

Notes on How to Prove It: A Structured Approach

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Chapter 1

Introduction

1.1 Prime Numbers

This is a book about deductive reasoning in mathematics and mathematical proofs. Fundamental theorems regarding prime numbers are introduced in this chapter to provide an idea of what to expect later in the book.

Definition 1.1.1

A natural number $n > 1$ is considered prime if and only if its only positive divisors are 1 and p .

As a first proof, let's test the conjecture that there are infinitely many prime numbers.

Theorem 1.1.1

There are infinitely many prime numbers.

Proof: If we assume that there is a finite list of prime numbers denoted as p_1, p_2, \dots, p_n , we can create a number m as follows:

$$m = p_1 p_2 \dots p_n + 1$$

Note that m is not divisible by any of the primes p_n .¹ We acknowledge the fact that every integer larger than 1 is either a prime or can be written as a product of two or more primes. If m is a prime, it can't be on the list because that would contradict the first assumption that all primes are on the list. Now, assume m is a product of primes. Take q as one of the primes in that product and an element on the list, so m is divisible by q . Then again, we reach a contradiction because m is not divisible by any p_n .

Since the assumption that there is a finite amount of prime numbers reached a contradiction, there must be an infinite amount of prime numbers. ☺

Theorem 1.1.2

For every composite integer $n > 1$, $2^n - 1$ will also be composite

Proof: If n is composite then it can be expressed as $n = ab$. Let $x = 2^b - 1$ and $y = 1 + 2^b + 2^{2b} + \dots + 2^{(a-1)b}$. Then:

$$\begin{aligned} xy &= (2^b - 1) * (1 + 2^b + 2^{2b} + \dots + 2^{(a-1)b}) \\ &= 2^b * (1 + 2^b + 2^{2b} + \dots + 2^{(a-1)b}) - (1 + 2^b + 2^{2b} + \dots + 2^{(a-1)b}) \\ &= (2^b + 2^{2b} + 2^{3b} + \dots + 2^{ab}) - (1 + 2^b + 2^{2b} + \dots + 2^{(a-1)b}) \\ &= 2^{ab} - 1 \\ &= 2^n - 1. \end{aligned}$$

¹This is because the remainder when dividing m by any of the primes p_n is 1, which means m is not divisible by them.



1.2 Exercises

Question 1

*1.

- (a) Factor $2^n - 1 = 32,767$ into a product of two smaller positive integers.
 (b) Find an integer x such that $1 < x < 2^{32,767} - 1$ and $2^{32,767} - 1$ is divisible by x .

Solution: (a) $n = \log_2(32,767 + 1) = 15$, we know that if n is not prime then $2^n - 1$ is not prime, so 32,767 is a composite number. One possible solution is $4,681 * 7 = 32,767$.

Solution: (b) $2^{32,767} - 1$ is composite so there exist at least two x that can solve the problem. From 1.1, if n is not prime then $n = a * b$. Also $2^n - 1 = (2^b - 1) * (1 + 2^b + 2^{2b} + \dots + 2^{(a-1)b})$. So one possibility is $x = 2^7 - 1$.

Question 2

2. Make some conjectures about the values of n for which $3^n - 1$ is prime or the values of n for which $3^n - 2^n$ is prime. (You might start by making a table similar to Figure I.1)

Solution:

n	$3^n - 1$
1	2
2	8
3	26
4	80
5	242
6	728
7	2,186
8	6,560
9	19,682
10	59,408

Table 1.1: Values of $3^n - 1$ from 1 to 10

Conjecture 1.2.1

For every positive integer n , $3^n - 1$ is even.

n	$3^n - 2^n$
1	0
2	1
3	5
4	19
5	65
6	211
7	665
8	2,059
9	6,305
10	19,171

Table 1.2: Values of $3^n - 2^n$ from 1 to 10

Conjecture 1.2.2

For every positive integer $n > 1$, $3^n - 2^n$ is odd.

There are no aparent relationships involving primes.

Question 3

*3. The proof of Theorem 3 gives a method for finding a prime number different from any in a given list of prime numbers.

- (a) Use this method to find a prime different from 2, 3, 5, and 7.
- (b) Use this method to find a prime different from 2, 5, and 11.

Solution: (a) Given said proof, we have a list of primes $p_1, p_2 \dots p_n$. We can create a number $m = p_1 * p_2 * \dots * p_n + 1$, for the given list this results in $m = 211$.

Solution: (b) With the same process we reach $m = 111$.

Question 4

4. Find five consecutive integers that are not prime.

Solution: We can solve this with a simple python script:

```
L = []
for n in range (2, 100):
    prime = True

    for i in range (2, int(n/2+1)):
        if n%i == 0:
            prime = False
            break

    if prime:
        L = []
    else:
        L.append(n)

    if len(L) == 5:
        print(L)
        break
```

Wich outputs: 24, 25, 26, 27, 28.

Question 5

5. Use the table in Figure I.1 and the discussion on p. 5 to find two more perfect numbers.

Solution: We know that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect. For $n = 7$, $2^n - 1$ is prime so $2^{n-1}(2^n - 1) = 8,128$ is perfect. For $n = 5$, $2^n - 1$ is prime so $2^{n-1}(2^n - 1) = 496$ is perfect.

Question 6

6. The sequence 3, 5, 7 is a list of three prime numbers such that each pair of adjacent numbers in the list differ by two. Are there any more such “triplet primes”?

Solution: Again, this can be solved using python:

```
L = []
for n in range (2, 10000):
    prime = True

    for i in range (2, int(n/2+1)):
        if n%i == 0:
            prime = False
            break

    if prime:
        if len(L) == 0:
            L.append(n)
        else:
            if n == L[-1]+2:
                L.append(n)
            else:
                L = [n]

    if len(L) == 3:
        print(L)
        L.pop(0)
```

The script only prints the sequence 3, 5, 7, so it seems like there aren't any more such "triplet primes". ²

Question 7

7. A pair of distinct positive integers (m, n) is called amicable if the sum of all positive integers smaller than n that divide n is m , and the sum of all positive integers smaller than m that divide m is n . Show that $(220, 284)$ is amicable.

Solution: With python:

```
def find_divisors(number):
    divisors = []
    for i in range(1, number):
        if number % i == 0:
            divisors.append(i)
    return divisors

def check_amicable(m,n):
    divm = find_divisors(m)
    divn = find_divisors(n)

    if sum(divm) == n and sum(divn) == m:
        return True
    else:
        return False

print(check_amicable(220,284))
```

The output of the script shows that this pair is indeed amicable.

²According to number theory, prime triplets are of the form $(p, p+2, p+6)$ or $(p, p+4, p+6)$. Which differ from the definition given by the exercise.

Chapter 2

Chapter 1: Sequential Logic

2.1 Deductive Logic and Logical Connectives

Deductive reasoning is the foundation on which proofs are based.

Example 2.1.1 (Deductive Reasoning)

1. It will either rain or snow tomorrow.
It's too warm to snow.
Therefore, it will rain.
2. If today is Sunday, then I don't have to work today.
Today is Sunday.
Therefore, I don't have to work today.
3. I will work either tomorrow or today.
I'm not working today.
Therefore, I'm working tomorrow.

In each case we arrive at a conclusion given a set of premises which are assumed to be true. The conclusion can only be false if at least one of the premises is false. If none of them are false, then the conclusion must be true.

Definition 2.1.1: Valid Argument

A valid argument is one in which the premises cannot all be true without the conclusion being true as well.

Note:-

An invalid argument is also known as a fallacy, in this case a false conclusion can be reached even if all the premises are true.

Example 2.1.2 (Statements and their Logical Forms)

1. Either John went to the store, or we're out of eggs.
2. Joe is going to leave home and not come back.
3. Either Bill is at work and Jane isn't, or Jane is at work and Bill isn't.

Natural Language	Symbol
And	\wedge
Or	\vee
Not	\neg
If and only if	\iff
Implies	\rightarrow
Exclusive or	\oplus
True	\top
False	\perp
For all	\forall
There exists	\exists

Table 2.1: Logical Connectives

Solution:

1. $P \vee Q$
2. $P \wedge Q$
3. $(P \wedge \neg Q) \vee (\neg P \wedge Q)$

Note:-

\neg always applies only to the statement immediately after it. $\neg P \wedge Q$ means $(\neg P) \wedge Q$.

2.2 Exercises

Question 8

*1. Analyze the logical forms of the following statements:

- (a) We'll have either a reading assignment or homework problems, but we won't have both homework problems and a test.
- (b) You won't go skiing, or you will and there won't be any snow.
- (c) $\sqrt{7} \not\leq 2$

Solution: (a) $(P \vee Q) \wedge \neg(Q \wedge R)$

Solution: (b) $(\neg P) \vee (P \wedge \neg Q)$

Solution: (c) $\neg P \wedge \neg Q$

Question 9

2. Analyze the logical forms of the following statements:

- (a) Either John and Bill are both telling the truth, or neither of them is.
- (b) I'll have either fish or chicken, but I won't have both fish and mashed potatoes.
- (c) 3 is a common divisor of 6, 9, and 15.

Solution: (a) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

Solution: (b) $(P \vee Q) \wedge \neg(P \wedge R)$

Solution: (c) $P \wedge Q \wedge R$

Question 10

3. Analyze the logical forms of the following statements:

- (a) Alice and Bob are not both in the room.
- (b) Alice and Bob are both not in the room.
- (c) Either Alice or Bob is not in the room.
- (d) Neither Alice nor Bob is in the room.

Solution: (a) $\neg(P \wedge Q)$

Solution: (b) $\neg P \wedge \neg Q$

Solution: (c) $\neg P \vee \neg Q$

Solution: (d) $\neg P \wedge \neg Q$

Question 11

4. Analyze the logical forms of the following statements:

- (a) Either both Ralph and Ed are tall, or both of them are handsome.
- (b) Both Ralph and Ed are either tall or handsome.
- (c) Both Ralph and Ed are neither tall nor handsome.
- (d) Neither Ralph nor Ed is both tall and handsome.

Solution: (a) $(P \wedge Q) \vee (R \wedge S)$

Solution: (b) $(P \vee R) \wedge (Q \wedge S)$

Solution: (c) $(\neg P \wedge \neg R) \wedge (\neg Q \wedge \neg S)$

Solution: (d) $\neg(P \wedge R) \wedge \neg(Q \wedge S)$