# Multiple Linear Regression

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car's efficency:

$y$ = mpg City : response

Goal:

1° Important factors rel. mpg

2° New car designed.

3° Asian cars more efficient?? mpg benchmark?

Data: $y$ : mpg

Wt, air dynamic (shell), length, width, engine type: cylinders: V4, V6, V8, drive; trans, size of wheels, HP

# Case Study: Fuel Efficiency in Automobiles

Goal of the study: How to build a fuel efficient car?

- Effects of features on a car
- Given a set of features, we'd like to estimate the mean fuel efficiency as well as the efficiency of one car
- Are Asian cars more efficient than cars built in other regions?

Let us answer these questions with Multiple Regression

# Data

Dataset: `car_04_regular.csv` , $n = 226$ cars

| Feature | Description | Categorical vars |
|---|---|---|
| Continent | Continent the Car Company is From | : As, Am, E |
| Horsepower | Horsepower of the Vehicle | cont |
| Weight | Weight of the vehicle (thousand lb) | con |
| Length | Length of the Vehicle (inches) | con |
| Width | Width of the Vehicle (inches) | con |
| Seating | Number of Seats in the Vehicle | con / categorical |
| Cylinders | Number of Engine Cylinders | con / categorical |
| Displacement | Volume displaced by the cylinders, defined as $\pi/4 \times bore^2 \times stroke \times$ Number of Cylinders | con |
| Transmission | Type of Transmission (manual, automatic, continuous) | Categori |

# Goal of Study: Rephrased

1. Fuel efficiency is measured by Mileage per Gallon, $Y = \texttt{MPG\_City}$
2. Predictors: Effects of each feature on $Y$
3. Estimate - the mean `MPG_City` for all such cars specified below and - predict $Y$ for the particular car described below
4. Are cars built by Asian more efficient?
5. Investigate the `MPG_City` for this newly designed American car

- mean$(y \mid x_1 \ldots x_q)$
mpg

- $\hat{y}$
new design

| Feature | Value |
|---|---|
| Continent | America |
| Horsepower | 225 |
| Weight | 4 |
| Length | 180 |
| Width | 80 |
| Seating | 5 |
| Cylinders | 4 |
| Displacement | 3.5 |
| Transmission | automatic |

# A Quick Glimpse at the data

```
data1 <- read.csv("car_04_regular.csv", header=TRUE)

names(data1)
```

```
##  [1] "Make.Model"   "Continent"    "MPG_City"     "MPG_Hwy"      "Horsepower"
##  [6] "Weight"       "Length"       "Width"        "Seating"      "Cylinders"
## [11] "Displacement" "Make"         "Transmission"
```

```
dim(data1)   # 226 cars and 13 variables
```

```
## [1] 226  13
```

# A Quick Glimpse at the data

```
str(data1)
```

```
## 'data.frame':    226 obs. of  13 variables:
##  $ Make.Model  : chr  "Acura_RL" "Acura_TL" "Acura_TSX" "Acura_RSX" ...
##  $ Continent   : chr  "As" "As" "As" "As" ...
##  $ MPG_City    : int  18 20 23 25 17 17 20 18 17 16 ...
##  $ MPG_Hwy     : int  24 28 32 34 24 23 28 25 24 22 ...
##  $ Horsepower  : int  225 270 200 160 252 265 170 220 330 250 ...
##  $ Weight      : num  3.9 3.58 3.32 2.77 3.2 ...
##  $ Length      : num  197 189 183 172 174 ...
##  $ Width       : num  71.6 72.2 69.4 67.9 71.3 77 76.3 76.1 74.6 76.1 ...
##  $ Seating     : int  5 5 5 4 2 7 5 5 5 5 ...
##  $ Cylinders   : int  6 6 4 4 6 6 4 6 6 6 ...
##  $ Displacement: num  3.5 3.2 2.4 2 3 3.5 1.8 3 4.2 2.7 ...
##  $ Make        : chr  "Acura" "Acura" "Acura" "Acura" ...
##  $ Transmission: chr  "automatic" "automatic" "automatic" "automatic" ...
```

# A Quick Glimpse at the data

```r
head(data1)
```

```
##    Make.Model Continent MPG_City MPG_Hwy Horsepower Weight Length Width Seating
## 1    Acura_RL        As       18      24        225   3.90    197  71.6       5
## 2    Acura_TL        As       20      28        270   3.58    189  72.2       5
## 3   Acura_TSX        As       23      32        200   3.32    183  69.4       5
## 4   Acura_RSX        As       25      34        160   2.77    172  67.9       4
## 5   Acura_NSX        As       17      24        252   3.20    174  71.3       2
## 6   Acura_MDX        As       17      23        265   4.45    189  77.0       7
##    Cylinders Displacement  Make Transmission
## 1          6          3.5 Acura    automatic
## 2          6          3.2 Acura    automatic
## 3          4          2.4 Acura    automatic
## 4          4          2.0 Acura    automatic
## 5          6          3.0 Acura    automatic
## 6          6          3.5 Acura    automatic
```

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# Introduction to Multiple regression

Guiding Question: How does `Length` affect `MPG_City`?

- It depends on how we model the response. We will investigate **three models** with `Length`.
- For the ease of presentation, we define some predictors below that we will use in subsequent models:

$$x_1 = Length, \quad x_2 = Horsepower, \quad x_3 = Width, \quad x_4 = Seating$$

$$, \quad x_5 = Cylinders, \quad x_6 = Displacement$$

# Model Specification

**M1. Our first model will only contain one predictor, Length:**

$$y_i = \beta_0 + \beta_1 \cdot x_{i1} + \epsilon$$

**Interpretation of** $\beta_1$ is that in general, the mean $y$ will change by $\beta_1$ if a car is 1" longer. So we can't really peel off the effect of the `Length` over $y$.

Additive model: $\beta_1$?

# Model Specification

**M2. Next, we add the predictor `Horsepower` to our model**

$$y_i = \beta_0 + \beta_1 \cdot x_{i1} + \beta_2 \cdot x_{i2} + \epsilon$$

**Interpretation of** $\beta_1$ is that in general, the mean $y$ will change by $\beta_1$ if a car is 1" longer and the 'Horse Power's' are the same.

# Model Specification

**M3. Finally, we fit a model with multiple predictors**

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \beta_6 x_{i6} + \epsilon$$

**Interpretation of** $\beta_1$ is that in general, the mean $y$ will change by $\beta_1$ if a car is 1" longer and the rest of the features are the same.

# Model Specification

**Question**: Are all the $\beta_1$s same in the 3 models above?

**No**. The effect of `Length` $\beta_1$ depends on the rest of the features in the model!!!!

Notes $x_1$ = Length, $x_2$ = HP, $x_3$ = width, $x_4$ = Seating, $x_5$ = Cylinders, $x_6$ = Disp.

$y$: mpg   $n = 226$

m1: $y|x_1 = \beta_0 + \beta_1 x_1 + \varepsilon_1 \Rightarrow$ LS: $\underbrace{\hat{y}|x_1 = 45.3 - .14\, x_1}$ // (length)

$\text{mean}(\text{mpg}\,|\,\text{length}) = \underbrace{\beta_0 + \beta_1 \cdot \text{length}}$

lm(mpg ~ length, data)

Q1: $\hat{\beta}_1 = \underline{-.14}$ ??

Q2: $\underline{\text{length} \uparrow 1.}$ $\underline{\text{mpg}}$?

m2: find effect of length over mpg take HP take $\underline{\text{the same}}$

$\boxed{\text{Additive}}$ model

# Notes

$$\text{mpg} = \beta_0 + \beta_1 \overset{\text{length}}{X_1} + \beta_2 \overset{\text{HP}}{X_2}$$

Claim: $\beta_1$ : effect of $X_1$ over mpg cont. HP

$\rightarrow \text{mpg}_{x_1=180,\ x_2=240} = \beta_0 + \beta_1 \times 180 + \beta_2 \times 240$

$\text{mpg}_{x_1=181,\ x_2=240} = \beta_0 + \beta_1 \times 181 + \beta_2 \times 240$

diff $\quad \text{mpg}_{x_1=181,\ x_2=240} - \text{mpg}_{x_1=180,\ x_2=240}$

$= \beta_1$ ✓

$\underline{m2}$ : $\text{mpg} = \beta_0 + \beta_1\,\text{length} + \beta_3\,(\text{HP})^2$ $\quad x_2$

linear model !!

$\beta_1$ : effect of length over mpg controlling HP ?

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$M_2^3$ : $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$ $x_3'$

$+ \beta_3 \underline{x_1 \times x_2}$ $\underline{x_1 \times 2 40}$

Is $\beta_1$ still the effect of length over mpg cont. for $x_2$ ?

(model w/ interaction)

$M_3$ : $\underline{mpg = \beta_0 + \beta_1 x_1 + \cdots + \beta_6 x_6}$

Do you expect $\beta_1$'s in three models be the same??

Q!

# General linear models

m3 :

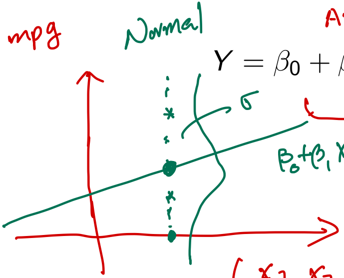
$x_1, x_2 \ldots x_p$

In general, We define a multiple regression as

$Ave(y \mid x_1, x_2 \ldots x_p)$

$$Y = \beta_0 + \underbrace{\beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}}_{\beta_0+\beta_1 x_1+\beta_2 x_2+\cdots+\beta_6 x_6} + \epsilon_i$$

$p = 6$




$(x_1, x_2 \ldots, x_6)$

m2 : $p = 2$

m1 : $p = 1$

m3 :

# General linear models: Assumptions

M3

How to est.

$\beta_0, \beta_1, \dots \beta_6$

$Y$: response; $X_1, X_2, \dots, X_p$: explanatory variables

- Linearity Assumption for this model is $\Rightarrow$ LS: min RSS $\beta_0, \beta_1 \dots \beta_6$

$$\underline{\mathbf{E}(y_i|\tilde{x}_i)} = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}$$

- The homoscedasticity assumption is

$$\mathbf{Var}(y_i|x_{i1}, x_{i2}, \dots, x_{ip}) = \sigma^2$$

- Normality assumption

$$y_i|x_i \overset{iid}{\sim} \mathcal{N}(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip}, \sigma^2)$$

# OLS and its Properties

These $\beta$ parameters are estimated using the same approach as simple regression, specifically by minimizing the sum of squared residuals ($RSS$):

$$\min_{b_0,\, b_1,\, b_2, \dots, b_p} \sum_{i=1}^{n=226} (y_i - \underset{?}{b_0} - \underset{?}{b_1} x_{i1} - \underset{?}{b_2} x_{i2} - \cdots - \overset{p=6}{\underset{?}{b_p}} x_{ip})^2$$

lm(y ~ x1 + .. + xp, data )

# OLS Estimates

- Each $\hat{\beta}_i$ is normal with mean $\beta_i$
- Produce $se(\hat{\beta}_i)$
- $z$ or $t$ interval for $\beta_i$ based on $\hat{\beta}_i$.

# OLS Estimates: Hypothesis Test

To test that

$$\beta_i = 0 \quad \text{vs.} \quad \beta_i = 0$$

which means that given other variables in the model, there is no $x_i$ effect. We carry out a t-test:

$$t\text{-stat} = \frac{\hat{\beta}_i - 0}{\text{se}(\hat{\beta}_i)}$$

The p-value is:

$$\text{p-value} = 2 \times P(\text{T variable} > \text{tstat})$$

.

We reject the null hypothesis at an $\alpha$ level if the p-value is $< \alpha$.

# OLS Prediction

A 95% Confidence interval for the mean given a set of predictors:

$$\hat{y} \pm 2 \times se(\hat{y})$$

**A** 95% **prediction interval for a future** $y$ **given a set of predictors:**

$$\hat{y} \pm 2 \times \hat{\sigma}.$$

# RSS, MSE, RSE

For multiple regression, *RSS* is estimated as:

$$\underline{RSS} = \sum_{i=1}^{n} \hat{\epsilon}_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \cdots + \hat{\beta}_p x_{ip}))^2$$

m3: $x_1, x_2, ..X_6$

$$MSE = \frac{RSS}{\underline{n-p-1}} = \hat{\sigma}^2 \qquad = \frac{RSS}{n-(\underline{\#}\ \beta's)}$$

$$\frac{RSS}{226-7}$$

$$\hat{\sigma} = RSE = \sqrt{MSE}$$

$$\underbrace{\beta_0, \beta_1, \beta_2 \cdots \beta_7}_{7\ \beta's \text{ to be estimated}}$$

# Goodness of Fit: $R^2$

TSS measures the total variance in the response $Y$.

$$TSS = \sum_{i=1}^{n}(y_i - \bar{y})^2$$

How much variability is captured in the linear model using this set of predictors? $R^2$ measures the proportion of variability in $Y$ that can be explained using this set of predictors in the model.

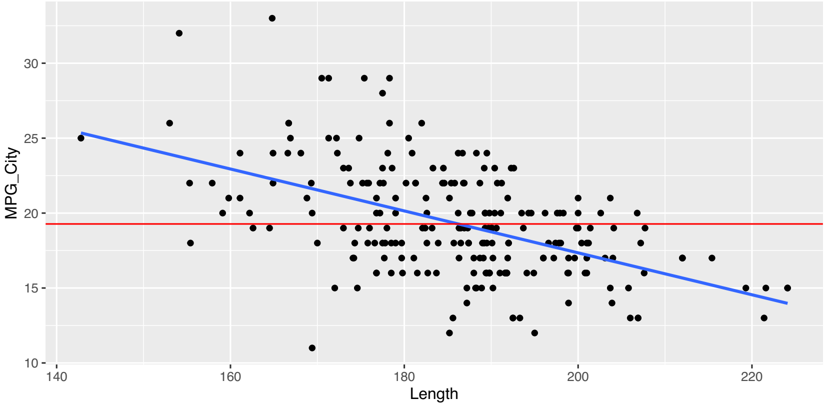
$$R^2 = \frac{TSS - RSS}{TSS}$$

# R function ‘lm()‘

- Linear models are so popular due to their nice interpretations as well as clean solutions
- R-function `lm()` takes a model specification together with other options, outputs all the estimators, summary statistics such as varies sum of squares, standard errors of estimators, testing statistics and p-values.
- The model also outputs the predicted values with margin of errors; confidence intervals and prediction intervals can also be called.

# Model 1: MPG_City ∼ Length

```
fit1 <- lm(MPG_City ~ Length, data = data1)    # model specification response ~ x1,..
ggplot(data1, aes(x = Length , y = MPG_City)) +
  geom_point() +
  geom_smooth(method="lm",formula = 'y~x', se=F) +
geom_hline(aes(yintercept = mean(MPG_City)), color = "red")
```



# Model 1: MPG_City ∼ Length

- We now create a model with `lm()`
- Note from the summary below, the $\hat{\beta}$ for Length is estimated as -0.14.
- We say on average MPG drops .13983 if a car is 1'' longer.

```r
fit1 <- lm(MPG_City ~ Length, data = data1)  # model one
summary(fit1)
```

y x1

m1 :

```
##
## Call:
## lm(formula = MPG_City ~ Length, data = data1)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -10.626  -2.279  -0.151   1.977  10.731
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  45.3138     2.8975   15.64   <2e-16 ***
## Length       -0.1398     0.0155   -9.01   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.18 on 224 degrees of freedom
## Multiple R-squared:  0.266,  Adjusted R-squared:  0.263
## F-statistic: 81.2 on 1 and 224 DF,  p-value: <2e-16
```

Get out :

1. $\hat{\beta}_1 = -.1398$

$\begin{cases} H_0: \beta_1 = 0 \\ H_1: \beta_1 \neq 0 \end{cases} \Rightarrow t = \frac{-.14}{.0155} = -9$

p-value $\Rightarrow$ reject $H_0: \beta_1 = 0$

$\Rightarrow \beta_1 \neq 0$

2: $\hat{\beta}_1 = -.1398$ ?

3. $\hat{y}_{x_1} = 45.3 - .14\,\text{length}$

4. $R_{m_1} = .266$

# Model 2: MPG_City ∼ Length + Horsepower

Recall: m1. $\hat{\beta}_1 = -.14$ /

```
fit2 <- lm(MPG_City ~ Length + Horsepower, data = data1)
summary(fit2)  #sum((fit2$res)^2)
```

```
##
## Call:
## lm(formula = MPG_City ~ Length + Horsepower, data = data1)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -7.152 -1.558  0.154  1.492  8.563
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 38.62587    2.22525   17.36  < 2e-16 ***
## Length      -0.06191    0.01300   -4.76  3.5e-06 ***
## Horsepower  -0.03690    0.00277  -13.31  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.38 on 223 degrees of freedom
## Multiple R-squared:  0.591,  Adjusted R-squared:  0.587
## F-statistic:  161 on 2 and 223 DF,  p-value: <2e-16
```

① $\hat{y}_{x_1,x_2} = 38.6 - .06$ length $- .037$ HP

② a) $H_0: \beta_1 = 0$
$H_1: \beta_1 \neq 0$

p-value $= .00 \Rightarrow$ reject $H_0 \Rightarrow$ length is needed after controlling for HP

similarly

b) : $\begin{cases} H_0: \beta_2 = 0 \\ H_1: \beta_2 \neq 0 \end{cases} \Rightarrow$

reject $H_0$? Yes

Con. HP is useful after controlling for Length.

③ Interpretation of $\hat{\beta}_1 = -.06$ / better est. effect of lengths over mpg

# Model 3: Several continuous variables

④ $R^2_{m_2} = .591$

Notice

$R^2_{m_1} = .26 \geq R^2_{m_2} = .591$

!! $RSS_{m_1} \geq RSS_{m_2}$

Provided

$m_1 \supset m_2$

$x_1$ $\quad$ $x_1, x_2$

⑤ $R^2$ is not a good criterion to choose a model

```
fit3 <- lm(MPG_City ~ Length + Horsepower + Width + Seating +
           Cylinders + Displacement, data = data1)
summary(fit3)
```

```
##
## Call:
## lm(formula = MPG_City ~ Length + Horsepower + Width + Seating +
##     Cylinders + Displacement, data = data1)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -4.877 -1.462  0.073  1.149  8.261
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  45.63372    4.02793   11.33  < 2e-16 ***
## Length        0.04909    0.01730    2.84    0.005 **
## Horsepower   -0.02000    0.00408   -4.90  1.8e-06 ***
## Width        -0.35358    0.06879   -5.14  6.1e-07 ***
## Seating      -0.24135    0.14955   -1.61    0.108
## Cylinders    -0.27169    0.23292   -1.17    0.245
## Displacement -0.93813    0.37166   -2.52    0.012 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.04 on 219 degrees of freedom
## Multiple R-squared:  0.705, Adjusted R-squared:  0.697
## F-statistic: 87.2 on 6 and 219 DF,  p-value: <2e-16
```

m3: ① $\hat{\beta}_1 = .05$ $\quad \begin{cases} H_0: \beta_1 = 0 \\ H_1: \beta_1 \neq 0 \end{cases}$ $\quad p = .005 \Rightarrow$ reject $H_0$

$\Rightarrow \beta_1 \neq 0$, positive.

# Compare 3 Models

② $R^2_{m_3} = .70$

Table 3:

| | Dependent variable: | | |
|---|---|---|---|
| | MPG_City | | |
| | (1) | (2) | (3) |
| | m1 | | |
| Length | $-0.140^{***}$ | $-0.062^{***}$ | $0.049^{***}$ |
| | (0.016) | (0.013) | (0.017) |
| | $\hat{\beta}_1$ m1 | $\hat{\beta}_{1m2}$ | $\hat{\beta}_1$ m3 |
| Horsepower | | $-0.037^{***}$ | $-0.020^{***}$ |
| | | (0.003) | (0.004) |
| Width | | | $-0.354^{***}$ |
| | | | (0.069) |
| Seating | | | $-0.241$ |
| | | | (0.150) |
| Cylinders | | | $-0.272$ |
| | | | (0.233) |
| Displacement | | | $-0.938^{**}$ |
| | | | (0.372) |
| Constant | $45.300^{***}$ | $38.600^{***}$ | $45.600^{***}$ |
| | (2.900) | (2.230) | (4.030) |
| | $\hat{\beta}_0$ m1 | | |
| Observations | 226 | 226 | 226 |
| $R^2$ | 0.266 | 0.591 | 0.705 |
| Residual Std. Error | 3.170 (df = 224) | 2.380 (df = 223) | 2.040 (df = 219) |
| *Note:* | | | $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$ |

$R^2 = 0$

No vars

$y = \beta_0$ $\leq$ 0.266 $\leq$ 0.591 $\leq$ 0.705

# Compare 3 Models

- They are different as expected
- Each one has its own meaning!

Question: what does $\hat{\beta}_1$ mean in 3 models

# Focus on Model 3

Concentrate on m3

```r
fit3 <- lm(MPG_City ~ Length + Horsepower + Width + Seating +
           Cylinders + Displacement, data = data1)
summary(fit3)
```

```
##
## Call:
## lm(formula = MPG_City ~ Length + Horsepower + Width + Seating +
##     Cylinders + Displacement, data = data1)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -4.877 -1.462  0.073  1.149  8.261
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  45.63372    4.02793   11.33  < 2e-16 ***
## Length        0.04909    0.01730    2.84    0.005 **
## Horsepower   -0.02000    0.00408   -4.90  1.8e-06 ***
## Width        -0.35358    0.06879   -5.14  6.1e-07 ***
## Seating      -0.24135    0.14955   -1.61    0.108
## Cylinders    -0.27169    0.23292   -1.17    0.245
## Displacement -0.93813    0.37166   -2.52    0.012 *
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## F-statistic: 87.2 on 6 and 219 DF,  p-value: <2e-16
```

Questions of interests

① diff. between stat. sig (smaller p-values) vs. practical sig: How useful each $\hat{\beta}_i$ to be??

# Notes

# Questions of interests

1. Write down the final OLS equation of `MPG_City` given the rest of the predictors.
2. What does each z (or t)-interval and z (or t)-test do?
3. Is `Width` **THE** most important variable, `HP` the second, etc since they each has the smallest p-value,...
4. Is `Width` most useful variable due to its largest coefficient in magnitude?
5. What is the standard error from the output? Precisely what does it measure?
6. Interpret the $R^2$ reported for this model. Do you feel comfortable using the output for the following questions based on this $R^2$ value?
7. Should we take `Seating` or `Cylinders` out?

# Confidence Interval for the Mean

Base on Model 3, the mean of `MPG_City` among all cars with the same features as the new design: length=180, HP=225, width=80, seating=5, cylinders=4, displacement=3.5, transmission="automatic", continent="Am" is

$$\hat{y} = 45.63 + 0.05 \times 180 - 0.02 \times 225 - 0.35$$
$$\times 80 - 0.24 \times 5 - 0.27 \times 4 - 0.94 \times 3.5 = 16.17,$$

# Confidence Interval for the Mean

```
predict(fit3, newcar, interval = "confidence", se.fit = TRUE)
```

```
## $fit
##    fit  lwr  upr
## 1 16.1 14.6 17.7
##
## $se.fit
## [1] 0.784
##
## $df
## [1] 219
##
## $residual.scale
## [1] 2.04
```

**Q: What assumptions are needed to make this a valid confidence interval?**

# Prediction Interval

Base on Model 3, `MPG_City` for this particular new design is

$$\hat{y} = 45.63 + 0.05 \times 180 - 0.02 \times 225 - 0.35$$
$$\times 80 - 0.24 \times 5 - 0.27 \times 4 - 0.94 \times 3.5 = 16.17$$

with a 95% prediction interval approximately to be

$$\hat{y} \pm 2 \times RSE = 16.17 \pm 2 \times 2.036.$$

# Prediction Interval

```r
# future prediction intervals
predict(fit3, newcar,  interval = "predict", se.fit = TRUE)
```

```
## $fit
##    fit  lwr  upr
## 1 16.1 11.8 20.4
##
## $se.fit
## [1] 0.784
##
## $df
## [1] 219
##
## $residual.scale
## [1] 2.04
```

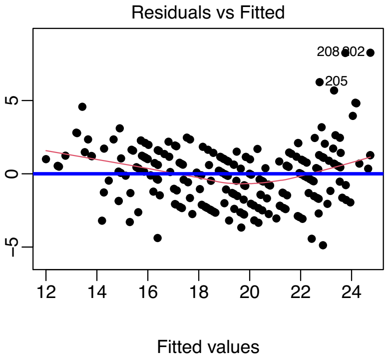
**Q: What assumptions are needed to make this a valid prediction interval?**

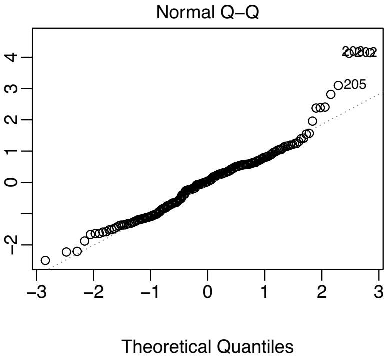
# Model Diagnoses

To check the model assumptions are met, we examine the residual plot and the qqplot of the residuals.

We use the first and second plots of `plot(fit)`.

```
par(mfrow=c(1,2), mar=c(5,2,4,2), mgp=c(3,0.5,0)) # plot(fit3) produces several plots
plot(fit3, 1, pch=16) # residual plot. try pch=1 to 25
abline(h=0, col="blue", lwd=3)
plot(fit3, 2) # qqplot
```



# Model Diagnoses

Are the linear model assumptions met for the model fit here (fit3)? What might be violated?

- linearity?
- Equal variances?
- Normality?

# Categorical Predictors

Let's use `Continent` as one variable. It has three categories. We explore the following questions:

1. Are Asian cars more efficient?
2. How does `Continent` affect the `MPG`?

```
unique(data1$Continent)    #data1$Continent
```

```
## [1] "As" "E"  "Am"
```

# Categorical Predictors

First, we explore the sample means and sample standard error of MPG for each continent.

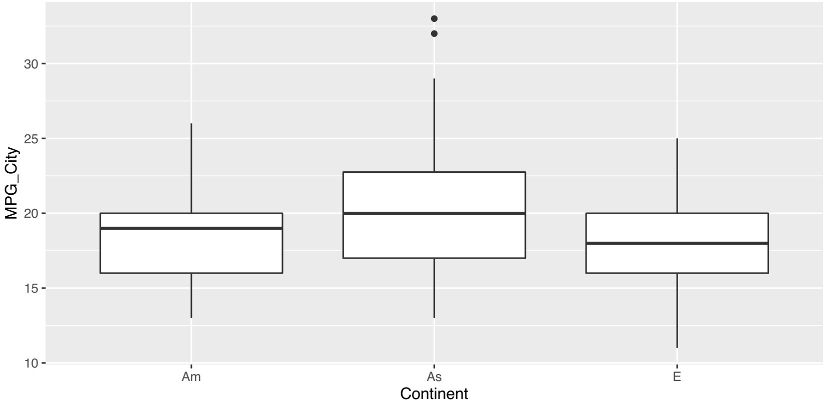
```
data1 %>%
group_by(Continent) %>%
  summarise(
    mean = mean(MPG_City),
    sd  = sd(MPG_City),
    n = n()
  )
```

```
## # A tibble: 3 x 4
##   Continent  mean    sd     n
##   <chr>     <dbl> <dbl> <int>
## 1 Am         18.7  2.99    74
## 2 As         20.2  4.20    98
## 3 E          18.3  3.22    54
```

# Categorical Predictors

Now we plot the boxplot of MPG by Continent.

```
ggplot(data1) + geom_boxplot(aes(x = Continent, y = MPG_City))
```



# 'lm()' with Categorical Predictors

```r
fit.continent <- lm(MPG_City ~ Continent, data1)
summary(fit.continent)
```

```
##
## Call:
## lm(formula = MPG_City ~ Continent, data = data1)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -7.259 -2.730 -0.245  1.755 12.755
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   18.730      0.420   44.62   <2e-16 ***
## ContinentAs    1.515      0.556    2.72   0.0069 **
## ContinentE    -0.470      0.646   -0.73   0.4673
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.61 on 223 degrees of freedom
## Multiple R-squared:  0.0552, Adjusted R-squared:  0.0467
## F-statistic: 6.52 on 2 and 223 DF,  p-value: 0.00178
```

# Notes

# 'Anova()'

To test whether `Continent` is significant, use `Anova()` from the car package.

```
Anova(fit.continent)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##           Sum Sq  Df F value Pr(>F)
## Continent    170   2    6.52 0.0018 **
## Residuals   2907 223
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# ‘Anova()‘

Let’s also control `Horsepower` in addition to `Continent`. We test whether `Continent` is significant after controlling for `Horsepower`.

```
fit.continent.hp <- lm(MPG_City ~ Horsepower + Continent, data1)
Anova(fit.continent.hp)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##            Sum Sq  Df F value Pr(>F)
## Horsepower   1567   1  259.47 <2e-16 ***
## Continent      46   2    3.81  0.024 *
## Residuals    1340 222
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Full model

Now we are ready to build a model using all predictors.

```r
# select useful predictors
data2 <- data1 %>% select(-Make.Model, -MPG_Hwy, -Make)
# fit all variables
fit.all <- lm(MPG_City ~., data2)
Anova(fit.all)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##              Sum Sq  Df F value  Pr(>F)
## Continent        10   2    1.79 0.16972
## Horsepower       33   1   11.71 0.00075 ***
## Weight          229   1   80.95 < 2e-16 ***
## Length           14   1    4.93 0.02737 *
## Width             0   1    0.03 0.85749
## Seating          14   1    4.94 0.02724 *
## Cylinders         3   1    0.95 0.33166
## Displacement      1   1    0.18 0.66988
## Transmission      5   2    0.89 0.41282
## Residuals       606 214
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

However, many of the predictors are NOT significant!

# Backward selection

We can perform backward selection:

- remove the predictor with largest $p$-value one by one
- until all the variables are significant.

Use `update()` to refit a model.

# Backward selection

## 'update()'

Step 1: Width has the largest $p$-value from `Anova(fit.all)`, so we remove Width first.

```
# . means keeping all the variables in the lm formula
# - means remove the predictor
fit.backward.1 <- update(fit.all, .~. - Width)
Anova(fit.backward.1)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##              Sum Sq  Df F value  Pr(>F)
## Continent        10   2    1.84 0.16171
## Horsepower       33   1   11.78 0.00072 ***
## Weight          318   1  112.59 < 2e-16 ***
## Length           15   1    5.38 0.02133 *
## Seating          14   1    5.00 0.02633 *
## Cylinders         3   1    0.98 0.32249
## Displacement      0   1    0.17 0.67726
## Transmission      5   2    0.88 0.41498
## Residuals       607 215
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Backward selection

Step 2: Displacement has the largest *p*-value from `fit.backward.1`, we remove it.

```
fit.backward.2 <- update(fit.backward.1, .~. - Displacement)
Anova(fit.backward.2)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##              Sum Sq  Df F value  Pr(>F)
## Continent        11   2    1.89 0.15364
## Horsepower       43   1   15.36 0.00012 ***
## Weight          417   1  148.23 < 2e-16 ***
## Length           15   1    5.27 0.02267 *
## Seating          15   1    5.31 0.02217 *
## Cylinders         8   1    2.82 0.09428 .
## Transmission      5   2    0.87 0.42198
## Residuals       607 216
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Backward selection

Step 3: Transmission has the largest *p*-value from `fit.backward.2`, we remove it.

```
fit.backward.3 <- update(fit.backward.2, .~. - Transmission)
Anova(fit.backward.3)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##            Sum Sq  Df F value  Pr(>F)
## Continent      10   2    1.82   0.165
## Horsepower     52   1   18.37 2.7e-05 ***
## Weight        414   1  147.62 < 2e-16 ***
## Length         14   1    5.16   0.024 *
## Seating        15   1    5.47   0.020 *
## Cylinders       8   1    2.68   0.103
## Residuals     612 218
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Backward selection

Step 4: Continent has the largest *p*-value from `fit.backward.3`, we remove it.

```
fit.backward.4 <- update(fit.backward.3, .~. - Continent)
Anova(fit.backward.4)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##            Sum Sq  Df F value Pr(>F)
## Horsepower     47   1   16.75  6e-05 ***
## Weight        444   1  157.06 <2e-16 ***
## Length         12   1    4.22 0.0412 *
## Seating        20   1    7.00 0.0088 **
## Cylinders      10   1    3.55 0.0607 .
## Residuals     622 220
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Now all the predictors are significant at 0.1 level. And we use it as the final model.

```
fit.final <- fit.backward.4
```

# Final model

Here is the summary of the final model.

```
summary(fit.final)
```

```
##
## Call:
## lm(formula = MPG_City ~ Horsepower + Weight + Length + Seating +
##     Cylinders, data = data2)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -4.078 -1.031  0.010  0.895  7.011
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 30.27666    1.80112   16.81   <2e-16 ***
## Horsepower  -0.01363    0.00333   -4.09    6e-05 ***
## Weight      -3.61334    0.28832  -12.53   <2e-16 ***
## Length       0.02655    0.01292    2.05   0.0412 *
## Seating      0.36312    0.13729    2.64   0.0088 **
## Cylinders   -0.28054    0.14879   -1.89   0.0607 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.68 on 220 degrees of freedom
## Multiple R-squared:  0.798,  Adjusted R-squared:  0.793
## F-statistic:  174 on 5 and 220 DF,  p-value: <2e-16
```

# Final model

Questions:

1. Given the summary of the final model, how would you interpret it?
2. Can we remove all the insignificant predictors at once at step 1?
3. Now we are treating Cylinders as a continuous variable. What if we treat it as a categorical variable?

# Goodness of Fit: $R^2$

**Remark 1:**

- $TSS \geq RSS$. Why so???
- $R^2 \leq 1$.
- $TSS = RSS + \sum(\hat{y}_i - \bar{y})^2$.
- $(corr(y, \hat{y}))^2$

An $R^2$ statistic that is close to 1 indicates that a large proportion of the variability in the response has been explained by the regression.

# Goodness of Fit: $R^2$

**Remark2**:

- How large $R^2$ needs to be so that you are comfortable to use the linear model?
- Though $R^2$ is a very popular notion of goodness of fit, but it has its limitation. Mainly all the sum of squared errors defined so far are termed as `Training Errors`. It really only measures how good a model fits the data that we use to build the model. It may not generate well to unseen data.