Notes on How to Prove It: A Structured Approach

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Chapter 1

Introduction

1.1 Prime Numbers

This is a book about deductive reasoning in mathematics and mathematical proofs. Fundamental theorems regarding prime numbers are introduced in this chapter to provide an idea of what to expect later in the book.

Definition 1.1.1

A natural number $n > 1$ is considered prime if and only if its only positive divisors are 1 and p.

As a first proof, let's test the conjecture that there are infinitely many prime numbers.

Theorem 1.1.1

There are infinitely many prime numbers.

Proof: If we assume that there is a finite list of prime numbers denoted as $p_1, p_2, \ldots, p_n$, we can create a number m as follows:

$$m = p_1 p_2 \ldots p_n + 1$$

Note that m is not divisible by any of the primes p_n. [1] We acknowledge the fact that every integer larger than 1 is either a prime or can be written as a product of two or more primes. If m is a prime, it can't be on the list because that would contradict the first assumption that all primes are on the list. Now, assume m is a product of primes. Take q as one of the primes in that product and an element on the list, so m is divisible by q. Then again, we reach a contradiction because m is not divisible by any p_n.

Since the assumption that there is a finite amount of prime numbers reached a contradiction, there must be an infinite amount of prime numbers. 😄

Theorem 1.1.2

For every composite integer $n > 1$, $2^n - 1$ will also be composite

Proof: If n is composite then it can be expressed as $n = ab$. Let $x = 2^b - 1$ and $y = 1 + 2^b + 2^{2b} + \ldots + 2^{(a-1)b}$. Then:

$$\begin{aligned}
xy &= (2^b - 1) * (1 + 2^b + 2^{2b} + \ldots + 2^{(a-1)b}) \\
&= 2^b * (1 + 2^b + 2^{2b} + \ldots + 2^{(a-1)b}) - (1 + 2^b + 2^{2b} + \ldots + 2^{(a-1)b}) \\
&= (2^b + 2^{2b} + 2^{3b} + \ldots + 2^{ab}) - (1 + 2^b + 2^{2b} + \ldots + 2^{(a-1)b}) \\
&= 2^{ab} - 1 \\
&= 2^n - 1.
\end{aligned}$$

[1] This is because the remainder when dividing m by any of the primes p_n is 1, which means m is not divisible by them.

😊

1.2 Exercises

Question 1

*1. (a) Factor $2^n - 1 = 32,767$ into a product of two smaller positive integers.
*1. (b) Find an integer x such that $1 < x < 2^{32,767} - 1$ and $2^{32,767} - 1$ is divisible by x.

Solution: **(a)** $n = log_2(32,767 + 1) = 15$, we know that if n is not prime then $2^n - 1$ is not prime, so 32,767 is a composite number. One possible solution is $4,681 * 7 = 32,767$.
Solution: **(b)** $2^{32,767} - 1$ is composite so there exist at least two x that can solve the problem. From 1.1, if n is not prime then $n = a * b$. Also $2^n - 1 = (2^b - 1) * (1 + 2^b + 2^{2b} + \ldots + 2^{(a-1)b})$. So one possibility is $x = 2^7 - 1$.

Question 2

2. Make some conjectures about the values of n for which $3^n - 1$ is prime or the values of n for which $3^n - 2^n$ is prime. (You might start by making a table similar to *Figure I.1*)

Solution:

n	$3^n - 1$
1	2
2	8
3	26
4	80
5	242
6	728
7	2,186
8	6,560
9	19,682
10	59,408

Table 1.1: Values of $3^n - 1$ fom 1 to 10

Conjecture 1.2.1
For every positive integer n, $3^n - 1$ is even.

n	$3^n - 2^n$
1	0
2	1
3	5
4	19
5	65
6	211
7	665
8	2,059
9	6,305
10	19,171

Table 1.2: Values of $3^n - 2^n$ fom 1 to 10

Conjecture 1.2.2
For every positive integer $n > 1$, $3^n - 2^n$ is odd.

There are no aparent relationships involving primes.

Question 3

*3. The proof of Theorem 3 gives a method for finding a prime number different from any in a given list of prime numbers.
(a) Use this method to find a prime different from 2, 3, 5, and 7.
(b) Use this method to find a prime different from 2, 5, and 11.

Solution: **(a)** Given said proof, we have a list of primes $p_1, p_2 \dots p_n$. We can create a number $m = p_1 * p_2 * \dots * p_n + 1$, for the given list this results in $m = 211$.
Solution: **(b)** With the same process we reach $m = 111$.

Question 4

4. Find five consecutive integers that are not prime.

Solution: We can solve this with a simple python script:

```
L = []
for n in range (2, 100):
    prime = True

    for i in range (2, int(n/2+1)):
        if n%i == 0:
            prime = False
            break

    if prime:
        L = []
    else:
        L.append(n)

    if len(L) == 5:
        print(L)
        break
```

Wich outputs $L = [24, 25, 26, 27, 28]$.

Question 5

5. Use the table in Figure I.1 and the discussion on p. 5 to find two more perfect numbers.

Solution: We know that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect. For $n = 7$, $2^n - 1$ is prime so $2^{n-1}(2^n - 1) = 8,128$ is perfect. For $n = 5$, $2^n - 1$ is prime so $2^{n-1}(2^n - 1) = 496$ is perfect.

Question 6

6. The sequence 3, 5, 7 is a list of three prime numbers such that each pair of adjacent numbers in the list differ by two. Are there any more such "triplet primes"?

Solution: Again, this can be solved using python:

```
L = []
for n in range (2, 10000):
    prime = True

    for i in range (2, int(n/2+1)):
        if n%i == 0:
            prime = False
            break

    if prime:
        if len(L) == 0:
            L.append(n)
        else:
            if n == L[-1]+2:
                L.append(n)
            else:
                L = [n]

    if len(L) == 3:
        print(L)
        L.pop(0)
```

The script only prints the sequence $3, 5, 7$, so it seems like there aren´t any more such "triplet primes". [2]

Question 7

7. A pair of distinct positive integers (m, n) is called amicable if the sum of all positive integers smaller than n that divide n is m, and the sum of all positive integers smaller than m that divide m is n. Show that (220, 284) is amicable.

[2] According to number theory, prime triplets are of the form $(p, p+2, p+6)$ or $(p, p+4, p+6)$. Wich differ from the definition given by the exercise.

Chapter 2

Chapter 1: Sequential Logic

2.1 Deductive Logic and Logical Connectives