# Lab 1 - Exercise 6

Andrei Cristian
mailto:andrei.cristian1@info.uaic.ro

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## 1 Definition of bisimulation

A relation $Z \subseteq W \times W'$ is said to be *bisimulation* between $M$ and $M'$ if the following conditions are satisfied:

1. For all $w \in W$ and $w' \in W'$ such that $(w, w') \in Z$ and each $p \in AP$, it holds $p \in L(w)$ if and only if $p \in L'(w')$;
2. If $(w, w') \in Z$ and $(w, v) \in R$, then there exists $v' \in W'$ such that $(v, v') \in Z$ and $(w', v') \in R'$ (*forth condition*);
3. If $(w, w') \in Z$ and $(w', v') \in R$, then there exists $v \in W$ such that $(v, v') \in Z$ and $(w, v) \in R$ (*back condition*).

Two worlds $w$ and $w'$ are called bisimilar, $w \leftrightsquigarrow_{M,M'} w'$, if there is a bisimulation $Z$ between $M$ and $M'$ with $(w, w') \in Z$.

## 2 Proposition 1

Let $M =< W, R, L >$ and $M' =< W', R', L' >$ be two Kripke structures. Let $w \in W'$ such as $w \leftrightsquigarrow_{M,M'} w'$.Then $w$ and $w'$ are modally equivalent, i.e., for each formula Modal Logic $\varphi$,

$$M, w \models \varphi \text{ iff. } M', w' \models \varphi$$

## 3 Structural induction proof

Demonstration using structural induction:

- $\varphi = p \in AP$, $M, w \models p$ iff. $p \in L(w)$
  $w \leftrightsquigarrow_{M,M'} w' \xrightarrow{1.} L(w) = L(w') \Rightarrow p \in L(w') \Rightarrow M', w' \models p$; **(a)**
  $M', w' \models p$ iff. $p \in L(w')$
  $w \leftrightsquigarrow_{M,M'} w' \xrightarrow{1.} L(w') = L(w) \Rightarrow p \in L(w) \Rightarrow M, w \models p$; **(b)**
  $(a), (b) \Rightarrow M, w \models p$ iff. $M, w' \models p$.
  $\Rightarrow M, w \models p$ iff. $M', w' \models p$.

- $\varphi = \neg\varphi_1$, $M, w \models \neg\varphi_1$ iff. $M, w \not\models \varphi_1$; **(a)**
  $M, w \models \varphi_1$ iff. $M', w' \models \varphi_1$; **(b)** – demonstrated previously
  $(a), (b) \Rightarrow M', w' \not\models \varphi_1 \Rightarrow M', w' \models \neg\varphi_1$.
  $M', w' \models \neg\varphi_1$ iff. $M', w' \not\models \varphi_1$; **(c)**
  $M', w' \models \varphi_1$ iff. $M, w \models \varphi_1$; **(d)** – demonstrated previously
  $(c), (d) \Rightarrow M, w \not\models \varphi_1 \Rightarrow M, w \models \neg\varphi_1$.
  $\Rightarrow M, w \models \varphi$ iff. $M', w' \models \varphi$.

- $\varphi = \Diamond\varphi_1$, $M, w \models \Diamond\varphi_1$ iff. $\exists t \in W, (w, t) \in R, M, t \models \varphi_1$; **(a)**
  $w \leftrightsquigarrow_{M,M'} w' \xrightarrow{2.}$ if exists $(w, t) \in R$, then $\exists t' \in W', (w', t') \in R'$ and $t \leftrightsquigarrow_{M,M'} t'$; **(b)**
  $(a), (b) \xrightarrow{IH} \exists t', (w', t') \in R', M', t' \models \varphi_1 \Rightarrow M', w' \models \Diamond\varphi_1$;
  $M', w' \models \Diamond\varphi_1$ iff. $\exists t' \in W', (w', t') \in R', M', t' \models \varphi_1$; **(c)**
  $w \leftrightsquigarrow_{M,M'} w' \xrightarrow{3.}$ if exists $(w', t') \in R'$, then $\exists t \in W, (w, t) \in R$ and $t \leftrightsquigarrow_{M,M'} t'$; **(d)**
  $(c), (d) \xrightarrow{IH} \exists t, (w, t) \in R, M, t \models \varphi_1 \Rightarrow M, w \models \Diamond\varphi_1$.
  $\Rightarrow M, w \models \varphi$ iff. $M', w' \models \varphi$.

- $\varphi = \Box\varphi_1$, $M, w \models \Box\varphi_1$ iff. $\forall t \in W, (w, t) \in R, M, t \models \varphi_1$; **(a)**
  $w \leftrightsquigarrow_{M,M'} w' \xrightarrow{2.}$ if exists $(w, t) \in R$, then $\exists t' \in W', (w', t') \in R'$ and $t \leftrightsquigarrow_{M,M'} t'$; **(b)**
  $(a), (b) \xrightarrow{IH} \forall t', (w', t') \in R', M', t' \models \varphi_1 \Rightarrow M', w' \models \Box\varphi_1$;
  $M', w' \models \Box\varphi_1$ iff. $\forall t' \in W', (w', t') \in R', M', t' \models \varphi_1$; **(c)**
  $w \leftrightsquigarrow_{M,M'} w' \xrightarrow{3.}$ if exists $(w', t') \in R'$, then $\exists t \in W, (w, t) \in R$ and $t \leftrightsquigarrow_{M,M'} t'$; **(d)**
  $(c), (d) \xrightarrow{IH} \forall t, (w, t) \in R, M, t \models \varphi_1 \Rightarrow M, w \models \Box\varphi_1$.
  $\Rightarrow M, w \models \varphi$ iff. $M', w' \models \varphi$.

In conclusion, $\forall \varphi \in BML$, $M, w \models \varphi$ iff. $M', w' \models \varphi$