2.3 埃尔米特 (Hermite) 插值

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重节点均差

根据均差性质，易证

定理 1

设 $f(x) \in C^n[a, b]$，$x_0, x_1, \cdots, x_n$ 为 $[a, b]$ 上互异节点，则 $f[x_0, x_1, \cdots, x_n]$ 是各变量的多元连续函数。

定义重节点的一阶均差

$$f[x_0, x_0] := \lim_{x \to x_0} f[x_0, x] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

定义重节点的二阶均差

$$f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0}$$

当 $x_1 \to x_0$ 时，由性质 $f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}$ 得

$$f[x_0, x_0, x_0] = \lim_{\substack{x_1 \to x_0 \\ x_2 \to x_0}} f[x_0, x_1, x_2] = \frac{1}{2} f''(x_0).$$

泰勒（Taylor）插值多项式

定义重节点的 n 阶均差

$$f[x_0,x_0,\cdots,x_0]=\lim_{x_i\to x_0}f[x_0,x_1,\cdots,x_n]=\frac{1}{n!}f^{(n)}(x_0).$$

Newton 均差插值多项式中若令 $x_i\to x_0(i=1,2,\cdots,n)$ 则得泰勒插值多项式

$$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

满足插值条件

$$P_n^{(k)}(x_0)=f^{(k)}(x_0),\quad k=0,1,\cdots,n$$

余项为

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},\quad \xi\in(a,b).$$

Lagrange 插值 $\longleftrightarrow$ Newton 插值 $\longleftrightarrow$ Taylor 插值

Hermite 插值

$n+1$个含函数值和导数值的插值条件可构造次数 $\leqslant n$ 的 Hermite 插值多项式

- 三点三次 Hermite 插值多项式
- 两点三次 Hermite 插值多项式

三点三次 Hermite 插值多项式: 构造插值多项式 $P=P(x)$ 满足插值条件

$$\boxed{P(x_i)=f(x_i)\ (i=0,1,2)\ \text{且} P'(x_1)=f'(x_1)}$$

f 的二次 Newton 插值多项式

$$P_2(x)=f(x_0)+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)$$

满足 $P_2(x_i)=f(x_i)(i=0,1,2)$ 。因而

$$(P-P_2)(x_i)=0 \Rightarrow P(x)-P_2(x)=A(x-x_0)(x-x_1)(x-x_2)$$

插值余项

由 $P'(x_1) = f'(x_1)$ 得

$$A = \frac{f'(x_1) - f[x_0, x_1] - (x_1 - x_0) f[x_0, x_1, x_2]}{(x_1 - x_0)(x_1 - x_2)}$$

设

$$R(x) := f(x) - P(x) = k(x)(x - x_0)(x - x_1)^2(x - x_2),$$

其中 $k(x)$ 为待定函数。类似 Lagrange 插值余项的推导得

定理 2

若 $f(x) \in C^4[a, b]$，则对任何 $x \in [a, b]$，插值余项

$$\boxed{R(x) = \frac{f^{(4)}(\xi)}{4!}(x - x_0)(x - x_1)^2(x - x_2),}$$

$\xi \in (a, b)$ 且依赖于 x。

例：给定 $f(x)=x^{3/2}, x_0=\frac{1}{4}, x_1=1, x_2=\frac{9}{4}$，试求三次 Hermite 插值多项式，并求出余项表达式。

解

$$f_0=f\left(\frac{1}{4}\right)=\frac{1}{8},\quad f_1=f(1)=1,\quad f_2=f\left(\frac{9}{4}\right)=\frac{27}{8},$$

$$f[x_0,x_1]=\frac{7}{6},\ f[x_0,x_1,x_2]=\frac{11}{30}$$

令

$$\begin{aligned}P(x)=&\frac{1}{8}+\frac{7}{6}\left(x-\frac{1}{4}\right)+\frac{11}{30}\left(x-\frac{1}{4}\right)(x-1)\\&+A\left(x-\frac{1}{4}\right)(x-1)\left(x-\frac{9}{4}\right)\end{aligned}$$

$$\text{由条件}P'(1)=f'(1)=\frac{3}{2}\ \text{可得}$$

$$A=-\frac{14}{225}.$$

所求 Hermite 插值多项式

$$P(x) = \frac{1}{8} + \frac{7}{6}\left(x - \frac{1}{4}\right) + \frac{11}{30}\left(x - \frac{1}{4}\right)(x-1) - \frac{14}{225}\left(x - \frac{1}{4}\right)(x-1)\left(x - \frac{9}{4}\right)$$

余项

$$\begin{aligned} R(x) = f(x) - P(x) &= \frac{f^{(4)}(\xi)}{4!}\left(x - \frac{1}{4}\right)(x-1)^2\left(x - \frac{9}{4}\right). \\ &= \frac{1}{4!}\frac{9}{16}\xi^{-5/2}\left(x - \frac{1}{4}\right)(x-1)^2\left(x - \frac{9}{4}\right), \quad \xi \in \left(\frac{1}{4}, \frac{9}{4}\right). \end{aligned}$$

两点三次 Hermite 插值多项式

已知插值节点 x_k，x_{k+1} 及节点上函数值和导数值求插值多项式 $H_3(x)$，满足

$$
\begin{aligned}
H_3(x_k) = y_k, &\quad H_3(x_{k+1}) = y_{k+1}, \\
H_3'(x_k) = m_k, &\quad H_3'(x_{k+1}) = m_{k+1}.
\end{aligned}
$$

插值基函数方法: 基函数 $\alpha_k(x)$, $\alpha_{k+1}(x)$, $\beta_k(x)$, $\beta_{k+1}(x)$

$$
\boxed{H_3(x) = \alpha_k(x) y_k + \alpha_{k+1}(x) y_{k+1} + \beta_k(x) m_k + \beta_{k+1}(x) m_{k+1}},
$$

满足插值条件

$$
\begin{aligned}
&\alpha_k(x_k) = 1, \quad \alpha_k(x_{k+1}) = 0, \quad \alpha_k'(x_k) = \alpha_k'(x_{k+1}) = 0 \\
&\alpha_{k+1}(x_k) = 0, \quad \alpha_{k+1}(x_{k+1}) = 1, \quad \alpha_{k+1}'(x_k) = a_{k+1}'(x_{k+1}) = 0;
\end{aligned}
$$

$$
\begin{aligned}
&\beta_k(x_k) = \beta_k(x_{k+1}) = 0, \quad \beta_k'(x_k) = 1, \quad \beta_k'(x_{k+1}) = 0; \\
&\beta_{k+1}(x_k) = \beta_{k+1}(x_{k+1}) = 0, \quad \beta_{k+1}'(x_k) = 0, \quad \beta_{k+1}'(x_{k+1}) = 1
\end{aligned}
$$

由 $\alpha_k(x_{k+1})=0, \alpha_k'(x_{k+1})=0$, 可设

$$\alpha_k(x)=(ax+b)(x-x_{k+1})^2$$

又由

$$1=\alpha_k(x_k)=(ax_k+b)(x_k-x_{k+1})^2$$

$$0=\alpha_k'(x_k)=2(ax_k+b)(x_k-x_{k+1})+a(x_k-x_{k+1})^2$$

解得

$$a=-\frac{2}{(x_k-x_{k+1})^3}, \quad b=\left(1+\frac{2x_k}{x_k-x_{k+1}}\right)\frac{1}{(x_k-x_{k+1})^2},$$

$$\alpha_k(x)=\left(1+2\frac{x-x_k}{x_{k+1}-x_k}\right)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2$$

同理

$$\alpha_{k+1}(x)=\left(1+2\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2$$

求 $\beta_k(x)$: 可令

$$\beta_k(x) = a\,(x - x_k)\,(x - x_{k+1})^2$$

由 $\beta_k'(x_k) = 1$ 得 $a = 1/(x_k - x_{k+1})^2$，从而

$$\beta_k(x) = (x - x_k)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2$$

同理，

$$\beta_{k+1}(x) = (x - x_{k+1})\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2$$

$$\begin{aligned} H_3(x) = & \left(1 + 2\frac{x - x_k}{x_{k+1} - x_k}\right)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2 y_k + \left(1 + 2\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2 y_{k+1} \\ & + (x - x_k)\left(\frac{x - x_{k+1}}{x_k - x_{k+1}}\right)^2 m_k + (x - x_{k+1})\left(\frac{x - x_k}{x_{k+1} - x_k}\right)^2 m_{k+1} \end{aligned}$$

余项 $R_3(x) = f(x) - H_3(x)$

$$R_3(x) = \frac{1}{4!} f^{(4)}(\xi)\,(x - x_k)^2\,(x - x_{k+1})^2, \quad \xi \in (x_k, x_{k+1})$$