

Multiple Linear Regression

Table of Content

- Linear model
 - ▶ Additive
 - ▶ Interpretation
- Model specification
- Model assumptions
- LS estimates
 - ▶ Mean estimates
 - ▶ CI's for each
 - ▶ Predictions for y
- Goodness of fit
- Model diagnosis
- Categorical X 's
- Variable selection

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

Case Study: Fuel Efficiency in Automobiles

Goal of the study: How to build a fuel efficient car?

- Effects of features on a car
- Given a set of features, we'd like to estimate the mean fuel efficiency as well as the efficiency of one car
- Are Asian cars more efficient than cars built in other regions?

Let us answer these questions with Multiple Regression

Data

Dataset: `car_04_regular.csv` , $n = 226$ cars

Feature	Description
Continent	Continent the Car Company is From
Horsepower	Horsepower of the Vehicle
Weight	Weight of the vehicle (thousand lb)
Length	Length of the Vehicle (inches)
Width	Width of the Vehicle (inches)
Seating	Number of Seats in the Vehicle
Cylinders	Number of Engine Cylinders
Displacement	Volume displaced by the cylinders, defined as $\pi/4 \times bore^2 \times stroke \times \text{Number of Cylinders}$
Transmission	Type of Transmission (manual, automatic, continuous)

Goal of Study: Rephrased

- 1 Fuel efficiency is measured by Mileage per Gallon, $Y = \text{MPG_City}$
- 2 Predictors: Effects of each feature on Y
- 3 Estimate - the mean MPG_City for all such cars specified below and - predict Y for the particular car described below
- 4 Are cars built by Asian more efficient?
- 5 Investigate the MPG_City for this newly designed American car

Feature	Value
Continent	America
Horsepower	225
Weight	4
Length	180
Width	80
Seating	5
Cylinders	4
Displacement	3.5
Transmission	automatic

A Quick Glimpse at the data

```
data1 <- read.csv("car_04_regular.csv", header=TRUE)
```

```
names(data1)
```

```
## [1] "Make.Model" "Continent" "MPG_City" "MPG_Hwy" "Horsepower"  
## [6] "Weight" "Length" "Width" "Seating" "Cylinders"  
## [11] "Displacement" "Make" "Transmission"
```

```
dim(data1) # 226 cars and 13 variables
```

```
## [1] 226 13
```

A Quick Glimpse at the data

```
str(data1)
```

```
## 'data.frame':    226 obs. of  13 variables:
##  $ Make.Model   : chr  "Acura_RL" "Acura_TL" "Acura_TSX" "Acura_RSX" ...
##  $ Continent    : chr  "As" "As" "As" "As" ...
##  $ MPG_City     : int   18 20 23 25 17 17 20 18 17 16 ...
##  $ MPG_Hwy      : int   24 28 32 34 24 23 28 25 24 22 ...
##  $ Horsepower   : int  225 270 200 160 252 265 170 220 330 250 ...
##  $ Weight       : num   3.9 3.58 3.32 2.77 3.2 ...
##  $ Length       : num  197 189 183 172 174 ...
##  $ Width        : num  71.6 72.2 69.4 67.9 71.3 77 76.3 76.1 74.6 76.1 ...
##  $ Seating      : int   5 5 5 4 2 7 5 5 5 5 ...
##  $ Cylinders     : int   6 6 4 4 6 6 4 6 6 6 ...
##  $ Displacement: num   3.5 3.2 2.4 2 3 3.5 1.8 3 4.2 2.7 ...
##  $ Make         : chr  "Acura" "Acura" "Acura" "Acura" ...
##  $ Transmission: chr  "automatic" "automatic" "automatic" "automatic" ...
```


A Quick Glimpse at the data

```
head(data1)
```

```
##   Make.Model Continent MPG_City MPG_Hwy Horsepower Weight Length Width Seating
## 1   Acura_RL        As      18      24        225   3.90   197   71.6      5
## 2   Acura_TL        As      20      28        270   3.58   189   72.2      5
## 3   Acura_TSX       As      23      32        200   3.32   183   69.4      5
## 4   Acura_RSX       As      25      34        160   2.77   172   67.9      4
## 5   Acura_NSX       As      17      24        252   3.20   174   71.3      2
## 6   Acura_MDX       As      17      23        265   4.45   189   77.0      7
##   Cylinders Displacement  Make Transmission
## 1         6          3.5 Acura   automatic
## 2         6          3.2 Acura   automatic
## 3         4          2.4 Acura   automatic
## 4         4          2.0 Acura   automatic
## 5         6          3.0 Acura   automatic
## 6         6          3.5 Acura   automatic
```

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

Introduction to Multiple regression

Guiding Question: How does Length affect MPG_City?

- It depends on how we model the response. We will investigate **three models** with Length.
- For the ease of presentation, we define some predictors below that we will use in subsequent models:

$x_1 = \text{Length}, \quad x_2 = \text{Horsepower}, \quad x_3 = \text{Width}, \quad x_4 = \text{Seating}$

, $x_5 = \text{Cylinders}, \quad x_6 = \text{Displacement}$

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

• Model Specification

- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

Model Specification

M1. Our first model will only contain one predictor, Length:

$$y_i = \beta_0 + \beta_1 \cdot x_{i1} + \epsilon$$

Interpretation of β_1 is that in general, the mean y will change by β_1 if a car is 1" longer. So we can't really peel off the effect of the Length over y .

Additive model: β_1 ?

Model Specification

M2. Next, we add the predictor Horsepower to our model

$$y_i = \beta_0 + \beta_1 \cdot x_{i1} + \beta_2 \cdot x_{i2} + \epsilon$$

Interpretation of β_1 is that in general, the mean y will change by β_1 if a car is 1" longer and the 'Horse Power's' are the same.

Model Specification

M3. Finally, we fit a model with multiple predictors

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \beta_6 x_{i6} + \epsilon$$

Interpretation of β_1 is that in general, the mean y will change by β_1 if a car is 1" longer and the rest of the features are the same.

Model Specification

Question: Are all the β_1 s same in the 3 models above?

No. The effect of Length β_1 depends on the rest of the features in the model!!!!

Notes

Notes

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

General linear models

In general, We define a multiple regression as

$$Y = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \epsilon_i$$

General linear models: Assumptions

Y : response; X_1, X_2, \dots, X_p : explanatory variables

- Linearity Assumption for this model is

$$\mathbf{E}(y_i|x_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_p x_{ip}$$

- The homoscedasticity assumption is

$$\mathbf{Var}(y_i|x_{i1}, x_{i2}, \dots, x_{ip}) = \sigma^2$$

- Normality assumption

$$y_i|x_i \stackrel{iid}{\sim} \mathcal{N}(\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_p x_{ip}, \sigma^2)$$

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

OLS and its Properties

These β parameters are estimated using the same approach as simple regression, specifically by minimizing the sum of squared residuals (RSS):

$$\min_{b_0, b_1, b_2, \dots, b_p} \sum_{i=1}^n (y_i - b_0 - b_1 x_{i1} - b_2 x_{i2} - \dots - b_p x_{ip})^2$$

OLS Estimates

- Each $\hat{\beta}_i$ is normal with mean β_i
- Produce $se(\hat{\beta}_i)$
- z or t interval for β_i based on $\hat{\beta}_i$.

OLS Estimates: Hypothesis Test

To test that

$$\beta_i = 0 \quad \text{vs.} \quad \beta_i \neq 0$$

which means that given other variables in the model, there is no x_i effect.
We carry out a t-test:

$$t\text{-stat} = \frac{\hat{\beta}_i - 0}{\text{se}(\hat{\beta}_i)}$$

The p-value is:

$$\text{p-value} = 2 \times P(T \text{ variable} > t\text{stat})$$

.

We reject the null hypothesis at an α level if the p-value is $< \alpha$.

OLS Prediction

A 95% Confidence interval for the mean given a set of predictors:

$$\hat{y} \pm 2 \times se(\hat{y})$$

A 95% prediction interval for a future y given a set of predictors:

$$\hat{y} \pm 2 \times \hat{\sigma}.$$

RSS, MSE, RSE

For multiple regression, RSS is estimated as:

$$RSS = \sum_{i=1}^n \hat{\epsilon}_i^2 = \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \sum_{i=1}^n (y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \cdots + \hat{\beta}_p x_{ip}))^2$$

$$MSE = \frac{RSS}{n - p - 1} = \hat{\sigma}^2$$

$$\hat{\sigma} = RSE = \sqrt{MSE}$$

Goodness of Fit: R^2

TSS measures the total variance in the response Y .

$$TSS = \sum_{i=1}^n (y_i - \bar{y})^2$$

How much variability is captured in the linear model using this set of predictors? R^2 measures the proportion of variability in Y that can be explained using this set of predictors in the model.

$$R^2 = \frac{TSS - RSS}{TSS}$$

R function 'lm()'

- Linear models are so popular due to their nice interpretations as well as clean solutions
- R-function `lm()` takes a model specification together with other options, outputs all the estimators, summary statistics such as varies sum of squares, standard errors of estimators, testing statistics and p-values.
- The model also outputs the predicted values with margin of errors; confidence intervals and prediction intervals can also be called.

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- **Compare three models**
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

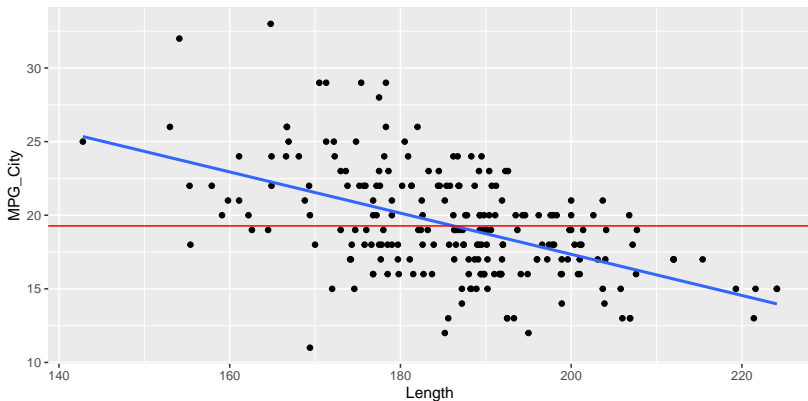
3 Categorical Predictors

4 Backward selection

5 Appendix

Model 1: $\text{MPG_City} \sim \text{Length}$

```
fit1 <- lm(MPG_City ~ Length, data = data1) # model specification response ~ x1,..
ggplot(data1, aes(x = Length, y = MPG_City)) +
  geom_point() +
  geom_smooth(method="lm", formula = 'y~x', se=F) +
  geom_hline(aes(yintercept = mean(MPG_City)), color = "red")
```



Model 1: MPG_City \sim Length

- We now create a model with `lm()`
- Note from the summary below, the $\hat{\beta}$ for Length is estimated as -0.14.
- We say on average MPG drops .13983 if a car is 1' longer.

```
fit1 <- lm(MPG_City ~ Length, data = data1) # model one
summary(fit1)
```

```
##
## Call:
## lm(formula = MPG_City ~ Length, data = data1)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -10.626  -2.279  -0.151   1.977  10.731
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  45.3138     2.8975   15.64  <2e-16 ***
## Length       -0.1398     0.0155   -9.01  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.18 on 224 degrees of freedom
## Multiple R-squared:  0.266, Adjusted R-squared:  0.263
## F-statistic: 81.2 on 1 and 224 DF,  p-value: <2e-16
```


Model 2: MPG_City \sim Length + Horsepower

```
fit2 <- lm(MPG_City ~ Length + Horsepower, data = data1)
summary(fit2) #sum((fit2$res)^2)
```

```
##
## Call:
## lm(formula = MPG_City ~ Length + Horsepower, data = data1)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -7.152 -1.558  0.154  1.492  8.563
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  38.62587    2.22525   17.36 < 2e-16 ***
## Length       -0.06191    0.01300    -4.76 3.5e-06 ***
## Horsepower   -0.03690    0.00277   -13.31 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.38 on 223 degrees of freedom
## Multiple R-squared:  0.591, Adjusted R-squared:  0.587
## F-statistic: 161 on 2 and 223 DF, p-value: <2e-16
```

Model 3: Several continuous variables

```
fit3 <- lm(MPG_City ~ Length + Horsepower + Width + Seating +  
           Cylinders + Displacement, data = data1)  
summary(fit3)
```

```
##  
## Call:  
## lm(formula = MPG_City ~ Length + Horsepower + Width + Seating +  
##     Cylinders + Displacement, data = data1)  
##  
## Residuals:  
##      Min       1Q   Median       3Q      Max   
## -4.877 -1.462  0.073  1.149  8.261   
##  
## Coefficients:  
##              Estimate Std. Error t value Pr(>|t|)      
## (Intercept)  45.63372    4.02793   11.33 < 2e-16 ***  
## Length       0.04909    0.01730    2.84  0.005 **   
## Horsepower   -0.02000    0.00408   -4.90 1.8e-06 ***  
## Width        -0.35358    0.06879   -5.14 6.1e-07 ***  
## Seating      -0.24135    0.14955   -1.61  0.108   
## Cylinders     -0.27169    0.23292   -1.17  0.245   
## Displacement -0.93813    0.37166   -2.52  0.012 *    
## ---  
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
##  
## Residual standard error: 2.04 on 219 degrees of freedom  
## Multiple R-squared:  0.705, Adjusted R-squared:  0.697   
## F-statistic: 87.2 on 6 and 219 DF,  p-value: <2e-16
```

Compare 3 Models

Table 3

	<i>Dependent variable:</i>		
	MPG_City		
	(1)	(2)	(3)
Length	-0.140*** (0.016)	-0.062*** (0.013)	0.049*** (0.017)
Horsepower		-0.037*** (0.003)	-0.020*** (0.004)
Width			-0.354*** (0.069)
Seating			-0.241 (0.150)
Cylinders			-0.272 (0.233)
Displacement			-0.938** (0.372)
Constant	45.300*** (2.900)	38.600*** (2.230)	45.600*** (4.030)
Observations	226	226	226
R ²	0.266	0.591	0.705
Residual Std. Error	3.170 (df = 224)	2.380 (df = 223)	2.040 (df = 219)

Compare 3 Models

- They are different as expected
- Each one has its own meaning!

Question: what does $\hat{\beta}_1$ mean in 3 models

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

Focus on Model 3

```
fit3 <- lm(MPG_City ~ Length + Horsepower + Width + Seating +
           Cylinders + Displacement, data = data1)
summary(fit3)

##
## Call:
## lm(formula = MPG_City ~ Length + Horsepower + Width + Seating +
##     Cylinders + Displacement, data = data1)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -4.877 -1.462  0.073  1.149  8.261
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  45.63372    4.02793   11.33 < 2e-16 ***
## Length       0.04909    0.01730    2.84  0.005 **
## Horsepower   -0.02000    0.00408   -4.90 1.8e-06 ***
## Width        -0.35358    0.06879   -5.14 6.1e-07 ***
## Seating      -0.24135    0.14955   -1.61  0.108
## Cylinders     -0.27169    0.23292   -1.17  0.245
## Displacement -0.93813    0.37166   -2.52  0.012 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.04 on 219 degrees of freedom
## Multiple R-squared:  0.705, Adjusted R-squared:  0.697
## F-statistic: 87.2 on 6 and 219 DF, p-value: <2e-16
```

Notes

Questions of interests

- 1 Write down the final OLS equation of `MPG_City` given the rest of the predictors.
- 2 What does each z (or t)-interval and z (or t)-test do?
- 3 Is `Width` **THE** most important variable, `HP` the second, etc since they each has the smallest p -value,...
- 4 Is `Width` most useful variable due to its largest coefficient in magnitude?
- 5 What is the standard error from the output? Precisely what does it measure?
- 6 Interpret the R^2 reported for this model. Do you feel comfortable using the output for the following questions based on this R^2 value?
- 7 Should we take `Seating` or `Cylinders` out?

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- **Confidence Interval**
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

Confidence Interval for the Mean

Base on Model 3, the mean of MPG_City among all cars with the same features as the new design: length=180, HP=225, width=80, seating=5, cylinders=4, displacement=3.5, transmission="automatic", continent="Am" is

$$\begin{aligned}\hat{y} &= 45.63 + 0.05 \times 180 - 0.02 \times 225 - 0.35 \\ &\times 80 - 0.24 \times 5 - 0.27 \times 4 - 0.94 \times 3.5 = 16.17,\end{aligned}$$

Confidence Interval for the Mean

```
predict(fit3, newcar, interval = "confidence", se.fit = TRUE)
```

```
## $fit
##      fit   lwr   upr
## 1 16.1 14.6 17.7
##
## $se.fit
## [1] 0.784
##
## $df
## [1] 219
##
## $residual.scale
## [1] 2.04
```

Q: What assumptions are needed to make this a valid confidence interval?

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- **Prediction Interval**
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

Prediction Interval

Base on Model 3, MPG_City for this particular new design is

$$\begin{aligned}\hat{y} &= 45.63 + 0.05 \times 180 - 0.02 \times 225 - 0.35 \\ &\times 80 - 0.24 \times 5 - 0.27 \times 4 - 0.94 \times 3.5 = 16.17\end{aligned}$$

with a 95% prediction interval approximately to be

$$\hat{y} \pm 2 \times RSE = 16.17 \pm 2 \times 2.036.$$

Prediction Interval

```
# future prediction intervals
predict(fit3, newcar, interval = "predict", se.fit = TRUE)

## $fit
##      fit   lwr   upr
## 1 16.1 11.8 20.4
##
## $se.fit
## [1] 0.784
##
## $df
## [1] 219
##
## $residual.scale
## [1] 2.04
```

Q: What assumptions are needed to make this a valid prediction interval?

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

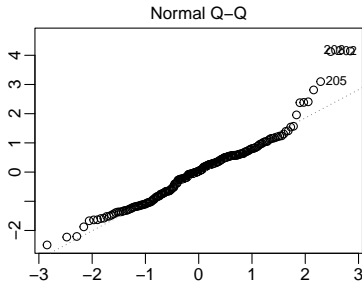
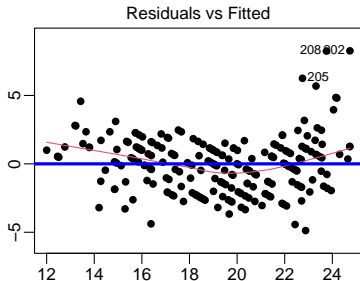
5 Appendix

Model Diagnoses

To check the model assumptions are met, we examine the residual plot and the qqplot of the residuals.

We use the first and second plots of `plot(fit)`.

```
par(mfrow=c(1,2), mar=c(5,2,4,2), mgp=c(3,0.5,0)) # plot(fit3) produces several plots
plot(fit3, 1, pch=16) # residual plot. try pch=1 to 25
abline(h=0, col="blue", lwd=3)
plot(fit3, 2) # qqplot
```



Fitted values

Theoretical Quantiles

Model Diagnoses

Are the linear model assumptions met for the model fit here (fit3)? What might be violated?

- linearity?
- Equal variances?
- Normality?

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

Categorical Predictors

Let's use `Continent` as one variable. It has three categories. We explore the following questions:

- 1 Are Asian cars more efficient?
- 2 How does `Continent` affect the MPG?

```
unique(data1$Continent)    #data1$Continent
```

```
## [1] "As" "E"  "Am"
```

Categorical Predictors

First, we explore the sample means and sample standard error of MPG for each continent.

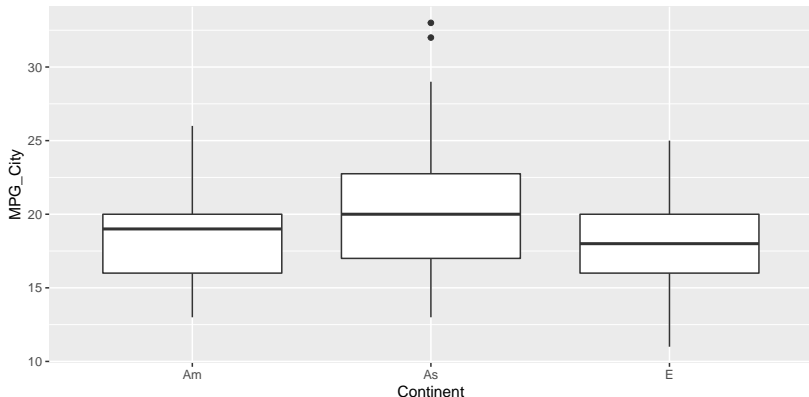
```
data1 %>%  
  group_by(Continent) %>%  
  summarise(  
    mean = mean(MPG_City),  
    sd = sd(MPG_City),  
    n = n()  
  )
```

```
## # A tibble: 3 x 4  
##   Continent mean    sd    n  
##   <chr>    <dbl> <dbl> <int>  
## 1 Am      18.7  2.99   74  
## 2 As      20.2  4.20   98  
## 3 E       18.3  3.22   54
```

Categorical Predictors

Now we plot the boxplot of MPG by Continent.

```
ggplot(data1) + geom_boxplot(aes(x = Continent, y = MPG_City))
```



'lm()' with Categorical Predictors

```
fit.continent <- lm(MPG_City ~ Continent, data1)
summary(fit.continent)
```

```
##
## Call:
## lm(formula = MPG_City ~ Continent, data = data1)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -7.259 -2.730 -0.245  1.755 12.755
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   18.730     0.420   44.62  <2e-16 ***
## ContinentAs    1.515     0.556    2.72  0.0069 **
## ContinentE   -0.470     0.646   -0.73  0.4673
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.61 on 223 degrees of freedom
## Multiple R-squared:  0.0552, Adjusted R-squared:  0.0467
## F-statistic: 6.52 on 2 and 223 DF,  p-value: 0.00178
```

Notes

'Anova()'

To test whether Continent is significant, use Anova() from the car package.

```
Anova(fit.continent)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##           Sum Sq Df F value Pr(>F)
## Continent    170  2    6.52 0.0018 **
## Residuals   2907 223
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

'Anova()'

Let's also control Horsepower in addition to Continent. We test whether Continent is significant after controlling for Horsepower.

```
fit.continent.hp <- lm(MPG_City ~ Horsepower + Continent, data1)
Anova(fit.continent.hp)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##           Sum Sq Df F value Pr(>F)
## Horsepower  1567  1  259.47 <2e-16 ***
## Continent    46  2    3.81  0.024 *
## Residuals   1340 222
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

Full model

Now we are ready to build a model using all predictors.

```
# select useful predictors
data2 <- data1 %>% select(-Make.Model, -MPG_Hwy, -Make)
# fit all variables
fit.all <- lm(MPG_City ~., data2)
Anova(fit.all)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##           Sum Sq Df F value  Pr(>F)
## Continent      10  2    1.79 0.16972
## Horsepower     33  1   11.71 0.00075 ***
## Weight        229  1   80.95 < 2e-16 ***
## Length         14  1    4.93 0.02737 *
## Width           0  1    0.03 0.85749
## Seating         14  1    4.94 0.02724 *
## Cylinders        3  1    0.95 0.33166
## Displacement    1  1    0.18 0.66988
## Transmission    5  2    0.89 0.41282
## Residuals      606 214
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

However, many of the predictors are NOT significant!

Backward selection

We can perform backward selection:

- remove the predictor with largest p -value one by one
- until all the variables are significant.

Use `update()` to refit a model.

Backward selection

'update()'

Step 1: Width has the largest p -value from `Anova(fit.all)`, so we remove Width first.

```
# . means keeping all the variables in the lm formula
# - means remove the predictor
fit.backward.1 <- update(fit.all, .~. - Width)
Anova(fit.backward.1)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##
```

	Sum Sq	Df	F value	Pr(>F)
## Continent	10	2	1.84	0.16171
## Horsepower	33	1	11.78	0.00072 ***
## Weight	318	1	112.59	< 2e-16 ***
## Length	15	1	5.38	0.02133 *
## Seating	14	1	5.00	0.02633 *
## Cylinders	3	1	0.98	0.32249
## Displacement	0	1	0.17	0.67726
## Transmission	5	2	0.88	0.41498
## Residuals	607	215		

```
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Backward selection

Step 2: Displacement has the largest p -value from `fit.backward.1`, we remove it.

```
fit.backward.2 <- update(fit.backward.1, .~. - Displacement)
Anova(fit.backward.2)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##           Sum Sq Df F value  Pr(>F)
## Continent      11  2    1.89 0.15364
## Horsepower     43  1   15.36 0.00012 ***
## Weight       417  1  148.23 < 2e-16 ***
## Length        15  1    5.27 0.02267 *
## Seating        15  1    5.31 0.02217 *
## Cylinders       8  1    2.82 0.09428 .
## Transmission   5  2    0.87 0.42198
## Residuals     607 216
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Backward selection

Step 3: Transmission has the largest p -value from `fit.backward.2`, we remove it.

```
fit.backward.3 <- update(fit.backward.2, ~. - Transmission)
Anova(fit.backward.3)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##
```

	Sum Sq	Df	F value	Pr(>F)
Continent	10	2	1.82	0.165
Horsepower	52	1	18.37	2.7e-05 ***
Weight	414	1	147.62	< 2e-16 ***
Length	14	1	5.16	0.024 *
Seating	15	1	5.47	0.020 *
Cylinders	8	1	2.68	0.103
Residuals	612	218		

```
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```


Backward selection

Step 4: Continent has the largest p -value from `fit.backward.3`, we remove it.

```
fit.backward.4 <- update(fit.backward.3, ~. - Continent)
Anova(fit.backward.4)
```

```
## Anova Table (Type II tests)
##
## Response: MPG_City
##           Sum Sq Df F value Pr(>F)
## Horsepower    47  1  16.75  6e-05 ***
## Weight       444  1 157.06 <2e-16 ***
## Length        12  1   4.22  0.0412 *
## Seating        20  1   7.00  0.0088 **
## Cylinders       10  1   3.55  0.0607 .
## Residuals     622 220
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Now all the predictors are significant at 0.1 level. And we use it as the final model.

```
fit.final <- fit.backward.4
```

Final model

Here is the summary of the final model.

```
summary(fit.final)
```

```
##
## Call:
## lm(formula = MPG_City ~ Horsepower + Weight + Length + Seating +
##     Cylinders, data = data2)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -4.078 -1.031  0.010   0.895   7.011
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  30.27666    1.80112   16.81  <2e-16 ***
## Horsepower   -0.01363    0.00333    -4.09   6e-05 ***
## Weight       -3.61334    0.28832   -12.53  <2e-16 ***
## Length        0.02655    0.01292    2.05   0.0412 *
## Seating       0.36312    0.13729    2.64   0.0088 **
## Cylinders     -0.28054    0.14879    -1.89   0.0607 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.68 on 220 degrees of freedom
## Multiple R-squared:  0.798, Adjusted R-squared:  0.793
## F-statistic: 174 on 5 and 220 DF,  p-value: <2e-16
```

Final model

Questions:

- ➊ Given the summary of the final model, how would you interpret it?
- ➋ Can we remove all the insignificant predictors at once at step 1?
- ➌ Now we are treating Cylinders as a continuous variable. What if we treat it as a categorical variable?

1 Case Study: Fuel Efficiency in Automobiles

2 Multiple regression

- Model Specification
- General linear models
- OLS and its Properties
- Compare three models
- Inferences for coefficients
- Confidence Interval
- Prediction Interval
- Model Diagnoses

3 Categorical Predictors

4 Backward selection

5 Appendix

Goodness of Fit: R^2

Remark 1:

- $TSS \geq RSS$. Why so???
- $R^2 \leq 1$.
- $TSS = RSS + \sum(\hat{y}_i - \bar{y})^2$.
- $(\text{corr}(y, \hat{y}))^2$

An R^2 statistic that is close to 1 indicates that a large proportion of the variability in the response has been explained by the regression.

Goodness of Fit: R^2

Remark2:

- How large R^2 needs to be so that you are comfortable to use the linear model?
- Though R^2 is a very popular notion of goodness of fit, but it has its limitation. Mainly all the sum of squared errors defined so far are termed as Training Errors. It really only measures how good a model fits the data that we use to build the model. It may not generate well to unseen data.