Importance of Cauchy Distribution in the Field of Statistics
A Descriptive Study

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Outline of talk

Objectives

Our aim of this project is to -

- study some unique features of Cauchy Distribution
- understand the difference between Cauchy and Normal
- learn the behaviour of Cauchy in Statistical Inference

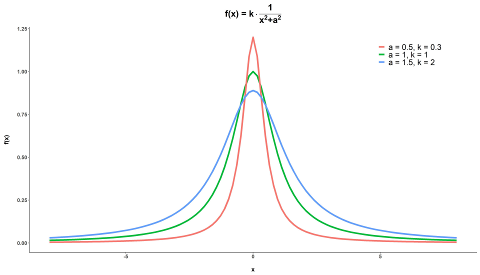
Importance

- Provides counter examples to some well known results
- Testing problems apart from Normality assumptions
- It has applications in mechanical and electrical theory, physical anthropology, measurement problems, risk and financial analysis.

Next Topic

Background

- First idea developed by Siméon Denis Poisson
- Named after Augustin-Louis Cauchy
- Special case of ‘Witch of Agnesi’



Graph of the function $\frac{k}{x^2+a^2}$ for different values of ‘k’ and ‘a’

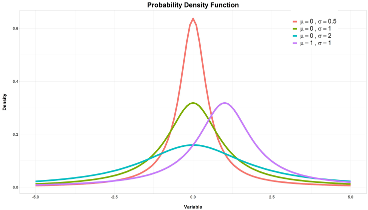
Next Topic

Density and Distribution Function

The Cauchy distribution with location parameter 'μ' ($\mu \in \mathbb{R}$) and scale parameter 'σ' ($\sigma > 0$) is denoted by $C(\mu, \sigma)$.

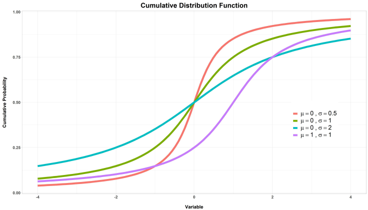
PDF of $C(\mu, \sigma)$

$$\frac{1}{\pi} \cdot \frac{\sigma}{\{\sigma^2 + (x-\mu)^2\}}, \quad x \in \mathbb{R}$$



CDF of $C(\mu, \sigma)$

$$\frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R}$$

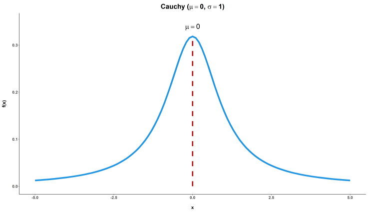


Next Topic

Shape and Graphical Behaviour

If we plot the density of the Cauchy PDF, it can be observed that the graph is -

- Bell shaped
- Symmetric
- The graph has **Thick Tails**



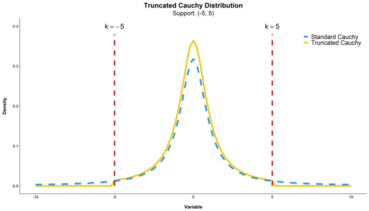
Also, it is symmetric about the line $x = \mu$.

Next Topic

There is no 'moment' for Cauchy!

The reasons behind non-existence of moments -

- High propensity of producing outliers
- The probability in the tails are higher



Under truncated set-up the probability becomes null beyond some cut-point. That is why the moments become finite.

Next Topic

Mode and Quantiles

Consider, $C(\mu, \sigma)$ distribution.

Mode

- Unimodal distribution.
- Mode is at $x = \mu$
- Modal value: $\frac{1}{\pi\sigma}$

Quantiles

- Symmetric distribution.
- $\xi_p = \mu + \sigma \tan\left[\pi\left(p - \frac{1}{2}\right)\right]$
- Median: $\xi_{\frac{1}{2}} = \mu$

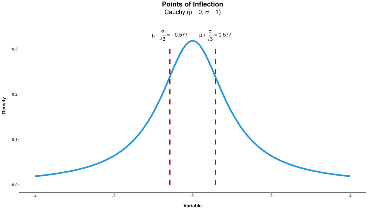
For the above Cauchy distribution, $Q_1 = \mu - \sigma$ and $Q_3 = \mu + \sigma$. So, the quartile deviation becomes σ.

Hence, the parameter 'σ' can also be interpreted from this point.

Points of Inflection

The point on a curve where it changes from concavity to convexity or vice-versa is called a point of inflection.

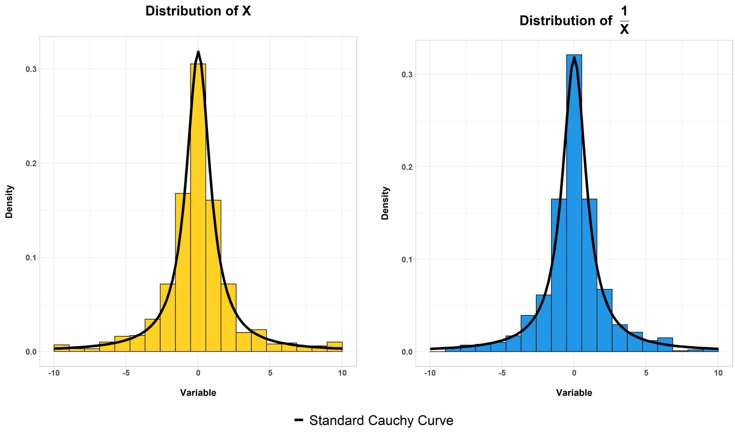
For $C(\mu, \sigma)$ the points of inflection are $\left(\mu - \frac{\sigma}{\sqrt{3}}, \frac{3}{4\pi\sigma}\right)$ and $\left(\mu + \frac{\sigma}{\sqrt{3}}, \frac{3}{4\pi\sigma}\right)$.



For Standard Cauchy distribution these points are (-0.577, 0.239) and (0.577, 0.239).

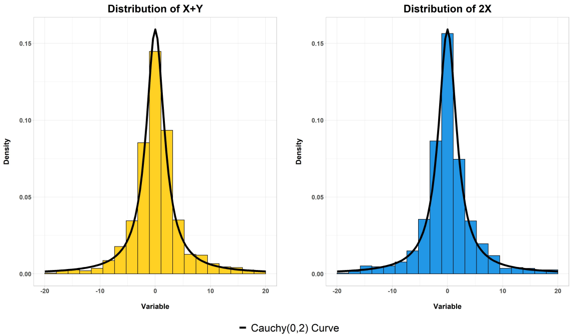
Inverse of a Cauchy is also a Cauchy!

$$X \sim C(0,1) \implies \frac{1}{X} \sim C(0,1)$$



A Unique Property

$$X, Y \overset{iid}{\sim} C(0,1) \implies X + Y \overset{D}{\equiv} 2X \sim C(0,2)$$

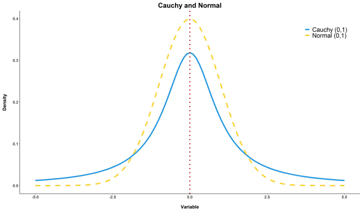


This is also a justification for the non-existence of variance of this distribution.

Graphical Approach

Let us consider two random variables X and Y where, $X \sim N(0,1)$ and $Y \sim C(0,1)$ with PDFs f_X and f_y, respectively.

- $\max\limits_{t\in\mathbb{R}} f_X(t) = \frac{1}{\sqrt{2\pi}} = f_X(0)$
  $\max\limits_{t\in\mathbb{R}} f_Y(t) = \frac{1}{\pi} = f_Y(0)$
  $\implies f_X(0) > f_Y(0)$
- Median and Mode exists for both but Mean does not.
- $f_X, f_y \to 0$ but f_X goes more rapidly as $|t| \to \infty$
- $P(|X| > k) < P(|Y| > k)$ $(\forall\, k > 0)$



Purpose

Consider, $C(\mu, 1)$. Now to test, $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$

Result

Table: **Empirical Levels**

Size	Cauchy	Normal
5	0.529	0.037
10	0.661	0.054
25	0.745	0.055
50	0.833	0.046
100	0.878	0.055
200	0.908	0.051
500	0.949	0.042
1000	0.960	0.050

From the Table, it seems that, under Normal setup, the Cauchy distribution is behaving badly.

Next Topic

Unbiased Estimator of Location parameter

In usual method, we cannot find any unbiased estimator of the location parameter μ for the $C(\mu, \sigma)$ distribution though there are some estimators which are unbiased. They are -

- Sample median $\left(X_{(k)}\right)$ when sample size is odd $[n = 2k + 1]$
- Sample mid-range $\left(\frac{X_{(1)} + X_{(n)}}{2}\right)$

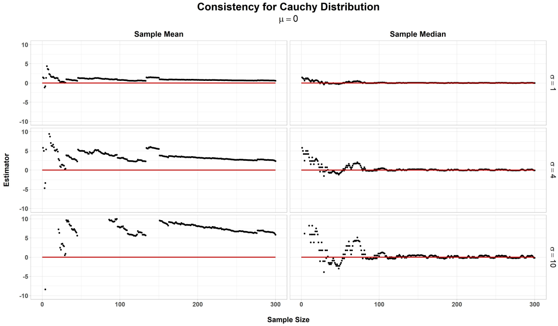
Some other estimators which are unbiased for location are **Quick Estimator** which is the weighted average of some suitable order statistics, **Sample Trimmed Mean** (using central 24% of total observations).

In all the above cases our target is to minimize the Asymptotic Relative Efficiency.

Next Topic

Consistent Estimator for the Location parameter

As the population mean does not exist, sample mean do not converge to a finite value.



From the graph, it follows that sample median is consistent for location.

Next Topic

M.L.E. and C.R. Inequality

Maximum Likelihood Estimation

Under Cauchy population, the location parameter cannot be estimated in a closed form using method of Maximum Likelihood.

Hence, we find the M.L.E. by iterative method to a specified level of convergence.

Cramer-Rao Inequality

The Cramer-Rao lower bound for Cauchy is $\frac{2}{n}$.

As the Cauchy distribution does not belong to the 'One Parameter Exponential Family', the Cramer-Rao lower bound is not attainable lower bound and hence no MVBUE for any parametric function does not exist.

Conclusion

- Estimation of the location parameter of the Cauchy distribution is not an easy task
- Thick tails of this distribution make it an 'outlier producing distribution'

Future Aspects

- Indulging ourselves in quantile measures only and study their properties thoroughly
- Diving deep into the estimation theory for Cauchy distribution
- Estimating the scale parameter, multivariate Cauchy parameters, testing procedures and confidence intervals

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Thank You