高等代数（荣誉）II
第四次习题课

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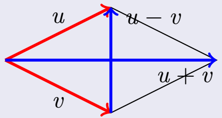
6.A-4

Suppose V is a real inner product space.

a. Show that $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2$ for every $u, v \in V$.
b. Show that if $u, v \in V$ have the same norm, then $u+v$ is orthogonal to $u-v$.
c. Use part (b) to show that the diagonals of a rhombus are perpendicular to each other.

Hint

a. $\langle u+v, u-v\rangle = \langle u, u\rangle + \langle v, u\rangle - \langle u, v\rangle - \langle v, v\rangle = \|u\|^2 - \|v\|^2$.
b. By (a), $\langle u+v, u-v\rangle = \|u\|^2 - \|v\|^2 = 0$.
c. 

6.A-5

Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

If dim $V < \infty$, it suffices to show that $T - \sqrt{2}I$ is injective.
Assume that $(T - \sqrt{2}I)v = 0$, i.e., $Tv = \sqrt{2}v$.
Then we have $\|v\| \geq \|Tv\| = \|\sqrt{2}v\|$, so $\|v\| \leq 0$, and hence $v = 0$.

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Infinite dimensional case

It suffices to show that $I - (\sqrt{2})^{-1}T$ is invertible.
For brevity, denote $S = (\sqrt{2})^{-1}T$. Then we have $\|Sv\| \leq (\sqrt{2})^{-1}\|v\|$.
Note that

$$(1 - x)(1 + x + x^2 + x^3 + \cdots) = 1.$$

Consider the opearator

$$R : V \to V, \ Rv = v + Sv + S^2v + S^3v + \cdots .$$

We claim that $R \in \mathcal{L}(V)$ and $(I - S)R = R(I - S) = I$, which shows that $I - S$ is invertible.

Infinite dimensional case (continued)

We have $\|Sv\| \leq (\sqrt{2})^{-1}\|v\|$. Consider the opearator

$$R : V \to V, \; Rv = v + Sv + S^2v + S^3v + \cdots .$$

We claim that $R \in \mathcal{L}(V)$ (well-defined and linear) and $(I - S)R = R(I - S) = I$.

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We claim that $R \in \mathcal{L}(V)$ (well-defined and linear) and $(I - S)R = R(I - S) = I$.
But, this is **NOT TRUE**.
Here is a counterexample:
Let $V = C_{00} = \{(x_1, x_2, x_3, \cdots) : \exists N,\ \forall n > N,\ x_n = 0\}$. Define $\langle x, y\rangle = \sum x_i y_i$.
Consider the right shift operator $T \in \mathcal{L}(V)$, i.e., $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots)$.

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However, it would be fine if assuming that V is a Banach space.

Banach space

Banach space: a complete normed vector space.
Complete: every Cauchy sequence converges.
Norm of a linear operator: $T \in \mathcal{L}(X)$: $\|T\| := \inf\{M > 0 : \|Tx\| \leq M\|x\|,\ \forall x \in X\}$.
Boundedness of a linear operator: $\|T\| < \infty$.
Assume that X is a Banach space and $S, T \in \mathcal{L}(X)$ are bounded.
Theorem: If $\|T\| < 1$, then $1 - T$ is invertible.
Corollary: If S^{-1} is bounded and $\|T\| \leq \|S^{-1}\|^{-1}$, then $S + T$ is invertible.

Infinite dimensional case (continued)

We have $\|Sv\| \le (\sqrt{2})^{-1}\|v\|$. Consider the opearator

$$R: V \to V,\ Rv = v + Sv + S^2v + S^3v + \cdots.$$

We claim that $R \in \mathcal{L}(V)$ (well-defined and linear) and $(I - S)R = R(I - S) = I$.
But, this is **NOT TRUE**.
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Note that the discussion above does not imply 6.A-5 is wrong for infinite dimensional case.
Even for the operator $T \in \mathcal{L}(C_{00})$ defined above, it has an inverse by taking Hamel basis.

6.A-6

Suppose $u, v \in V$.
Prove that $\langle u, v\rangle = 0$ if and only if $\|u\| \le \|u + av\|$ for all $a \in \mathbb{F}$.

Hint

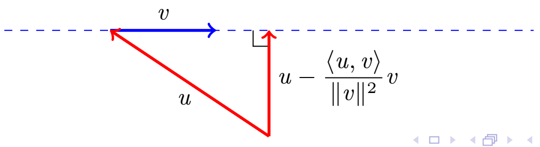
Note that $\|u+av\|^2 - \|u\|^2 = 2\,\mathrm{Re}(a\langle v, u\rangle) + \|av\|^2$.

- $\Rightarrow$ Clear by the formula above.
- $\Leftarrow$ Assume that $\langle u, v\rangle \neq 0$, it is clear that $v \neq 0$. Taking

$$a = -\frac{\langle u, v\rangle}{\|v\|^2},$$

we have

$$\|u+av\|^2 - \|u\|^2 = -2\frac{|\langle u, v\rangle|^2}{\|v\|^2} + \frac{|\langle u, v\rangle|^2}{\|v\|^2} = -\frac{|\langle u, v\rangle|^2}{\|v\|^2} < 0.$$



6.A-6

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Analysis way

Let t be in $\mathbb{R}$. Consider the function $f(t) = \|u + t\langle u, v\rangle v\|^2 - \|u\|^2$.
Then we have

$$f'(t) = 2t|\langle u, v\rangle|^2\|v\|^2 + 2|\langle u, v\rangle|^2.$$

Note that $f(0) = 0$ and $f(t) \leq 0$ for all $t \in \mathbb{R}$.

6.A-9

Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2} \leq 1 - |\langle u, v\rangle|.$$

Hint

It suffice to show that $(1-\|u\|^2)(1-\|v\|^2) \leq (1-\|u\|\cdot\|v\|)^2$, that is, $(1-a^2)(1-b^2) \leq (1-ab)^2$ for $a, b \in [0,1]$.

6.A-11

Prove that

$$16 \leq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

for all positive numbers a, b, c, d.

Hint

Let $u = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$ and $v = (\sqrt{a^{-1}}, \sqrt{b^{-1}}, \sqrt{c^{-1}}, \sqrt{d^{-1}})$.
Then the inequality can be transformed into

$$4 = \langle u, v\rangle \leq \|u\| \cdot \|v\|.$$

6.A-18

Suppose $p > 0$. Prove that there is an inner product on $\mathbb{R}^2$ such that the associated norm is given by

$$\|(x, y)\| = (x^p + y^p)^{1/p}$$

for all $(x, y) \in \mathbb{R}^2$ if and only if $p = 2$.

The inner product induced from the norm should be

$$\langle u, v \rangle = \frac{1}{2}(\|u + v\|^2 - \|u\|^2 - \|v\|^2).$$

By the additivity in first slot, we have

$$2^{2/p} - 2 = 2 \times \langle (0, 1), (1, 0) \rangle = \langle (0, 2), (1, 0) \rangle = \frac{1}{2}((2^p + 1)^{2/p} - 5),$$

i.e,

$$2^{2/p+1} + 1 = (2^p + 1)^{2/p}.$$

However, it is not easy to show that $p = 2$ by this formula.

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Hint

Alternatively, consider the parallelogram equality

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

with $u = (0, 1)$ and $v = (0, 1)$. Then we have

$$2^{2/p} + 2^{2/p} = 2 \times (1 + 1).$$

6.A-19

Suppose V is a real inner product space. Prove that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$

for all $u, v \in V$.

Hint

Clear.

6.A-24

Suppose $S \in \mathcal{L}(V)$ is an injective operator on V. Define $\langle \cdot, \cdot \rangle_1$ by

$$\langle u, v \rangle_1 = \langle Su, Sv \rangle$$

for $u, v \in V$. Show that $\langle \cdot, \cdot \rangle_1$ is an inner product on V.

Hint

A straightforward verification.

6.A-26

Suppose f, g are differentiable functions from $\mathbb{R}$ to $\mathbb{R}^n$.

- (a) Show that
$$\langle f(t), g(t)\rangle' = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle.$$
- (b) Suppose $c > 0$ and $\|f(t)\| = c$ for every $t \in \mathbb{R}$. Show that $\langle f'(t), f(t)\rangle = 0$ for every $t \in \mathbb{R}$.
- (c) Interpret the result in part (b) geometrically in terms of the tangent vector to a curve lying on a sphere in $\mathbb{R}^n$ centered at the origin.

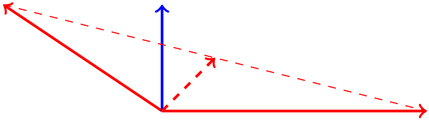
Hint

- (a) Note that
$$\begin{aligned}\langle f(t), g(t)\rangle' &= \lim_{s\to 0} \frac{1}{s}(\langle f(t+s), g(t+s)\rangle - \langle f(t), g(t)\rangle) \\ &= \lim_{s\to 0} \frac{1}{s}(\langle f(t+s), g(t+s)\rangle - \langle f(t), g(t)\rangle) \\ &= \lim_{s\to 0} \frac{1}{s}(\langle f(t+s), g(t+s)\rangle - \langle f(t), g(t+s)\rangle) \\ &\quad + \lim_{s\to 0} \frac{1}{s}(\langle f(t), g(t+s)\rangle - \langle f(t), g(t)\rangle) = \quad \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle.\end{aligned}$$
- (b) By (a) with $f = g$.
- (c) Consider the canonical inner product on $\mathbb{R}^n$.
  Let $f(t)$ be a curve lying on a sphere in $\mathbb{R}^n$ with radius c and centered at the origin.
  Then $f'(t)$ is the corresponding tangent vector.

6.A-27

Suppose that $u, v, w \in V$. Prove that

$$\left\|w - \tfrac{1}{2}(u+v)\right\|^2 = \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4}.$$



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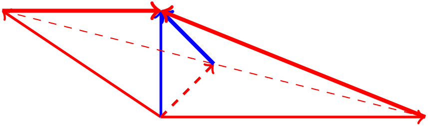

6.A-27

$$\left\|w - \tfrac{1}{2}(u+v)\right\|^2 = \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4}.$$

6.A-28

Suppose C is a subset of V with the property that $u, v \in C$ implies $\frac{1}{2}(u+v) \in C$. (convexity)
Let $w \in V$. Show that there is at most one point in C that is closest to w.
In other words, show that there is at most one $u \in C$ such that

$$\|w-u\| \leq \|w-v\| \quad \text{for all } v \in C.$$

Hint

Suppose that there are two points $u, v \in C$ that is closest to w.
Then by 6.A-27, $\frac{u+v}{2} \in C$ is closer to w than u and v.

6.A-29

For $u, v \in V$, define $d(u, v) = \|u - v\|$.

a. Show that d is a metric on V.
b. Show that if V is finite-dimensional, then d is a complete metric on V, i.e., every Cauchy sequence converges.
c. Show that every finite-dimensional subspace of V is closed subset of V w.r.t d.

Hint

a. Omitted.
b. Assume that $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence in V, where $u_k = (u_k^{(i)})_{i=1}^{n}$.

   Then for each i, the sequence $\{u_k^{(i)}\}_{k=1}^{\infty}$ is also a Cauchy sequence, since

$$|u_{k_1}^{(i)} - u_{k_2}^{(i)}| \leq d(u_{k_1}, u_{k_2}).$$

   Denote the limit of $\{u_k^{(i)}\}_{k=1}^{\infty}$ by $u^{(i)}$.

   Let $u := (u^{(1)}, \ldots, u^{(n)}) \in V$. Then u is the limit of $\{u_k\}_{k=1}^{\infty}$ since

$$d(u_k, u) \leq \sum_{i=1}^{n} |u_k^{(i)} - u^{(i)}|.$$

c. Any complete subset of a metric space is closed.

6.A-30

Fix a positive integer n. The **Laplacian** Δp of a twice differentiable function p on $\mathbb{R}^n$ is the function on Rn defined by

$$\Delta p = \frac{\partial^2 p}{\partial x_1^2} + \cdots + \frac{\partial^2 p}{\partial x_n^2}.$$

The function p is called **harmonic** if $\Delta p = 0$.
A **polynomial** on $\mathbb{R}^n$ is a linear combination of functions of the form $x_1^{m_1} \cdots x_n^{m_n}$, where $m_1, \ldots, m_n$ are nonnegative integers.
Suppose q is a polynomial on $\mathbb{R}^n$. Prove that there exists a harmonic polynomial p on $\mathbb{R}^n$ such that $p(x) = q(x)$ for every $x \in \mathbb{R}^n$ with $\|x\| = 1$.

Lemma (Maximum Principle)

If p is a harmonic function on $\mathbb{R}^n$ and $p(x) = 0$ for all $x \in \mathbb{R}^n$ with $\|x\| = 1$, then $p = 0$.

Hint

Let V be the vector space spanned by $x_1^{m_1} \cdots x_n^{m_n}$ with $m_1 + \cdots + m_n \leq \deg q$.
Define an operator T on V by

$$Tr = \Delta((1 - \|x\|^2)r).$$

The operator T is injective by Lemma. Then T is surjective since $\dim V < \infty$.
So $Tr_1 = -\Delta q$ for some $r_1 \in V$. Let $p = q + (1 - \|x\|^2)r_1$.
Then we have $p(x) = q(x)$ for every $x \in \mathbb{R}^n$ with $\|x\| = 1$, and

$$\Delta p = \Delta q + Tr_1 = 0.$$

6.B-11

Suppose $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are inner products on V such that $\langle v,w\rangle_1 = 0$ if and only if $\langle v,w\rangle_2 = 0$. Prove that there is a positive number c such that $\langle\cdot,\cdot\rangle_1 = c\langle\cdot,\cdot\rangle_2$ for every $v,w \in V$.

Define $\phi_i : V \times V \to \mathbb{R}$ by $\phi_i(v,w) = \langle v,w\rangle_i (i = 1,2)$.
Then we have $\phi_1, \phi_2 \in (V \times V)^*$ and $\operatorname{Ker}\phi_1 = \operatorname{Ker}\phi_2$.
So ϕ_1 and ϕ_2 are proportional (习题 1.6-7).

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Define $\phi_i : V \otimes V \to \mathbb{R}$ by $\phi_i(v \otimes w) = \langle v,w\rangle_i (i = 1,2)$.
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However, it is not easy to show that $\operatorname{Ker}\phi_1 = \operatorname{Ker}\phi_2$.

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However, it is not easy to show that Ker ϕ_1 = Ker ϕ_2.

Hint

For any nonzero $w \in V$, define $\phi_{i,w} : V \to \mathbb{R}$ by $\phi_{i,w}(v) = \langle v, w \rangle_i (i = 1, 2)$.
Then we have $\phi_{1,w} = c_w \phi_{2,w}$ for some nonzero c_w.
Note that $c_w \langle v, w \rangle_2 = \langle v, w \rangle_1 = \langle w, v \rangle_1 = c_v \langle w, v \rangle_2$, so $c_w = c_v$ if $\langle v, w \rangle_2 \neq 0$.

6.B-12

Suppose V is finite-dimensional and $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2$ are inner products on V with corresponding norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Prove that there exists a positive number c such that

$$\|v\|_1 \leq c\|v\|_2$$

for every $v \in V$.

Hint

Lemma: a countinuous function attains the maximum on any compact set. Show that any norm $\|\cdot\|$ on a finite-dimensional space V attains the maximum and the nonzero minimum on the set $\{v = \sum k_i e_i \in V : \sum |k_i| = 1\}$.

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Counterexample for Infinite-dimensional Case

Consider $V = C_{00}$. For any $x = (x_1, x_2, x_3, \cdots)$, define

$$\|x\|_1 = \left(\sum_{k=1}^{\infty} |kx_k|^2\right)^{1/2}, \ \|x\|_2 = \left(\sum_{k=1}^{\infty} |x_k|^2\right)^{1/2}.$$

6.B-13

Suppose $v_1, \ldots, v_m$ is a linearly independent list in V. Show that there exists $w \in V$ such that $\langle w, v_j \rangle > 0$ for all $j \in \{1, \ldots, m\}$.

Hint

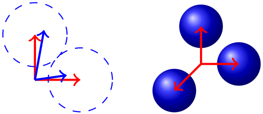
Taking $A = (\langle v_i, v_j \rangle)_{i=1}^m$, it suffices to show that there exists x such that $Ax > 0$.

6.B-14

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $v_1, \ldots, v_n$ are vectors in V such that

$$\|e_j - v_j\| < \frac{1}{\sqrt{n}}$$

for each j. Prove that $v_1, \ldots, v_n$ is a basis of V.



Assume that $k_1 v_1 + \cdots + k_n v_n = 0$ with some k_i nonzero. Then we have

$$k_1 e_1 + \cdots + k_n e_n = k_1(e_1 - v_1) + \cdots + k_n(e_n - v_n),$$

so

$$\begin{aligned} k_1^2 + \cdots + k_n^2 =& \|k_1(e_1 - v_1) + \cdots + k_n(e_n - v_n)\|^2 \\ \leq& (\|k_1(e_1 - v_1)\| + \cdots + \|k_n(e_n - v_n)\|)^2 \\ <& \tfrac{1}{n}(|k_1| + \cdots + |k_n|)^2 \leq k_1^2 + \cdots + k_n^2, \end{aligned}$$

which contradicts.

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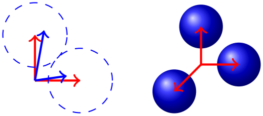

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$$\begin{aligned} k_1^2 + \cdots + k_n^2 =& \|k_1(e_1 - v_1) + \cdots + k_n(e_n - v_n)\|^2 \\ \leq& (\|k_1(e_1 - v_1)\| + \cdots + \|k_n(e_n - v_n)\|)^2 \\ <& \tfrac{1}{n}(|k_1| + \cdots + |k_n|)^2 \leq k_1^2 + \cdots + k_n^2, \end{aligned}$$

which contradicts.

Indeed, the critical case could be chosen as $v_j = e_j - \dfrac{1}{n}(e_1 + \cdots + e_n)$.

6.B-16

Suppose $\mathbb{F} = \mathbb{C}$, V is finite-dimensional, $T \in \mathcal{L}(V)$, all the eigenvalues of T have absolute value less than 1, and $\epsilon > 0$. Prove that there exists a positive integer m such that $\|T^m v\| \leq \epsilon \|v\|$ for every $v \in V$.

Hint

Assume that A is the matrix of T with respect to some orthonormal basis of V. It suffices to show that for such an A, each entry of A^m tends to zero as $m \to \infty$.

Note that the diagonal elements of the Jordan form B are the eigenvalues of T. So we could show that B^m tends to zero as $m \to \infty$.

Then $A^m = (PBP^{-1})^m = PB^m P^{-1} \to O$ as $m \to \infty$.

6.C-5

Suppose V is finite-dimensional and U is a subspace of V. Show that $P_{U^\perp} = I - P_U$, where I is the identity operator on V.

Hint

For all $v \in V$, $(P_U + P_{U^\perp})(v) = v$ since $V = U \oplus U^\perp$.

6.C-6

Suppose U and W are finite-dimensional subspaces of V. Prove that $P_U P_W = 0$ if and only if $\langle u, w\rangle = 0$ for all $u \in U$ and all $w \in W$.

Hint

If $\langle u, w\rangle = 0$ for all $u \in U$ and all $w \in W$, $W \subseteq U^\perp$, $P_U P_W(V) = P_U(W) = 0$.
If $\langle u, w\rangle \neq 0$ for some $u \in U$ and some $w \in W$, $w \notin U^\perp$, $P_U P_W(w) = P_U(w) \neq 0$.

6.C-9

Suppose $T \in \mathcal{L}(V)$ and U is a finite-dimensional subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

Hint

Note that $TP_U(V) = T(U)$ and $P_U(w) = w$ iff $w \in U$. So

$$P_U T P_U = T P_U \iff P_U(w) = w, \forall w \in TP_U(V) = T(U) \iff T(U) \subseteq U.$$

6.C-10

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V. Prove that U and $U^\perp$ are both invariant under T if and only if $P_U T = TP_U$.

Hint

Consider the matrix of T w.r.t. U and $U^\perp$, say $\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$.

Since the matrix of P_U w.r.t. U and $U^\perp$ is $\begin{pmatrix} I_U & O \\ O & O \end{pmatrix}$, $P_U T = TP_U$ is equivalent to $T_2 = T_3 = O$.