RC\dfrac{de_o}{dt} + e_o = 0$

$$\therefore \quad e_o(t) = e_o(0)e^{-t/RC}$$

(zero input response)

Case 2: $e_o(0) = 0$ and $e_i = A$ (constant) $\quad \therefore RC\dfrac{de_o}{dt} + e_o = A \;;\; e_o(0) = 0$

$$\therefore \quad e_o(t) = A\left(1 - e^{-t/RC}\right)$$

(zero state response)

Case 3:
$$e_o(t) = e_o(0)e^{-t/RC} + A\left(1 - e^{-t/RC}\right)$$

(general response)

Relaxed (at rest): Condition for defining the I/O

A system is ***relaxed*** (or at rest), if the initial conditions (i.c.) are set to zero.

1.5 Definition: Linearity

A relaxed system is said to be ***linear***, ***if and only if***

$$L\{a_1u_1(t) + a_2u_2(t)\} = a_1L\{u_1(t)\} + a_2L\{u_2(t)\}$$

(Superposition principle)

2 Lecture 2

Objectives:

1) ***Examples***
2) ***Introduction to causality and time-invariance system***

Examples of linear systems :

Example 1 : RC net.

$$RC\frac{de_o}{dt}+e_o=e_i \quad \Rightarrow \frac{de_o}{dt}+\frac{1}{RC}e_o=\frac{1}{RC}e_i \quad (*)$$

Since Integrating factor $\frac{d}{dt}\left[e^{t/RC}e_o\right]=e^{t/RC}\frac{de_o}{dt}+\frac{1}{RC}e^{t/RC}e_o \quad (**)$

$$(*)(**)\Rightarrow \frac{d}{dt}\left[e^{t/RC}e_o\right]=\frac{1}{RC}e^{t/RC}e_i \Rightarrow \int_0^{\tau}\frac{d}{dt}\left(e^{t/RC}e_o\right)dt=\int_0^{\tau}\frac{1}{RC}e^{t/RC}e_i dt$$

$$\Rightarrow \quad e^{\tau/RC}e_o(\tau)-e^0e_o(0)=\int_0^{\tau}\frac{1}{RC}e^{t/RC}e_i dt$$

Since the system is relaxed $\Rightarrow \ e_o(0)=0$

$$e^{\tau/RC}e_o(\tau)=\frac{1}{RC}\int_0^{\tau}e^{t/RC}e_i dt \quad \Rightarrow e_o(\tau)=\frac{1}{RC}e^{-\tau/RC}\int_0^{\tau}e^{t/RC}e_i dt$$

$$\Rightarrow \quad e_o(\tau)=\frac{1}{RC}\int_0^{\tau}e^{(t-\tau)/RC}e_i dt$$

e_i(t)

(I/O representation)

Q: Show if the above I/O representation is linear?

To show this, we use superposition principal

$$e_1 \xrightarrow{L} e_0^1 \;;\; e_2 \xrightarrow{L} e_0^2 \;\Rightarrow$$

$$e_0^1(\tau) = \frac{1}{RC}\int_0^{\tau} e^{(t-\tau)/RC} e_1 dt \quad ; \quad e_0^2(\tau) = \frac{1}{RC}\int_0^{\tau} e^{(t-\tau)/RC} e_2 dt$$

$$a_1 e_1 + a_2 e_2 \xrightarrow{L} e_0 \;\Rightarrow$$

$$e_0(\tau) = \frac{1}{RC}\int_0^{\tau} e^{(t-\tau)/RC}(a_1 e_1 + a_2 e_2)dt$$

$$= \frac{1}{RC}\int_0^{\tau} e^{(t-\tau)/RC} a_1 e_1 dt \; + \frac{1}{RC}\int_0^{\tau} e^{(t-\tau)/RC} a_2 e_2 dt$$

$\therefore \quad e_0 = a_1 e_0^1 + a_2 e_0^2 \;\Rightarrow$ The system is linear.

Example 2: System $y(t) = \int_0^t u(\tau)d\tau$, is this system linear?

$$u_1 \to y_1 \;;\; u_2 \to y_2 \;\Rightarrow\; y_1 = \int_0^t u_1 d\tau \;;\; y_2 = \int_0^t u_2 d\tau$$

$$u = a_1 u_1 + a_2 u_2 \to y \;\Rightarrow$$

$$y = \int_0^t (a_1 u_1 + a_2 u_2)d\tau = a_1\int_0^t u_1 d\tau + a_2\int_0^t u_2 d\tau = a_1 y_1 + a_2 y_2$$

Therefore, the system is linear.

Example 3: $y(t) = \int_0^t u^2(\tau)d\tau$ is nonlinear

2.1 Definition: Causality

The system is said to be ***causal*** if the output at time t does not depend on the input at times after t.

A system which is not causal is said to be ***noncausal*** or ***anticipatory***.

2.2 Definition: Time-Invariance

Time-varying vs time-invariant

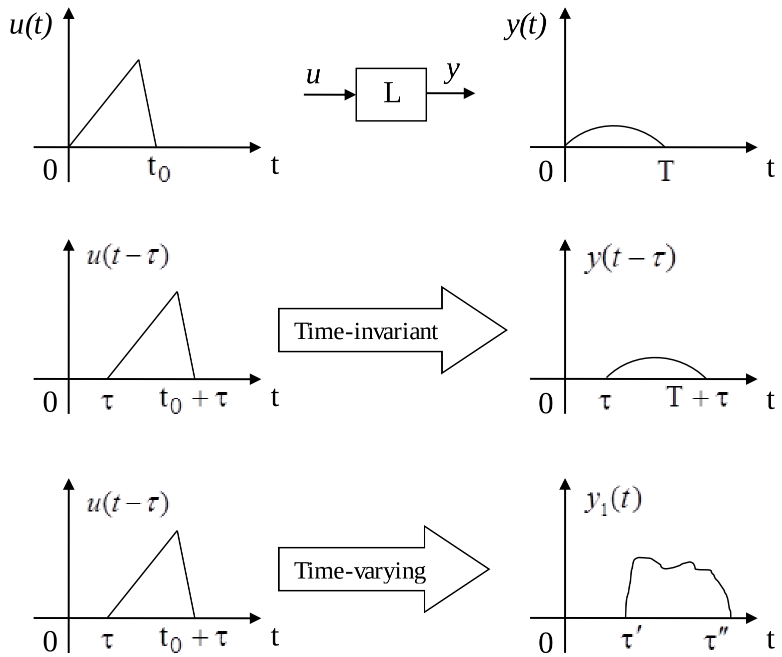


Figure 2-1. Time varying systems.

t+1

Example 4: RC net. $$e_0(t) = \frac{1}{RC}\int_0^t e^{-(t-\tau)/RC} e_i(\tau)\, d\tau$$

with t is causal with t+1 is noncausal

Show if this system is time varying or not.

Change $e_i(\tau) \rightarrow e_i(\tau - T) \;\Rightarrow\; e_0'(t) = \dfrac{1}{RC}\displaystyle\int_0^t e^{-(t-\tau)/RC} e_i(\tau - T)\, d\tau$

Let $\tau' = \tau - T \;\Rightarrow\; \tau = \tau' + T$, $d\tau = d\tau'$

$$\therefore\; e_0'(t) = \frac{1}{RC}\int_{-T}^{t-T} e^{-(t-\tau'-T)/RC} e_i(\tau')\, d\tau'$$

Since $e_i(\tau') = 0$ for all $\tau' < 0$!

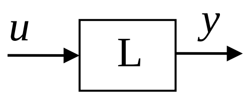
$$\Rightarrow e_0'(t) = \frac{1}{RC}\int_{0}^{t-T} e^{-(t-\tau'-T)/RC} e_i(\tau')\, d\tau'$$

$$\therefore\; e_0'(t) = e_o(t - T)$$

The system is time-invariant

Example 5: $\dot{y}(t)+\frac{1}{t}y(t)=u(t)$

- Is this system time-invariant or time-varying?
- Is this system linear or nonlinear?



$y(t)=\mathrm{L}\{u(t)\}$

In general, for $\dot{y}+a(t)y$, by using integrating factor $e^{\int a(\tau)d\tau}$,

$$\text{since}\begin{cases}\dfrac{d}{dt}\left\{y(t)e^{\int a(t)dt}\right\}=y(t)a(t)e^{\int a(t)dt}+\dot{y}(t)e^{\int a(t)dt} \\ \dot{y}(t)+ay(t)=u(t)\ \Rightarrow \dot{y}(t)=u(t)-ay(t)\end{cases}\Rightarrow$$

$$\frac{d}{dt}\left\{y(t)e^{\int a(t)dt}\right\}=ue^{\int a(t)dt}$$

For this example $a(t)=\dfrac{1}{t}\ \Rightarrow\ e^{\int a(t)dt}=e^{\int\frac{dt}{t}}=e^{\ln t}=t$

$$\because \frac{d}{dt}\{t\cdot y(t)\}=1\cdot y(t)+t\cdot\dot{y}=t\cdot u(t)$$

so, $\displaystyle\int_b^\tau \frac{d}{dt}\{y\,t\}dt=\int_b^\tau t\,u\,\mathrm{dt}$

b is the initial time

$$\Rightarrow y(\tau)\tau-y(b)b=\int_b^\tau t\,u\,\mathrm{dt}$$

For a relaxed system, $y(b)=0$

$$\therefore\quad y(\tau)=\frac{1}{\tau}\int_b^\tau t\,u\,\mathrm{dt}=\int_b^\tau\frac{t}{\tau}u\,\mathrm{dt}$$

Q: Linear or nonlinear?

$$y(t)=\int_b^t \frac{\tau}{t}u(\tau)d\tau$$ Causal

$$u_1 \to y_1 \Rightarrow y_1=\int_b^t \frac{\tau}{t}u_1 d\tau\,; \quad u_2 \to y_2 \Rightarrow y_2=\int_b^t \frac{\tau}{t}u_2 d\tau$$

$$a_1u_1+a_2u_2 \to y \Rightarrow$$

$$y=\int_b^t \frac{\tau}{t}(a_1u_1+a_2u_2)d\tau=\int_b^t \frac{\tau}{t}a_1u_1d\tau+\int_b^t \frac{\tau}{t}a_2u_2d\tau=a_1y_1+a_2y_2$$

Therefore, the system is linear.

Q: Time-invariant or time-varying?

$$y(t)=\int_b^t \frac{\tau}{t}u(\tau)d\tau\ ,\ y(t)=\mathrm{L}\{u(t)\}$$

$u(\tau)\to u(\tau-T)$ Due to the shifted input, the resulting output y_1 is

$$y_1(t)=\int_b^t \frac{\tau}{t}u(\tau-T)d\tau$$ Question: Is $y_1(t)=y(t-T)$?

$$y(t-T)=\int_b^{t-T} \frac{\tau}{t-T}u(\tau)d\tau$$

Chang of variable $\tau'=\tau-T \Rightarrow \tau=\tau'+T$, $d\tau'=d\tau$ (T is constant)

at $\tau=b \Rightarrow \tau'=b-T$; at $\tau=t \Rightarrow \tau'=t-T$

$$y_1(t)=\int_{b-T}^{t-T} \frac{\tau'+T}{t}u(\tau')d\tau'$$

$$\therefore\quad y_1(t)\neq y(t-T)$$

The system is time-varying

3 Lecture 3

Objectives

1) ***Impulse Response of LTI and LTV systems***
2) ***Computation of the Convolution Integral for Linear Systems***

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_0 u$$

This is a linear system since $y \ldots\ y^{(n)}$ and $u \ldots\ u^{(m)}$ enter linearly.

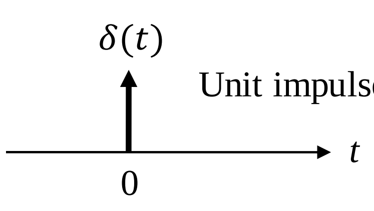
If the coefficients $a_n \ldots a_0$ **and** $b_m \ldots b_0$ **are constant**, then the system is **time-invariant**.

If they are time dependent, then the system is **time-varying**.

A nonlinear example is $y\dot{y}$, y^2, $\dot{y}^2$, uy, $\dot{u}y$, ...

3.1 Impulse Response

Let $h(t,\tau)$ be the response of the system at time t due to an impulse applied at time τ. $h(t,\tau)$ is called the impulse response of the system.



Unit impulse $\int_{-\infty}^{\infty} \delta(t)dt = 1$

Figure 3-1. Delta function.

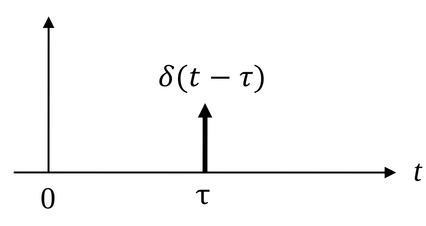


Figure 3-2. Impulse at t = τ.

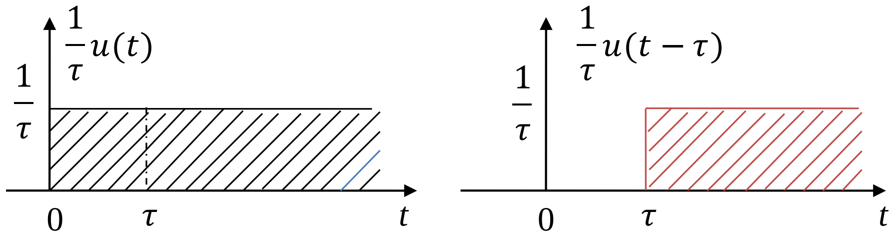


Figure 3-3. Property of step function.

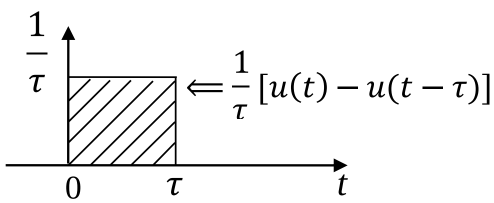


Figure 3-4. Behavior of rectangular pulse.

In the limit as $\tau \to 0 \;\Rightarrow\; \delta(t) = \lim_{\tau\to 0}\frac{1}{\tau}[u(t)-u(t-\tau)]$

For a given u , we can always write $u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau$, since

$$f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)$$

$$\Rightarrow u(t) = \int_{-\infty}^{\infty} u(t)\delta(t-\tau)d\tau$$

$$= u(t)\underbrace{\int_{-\infty}^{\infty} \delta(t-\tau)d\tau}_{1}$$

$$= u(t)$$

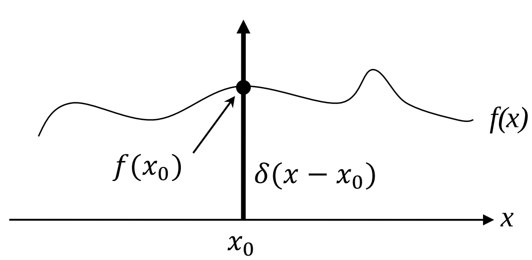


Figure 3-5. Multiplication of a function with a delta function.

3.2 Computation of the Convolution Integral for Linear Systems

The output *y(t)* due to *u(t)* is $y(t) = L\{u(t)\}$

$$y(t) = L\left\{ \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau \right\}$$ L is an operator in time *t*

$$y(t) = \int_{-\infty}^{\infty} L\{u(\tau)\delta(t-\tau)d\tau\} = \int_{-\infty}^{\infty} \underbrace{L\{\delta(t-\tau)\}}_{h(t,\tau)} u(\tau)d\tau$$

$$\therefore \; y(t) = \int_{-\infty}^{\infty} u(\tau)h(t,\tau)d\tau$$

(General Convolution Integral for a Linear, Relaxed LTI or LTV systems).

3.2.1 For a Causal, Linear, Relaxed, and Time-Varying System

For a causal, linear, relaxed, and time-varying system we now get

$$y(t) = \int_{-\infty}^{t} u(\tau)h(t,\tau)d\tau + \underbrace{\int_{t}^{\infty} u(\tau)h(t,\tau)d\tau}_{=0}$$

$$\Rightarrow \; y(t) = \int_{-\infty}^{t} u(\tau)h(t,\tau)d\tau$$

For a causal system $h(t,\tau) = 0$ for all $\tau > t$

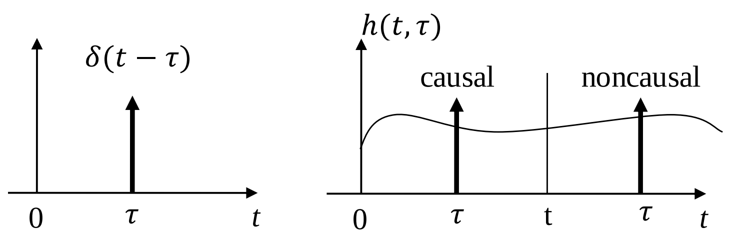


Figure 3-6. Causal versus non-causal system.

If the input is applied at time $t = t_0$ and is zero before that, then

$$y(t) = \int_{t_0}^{t} u(\tau)h(t,\tau)d\tau$$

Let *h(t)* be the impulse response due to $\delta(t)$, $\delta(t-\tau) \to h(t,\tau)$

$\delta(t) \to h(t,0) \equiv h(t)$ $\quad$ $\delta(t-\tau) \to h(t,\tau)$

Figure 3-7. Delta function and its shifted value.

For a time-invariant system $\delta(t-\tau) \to h(t-\tau,0) \equiv h(t-\tau)$

<u>Example 1</u>: $h(t,\tau) = e^{-\frac{t-\tau}{t}}, \sin t\,, e^{\sin t}$ are **time-varying**

While $h(t,\tau) = e^{-(t-\tau)^2}, \sin(t-\tau), e^{\sin(t-\tau)}$, are **time-invariant**

3.2.2 For a Linear, Relaxed, Causal, Time-Invariant System

For a linear, relaxed, causal, time-invariant system, we now get

$$y(t) = \int_{0}^{t} u(\tau)h(t-\tau)d\tau$$

We want to show that alternatively this is the same as $y(t) = \int_{0}^{t} h(\tau)u(t-\tau)d\tau$

Let $\tau' = t - \tau \;\; \Rightarrow d\tau' = -d\tau, \;\; \tau = t - \tau'$

$$y(t) = \int_{t}^{0} u(t-\tau')h(\tau')(-d\tau') = -\int_{t}^{0} h(\tau')u(t-\tau')d\tau'$$

$$= \int_{0}^{t} h(\tau')u(t-\tau')d\tau'$$

4 Lecture 4

Objectives

1) ***Impulse response computation using differential equation***
2) ***Impulse response computation using Laplace transform***

4.1 Computation of Impulse Response Using Differential Equation

For a given general system given by,

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_0 u$$

to find the impulse response, let *u(t)* be an impulse.

Assuming *a*'s and *b*'s are constant $\Rightarrow$ this is a linear time-invariant system

$$\therefore u(t) = \delta(t) \Longrightarrow y(t) = h(t)$$

System is assumed to be relaxed, that is initial conditions (I.C.'s) are zero at $t = 0^-$.

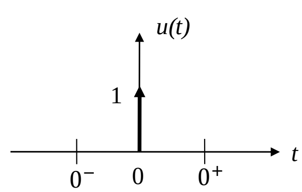


Figure 4-1. Unit impulse at t = 0^-, 0, 0^+.

Example 1: For RC network we have: $RC\frac{de_o}{dt}+e_o=e_i$. Let us find the impulse response:

For finding the impulse response $\therefore\ e_i=\delta(t)\ \Rightarrow e_0=h(t)$

$$\begin{cases} for \quad \mathrm{t}\geq 0 \quad \mathrm{RC}\frac{\mathrm{dh}}{\mathrm{dt}}+h=\delta(t) \quad (*) \\ \\ for \quad \mathrm{t}<0 \quad \mathrm{RC}\frac{\mathrm{dh}}{\mathrm{dt}}+h=0 \quad (**) \end{cases}$$

From (**) $\Rightarrow$ the system is relaxed $\therefore e_o(0^-)=0$

$$\Rightarrow h(0^-)=0\ \Rightarrow h(t)=0\ \ (t<0)$$

from (*), integrate it from $t=0^-$ to $t=0^+$. We do this to get the initial condition at $t=0^+$, in other words, we need to find $h(0^+)$.

$$\int_{0^-}^{0^+} RC\frac{dh}{dt}dt+\underbrace{\int_{0^-}^{0^+} hdt}_{=0}=\int_{0^-}^{0^+}\delta(t)dt \Rightarrow RC\left[h(0^+)-h(0^-)\right]+0=1$$

$$\because RCh(0^-)=0\ \ \Rightarrow RCh(0^+)=1\ \ \therefore h(0^+)=\frac{1}{RC}$$

For t > 0, the system becomes $RC\frac{dh}{dt}+h=0$ with $h(0^+)=\frac{1}{RC}$

Therefore, $h(t)=\frac{1}{RC}e^{-t/RC} \quad t>0$

Since *h(t)* does not contain an impulse, therefore $\int_{0^-}^{0^+} h\,dt\equiv 0$

So, $h(t,\tau)=\begin{cases}\frac{1}{RC}e^{\frac{-(t-\tau)}{RC}} & t\geq\tau \\ 0 & t<\tau\end{cases}$

<u>Example 2</u>: For the system $M\ddot{x}+B\dot{x}+Kx=F$ **,**

find impulse response.

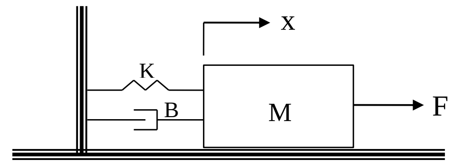


Figure 4-2. Mass-spring-dashpot system.

Since this system is relaxed

$\therefore\ x(0^-)=\dot{x}(0^-)=0$

We need $x(0^+)$ and $\dot{x}(0^+)$ for $F=\delta(t)$

$$\begin{cases} for\ \ t\geq 0 & M\ddot{h}+B\dot{h}+Kh=\delta(t) \\ for\ \ t<0 & M\ddot{h}+B\dot{h}+Kh=0 \end{cases}$$

for $t<0 \quad \rightarrow \quad h(0^-)=\dot{h}(0^-)=0 \quad \Rightarrow \quad h(t)=0,\ \ t<0$

for $t\geq 0$ integrate from $t=0^-$ to $t=0^+$

$$\int_{0^-}^{0^+} M\ddot{h}\,dt+\int_{0^-}^{0^+} B\dot{h}\,dt+\int_{0^-}^{0^+} Khdt=\int_{0^-}^{0^+} \delta dt$$

$$\Rightarrow M\left[\dot{h}(0^+)-\dot{h}(0^-)\right]+B\left[h(0^+)-h(0^-)\right]+0=1$$

Since $\dot{h}(0^-)=0, h(0^-)=0 \quad \therefore M\dot{h}(0^+)+Bh(0^+)=1$

By double integrating the equation from $t=0^-$ to $t=0^+$

$$\int_{0^-}^{0^+}\int_{0^-}^{0^+} M\ddot{h}dtdt+\int_{0^-}^{0^+}\int_{0^-}^{0^+} B\dot{h}dtdt+\int_{0^-}^{0^+}\int_{0^-}^{0^+} Khdtdt=\int_{0^-}^{0^+}\int_{0^-}^{0^+} \delta dtdt$$

$$\Rightarrow M\left[h(0^+)-h(0^-)\right]+0+0=0$$

$\therefore\ h(0^+)=0$, now since $M\dot{h}(0^+)+Bh(0^+)=1 \therefore\ \dot{h}(0^+)=\dfrac{1}{M}$

For $t>0$ $\ M\ddot{h}+B\dot{h}+Kh=0$ with $h(0^+)=0 \quad \dot{h}(0^+)=\dfrac{1}{M}$

$$\therefore h(t)=\begin{cases} \cdots & t\geq 0 \\ 0 & t<0 \end{cases}$$

The explicit solution depends on particular value of M, B and K.

4.2 Computation of Impulse Response Using Laplace Transform

This method is only applicable to LTI systems.

<u>Example 3</u>: $RC\frac{de_o}{dt}+e_o=e_i$ Find impulse response using Laplace transform (L.T.) method.

Let $e_i=\delta(t)\ \rightarrow\ e_o=h(t)\quad \therefore RC\frac{dh}{dt}+h=\delta \quad \forall t$

L{R(t)C(t)dh/dt}?=
L(R(t)C(t))L(dh/dt) X

Take L.T. of both sides $L\left\{RC\frac{dh}{dt}+h=\delta\right\}$

H(s)=...

$$x(t)\xleftrightarrow{L}X(s)=\int_{0^-}^{\infty}x(t)e^{-st}dt \qquad RC\left\{sH(s)-h(0^-)\right\}+H(s)=1$$

Since the system is relaxed $h(0^-)=0\ \ \therefore RCsH+H=1\Rightarrow\ H(s)=\frac{1}{1+RCs}$ Transfer function

$$\therefore h(t)=L^{-1}\{H(s)\}=\frac{1}{RC}e^{-t/RC} \qquad t\geq 0$$

L(f(t)g(t))=F(s)*G(s)
L(R(t)C(t))*sH(s)+H(s)=1
H(s)=?

For system $y(t)=\int_0^t h(t-\tau)u(\tau)d\tau$, take L.T.

$$L\{y(t)\}=L\left\{\int_0^t h(t-\tau)u(\tau)d\tau\right\}\Rightarrow Y(s)=\int_{0^-}^{\infty}e^{-st}\left[\int_0^t h(t-\tau)u(\tau)d\tau\right]dt$$

Since the system is causal

$$Y(s)=\int_{0^-}^{\infty}e^{-st}\left[\int_{0^-}^{\infty}h(t-\tau)u(\tau)d\tau\right]dt=\int_{0^-}^{\infty}\underbrace{\left[\int_{0^-}^{\infty}h(t-\tau)e^{-st}dt\right]}_{H(s)e^{-s\tau}}u(\tau)d\tau$$

$$=\int_{0^-}^{\infty}H(s)e^{-s\tau}u(\tau)d\tau=H(s)\underbrace{\int_{0^-}^{\infty}e^{-s\tau}u(\tau)d\tau}_{U(s)}$$

$$\therefore\ \ \mathrm{Y(s) = H(s)\cdot U(s)}\ \ \Rightarrow H(s)=\frac{Y(s)}{U(s)}$$

Example 4: For system $\ddot{y}+4\dot{y}+3y=4\dot{u}+u$

5u

1) Find T.F. and impulse response.

Taking the L.T of the system,

$$s^2Y(s)+4sY(s)+3Y(s)=4sU(s)+5U(s)$$

$$Y(s)\left[s^2+4s+3\right]=U(s)\left[4s+5\right]$$

$$\therefore T.F.=H(s)=\frac{Y(s)}{U(s)}=\frac{4s+5}{s^2+4s+3}$$

Impulse response $\quad h(t)=L^{-1}\{H(s)\}$

$$\therefore h(t)=L^{-1}\left\{\frac{4s+5}{s^2+4s+3}\right\}$$

Using partial fraction expansion

$$h(t)=L^{-1}\left\{\frac{7/2}{s+3}+\frac{1}{s+1}\right\}$$

$$\therefore h(t)=\frac{7}{2}e^{-3t}+\frac{1}{2}e^{-t} \qquad t\geq 0$$

2) Find the output for a step input using the convolution integral for a relaxed system.

$$y(t)=\int_0^t h(t-\tau)u(\tau)d(\tau)$$

$$\left\{\begin{array}{l} h(t)=\frac{7}{2}e^{-3t}+\frac{1}{2}e^{-t} \qquad t\geq 0 \\ \\ u(t)=u_s(t)=\begin{cases}1 & t>0\\ 0 & t\leq 0\end{cases}\end{array}\right.$$

$$y(t)=\int_0^t\left[\frac{7}{2}e^{-3(t-\tau)}+\frac{1}{2}e^{-(t-\tau)}\right]\cdot 1\cdot d\tau$$

$$\therefore y(t) = \frac{5}{3} - \frac{7}{6}e^{-3t}\,\frac{1}{2}e^{-t}$$ **Step Response**

3) Find the output for a step input using T.F. for a relaxed system.

Y(s) = H(s)U(s)

$$H(s) = \frac{4s+5}{s^2+4s+3} \;;\; U(s) = \frac{1}{s}$$

$$Y(s) = \frac{4s+5}{s\left(s^2+4s+3\right)} \Rightarrow y(t) = \frac{5}{3} - \frac{7}{6}e^{-3t} - \frac{1}{2}e^{-t}$$

4) If *y(0)* = 1 and $\dot{y}(0) = 0$, find *y(t)* for a step input.

$$L\{\ddot{y} + 4\dot{y} + 3y = 4\dot{u} + 5u\}$$

$$L\{\ddot{y}\} = s^2Y(s) - sy(0^-) - \dot{y}(0^-)$$

$$L\{\dot{y}\} = sY(s) - y(0^-)$$

$$\Rightarrow s^2Y(s) - s\cdot 1 - 0 + 4\left[sY(s) - 1\right] + 3Y(s) = 4sU(s) + 5U(s)$$

$$Y(s)\left[s^2 + 4s + 3\right] - s - 4 = U(s)\left[4s + 5\right]$$

$$Y(s) = \frac{4s+5}{s^2+4s+3}U(s) + \underbrace{\frac{s+4}{s^2+4s+3}}_{due\ to\ i.c.\neq 0}$$ Y(s)/U(s)=?

$$Y(s) = \frac{4s+5}{s\left(s^2+4s+3\right)} + \frac{s+4}{s^2+4s+3}$$

$$\therefore y(t) = \frac{5}{3} - \frac{5}{3}e^{-3t} + e^{-t}$$

5 Lecture 5

Objectives

1) ***State Diagram***
2) ***State Space Representation***

5.1 State Diagram

State diagram basic mathematical operations are shown in figure 5-1.

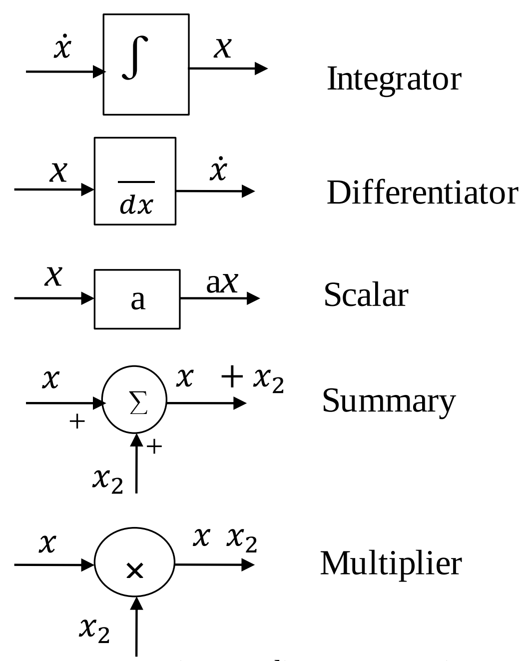


Figure 5-1. Basic state diagram operations.

<u>Example1</u>: Draw state diagram for system

$$\frac{dy}{dt}+ay=b\frac{du}{dt}+cu\,.$$

Solve for $\frac{dy}{dt}$ in terms of the rest.

Minimal Representation:

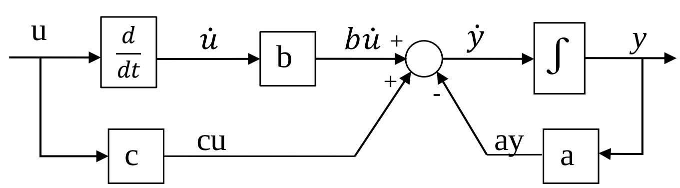
$$\frac{dy}{dt}=b\frac{du}{dt}+cu-ay$$

Integrate both sides of the equation to get

$$y=\int\frac{dy}{dt}dt=\int b\frac{du}{dt}dt+\int cu\,dt-\int ay\,dt$$

$$\therefore y=bu+\int(cu-ay)dt$$

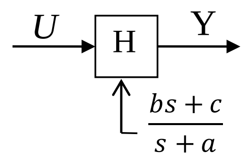
state diagram is shown as below.



Cause & Effect
Effet=CauseXGain

Signal to Noise ratio
u=sint+10^-3cos10^6t
du/dt=cost-10^3sin10^6t

Figure 5-2. Non-minimal internal state representation.



Y(s)=H(s)U(s)

y(t)=L{u(t)}

Figure 5-3. External representation.

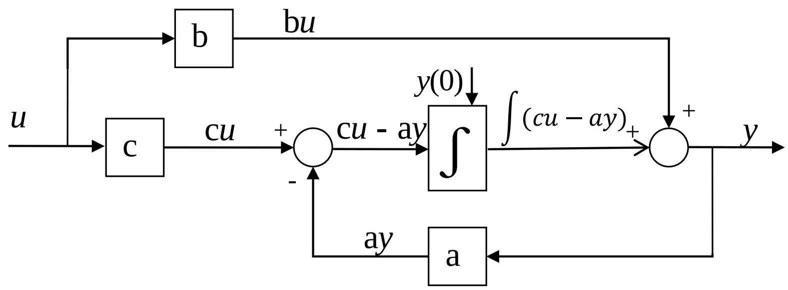


Figure 5-4. Minimal internal state representation.

Example 2: Single Input Single Output (SISO) $\ddot{y}+3\dot{y}+5y=8\dot{u}-4u$

$$\ddot{y}=-3\dot{y}-5y+8\dot{u}-4u$$

$$\therefore y=-\iint 3\dot{y}-\iint 5y+\iint 8\dot{u}-\iint 4u$$

$$y=\iint(-4u-5y)+\int(8u-3y)$$

State diagram is drawn as below:

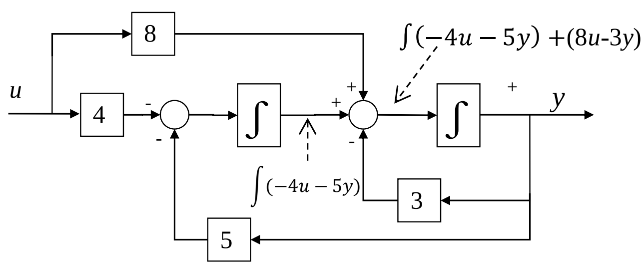


Figure 5-5. System state diagram for example 2.

Example 3: For a multi - input - multi - output(MIMO) system

$$\begin{cases}\ddot{y}_1+2\dot{y}_1+3y_1-5\dot{y}_2+8y_y=6\dot{u}_1-3u_1\\ \ddot{y}_2+9\dot{y}_2-6y_2+\dot{y}_1-y_1=u_2\end{cases}$$

draw state diagram for it.

$$\begin{cases} \ddot{y}_1 = -2\dot{y}_1 - 3y_1 + 5\dot{y}_2 - 8y_2 + 6\dot{u}_1 - 3u_1 \\ \ddot{y}_2 = -9\dot{y}_2 + 6y_2 - \dot{y}_1 + y_1 + u_2 \end{cases}$$

$$\Rightarrow \begin{cases} y_1 = -\iint 2\dot{y}_1 - \iint 3y_1 + \iint 5\dot{y}_2 - \iint 8y_2 + \iint 6\dot{u}_1 - \iint 3u_1 \\ \\ y_2 = -\iint 9\dot{y}_2 + \iint 6y_2 - \iint \dot{y}_1 + \iint y_1 + \iint u_2 \end{cases}$$

$$\therefore \begin{cases} y_1 = \iint (-3y_1 - 8y_2 - 3u_1) + \int (-2y_1 + 5y_2 + 6u_1) \\ \\ y_2 = \iint (y_1 + u_2 + 6y_2) + \int (-9y_2 - y_1) \end{cases}$$

Therefore, state diagram of this system is drawn below:

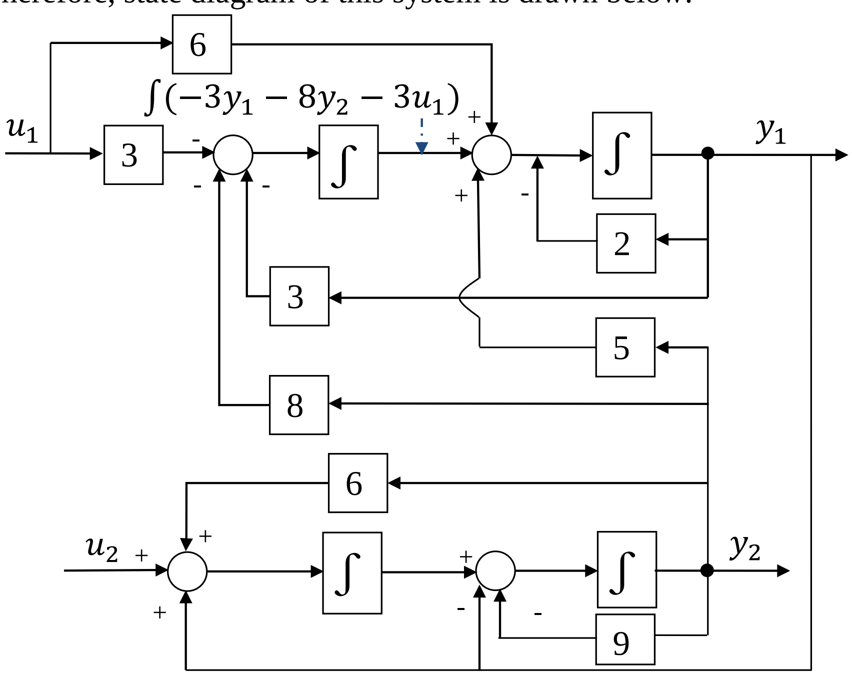


Figure 5-6. System state diagram for example 3.

<u>Example 4</u>: $\dot{y} + \sin(y) = u \rightarrow \dot{y} = u - \sin(y)$ (this is a nonlinear system)

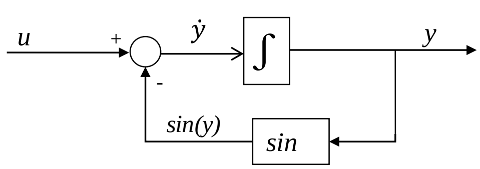


Figure 5-7. System state diagram for example 4.

5.1.1 Concept of State

- Internal Representation $\longrightarrow$ (Time) State Equations
- External Representation $\longrightarrow$ I/O Representation

I/O is partitioned into (a) Convolution Integral (time) and (b) Transfer Function (frequency)

State: The minimal number of variables *x(t)* that their knowledge at time $t = t_0$, and the input *u(t)* for $t \geq t_0$ will completely characterize the response of the system.

Example 5: $RC\dfrac{de_o}{dt} + e_o = e_i ; e_o(t_0)$ $\quad$ e_o is the state

$$\therefore e_o(t) = \underbrace{e_o(t_0)e^{-t/RC}}_{zero\ \text{input response}} + \underbrace{\frac{1}{RC}\int_{t_0}^{t} e^{-(t-\tau)/RC} e_i(\tau)d\tau}_{\text{zero state response}}$$

Example 6:

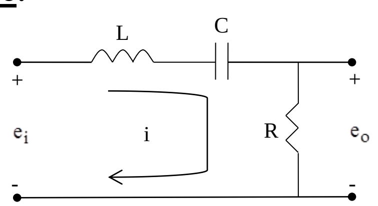


Figure 5-8. RLC network.

$$\begin{cases} e_i = L\dfrac{di}{dt} + \underbrace{\dfrac{1}{C}\displaystyle\int i\, \mathrm{dt}}_{v} + Ri \\ e_o = Ri \end{cases}$$

$$L\frac{d^2 i}{dt^2} + \frac{1}{C}i + R\frac{di}{dt} = \frac{de_i}{dt} \quad e_i(t) \quad (t \geq t_0), \quad i(t_0), \ \frac{di}{dt}(t_0) \ \text{should be specified !}$$

$$\therefore \quad \text{States are } i(t) \text{ and } \frac{di}{dt} \quad (*)$$

$$\frac{de_i}{dt} = \frac{L}{R}\frac{d^2 e_0}{dt^2} + \frac{1}{RC}e_o + \frac{de_o}{dt} \therefore \text{ States are } e_o \text{ and } \frac{de_o}{dt} \quad (**)$$

Case (*) and (**) reveal that ***states are NOT UNIQUE*** , there is another choice for the states !

Let $\quad v = \dfrac{1}{C}\displaystyle\int i\ \mathrm{dt} \Rightarrow \dfrac{dv}{dt} = \dfrac{i}{C} \qquad \therefore e_i = LC\dfrac{d^2 v}{dt^2} + v + RC\dfrac{dv}{dt}$

In order to solve for $v(t)$, we need $v(t_0)$, $\frac{dv}{dt}(t_0)$ and e_i, since $\frac{dv}{dt}(t_0)=\frac{i(t_0)}{C}$

so, to solve the system we need $v(t_0)$ and $i(t_0)$.

Therefore, for an RLC network the ***voltage*** across the capacitor and the ***current*** through an inductor are the **states**.

5.1.2 State Vector

Given n states we define the state vector as an (n×1)-dimensional column vector

$$x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1} \quad , \quad x(t)\in R^n$$

5.1.3 State Space

The space where the states are evolving in.

5.1.4 State Trajectories

$$x(t)=f\left(t\,;\;\; t_0,\; x(t_0),\; u(t)\right)$$

Example 7: $\ddot{y}+3\dot{y}+2y=0$

Characteristic polynomial:
s^2+3s+2=0 -> s_1=-2,s_2=-1

Let $\begin{cases} x_1(t)=y \\ x_2(t)=\dot{y} \end{cases}$, $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the state vector, $x\in R^2$ is the state space.

$y(t)=c_1e^{-2t}+c_2e^{-t}\equiv x_1(t)$ with c_1 and c_2 depending on $y(t_0)$ and $\dot{y}(t_0)$

$$\therefore \quad \dot{y}(t)=-2c_1e^{-2t}-c_2e^{-t}\equiv x_2(t)$$

$$\begin{cases} x_1(t)=c_1e^{-2t}+c_2e^{-t} \\ x_2(t)=-2c_1e^{-2t}-c_2e^{-t} \end{cases}$$

let $t_0=0$, $y(0)=0$ and $\dot{y}(0)=1$

parametrized by time

$$\Rightarrow \begin{cases} x_1(t)=e^{-t}-e^{-2t} \\ x_2(t)=2e^{-2t}-e^{-t} \end{cases}$$

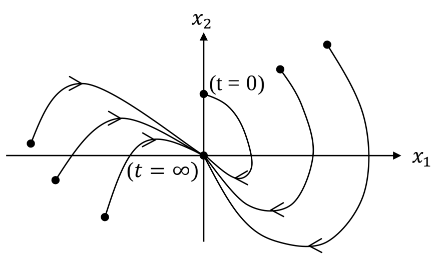


Figure 5-9. System state trajectory for example 1.

5.2 General State Space Representation

$$\begin{cases} \dot{x}(t) = f(x(t),u(t),t) & \leftarrow \textit{State} \ \text{ Equations} \\ y(t) = g(x(t),u(t),t) & \leftarrow \textit{Output} \ \text{ Equations} \end{cases}$$

$x \in R^n$ n_dim state

$y \in R^m$ m_dim output

$u \in R^q$ $q\_$ dim input

This is a nonlinear, time varying, causal system.

Nonlinear, time invariant, causal: $\begin{cases} \dot{x}(t) = f(x(t),u(t)) \\ y(t) = g(x(t),u(t)) \end{cases}$

Linear, time varying system:

Ax(t) \ne x(t)A -> (nx1)(nxn)?

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$$

MIMO system

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \rightarrow n \times 1 \ \text{vector} \qquad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \underbrace{\underbrace{\overbrace{\begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}}^{A(t)}}_{n\times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1}}_{n\times 1} + \underbrace{\underbrace{\overbrace{\begin{bmatrix} b_{11}(t) & b_{12}(t) & \cdots & b_{1q}(t) \\ b_{21}(t) & b_{22}(t) & \cdots & b_{2q}(t) \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}(t) & b_{n2}(t) & \cdots & b_{nq}(t) \end{bmatrix}}^{B(t)}}_{n\times q} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{bmatrix}}_{q\times 1}}_{n\times 1}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \underbrace{\underbrace{\overbrace{\begin{bmatrix} c_{11}(t) & c_{12}(t) & \cdots & c_{1n}(t) \\ c_{21}(t) & c_{22}(t) & \cdots & c_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1}(t) & c_{m2}(t) & \cdots & c_{mn}(t) \end{bmatrix}}^{C(t)}}_{m\times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1}}_{m\times 1} + \underbrace{\underbrace{\overbrace{\begin{bmatrix} d_{11}(t) & d_{12}(t) & \cdots & d_{1q}(t) \\ d_{21}(t) & d_{22}(t) & \cdots & d_{2q}(t) \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1}(t) & d_{m2}(t) & \cdots & d_{mq}(t) \end{bmatrix}}^{D(t)}}_{m\times q} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{bmatrix}}_{q\times 1}}_{m\times 1}$$

6 Lecture 6

Objectives

1) ***State Models***

2) ***Canonical Representations:***

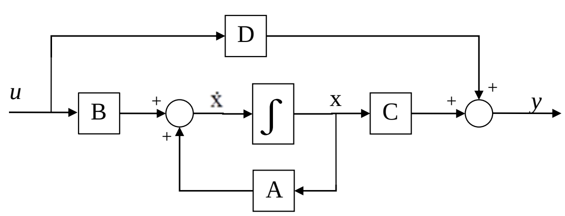
Observable Canonical Form (OCF)

6.1 State Models

Linear Time Invariant System
$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \leftarrow \quad \textit{State} \text{ Equations} \\ y(t) = Cx(t) + Du(t) & \leftarrow \quad \textit{Output} \text{ Equations} \end{cases}$$

A, B, C and D are now constant matrices.

State Diagram



D is called throughput

Figure 6-1. State diagram for a linear time invariant system.

Example 1: $\ddot{y} + (t+1)\dot{y} + t^2 y = u$, find the state space representation, in another word, state equation and output equation

Let $x_1 = y$, $x_2 = \dot{y} \Rightarrow \dot{x}_1 = x_2$, now we need $\dot{x}_2 = ?$

$$\dot{x}_2 = \ddot{y} = u - t^2 y - (t+1)\dot{y} \Rightarrow \dot{x}_2 = u - t^2 x_1 - (t+1)x_2$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u - t^2 x_1 - (t+1)x_2 \end{cases} \qquad \text{State Equation}$$

$$\{ y = x_1 \qquad \text{Output Equation}$$

In order to get A, B, C and D, define $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}_{2\times1} = \underbrace{\begin{bmatrix} 0 & 1 \\ -t^2 & -(t+1) \end{bmatrix}_{2\times2}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times1} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2\times1}}_{B} u_{1\times1}$$

$$\underbrace{[1 \quad 0]_{1\times2}}_{C} \begin{bmatrix} x1 \\ x2 \end{bmatrix}_{2\times1} + \underbrace{[0]_{1\times1}}_{D} u_{1\times1}$$

<u>Example 2</u>: For system shown as below, find the state and output equations.

1) $u = Ri_1 + \dfrac{1}{C}\int (i_1 - i_2)dt$

2) $Ri_2 + \dfrac{1}{C}\int i_2 dt + \dfrac{1}{C}\int (i_2 - i_1)dt = 0$

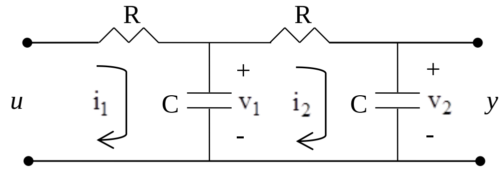


Figure 6-2. RC network.

Let $x_1 = \dfrac{1}{C}\int i_1 dt$, $x_2 = \dfrac{1}{C}\int i_2 dt$ $\quad \therefore c\dot{x}_1 = i_1$, $c\dot{x}_2 = i_2$

$$u = RC\dot{x}_1 + x_1 - x_2$$
$$0 = RC\dot{x}_2 + x_2 + x_2 - x_1$$

$$\therefore \begin{cases} \dot{x}_1 = \dfrac{1}{RC}x_2 - \dfrac{1}{RC}x_1 + \dfrac{1}{RC}u \\ \\ \dot{x}_2 = \dfrac{1}{RC}x_1 - \dfrac{2}{RC}x_2 \end{cases}$$ State Equation

$$y = \frac{1}{C}\int i_2 dt = x_2 \qquad \therefore \{ y = x_2$$ Output Equation

Find A, B, C and D

$$A = \begin{bmatrix} -\dfrac{1}{RC} & \dfrac{1}{RC} \\ \\ \dfrac{1}{RC} & \dfrac{-2}{RC} \end{bmatrix}_{2\times2} \quad ; \quad B = \begin{bmatrix} \dfrac{1}{RC} \\ \\ 0 \end{bmatrix}_{2\times1}$$

$$C = [0 \quad 1]_{1\times2} \quad ; \quad D = [0]_{1\times1}$$

Do not write D=0!
D=[.]nxm

This system is Time invariant since A, B, C and D are constant.

Now with the voltages across the capacitors as states, we get

$$\left.\begin{aligned} x_1 = v_1 = \frac{1}{C}\int (i_1 - i_2)dt \;\Rightarrow\; i_1 - i_2 = C\dot{x}_1 \\ x_2 = v_2 = \frac{1}{C}\int i_2 dt \;\Rightarrow\; i_2 = C\dot{x}_2 \end{aligned}\right\} \Rightarrow i_1 = C\dot{x}_1 + C\dot{x}_2$$

$$\therefore \begin{cases} u = RC(\dot{x}_1 + \dot{x}_2) + x_1 & \textit{from } 1) \quad (*) \\ 0 = RC\dot{x}_2 + x_2 - x_1 & \textit{from } 2) \quad (**) \end{cases}$$

$$\dot{x}_2 = \frac{1}{RC}x_1 - \frac{1}{RC}x_2$$

$$(*) - (**) \Rightarrow u = RC\dot{x}_1 - x_2 + 2x_1 \qquad \therefore \dot{x}_1 = \frac{1}{RC}u + \frac{1}{RC}x_2 - \frac{2}{RC}x_1$$

$$\therefore \begin{cases} \dot{x}_1 = \dfrac{-2}{RC}x_1 + \dfrac{1}{RC}x_2 + \dfrac{1}{RC}u \\ \dot{x}_2 = \dfrac{1}{RC}x_1 - \dfrac{1}{RC}x_2 \end{cases} \qquad \text{State Equation}$$

$$y = v_2 = x_2 \quad \therefore \{ y = x_2 \qquad \text{Output Equation}$$

$$A = \begin{bmatrix} \dfrac{-2}{RC} & \dfrac{1}{RC} \\ \dfrac{1}{RC} & \dfrac{-1}{RC} \end{bmatrix} \;;\; B = \begin{bmatrix} \dfrac{1}{RC} \\ 0 \end{bmatrix} ; \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \;; \quad D = [0]$$

<u>Example 3</u>: System as below:

$$\begin{cases} m\ddot{y}_1 + ky_1 + k(y_1 - y_2) = F \\ m\ddot{y}_2 + ky_2 + k(y_2 - y_1) = 0 \end{cases}$$

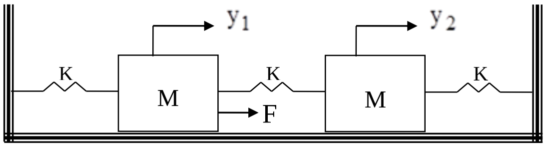


Figure 6-3. Mass-Spring system.

Choose the states as $x_1 = y_1,\; x_2 = \dot{y}_1\,,\; x_3 = y_2\,,\; x_4 = \dot{y}_2$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{y}_1 = \dfrac{F}{m} - \dfrac{2k}{m}x_1 + \dfrac{k}{m}x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \ddot{y}_2 = -\dfrac{2k}{m}x_3 + \dfrac{k}{m}x_1 \end{cases} \qquad \text{State Equations, let } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(i) $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$ Output Equations or

(ii) $y = y_1 = x_1$ Output Equation (valid) or

(iii) $y = y_2 = x_3$ Output Equation (valid) or

(iv) $y = y_1 - y_2 = x_1 - x_3$ Output Equation (valid) etc.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-2k}{m} & 0 & \frac{k}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m} & 0 & \frac{-2k}{m} & 0 \end{bmatrix}_{4\times 4} \quad ; \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \\ 0 \end{bmatrix}_{4\times 1}$$

$$\dot{x}_{4\times 1} = A_{4\times 4} x_{4\times 1} + B_{4\times 1} F_{1\times 1} \qquad y_{2\times 1} = C_{2\times 4} x_{4\times 1} + D_{2\times 1} F_{1\times 1}$$

(i) $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{2\times 4}$, $D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1}$, $y \to 2\times 1 \therefore C \to 2\times 4 \therefore D \to 2\times 1$

(ii) $C = [1 \quad 0 \quad 0 \quad 0]_{1\times 4}$, $D = [0]_{1\times 1}$, $y \to 1\times 1 \therefore C \to 1\times 4 \therefore D \to 1\times 1$

(iii) $C = [0 \quad 0 \quad 1 \quad 0]_{1\times 4}$, $D = [0]_{1\times 1}$

(iv) $C = [1 \quad 0 \quad -1 \quad 0]_{1\times 4}$, $D = [0]_{1\times 1}$

Now choose the states as follows: $x_1 = y_1$, $x_2 = \dot{y}_1$, $x_3 = y_1 - y_2$, $x_4 = \dot{y}_1 - \dot{y}_2$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{y}_1 = \frac{F}{m} - \frac{k}{m} x_1 - \frac{k}{m} x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \ddot{y}_1 - \ddot{y}_2 = \frac{F}{m} - \frac{3k}{m} x_3 \end{cases} \qquad \text{State Equations}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k}{m} & 0 & \frac{-k}{m} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-3k}{m} & 0 \end{bmatrix}_{4\times 4} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \\ \frac{1}{m} \end{bmatrix}_{4\times 1}$$

Let $y = y_1$ as an example,

$C = [1 \quad 0 \quad 0 \quad 0]$

$D = [0]$ (Output Equation)

6.2 Canonical Representations

a) Observable Canonical Form (O.C.F)

b) Controllable Canonical Form (C.C.F) (also known as Phase Variable CF)

c) Jordan Canonical Form (J.C.F)

6.2.1 Observable Canonical Form

Characteristic polynomial: Laplace transform of the LHS s^n+a_(n-1)s^(n-1) +...+a_0=0

For system $\dfrac{d^n y}{dt^n}+a_{n-1}\dfrac{d^{n-1}y}{dt^{n-1}}+\cdots+a_0 y=b_n\dfrac{d^n u}{dt^n}+b_{n-1}\dfrac{d^{n-1}u}{dt^{n-1}}+\cdots+b_0 u$

we define $D\equiv\dfrac{d}{dt}$, then $D^n y+a_{n-1}D^{n-1}y+\cdots+a_0 y=b_n D^n u+b_{n-1}D^{n-1}u+\cdots+b_0 u$

$$\Rightarrow D^n y=-a_{n-1}D^{n-1}y-\cdots-a_0 y+b_n D^n u+b_{n-1}D^{n-1}u+\cdots+b_0 u$$

$$\Rightarrow y=b_n u+\frac{1}{D}\left[b_{n-1}u-a_{n-1}y\right]+\cdots+\frac{1}{D^n}\left[b_0 u-a_0 y\right]$$

State Diagram:

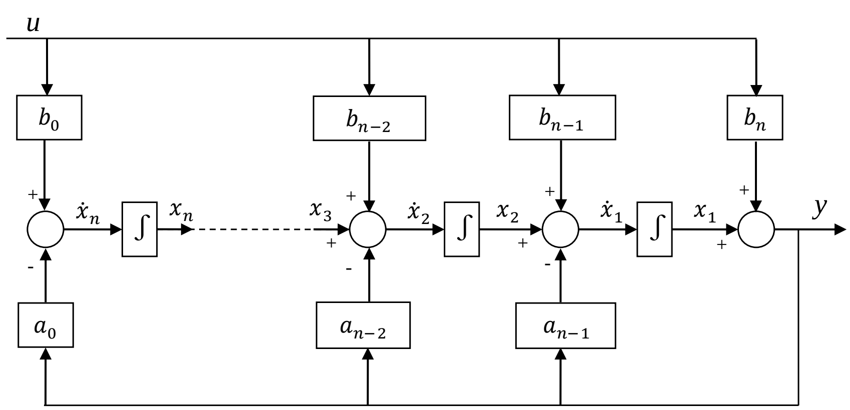


Figure 6-4. State diagram.

Define the states as the output of each integrator.

$$\begin{cases}\dot{x}_1=x_2+b_{n-1}u-a_{n-1}y\\ \dot{x}_2=x_3+b_{n-2}u-a_{n-2}y\\ \vdots\\ \dot{x}_{n-1}=x_n+b_1u-a_1y\\ \dot{x}_n=b_0u-a_0y\end{cases}\quad\text{and } y=b_n u+x_1\quad\therefore\begin{cases}\dot{x}_1=-a_{n-1}x_1+x_2+(b_{n-1}-a_{n-1}b_n)u\\ \dot{x}_2=-a_{n-2}x_1+x_3+(b_{n-2}-a_{n-2}b_n)u\\ \vdots\\ \dot{x}_n=-a_0x_1+(b_0-a_0b_n)u\end{cases}$$

State Equations

The coefficients of the characteristic polynomial

$I_{(n-1)\times(n-1)}$

$$A=\left[\begin{array}{c|ccccc}
-a_{n-1} & 1 & 0 & 0 & \cdots & 0\\
-a_{n-2} & 0 & 1 & 0 & \cdots & 0\\
-a_{n-3} & 0 & 0 & 1 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
-a_{1} & 0 & 0 & 0 & \cdots & 1\\
\hline
-a_{0} & 0 & 0 & 0 & \cdots & 0
\end{array}\right]_{n\times n}$$

n-1 (columns of the identity block), n-1 (rows of the identity block)

$$B=\begin{bmatrix} b_{n-1}-a_{n-1}b_n \\ b_{n-2}-a_{n-2}b_n \\ \vdots \\ b_1-a_1b_n \\ b_0-a_0b_n \end{bmatrix}_{n\times 1}$$

$$C=\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}_{1\times n}$$

$$D=\begin{bmatrix}b_n\end{bmatrix}_{1\times 1}$$

$$y=x_1+b_nu \qquad Output$$

The pair (A, C) is known as one of the **Observable Canonical** forms.

Alternatively, from the state diagram, **reverse** the assignment for the states, i.e. the output of the left integral is denoted by x_1 and consecutively increased to x then we get,

$$A=\left[\begin{array}{ccccc|c}
0 & 0 & \cdots & 0 & 0 & -a_0\\
\hline
1 & 0 & \cdots & 0 & 0 & -a_1\\
0 & 1 & \cdots & 0 & 0 & -a_2\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & 0 & -a_{n-2}\\
0 & 0 & \cdots & 0 & 1 & -a_{n-1}
\end{array}\right]_{n\times n} \qquad C=\begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}.$$

n − 1 (rows), n − 1 (columns)

The pair (A, C) is now considered as the selected **Observable Canonical Form (OCF)** in this course.

7 Lecture 7

Objectives

1) ***Controllable Canonical Forms (CCF)***

2) ***Jordan Canonical Form (JCF)***

7.1 Controllable Canonical Form

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s^1 + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s^1 + a_0}$$

Y/U=b_n+N(s)/D(s)
D(s)=Char. Poly.

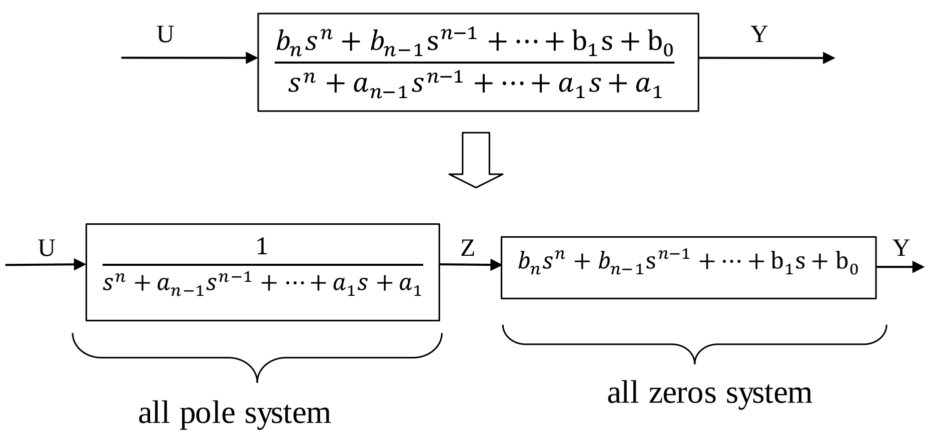


Figure 7-1. Decomposition of the transfer function.

$$\frac{Z(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

$$\therefore z^{(n)} + a_{n-1}z^{(n-1)} + \cdots + a_0 z = u$$

$$\text{Let} \begin{cases} x_1 = z \\ x_2 = \dot{z} \\ \vdots \\ x_{n-1} = z^{(n-2)} \\ x_n = z^{(n-1)} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = z^{(n)} \end{cases}$$

$$\begin{aligned} z^{(n)} &= u - a_0 z - a_1 \dot{z} - \cdots - a_{n-1} z^{(n-1)} \\ &= u - a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n \end{aligned}$$

Therefore, $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = u - a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n \end{cases}$ State Equation

Output Equation

$$\frac{Y}{Z} = b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0 \Rightarrow y = b_n z^{(n)} + b_{n-1} z^{(n-1)} + \cdots + b_0 z$$

$$\therefore y = b_n \left(u - a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n\right) + b_{n-1} x_n + \cdots + b_0 x_1$$

$$y = (b_0 - b_n a_0) x_1 + (b_1 - b_n a_1) x_2 + \cdots + (b_{n-1} - b_n a_{n-1}) x_n + b_n u$$ Output equation

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}_{n \times n} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \left[(b_0 - b_n a_0) \quad (b_1 - b_n a_1) \quad \cdots \quad (b_{n-1} - b_n a_{n-1})\right] \qquad D = [b_n]$$

The pair (A,B) is in the **Controllable Canonical Form (CCF)** also known as the **Phase Variable Canonical Form**

7.2 Jordan Canonical Form

$$Y(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s^1 + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s^1 + a_0} U(s) = b_n U(s) + \frac{N(s)}{D(s)} U(s)$$

Where $N(s) = (b_{n-1} - a_{n-1} b_n) s^{n-1} + (b_{n-2} - a_{n-2} b_n) s^{n-2} + \cdots + (b_0 - a_0 b_n)$ and

$$D(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_0$$

7.2.1 Characteristic Polynomial D(s) = 0 Contains N Distinct Poles

$$D(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$$

then $Y(s) = b_n U(s) + \left[\dfrac{\gamma_1}{s-\lambda_1} + \dfrac{\gamma_2}{s-\lambda_2} + \cdots + \dfrac{\gamma_n}{s-\lambda_n} \right] U(s)$ Partial Fraction Expansion

Construct the state diagram.

<u>Example 1</u>: Consider $Y = \dfrac{\gamma}{s-\lambda} U$ LTI

$$\dot{y} - \lambda y = \gamma u \Longrightarrow \dot{y} = \lambda y + \gamma u$$

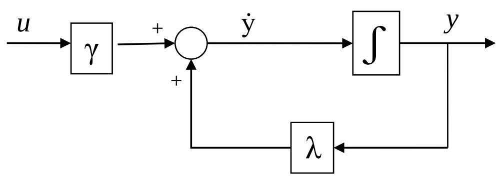


Figure 7-3. State diagram.

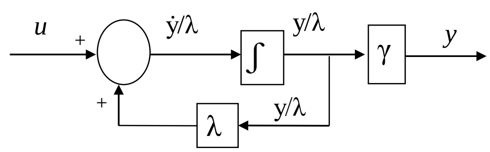


Figure 7-2. Alternative state diagram.

$$\frac{\dot{y}}{\gamma} = u + \lambda \frac{y}{\gamma} \Rightarrow \dot{y} = \gamma u + \lambda y$$

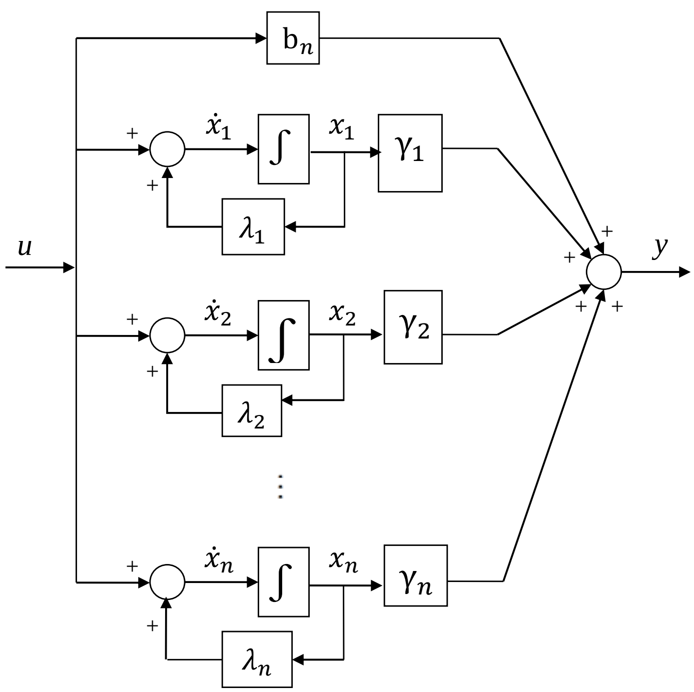


Figure 7-4. Parallel configuration.

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + u \\ \dot{x}_2 = \lambda_2 x_2 + u \\ \vdots \\ \dot{x}_{n-1} = \lambda_{n-1} x_{n-1} + u \\ \dot{x}_n = \lambda_n x_n + u \end{cases} \qquad \text{Sate Equation}$$

$$\{ y = b_n u + \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n \qquad \text{Output Equation}$$

$$\underbrace{A = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} ; \; B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} ; \; C = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_n \end{bmatrix} ; \; D = [n_n]}_{\textbf{Jordan Canonical Form (JCF)}}$$

D=[b_n]

7.2.2 Characteristic Polynomial D(s) = 0 Contains Multiple Repeated Poles

Suppose that D(s) can be expressed as

$$D(s) = (s - \lambda_1)^{\mu} \times (s - \lambda_2)^{\rho} \times (s - \lambda_3)(s - \lambda_4) \cdots (s - \lambda_{n-\mu-\rho})$$

$$Y = b_n U + \left[\frac{\gamma_1}{(s-\lambda_1)^{\mu}} + \frac{\gamma_2}{(s-\lambda_1)^{\mu-1}} + \cdots + \frac{\gamma_{\mu}}{(s-\lambda_1)} \right] U$$

This bracket is represented in the next state diagram

$$+ \left[\frac{\gamma_{\mu+1}}{(s-\lambda_2)^{\rho}} + \frac{\gamma_{\mu+2}}{(s-\lambda_2)^{\rho-1}} + \cdots + \frac{\gamma_{\mu+\rho}}{(s-\lambda_2)} \right] U$$

$$+ \left[\frac{\gamma_{\mu+\rho+1}}{s-\lambda_3} + \frac{\gamma_{\mu+\rho+2}}{s-\lambda_4} + \cdots + \frac{\gamma_n}{s-\lambda_{n-\mu-\rho}} \right] U$$

First, construct sate diagram for $(s-\lambda_1)^\mu$ terms in $\dfrac{Y}{U}=\dfrac{\gamma_1}{(s-\lambda_1)^\mu}$ as shown below..

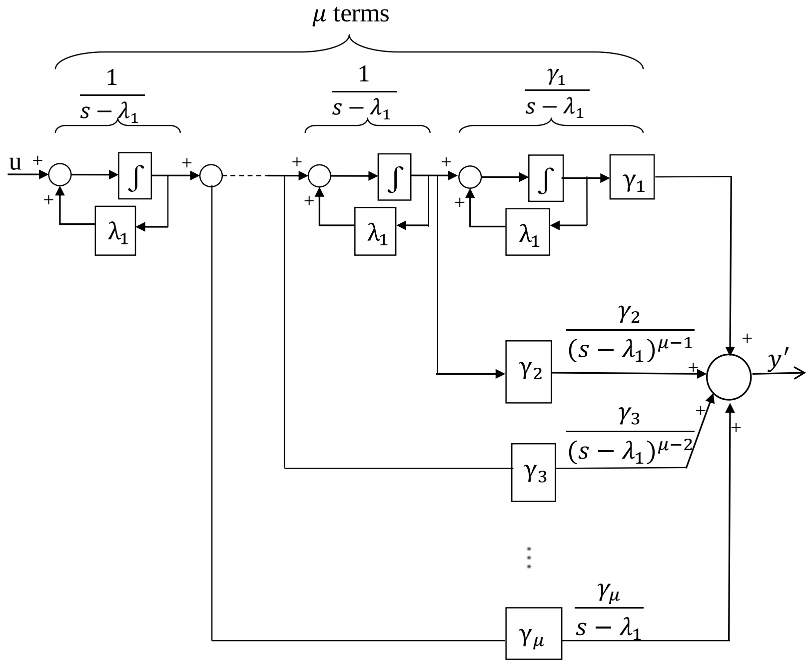


Figure 7-5. State diagram of the term in bracket.

Then, similarly construct state diagram for $(s-\lambda_2)^\rho$ and $(s-\lambda_3),(s-\lambda_4)\ldots(s-\lambda_{n-\mu-\rho})$ parallel with the above state diagram (detail omitted).

Finally, the combined state diagram leads to the following A and B matrices.

$$A=\begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-\mu-\rho} \end{bmatrix},\quad B=\begin{bmatrix} B_1 \\ B_2 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

where $\underbrace{J_i=\begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}}_{\text{Jordan Block}}$, $B_i=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ for i = 1 and i = 2

In other words, $J_1 = \begin{bmatrix} \lambda_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_1 \end{bmatrix}_{\mu\times\mu}$, $B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\mu\times 1}$ and

$$J_2 = \begin{bmatrix} \lambda_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_2 \end{bmatrix}_{\rho\times\rho}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{\rho\times 1}$$

8 Lecture 8

Objectives

1) ***Transformation from state equations to Transfer Function***
2) ***Examples for C.C.F., O.C.F., J.C.F***

8.1 Transformation From State Equations to Transfer Function

Given $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$, find the transfer function (TF) $\dfrac{Y(s)}{U(s)} = ?$

Take Laplace transform (L.T.) ($x(0) = 0$) $\begin{cases} sX(s) = AX(s) + BU(s) & (*) \\ Y(s) = CX(s) + DU(s) & (**) \end{cases}$

$(*) \Rightarrow sX(s) - AX(s) = BU(s) \Rightarrow \underbrace{(sI - A)}_{n\times n} X(s) = BU(s)$

Assuming $(sI - A)$ is nonsingular, then $(sI - A)^{-1}$ exists.

$$\therefore X(s) = (sI - A)^{-1} BU(s)$$

$$\therefore Y(s) = C(sI - A)^{-1} BU(s) + DU(s)$$

$$\underbrace{\overbrace{Y(s)}^{p\times 1}}_{1\times 1} = \underbrace{\overbrace{\boxed{C(sI - A)^{-1} B + D}}^{p\times q}}_{1\times 1} \underbrace{\overbrace{U(s)}^{q\times 1}}_{1\times 1}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \underbrace{\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix}}_{TransferMatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix} \quad \text{u\_q}$$

$$\therefore \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D \qquad (T.F.)$$

T.F. for SISO system

Transfer Matrix for MIMO system

$Y = GU$

Example 1: Find T.F. for system $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u; y=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+u$

- First approach: **Brute Force**

Write down equations for system: $\begin{cases}\dot{x}_1 = x_1 + u\\ \dot{x}_2 = 2x_1 + x_2\\ y = x_1 - x_2 + u\end{cases}$

Since $y = x_1 - x_2 + u \Rightarrow \dot{y} = \dot{x}_1 - \dot{x}_2 + \dot{u}$

$$\dot{y} = x_1 + u - 2x_1 - x_2 + \dot{u} = -x_1 - x_2 + u + \dot{u}$$

$$\Rightarrow \ddot{y} = -\dot{x}_1 - \dot{x}_2 + \dot{u} + \ddot{u} = -x_1 - u - 2x_1 - x_2 + \dot{u} + \ddot{u}$$

$$\therefore \ddot{y} = -3x_1 - x_2 - u + \dot{u} + \ddot{u}$$

Since $y + \dot{y} = -2x_2 + 2u + \dot{u} \Rightarrow x_2 = u + \frac{1}{2}\dot{u} - \frac{1}{2}(y + \dot{y})$

Since $y = x_1 - x_2 + u \Rightarrow x_1 = y + x_2 - u$

$$\therefore x_1 = y + 0.5\dot{u} - 0.5y - 0.5\dot{y}$$

$$\ddot{y} = -3(y + 0.5\dot{u} - 0.5y - 0.5\dot{y}) - (u + 0.5\dot{u} - 0.5y - 0.5\dot{y}) - u + \dot{u} + \ddot{u}$$

$$\ddot{y} = 2\dot{y} - y + \ddot{u} - 2u - \dot{u}$$

$$\therefore \ddot{y} - 2\dot{y} + y = \ddot{u} + -2u - \dot{u}$$

Therefore, Transfer Function (T.F.):

$$s^2 y - 2sy + y = s^2 u - su - 2u$$

$$\frac{Y(s)}{U(s)} = \frac{s^2 - s - 2}{s^2 - 2s + 1}$$

- Second Approach: Using formula $\dfrac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$

With $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 \end{bmatrix}$, $D = [1]$ to find $(sI - A) = ?$

$$sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix},\ sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -2 & s-1 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s-1)(s-1) - 0} \begin{bmatrix} s-1 & 0 \\ 2 & s-1 \end{bmatrix}$$

$$\begin{bmatrix} s-1 & 0 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} \dfrac{s-1}{(s-1)(s-1)} & 0 \\ \dfrac{2}{(s-1)(s-1)} & \dfrac{s-1}{(s-1)(s-1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(sI - A)^{-1}B = \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 0 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(s-1)^2} \begin{bmatrix} s-1 \\ 2 \end{bmatrix}$$

$$C(sI - A)^{-1}B = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \dfrac{(s-1)}{(s-1)^2} \\ \dfrac{2}{(s-1)^2} \end{bmatrix} = \frac{s-1}{(s-1)^2} - \frac{2}{(s-1)^2} = \frac{s-3}{(s-1)^2}$$

$$\frac{Y(s)}{U(s)} = \frac{s-3}{(s-1)^2} + 1 = \frac{s-3+(s-1)^2}{(s-1)^2}$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{s-3+s^2-2s+1}{s^2-2s+1} = \frac{s^2-s-2}{s^2-2s+1}$$

8.2 Examples for C.C.F., O.C.F., J.C.F.

8.2.1 Observable C.F.

<u>Example 2</u>: $\ddot{y}+5\dot{y}+6y=2\dot{u}-u$, for observable C.F. draw the state diagram

$$\ddot{y}=-5\dot{y}-6y+2\dot{u}-u \Rightarrow y=-\int 5y-\iint 6y+\int 2u-\iint u$$

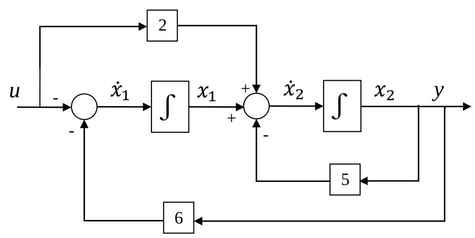


Figure 8-1. State diagram for example 2.

$$\begin{cases}\dot{x}_1=-u-6x_2\\ \dot{x}_2=x_1+2u-5x_2\end{cases}$$ State Equations $\{y=x_2$ Output Equation

$$x=\begin{bmatrix}x_1\\ x_2\end{bmatrix};\quad \begin{cases}\dot{x}=A_o x+B_o u\\ y=C_o x+D_o u\end{cases}$$

$$A_o=\begin{bmatrix}0 & -6\\ 1 & -5\end{bmatrix},\ B_o=\begin{bmatrix}-1\\ 2\end{bmatrix},\ C_o=\begin{bmatrix}0 & 1\end{bmatrix},\ D_o=0$$ <u>Observable C.F.</u>

8.2.2 Controllable C.F.

<u>Example 3</u>: $\dot{x}=\underbrace{\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}}_{A_c}x+\underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B_c}u \qquad y=\underbrace{\begin{bmatrix}-1 & 2\end{bmatrix}}_{C_c}x+\underbrace{\begin{bmatrix}0\end{bmatrix}}_{D_c}u$

Duality Principle

$$A_c=A_o^T\ ,\quad B_c=C_o^T\ ,\quad C_c=B_o^T\ ,\quad D_c=D_o$$

To get C.C.F, directly from the differential equation, we find the T.F.

$$\frac{Y(s)}{U(s)}=\frac{2s-1}{s^2+5s+6}$$

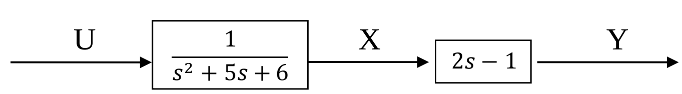


Figure 8-2. Transfer function decomposition for example 3.

$$\therefore \frac{X}{U} = \frac{1}{s^2 + 5s + 6} \Rightarrow \ddot{x} + 5\dot{x} + 6x = u$$

Let $\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases}$, then $\begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{x} \end{matrix}$ $\quad \therefore \dot{x}_2 = \ddot{x} = u - 5\dot{x} - 6x = u - 5x_2 - 6x_1$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -6x_1 - 5x_2 + u \end{cases}$$ State Equations

Since $y = 2\dot{x} - x \Rightarrow \frac{Y}{X} = \frac{(2s-1)}{1} \Rightarrow Y = (2s-1)X = 2sX - X$

$$\therefore \{ y = 2x_2 - x_1$$ Output Equation

$$A_c = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_c = \begin{bmatrix} -1 & 2 \end{bmatrix}, \quad D_c = 0$$

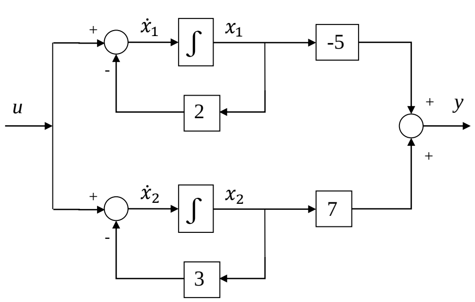
8.2.3 Jordan Canonical Form

For system $\frac{Y}{U} = \frac{2s-1}{s^2 + 5s + 6}$, in order to find the J.C.F., follow the steps listed below:

1) Using partial fraction expansion, $\frac{Y}{U} = \frac{-5}{s+2} + \frac{7}{s+3}$

Strictly Proper TF
degree of num<degree of den
if deg of num=deg of den
->Proper TF

2) Draw the state diagram



$$\begin{cases} \dot{x}_1 = -2x_1 + u \\ \dot{x}_2 = -3x_2 + u \\ y = -5x_1 + 7x_2 \end{cases}$$

Figure 8-3. System state diagram.

Therefore, for Jordan C.F. $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} -5 & 7 \end{bmatrix}$, $D = 0$

Example 4: $\ddot{y}+2\dot{y}+y=\dot{u}-2u$, show

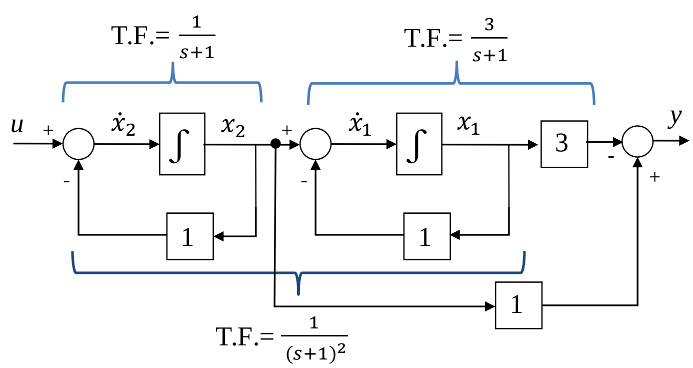
$$A_c=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix},\quad B_c=\begin{bmatrix}0\\1\end{bmatrix},\quad C_c=\begin{bmatrix}-2 & 1\end{bmatrix},\quad D_c=0$$

$$A_o=\begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix},\quad B_o=\begin{bmatrix}-2\\1\end{bmatrix},\quad C_o=\begin{bmatrix}0 & 1\end{bmatrix},\quad D_o=0$$

For Jordan Canonical Form

$$\frac{Y(s)}{U(s)}=\frac{s-2}{s^2+2s+1}=\frac{s-2}{(s+1)^2}=\frac{a}{(s+1)}+\frac{b}{(s+1)^2}=\frac{-3}{(s+1)^2}+\frac{a}{(s+1)}=\frac{-3+as+a}{(s+1)^2}$$

$$\therefore a=1\Rightarrow\frac{Y}{U}=\frac{1}{s+1}+\frac{-3}{(s+1)^2}$$



$$\begin{cases}\dot{x}_1=x_2-x_1\\ \dot{x}_2=u-x_2\\ y=-3x_1+x_2\end{cases}$$

Figure 8-4. State diagram.

Jordan Block:

$$A=\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix},\quad B=\begin{bmatrix}0\\1\end{bmatrix},\quad C=\begin{bmatrix}-3 & 1\end{bmatrix},\quad D=[0]$$

Example 5: Given poles: -1, -1, 3， what is A and B?

$$A=\left[\begin{array}{cc:c}-1 & 1 & 0\\ 0 & -1 & 0\\ \hdashline 0 & 0 & 3\end{array}\right],\quad B=\left[\begin{array}{c}0\\1\\ \hdashline 1\end{array}\right]$$

Example 6: Armature Controlled DC Motor

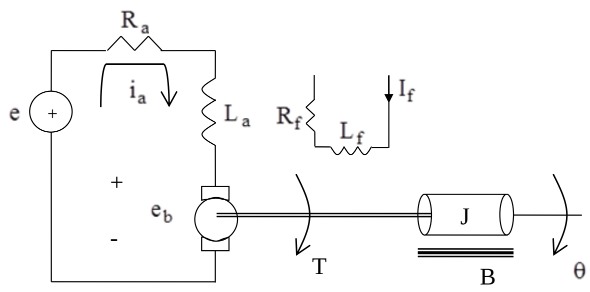


Figure 8-5. Armature controlled DC motor.

From the above figure, the governing equations are

$$\begin{cases} J\ddot{\theta}+B\dot{\theta}=T \\ T=K_T i_a \\ e=R_a i_a+L_a\dfrac{di_a}{dt}+e_b \\ e_b=K_b\dot{\theta} \end{cases}$$

This is a 3rd order system,

let $x_1=i_a, x_2=\theta, x_3=\dot{\theta}$

$$\begin{cases} J\ddot{\theta}+B\dot{\theta}=K_T i_a \\ L_a\dfrac{di_a}{dt}+R_a i_a+K_b\dot{\theta}=e \end{cases} \Rightarrow \frac{di_a}{dt}=-\frac{R_a}{L_a}i_a-\frac{K_b}{L_a}\dot{\theta}+\frac{1}{L_a}e$$

$$\therefore \begin{cases} \dot{x}_1=-\dfrac{R_a}{L_a}x_1-\dfrac{K_b}{L_a}x_3+\dfrac{1}{L_a}e \\ \dot{x}_2=x_3 \\ \dot{x}_3=\dfrac{K_T}{J}x_1-\dfrac{B}{J}x_3 \end{cases}$$

Defining the output as the angular position of the motor

$\therefore \{y=\theta=x_2$, Jordan Canonical Form is:

$$A=\begin{bmatrix} -\dfrac{R_a}{L_a} & 0 & -\dfrac{K_b}{L_a} \\ 0 & 0 & 1 \\ \dfrac{K_T}{J} & 0 & -\dfrac{B}{J} \end{bmatrix},\quad B=\begin{bmatrix} \dfrac{1}{L_a} \\ 0 \\ 0 \end{bmatrix},\quad C=\begin{bmatrix} 0 & 1 & 0 \end{bmatrix},\quad D=[0]$$

9 Lecture 9

Objectives

1) Solution of State Equation

2) Fundamental Matrix

3) State Transition Matrix

9.1 Solution of State Equation

Given $\begin{cases} \dot{x} = A(t)x + B(t)u \;, & x \in R^n \;, \quad \mathrm{u} \in R^m \\ y = C(t)x + D(t)u \;, & y \in R^q \end{cases}$

<u>Example 1</u>: $x \in R \quad , \quad \mathrm{u} \in R \quad , \quad y \in R$; $\dot{x} = a(t)x + b(t)u$, $x(t_0)$ is given

$$\frac{d}{dt}\left[e^{-\int_{t_0}^{t} a(\tau)d\tau} x(t)\right] = e^{-\int_{t_0}^{t} a(\tau)d\tau} b(t)u$$

Integrating factor $e^{-\int_{t_0}^{t} a(\tau)d\tau}$

$$\int_{t_0}^{t} \frac{d}{d\tau}\left[e^{-\int_{t_0}^{\tau} a(\tau')d\tau'} x(\tau)\right] d\tau = \int_{t_0}^{t} e^{-\int_{t_0}^{\tau} a(\tau')d\tau'} b(\tau)ud\tau$$

$$e^{-\int_{t_0}^{t} a(\tau)d\tau} x(t) - e^{-\int_{t_0}^{t_0} a(\tau)d\tau} x(t_0) = \int_{t_0}^{t} e^{-\int_{t_0}^{\tau} a(\tau')d\tau'} b(\tau)ud\tau$$

$$e^{-\int_{t_0}^{t} a(\tau)d\tau} x(t) - x(t_0) = \int_{t_0}^{t} e^{-\int_{t_0}^{\tau} a(\tau')d\tau'} b(\tau)ud\tau$$

$$x(t) = e^{\int_{t_0}^{t} a(\tau)d\tau} x(t_0) + \int_{t_0}^{t} e^{\int_{t_0}^{t} a(\tau)d\tau} \cdot e^{-\int_{t_0}^{\tau} a(\tau')d\tau'} bud\tau$$

$$x(t) = \underbrace{e^{\int_{t_0}^{t} a(\tau)d\tau} x(t_0)}_{zero\ input\ response} + \underbrace{\int_{t_0}^{t} e^{\int_{\tau}^{t} a(\tau')d\tau'} b(\tau)u(\tau)d\tau}_{zero\ state\ response}$$

Example 2: Let $a(t) = a$ (constant) $\therefore \dot{x} = ax + b(t)u$

$$x(t) = e^{a(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)} b(\tau)ud\tau$$

Example 3: Let $t_0 = 0 \Rightarrow x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}b(\tau)ud\tau$

Show this by using Laplace Transform method !

We have $\dot{x} = ax + b(t)u$, $t = 0$,

$$L\{\dot{x} = ax + b(t)u\} \;\Rightarrow\; \mathrm{sX(s)\text{-}x(0) = aX(s) + L\{b(t)u(t)\}}$$

$$X(s) = \frac{1}{-a+s}x(0) + \frac{1}{s-a}L\{b(t)u(t)\} \qquad \text{taking } \mathrm{L}^{-1}$$

$$x(t) = e^{at}x(0) + L^{-1}\left\{\frac{1}{s-a}L\{b(t)u(t)\}\right\}$$

$$L^{-1}\{G_1(s)G_2(s)\} = \int_0^t g_1(t-\tau)g_2(\tau)d\tau = g_1(t) * g_2(t)$$

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}b(\tau)u(\tau)d\tau$$

Example 4: $\dot{x} = \frac{-1}{t}x + tu$ with $u = t$, $x(1) = 1$

We know $x(t) = e^{\int_{t_0}^{t} a(\tau)d\tau}x(t_0) + \int_{t_0}^{t} e^{\int_{\tau}^{t} a(\tau')d\tau'} b(\tau)u(\tau)d\tau$

$$a(t) = \frac{-1}{t},\quad b(t) = t \Rightarrow \int_{t_0}^{t} a(\tau)d\tau \Rightarrow \int_1^t \frac{-1}{\tau}d\tau = -\int_1^t \frac{d\tau}{\tau} = -\ln\tau\Big|_1^t = -\ln t$$

$$e^{\int_{t_0}^{t} a(\tau)d\tau} \Rightarrow e^{-\ln t} = e^{\ln\frac{1}{t}} = \frac{1}{t}\,, \qquad x(t) = \frac{1}{t}\cdot 1 + \int_1^t (e^{-\ln t + \ln\tau})\cdot\tau\cdot\tau d\tau$$

$$x(t) = \frac{1}{t} + \int_1^t e^{\ln\frac{\tau}{t}}\cdot\tau^2 d\tau = \frac{1}{t} + \int_1^t \frac{\tau^3}{t}d\tau = \frac{3}{4t} + \frac{t^3}{4} \quad \text{for } t \geq 1$$

9.2 Fundamental Matrix

9.2.1 Definition:

Homegenous state space representation

Given $\dot{x} = A(t)x,\ x(t_0) = x_0,\ x \in R^n$, an n×n matrix M(t) is said to be a fundamental matrix of $\dot{x} = A(t)x$, ***if and only if*** the n columns of M(t), denoted as $M_1(t), M_2(t), \ldots, M_n(t)$ are ***any*** n linearly independent solutions of $\dot{x} = A(t)x$.

9.2.2 Examples:

Example 5: Given $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2t \end{bmatrix} x$, find M(t).

Let $M(t) = \begin{bmatrix} \underbrace{M_1(t)}_{2\times1} & \underbrace{M_2(t)}_{2\times1} \end{bmatrix}$, $M_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$, since M_1 is a solution, it has to satisfy the dynamic equation, namely $\dot{M}_1 = A(t)M_1$.

M_2=[x_21 x_22]^{T}

$$\begin{cases} \dot{x}_{11} = x_{11} \Rightarrow x_{11} = e^t, x_{11} = 0 \\ \dot{x}_{12} = 2tx_{12} \Rightarrow x_{12} = e^{t^2}, x_{12} = 0 \end{cases} \quad \text{let } M_1 = \begin{bmatrix} e^t \\ 0 \end{bmatrix};\ M_2 = \begin{bmatrix} 0 \\ e^{t^2} \end{bmatrix}$$

a_1v_1+a_2v_2=0
==> a_1=a_2=0
=>LI
if a_1 OR a_2 \ne 0
then LD

Why not $M_1 = \begin{bmatrix} e^t \\ e^{t^2} \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$? Linear combination of M_1 and M_2 is

$$\alpha_1 \begin{bmatrix} e^t \\ e^{t^2} \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$, since $\alpha_2 \neq 0 \Rightarrow$ these vectors are linearly dependent

NOT ACCEPTABLE FOR M(t) !

Now $\alpha_1 \begin{bmatrix} e^t \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ e^{t^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ since $\alpha_1 = \alpha_2 = 0 \Rightarrow$ linearly independent $\Rightarrow$**M(t)** √

$\therefore M(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{t^2} \end{bmatrix}$ is the fundamental matrix.

Fundamental matrix is ***no*t unique**! Why?

Choose $M(t)=\begin{bmatrix} 2e^{t} & 0 \\ 0 & 4e^{t^2} \end{bmatrix}$, now it is still an acceptable fundamental matrix as it satisfies the differential equations.

Example 6: Given $\dot{x}=\begin{bmatrix} \frac{1}{t} & 0 \\ -\frac{2}{t^2} & \frac{2}{t} \end{bmatrix}x$, find M(t).

Let $M(t)=[M_1(t) \quad M_2(t)]$, $M_1=\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$ \dot{x_12}=(-2/t^2) x_11+(2/t)x_12

$$\dot{x}_{11}=\frac{1}{t}x_{11} \Rightarrow x_{11}=t, x_{11}=0$$

with $x_{11}=t \Rightarrow \dot{x}_{12}=\frac{2}{t}x_{12}-\frac{2}{t} \Rightarrow x_{12}=1$

with $x_{11}=0 \Rightarrow \dot{x}_{12}=\frac{2}{t}x_{12} \Rightarrow x_{12}=t^2$

$$M_1=\begin{bmatrix} t \\ 1 \end{bmatrix}, \quad M_2=\begin{bmatrix} 0 \\ t^2 \end{bmatrix} \quad \therefore M(t)=\begin{bmatrix} t & 0 \\ 1 & t^2 \end{bmatrix}$$

Check to see if M(t) satisfies the dynamic equation: $\dot{M}(t)=A(t)M$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2t \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} \frac{1}{t} & 0 \\ \frac{-2}{t^2} & \frac{2}{t} \end{bmatrix}\begin{bmatrix} t & 0 \\ 1 & t^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 2t \end{bmatrix} \qquad \therefore \quad \surd$$

9.3 State Transition Matrix

9.3.1 Definition

Given $\dot{x} = A(t)x$, an $n \times n$ matrix $\Phi(t,t_0)$ is a **state transition matrix** ***if and only if*** $x(t) = \Phi(t,t_0)x(t_0)$ is a solution of the state equation with $x(t_0)$ given.

9.3.2 Proposition

Let M(t) be the fundamental matrix of $\dot{x} = A(t)x$, then

$$\underbrace{\Phi(t,t_0)}_{unique} = \underbrace{M(t)}_{not\ unique} M^{-1}(t_0) \qquad \forall t$$

Example 7: $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2t \end{bmatrix} x \Rightarrow M(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{t^2} \end{bmatrix}$, find $\Phi(t,t_0)$.

We need $M(t_0) = \begin{bmatrix} e^{t_0} & 0 \\ 0 & e^{t_0^2} \end{bmatrix}$, then $M^{-1}(t_0) = \begin{bmatrix} e^{-t_0} & 0 \\ 0 & e^{-t_0^2} \end{bmatrix}$

$$\therefore \Phi(t,t_0) = \begin{bmatrix} e^t & 0 \\ 0 & e^{t^2} \end{bmatrix} \begin{bmatrix} e^{-t_0} & 0 \\ 0 & e^{-t_0^2} \end{bmatrix} = \begin{bmatrix} e^{t-t_0} & 0 \\ 0 & e^{t^2-t_0^2} \end{bmatrix}$$

Moreover, $x(t) = \Phi(t,t_0)x(t_0)$

$$\therefore x(t) = \begin{bmatrix} e^{t-t_0} & 0 \\ 0 & e^{t^2-t_0^2} \end{bmatrix} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \qquad \begin{cases} x_1(t) = e^{t-t_0} x_{10} \\ x_2(t) = e^{t^2-t_0^2} x_{20} \end{cases}$$

9.3.3 Properties of $\Phi(t,t_0)$

1) $$\frac{\partial \Phi(t,t_0)}{\partial t} = A(t)\Phi(t,t_0)$$

2) $$\Phi(t_0,t_0) = I$$

3) $\left.\dfrac{\partial\Phi(t,t_0)}{\partial t}\right|_{t=t_0} = A(t_0)$ To get A(t), replace t_0 with t in A(t_0)

4) Transition property: $\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0)$

5) Inversion property: $\Phi^{-1}(t_1,t_0) = \Phi(t_0,t_1)$

1) $\dot{\Phi} = \dfrac{\partial\Phi}{\partial t} = A(t)\Phi$

Since $x(t) = \Phi(t,t_0)x(t_0)$ and *x(t)* is a solution, therefore it should satisfy the differential equation (d.e.), i.e.

$$\dot{x} = \dot{\Phi}(t,t_0)x(t_0) = A(t)\Phi(t,t_0)x(t_0)$$

$$\therefore \dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0) \quad \surd$$

2) $\Phi(t_0,t_0) = I$

Since $\Phi(t,t_0) = M(t)M^{-1}(t_0)$

$$at \;\; t = t_0, \quad \Phi(t_0,t_0) = M(t_0)M^{-1}(t_0) = I \quad \surd$$

3) Follows from (1) and (2), since $\dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0)$

$$at \;\; t = t_0, \quad \dot{\Phi}(t_0,t_0) = A(t_0)\underbrace{\Phi(t_0,t_0)}_{I} \Rightarrow \left.\dot{\Phi}(t,t_0)\right|_{t=t_0} = A(t_0) \quad \surd$$

4) $\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0)$

$x(t) = \Phi(t,t_0)x(t_0) \quad \forall t \geq t_0 \;\; (*) \quad \Rightarrow x(t_1) = \Phi(t_1,t_0)x(t_0)$ for $t_1 \geq t_0$

$x(t) = \Phi(t,t_1)x(t_1) \quad \forall t \geq t_1 \;\; (**) \Rightarrow x(t_2) = \Phi(t_2,t_1)x(t_1)$ for $t_2 \geq t_1$

$\therefore x(t_2) = \Phi(t_2,t_1)\Phi(t_1,t_0)x(t_0)$

From $(*) \Rightarrow x(t_2) = \Phi(t_2,t_0)x(t_0)$

$\therefore \Phi(t_2,t_1)\Phi(t_1,t_0)x(t_0) = \Phi(t_2,t_0)x(t_0) \quad \surd$

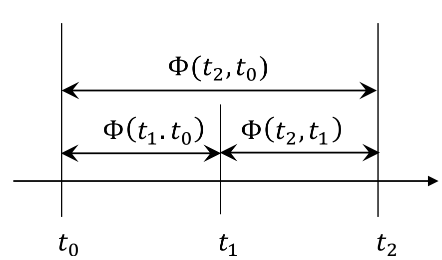


Figure 9-1. Transition property.

5) $\Phi^{-1}(t_1,t_0)=\Phi(t_0,t_1)$

Since $\Phi^{-1}(t_1,t_0)\Phi(t_1,t_0)=I=\Phi(t_0,t_0)=\Phi(t_0,t_1)\Phi(t_1,t_0)$

$$\Rightarrow \Phi^{-1}(t_1,t_0)\underline{\Phi(t_1,t_0)}=\Phi(t_0,t_1)\underline{\Phi(t_1,t_0)}$$

$$\therefore \Phi^{-1}(t_1,t_0)=\Phi(t_0,t_1)\ \surd$$

Example 8: $\dot{x}=\begin{bmatrix}1 & 0\\ 0 & 2t\end{bmatrix}x \rightarrow \Phi(t,t_0)=\begin{bmatrix}e^{t-t_0} & 0\\ 0 & e^{t^2-t_0^2}\end{bmatrix}$, check the above properties.

- $\dot{\Phi}(t,t_0)=A(t)\Phi(t,t_0)$

$$\begin{bmatrix}e^{t-t_0} & 0\\ 0 & 2te^{t^2-t_0^2}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 2t\end{bmatrix}\begin{bmatrix}e^{t-t_0} & 0\\ 0 & 2te^{t^2-t_0^2}\end{bmatrix}\qquad \surd$$

Delete 2t

- $\Phi(t_0,t_0)=I$

$$\Phi(t_0,t_0)=\begin{bmatrix}e^{t_0-t_0} & 0\\ 0 & e^{t_0^2-t_0^2}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\quad \surd$$

- $\dot{\Phi}(t,t_0)\big|_{t=t_0}=A(t_0)$

$$\left.\begin{bmatrix}e^{t-t_0} & 0\\ 0 & 2te^{t^2-t_0^2}\end{bmatrix}\right|_{t=t_0}=\begin{bmatrix}e^{0} & 0\\ 0 & 2t_0e^{0}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 2t_0\end{bmatrix}=A(t_0)\ \surd$$

A(t)=? replace t_0 with t

10 Lecture 10

Objectives

1) ***State Transition Matrix for LTI Systems***
2) ***Cayley-Hamilton technique***

10.1 State Transition Matrix for LTI Systems

10.1.1 Definition

Assuming system $\dot{x} = Ax$, $x(t_0)$ is given, for a scalar system $\dot{x} = ax$, $x \in R$, $x(t_0)$ $\therefore x(t) = e^{a(t-t_0)}x(t_0)$, generalization to an *n-th* order system $\dot{x} = Ax$, $x(t_0)$, $x \in R^n$, suggests that the solution can be expressed as $x(t) = e^{A(t-t_0)}x(t_0)$. Check to see if x satisfies the differential equation $\dot{x} = \underline{\underline{Ae^{A(t-t_0)}x(t_0)}} = \underline{\underline{Ax(t)}}$.

Since $x(t) = \Phi(t,t_0)x(t_0)$ \phi^(-1)x(t)=\Phi^(-1)\Phix(t_0)=x(t_0)

$\therefore$ $\Phi(t,t_0) = e^{A(t-t_0)}$ **is a state transition matrix for a LTI system**

$$\Phi(t,t_0) = \Phi(t-t_0)$$

Now at $t_0 = 0$, $\Phi(t,0) = \Phi(t) = e^{At}$ $(t_0 = 0)$, from property (5), $\Phi^{-1}(t) = ?$

$\Phi^{-1}(t,t_0) = \Phi(t_0,t), \Phi^{-1}(t-t_0) = \Phi(t_0-t)$ for $t_0 = 0 \Rightarrow \Phi^{-1}(t) = \Phi(-t)$

$$\Phi^{-1}(t) = \Phi(-t)$$

10.1.2 Approaches to Compute $\Phi(t)$

1) $\Phi(t) = e^{At} = L^{-1}\{(sI - A)^{-1}\}$

2) Cayley-Hamilton technique

3) The Jordan Form technique

4) $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$ Sparse matrix

For approach 1), given $\dot{x} = Ax$ with $x(t_0)$, take L-transform

$$sX(s) - x(t_0) = AX(s) \Rightarrow (sI - A)X(s) = x(t_0)$$

$$\therefore \underbrace{X(s)}_{n\times 1} = \underbrace{(sI - A)^{-1}}_{n\times n}\underbrace{x(t_0)}_{n\times 1}$$

$$\therefore x(t) = L^{-1}\{(sI - A)^{-1}\}x(t_0)$$

Since $x(t) = \Phi(t,t_0)x(t_0)$ $\therefore$ $\Phi(t,t_0) = L^{-1}\{(sI - A)^{-1}\}$

$$\therefore e^{At} = L^{-1}\{(sI - A)^{-1}\}$$

Computation of $(sI - A)^{-1}$, using **Leverrier's algorithm**:

$$(sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)} = \frac{P_{n-1}s^{n-1} + P_{n-2}s^{n-2} + \cdots + P_1 s + P_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

where P_i's are $n \times n$ matrix, a_i's are scalars

Algorithm:

1) $P_{n-1} = I_{n\times n}$

2) $a_{n-1} = -tr(A)$ tr=trace=\sum a_ii, i=1,...,n

3) $P_{n-2} = P_{n-1}A + a_{n-1}I$

$$a_{n-2} = \frac{-1}{2} tr(P_{n-2}A)$$

$$\vdots$$

4)

$$P_k = P_{k+1}A + a_{k+1}I$$

$$a_k = \frac{-1}{n-k} tr(P_k A)$$

Final verification test is $P_0 A + a_o I = 0$.

Example 1: $A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}, n = 3$

$$\begin{cases} P_2 = I \\ a_2 = 5 \end{cases} \Rightarrow P_1 = P_2 A + a_2 I = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 4 \end{bmatrix}$$

$$a_1 = \frac{-1}{2} tr(P_1 A) = 7 \Rightarrow P_0 = P_1 A + a_1 I = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$$a_0 = \frac{-1}{3} tr\left(P_0 A\right) = 1 \Rightarrow \left(sI - A\right)^{-1} = \frac{P_2 s^2 + P_1 s + P_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{1}{s^3 + 5s^2 + 7s + 1} \begin{bmatrix} s^2 + 3s + 2 & 1 & s + 2 \\ s + 1 & s^2 + 3s + 1 & 1 \\ s + 3 & s + 2 & s^2 + 4s + 4 \end{bmatrix}$$

Example 2: System $\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} x$, $y = \begin{bmatrix} 1 & 5 \end{bmatrix} x$, i.c. $y(0) = 0$, $\dot{y}(0) = 1$, find $y(t)$.

$$\begin{cases} y(t) = x_1(t) + 5x_2(t) \\ \dot{y}(t) = \dot{x}_1(t) + 5\dot{x}_2(t) \end{cases} \forall t \quad \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -4x_1 - 4x_2 \\ \dot{y} = x_2 + 5(-4x_1 - 4x_2) = -20x_1 - 19x_2 \end{cases} \forall t$$

$$\begin{cases} y(0) = x_1(0) + 5x_2(0) = 0 \\ \dot{y}(0) = -20x_1(0) - 19x_2(0) = 1 \end{cases} \Rightarrow \begin{cases} x_1(0) = -\dfrac{5}{81} \\ x_2(0) = \dfrac{1}{81} \end{cases},$$

$$e^{At} = L^{-1}\{(sI-A)^{-1}\}$$

$$(sI-A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 4 & s+4 \end{bmatrix} \Rightarrow (sI-A)^{-1} = \begin{bmatrix} \frac{s+4}{(s+2)^2} & \frac{1}{(s+2)^2} \\ \frac{-4}{(s+2)^2} & \frac{s}{(s+2)^2} \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} 2te^{-2t} + e^{-2t} & te^{-2t} \\ -4te^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix}$$

$$x(t) = e^{At}x(0) \to \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2te^{-2t} + e^{-2t} & te^{-2t} \\ -4te^{-2t} & e^{-2t} - 2te^{-2t} \end{bmatrix} \begin{bmatrix} -5/81 \\ 1/81 \end{bmatrix}$$

$$\Rightarrow y(t) = x_1(t) + 5x_2(t) \Rightarrow y(t) = te^{-2t}$$

10.2 Cayley-Hamilton Technique

10.2.1 Cayley-Hamilton Theorem

For a given $n \times n$ matrix A with characteristic polynomial $\Pi_A(\lambda) = \det(\lambda I - A) = 0$, the matrix A satisfies its characteristic polynomial (C.P.), i.e. $\Pi_A(A) = 0$.

In other words, if $\Pi_A(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n = 0$ then

$A^n + \alpha_1 A^{n-1} + \cdots + \alpha_{n-1}A + \alpha_n I = 0$.

<u>Example 3</u>: $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \Pi_A(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{bmatrix} = \lambda^2 + 2\lambda + 1 = 0$

We want to show that (from C.H. Theorem): $A^2 + 2A + 1 = 0$

$$\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}^2 + \begin{bmatrix} 0 & 2 \\ -2 & -4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \surd$$

10.2.2 Cayley-Hamilton Technique

Given an $n \times n$ matrix A with characteristic polynomial (C.P.) $\Pi_A(\lambda) = 0$, the matrix polynomial P(A) can be computed by considering the scalar polynomial $P(\lambda)$.

We compute $\frac{P(\lambda)}{\Pi_A(\lambda)} = Q(\lambda) + \frac{R(\lambda)}{\Pi_A(\lambda)}$

Polynomial of degree n-1
with unknown coefficients

with n unknown coefficients

$$\therefore P(\lambda) = \Pi_A(\lambda) Q(\lambda) + R(\lambda)$$

at $\lambda = \lambda_i$ (λ_i is the eigenvalue of A or the roots of the C.P.)

$$P(\lambda_i) = \underbrace{\Pi_A(\lambda_i)}_{0} Q(\lambda_i) + R(\lambda_i)$$

$$\therefore P(\lambda_i) = R(\lambda_i) \quad (1) \quad i = 1, \ldots, n \ (\text{distinct } \lambda_i\text{'s})$$

Now, from C.H. Theorem $P(A) = \underbrace{\Pi_A(A)}_{0} Q(A) + R(A)$

$$\therefore P(A) = R(A)$$

10.2.3 Algorithm to Apply Cayley-Hamilton Technique

1) Find the eigenvalues of A.

2) If the eigenvalues are distinct, use equation (1) for the coefficients of $R(\lambda)$ by solving n simultaneous equations. If the eigenvalues are repeated, let's say for an eigenvalue with multiplicity m, then instead of (1) use:

$$P(\lambda_i) = R(\lambda_i)$$

\lamda_i has multiplicity m

$$\left.\frac{dP}{d\lambda}\right|_{\lambda=\lambda_i} = \left.\frac{dR}{d\lambda}\right|_{\lambda=\lambda_i}$$

$$\left.\frac{d^2P}{d\lambda^2}\right|_{\lambda=\lambda_i} = \left.\frac{d^2R}{d\lambda^2}\right|_{\lambda=\lambda_i}$$

$$\vdots$$

$$\left.\frac{d^{m-1}P}{d\lambda^{m-1}}\right|_{\lambda=\lambda_i} = \left.\frac{d^{m-1}R}{d\lambda^{m-1}}\right|_{\lambda=\lambda_i}$$

3) Use the constants found in step (2) to get $P(A) = R(A)$.

[Examples of $P(A) \rightarrow \ e^{At}; A^{101}; A^2; e^{A^2 t}$]

Example 4: Find A^{101} for $A=\begin{bmatrix}0 & 1\\ -3 & -4\end{bmatrix}$

1) $P(A)=A^{101}$; $P(\lambda)=\lambda^{101}$

2) $n=2 \Rightarrow R(\lambda)=\alpha\lambda+\beta$

3) $\Pi_A(\lambda)=\det(\lambda I-A)=0 \Rightarrow \lambda=-1,-3$

4) Apply $P(\lambda_i)=R(\lambda_i)$ *for* $i=1,2,\ldots,$ *i.e.*

$$\begin{cases}\lambda=-1 \Rightarrow (-1)^{101}=-\alpha+\beta\\ \lambda=-3 \Rightarrow (-3)^{101}=-3\alpha+\beta\end{cases}$$

$$\therefore \quad \alpha=\frac{(-1)^{101}-(-3)^{101}}{2} \quad ; \quad \beta=\frac{3(-1)^{101}-(-3)^{101}}{2}$$

5) Apply $P(A)=R(A)$

$$\therefore \quad P(A)=A^{101}=R(A)=\alpha A+\beta I$$

$$\therefore \quad A^{101}=\frac{(-1)^{101}-(-3)^{101}}{2}\begin{bmatrix}0 & 1\\ -3 & -4\end{bmatrix}+\frac{3(-1)^{101}-(-3)^{101}}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$A^{101}=\begin{bmatrix}\frac{3(-1)^{101}-(-3)^{101}}{2} & \frac{(-1)^{101}-(-3)^{101}}{2}\\ \frac{-3(-1)^{101}+3(-3)^{101}}{2} & \frac{-1(-1)^{101}+3(-3)^{101}}{2}\end{bmatrix} \quad \surd$$

Example 5: Find e^{At} for $A=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$

$n=2,\ P(A)=e^{At},\ P(\lambda)=e^{\lambda t},\ R(\lambda)=\alpha\lambda+\beta$

$\Pi_A(\lambda)=(\lambda+1)^2=0 \Rightarrow \lambda=-1,-1$ (Repeated) -1 has multiplicity 2 -> m=2

$$@\ \lambda=-1 \quad apply \quad \begin{cases} P(\lambda)=R(\lambda) \ \Rightarrow e^{-t}=-\alpha+\beta \\ \dfrac{dP}{d\lambda}=\dfrac{dR}{d\lambda} \qquad \Rightarrow te^{\lambda t}=\alpha \Rightarrow at\ \lambda=-1 \Rightarrow \alpha=te^{-t} \end{cases}$$

$$\therefore \alpha=te^{-t}, \quad \beta=e^{-t}+te^{-t}$$

Apply P(A) = R(A) , hence

$$e^{At}=\alpha A+\beta I=te^{-t}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}+(te^{-t}+e^{-t})\cdot I$$

$$\therefore e^{At}=\begin{bmatrix} e^{-t}+te^{-t} & te^{-t} \\ -te^{-t} & e^{-t}-te^{-t} \end{bmatrix} \qquad \surd$$

<u>Example 6</u>: $A=\begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix}$, find e^{At}.

$$\Pi_A(\lambda)=\det(\lambda I-A)=\det\begin{bmatrix} \lambda & -1 \\ 2 & \lambda-2 \end{bmatrix}=\lambda^2-2\lambda+2=0 \Rightarrow \lambda=1\pm j$$

$$A=\begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} \Rightarrow \lambda=+1\pm j \quad \text{(distinct)}$$

$$P(A)=e^{At}, \quad P(\lambda)=e^{\lambda t}, \quad R(\lambda)=\alpha\lambda+\beta$$

Apply $P(\lambda)=R(\lambda)\ \ at\ \ \lambda=1\pm j$
$$\begin{cases} e^{(1+j)t}=\alpha(1+j)+\beta \\ e^{(1-j)t}=\alpha(1-j)+\beta \end{cases}$$

$$\Rightarrow \begin{cases} e^t(\cos t + j\sin t) = \alpha\sqrt{2}e^{j\pi/4} + \beta \\ e^t(\cos t - j\sin t) = \alpha\sqrt{2}e^{-j\pi/4} + \beta \end{cases}$$

$$e^{jx} = \cos x + j\sin x$$

$$1 \pm j = \sqrt{2}e^{\pm j\pi/4}$$

$$\Rightarrow \begin{cases} e^{jt} = \dfrac{\alpha}{e^t}\sqrt{2}e^{j\pi/4} + \dfrac{\beta}{e^t} \\ e^{-jt} = \dfrac{\alpha}{e^t}\sqrt{2}e^{-j\pi/4} + \dfrac{\beta}{e^t} \end{cases}$$

Use $\alpha = \frac{e^t}{\sqrt{2}} sin\ t$ to find β, it will be a real number

$$e^{jt} - e^{-jt} = \frac{\alpha}{e^t}\sqrt{2}\left(e^{j\pi/4} - e^{-j\pi/4}\right) \Rightarrow 2j\sin t = \frac{\alpha}{e^t}\sqrt{2}(2j)$$

$$\therefore \alpha = \frac{e^t}{\sqrt{2}}\sin t$$

$$e^{At} = \left(\frac{e^t}{\sqrt{2}}\sin t\right)\begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} + \beta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

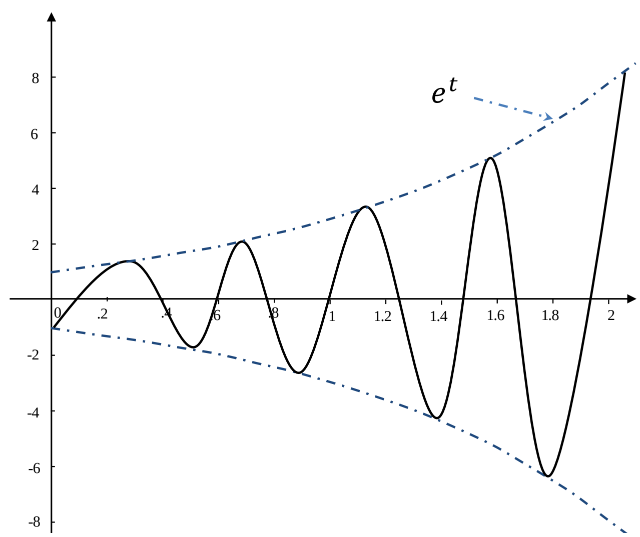


Figure 10-1. System plot.

11 Lecture 11

Objectives

1) ***Using Cayley Hamilton technique to calculate $\Phi(t, t_0)$ for LTI Systems***
2) ***Using Cayley Hamilton technique to calculate $\Phi(t, t_0)$ for LTV Systems***
3) ***Solution of the state equations and output equation***

11.1 $\Phi(t, t_0)$ for LTI System

From the scalar case: $e^{\lambda t} = 1 + t\lambda + \frac{t^2}{2!}\lambda^2 + \cdots$, we get

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \qquad (*)$$

Some of the properties of e^{At} are :

1) $e^0 = I$

2) $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$ $\quad t_1, t_2$ scalars

3) $e^{(A+B)t} = e^{At}e^{Bt}$ $\quad$ ***if and only if (iff.)*** AB = BA (A & B commute)

4) $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$ $\quad$ A and exp^{At} commute.

✓ **<u>Proof of (1):</u>** from (*) with either $t = 0$ or A= 0!

✓ **Proof of (2):** $e^{A(t_1+t_2)} = I + A(t_1+t_2) + \frac{A^2}{2!}(t_1+t_2)^2 + \cdots$

$$= I + At_1 + At_2 + \frac{A^2}{2!}t_1^2 + \frac{A^2}{2!}t_2^2 + A^2 t_1 t_2 + \cdots \quad (*)$$

$$e^{At_1}e^{At_2} = \left(I + At_1 + \frac{A^2t_1^2}{2!} + \cdots\right)\left(I + At_2 + \frac{A^2t_2^2}{2!} + \cdots\right)$$

$$= I + At_2 + \frac{A^2t_2^2}{2!} + At_1 + A^2t_1t_2 + \frac{A^2t_1^2}{2!} + \cdots \quad (**)$$

Since (*) = (**) $\Rightarrow e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$

✓ **Proof of (3):** $e^{(A+B)t} = I + (A+B)t + (A+B)^2\frac{t^2}{2!} + \cdots$

$$e^{At}e^{Bt} = \left(I + At + \frac{A^2t^2}{2!} + \cdots\right)\left(I + Bt + \frac{B^2t^2}{2!} + \cdots\right)$$

$$= I + Bt + \frac{B^2t^2}{2!} + At + ABt^2 + \frac{AB^2t^3}{2!} + \frac{A^2t^2}{2!} + \frac{A^2Bt^3}{2!} + \cdots$$

$$= I + (A+B)t + t^2\left[\frac{A^2}{2!} + \frac{B^2}{2!} + AB\right] + \cdots$$

$$\frac{(A+B)^2}{2} \overset{?}{=} \frac{A^2}{2} + \frac{B^2}{2} + AB$$ (A+B)^2=(A+B)(A+B) ->

$$\frac{1}{2}(A^2 + B^2 + AB + BA) \overset{?}{=} \frac{1}{2}(A^2 + B^2 + 2AB)$$

If and only if AB = BA $\Rightarrow$ AB+BA=2AB $\Rightarrow$ that is, **A & B should commute!**

✓ **Proof of (4):** $\frac{d}{dt}e^{At} = \frac{d}{dt}\left(\sum_{k=0}^{\infty}\frac{t^k A^k}{k!}\right) = \sum_{k=1}^{\infty}\frac{kt^{k-1}A^k}{k!} = \sum_{k=1}^{\infty}\frac{t^{k-1}A^k}{(k-1)!}$ k'=k-1

$$= \sum_{k=0}^{\infty}\frac{t^k A^{k+1}}{k!} = \sum_{k=0}^{\infty}\frac{t^k A^k A}{k!} = \sum_{k=0}^{\infty}\frac{t^k A A^k}{k!} = \left(\sum_{k=0}^{\infty}\frac{t^k A^k}{k!}\right)A$$

$$= A\left(\sum_{k=0}^{\infty}\frac{t^k A^k}{k!}\right) = e^{At}A = Ae^{At}$$ **A & e^{At} commute!**

Jordan Canonical Representation $\Leftrightarrow e^{At}$

1) A has distinct eigenvalues

2) A has repeated eigenvalues

Case 1: (Distinct λ) For LTI System $A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}_{n\times n}$, find e^{At} =?

$e^{At} = L^{-1}\{(sI - A)^{-1}\}$,

$$sI - A = \begin{bmatrix} s-\lambda_1 & 0 & \cdots & 0 \\ 0 & s-\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s-\lambda_n \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s-\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{s-\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s-\lambda_n} \end{bmatrix}$$

$$\therefore e^{At} = L^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}_{n\times n}$$

Case 2: (Repeated λ) $J = \begin{bmatrix} \lambda_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_1 \end{bmatrix}_{n\times n}$, find e^{Jt}

n= multiplicity of \lambda_1

Using Cayley-Hamilton technique:

1) $\Pi_J(\lambda) = \det(\lambda I - J) = (\lambda - \lambda_1)^n$

2) $P(J) = e^{Jt}$, $P(\lambda) = e^{\lambda t}$

R(\lamda)=\alpha_0+\alpha_1 \lambda+\alpha_2 \lambda^{2}+....

$$R(\lambda) = \alpha_0 + \alpha_1(\lambda - \lambda_1) + \alpha_2(\lambda - \lambda_1)^2 + \cdots + \alpha_{n-1}(\lambda - \lambda_1)^{n-1}$$

3) $P(\lambda_1) = R(\lambda_1) \quad \Rightarrow e^{\lambda_1 t} = \alpha_0 \quad \surd$

$$\left.\frac{dP}{d\lambda}\right|_{\lambda=\lambda_1} = \left.\frac{dR}{d\lambda}\right|_{\lambda=\lambda_1} \quad \Rightarrow te^{\lambda_1 t} = \alpha_1 \quad \surd$$

$$\left.\frac{d^2P}{d\lambda^2}\right|_{\lambda=\lambda_1} = \left.\frac{d^2R}{d\lambda^2}\right|_{\lambda=\lambda_1} \quad \Rightarrow \frac{t^2}{2}e^{\lambda_1 t} = \alpha_2 \quad \surd$$

$$\vdots \qquad \vdots$$

$$\left.\frac{d^{n-1}P}{d\lambda^{n-1}}\right|_{\lambda=\lambda_1} = \left.\frac{d^{n-1}R}{d\lambda^{n-1}}\right|_{\lambda=\lambda_1} \quad \Rightarrow \frac{t^{n-1}}{(n-1)!}e^{\lambda_1 t} = \alpha_{n-1} \quad \surd$$

4) $e^{Jt} = \alpha_0 I + \alpha_1 (J-\lambda_1 I) + \cdots + \alpha_{n-1}(J-\lambda_1 I)^{n-1}$

$$= \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2!}e^{\lambda_1 t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda_1 t} & \frac{t^{n-1}}{(n-1)!}e^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & \cdots & \frac{t^{n-3}}{(n-3)!}e^{\lambda_1 t} & \frac{t^{n-2}}{(n-2)!}e^{\lambda_1 t} \\ 0 & 0 & e^{\lambda_1 t} & \cdots & \frac{t^{n-4}}{(n-4)!}e^{\lambda_1 t} & \frac{t^{n-3}}{(n-3)!}e^{\lambda_1 t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & 0 & 0 & \cdots & 0 & e^{\lambda_1 t} \end{bmatrix}$$

11.2 $\Phi(t, t_0)$ for LTV Systems

\dot{x}=A(t)x LTV
Nonlinear TI
\dot{y}=1 ->y=t
X=[x y]^{T}

1) $\Phi(t,t_0) = M(t)M^{-1}(t)$0

2) Direct solution of $\dot{x} = A(t)x$ to get $x(t) = \Phi(t,t_0)x(t_0)$. **Brute Force** (Hyperlink) reference to lecture 8 Example 1.

[x
y]
\dot{X}=F(X)
F=[A(y)x 1]^{T}

3) If A(t) and $A(t_0)$ commute, then $\Phi(t,t_0) = e^{\int_{t_0}^{t} A(\tau)d\tau}$

4) If $A(t) = \sum_{i=1}^{k} \alpha_i(t) M_i$, M_i are constant matrices where $\alpha_i{}'\ s$ are scalars and $M_i and\ M_j\ \forall\ i,j\ commute$, then $\Phi(t,t_0) = \prod_{i=1}^{k} e^{M_i \int_{t_0}^{t} \alpha_i(\tau)d\tau}$

Example 3: $A(t)=\begin{bmatrix} 2t & 1 \\ 1 & 2t \end{bmatrix},\ find\ \Phi(t,t_0)$

$$A(t)=\underbrace{2t}_{\alpha_1}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{M_1}+\underbrace{1}_{\alpha_2}\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{M_2},$$

Since $M_1M_2=M_2M_1$ $\therefore \Phi(t,t_0)=e^{M_1(t^2-t_0^2)}e^{M_2(t-t_0)}$

$$e^{M_1(t^2-t_0^2)}=e^{\begin{bmatrix} t^2-t_0^2 & 0 \\ 0 & t^2-t_0^2 \end{bmatrix}}=\begin{bmatrix} e^{t^2-t_0^2} & 0 \\ 0 & e^{t^2-t_0^2} \end{bmatrix}$$ Cay-Ham.

$$e^{M_2(t-t_0)}=e^{\begin{bmatrix} 0 & t-t_0 \\ t-t_0 & 0 \end{bmatrix}}=\begin{bmatrix} \cos h(t-t_0) & \sin h(t-t_0) \\ \sin h(t-t_0) & \cos h(t-t_0) \end{bmatrix}$$

(This can be shown by applying Cayley-Hamilton technique -- details not shown)

Therefore, $\Phi(t,t_0)=e^{t^2-t_0^2}\begin{bmatrix} \cos h(t-t_0) & \sin h(t-t_0) \\ \sin h(t-t_0) & \cos h\,(t-t_0) \end{bmatrix}$

Example 4: $A(t)=\begin{bmatrix} \frac{1}{t} & 0 \\ \frac{-2}{t^2} & \frac{2}{t} \end{bmatrix}$

Check to see that $A(t)$ and $A(t_0)$ do not commute, therefore #3 does not apply.

$\therefore$ Use method #2 $\dot{x}_1=\frac{1}{t}x_1$; $\dot{x}_2=\frac{-2}{t^2}x_1+\frac{2}{t}x_2$

$$x_1(t)=e^{\int_{t_0}^{t}\frac{1}{\tau}d\tau}x_1(t_0)=\frac{t}{t_0}x_1(t_0)\Rightarrow \dot{x}_2(t)=\underbrace{\frac{2}{t}}_{a(t)}x_2\underbrace{-\frac{2}{tt_0}x_1(t_0)}_{b(t)u(t)}$$

$$x_2(t) = e^{\int_{t_0}^{t}\frac{2}{\tau}d\tau} x_2(t_0) + \int_{t_0}^{t} e^{\int_{\tau}^{t}\frac{2}{\tau'}d\tau'}\left[\frac{-2}{\pi t_0}x_1(t_0)\right]d\tau$$

$$= e^{\ln t^2 - \ln t_0^2} x_2(t_0) + \int_{t_0}^{t}\left(e^{\ln t^2 - \ln \tau^2}\right)\left(\frac{-2}{\pi t_0}x_1(t_0)\right)d\tau$$

$$= \frac{t^2}{t_0^2}x_2(t_0) + \int_{t_0}^{t}\frac{t^2}{\tau^2}\left(\frac{-2}{\tau}\right)\left(\frac{x_1(t_0)}{t_0}\right)d\tau$$

$$= \frac{t^2}{t_0^2}x_2(t_0) + \frac{t^2}{t_0}x_1(t_0)\int_{t_0}^{t}\frac{-2}{\tau^3}d\tau$$

$$x_2(t) = \frac{t^2}{t_0^2}x_2(t_0) + \frac{t^2}{t_0}x_1(t_0)\left(\frac{1}{t^2} - \frac{1}{t_0^2}\right)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{t}{t_0} & 0 \\ \frac{1}{t_0} - \frac{t^2}{t_0^3} & \frac{t^2}{t_0^2} \end{bmatrix}}_{\Phi(t,t_0)} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$$

__Example 5__: $A(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2t \end{bmatrix}$

$A(t_0) = \begin{bmatrix} 1 & 0 \\ 0 & 2t_0 \end{bmatrix} \Rightarrow A(t)A(t_0) = A(t_0)A(t)$, therefore, use method #3

$$\Phi(t, t_0) = e^{\int_{t_0}^{t}\begin{bmatrix} 1 & 0 \\ 0 & 2\tau \end{bmatrix}d\tau}$$

$$\int_{t_0}^{t}\begin{bmatrix} 1 & 0 \\ 0 & 2\tau \end{bmatrix}d\tau = \begin{bmatrix} t - t_0 & 0 \\ 0 & t^2 - t_0^2 \end{bmatrix}$$

$$e^{\begin{bmatrix} t-t_0 & 0 \\ 0 & t^2-t_0^2 \end{bmatrix}} = \begin{bmatrix} e^{t-t_0} & 0 \\ 0 & e^{t^2-t_0^2} \end{bmatrix} \quad \surd$$

This follows from Cayley-Hamilton technique.

Example 6: $A(t)=\begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix}$

$$A(t_0)=\begin{bmatrix} t_0 & 1 \\ 1 & t_0 \end{bmatrix} \quad A(t)A(t_0)=\begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix}\begin{bmatrix} t_0 & 1 \\ 1 & t_0 \end{bmatrix}=\begin{bmatrix} tt_0+1 & t+t_0 \\ t+t_0 & 1+tt_0 \end{bmatrix} \overset{(shows)}{=} A(t_0)A(t) \surd$$

$\therefore$ Method #3 applies $\quad \therefore \Phi(t,t_0)=e^{\int_{t_0}^{t} A(\tau)d\tau}$

$$\int_{t_0}^{t}\begin{bmatrix} \tau & 1 \\ 1 & \tau \end{bmatrix} d\tau = \underbrace{\begin{bmatrix} \frac{t^2-t_0^2}{2} & t-t_0 \\ t-t_0 & \frac{t^2-t_0^2}{2} \end{bmatrix}}_{B}, \quad \Rightarrow e^{\begin{bmatrix} \frac{t^2-t_0^2}{2} & t-t_0 \\ t-t_0 & \frac{t^2-t_0^2}{2} \end{bmatrix}} = ?$$

$$e^B = ?$$

$$P(B)=e^B, P(\lambda)=e^{\lambda}, R(\lambda)=\alpha_0+\alpha_1\lambda$$

Find eigenvalues of B

$$\det(\lambda I-B)=\det\begin{bmatrix} \lambda-\frac{t^2-t_0^2}{2} & -t+t_0 \\ -t+t_0 & \lambda-\frac{t^2-t_0^2}{2} \end{bmatrix}=0 \Rightarrow \lambda_{1,2}=\frac{t^2-t_0^2}{2}\pm(t-t_0)$$

$$\begin{cases} P(\lambda_1)=R(\lambda_1) \\ P(\lambda_2)=R(\lambda_2) \end{cases} \Rightarrow \begin{cases} \alpha_0 = e^{\frac{t^2-t_0^2}{2}}\left[e^{t-t_0}-\sinh(t-t_0)\left(\frac{t+t_0}{2}+1\right)\right] \\ \alpha_1 = e^{\frac{t^2-t_0^2}{2}}\left[\frac{\sinh(t-t_0)}{t-t_0}\right] \end{cases}$$

$$e^B=\alpha_0 I+\alpha_1 B$$

$$\Phi(t,t_0)=e^B=\begin{bmatrix} \cosh(t-t_0) & \sinh(t-t_0) \\ \sinh(t-t_0) & \cosh(t-t_0) \end{bmatrix} e^{\frac{t^2-t_0^2}{2}} \qquad \surd$$

11.3 Solution of the State Equations and Output Equation

For system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$,

the complete solution is given by $x(t) = \underbrace{\Phi(t, t_0)x(t_0)}_{\text{zero input respond}} + \underbrace{\int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau}_{\text{zero state respond}}$

Proof:

$$\dot{\Phi}(t, t_0)x(t_0) + \frac{\partial}{\partial t}\int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau \overset{?}{=}$$

$$A(t)\Phi(t, t_0)x(t_0) + A(t)\int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau + B(t)u(t)$$

Left= $A(t)\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \frac{\partial}{\partial t}\{\Phi(t, \tau)B(\tau)u(\tau)d\tau\} + \underbrace{\Phi(t, t)}_{=I} B(t)u(t) - 0$

Since $\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau)$, $\int_{t_0}^{t} \frac{\partial}{\partial t}\{\Phi(t, \tau)\}B(\tau)u(\tau)d\tau = A(t)\int_{t_0}^{t} \Phi Bud\tau$

$$Left = A(t)\Phi(t, t_0)x(t_0) + A(t)\int_{t_0}^{t} \Phi Bud\tau + B(t)u(t) = Right$$

Leibnitz's Rule

$$\frac{\partial}{\partial t}\int_{u(t)}^{v(t)} f(x,t)dx = \int_{u(t)}^{v(t)} \frac{\partial f(x,t)}{\partial t}dx + f(v,t)\frac{\partial v}{\partial t} - f(u,t)\frac{\partial u}{\partial t}$$

$$\therefore y(t) = C(t)\Phi(t, t_0)x(t_0) + C(t)\int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

Given $\dot{x} = Ax + Bu$, A and B are constant

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

12 Lecture 12

Objectives

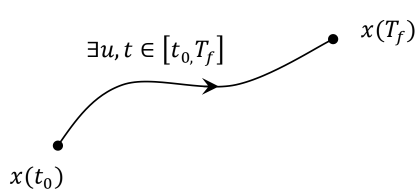
1) ***Controllability; Observability***
2) ***Canonical Transformations: C.C.F., O.C.F***

12.1 Definitions

12.1.1 Controllability

Given $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$, $x \in R^n, y, u \in R$ (SISO), the system is **controllable *if and only if*** the $n \times n$ matrix $C_x = [B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B]_{n \times n}$ (known as the controllability matrix) is nonsingular; i.e. $Rank\{C_x\} = n$.

A^{2}B=A(AB)



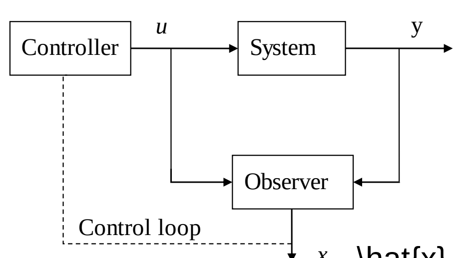
Pair (A,B)

Figure 12-1. State trajectory as a result of the control u.

12.1.2 Observability

Given the above system, we say it is **observable *if and only if*** the $n \times n$ matrix O_x (known as the observability matrix) is nonsingular, in another word, $Rank\{O_x\} = n$, where

$$O_x = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}_{n \times n}$$



Pair (A,C)

\hat{x} estimate of x

Figure 12-2. System diagram.

12.1.3 Examples for Controllability and Observability

<u>Example 1</u>: $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}=\begin{bmatrix}B_1 \vdots & B_2\end{bmatrix}$, is this system controllable ?

$$C_x = [B \vdots \quad AB] = \begin{bmatrix}1 & 1 & \vdots & 1 & -1\\ 1 & -1 & \vdots & 1 & 1\end{bmatrix}$$

$Rank(C_x)=2 \Rightarrow$ full rank $\Rightarrow$ **controllable !**

Q: check to see if the system is controllable from individual inputs.

$C_x^1 = [B_1 \vdots \quad AB_1] = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ $Rank(C_x^1)=1 \neq 2 \Rightarrow$ not controllable from u_1

$C_x^2 = [B_2 \vdots \quad AB_2] = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$ $Rank(C_x^2)=1 \neq 2 \Rightarrow$ not controllable from u_2

<u>Example 2</u>: $A=\begin{bmatrix}1 & 2\\ 0 & 4\end{bmatrix}$, $C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}C_1\\ C_2\end{bmatrix}$, is this system observable?

$$O_x=\begin{bmatrix}C\\ CA\end{bmatrix} \quad O_x=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 2\\ 0 & 4\end{bmatrix} \Rightarrow Rank(O_x)=2 \Rightarrow \textbf{Observable}$$

Q: is the system observable from each output alone?

a_1v_1+a_2v_2=0
v_1=[1 1]^{T}

$O_x^1=\begin{bmatrix}C_1\\ C_1A\end{bmatrix}=\begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix} \Rightarrow Rank(O_x^1)=2 \Rightarrow$ **Observable from** y_1

$O_x^2=\begin{bmatrix}C_2\\ C_2A\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 4\end{bmatrix} \Rightarrow Rank(O_x^2)=1 \neq 2 \Rightarrow$ **Unobservable from** y_2

12.2 Canonical Transformations

12.2.1 A Canonical Decomposition of Dynamic System

NO=Not observable
NC=Not controllable

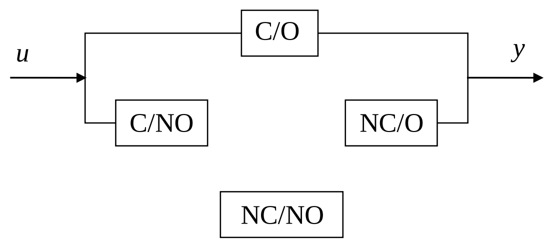


Figure 12-3. System classification.

If O_x singular $\Longrightarrow$ system is not observable

If C_x singular $\Longrightarrow$ system is not controllable

Example 3: $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \; y = \begin{bmatrix} 1 & -1 \end{bmatrix} x$ (n = 2)

- **For Controllability**

$$C_x = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow Rank(C_x) = 1 \neq 2 \Rightarrow \textbf{ Uncontrollable}$$

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = 2x_2 \Rightarrow x_2(t) \longrightarrow \infty \; as \; t \to \infty \quad \nexists u \; such \text{ that y is bounded!} \\ y = x_1 - x_2 \Longrightarrow y \to \infty \; as \; t \to \infty \end{cases}$$

- **For Observability**

$$O_x = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} \Rightarrow Rank(O_x) = 2 \Rightarrow \textbf{Observable}$$

Let us set C as $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, then $O_x = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \Rightarrow Rank(O_x) = 1 \neq 2 \Rightarrow$ **Unobservable**

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = 2x_2 \\ y = x_1 \end{cases}$$

Uncontrollable mode is also unobservable
(in addition to being unstable)

12.2.2 Controllable Canonical Form (C.C.F.)

12.2.2.1 Theorem

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$ $x \in R^n, y, u \in R$, assume that the system is **controllable** i.e.

$C_x = [B \vdots\ AB \vdots \cdots\ A^{n-1}B]$ is nonsingular. We want to find a nonsingular transformation

$x_c = T_c^{-1}x$, so that in the new coordinate system we have $\begin{cases} \dot{x}_c = A_c x_c + B_c u \\ y = C_c x_c + D_c u \end{cases}$

where $A_c = \begin{bmatrix} 0 & \vdots & & & & \\ \vdots & \vdots & & & & \\ 0 & \vdots & & I_{(n-1)\times(n-1)} & & \\ 0 & \vdots & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_0 & \vdots & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}_{n\times n}$, $B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \cdots \\ 1 \end{bmatrix}_{n\times 1}$,

C_c does not have any particular form and $D_c = D$.

For $x_c = T_c^{-1}x$, $\dot{x}_c = T_c^{-1}\dot{x} = T_c^{-1}(Ax + Bu) = T_c^{-1}Ax + T_c^{-1}Bu$.

Since $x = T_c x_c \therefore \dot{x}_c = \underbrace{T_c^{-1}AT_c}_{A_c} x_c + \underbrace{T_c^{-1}B}_{B_c} u$, $y = \underbrace{CT_c}_{C_c} x_c + \underbrace{D}_{D_c} u$ $\quad \therefore \begin{cases} A_c = T_c^{-1}AT_c \\ B_c = T_c^{-1}B \\ C_c = CT_c \\ D_c = D \end{cases}$

The controllability matrix for the new system

$$C_{x_c} = [B_c \vdots\ A_c B_c \vdots \cdots\ A_c^{n-1}B_c] = \left[T_c^{-1}B \vdots\ (T_c^{-1}A\overbrace{T_c)(T_c^{-1}}^{I}B) \vdots \cdots\ (T_c^{-1}AT_c)^{n-1}(T_c^{-1}B) \right]$$

$$= [T_c^{-1}B \vdots\ T_c^{-1}AB \vdots \cdots\ T_c^{-1}A^{n-1}B)] = T_c^{-1}\underbrace{[B \quad AB \quad \cdots \quad A^{n-1}B]}_{C_x} = T_c^{-1}C_x$$

$$(T_c^{-1}AT_c)^{n-1}(T_c^{-1}B) = (T_c^{-1}\overbrace{A\underbrace{T_c)(T_c^{-1}}_{I}A\underbrace{T_c)\cdots\cdots\cdots(T_c^{-1}}_{I}A\underbrace{T_c)(T_c^{-1}}_{I}}^{A^{n-1}}B)$$

By assumption, original system is controllable, $Rank(C_x) = n$

$$Rank\left(C_{x_c}\right) = Rank(T_c^{-1}C_c) = n \Rightarrow C_x^{-1} \text{ and } C_{x_c}^{-1} \text{ exist}$$

rank(AB), A & B full rank

C_x

$$\therefore T_c = C_x C_{x_c}^{-1} \text{ or } T_c^{-1} = C_{x_c} C_x^{-1}$$

(AB)^(-1)=B^(-1)A^(-1)

12.2.2.2 Procedure for Controllable Canonical Form

0) Check to see if the system is controllable.

   If yes, proceed to the following steps; if not, stop as there is no CCF possible.

1) Determine the coefficients of the C.P. of matrix A.
2) Form A_c and B_c.
3) Calculate C_x and C_{x_c}, using A, B and A_c, B_c.
4) Find the transformation $T_c = C_x C_{x_c}^{-1}$.
5) Use $C_c = CT_c$ to get C_c.
6) As a check, verify that $A_c = T_c^{-1} A T_c$ and $B_c = T_c^{-1} B$.
7) $D_c = D$.

12.2.3 Observable Canonical Form (O.C.F.)

12.2.3.1 Theorem

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Given this system, find a transformation that takes the system into an observable canonical form. The transformed system takes the form $\begin{cases} \dot{x}_o = A_o x_o + B_o u \\ y = C_o x_o + D_o u \end{cases}$

Where $A_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & \vdots & -a_0 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ & & & & \vdots & -a_1 \\ & & I_{(n-1)\times(n-1)} & & \vdots & -a_2 \\ & & & & \vdots & \vdots \\ & & & & \vdots & -a_{n-1} \end{bmatrix}_{n\times n}$, $C_o = [0 \quad 0 \quad \cdots \quad 0 \quad \vdots \quad 1]_{1\times n}$

B_o does not have any particular structure, $D_o = D$

Define $x_o = T_o^{-1}x$, $x = T_o x_o$ $\quad \Rightarrow \dot{x}_o = T_o^{-1}\dot{x} = T_o^{-1}(Ax + Bu) = T_o^{-1}AT_o x_o + T_o^{-1}Bu$

$$y = Cx + Du = CT_o x_o + Du \Rightarrow \begin{cases} \dot{x}_o = T_o^{-1}AT_o x_o + T_o^{-1}Bu \\ y = CT_o x_o + Du \end{cases} \quad \therefore \begin{cases} A_o = T_o^{-1}AT_o \\ B_o = T_o^{-1}B \\ C_o = CT_o \\ D_o = D \end{cases}$$

Form O_x and O_{x_o} as $O_x = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ and $O_{x_o} = \begin{bmatrix} C_o \\ C_o A_o \\ \vdots \\ C_o A_o^{n-1} \end{bmatrix}$, it follows that $O_{x_o} = O_x T_o$.

$\therefore Rank(O_{x_o}) = Rank(O_x T_o)$. Since T_o is full rank, O_x is full rank $\therefore Rank(O_{x_o}) = n$

$\therefore O_{x_o}$ **is nonsingular!** $\therefore T_o^{-1} = O_{x_o}^{-1}O_x \quad or \quad T_o = O_x^{-1}O_{x_o}$

12.2.3.2 Procedure for Observable Canonical Form

0) Check to see if the system is observable.
   If yes, proceed to the following steps; if not, stop as there is no possible OCF.

1) Find the coefficients of the C.P. of A.

2) Form A_o and C_o.

3) Calculate O_x and O_{x_0} using A, C and A_o , C_o.

4) Find the transformation $T_o^{-1} = O_{x_o}^{-1}O_x$.

5) Use $T_o^{-1}B$ to get $B_o = T_o^{-1}B$.

6) As a check, verify that $A_o = T_o^{-1}AT_o$ and $C_o = CT_o$.

7) $D_o = D$.

Example 4: System given as $\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u$, $y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$

transform it into C.C.F and O.C.F.

- Controllable form

0) Check that C_x is nonsingular? √

1) C.P. = $\det(\lambda I - A) = 0 \Rightarrow \lambda^3 - 9\lambda + 2 = 0$

2) $A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 9 & 0 \end{bmatrix}$; $B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then $C_c = ?$

$$C_x = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 2 & 4 & 16 \\ 1 & 6 & 8 \\ 1 & 2 & 12 \end{bmatrix};\ C_{x_c} = \begin{bmatrix} B_c & A_cB_c & A_c^2B_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 9 \end{bmatrix}$$

$$\therefore\ T_c = C_x C_{x_c}^{-1} = \begin{bmatrix} -2 & 4 & 2 \\ 1 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\therefore\ C_c = CT_c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 6 & 1 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

$$\begin{cases} \dot{x}_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 9 & 0 \end{bmatrix} x_c + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} x_c \end{cases}$$

$$T.F. = \frac{s^2 + 2s + 3}{s^3 - 9s + 2}$$

- Observable Canonical Form

0) $O_x = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 6 & -2 & 1 \end{bmatrix}$ $\Longrightarrow$ nonsingular $\Longrightarrow$ **Observable**

$$\therefore A_o = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 9 \\ 0 & 1 & 0 \end{bmatrix} = A_c^T \quad ; \quad C_o = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = B_c^T$$

$$O_{x_o} = \begin{bmatrix} C_o \\ C_o A_o \\ C_o A_o^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 9 \end{bmatrix}$$

$$\therefore T_o = O_x^{-1} O_{x_o} \Rightarrow B_o = T_o^{-1} B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = C_c^T$$

13 Lecture 13

Objectives

1) Diagonalization of a matrix – Jordan canonical form

2) Controllability of J.C.F; Observability of J.C.F.

13.1 Diagonalization of a Matrix

13.1.1 Jordan Canonical Form (J.C.F.) – Distinct Eigenvalues Case

Given an *nxn* matrix A with distinct eigenvalues, the diagonal form of A denoted by $\hat{A}$ takes the form $\hat{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$ where $\hat{A} = M^{-1}AM$, with $M = \underbrace{\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}}$

Eigenvectors corresponding to eigenvalues $\lambda_1, \cdots, \lambda_n$

(known as modal transformation).

Given $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$, transformation into Jordan Canonical Form is achieved through

$x_J = M^{-1}x$, $x = Mx_J$, $\dot{x}_J = M^{-1}\dot{x} = M^{-1}Ax + M^{-1}Bu = M^{-1}AMx_J + M^{-1}Bu \Rightarrow$

$\begin{cases} \dot{x}_J = M^{-1}AMx_J + M^{-1}Bu \\ y = CMx_J + Du \end{cases}$, if A has n **distinct** eigenvalues, then choose M as the modal matrix, then

$$\begin{cases} \dot{x}_J = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} x_J + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u \\ y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} x_J + Du \end{cases}$$

The above system in Jordan Canonical Form is

Controllable *if and only if* $b_i \neq 0 \;\; \forall i, \;\; i = 1,2,...,n$ and

Observable *if and only if* $c_i \neq 0 \;\; \forall i, \;\; i = 1,2,...,n$

Example 1: $\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$, $y = [\sqrt{2} \quad \sqrt{2}\,]\, x$

Transform it into J.C.F. and also check its controllability and observability

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \Rightarrow \det(\lambda I - A) = 0 \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -3 \end{cases}$$

Given a matrix A and eigenvalue λ, the eigenvalue and eigenvector v relations are given by: $Av = \lambda v,$ *where* v *is eigenvector*

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{v_1} = \underbrace{-1}_{\lambda_1} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{v_1} \Rightarrow \begin{cases} x_1 + x_2 = 0 \\ -3x_1 - 3x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = -1 \end{cases} \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Similarly,

Without loss of generality, normalize v_1 to a unit length, named $nv_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{v_2} = \underbrace{-3}_{\lambda_2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{v_2} \Rightarrow \begin{cases} 3x_1 + x_2 = 0 \\ -3x_1 - x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = -3 \end{cases} \Rightarrow v_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Normalizing v_2 to unit length $\Rightarrow nv_2 = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} \end{bmatrix}$

$$\therefore \lambda_1 = -1 \Rightarrow nv_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}^T, \quad \lambda_2 = -3 \Rightarrow nv_2 = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}^T$$

Modal matrix:

$$M=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}}\\ -\frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{10}}\end{bmatrix}\Rightarrow M^{-1}=\frac{-\sqrt{20}}{2}\begin{bmatrix}\frac{-3}{\sqrt{10}} & \frac{-1}{\sqrt{10}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\frac{3\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{10}}{2} & \frac{-\sqrt{10}}{2}\end{bmatrix}$$

$$M^{-1}AM=\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}$$

$$\therefore \dot{x}_J=\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}x_J+\begin{bmatrix}\frac{3\sqrt{2}}{2}\\ \frac{\sqrt{10}}{2}\end{bmatrix}u$$

$\neq 0 \Rightarrow \lambda=-1$ *is controllable*

$\neq 0 \Rightarrow \lambda=-3$ *is controllable*

$$y=\begin{bmatrix}0 & \frac{-2\sqrt{2}}{\sqrt{10}}\end{bmatrix}x_J$$

$=0 \Rightarrow \lambda=-1$ *is unobservable*

$\neq 0 \Rightarrow \lambda=-3$ *is observable*

$$A=\begin{bmatrix}0 & 1\\ -3 & -4\end{bmatrix},\ B=\begin{bmatrix}1\\ 0\end{bmatrix},\ C=[\sqrt{2}\quad \sqrt{2}]$$

$$C_x=[B\vdots\ \ AB]=\begin{bmatrix}1\vdots & 0\\ 0\vdots & -3\end{bmatrix}\Rightarrow \text{rank}=2\Rightarrow \textbf{Controllable}$$

$$O_x=\begin{bmatrix}C\\ CA\end{bmatrix}=\begin{bmatrix}\sqrt{2} & \sqrt{2}\\ -3\sqrt{2} & -3\sqrt{2}\end{bmatrix}\Rightarrow \text{rank}=1\Rightarrow \textbf{Unobservable}$$

13.1.2 Jordan Canonical Form (J.C.F.) – Repeated Eigenvalues Case

13.1.2.1 Diagonalization of Matrices

An $n \times n$ matrix A may be diagonalized by a similarity transformation where $\hat{A} = M^{-1}AM$ ***if and only if*** nullity of $(A - \lambda_i I)$ (which is $n - rank(A - \lambda_i I)$) is equal to the multiplicity of the eigenvalues, and moreover the modal matrix M is the matrix of eigenvectors. Therefore, if $\exists$ multiple eigenvalues, then $\exists$ same number of linearly independent eigenvectors. If this is not the case, then a diagonal form is **not** possible, and Jordan form is used as the standard form.

Example 2: $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \therefore \det(A - \lambda I) = (\lambda + 1)^2 = 0 \Rightarrow \lambda = -1, -1$

$$rank(A - \lambda I) = rank \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = 1,$$

therefore, nullity of $A - \lambda I = 1 \Longrightarrow \exists$ **one** linearly independent eigenvector.

For $\lambda = -1 \Longrightarrow$ A cannot be diagnolized.

We will show later in example3 that the proper form is $\hat{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

13.1.2.2 Generalized Eigenvectors

A nonzero vector x for which $(A - \lambda I)^{k-1} x \neq 0$ but $(A - \lambda I)^k x = 0$ is called a **generalized eigenvector** of **rank *k*** of A associated with the eigenvalue λ.

Generalized eigenvectors are defined as

$$x_k = x$$

$$x_{k-1} = (A - \lambda I)x = (A - \lambda I)x_k$$

$$x_{k-2} = (A - \lambda I)^2 x = (A - \lambda I)x_{k-1}$$

$$\vdots$$

$$x_1 = (A - \lambda I)^{k-1} x = (A - \lambda I)x_2$$

$\therefore$ generalized eigenvectors $\{x_1, x_2, \cdots, x_k\}$

Theorem:

- The set of generalized eigenvectors $x_1, x_2, ..., x_k$ for the same eigenvalue are linearly independent.
- The generalized eigenvectors of A associated with different eigenvalues are linearly independent.

Example 3: $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $\lambda = -1, -1$

Here $k = 2$, $x = x_2$ is computed from $(A - \lambda I)x_2 \neq 0$ and $(A - \lambda I)^2 x_2 = 0$

$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}^2 x_2 = 0 \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x_2 = 0 \Rightarrow$ Any vector that satisfies $(A - \lambda I)x_2 \neq 0$ can be used. Choose $x_2 = [1 \quad 0]^T$

$$\therefore (A - \lambda I)x_2 = [1 \quad -1]^T \neq 0$$

Calculate $x_1 = (A - \lambda I)^{2-1} x_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore M = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = [x_1 \quad x_2]$$

$$\hat{A} = M^{-1} A M \Rightarrow \hat{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Example 4: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

$$\det(A - \lambda I) = (1 - \lambda)^2 (2 - \lambda) = 0 \Rightarrow \lambda = 2, 1, 1$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{bmatrix}$$

Q : Nullity $A-\lambda_i I = ?$ when $\lambda_i = 1$

$$A-I=\begin{bmatrix}0&0&0\\0&0&0\\-1&0&1\end{bmatrix},\ rank(A-I)=1,$$

$$n-rank(A-I)=3-1=2\equiv multiplicity\ \text{of}\ \lambda=1$$

$\therefore \exists$ 2 linearly independent eigenvectors for $\lambda=1$

$$(A-\lambda I)x_1=\begin{bmatrix}0&0&0\\0&0&0\\-1&0&1\end{bmatrix}x_1=0\Rightarrow x_1=\begin{bmatrix}1\\0\\1\end{bmatrix},\ x_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\text{For } \lambda=2\Rightarrow\begin{bmatrix}-1&0&0\\0&-1&0\\-1&0&0\end{bmatrix}x_3=0\Rightarrow x_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\therefore M=[x_1\quad x_2\quad x_3]=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\Rightarrow \hat{A}=M^{-1}AM=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$$

13.1.3 Matrix Reduction to Jordan Form

13.1.3.1 Definition

The Jordan form J of an $n\times n$ matrix A is an upper triangular matrix whose diagonal entries

are of the matrix form $L_{ij}(\lambda_j)=\begin{bmatrix}\lambda_j&1&&&&\\&\lambda_j&1&&0&\\&&\lambda_j&1&&\\&&&\ddots&&\\&0&&&\lambda_j&1\\&&&&&\lambda_j\end{bmatrix}$

The general case of the Jordan Matrix J is

$$J=\begin{bmatrix} L\ \ (\lambda\ ) & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & L_{21}(\lambda\ ) & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & L_k\ (\lambda\ ) & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & L_{12}(\lambda_2) & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & L_{mp}(\lambda_\rho) \end{bmatrix}$$

$L_{ij}(\lambda_j)$ matrices are called **Jordon blocks**.

13.1.3.2 General Rules for Constructing the Jordan Matrices

1) There is ***one and only one*** linearly independent eigenvector associated with each Jordan block. In general, there are $n-rank(A-\lambda_i I)$ linearly independent eigenvectors for a general $n\times n$ matrix A with its associated eigenvalue λ_i.

2) The number of 1's directly above the diagonal terms equal the order of A minus the number of linearly independent eigenvectors r, in another words, $n_{1's}=n-r$

3) The number of Jordan blocks associated with each eigenvalue is equal to the number of linearly independent eigenvectors associated with each eigenvalue r.

Example 5:

1 generalized eigenvector; 1 linearly independent eigenvector for λ

$$J=\begin{bmatrix} \lambda & 1 & \vdots & & \vdots & \\ 0 & \lambda & \vdots & & \vdots & \\ \cdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ & & \vdots & \lambda & \vdots & \\ \cdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ & & \vdots & & \vdots & \lambda_2 \end{bmatrix},\ J_2=\begin{bmatrix} \lambda & 1 & & \vdots & \\ & \lambda & 1 & \vdots & \\ & & \lambda & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \vdots & \lambda_2 \end{bmatrix}$$

if nulity is 3 ->three blocks
J is perfectly diagnolizable
n-r=0

1 $l.i.e.v.$ for λ_2

1 $l.i.e.v.$ for λ

1 $l.i.e.v.$ for λ_2

2 $g.e.v.$;
1 $l.$ $i.e.v.$ $for\lambda$

For J_1(with $r=\#\ of\ blocks)\Rightarrow n-r=4-3=1\Rightarrow only$ 1 one in JCF.

For $J_2(with\ r=\#\ of\ blocks)\Rightarrow n-r=4-2=2\Rightarrow only$ 2 one's in JCF.

Example 6: Transform A into Jordan form $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -7 & -9 & -5 \end{bmatrix}$

$$\det(A-\lambda I) = \lambda^4 + 5\lambda^3 + 9\lambda^2 + 7\lambda + 2 = 0 \Rightarrow \lambda = -1,-1,-1,-2$$

$$\#\ of\ \ l.i.e.v. = n - rank(A-\lambda_i I)$$

$$rank(A-\lambda_1 I) = \text{rank} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -2 & -7 & -9 & -4 \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 3$$

$\therefore \exists$ only $4 - 3 = 1$ l.i.e.v. associated with $\lambda = -1$.

Thus $J = \begin{bmatrix} -1 & 1 & 0 & \vdots & 0 \\ 0 & -1 & 1 & \vdots & 0 \\ 0 & 0 & -1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & -2 \end{bmatrix}$ (# of 1's in J are n - 2 = 4-2 = 2, where A has 2 *l.i.e.v.*)

Check: Let $M = [x_1 \quad x_2 \quad x_3 \quad x_4]$

Define x_3 by $\begin{cases} (A-\lambda_1 I)^3 x_3 = 0 \\ (A-\lambda_1 I)^2 x_3 \neq 0 \end{cases}$ and subsequently

$x_2 = (A-\lambda_1 I)x_3$, $x_1 = (A-\lambda_1 I)x_2$ and $(A-\lambda_2 I)x_4 = 0$.

For x_3, we have: $(A+I)^3 x_3 = 0 \Rightarrow \begin{bmatrix} 1 & 3 & 3 & 1 \\ -2 & -6 & -6 & -2 \\ 4 & 12 & 12 & 4 \\ -8 & -24 & -24 & -8 \end{bmatrix} x_3 = 0$

$\exists$ only 1 linearly independent vector

$\therefore a + 3b + 3c + d = 0 \Rightarrow x_3 = [1 \quad 0 \quad 0 \quad -1]^T$

$\therefore x_2 = (A+I)x_3 = [1 \quad 0 \quad -1 \quad 2]^T$,

$x_1 = (A+I)x_2 = [1 \quad -1 \quad 1 \quad -1]^T$

x_4 is calculated from $(A+2I)x_4 = 0 \Rightarrow x_4 = [1 \quad -2 \quad 4 \quad -8]^T$

$\therefore M = [x_1 \quad x_2 \quad x_3 \quad x_4] \Rightarrow \hat{A} = M^{-1}AM$ $\quad \surd$

13.1.3.3 A Procedure for Generating a JCF

1) Compute the eigenvalues of A by solving $\det(A-\lambda I)=0$. Let $\lambda_1,\lambda_2,\cdots\lambda_n$ be these **distinct** eigenvalues with multiplicity $n_1,n_2,\cdots n_n$, respectively.

2) Compute the number of linearly independent eigenvectors for λ_1. This is also the number of Jordan blocks for λ_1.

3) For eigenvalue λ_1 determine its Jordan blocks from the information in (1) and (2) and the general rules.

4) Determine the linearly independent eigenvector for λ_1 associated with each Jordan block and the remaining generalized eigenvectors for each block. Denote the linearly independent one with the subscript of the order of the Jordan block. Thus if the Jordan block were of order 3, then denote the linearly independent vector as x_3 and the generalized eigenvectors as x_2 and x_1..

5) Repeat steps (2), (3), and (4) for eigenvectors $\lambda_2,\lambda_3,\cdots\lambda_n$ to generate a Modal matrix by placing the eigenvectors as columns. The order of the column placement is important and is directly related to how they are numbered in step (4).

As an example, suppose that A is 5×5 with eigenvalues $\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_3$. Also, suppose there are two linearly independent eigenvectors associated with λ_1, that is $rank(A-\lambda_1 I)=3 \Rightarrow$ nullity of $(\mathrm{A}-\lambda_1 I)=2$. Therefore, there are two Jordan blocks associated with λ_1 as $J_{11}(\lambda_1)=\lambda_1$ & $J_{21}(\lambda_1)=\begin{bmatrix}\lambda_1 & 1\\ 0 & \lambda_1\end{bmatrix}$

The Jordan form is $J=\left[\begin{array}{c:cc:c:c}\lambda_1 & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & \lambda_1 & 1 & & \\ & 0 & \lambda_1 & & \\ \cdots & \cdots & \cdots & \cdots\ \cdots & \cdots \\ & & & \lambda_2 & \\ \cdots & \cdots & \cdots & \cdots\ \cdots & \cdots \\ & & & & \lambda_3\end{array}\right]=\begin{bmatrix}J_{11}(\lambda_1) & & & \\ & J_{21}(\lambda_1) & & \\ & & J_{12}(\lambda_2) & \\ & & & J_{13}(\lambda_3)\end{bmatrix}$

Assume we have found for block $J_{11}(\lambda_1)$ an eigenvector x_1 from $Ax_1 = \lambda x_1$.For block $J_{21}(\lambda_1)$, we need two generalized eigenvectors x_3 from $(A-\lambda_1 I)^2 x_3 = 0$ and $(A-\lambda_1 I)x_3 = x_2$. Eigenvalues λ_2 and λ_3 will contribute the two eigenvectors x_4 and x_5, respectively from $Ax_4 = \lambda_2 x_4$ and $Ax_5 = \lambda_3 x_5$.

The Modal matrix is formed by $M = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}$

13.2 Controllability of JCF and Observability of JCF

13.2.1 Theorem

C=[B AB A^2B ...A^(n-1)B]

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad x \in R^n, u \in R^p, y \in R^q$$

$$A_{n\times n} = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{bmatrix}, \; B_{n\times p} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}, C_{q\times n} = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix}$$

A has m distinct eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_m$; A_i denotes the Jordan blocks associated with λ_i.

r(i) is the number of Jordan blocks in A_i; A_{ij} is the *j-th* Jordan block in A_i.

$$A = diagn\left(A_1, A_2, \ldots, A_m\right); \; A_i = diagn\left(A_{i1}, A_{i2}, \ldots, A_{ir(i)}\right)$$

r(i) \equiv nulity

$$A_{i(n_i\times n_i)} = \begin{bmatrix} A_{i1} & & & \\ & A_{i2} & & \\ & & \ddots & \\ & & & A_{ir(i)} \end{bmatrix} \quad ; \quad B_{i(n_i\times p)} = \begin{bmatrix} B_{i1} \\ B_{i2} \\ \vdots \\ B_{ir(i)} \end{bmatrix}$$

$$C_{i(q\times n_i)} = \begin{bmatrix} C_{i1} & C_{i2} & \cdots & C_{ir(i)} \end{bmatrix} \qquad n = \sum_{i=1}^{m} n_i = \sum_{i=1}^{m} \sum_{j=1}^{r(i)} n_{ij}$$

$$A_{ij(n_{ij}\times n_{ij})} = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}, \quad B_{ij(n_{ij}\times p)} = \begin{bmatrix} b_{1ij} \\ b_{2ij} \\ \vdots \\ b_{lij} \end{bmatrix}, \quad C_{ij(n_{ij}\times p)} = \begin{bmatrix} c_{1ij} \\ c_{2ij} \\ \vdots \\ c_{lij} \end{bmatrix}^T$$

The n-dimensional LTI Jordan form is **controllable *if and only if*** for each i =1, 2,…,m,

the rows of the $r(i)\times p$ matrix $B_i^l = \begin{bmatrix} b_{li1} \\ b_{li2} \\ \vdots \\ b_{lir(i)} \end{bmatrix}$ are linearly independent.

O=[C
CA
CA^2
....
CA^(n-1)]

and the system is **observable *if and only if*** for each i = 1, 2,…,m the columns of the $q\times r(i)$ matrix $C_i^1 = [C_{1i1} \quad C_{1i2} \quad \cdots \quad C_{1ir(i)}]$ are linearly independent.

13.2.2 Examples

Example 7:

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & \vdots & & \vdots & & \vdots & & & \\ 0 & \lambda_1 & \vdots & & \vdots & & \vdots & & & \\ \cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ & & \vdots & \lambda_1 & \vdots & & \vdots & & & \\ \cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ & & \vdots & & \vdots & \lambda_1 & \vdots & & & \\ \cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ & & \vdots & & \vdots & & \vdots & \lambda_2 & 1 & 0 \\ & & \vdots & & \vdots & & \vdots & 0 & \lambda_2 & 1 \\ & & \vdots & & \vdots & & \vdots & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 1 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 1 \\ \cdots & \cdots & \cdots \\ 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ b_{l11} \\ \\ b_{l12} \\ \\ b_{l13} \\ \\ \\ \\ b_{l14} \end{array} u$$

$$\begin{array}{cc} & \begin{array}{cccccccccc} c_{111} & & & c_{112} & & c_{113} & & c_{121} & & \end{array} \\ y = & \begin{bmatrix} 1 & 1 & \vdots & 2 & \vdots & 0 & \vdots & 0 & 2 & 0 \\ 1 & 0 & \vdots & 1 & \vdots & 2 & \vdots & 0 & 1 & 1 \\ 1 & 0 & \vdots & 2 & \vdots & 3 & \vdots & 0 & 2 & 2 \end{bmatrix} x \end{array}$$

$$B_1^l = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \textit{full rank } B_1^l \Rightarrow \text{mode } \lambda_1 \textit{ controllable}$$

$$B_2^l = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{linealry independent} \Rightarrow \text{mode } \lambda_2 \textit{ controllable}$$

$$C_1^1 = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow \text{full rank } C_1^1 \Rightarrow \text{mode } \lambda_1 \text{ observable}$$

$$C_2^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \text{rank} = 0 \Rightarrow \text{mode } \lambda_2 \text{ unobservable}$$

Block diagram of this system:

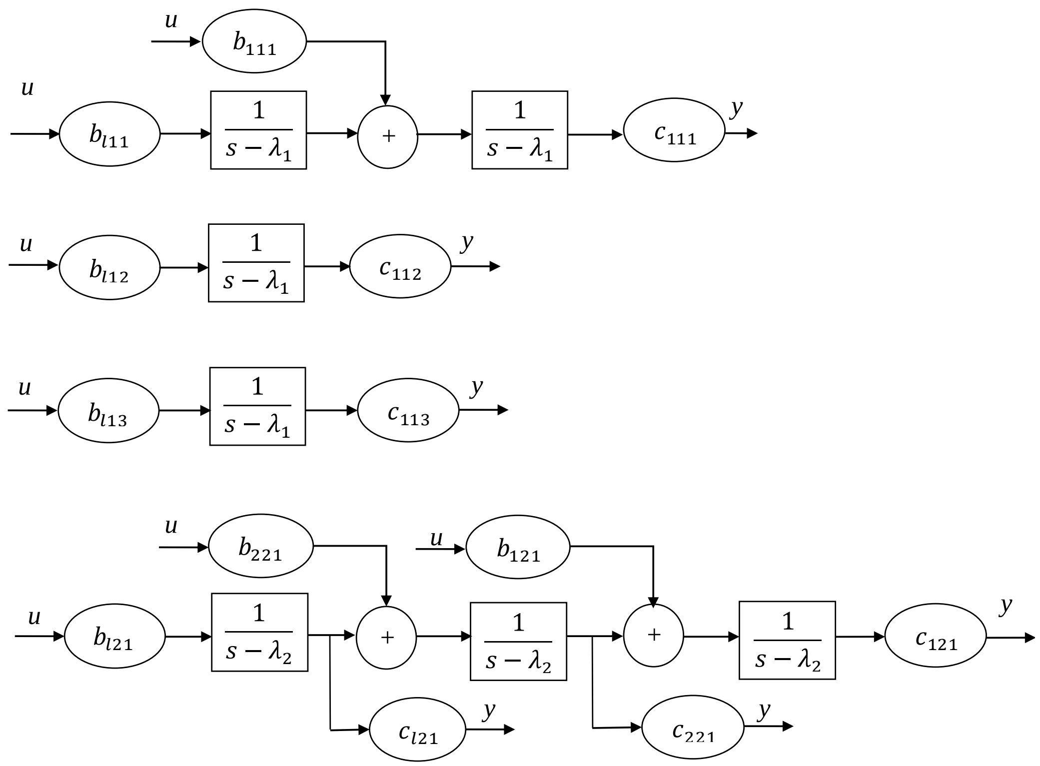


Figure 13-1. Block diagram.

If $b_{l21} \neq 0 \Rightarrow$ all modes in the chain can be controlled;

If $c_{121} \neq 0 \Rightarrow$ all modes in the chain can be observed.

$= 0 \Longrightarrow \lambda = 0$ uncontrollable.

<u>Example 8</u>: $\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix} x + \begin{bmatrix} 10 \\ 9 \\ 0 \\ \cdots \\ 1 \end{bmatrix} u$

$\neq 0 \Longrightarrow \lambda = 1$ controllable.

$$y = \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \end{bmatrix} x$$

$\neq 0 \Longrightarrow \lambda = 0$ observable.

$\neq 0 \Longrightarrow \lambda = 1$ observable.

What is the JCF for a system with complex conjugate eigenvalues?

Let $\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & \bar{A}_1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} b_1 \\ \bar{b}_2 \end{bmatrix} u$, $y = \begin{bmatrix} c_1 & \bar{c}_1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$

where A_1 is the Jordan block associated with λ and $\bar{A}_1$ is the complex conjugate of A_1, in another word, $\bar{A}_1$ is the Jordan block associated with $\bar{\lambda}_1$.

Introduce the equivalence transformation $\bar{x} = Px$

where $P = \begin{bmatrix} I & I \\ jI & -jI \end{bmatrix}$ and $P^{-1} = \frac{1}{2} \begin{bmatrix} I & -jI \\ I & jI \end{bmatrix}$

A=a+/-jb -> J=diag[a+jb a-jb]

J'=[a b
-b a]

It can be verified that $\begin{cases} \begin{bmatrix} \dot{\bar{x}}^1 \\ \dot{\bar{x}}^2 \end{bmatrix} = \begin{bmatrix} \operatorname{Re} A_1 & \operatorname{Im} A_1 \\ -\operatorname{Im} A_1 & \operatorname{Re} A_1 \end{bmatrix} \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix} + \begin{bmatrix} 2\operatorname{Re} b_1 \\ -2\operatorname{Im} b_1 \end{bmatrix} u \\ y = \begin{bmatrix} \operatorname{Re} c_1 & \operatorname{Im} c_1 \end{bmatrix} \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix} \end{cases}$

This system can now be used in computer simulations.

Example 9: $A \rightarrow 2\ ,\ 1 \pm 2j\ ,\ 1 \pm 2j$

$$\dot{x} = \begin{bmatrix} 1+2j & 1 & \vdots & & & \vdots & \\ 0 & 1+2j & \vdots & & & \vdots & \\ \cdots & \cdots & \vdots & \cdots & \cdots & \vdots & \\ & & \vdots & 1-2j & 1 & \vdots & \\ & & \vdots & 0 & 1-2j & \vdots & \\ \cdots & \cdots & \vdots & \cdots & \cdots & \vdots & \cdots \\ & & & & & \vdots & 2 \end{bmatrix} x + \begin{bmatrix} 2-3j \\ 1 \\ \cdots \\ 2+3j \\ 1 \\ \cdots \\ 2 \end{bmatrix} u$$

$$y = [1 \quad -j \quad \vdots \quad 1 \quad j \quad \vdots \quad 2]\, x$$

Let $\bar{x} = Px$, where $P = \begin{bmatrix} 1 & 0 & \vdots & 1 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 & 1 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots & \cdots & \vdots & \cdots \\ j & 0 & \vdots & -j & 0 & \vdots & 0 \\ 0 & j & \vdots & 0 & -j & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & 0 & 0 & \vdots & 1 \end{bmatrix}$

$$\therefore \dot{\bar{x}} = \begin{bmatrix} 1 & 1 & \vdots & 2 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 & 2 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots & \cdots & \vdots & \cdots \\ -2 & 0 & \vdots & 1 & 1 & \vdots & 0 \\ 0 & -2 & \vdots & 0 & 1 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & 0 & 0 & \vdots & 2 \end{bmatrix} \bar{x} + \begin{bmatrix} 4 \\ 2 \\ \cdots \\ 6 \\ 0 \\ \cdots \\ 2 \end{bmatrix} u$$

$y = [1 \quad 0 \quad \vdots \quad 0 \quad -1 \quad \vdots \quad 2]\,\bar{x}$ All the coefficients are real!

<u>Example 10</u>: $A \quad \rightarrow \quad 1 \pm j \ , \ 2 \pm 3j$

A

$$\dot{x} = \begin{bmatrix} 1+j & & \vdots & & \\ & 2+3j & \vdots & & \\ \cdots & \cdots & \vdots & \cdots & \cdots \\ & & \vdots & 1-j & \\ & & \vdots & & 2-3j \end{bmatrix} x$$

$\bar{A}$

$$\dot{\bar{x}} = \begin{bmatrix} 1 & 0 & \vdots & 1 & 0 \\ 0 & 2 & \vdots & 0 & 3 \\ \cdots & \cdots & \vdots & \cdots & \cdots \\ -1 & 0 & \vdots & 1 & 0 \\ 0 & -3 & \vdots & 0 & 2 \end{bmatrix} \bar{x}$$

14 Lecture 14

Objectives

State Space Control Method

1) ***State Feedback: General Case***
2) ***State Estimation: Full Order Observer***
3) ***State Estimation: Reduced Order Observer***

14.1 State ~~Stable~~ Feedback: General Case

State: $\dot{x} = Ax + Bu,\ \ x \in R^n, u \in R$

Control: $u = -Kx$ with $K = [K_1 \quad K_2 \quad \cdots \quad K_n]_{1\times n}$

r \eq 0

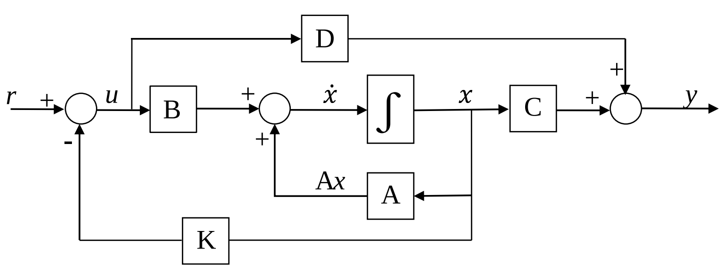


Figure 14-1. Closed-loop system state diagram.

$$\therefore \begin{cases} \dot{x} = (A - BK)x \overset{\Delta}{=} A_f x \\ A_f = A - BK \leftarrow \text{closed - loop matrix} \end{cases},$$

Pole Placement Problem

the characteristic polynomial for the closed-loop system is

$$\det(sI - A_f) = \det(sI - A + BK) = 0$$

Let the Design Specification require closed-loop eigenvalues at $-\lambda_1, -\lambda_2, \ldots, -\lambda_n$.

$$\therefore \alpha_c(s) = (s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0$$ Desired C.P.

Pole-placement design procedure is achieved by setting

$$\det(sI - A + BK) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

$\therefore$ Solve for n unknown $K_1,\ldots,K_n$. The equations are linear!

K can be computed directly from this formula ***if and only if*** (A,B) is controllable.

Consider the transfer function $G_p(s)=\dfrac{b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$

The controllable canonical form is

$$\dot{x}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y=\begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} x$$

The control law is $u=-Kx$

$\therefore$ Closed - loop A_f matrix is

$$A_f=A-BK=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0-K_1 & -a_1-K_2 & -a_2-K_3 & \cdots & -a_{n-1}-K_n \end{bmatrix}$$

Characteristic polynomial $\Rightarrow$ $\det(sI-A+BK)=0$

$$\therefore s^n+(a_{n-1}+K_n)s^{n-1}+\cdots+(a_1+K_2)s+(a_0+K_1)=0$$

Since desired characteristic polynomial is

$$\alpha_c(s)=s^n+\alpha_{n-1}s^{n-1}+\cdots+\alpha_1 s+\alpha_0=0$$

$$\therefore K_i=\alpha_{i-1}-a_{i-1} \qquad i=1,2,\ldots n$$

Alternatively, K can be computed according to the **Ackermann's Formula** as follows:

$$\begin{cases} K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \underbrace{B \quad AB \quad \cdots \quad A^{n-2}B \quad A^{n-1}B}_{C_x} \end{bmatrix}^{-1} \alpha_c(A) \\ where \quad \alpha_c(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I \end{cases}$$

Example 1: $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Let the desired characteristic polynomial be

$$\alpha_c(s) = s^2 + (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2 = 0$$

Based on Ackermann's formula, need to compute C_x

$$C_x = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow C_x^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\alpha_c(A) = A^2 + (\lambda_1 + \lambda_2)A + \lambda_1\lambda_2 I == \begin{bmatrix} \lambda_1\lambda_2 & \lambda_1 + \lambda_2 \\ 0 & \lambda_1\lambda_2 \end{bmatrix}$$

$$\therefore K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} B & AB \end{bmatrix}^{-1} \alpha_c(A) \Rightarrow K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} \lambda_1\lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}$$

Example 2: Let the design specifications require a critically damped system with a settling time of 1 sec., i.e, $4\tau = 1$.

\zeta =1 (damping ratio)
T_s=4/\zeta \omega_n

$$\because \xi\omega_n = \frac{1}{\tau} \Rightarrow \xi\omega_n = 4 \Rightarrow \xi = 1 \text{ (critically damped)}, \omega_n = 4$$

$$\therefore \alpha_c(s) = (s+4)(s+4) = s^2 + 8s + 16$$

$$\therefore K_1 = \lambda_1\lambda_2 = 16, \quad K_2 = \lambda_1 + \lambda_2 = 8$$

If we compute the closed-loop transfer function, we get the figure 14-2.

Also, $T(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} sI - A + BK \end{bmatrix}^{-1} B = \dfrac{1}{s^2 + 8s + 16}$ as desired!

As a different example, let $\xi = .707, \tau = .25 \;\therefore s = -4 \pm j4 \Rightarrow \alpha_c(s) = s^2 + 8s + 32$

$$\therefore \begin{cases} K_1 = \lambda_1\lambda_2 = 32 \\ K_2 = \lambda_1 + \lambda_2 = 8 \end{cases} \Rightarrow K = \begin{bmatrix} 32 & 8 \end{bmatrix}$$

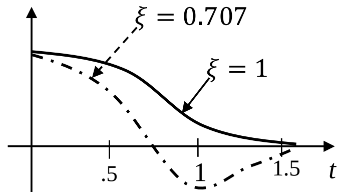


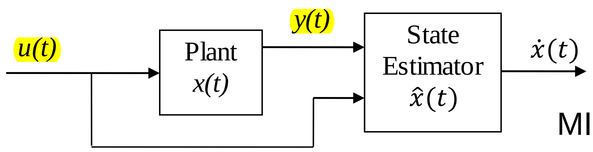
Figure 14-2. System plot for example 2.

14.2 State Estimation: Full-Order Observer

Luenberger Observer
vs. Kalman Filter
lim \hat{x} --> x as t --> \infty

14.2.1 Theorem

$$\begin{cases} \dot{x} = Ax + Bu \quad x \in R^n, u \in R \\ y = Cx \quad y \in R \end{cases}$$



Y=\hat{x}
\hat{x(t)}
MIMO sys.

The estimator equation is expressed as

$$\dot{\hat{x}} = F_{n\times n}\,\hat{x} + H_{n\times 1}\,u + G_{n\times 1}\,y$$

Figure 14-3. Block diagram of combined plant & observer.

We need to choose matrices F, G, and H such that $\hat{x}(t)$ is accurate estimate of *x(t)*. Then in the control system, the estimated states are used to generate the feedback control, i.e. $u = -K\hat{x}$. To do this, the transfer function from u to $\hat{x}_i$ must be equal to the transfer function from u to x_i, $i = 1,2,\dots,n$, that is

$$\frac{\hat{X}_i(s)}{U_i(s)} = \frac{X_i(s)}{U_i(s)} \qquad i = 1,2,\dots n$$

The Laplace Transform of state equations are

$$\begin{cases} sX(s) - x(0) = AX(s) + BU(s),\ x(0)\ \text{is unknown} \\ Y(s) = CX(s), A, B, C, \text{and } D\ \text{known} \end{cases}$$

By neglecting i.c. $\Rightarrow X(s) = (sI - A)^{-1}BU(s)$

$\therefore \dfrac{X_i(s)}{U(s)}$ can be obtained.

The Laplace Transform of estimator is (neglecting i.c.)

$$s\hat{X}(s) = F\hat{X}(s) + HU(s) + GY(s) = F\hat{X}(s) + HU(s) + GCX(s)$$

Since $Y(s) = CX(s)$

$$\therefore \hat{X}(s) = (sI - F)^{-1}\left[HU(s) + GCX(s)\right] = (sI - F)^{-1}\left[H + GC(sI - A)^{-1}B\right]U(s)$$

We want $\dfrac{X_i(s)}{U(s)} = \dfrac{\hat{X}_i(s)}{U(s)}$

$$\therefore (sI-A)^{-1}B = (sI-F)^{-1}\left[H + GC(sI-A)^{-1}B\right]$$

Collecting $(sI-A)^{-1}B$ terms

$$\left[I-(sI-F)^{-1}GC\right](sI-A)^{-1}B = (sI-F)^{-1}H$$

Since $(sI-F)^{-1}(sI-F) = I \;\therefore (sI-F)^{-1}\left[sI-F-GC\right](sI-A)^{-1}B = (sI-F)^{-1}H$

$$\therefore \left[sI-F-GC\right](sI-A)^{-1}B = H \Rightarrow (sI-A)^{-1}B = (sI-F-GC)^{-1}H$$

This equation is satisfied if we choose $H = B \; and \; F + GC = A$

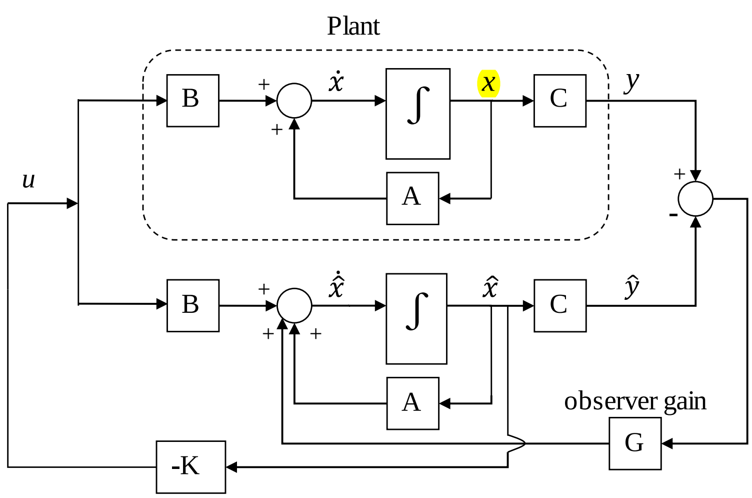
$$\therefore F = A - GC \quad and \quad H = B$$

G is chosen such that an acceptable transient response or frequency response for the state estimator is achieved.

$$\therefore \dot{\hat{x}} = A\hat{x} + Bu + Gy - GC\hat{x}$$

$\therefore \dot{\hat{x}} = (A-GC)\hat{x} + Bu + Gy, \;\; \hat{y} = C\hat{x}$ \hat{y} is estimate of output y

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + G(y-\hat{y}) \\ \hat{y} = C\hat{x} \end{cases}$$



y->y+noise

Output estimation error

Figure 14-4. Plant and observer diagram.

state estimation error

Let $e(t) = x(t) - \hat{x}(t) \;\therefore \dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - (A-GC)\hat{x} - Bu - Gy$

Since $y = Cx$, hence $\dot{e} = Ax - (A-GC)\,\hat{x} - GC\,x = (A-GC)\,(x-\hat{x})$

$$\therefore \dot{e} = (A-GC)\,e$$

The error in the estimation of the states is governed by the above dynamic equation with the characteristic polynomial: $\det(sI - A + GC) = 0$

The gain vector G is chosen to make the dynamics of the estimator faster than the open-loop system dynamics (~ 2 to 4 times faster).

14.2.2 Design of the State Estimator

1) $\dot{\hat{x}} = (A - GC)\hat{x} + Bu + Gy$, the characteristic polynomial is $\det(sI - A + GC) = 0$
2) Let the desired char. poly. be $\alpha_e(s)$, that reflects the desired transient behavior:
$\alpha_e(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0$
3) The gain matrix G is calculated to satisfy $\det(sI - A + GC) = \alpha_e(s)$
4) Using the transformation $x_o = T_o^{-1}x$ transforms the system into observable

$$\text{Canonical form}\begin{cases} \dot{x}_0 = \begin{bmatrix} 0 & \cdots & 0 & \vdots & -a_0 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ & & & \vdots & -a_1 \\ & I_{(n-1)\times(n-1)} & & \vdots & \vdots \\ & & & \vdots & -a_{n-1} \end{bmatrix} x_0 + \begin{bmatrix} b_0 \\ \cdots \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} u \\ y = [0 \quad \cdots \quad 0 \quad 1] \quad \text{x\_0} \end{cases}$$

G=[g_1,...,g_n]^{T}

Now $A - GC$ becomes $A - GC = \begin{bmatrix} 0 & \cdots & 0 & \vdots & -a_0 - g_1 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ & & & \vdots & -a_1 - g_2 \\ & I & & \vdots & \vdots \\ & & & \vdots & -a_{n-1} - g_n \end{bmatrix}$

$$\therefore \det(sI - A + GC) = \alpha_e(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

$$\therefore g_i = \alpha_{i-1} - a_{i-1} \quad i = 1,2,\ldots n$$

Now using Ackermann's formula we have [***if and only if*** (A, C) is observable]

$$\begin{cases} G_{n\times 1} = \alpha_e(A)\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ \alpha_e(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I \end{cases}$$

Example 3: $\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$

A controller was designed for the characteristic polynomial $\alpha_c(s) = s^2 + 8s + 32$,

which yielded a time constant $\tau = .25\sec,\ \xi = .707$

We choose the estimator to be critically damped with a $\tau = .1\sec$.

$$\therefore \alpha_e(s) = (s+10)^2 = s^2 + 20s + 100$$

$$\therefore \alpha_e(A) = A^2 + 20A + 100I = \begin{bmatrix} 100 & 20 \\ 0 & 100 \end{bmatrix}$$

Now $O_x = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $O_x^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore G = \alpha_e(A) \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$

$$\therefore g_1 = 20 \quad , \quad g_2 = 100 \quad ; \quad G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

Example 4: Combining previous two examples: $K = \begin{bmatrix} 32 & 8 \end{bmatrix}$, $G = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$

The estimator equations $\dot{\hat{x}} = (A - GC)\hat{x} + Bu + Gy = (A - GC - BK)\hat{x} + Gy$

Since $u = -K\hat{x}$, now $A - GC - BK = \begin{bmatrix} -20 & 1 \\ -132 & -8 \end{bmatrix}$

$$\therefore \quad \text{Observer/Controller} \quad \begin{cases} \dot{\hat{x}} = \begin{bmatrix} -20 & 1 \\ -132 & -8 \end{bmatrix} \hat{x} + \begin{bmatrix} 20 \\ 100 \end{bmatrix} y \\ u = \begin{bmatrix} -32 & -8 \end{bmatrix} \hat{x} \end{cases}$$

acting as input

acting as output

The T.F. from output u to input y is $-G_{ec}(s) = -K(sI - A + BK + GC)^{-1}G$

$$\therefore G_{ec}(s) = \frac{1440s + 3200}{s^2 + 28s + 292}$$

The transfer function of the controller-estimator is

$$G_{ec}(s) = K(sI - A + BK + GC)^{-1}G$$

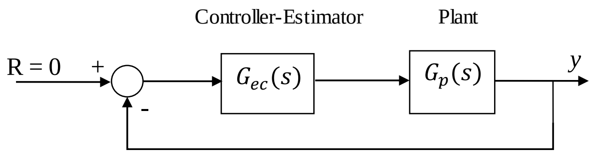


Figure 14-5. Block diagram for example 4.

The characteristic polynomial of the closed-loop system is

$$1+G_{ec}(s)G_p(s)=0$$

Example 5: $\dot{x}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x+\begin{bmatrix}0\\ 1\end{bmatrix}u, \quad y=\begin{bmatrix}1 & 0\end{bmatrix}x$

The controller-estimator is given by $\begin{cases}\dot{\hat{x}}=\begin{bmatrix}-20 & 1\\ -132 & -8\end{bmatrix}\hat{x}+\begin{bmatrix}20\\ 100\end{bmatrix}y\\ u=\begin{bmatrix}-32 & -8\end{bmatrix}\hat{x}\end{cases}$

Signal flow graph of the controller-estimator

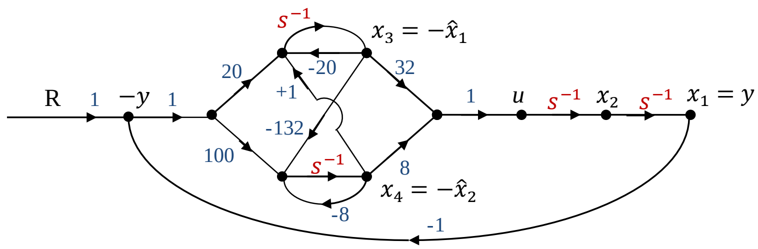


Figure 14-6. Signal flow graph.

Defining $x_3=-\hat{x}_1, x_4=-\hat{x}_2$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 32 & 8\\ -20 & 0 & -20 & 1\\ -100 & 0 & -132 & -8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 20\\ 100\end{bmatrix}r,$$

$$y=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}^T$$

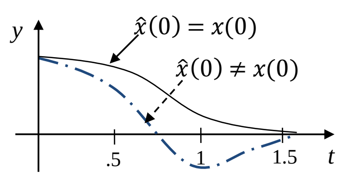


Figure 14-7. Estimator response.

14.2.3 Closed-Loop System Characteristics

The characteristic polynomial with full state feedback is $\alpha_c(s) = \det(sI - A + BK) = 0$ and the characteristic polynomial of a state estimator is $\alpha_e(s) = \det(sI - A + GC) = 0$

We now derive the characteristic polynomial of the closed-loop system.

x, \hat{x}, e

First consider $e = x - \hat{x}$, the plant equations are $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ $\quad with\ u = -K\hat{x}$

Hence, $\dot{x} = Ax - BK\hat{x} = Ax - BK(x - e) \Longrightarrow \dot{x} = (A - BK)x + BKe$

The error dynamic equation is $\dot{e} = (A - GC)e$, therefore the closed-loop system is

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - GC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

The characteristic polynomial of the closed-loop system is

$$det \begin{bmatrix} sI - A + BK & -BK \\ 0 & sI - A + GC \end{bmatrix} = 0$$

$$\therefore \det(sI - A + BK) \cdot \det(sI - A + GC) = 0$$

The **2*n*** roots of the closed-loop characteristic polynomial are then the ***n*** roots of the pole-placement design ***plus*** the ***n*** roots of the estimator. This is fortunate, since otherwise the roots from the pole-placement would have been shifted by the addition of the estimator.

This is called the **separation principle**.

Example 6: From the previous examples we have

$$\alpha_c(s)\alpha_e(s) = (s^2 + 8s + 32)(s^2 + 20s + 100)$$
$$= s^4 + 28s^3 + 292s^2 + 1440s + 3200$$

We have $1 + G_{ec}(s)G_p(s) = 1 + \frac{1440s+3200}{s^2+28s+292} \cdot \frac{1}{s^2} = 0$

$$\Longrightarrow s^4 + 28s^3 + 292s^2 + 1440s + 3200 = 0$$

or equivalently from $\det\left(sI - A_f\right) = det \begin{bmatrix} s & -1 & 0 & 0 \\ 0 & s & -32 & -8 \\ 20 & 0 & s + 20 & -1 \\ 100 & 0 & 132 & s + 8 \end{bmatrix} = 0$

The above characteristic polynomial is obtained.

14.3 Reduced-Order Estimators

Estimators developed so far are called full-order estimators since all states of the plant are estimated. However, usually from output measurement some states are directly available. Hence it is not logical to estimate states that are directly measured.

Assume without loss of generality that $y = x_1 = Cx = [1 \quad 0 \quad \cdots \quad 0]x$

Partition x into $x = \begin{bmatrix} x_1 \\ x_e \end{bmatrix}$, where x_e are the states to be estimated.

Partition also $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} a_{11} & A_{1e} \\ A_{e1} & A_{ee} \end{bmatrix} \begin{bmatrix} x_1 \\ x_e \end{bmatrix} + \begin{bmatrix} b_1 \\ B_e \end{bmatrix} u$

The equation of the states to be estimated $\dot{x}_e = A_{e1}x_1 + A_{ee}x_e + B_e u$

And the equation of the state that is measured $\dot{x}_1 = a_{11}x_1 + A_{1e}x_e + b_1 u$ where x_e is unknown and to be estimated and x_1 and u are known.

To derive the equation of the estimator observer that for the full state estimation

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}, \quad u, y \text{ are known} ; \ x \text{ is unknown}$$

For the reduced order observer we have

$$\begin{cases} \underbrace{\dot{x}_e}_{unknown} = A_{ee}x_e + (A_{e1}x_1 + B_e u) \\ (\underbrace{\dot{x}_1}_{known} - a_{11}\underbrace{x_1}_{known} - b_1 \underbrace{u}_{known}) = A_{1e}\underbrace{x_{1e}}_{unknown} \end{cases}$$

$$\text{Comparing we have} \begin{cases} x(t) \leftarrow x_e(t) \quad \text{replace } x \text{ with } x_e \\ A \leftarrow A_{ee} \\ Bu \leftarrow A_{e1}x + B_e u \\ y \leftarrow \dot{x}_1 - a_{11}x_1 - b_1 u \\ C \leftarrow A_{1e} \end{cases}$$

Making these substitutions for the full-order observer equations yield

$$[\dot{\hat{x}} = (A - GC)\hat{x} + Bu + Gy]$$

$$\dot{\hat{x}}_e = (A_{ee} - G_e A_{1e})\hat{x}_e + A_{e1}y + B_e u + G_e(\dot{y} - a_{11}y - b_1 u)$$

Thus the error dynamic is $\dot{e} = \dot{x}_e - \dot{\hat{x}}_e \Rightarrow \dot{e} = (A_{ee} - G_e A_{1e})e$

Thus, the characteristic polynomial of the reduced-order estimator is

$$\alpha_e(s) = \det(sI - A_{ee} + G_e A_{1e}) = 0$$

We then choose G_e to satisfy this equation where α_e is chosen to give the estimator desired dynamics. This can be achieved ***if and only if*** (A_{ee}, A_{ie}) is observable.

Ackermann's formula yields

$$G_e = \alpha_e(A_{ee}) \begin{bmatrix} A_{1e} \\ A_{1e}A_{ee} \\ \vdots \\ A_{1e}A_{ee}^{n-2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The above estimator requires measurement of $\dot{y}$. To relax this requirement, define a new variable $\hat{x}_{e1} = \hat{x}_e - G_e y \quad or \quad \hat{x}_e = \hat{x}_{e1} + G_e y$

Substituting in observer dynamic yields

$$\dot{\hat{x}}_{e1} + G_e \dot{y} = (A_{ee} - G_e A_{1e})(\hat{x}_{e1} + G_e y) + A_{e1} y + B_e u + G_e(\dot{y} - a_{11} y - b_1 u)$$

$$\therefore \dot{\hat{x}}_{e1} = (A_{ee} - G_e A_{1e})\hat{x}_{e1} + (A_{e1} - G_e a_{11} + A_{ee} G_e - G_e A_{1e} G_e) y + (B_e - G_e b_1) u$$

The control u is

$$u = -K_1 y - K_e \hat{x}_e = -K_1 y - K_e(\hat{x}_{e1} + G_e y) \qquad \text{K\_e=[K\_2 ...K\_n]}$$

Where $K = [K_1 \quad K_{2e}]$, K_1 *is scalar and* K_e *is* $1 \times (n-1)$ *vector*.

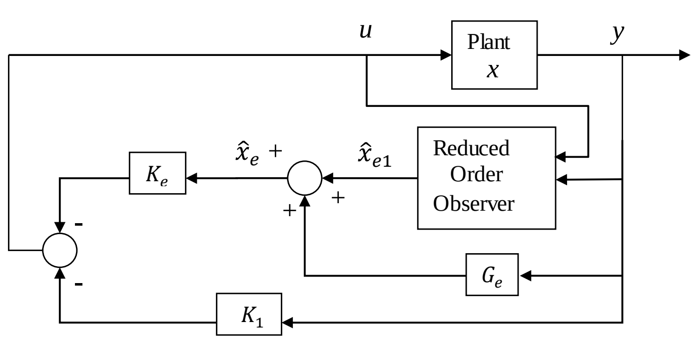


Figure 14-8. Block diagram of the combined plant and reduced order observer.

s^2+2\zeta \omega_n s+ \omega_n^{2}=0

Example 7: $\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = [1 \quad 0]x \end{cases}$, the controller designed earlier yielded

e^{-\zeta \omega_n t}

$$\alpha_c(s) = s^2 + 8s + 32 \; with \; \tau = 0.25 \; sec, \; \xi = .707$$

As for the estimator of reduced order (1st order), let $\tau = 0.1$ sec $and \; \xi = 1 \Longrightarrow$

$\alpha_e(s) = s + 10 \quad \therefore a_{11} = 0, \; A_{1e} = 1, \; b_1 = 0, \; A_{e1} = 0, \; A_{ee} = 0, \; B_e = 1$

From Ackermann's formula:

$$\alpha_e(A_{ee}) = 0 + 10 = 10 \quad \& \quad G_e = \alpha_e(A_{ee})[A_{1e}]^{-1}[1] = 10$$

$$\dot{\hat{x}}_{e1} = (A_{ee} - G_e A_{1e})\hat{x}_{e1} + (A_{e1} - G_e a_{11} + A_{ee} G_e - G_e A_{1e} G_e)y + (B_e - G_e b_1)u$$

$$\therefore \; \dot{\hat{x}}_{e1} = -10\hat{x}_{e1} - 100y + u$$

The estimated value of x_2 is then $\hat{x}_e = \hat{x}_{e1} + G_e y \Longrightarrow \hat{x}_e = \hat{x}_{e1} + 10y$

Example 8: If we combine plant equations with the estimator and the control we have

$$u = -K[y \quad \hat{x}_e]^T = [-32 \quad -8][y \quad \hat{x}_e]^T = -32y - 8\hat{x}_e$$

The signal flow graph becomes

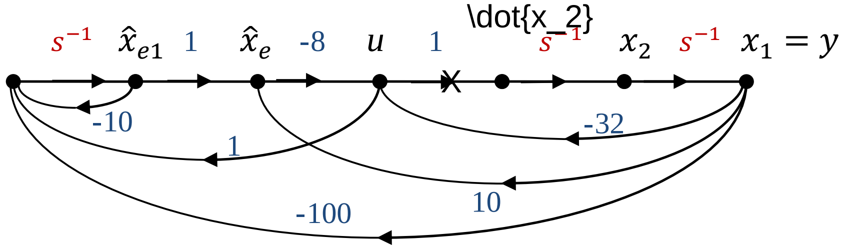


Figure 14-9. System signal flow graph.

Mason's Gain Formula

The characteristic polynomial is

$$s^3 + 18s^2 + 112s + 320 = \underbrace{(s+10)}_{\alpha_e(s)} \underbrace{(s^2 + 8s + 32)}_{\alpha_c(s)} = 0$$

Now opening the graph from u we calculate

$$\frac{U(s)}{Y(s)} = -G_{ec}(s) = \frac{-32(1 + 10s^{-1}) - 80(1 + 10s^{-1}) + 800s^{-1}}{1 + 10s^{-1} + 8s^{-1}}$$

$$= \frac{-(112s + 320)}{s + 18}$$

Phase Lead -> High Pass Filter

s+1 Phase Lag Low Pass

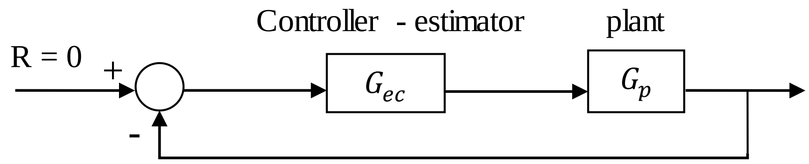


Figure 14-10. Closed-loop combined plant and controller estimator.

$$1 + G_{ec}(s)G_p(s) = 1 + \frac{112s + 320}{s + 18} \cdot \frac{1}{s^2} = 0$$

$$\therefore s^3 + 18s^2 + 112s + 320 = 0 \quad \surd$$

14.4 Reduced-Order State Estimator: General Case

Consider $\begin{cases} \dot{x} = Ax + Bu, & x \in R^n, \; u \in R^p \\ y = Cx & , y \in R^q \end{cases}$ $\qquad A \to n \times n; \;\; B \to n \times p; \;\; C \to q \times n$

Assumption: C has full rank, $Rank\ C = q$

Define: $P = \begin{bmatrix} C \\ R \end{bmatrix}$, where $R \to (n - q) \times n$ arbitrary as long as P is nonsingular.

Define $Q = P^{-1} = [Q_1 \quad Q_2]$ $where\ Q_1 \to n \times q$ & $Q_2 \to n \times (n - q)$

$$I_n = PQ = \begin{bmatrix} C \\ R \end{bmatrix} [Q_1 \quad Q_2] = \begin{bmatrix} CQ_1 & CQ_2 \\ RQ_1 & RQ_2 \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & I_{n-q} \end{bmatrix}$$

Define the equivalence transformation $\bar{x} = Px$

$$\begin{cases} \dot{\bar{x}} = PAP^{-1}\bar{x} + PBu \\ y = CP^{-1}\bar{x} = CQ\bar{x} = [I_q \quad 0]\bar{x} \end{cases}$$

Partition above system as

$$\begin{cases} \begin{bmatrix} \dot{\bar{x}}_{1(q\times 1)} \\ \dot{\bar{x}}_{2[(n-q)\times 1]} \end{bmatrix} = \begin{bmatrix} \bar{A}_{11(q\times q)} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22[(n-q)\times(n-q)]} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u \\ y = [I_q \quad 0]\bar{x} = \bar{x}_1 \end{cases}$$

Therefore, only the last (n - q) elements of $\bar{x}$ need to be estimated.

We need only an (n - q)-dimensional state estimator.

Using $\bar{x}_1 = y \;\therefore \begin{cases} \dot{\bar{x}}_1 = \dot{y} = \bar{A}_{11}y + \bar{A}_{12}\bar{x}_2 + \bar{B}_1 u \\ \dot{\bar{x}}_2 = \bar{A}_{22}\bar{x}_2 + \bar{A}_{21}y + \bar{B}_2 u \end{cases}$

Define $\bar{u} = \bar{A}_{21}y + \bar{B}_2 u$ & $w = \dot{y} - \bar{A}_{11}y - \bar{B}_1 u$

$$\therefore \dot{\bar{x}}_2 = \bar{A}_{22}\bar{x}_2 + \bar{u}\,, \qquad w = \bar{A}_{12}\bar{x}_2$$

Lemma:

The pair (A, C) or equivalently$(\bar{A},\ \bar{C})$ is observable, ***iff.*** the pair $(\bar{A}_{22}, \bar{A}_{12})$ is observable.

Therefore, ∃ a (n - q)-dimensional estimator of $\bar{x}_2$ in the form

$$\dot{\hat{\bar{x}}}_2 = (\bar{A}_{22} - \bar{G}\cdot\bar{A}_{12})\hat{\bar{x}}_2 + \bar{G}(\dot{y} - \bar{A}_{11}y - \bar{B}_1 u) + (\bar{A}_{21}y + \bar{B}_2 u)$$

such that the eigenvalues of $(\bar{A}_{22} - \bar{G}\cdot\bar{A}_{12})$ can be arbitrarily assigned by proper choice of $\bar{G}$.

To eliminate $\dot{y}$ define $z = \hat{\bar{x}}_2 - \bar{G}y$

$$\dot{z} = (\bar{A}_{22} - \bar{G}\cdot\bar{A}_{12})z + [(\bar{A}_{22} - \bar{G}\cdot\bar{A}_{12})\bar{G} + (\bar{A}_{21} - \bar{G}\cdot\bar{A}_{11})]y + (\bar{B}_2 - \bar{G}\cdot\bar{B}_1)u$$

Define $e = \bar{x}_2 - \hat{\bar{x}}_2 = \bar{x}_2 - (z + \bar{G}y) \Longrightarrow \dot{e} = (\bar{A}_{22} - \bar{G}A_{12})e \Longrightarrow e(t) \longrightarrow 0 \; as \;\; t \longrightarrow \infty$ by proper choice of $\bar{G}$.

Now $\hat{\bar{x}} = \begin{bmatrix} \hat{\bar{x}}_1 \\ \hat{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} y \\ \bar{G}y + z \end{bmatrix}$, since

$$\bar{x} = Px \Rightarrow x = P^{-1}\bar{x} = Q\bar{x} \quad \therefore \hat{x} = Q\hat{\bar{x}} = [Q_1 \quad Q_2]\begin{bmatrix} y \\ \tilde{\bar{G}}y + z \end{bmatrix} = [Q_1 \quad Q_2]\begin{bmatrix} I_q & 0 \\ \bar{G} & I_{n-q} \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix}$$

which is an estimate of the original vector x.

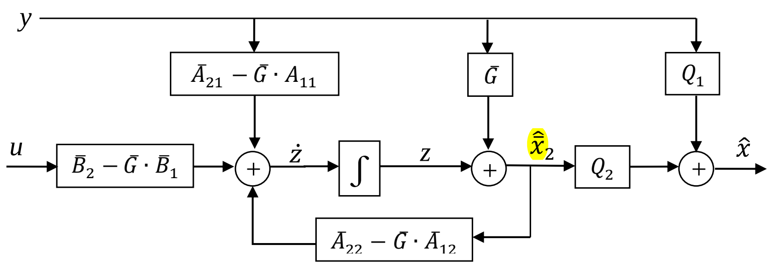


Figure 14-11. Block diagram for system with reduced order estimator.

15 Lecture 15

Objectives

1) ***Tracking Problem***
2) ***Closed-Loop Pole-Zero Assignment***
3) ***General Case Using Phase Variable Canonical Form***

15.1 Tracking Problem

Regulation/Stabilization Problem vs. Tracking Problem

$$\begin{cases} \dot{x} = Ax + Bu \\ u = -Kx + K_r r \\ y = x_1 \end{cases}$$

r= reference signal (time-varying)

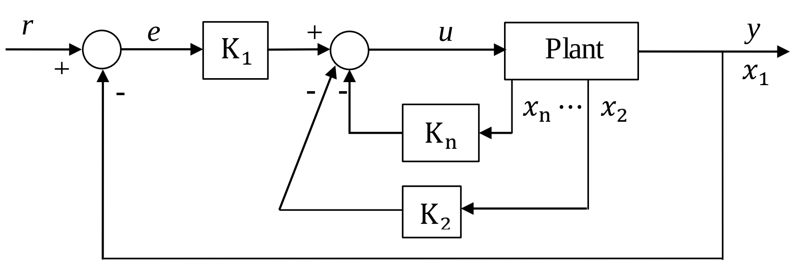


Figure 15-1. State feedback diagram.

It is assumed that K is designed to meet a desired closed-loop characteristics. Problem is to design K_r to satisfy the tracking requirement.

$$u = -K_1 x_1 - K_2 x_2 - \cdots - K_n x_n + K_r r$$

$u = K_1 \underbrace{(r - x_1)}_{e} - K_2 x_2 - \cdots - K_n x_n$, where a choice for K_r is to defined as $K_r = K_1$.

Example 1: System $\begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$ $with\ \ K = \begin{bmatrix} 32 & 8 \end{bmatrix}$ from lecture 14

Example 2. (Hyperlink)

$$\therefore u = 32(r - x_1) - 8x_2 = 32(r - y) - 8x_2$$

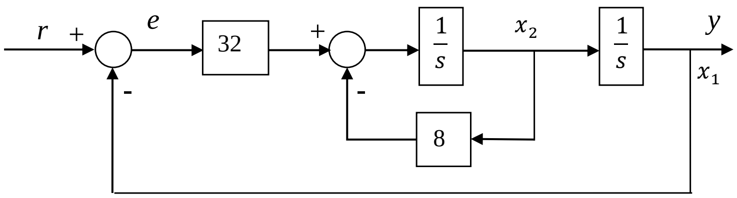


Figure 15-2. State diagram for example 1.

For the general case, if $y \neq x_1$, instead $y = c_1x_1 + c_2x_2 + \cdots + c_nx_n$,

then we express $u = -Kx + K_r r = K_a(r - y) - K_b x$

since we want the system to be driven by the difference between the input

and output.

$$\therefore u = K_a r - K_a y - K_b x = K_a r - K_a Cx - K_b x \Longrightarrow$$

$$u = K_a r - (K_a C + K_b)x$$

Comparing with $u = -Kx + K_r r$ $\therefore K_r = K_a$ & $K_a C + K_b = K$

$$\begin{cases} K_a C_1 + K_{1b} = K_1 \\ \vdots \\ K_a C_n + K_{nb} = K_n \end{cases}$$

$c_i's$ & $K_i's$ *are known and* $K_b's$ & K_a *are unknown*

n *equations in* $(n + 1)$ *unknowns*

Hence one gain has to be determined from other design criteria. (steady state requirements)

15.2 Closed-Loop Pole-Zero Assignment

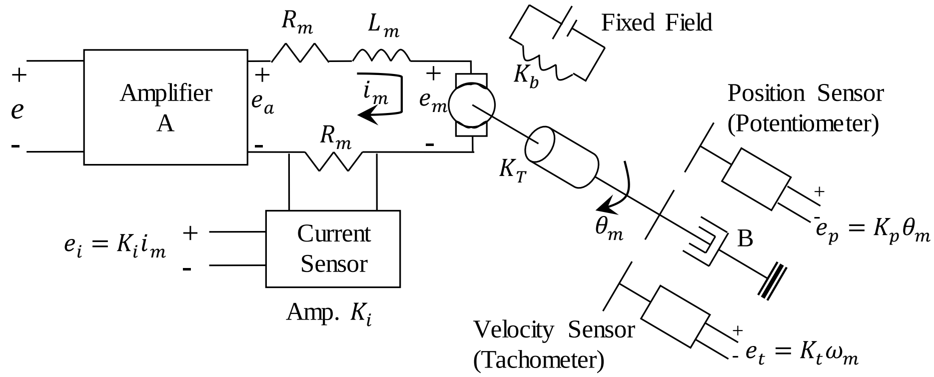


Figure 15-3. ADC motor system schematic.

$$
\begin{cases}
e_a = Ae; A > 0 \\
e_a - e_m = (R_m + L_m D)i_m \\
e_m = K_b\omega_m \\
T = K_T i_m = JD\omega_m + B\omega_m \\
\omega_m = D\theta_m \\
D = \frac{d}{dt}
\end{cases}
$$

Figure 15-4. System close-loop block diagram.

R-Kx

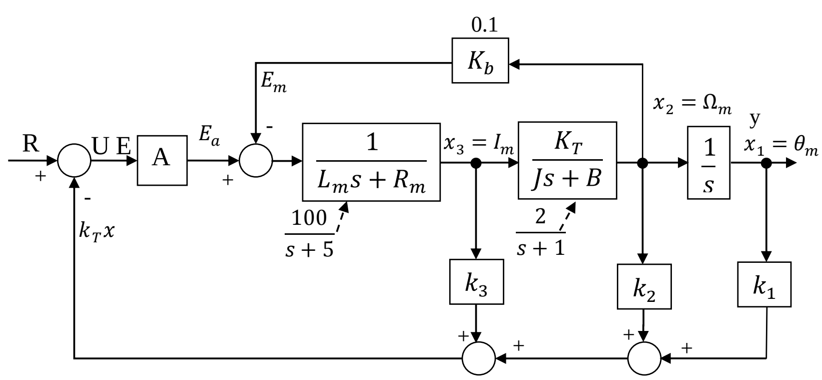


Figure 15-5. System block diagram.

G(s) is open loop plant TF

H_eq is the feedback block TF

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H_{eq}(s)}, \quad G(s) = \frac{200A}{s(s^2 + 6s + 5)}$$

$$\frac{Y(s)}{R(s)} = \frac{200A}{s^3 + (6 + 100k_3A)s^2 + (5 + 200k_2A + 100k_3A)s + 200k_1A}$$

Mason's Gain Formula

$$\therefore H_{eq}(s) = \frac{k_3 s^2 + (2k_2 + k_3)s + 2k_1}{2}$$ All zero Filter -> PD^2

H_{eq} must be designed such that $e_{ss}|_{step\ input} = 0$ and other design specifications are satisfied.

Observations:

1) To have zero steady state error due to $R(s) = \frac{R_0}{s}$

   $\therefore y_{ss} = \lim_{s \to 0} sy(s) = \frac{R_0}{k_1} = R_0 \Longrightarrow k_1 = 1$.

2) Poles of GH_{eq} are poles of G.

3) Zeros of GH_{eq} are zeros of H_{eq}, $GH_{eq} = \frac{100A[k_3 s^2 + (2k_2 + k_1)s + 2k_1]}{s(s^2 + 6s + 25)}$

4) Use of state feedback produces additional zeros in loop transfer function without adding poles! Additional zeros move the root locus to the left $\Longrightarrow$ system is more stable and time response is improved.

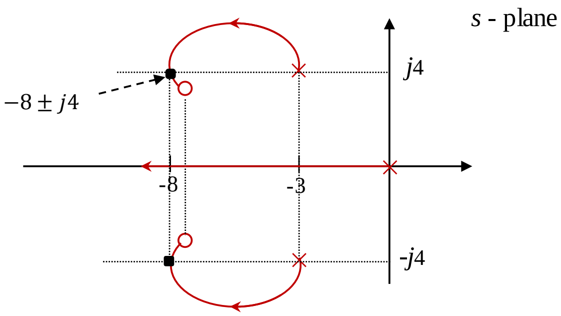


Figure 15-6. Root Locus.

Let the desired closed-loop poles be located at $-a \pm jb, -c$

$$\therefore (s + a \pm jb)(s + c) = s^3 + (6 + 100k_3 A)s^2 + (25 + 200k_2 A + 100k_3 A)s + 200A$$

Therefore, solve for $k_2, k_3\ and\ A$.

15.3 General Case Using Controllable Canonical Form

Let $G(s)=\dfrac{Y(s)}{U(s)}=\dfrac{K(s^m+c_{m-1}s^{m-1}+\cdots+c_1s+c_0)}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$, m < n

$$\therefore \begin{cases} \dot{x}=\begin{bmatrix} 0 & \vdots & 1 & 0 & \cdots & 0 \\ 0 & \vdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & 0 & 0 & \cdots & 1 \\ \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ -a_0 & \vdots & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x+\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \cdots \\ 1 \end{bmatrix} u \\ y=\begin{bmatrix} c_0 & c_1 & \cdots & c_m & \vdots & 0 & \cdots & 0 \end{bmatrix} x=Cx \end{cases}$$

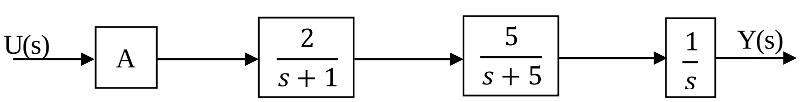
$$\therefore H_{eq}(s)=\frac{k^T X(s)}{Y(s)}=\frac{k^T X(s)}{CX(s)}=\frac{k_1X_1(s)+\cdots+k_nX_n(s)}{c_0X_1(s)+\cdots+X_{m+1}(s)}$$

Since $X_j(s)=s^{j-1}X_1(s)=s^{j-1}Y(s)\Rightarrow H_{eq}(s)=\dfrac{k_ns^{n-1}+k_{n-1}s^{n-2}+\cdots+k_2s+k_1}{s^n+c_{m-1}s^{m-1}+\cdots+c_1s+c_0}$

$$\therefore GH_{eq}(s)=\frac{K\left(k_ns^{n-1}+k_{n-1}s^{n-2}+\cdots+k_2s+k_1\right)}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=\frac{k_nK\left(s^{n-1}+\alpha_{n-2}s^{n-2}+\cdots+\alpha_1s+\alpha_0\right)}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$$

Now $\dfrac{Y(s)}{R(s)}=\dfrac{K(s^m+c_{m-1}s^{m-1}+\cdots+c_0)}{s^n+(a_{n-1}+Kk_n)s^{n-1}+\cdots+(a_0+Kk_1)}$

Example 2: Given system as below,



design a state feedback such that e_{ss} due to a unit step is zero and

%OS = 4.3% and $t_s=5.65s$ $(\xi\omega_n=.708)$

$$\frac{Y(s)}{R(s)}=\frac{10A}{s^3+(6+2Ak_3)s^2+[5+(k_2+k_3)\cdot 10A]s+10Ak_1}$$

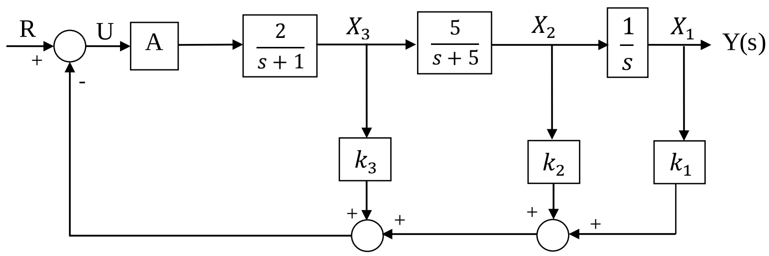


Figure 15-7. State diagram for example 1.

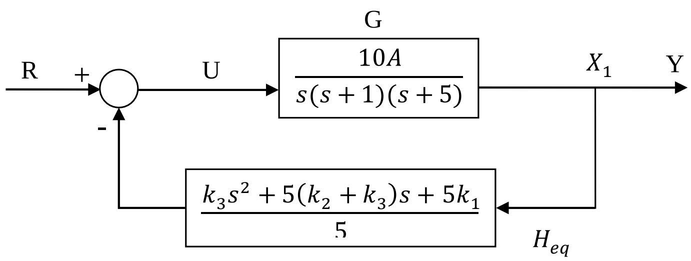


Figure 15-8. Block diagram of the closed-loop system for example 1.

To get $e_{ss}\big|_{unit\ step} = 0,\ with\ R = R_0 u_s(t) \Rightarrow k_1 = 1$

Given $\begin{cases} \%OS = 4.3 \Rightarrow s_{1,2} = -0.708 \pm j0.7064 \\ T_s = 5.65,\ \ \xi\omega_n = 0.708 \end{cases}$

Now we place the 3rd pole arbitrarily at $s_3 = -100$

Therefore, the desired closed–loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{10A}{(s+100)(s+0.708 \pm j0.7064)} = \frac{10A}{s^3 + 101.4s^2 + 142.6s + 100}$$

$$\begin{cases} 6 + 2Ak_3 = 101.4 \\ 5 + (k_2 + k_3)10A = 142.6 \\ 10A = 100 \end{cases} \Rightarrow \begin{cases} A = 10 \\ k_1 = 1 \\ k_2 = -3.393 \\ k_3 = 4.77 \end{cases}$$

$$\therefore G_{eq}(s) = \frac{100}{s(s^2 + 101.4s + 142.7)} \qquad (type\,1\,systm)$$

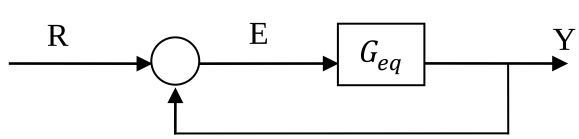


Figure 15-9. Closed-loop diagram 3 for a unity feedback of example 1.

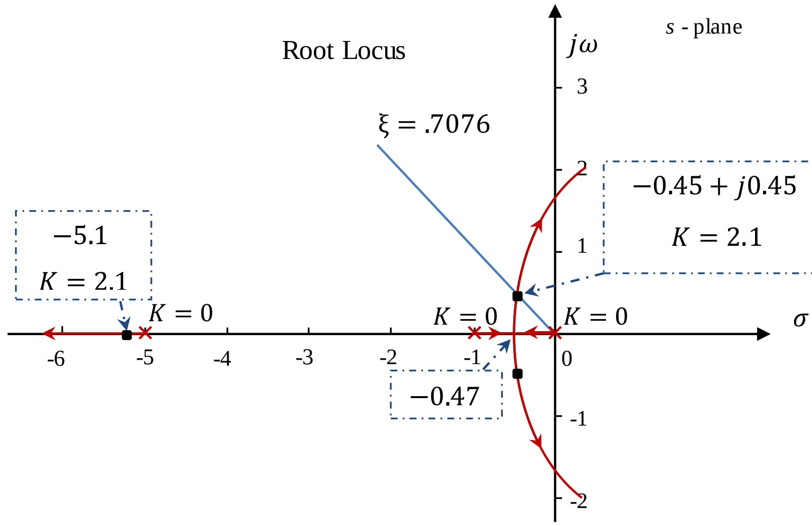


Figure 15-10. Root Locus for the standard feedback control system of example 2.

$$\begin{cases} 1+KGH=0 \\ GH=\dfrac{-1}{K}\Rightarrow |GH|=\dfrac{1}{|K|} \\ \angle GH=2k\pi, k<0 \\ \angle GH=(2k+1)\pi, k>0 \end{cases} \qquad G(s)=\frac{K}{s(s+1)(s+5)}$$

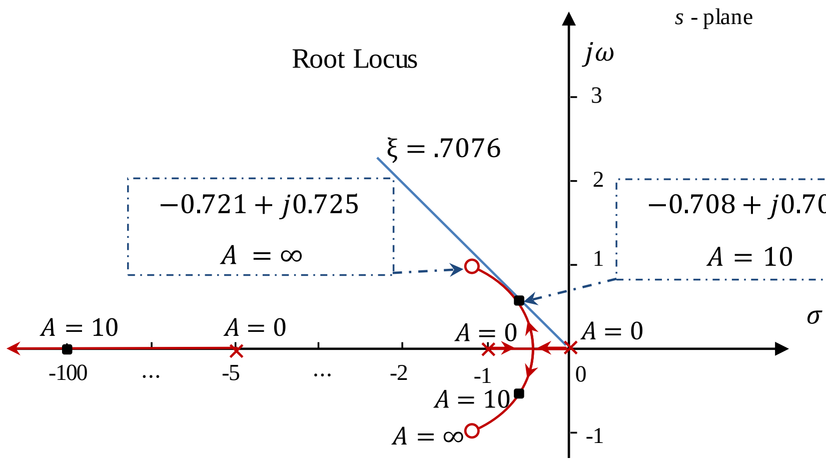


Figure 15-11. Root Locus for the state feedback control system of example 2.

$$GH_{eq}(s)=\frac{9.54A(s+0.721\pm j0.726)}{s(s+1)(s+5)}$$

As a result of vicinity of desired pole locations with the zero of the controller, a system is insensitive to variations in A.

Example 3: Given system as below,

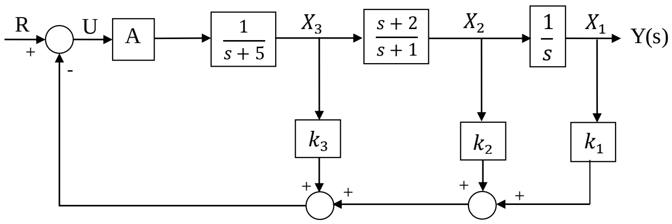


Figure 15-12. System sate diagram for example 3.

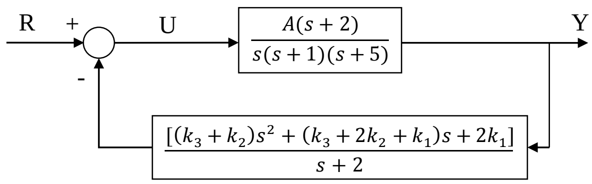


Figure 15-13. Closed-loop block diagram for example 3.

$$\frac{Y(s)}{R(s)} = \frac{A(s+2)}{s^3+[6+(k_3+k_2)A]s^2+[5+(k_3+2k_2+k_1)A]s+2k_1A}$$

A desired closed-loop T.F. is selected as

$$\frac{Y(s)}{R(s)} = \frac{2(s+2)}{(s^2+2s+2)(s+2)} = \frac{2}{s^2+2s+2};\ s_{1,2} = -1 \pm j, \omega_{n=}\sqrt{2}, \xi = 0.707$$

$$= \frac{2(s+2)}{s^3+4s^2+6s+4}$$

$$\therefore k_1 = 1, k_2 = 0.5, k_3 = -1.5$$

No non-dominant pole.

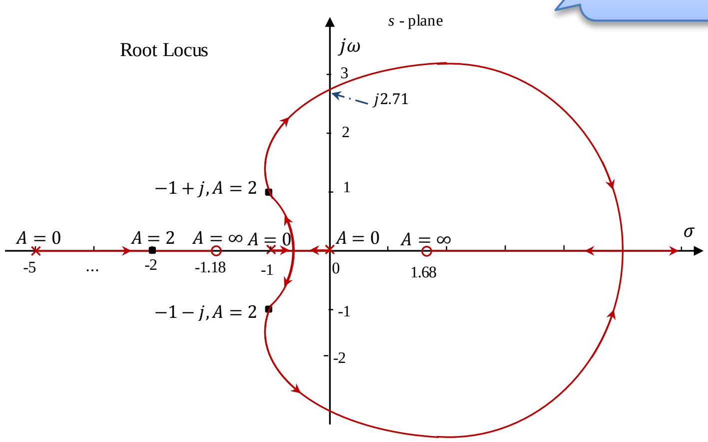


Figure 15-14. Root Locus for example 3.

$$GH_{eq}(s)=\frac{A(s^2-0.5s-2)}{s(s+1)(s+5)}$$

Unfortunately the system is sensitive to variations in A. In order to achieve this we must have one non-dominant pole. Let the desired closed-loop T.F.

be $\frac{Y(s)}{R(s)}=\frac{2(s+2)}{(s^2+2s+2)(s+100)}, \therefore k_1=50, k_2=0.5, k_3=47.5$

$$\therefore H_{eq}=\frac{48s^2+98.5s+100}{(s+2)},\ \ Hence,GH_{eq}=\frac{48s^2+98.5s+100}{s(s+1)(s+5)}$$

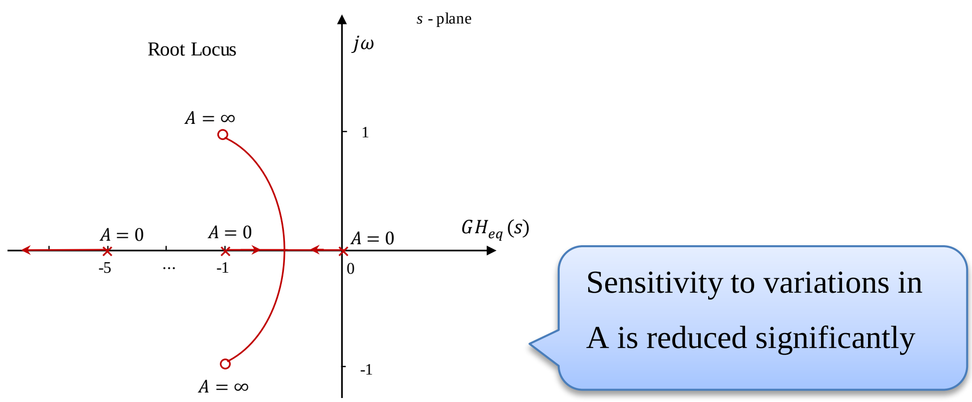


Figure 15-15. Root Locus for example 3.

<u>**Example 4**</u>**:** Alternatively, since in no dominant pole cases there are <u>3</u> dominant poles at $s_{1,2}=-1\pm j$ & $s_3=-2 \Longrightarrow H_{eq}(s)$ must have <u>3</u> zeros to be insensitive to A. However, to have <u>3</u> zeros in H_{eq} requires having <u>4</u> poles in G. Therefore, ***one pole*** is added to G by using a cascade compensator $G_c(s)=\frac{1}{s+a}$. Then $\frac{Y(s)}{R(s)}$ can have one non-dominant pole p and

$$\left.\frac{Y(s)}{R(s)}\right|_{desired}=\frac{A(s+2)}{(s^2+2s+2)(s+2)(s-p)}$$

Selection of A and p are independent since $e_{ss}=0 \therefore A=-2p$

The closer poles and zeros are the less sensitivity of changes in A.

Selecting $A=100 \Longrightarrow p=-50$ is non-dominant.

$$\frac{Y(s)}{R(s)}=\frac{100(s+2)}{s^4+54s^3+206s^2+304s+200}$$

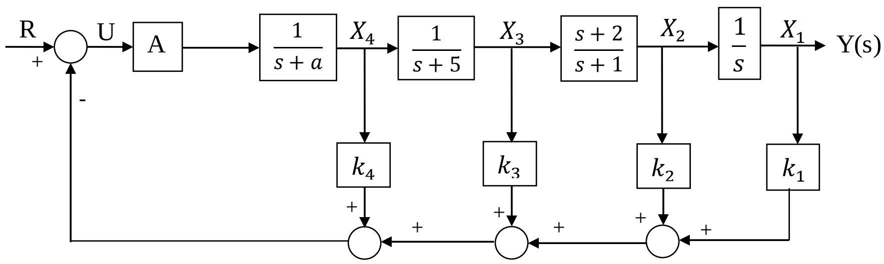


Figure 15-16. System state diagram for example 4.

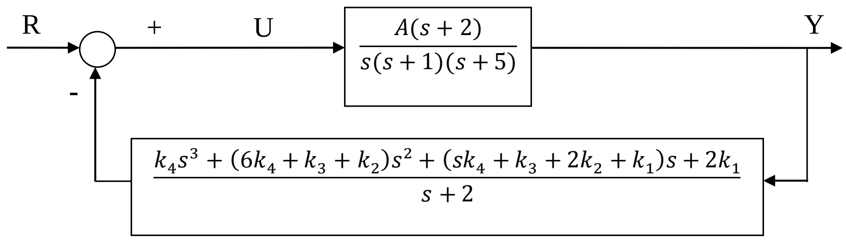


Figure 15-17. Closed-loop block diagram for example 4.

To guarantee a controllable and observable plant $a \neq 2$, let $a = 1$.

Therefore, with $A = 100$, $a = 1$

$$\frac{Y(s)}{R(s)} =$$

$$\frac{100(s+2)}{s^4+(7+100k_4)s^3+[11+100(6k_4+k_3+k_2)]s^2+[5+100(5k_4+k_3+k_2+k_1)]s+200k_1}$$

Due to a unit step reference input, with $k_1 = 1$

$\therefore k_2 = 0.51, k_3 = -1.38, k_4 = 0.47$

$$GH_{eq} = \frac{0.47A(s^3+4.149s^2+6.362s+4.255)}{s^4+7s^3+11s^2+5s}$$

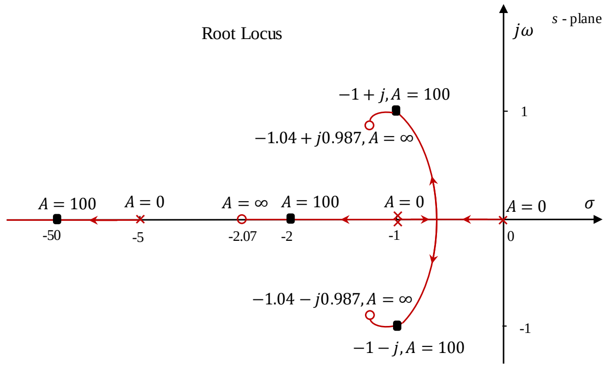


Figure 15-18. Root Locus for example 4.