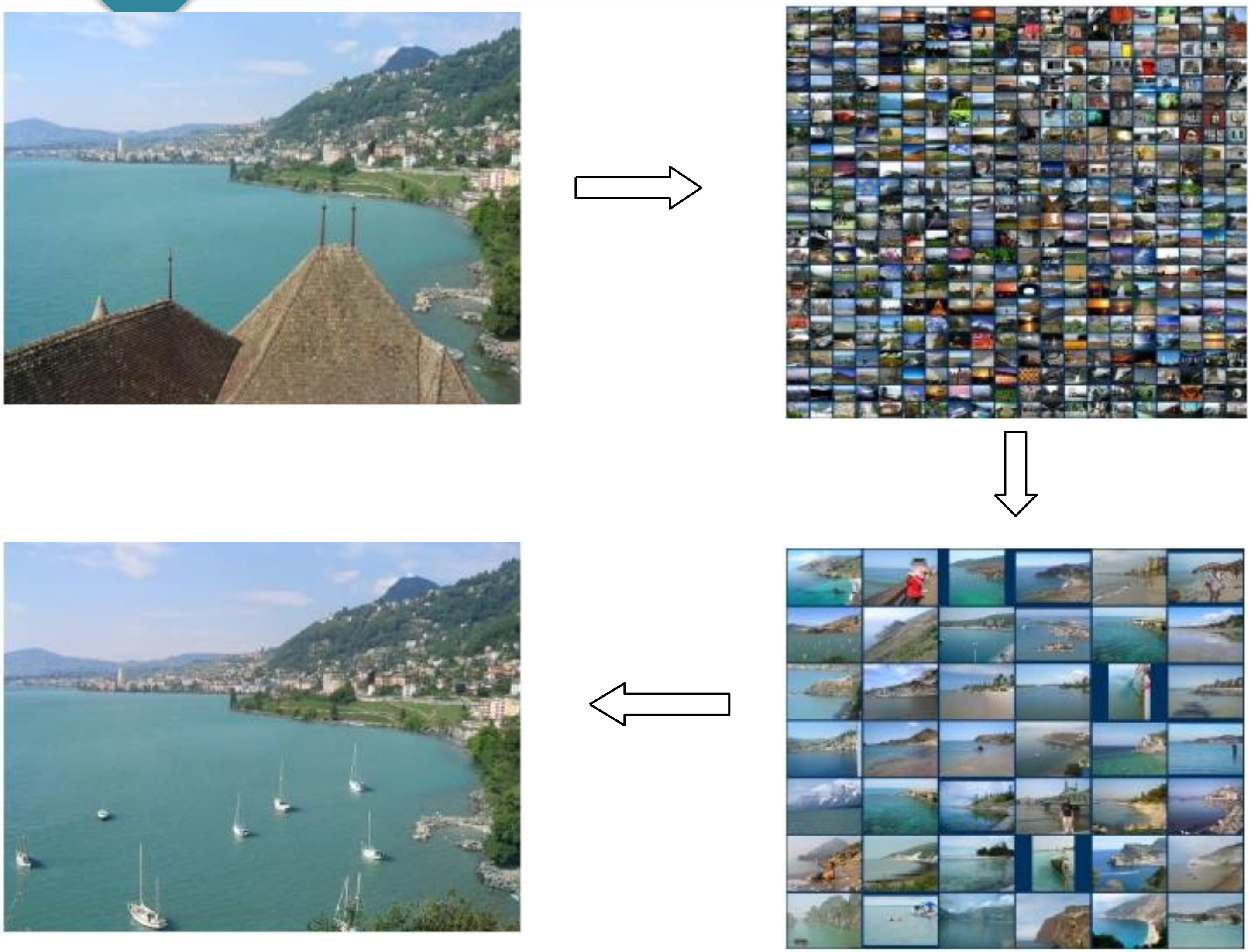
中国科学院大学
University of Chinese Academy of Sciences

Mining Massive datasets
Finding Similar Items:
Locality Sensitive Hashing

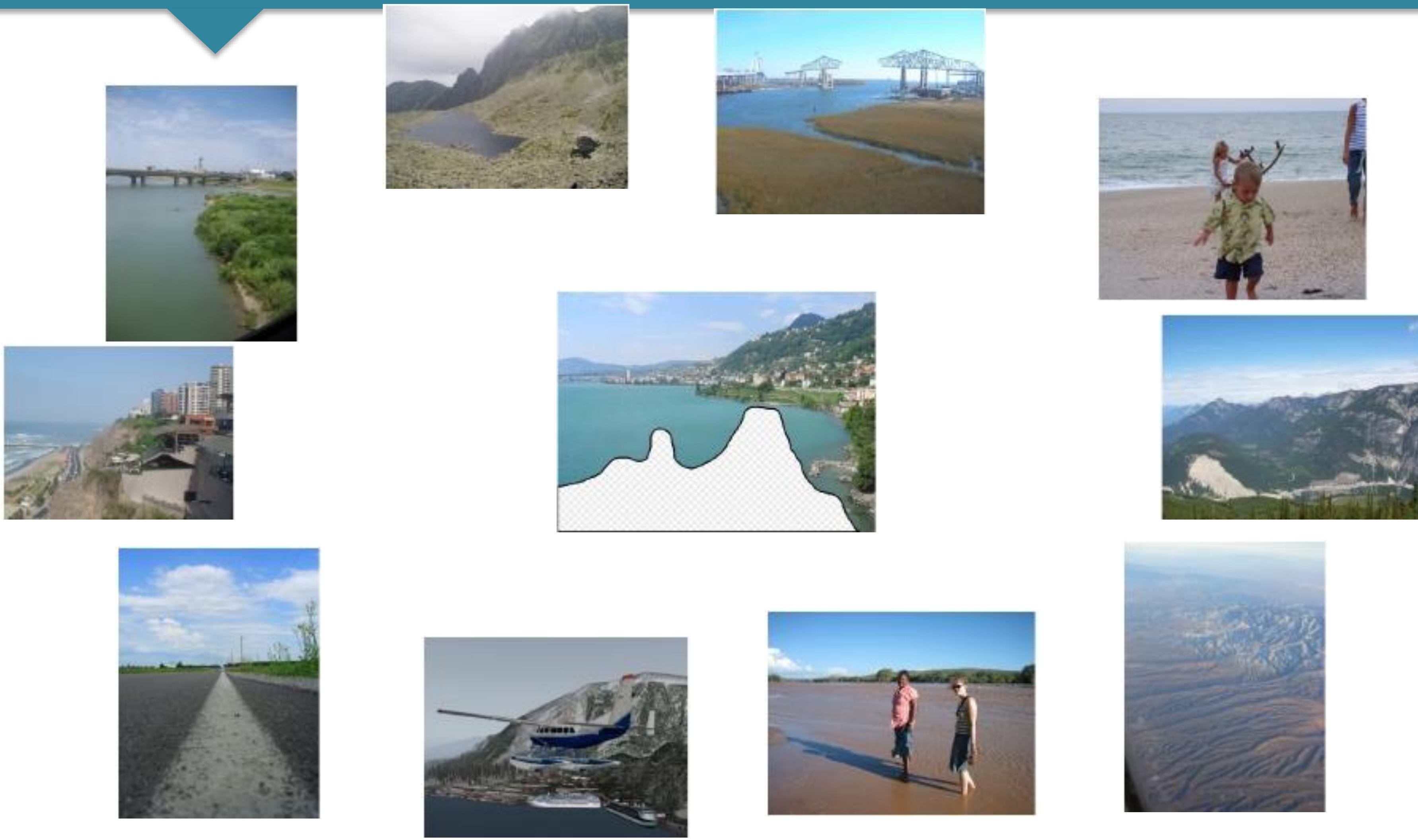
Yi Sun, UCAS 2017

Outline

Scene Completion Problem

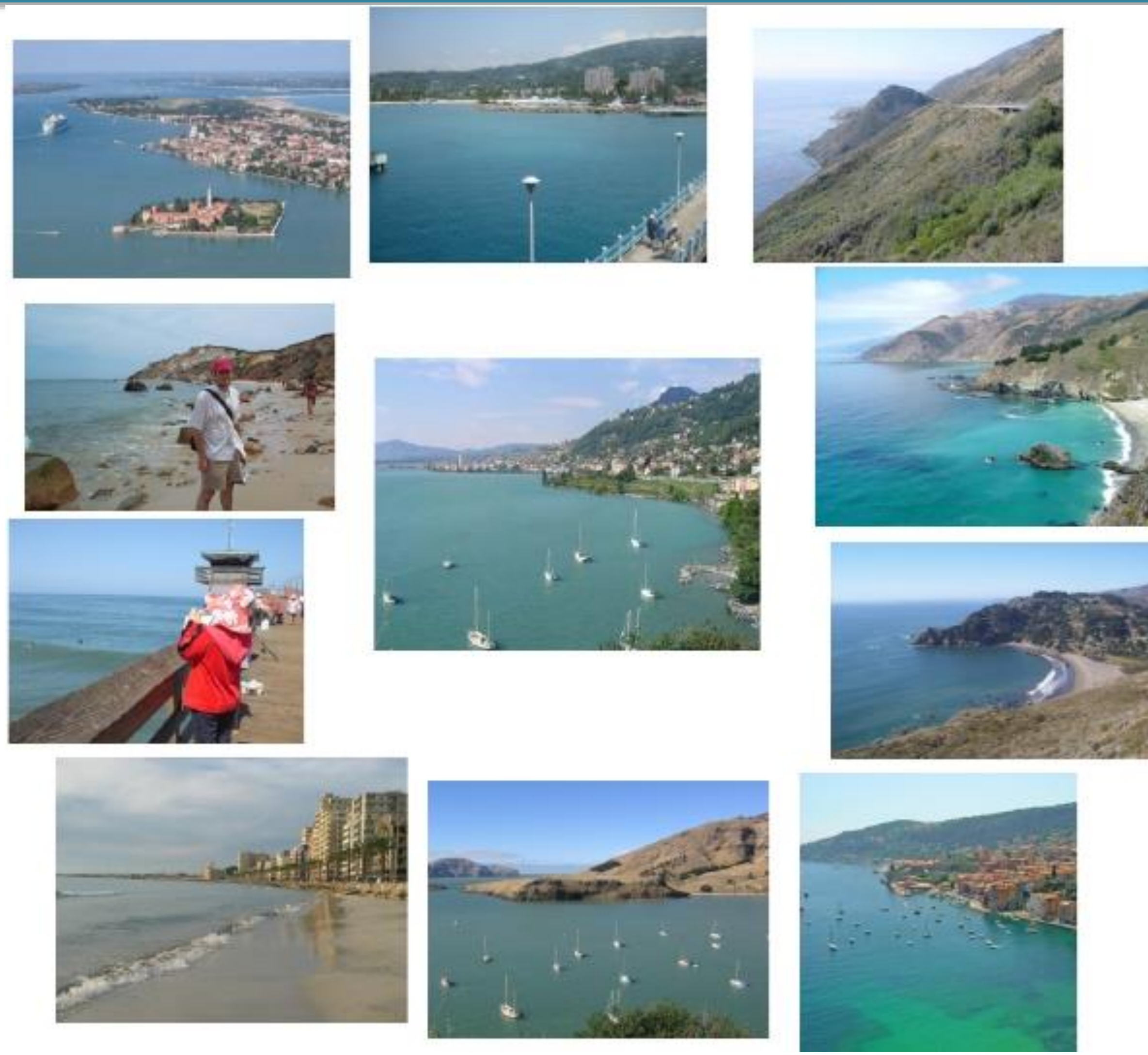


Scene Completion Problem



10 nearestneighbors from a collection of 20,000 images

Scene Completion Problem



10 nearestneighbors from a collection of 2 million images

Motivation

- Finding similar documents/webpages/images
  - (Approximate) mirror sites.

    Application: Don't want to show both when Google.
  - Plagiarism, large quotations

    Application: I am sure you know
  - Similar topic/articles from various places

    Application: Cluster articles by \same story".
  - Google image

- Social network analysis
  - Finding NetFlix users with similar tastes in movies.
  - e.g., for personalized recommendation systems.

A Common Metaphor

- Many problems can be expressed as finding "similar" sets:
  - Find near-neighbors in high-dim space
- Example
  - **Pages with similar words**
    - For duplicate detection, classification by topic
  - **Customers who purchased similar products**
    - Products with similar customer sets
  - **Images with similar features**
    - Users who visited similar websites

How can we compare similarity?

Convert the data (homework, webpages, images) into an object in an **abstract space** that we know how to measure distance, and how to do it efficiently.

What abstract space?

For example, the Euclidean space over R^d

L^1 space, L^∞ space, …

Problem for today's lecture

- Given: High dimensional data points x_1, x_2
  - For example: Image is a long vector of pixel colors
  - $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow [1\ 2\ 1\ 0\ 2\ 1\ 0\ 1\ 0]$
- And some distance function
  - Which quantifies the "distance" between x_1 and x_2
- Goal: Find all pairs of data points (x_i, x_j) that are within some distance threshold $d(x_i, x_j) \leq s$
- Note: Naïve solution would take $O(N^2)$ where N is the number of data points
- MAGIC: This can be done in O(N)!! How?

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3.0 Motivation

3.1 Finding Similar Items
- 3.1.1 Shingling
- 3.1.2 Min-Hashing
- 3.1.3 Locality-sensitive Hashing

3.2 Theory of LSH

3.3 Amplifying Hash Functions: AND and OR

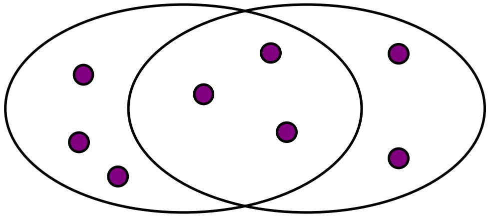
3.4 LSH for other distance metrics

Distance Measures

- Goal: Find near-neighbors in high-dim. space
  - We formally define "near neighbors" as points that are a "small distance" apart
- For each application, we first need to define what "distance" means
- Today: Jaccard distance/similarity
  - The Jaccard similarity of two sets is the size of their intersection divided by the size of their union:

$$sim(C1, C2) = |C1 \cap C2|/|C1 \cup C2|$$

  - Jaccard distance: $d(C1, C2) = 1 - C1 \cap C2|/|C1 \cup C2|$



3 in intersection
8 in union
Jaccard similarity= 3/8
Jaccard distance = 5/8

Jaccard similarity with clusters

- Consider two sets A = {0,1,2,5,6}, B = {0,2,3,5,7,9}. What is the Jaccard similarity of A and B?

- With clusters: We may have some items which basically represent the same thing. We group objects to clusters. E.g.,

C1= {0,1,2}; C2={3,4}; C3={5,6}; C4={7,8,9}

For instance: C1 may represent action movies, C2 may represent comedies, C3 may represent documentaries, C4 may represent horror movies.

Now we can represent A_{clu} = {C1, C3}, B_{clu} = {C1, C2, C3, C4}

JS_{clu}(A; B) = JS(Aclu; Bclu) = |{C1, C2}|/|{C1, C2, C3, C4}|=0.5

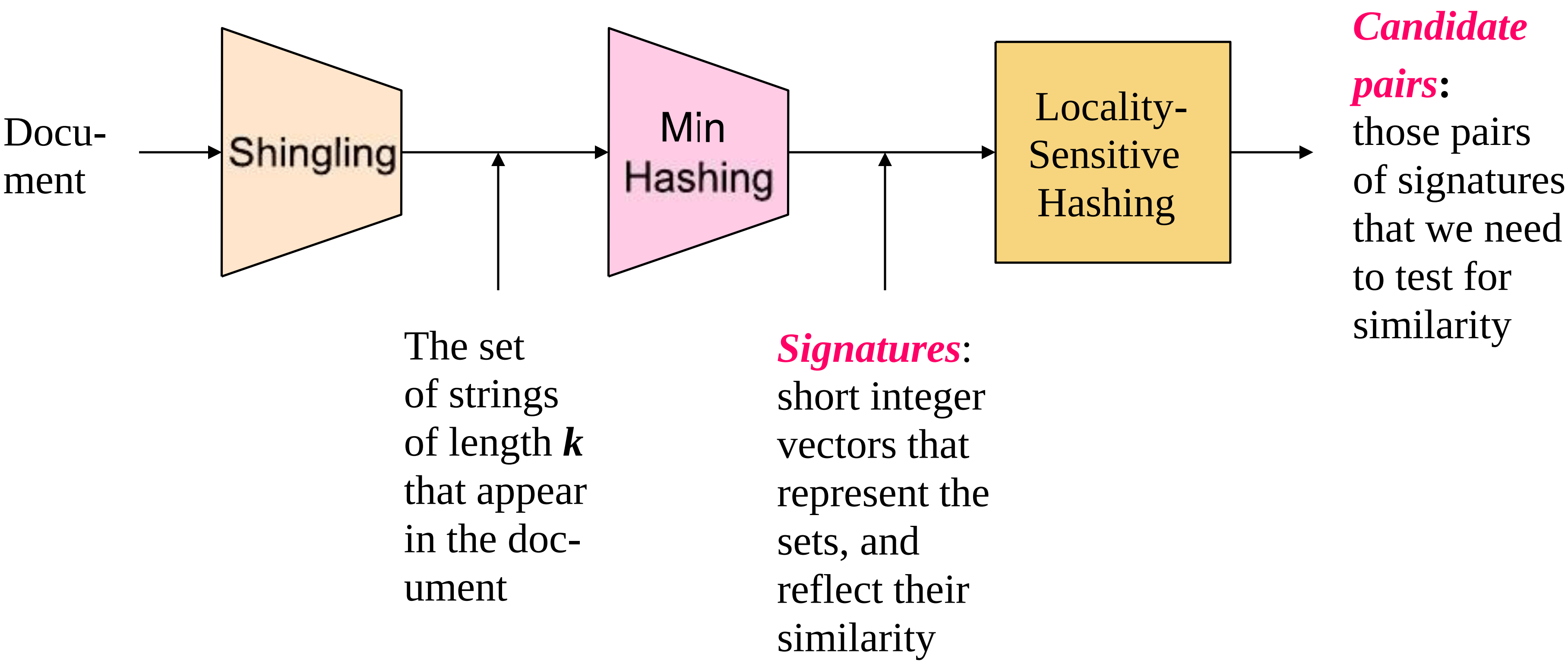
Finding Similar Documents

- Goal: Given a large number (N in the millions or billions) of documents, find "near duplicate" pairs
- Applications:
  - Mirror websites, or approximate mirrors
    - Don't want to show both in search results
  - Similar news articles at many news sites
    - Cluster articles by "same story"
- Problems:
  - Many small pieces of one document can appear out of order in another
  - Too many documents to compare all pairs
  - Documents are so large or so many that they cannot fit in main memory

3 Essential Steps for Similar Docs

1. **Shingling**: Convert documents to sets
2. **Min-Hashing**: Convert large sets to short signatures, while preserving similarity
3. **Locality-Sensitive Hashing**: Focus on pairs of signatures likely to be from similar documents
   - ***Candidate pairs!***

The Big Picture



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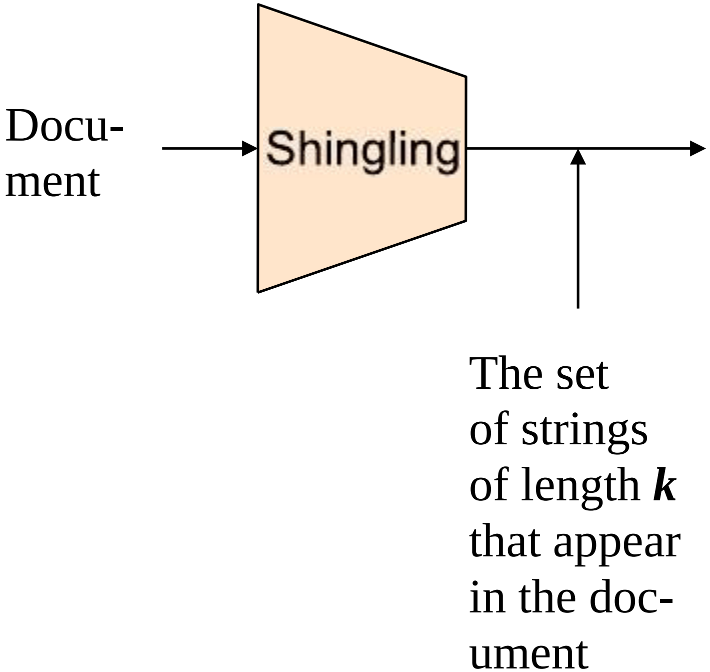
Shingling



Step 1: Shingling:

Convert documents to sets

Documents as High-Dim Data

- Step 1: Shingling: Convert documents to sets

- Simple approaches:
  - Document = set of words appearing in document
  - Document = set of "important" words
  - Don't work well for this application. Why?

- Need to account for ordering of words!
- A different way: Shingles!

Define: Shingles

- A k-shingle (or k-gram) for a document is a sequence of k tokens that appears in the doc
  - Tokens can be characters, words or something else, depending on the application
  - Assume tokens = characters for examples (ignoring back-space)

- Example: **k=2**; document **D1**= abcab
- Set of 2-shingles: **S(D1)** = {ab, bc, ca}
  - Option: Shingles as a bag (multiset), count ab twice: **S'(D1)** = {ab, bc, ca, ab}

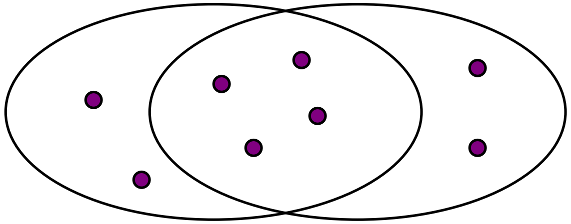
Compressing Shingles

- To compress long shingles, we can hash them to (say) 4 bytes
- Represent a document by the set of hash values of its k-shingles
  - Idea: Two documents could (rarely) appear to have shingles in common, when in fact only the hash-values were shared
- Example: k=2; document D1= abcab
- Set of 2-shingles: S(D1) = {ab, bc, ca}
- Hash the singles: h(D1) = {1, 5, 7}

Similarity Metric for Shingles

- Document D1 is a set of its k-shingles C1=S(D1)
- Equivalently, each document is a 0/1 vector in the space of k-shingles
  - Each unique shingle is a dimension
  - Vectors are very sparse
- **A natural similarity measure is the** Jaccard similarity:

$$\text{sim}(D1, D2) = |C1 \cap C2| / |C1 \cup C2|$$



Working Assumption

- **Documents that have lots of shingles in common have similar text, even if the text appears in different order**

- Careful: You must pick **k** large enough, or most documents will have most shingles
  - **k** = 5 is OK for short documents (e.g. e-mails and short essays)
  - **k** = 10 is better for long documents (e.g. novels and papers)

Motivation for Minhash/LSH

- Suppose we need to find near-duplicate documents among N = 1 million documents
- Naïvely, we would have to compute pairwise Jaccard similarities for every pair of docs
  - $N(N-1)/2 \approx 5*10^{11}$ comparisons
  - At 10^5 secs/day and 10^6 comparisons/sec it would take 5 days
- For N = 10 million, it takes more than a year…

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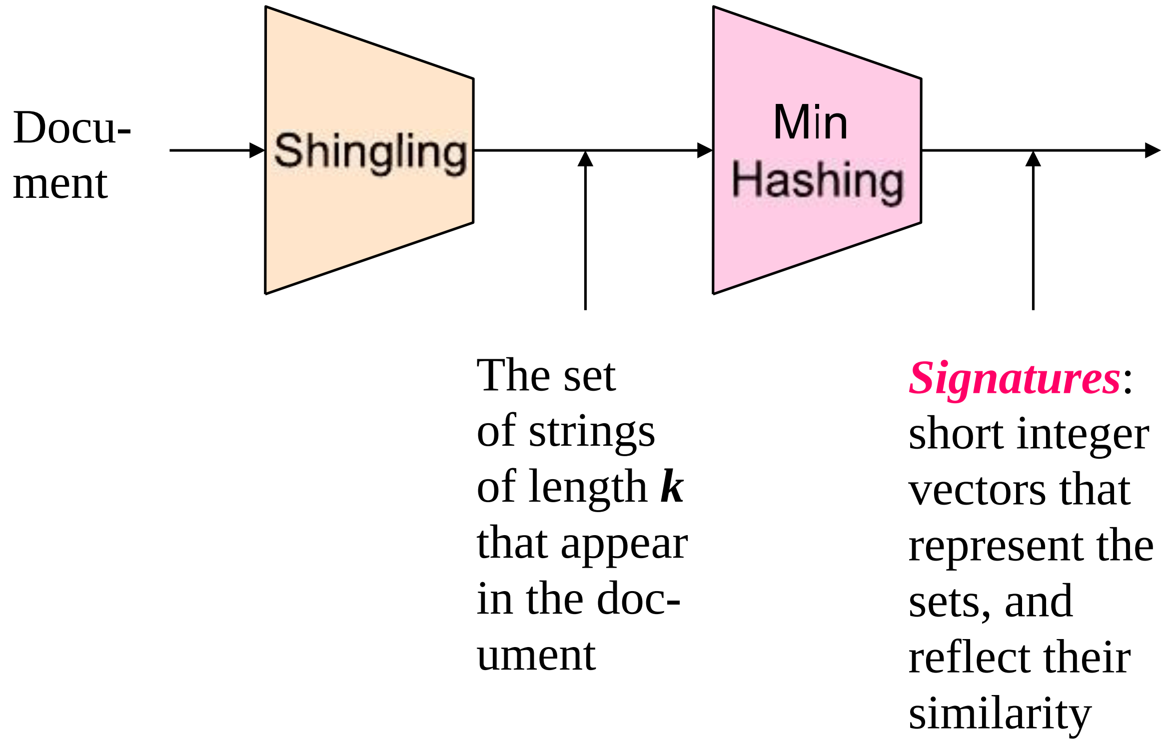
Min Hashing



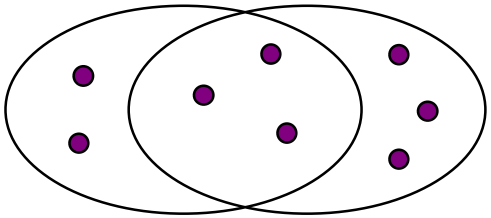
Step 2: Min-Hashing: Convert large sets to short signatures, while preserving similarity

Encoding Sets as Bit Vectors

- Many similarity problems can be formalized as **finding subsets that have significant intersection**

- Encode sets using 0/1 (bit, boolean) vectors
  - One dimension per element in the universal set

- Interpret set intersection as bitwise AND, and set union as bitwise OR



- **Example**: $C1 = 10111$; $C2 = 10011$
  - Size of intersection = 3; size of union = 4,
  - Jaccard similarity (not distance) = 3/4
  - Distance: d(C1,C2) = 1 – (Jaccard similarity) = 1/4

From Sets to Boolean Matrices

- Rows = elements (shingles)
- Columns = sets (documents)
  - 1 in row e and column s if and only if e is a member of s
  - Column similarity is the Jaccard similarity of the corresponding sets (rows with value 1)
  - Typical matrix is sparse!
- Each document is a column:
  - Example: sim(C1 ,C2) = ?
    - Size of intersection = 3; size of union = 6,
    - Jaccard similarity (not distance) = 3/6
    - d(C1,C2) = 1 – (Jaccard similarity) = 3/6

Documents

Shingles

1	1	1	0
1	1	0	1
0	1	0	1
0	0	0	1
1	0	0	1
1	1	1	0
1	0	1	0

Outline: Finding Similar Columns

- So far:
  - Documents → Sets of shingles
  - Represent sets as boolean vectors in a matrix
- Next goal: Find similar columns while computing small signatures
  - Similarity of columns == similarity of signatures

Outline: Finding Similar Columns

- Next Goal: Find similar columns, Small signatures
- Naïve approach:
  - 1) Signatures of columns: small summaries of columns
  - 2) Examine pairs of signatures to find similar columns
    - Essential: Similarities of signatures and columns are related
  - 3) Optional: Check that columns with similar signatures are really similar
- Warnings:
  - Comparing all pairs may take too much time: Job for LSH
    - These methods can produce false negatives, and even false positives
    - Optional check can cancel false positives

Hashing Columns (Signatures)

- Key idea: "hash" each column C to a small signature $h(C)$, such that:
  - (1) $h(C)$ is small enough that the signature fits in RAM
  - (2) $sim(C1, C2)$ is the same as the "similarity" of signatures $h(C1)$ and $h(C2)$

- Goal: Find a hash function h(·) such that:
  - If $sim(C1,C2)$ is high, then with high prob. $h(C1) = h(C2)$
  - If $sim(C1,C2)$ is low, then with high prob. $h(C1) \neq h(C2)$

- Hash docs into buckets. Expect that "most" pairs of near duplicate docs hash into the same bucket!

Min-Hashing

- Goal: Find a hash function $h(\cdot)$ such that:
  - if $sim(C1,C2)$ is high, then with high prob. $h(C1) = h(C2)$
  - if $sim(C1,C2)$ is low, then with high prob. $h(C1) \neq h(C2)$

- Clearly, the hash function depends on the similarity metric:
  - Not all similarity metrics have a suitable hash function

- **There is a suitable hash function for the Jaccard similarity:** It is called Min-Hashing

Min-Hashing

- Imagine the rows of the boolean matrix permuted under random permutation π

- Define a "hash" function $h_\pi(C)$ = the index of the first (in the permuted order π) row in which column C has value 1:

$$h_\pi(C) = \min \pi(C)$$

- Use several (e.g., 100) independent hash functions (that is, permutations) to create a signature of a column

Min-Hashing Example (one element)

$$h_{\pi}(C) = min\pi\ (C)$$

$$h_{\pi1}(C1) = \begin{bmatrix} 2 \\ 3 \\ 7 \\ 6 \\ 1 \\ 5 \\ 4 \end{bmatrix} .* \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 5 \\ 4 \end{bmatrix} \xrightarrow[\text{Ignore '0'}]{\text{Min Item}} 2$$

$\pi1 \quad .* \quad C1$

Min-Hashing Example

2nd element of the permutation is the first to map to a 1

Permutation π

π1	π2	π3
2	4	3
3	2	4
7	1	7
6	3	2
1	6	6
5	7	1
4	5	5

Input matrix (Shingles x Documents)

C1	C2	C3	C4
1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature matrix *M*

Sig1	Sig2	Sig3	Sig4
2	1	2	1
2	1	4	1
1	2	1	2

4th element of the permutation is the first to map to a 1

$$M_{i,j} = h_{\pi_i}(C_j)$$

The Min-Hash Property

- Choose a random permutation π
- Claim: $\Pr[h\pi(C1) = h\pi(C2)] = sim(C1, C2)$
- Why?
  - Let X be a doc (set of shingles), $y \in X$ is a shingle
  - Then: $\Pr[\pi(y) = \min(\pi(X))] = 1/|X|$
    - It is equally likely that any $y \in X$ is mapped to the min element
- Let y be s.t. $\pi(y) = \min(\pi(C1 \cup C2))$
- Then either: $\pi(y) = \min(\pi(C1))$ if $y \in C1$, or
  $\pi(y) = \min(\pi(C2))$ if $y \in C2$

One of the two cols had to have 1 at position ***y***

- So the prob. that **both** are true is the prob. $y \in C1 \cap C2$
- **$\Pr[\min(\pi(C1))=\min(\pi(C2))]$**

$= |C1 \cap C2| / |C1 \cup C2| =$ *$sim(C1, C2)$*

Similarity for Signatures

- We know: $\Pr[h_\pi(C1) = h_\pi(C2)] = sim(C1, C2)$
- Now generalize to multiple hash functions
- The similarity of two signatures is the fraction of the hash functions in which they agree
- Note: Because of the Min-Hash property, the similarity of columns is the same as the expected similarity of their signatures

Hence, sim(C1, C2) = sim(Sig1,Sig2) when the number of permutation function is large enough(e.g. 100)

Min-Hashing Example

Permutation □

2	4	3
3	2	4
7	1	7
6	3	2
1	6	6
5	7	1
4	5	5

Input matrix (Shingles x Documents)

1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature matrix *M*

2	1	2	1
2	1	4	1
1	2	1	2

Similarities:

	1-3	2-4	1-2	3-4
Col/Col	0.75	0.75	0	0
Sig/Sig	0.67	1.00	0	0

We achieved our goal! We “compressed” long bit vectors into short signatures

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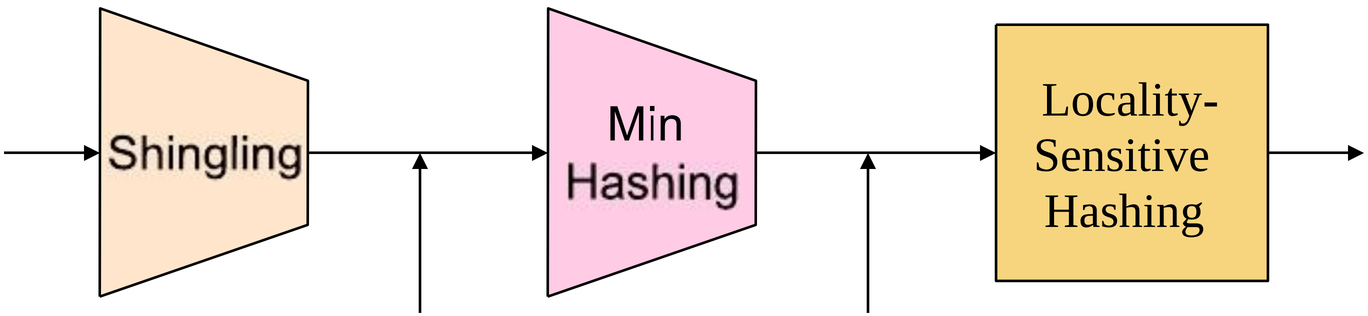
Locality Sensitive Hashing

Docu-
ment



Candidate pairs: those pairs of signatures that we need to test for similarity

The set of strings of length ***k*** that appear in the doc-ument

Signatures: short integer vectors that represent the sets, and reflect their similarity

Step 3: Locality-Sensitive Hashing:

Focus on pairs of signatures likely to be from similar documents

LSH: First Cut

- Goal: Find documents with Jaccard similarity at least s (for some similarity threshold, e.g., s=0.8)

- LSH – General idea: Use a function *f(x,y)* that tells whether x and y is a candidate pair: a pair of elements whose similarity must be evaluated

- For Min-Hash matrices:
  - Hash columns of signature matrix M to many buckets
  - Each pair of documents that hashes into the same bucket is a candidate pair

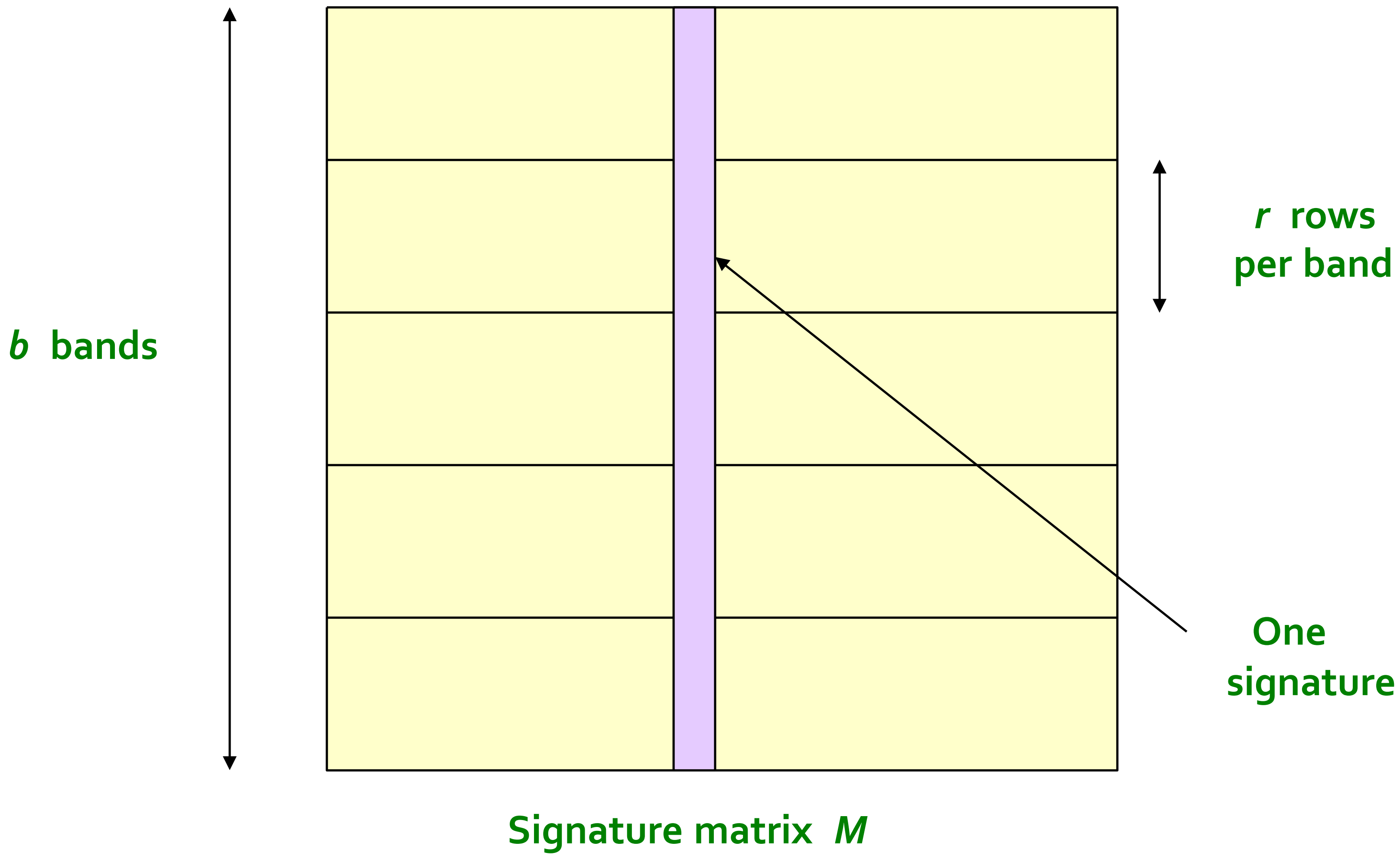
Candidates from Min-Hash

- Pick a similarity threshold s ($0 < s < 1$)
- Columns x and y of M are a candidate pair if their signatures agree on at least fraction s of their rows:
- $M(i, x) = M(i, y)$ for at least frac. s values of i
  - We expect documents x and y to have the same (Jaccard) similarity as their signatures

LSH for Min-Hash

- Big idea: Hash columns of signature matrix M several times instead of computing all couples of columns
- Arrange that (only) similar columns are likely to hash to the same bucket, with high probability
- Candidate pairs are those that hash to the same bucket

Partition M into b Bands



Partition M into Bands

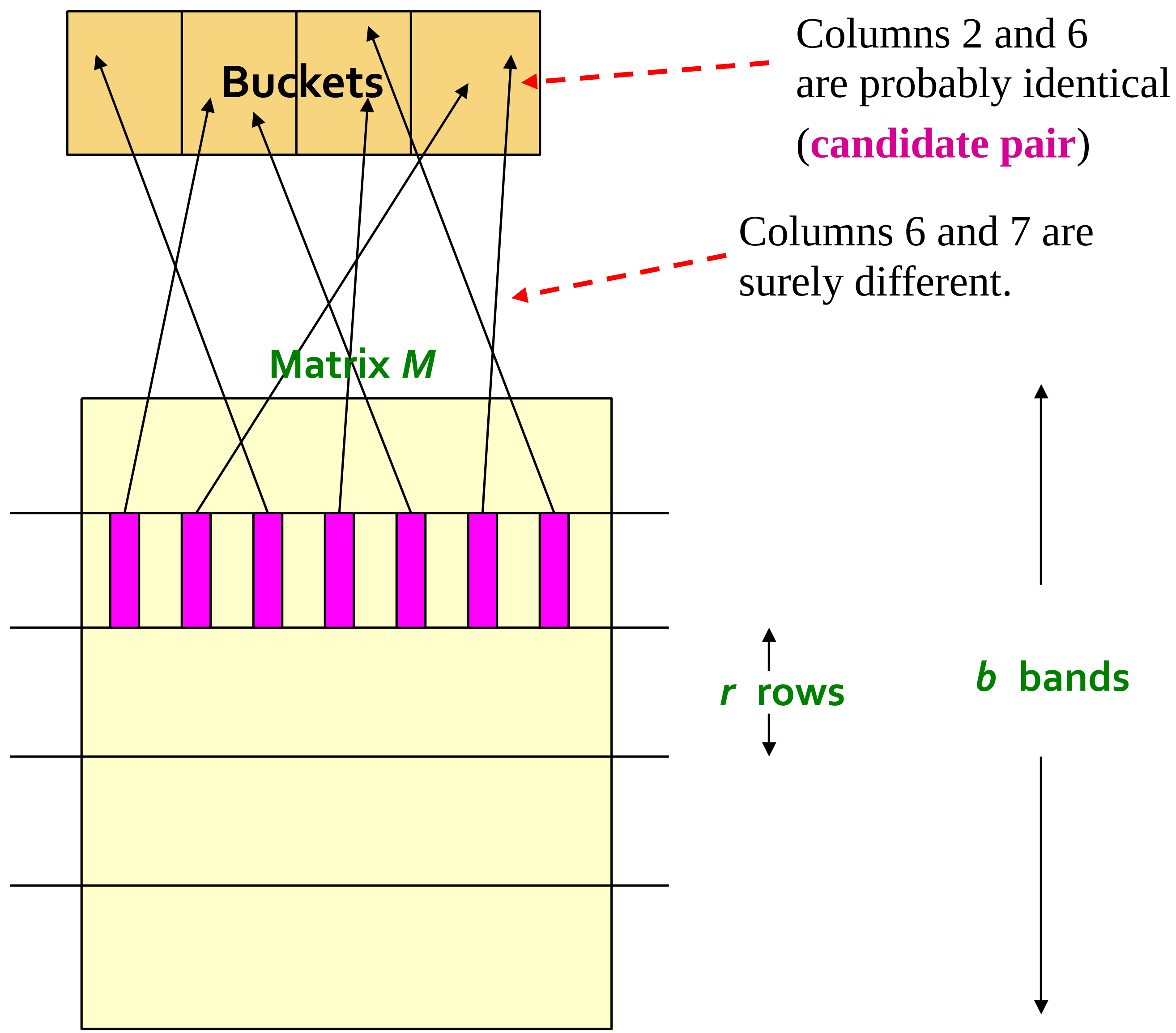
- Divide matrix M into b bands of r rows

- For each band, hash its portion of each column to a hash table with k buckets
  - Make k as large as possible (at least k >= number of documents)

- Candidate column pairs are those that hash to the same bucket for ≥ 1 band

- Tune b and r to catch most similar pairs, but few non-similar pairs

Hashing Bands



Simplifying Assumption

- There are enough buckets that columns are unlikely to hash to the same bucket unless they are identical in a particular band
- Hereafter, we assume that "***same bucket***" means "***identical in that band***"
- Assumption needed only to simplify analysis, not for correctness of algorithm

C1,C2 are 80% Similar

- Find pairs of $\geq$ s = 0.8 similarity, set b =20 and r =5
- Assume: sim(C1,C2) =0.8
  - Since Sim(C1,C2) $\geq$ s, we want C1,C2 to be candidate pair: We want them to hash to at least 1 common bucket (at least one band is identical)
- Probability C1, C2 identical in one particular band: $(0.8)^5$ = 0.328
- Probability C1, C2 are not similar in all of the 20 bands: $(1\text{-}0.328)^{20}$ = 0.00035
  - i.e., about 1/3000th of the 80%-similar column pairs are false negatives (we miss them)
  - We would find 99.965% pairs of truly similar documents

C1,C2 are 30% Similar

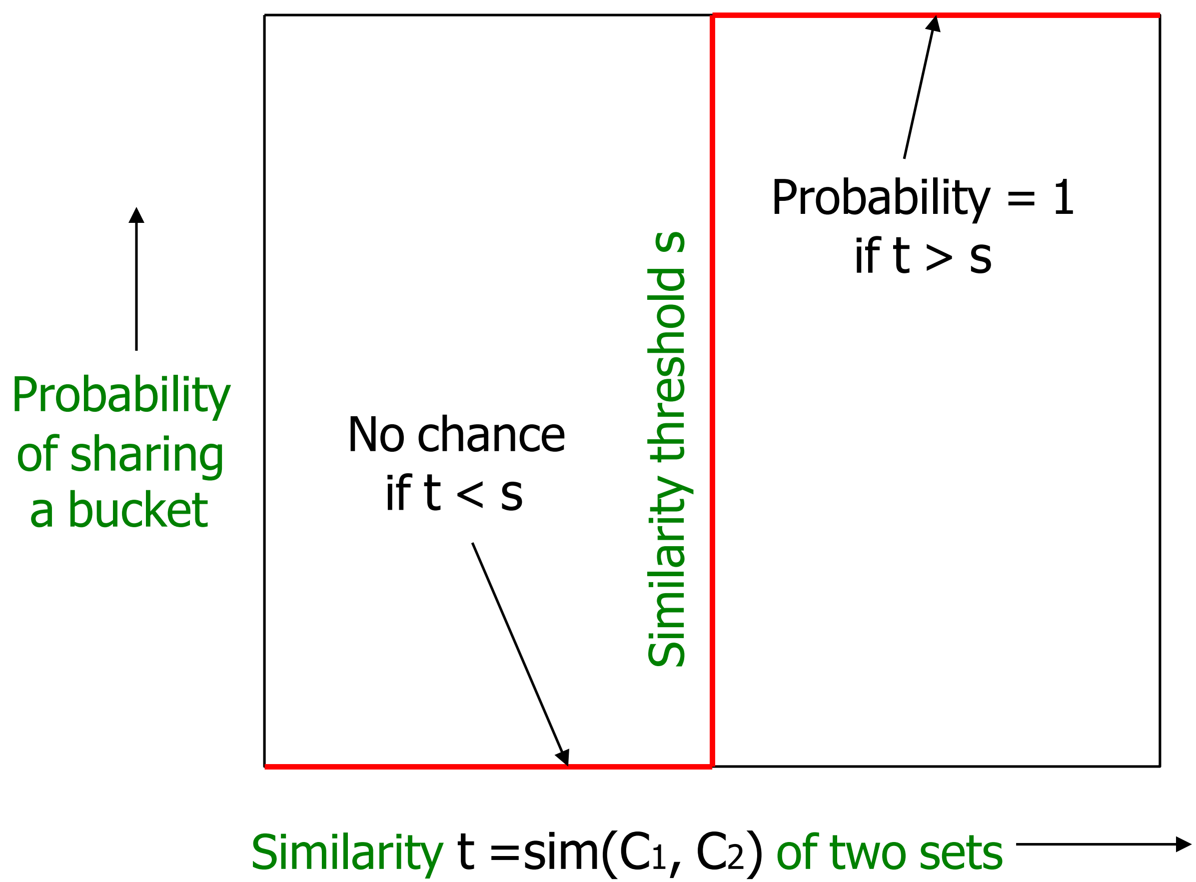
- Find pairs of $\geq s = 0.8$ similarity, set $b = 20$ and $r = 5$
- Assume: $sim(C1, C2) = 0.3$
  - Since $Sim(C1, C2) < s$, we want C1,C2 to be No candidate pair (all bands should be different)
- Probability C1, C2 identical in one particular band: $(0.3)^5 = 0.00243$
- Probability C1, C2 identical in at least 1 of 20 bands: $1-(1-0.00243)^{20} = 0.00474$
  - In other words, approximately 0.474% pairs of docs with similarity 0.3 end up becoming candidate pairs
  - They are false positives since we will have to examine them (they are candidate pairs) but then it will turn out their similarity is below threshold s

b Bands, r rows/band

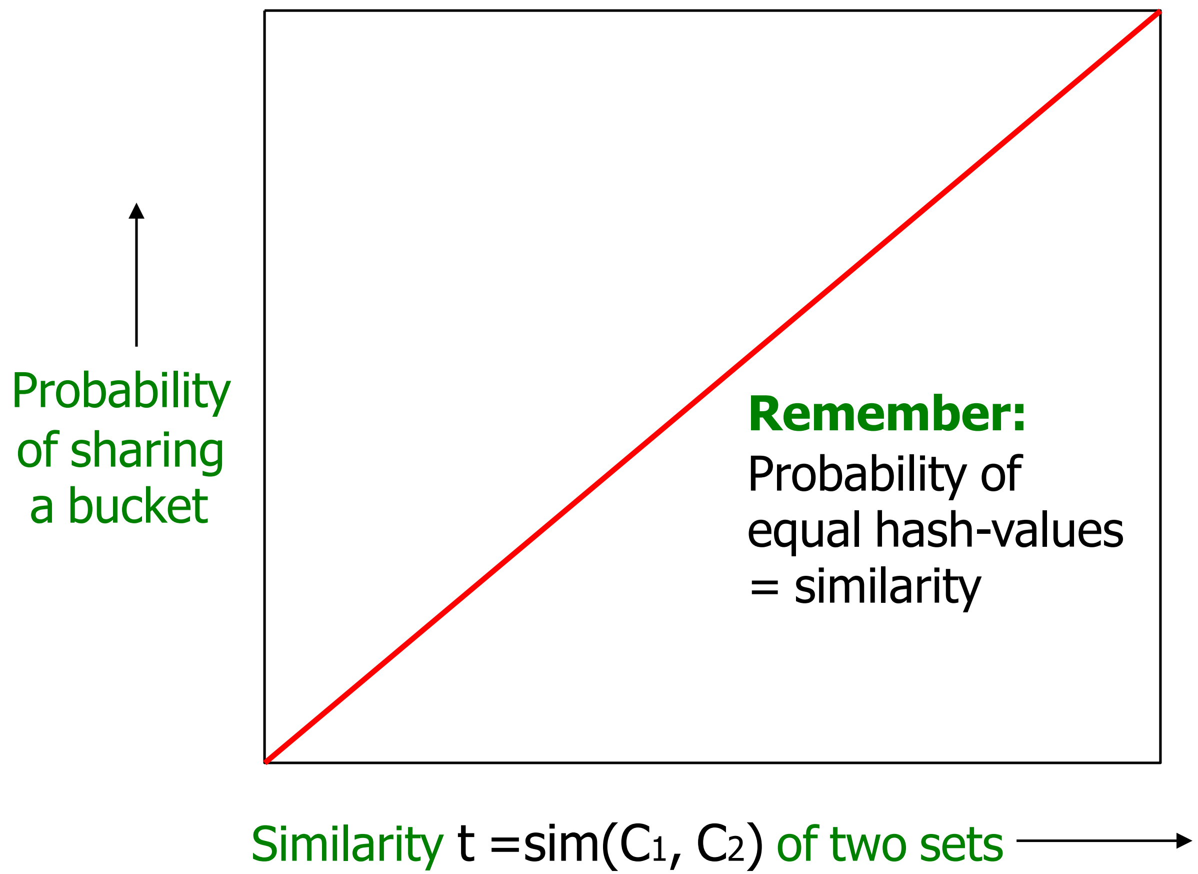
- Columns C_1 and C_2 have similarity t
- Pick any band (r rows)
  - Prob. that all rows in band equal = t^r
  - Prob. that some row in band unequal = $1 - t^r$
- Prob. that no band identical = $(1 - t^r)^b$
- Prob. that at least 1 band identical =

$$1 - (1 - t^r)^b$$

Analysis of LSH – What We Want

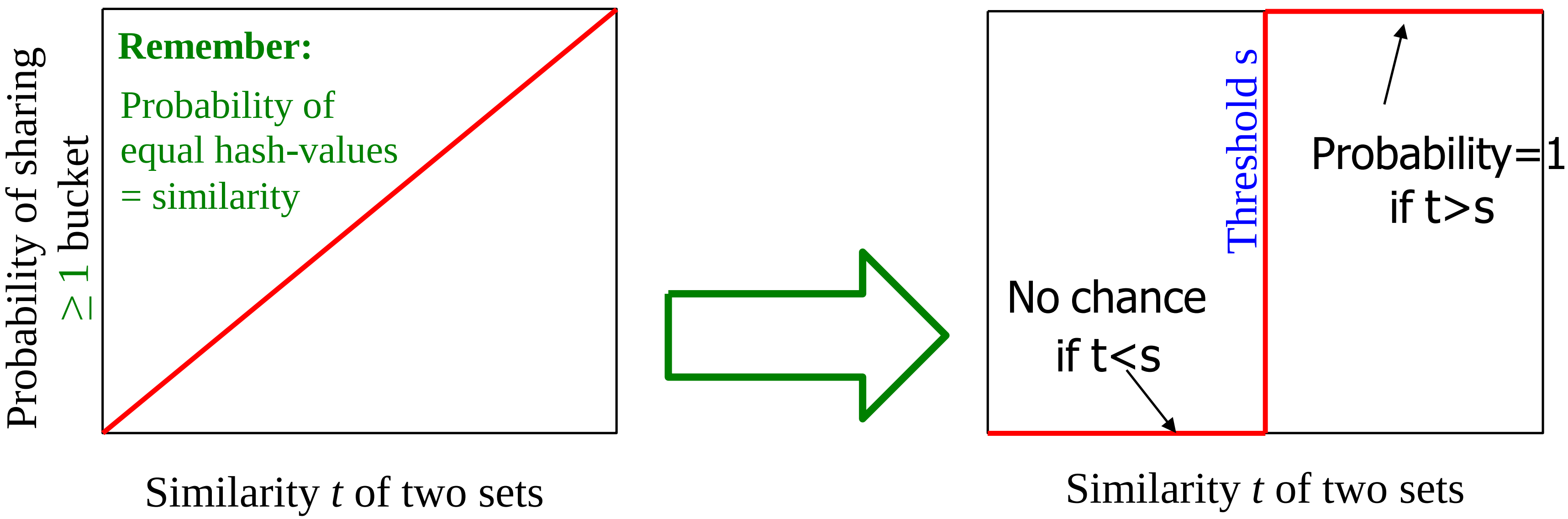


What 1 band of 1 Row Gives You



Recap: The S-Curve

- **The S-curve is where the "magic" happens**



This is what 1 hash-code gives you

$\Pr[h(C_1) = h(C_2)] = sim(D_1, D_2)$

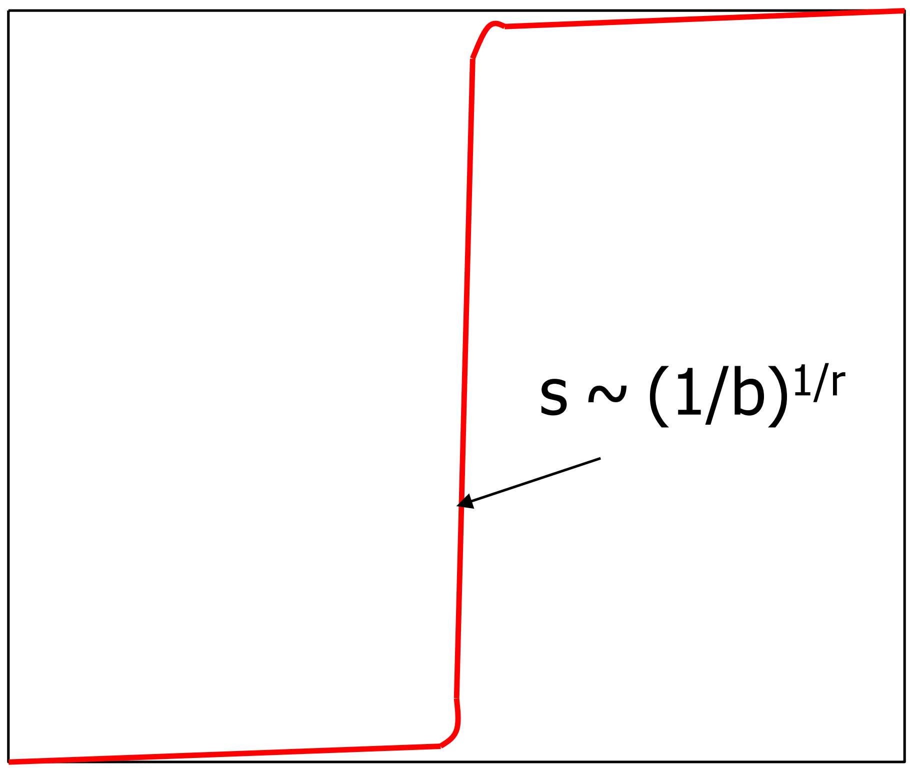
This is what we want!

How to get a step-function?

By choosing *r* and *b*!

What b Bands of r Rows Gives You

Probability of sharing a bucket



Similarity t=sim(C_1, C_2) of two sets

At least one band identical

No bands identical

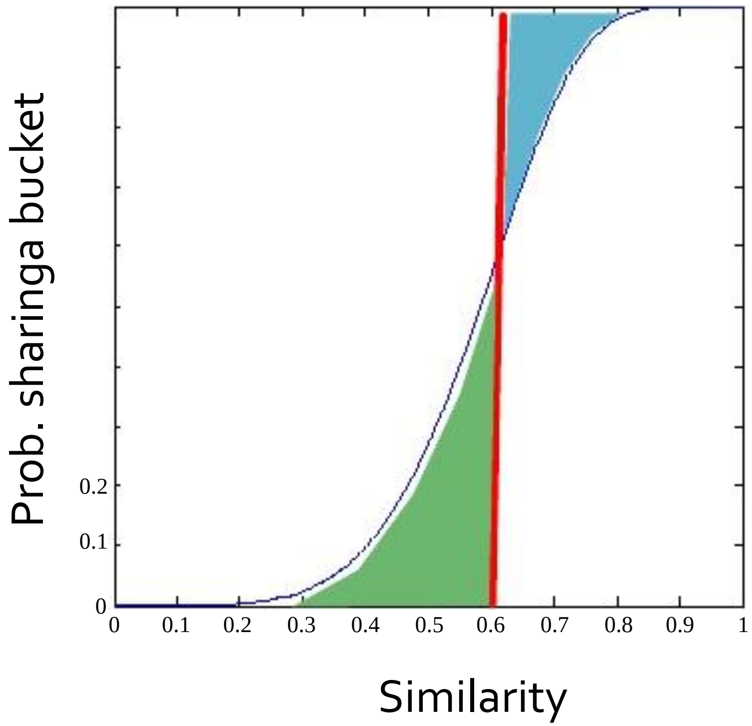
$$1 - (1 - t^r)^b$$

Some row of a band unequal

All rows of a band are equal

Picking r and b: The S-curve

- Picking r and b to get the best S-curve
  - 50 hash-functions (r=5, b=10)



Blue area: False Negative rate
Green area: False Positiverate

Optional Check

- Because of false positive/negative existing, we need the optional check work to solve the false positive
- Check in main memory that candidate pairs really do have similar signatures
- Optional: In another pass through data, check that the remaining candidate pairs really represent similar documents

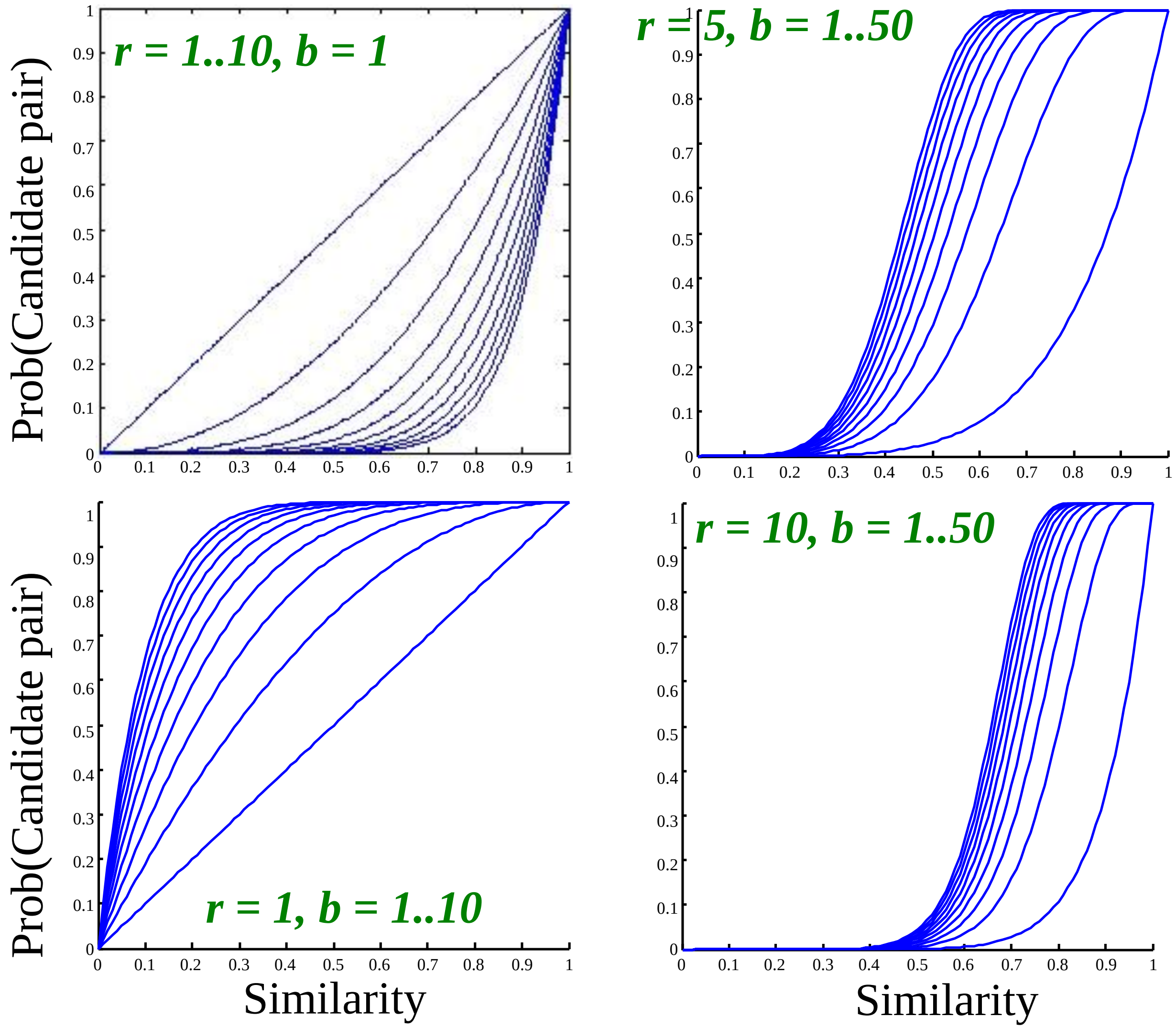
Summary: 3Steps

- Shingling: Convert documents to sets
  - We used hashing to assign each shingle an ID
- Min-Hashing: Convert large sets to short signatures, while preserving similarity
  - We used similarity preserving hashing to generate signatures with property $Pr[h_\pi(C1) = h_\pi(C2)] = sim(C1, C2)$
  - We used hashing to get around generating random permutations
- Locality-Sensitive Hashing: Focus on pairs of signatures likely to be from similar documents
  - We used hashing to find candidate pairs of similarity $\geq s$

S-curves as a func. of *b* and *r*

Given a fixed threshold ***s***.

We want choose ***r*** and ***b*** such that the ***P(Candidate pair)*** has a "step" right around ***s***.



***P(C1,C2 is a candidate pair)* = 1 - (1 - t^r)^b**

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Families of Hash Functions

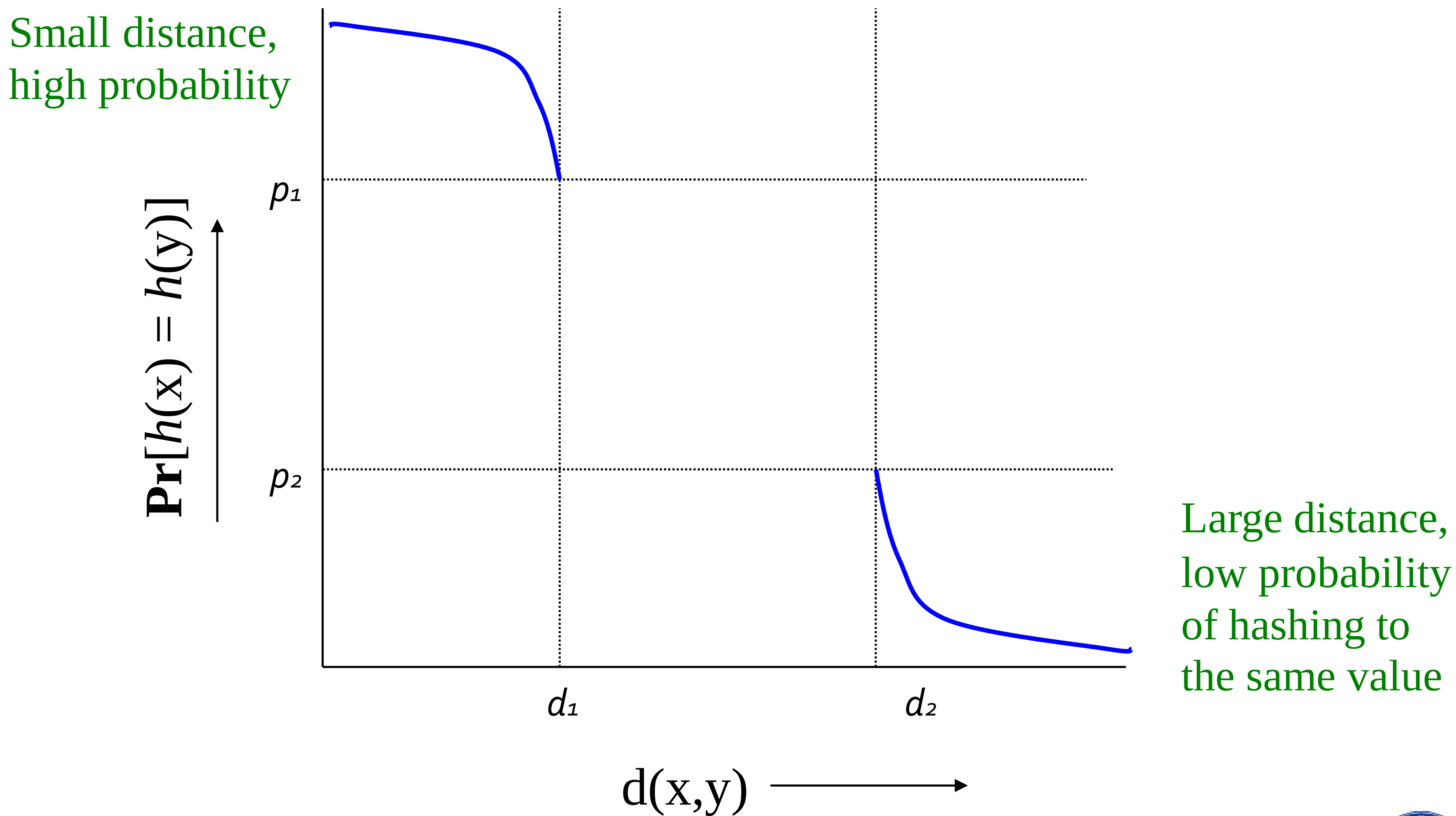
- For Min-Hashing signatures, we got a Min-Hash function for each permutation of rows
- A "hash function" is any function that takes two elements and says whether they are "equal"
  - Example: h(x)=h(y) means "h says x and y are equal"
- A family of hash functions is any set of hash functions from which we can pick one at random efficiently
  - Example: The set of Min-Hash functions generated from permutations of rows

Locality –Sensitive(LS) Families

- Suppose we have a space *S* of points with a distance measure *d(x,y)*

- A family *H* of hash functions is said to be *(d1, d2, p1, p2)-sensitive* if for any *x* and *y* in *S*:
  - 1. If $d(x, y) < d1$, then the probability over all $h \in H$, that $h(x) = h(y)$ is <u>at least</u> *p1*
  - 2. If $d(x, y) > d2$, then the probability over all $h \in H$, that $h(x) = h(y)$ is <u>at most</u> *p2*

A *(d1, d2, p1, p2)-sensitive* function



LS Family of Min-Hash

- For any hash function $h \in H$:

$$\Pr[h(x) = h(y)] = 1 - d(x, y)$$

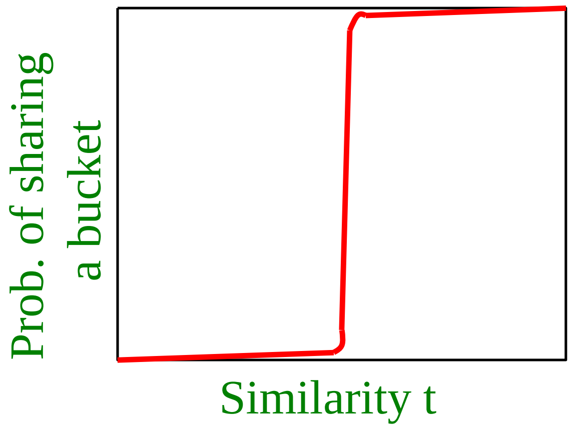
  - Simply restates theorem about Min-Hashing in terms of distances rather than similarities

- For Jaccard similarity, Min-Hashing gives a (d1,d2,(1-d1),(1-d2))-sensitive family for any d1<d2

- Theory leaves unknown what happens to pairs that are at distance between d1 and d2
  - Consequence: No guarantees about fraction of false positives in that range

Amplifying a LS-Family

- Can we reproduce the "S-curve" effect we saw before for any LS family?



- The "bands" technique we learned for signature matrices carries over to this more general setting
  - So we can do LSH with any (d1, d2, p1, p2)-sensitive family
- Two constructions:
  - AND construction like "rows in a band"
  - OR construction like "many bands"

Outline

3.0 Motivation

3.1 Finding Similar Items
- 3.1.1 Shingling
- 3.1.2 Min-Hashing
- 3.1.3 Locality-sensitive Hashing

3.2 Theory of LSH

3.3 Amplifying Hash Functions: AND and OR

3.4 LSH for other distance metrics

AND of Hash Functions

- Given family H, construct family H' consisting of r functions from H
- For $h = [h_1,\ldots,h_r]$ in H', we say $h(x) = h(y)$ if and only if $h_i(x) = h_i(y)$ for all i, $1 \leq i \leq r$
  - Note this corresponds to creating a band of size r
- Theorem: If H is (d_1, d_2, p_1, p_2)-sensitive, then H' is $(d_1, d_2, (p_1)^r, (p_2)^r)$-sensitive
- Proof: Use the fact that h_i's are independent

Subtlety Regarding Independence

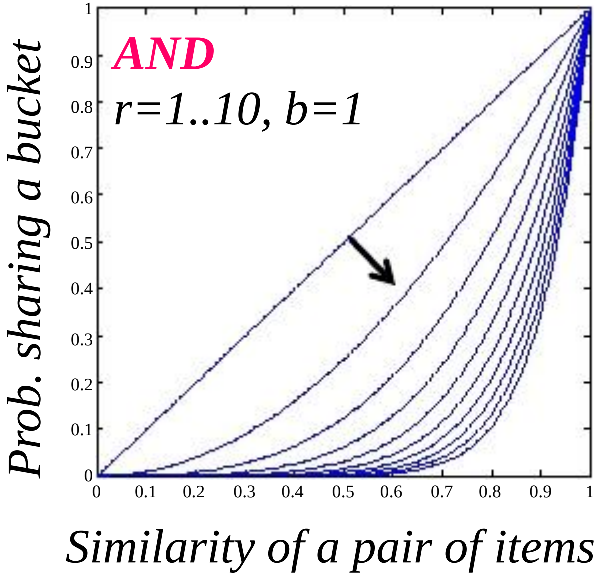
- Independence of hash functions (HFs) really means that the prob. of two HFs saying "yes" is the product of each saying "yes"
  - But two hash functions could be highly correlated
    - For example, in Min-Hash if their permutations agree in the first one million entries
  - However, the probabilities in definition of a LSH-family are over all possible members of H, H'

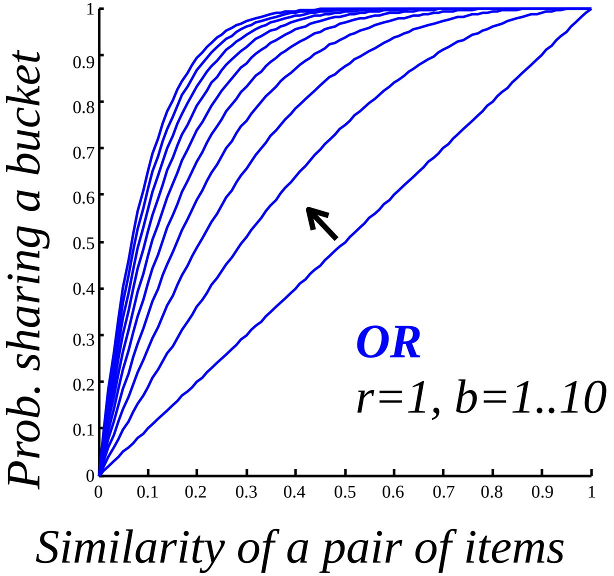
OR of Hash Functions

- Given family H, construct family H' consisting of b functions from H
- For $h = [h_1,\ldots,h_b]$ in H', we say $h(x) = h(y)$ if and only if $h_i(x) = h_i(y)$ for at least 1 i
- Theorem: If H is *(d_1, d_2, p_1, p_2)-sensitive*, then H' *is* $(d_1, d_2, 1-(1-p_1)^b, 1-(1-p_2)^b)$-sensitive
- Proof: Use the fact that h_i's are independent

Effect of AND and OR Constructions

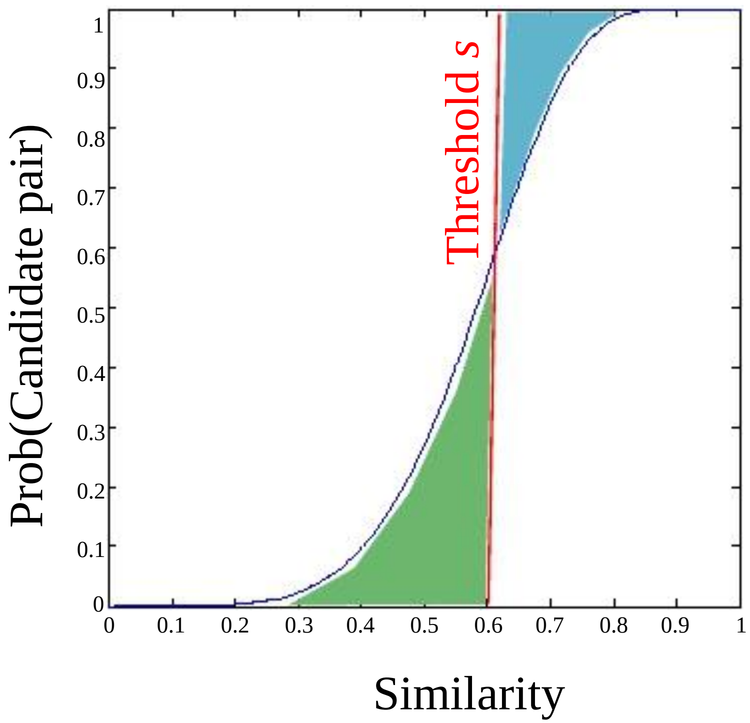
- AND makes all probs. shrink, but by choosing r correctly, we can make the lower prob. approach 0 while the higher does not
- OR makes all probs. grow, but by choosing b correctly, we can make the upper prob. approach 1 while the lower does not

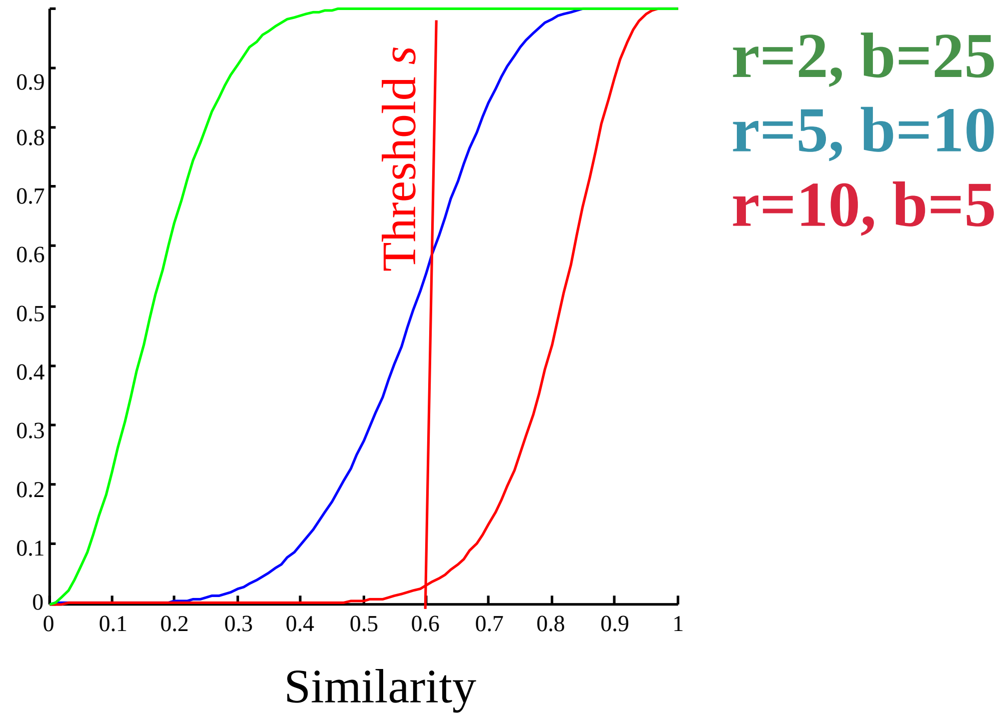




Picking r and b: the S-curve

- Picking r and b to get desired performance
  - 1. 50 hash-functions (r = 5, b = 10)
  - 2. 50 hash-functions (r * b = 50)



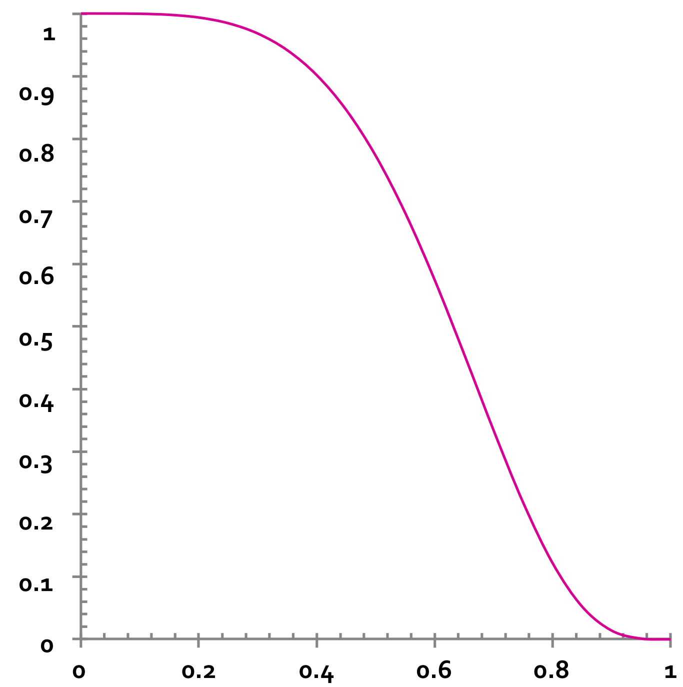


Composing Constructions

- Exactly what we did with Min-Hashing
  - If bands match in all r values hash to same bucket
  - Cols that are hashed into ≥ 1 common bucket -> Candidate
- Take points x and y s.t. Pr[h(x) = h(y)] = p
  - H will make (x,y) a candidate pair with prob. P
- Construction makes (x,y) a candidate pair with probability $1-(1-p^r)^b$
  - The S-Curve!
  - Example: Take H and construct H' by the AND construction with *r = 4*. Then, from H', construct H'' by the OR construction with *b = 4*

Table for Function $1-(1-p^4)^4$

p	$1-(1-p^4)^4$
.1	.0140
.2	.1215
.3	.3334
.4	.5740
.5	.7725
.6	.9015
.7	.9680
.8	.9936



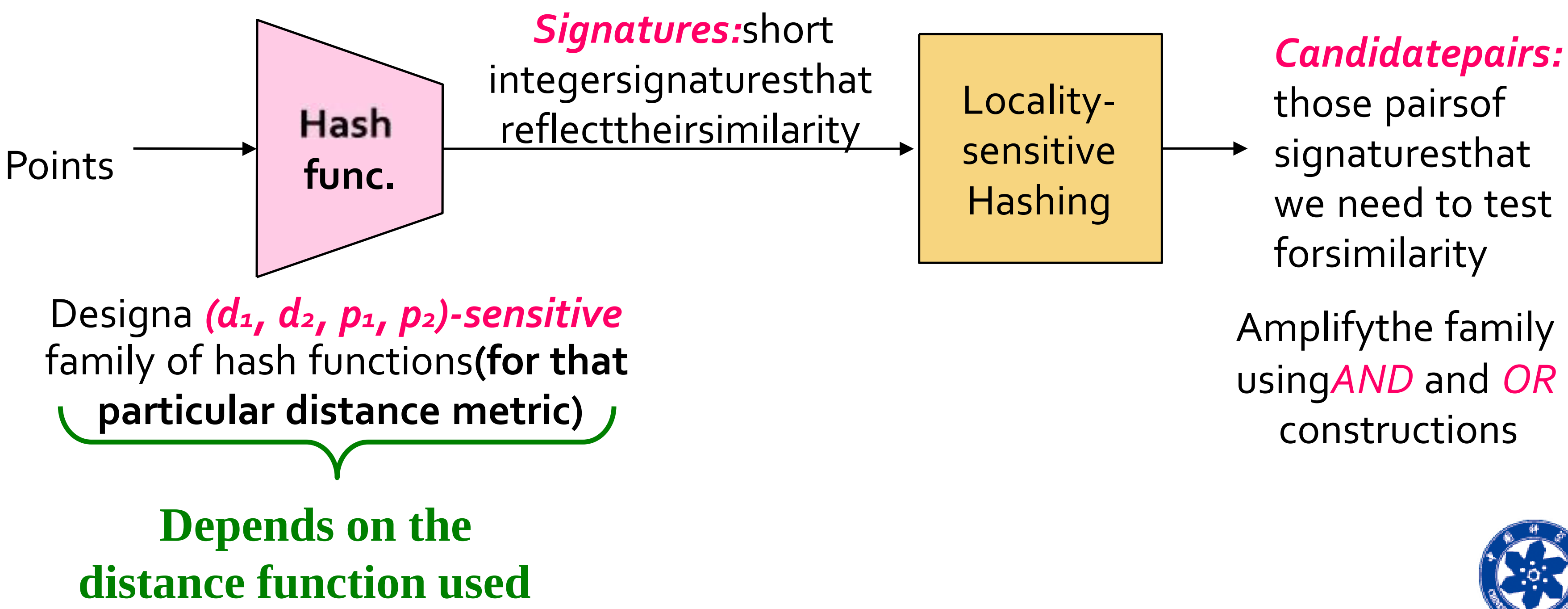
The example transforms a (.2,.8,.8,.2)-sensitive family into a (.2,.8,.9936,.1215)-sensitive family

Cascading Constructions

- Transforms a (.2, .8, .8, .2)-sensitive family into a (.2, .8, . 9936, . 1215)-sensitive family
  - Note this family uses 256 (=4*4*4*4) of the original hash functions
  - Note by using 256 hash functions, the (.2, .8, .8, .2)-sensitive family was transformed into a (.2, .8, .9999996, .0008715)-sensitive family
- The closer to 0 and 1 we get, the more hash functions must be used!

LSH for other Distance Metrics

- Problem: More generally, we found similar columns in large sparse matrices with high Jaccard similarity
- Can we use LSH for other distance measures?
  - E.g., cosine distance: Random hyperplanes
  - E.g., euclidean distance: Project on lines



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Distance Measures

- Generalized LSH is based on some kind of "distance" between points.
  - Similar points are "close."
  - Jaccard similarity is not a distance; 1 minus Jaccard similarity is.

- Two major classes of distance measure:

1. Euclidean
2. Non-Euclidean

Euclidean VS. Non- Euclidean

- A *Euclidean space* has some number of real valued dimensions and "dense" points.
  - There is a notion of "average" of two points.
  - A Euclidean distance is based on the locations of points in such a space.
- Any other space is Non-Euclidean.
  - Distance measures for non-Euclidean spaces are based on properties of points, but not their "location" in a space.

Axioms of a Distance Measure

- *d* is a *distance measure* if it is a function from pairs of points to real numbers such that:

1. $d(x,y) \geq 0$.
2. $d(x,y) = 0$ iff $x = y$.
3. $d(x,y) = d(y,x)$.
4. $d(x,y) \leq d(x,z) + d(z,y)$ (*triangle inequality*).

Some Euclidean Distances

- *L_2 norm*: d(x,y) = square root of the sum of the squares of the differences between *x* and *y* in each dimension.
  - The most common notion of "distance."
- *L_1 norm*: sum of the differences in each dimension.
  - *Manhattan distance* = distance if you had to travel along coordinates only.

Some Euclidean Distances

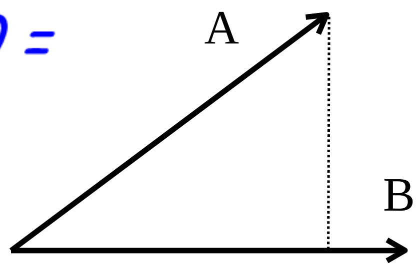
- L_∞ *norm*: d(x,y) = the maximum of the differences between *x* and *y* in any dimension.
- Note: the maximum is the limit as *r* goes to ∞ of the L_r norm: what you get by taking the r^{th} power of the differences, summing and taking the r^{th} root.

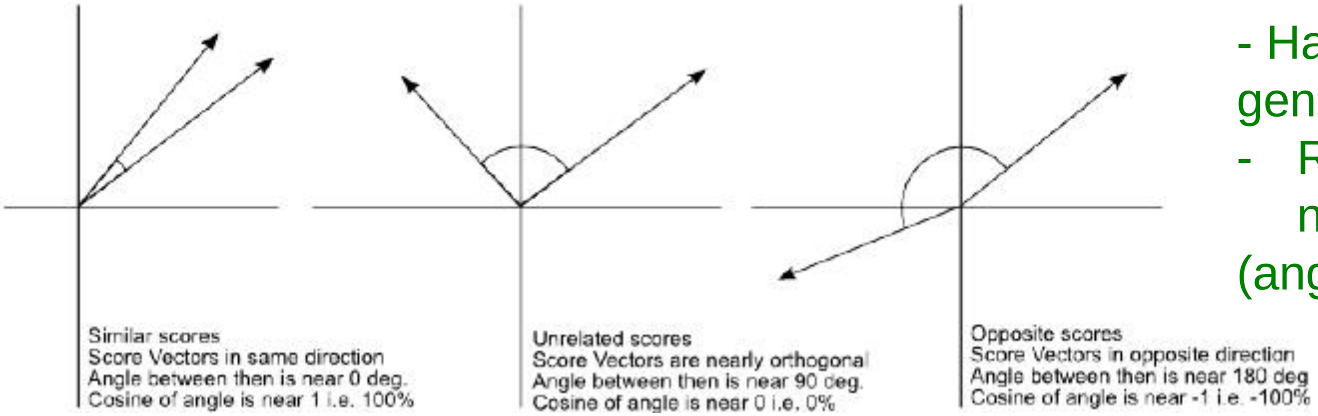
Axioms of a Distance Measure

- *Jaccard distance* for sets = 1 – Jaccard similarity.
- *Cosine distance* for vectors = angle between the vectors.
- *Edit distance* for strings = number of inserts and deletes to change one string into another.
- *Hamming Distance* for bit vectors = the number of positions in which they differ.

Cosine Distance

- Cosine distance = angle between vectors from the origin to the points in question $d(A, B) = \theta = \arccos(A \cdot B / \|A\| \cdot \|B\|)$
  - Has range $0 \ldots \pi$ (equivalently 0...180°)
  - Can divide θ by $\pi\pi$ to have distance in range 0...1
- Cosine similarity = 1-d(A,B)
  - But often defined as cosine sim: $\cos(\theta) = \dfrac{A \cdot B}{\|A\| \, \|B\|}$





- Has range -1…1 for general vectors
- Range 0..1 for non-negative vectors (angles up to 90°)

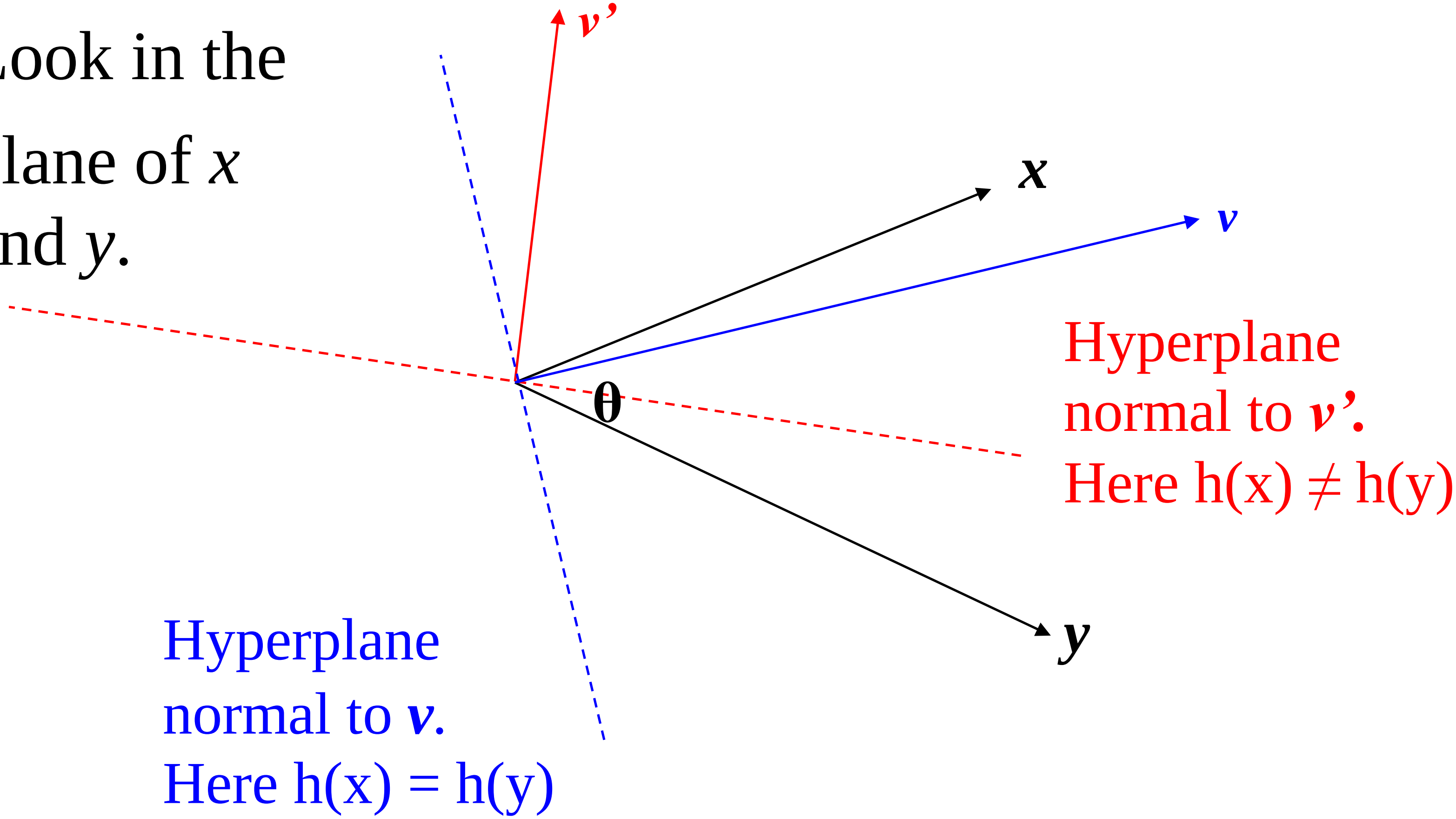
LSH for Cosine Distance

- For cosine distance, there is a technique called Random Hyperplanes
  - Technique similar to Min-Hashing
- Random Hyperplanes method is a $(d_1, d_2, (1\text{-}d_1/180), (1\text{-}d_2/180))$-sensitive family for any d_1 and d_2
- Pick a random vector v, which determines a hash function h_v with two buckets
- $h_v(x) = +1$ if $v \cdot x \geq 0$; $= -1$ if $v \cdot x < 0$
- LS-family H = set of all functions derived from any vector
- Claim: For points x and y,

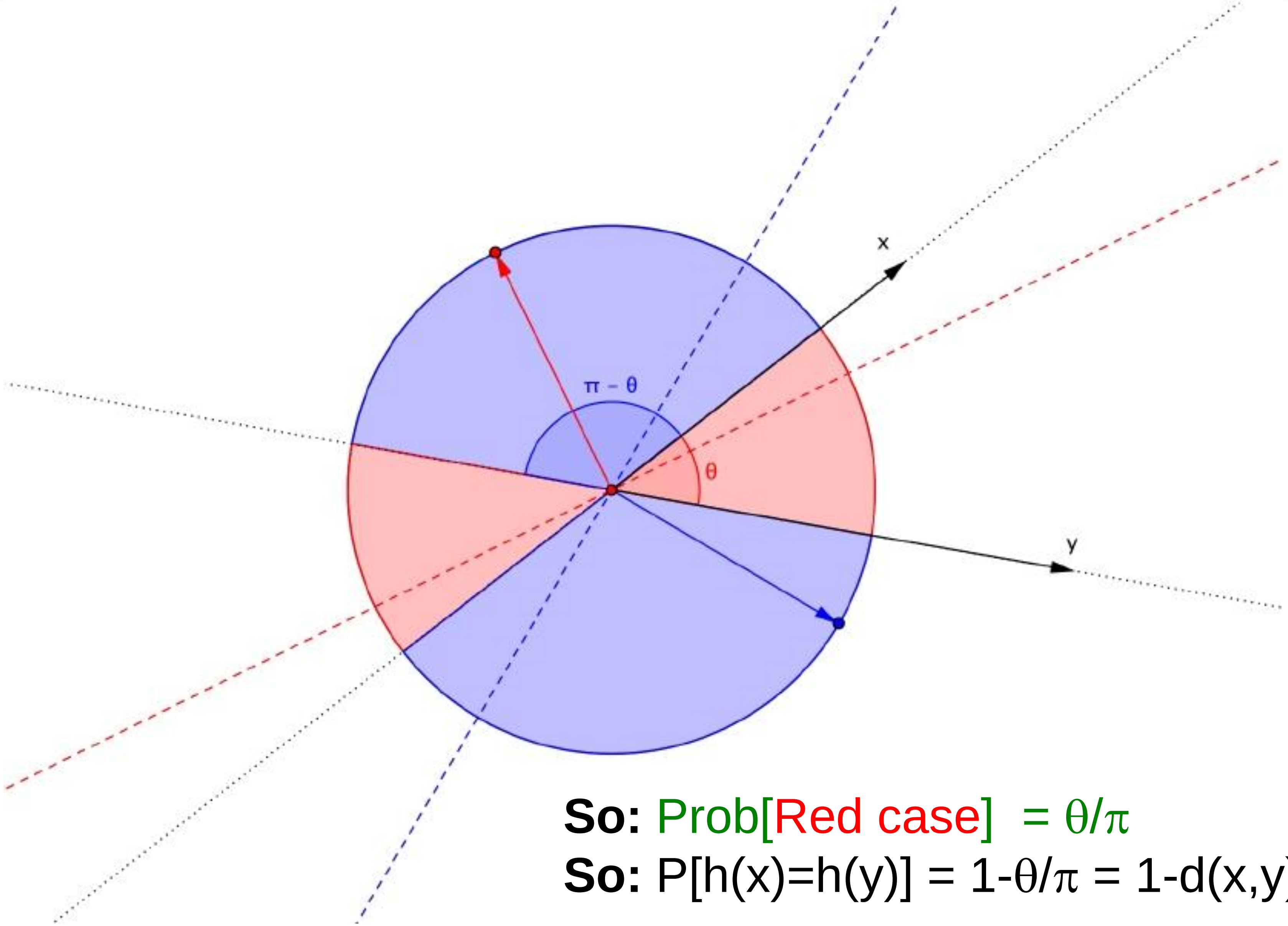
$$Pr[h_v(x) = h_v(y)] = 1 - d(x,y) / 180\ \pi$$

Proof of Claim

Look in the plane of *x* and *y*.



Proof of Claim



So: Prob[Red case] = θ/π

So: P[h(x)=h(y)] = $1-\theta/\pi$ = 1-d(x,y)

Signatures for Cosine Distance

Random vector

v1	v2	v3
+1	-1	0
-1	-1	0
0	0	0
+1	-1	-1
+1	0	+1
-1	0	+1
-1	0	-1

Input matrix (Shingles x Documents)

C1	C2	C3	C4
1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature matrix *M*

Sig1	*Sig2*	*Sig3*	*Sig4*
-1	+1	-1	+1
-1	-1	-1	-1
0	0	0	0

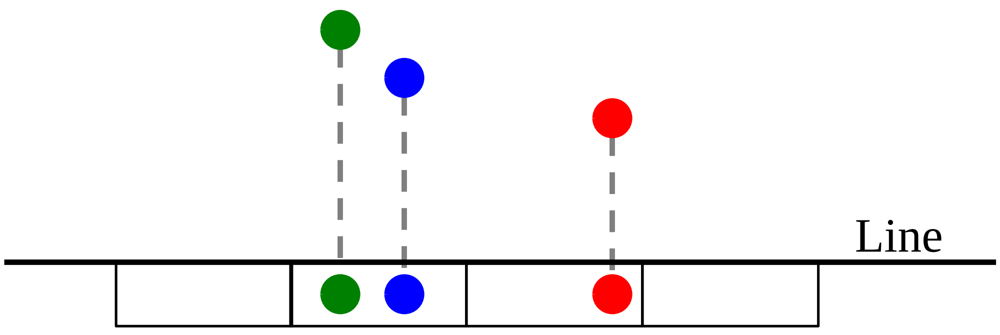
$$M_{i,j} = h_{v_i}(C_j)$$

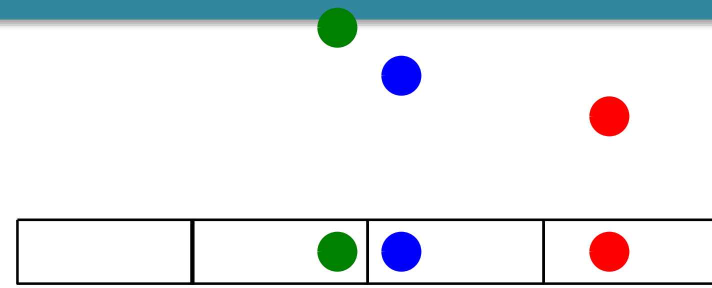
- LSH of cosine distance likes we used the Min-Hash signatures for Jaccard distance
- Amplify using **AND/OR** constructions(*b* bands and *r* rows)

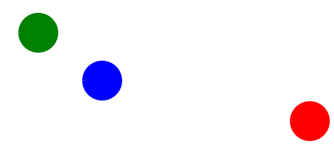
LSH for Euclidean Distance

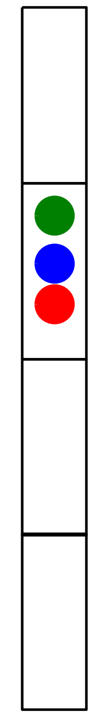
- Simple idea: Hash functions correspond to lines
- Partition the line into buckets of size a
- Hash each point to the bucket containing its projection onto the line
- **Nearby points are always close;**
  distant points are rarely in same bucket

Projection of Points



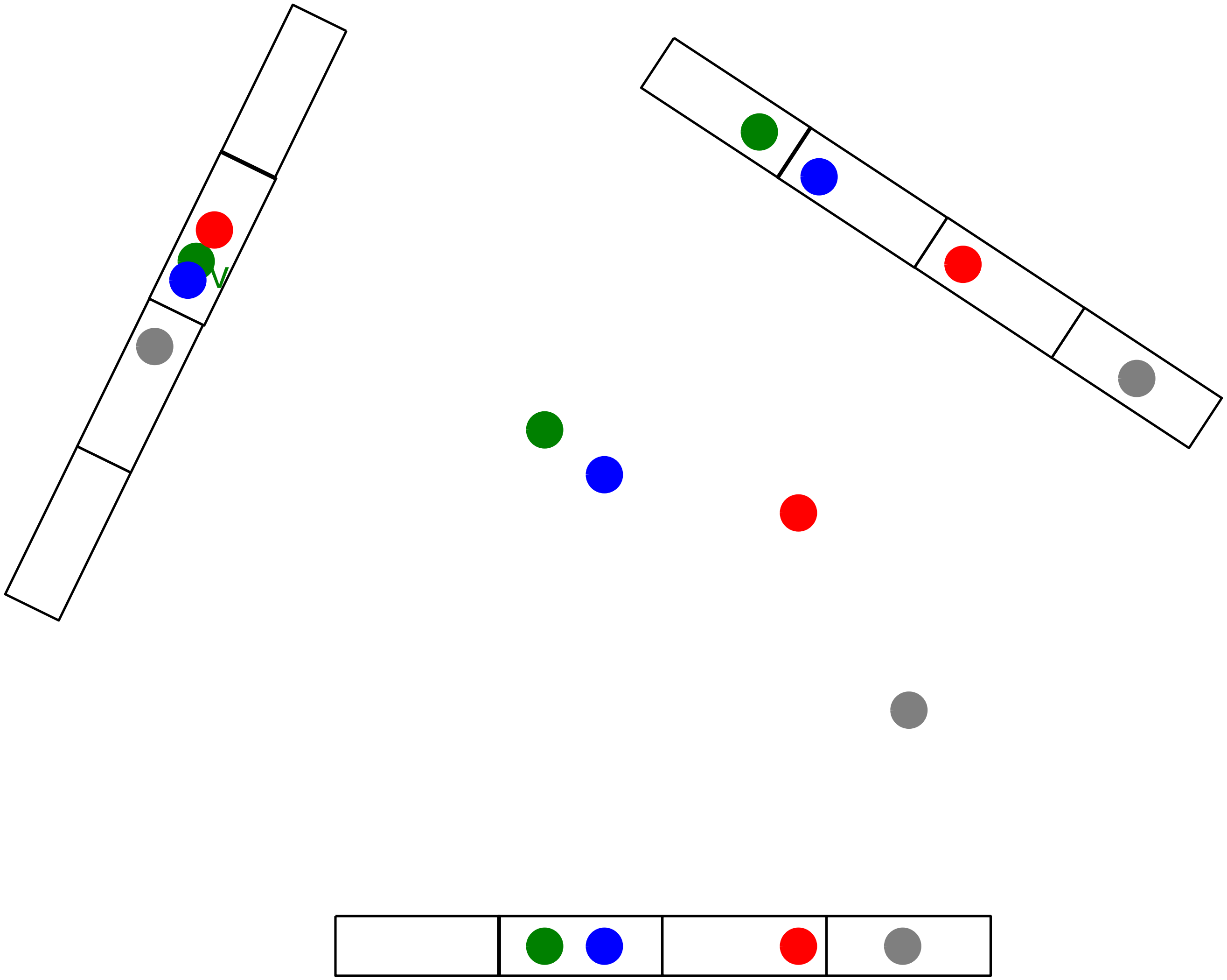




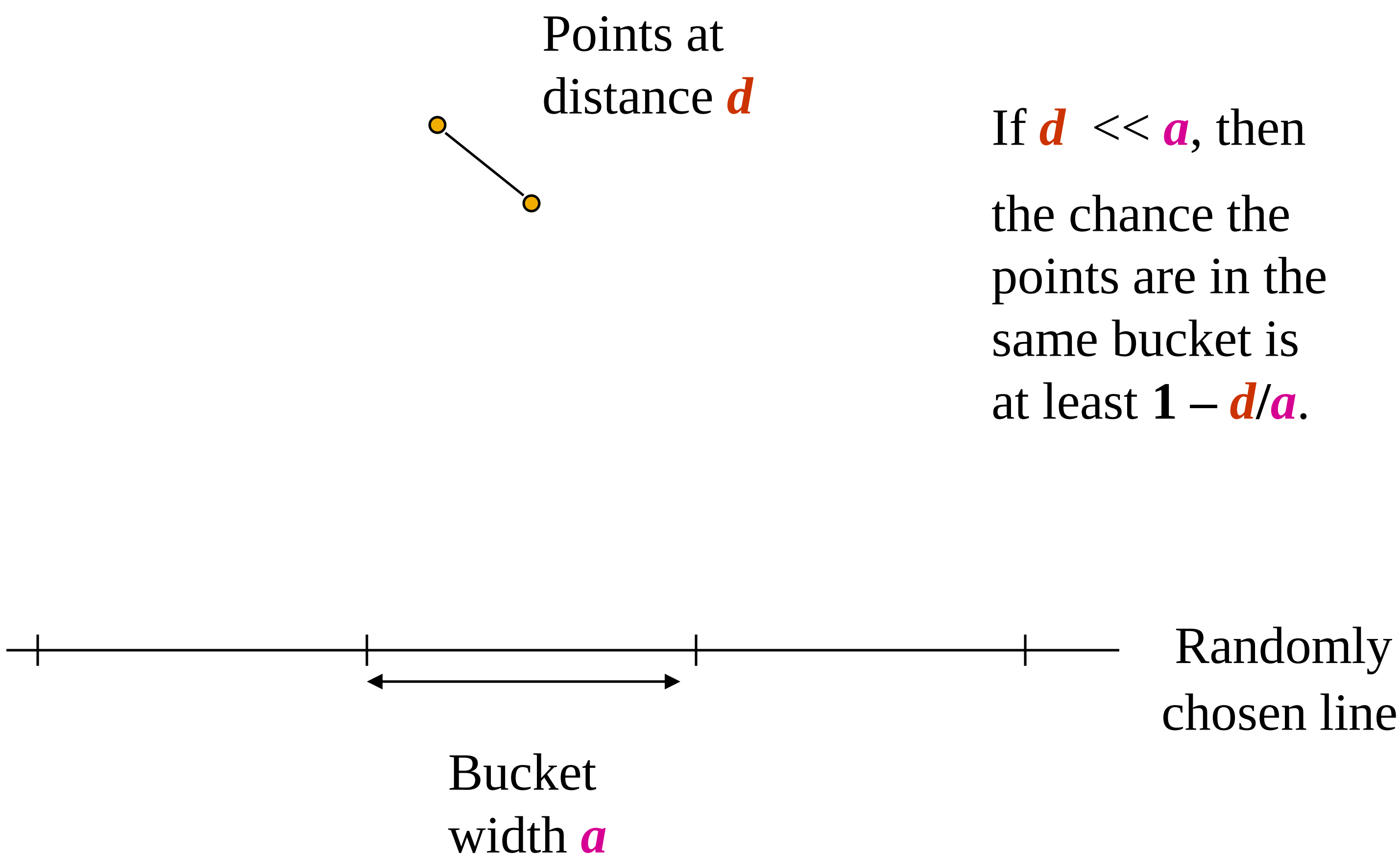


- **"Lucky" case:**
  - Points that are close hash in the same bucket
  - Distant points end up in different buckets
- **Two "unlucky" cases:**
  - **Top:** unlucky quantization
  - **Bottom:** unlucky projection

Multiple Projections



Projection of Points

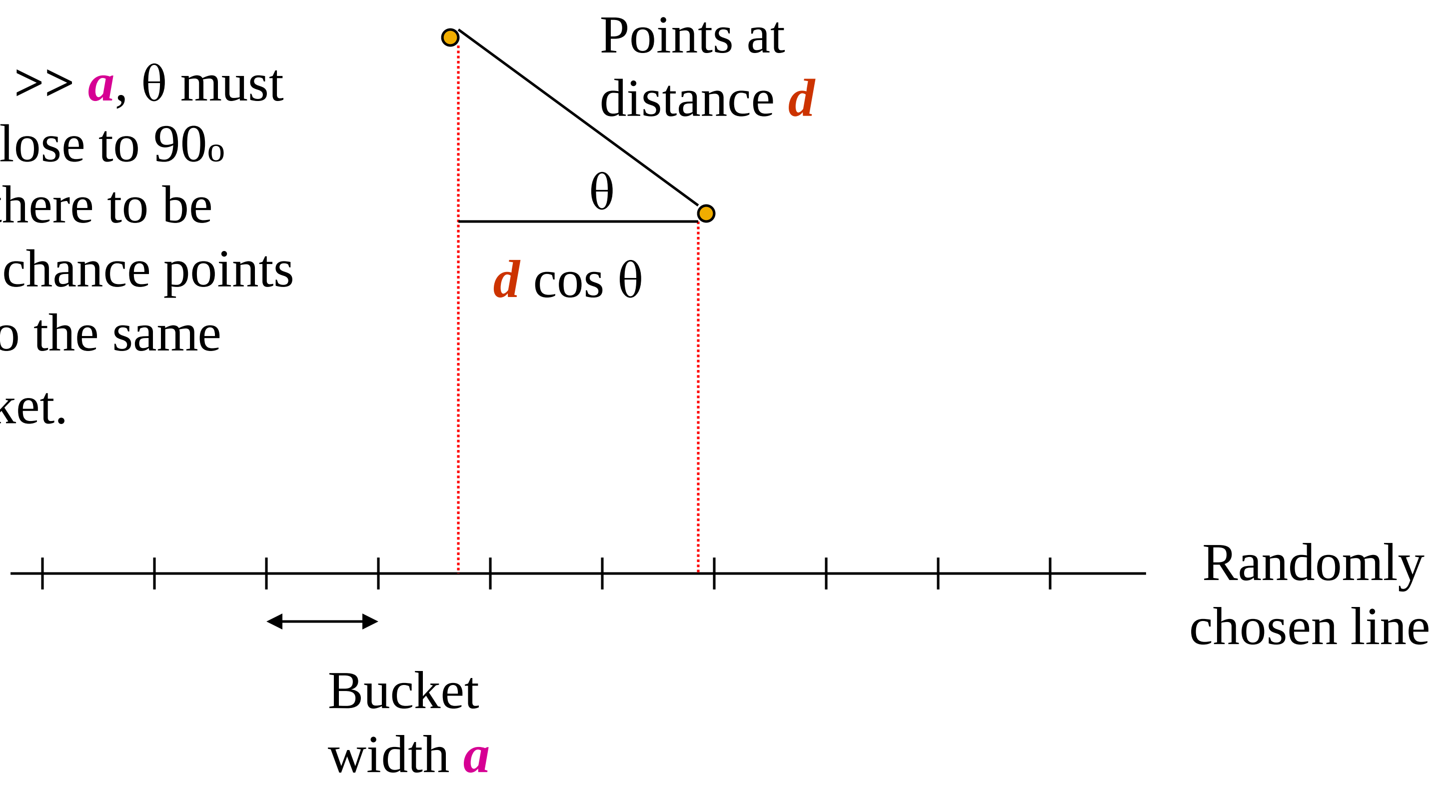


If ***d*** << ***a***, then

the chance the points are in the same bucket is at least **1** – ***d***/***a***.

Projection of Points

If ***d*** >> ***a***, θ must be close to 90° for there to be any chance points go to the same bucket.



An LS-Famlily for Euclidean Distance

- If points are distance $d < a/2$, prob. they are in same bucket $\geq 1 - d/a = \frac{1}{2}$
- If points are distance $d > 2a$ apart, then they can be in the same bucket only if $d \cos \theta \leq a$
  - $\cos \theta \leq \frac{1}{2}$
  - $60 < \theta < 90$, i.e., at most 1/3 probability
- Yields a (a/2, 2a, 1/2, 1/3)-sensitive family of hash functions for any a
- Amplify using AND-OR cascades

Fixup: Euclidean Distance

- Projection method yields a *(a/2, 2a, 1/2, 1/3)-sensitive* family of hash functions
- For previous distance measures, we could start with an *(d_1, d_2, p_1, p_2)-sensitive* family for any $d_1 < d_2$, and drive p_1 and p_2 to 1 and 0 by AND/OR constructions
- Note: Here, we seem to need $d_1 \leq 4\ d_2$
- In the calculation on the previous slide we only considered cases $d < a/2$ and $d > 2a$

Fixup-(2)

- But as long as $d_1 < d_2$, the probability of points at distance d_1 falling in the same bucket is greater than the probability of points at distance d_2 doing so

- Thus, the hash family formed by projecting onto lines is an (d_1, d_2, p_1, p_2)-sensitive family for some $p_1 > p_2$
  - Then, amplify by AND/OR constructions

Tell me and I forget.
Show me and I remember.
Involve me and I understand.

Thank you!   Q&A

Appendix: Efficiently Matching Sets of Features with Random Histograms

Motivation

- Set-of-feature representation is popular
  - Example: images, video, audio, .etc
  - Higher empirical results than global feature

- The set matching problem

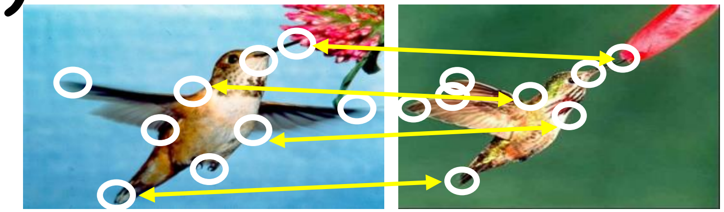
$$X = \{x_1, x_2, \ldots, x_{n1}\} \ Y = \{y_1, y_2, \ldots, y_{n_2}\}$$

Given feature similarity measure $s(x_i, y_j)$

  - How to define set similarity $S(X, Y)$ meaningfully ?
  - How to evaluate $S(X, Y)$ efficiently ?

Set Similarity Measures

- One-to-one match: $O(N^3)$



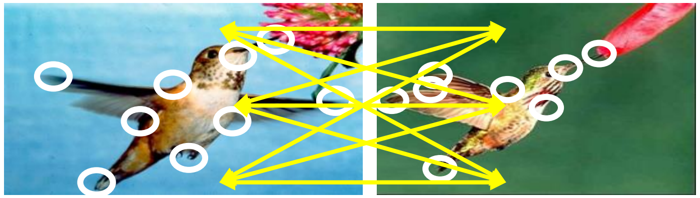
$$S_1(X,Y) = \max \sum_{<x,y>} s(x,y)$$

s.t. each feature used at most once

  - Bipartite graph matching
  - Weighted version: EMD[Rubner98]

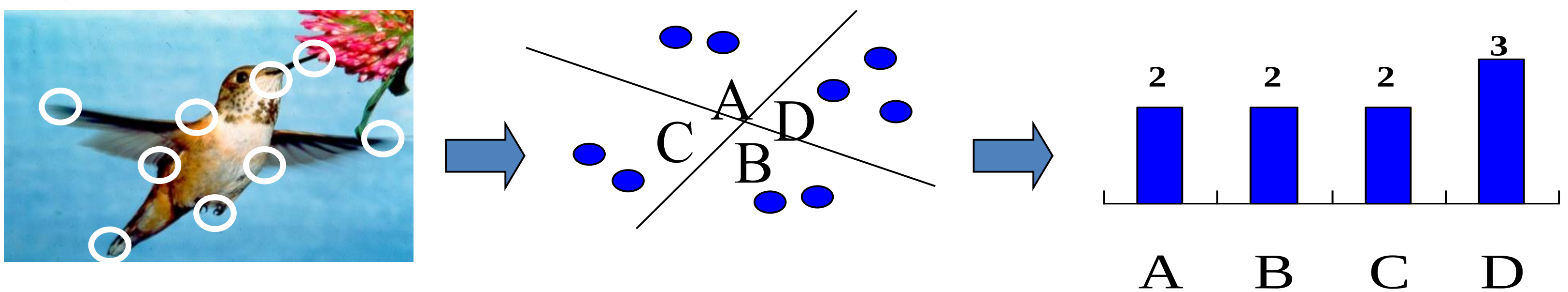
- One-to-many match: $O(N^2)$

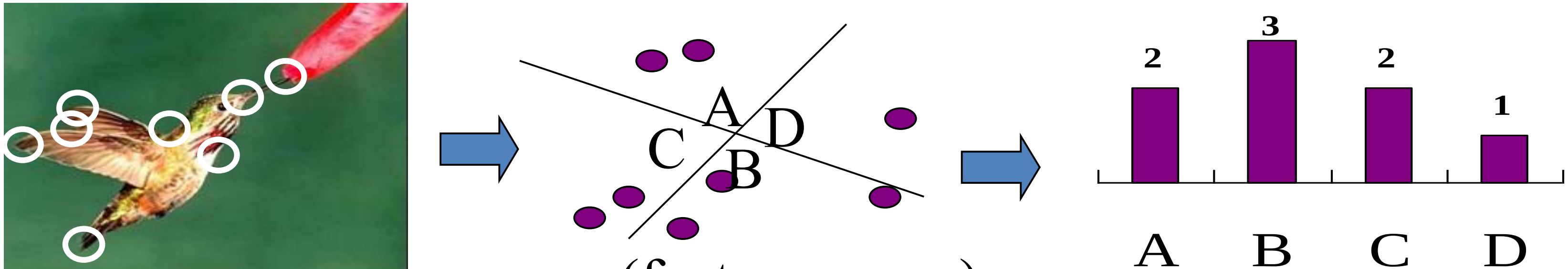
$$S_2(X,Y) = \sum_{x \in X} \sum_{y \in Y} s(x,y).$$



The Histogram Approach

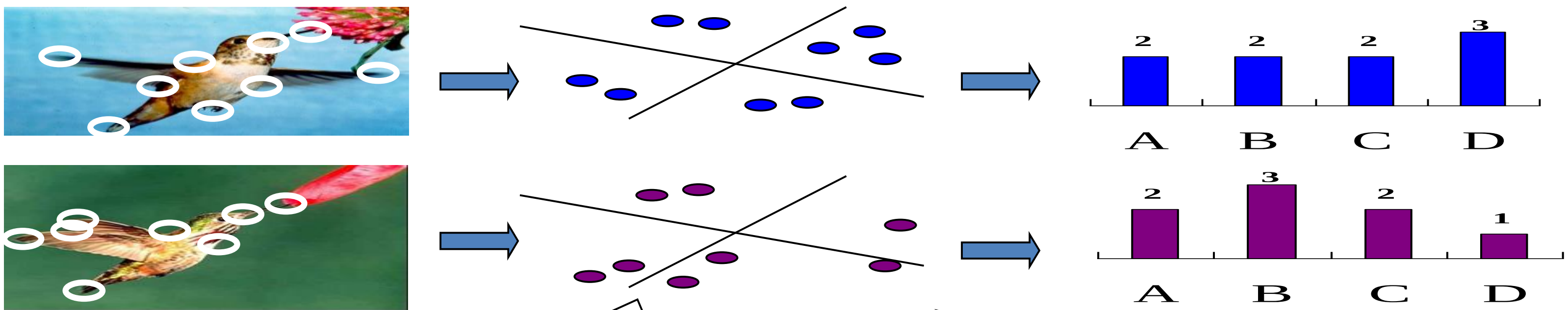
- Embed to histograms via feature space quantization
- Improve online matching speed significantly
- Potential precision loss, but good empirical performance





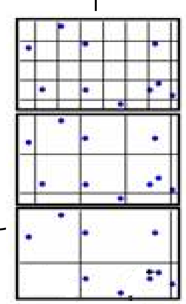
(feature space)

Existing Quantization Methods



Pyramid match

- Originally uses scalar quantization[Grauman 05]
  - Histograms are sparse
- New version with vector quantization [Grauman07]



Visual words

- Vector quantization
- Time consuming clustering
- Not incrementa

Contribution

- A randomized histogram construction
  - Quantization by Locality Sensitive Hashing
  - Compact histogram representation
  - Embedding is fast and parallelizable
  - Support various feature similarity measures

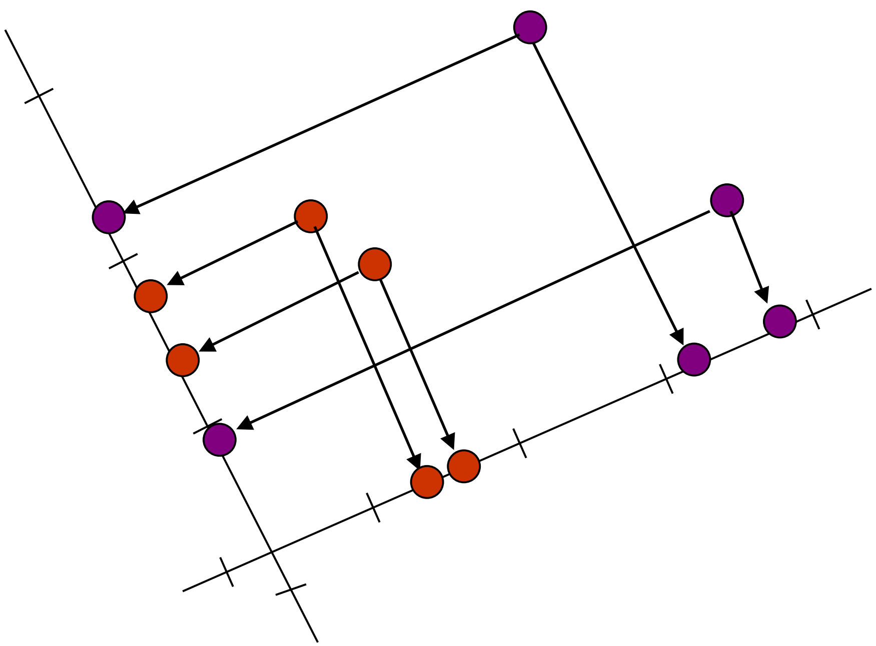
- Evaluation with three task settings
  - Very fast matching speed with good accuracy
  - E.g. 20x speedup over Pyramid Match on Caltech 101

Locality Sensitive Hashing

- Idea: hash functions that similar objects are more likely to have the same hash [Indyk98]

$$\Pr_{h \in \mathcal{H}}[h(x) = h(y)] = s_{\mathcal{H}}(x, y),$$

$s_{\mathcal{H}}$: similarity induced by the LSH family $\mathcal{H}$.

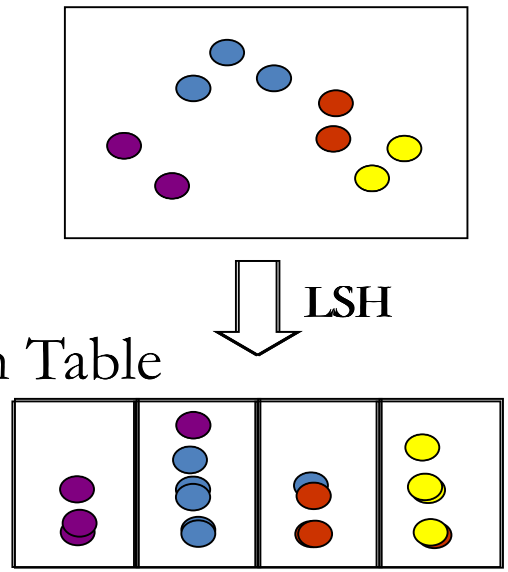


LSHs have been designed for

- L1 and L2 distance
- Cosine similarity
- Hamming distance
- Jaccard index for set similarity
- ……

Random Histogram Construction

Features of one object



Hash Table

Repeat M times

Histogram

$\langle 1, 4, 2, 3 \rangle$ $\cdots\cdots$

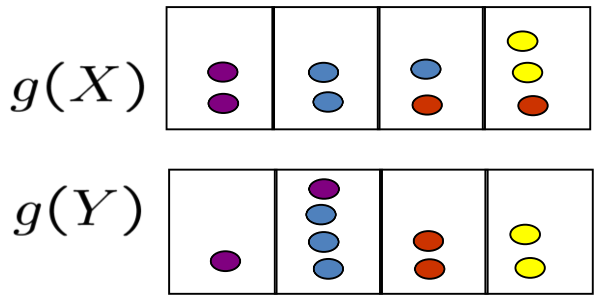
concatenate

$\langle 2, 2, 2, 3, 1, 4, 2, 2, \ldots \rangle$

Matching Random Histograms

- One-to-one match

$$S_1(X,Y) = \sum_i \mathrm{min}[g_i(X), g_i(Y)]$$
$$= 1 + 2 + 2 + 2 = 7$$



- One-to-many match

$$S_2(X,Y) = \sum_i g_i(X) \cdot g_i(Y)$$
$$= 2 + 8 + 4 + 6 = 20$$

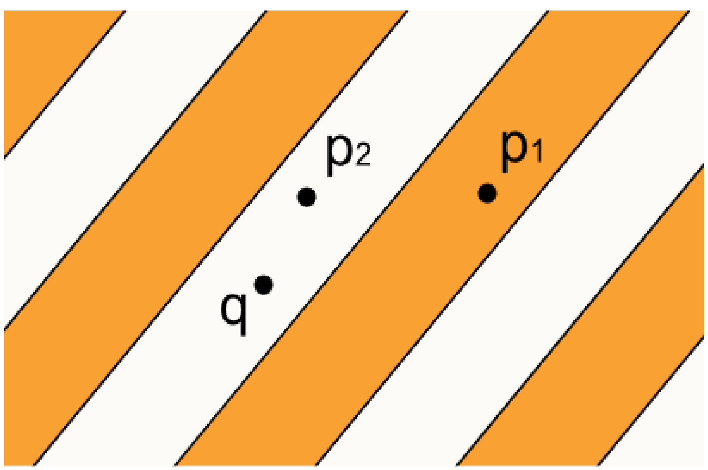
$$E[S_2(X,Y)] = \sum_{x \in X} \sum_{x \in Y} s_{\mathcal{H}}(x,y)$$

Choosing LSH: L2 Distance

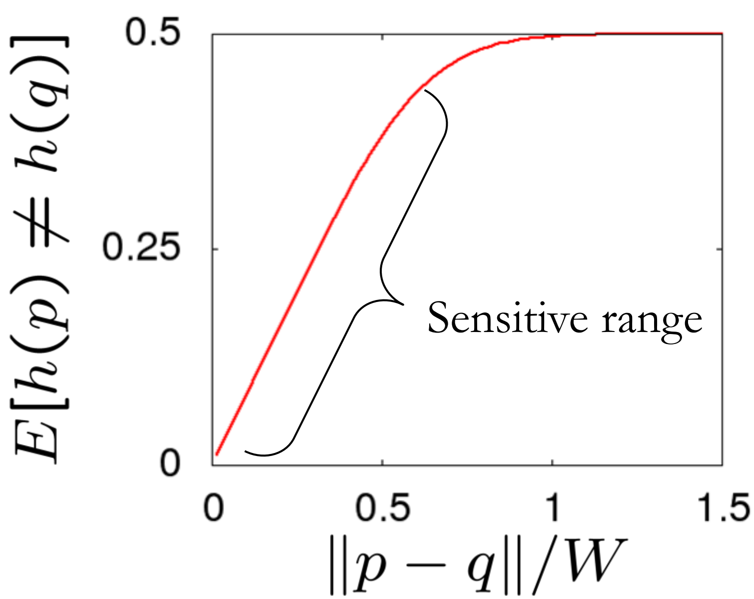
- Least significant bit of the p-stable LSH [Datar04]

$$h(p) = \lfloor \frac{A \cdot p + b}{W} \rfloor \, mod \, 2$$

  - LSB only so that hash values distribute evenly

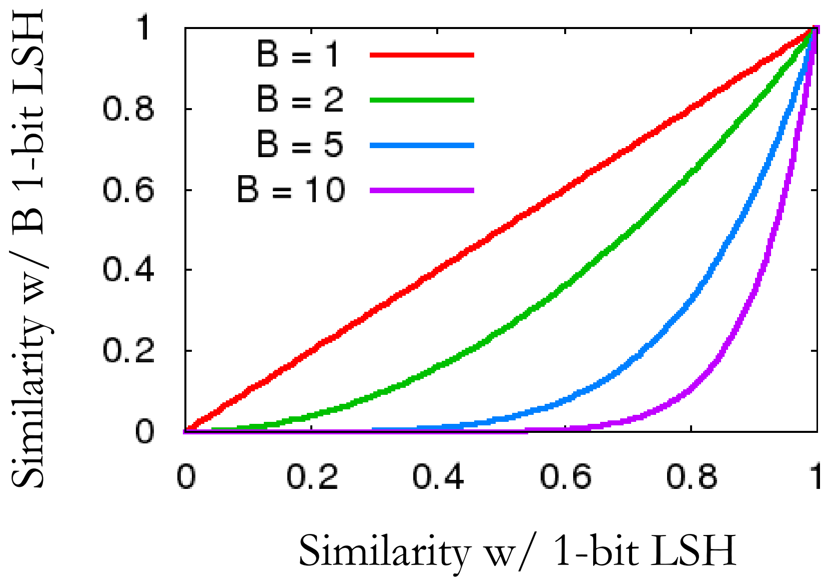


W: stripe width



Choosing LSH Scheme

- Concatenate *B* 1-bit LSH to make a B-bit LSH
  - Enhance discriminating power
  - Enlarge histogram size to 2^B
  - Tunable tradeoff between space vs. accuracy



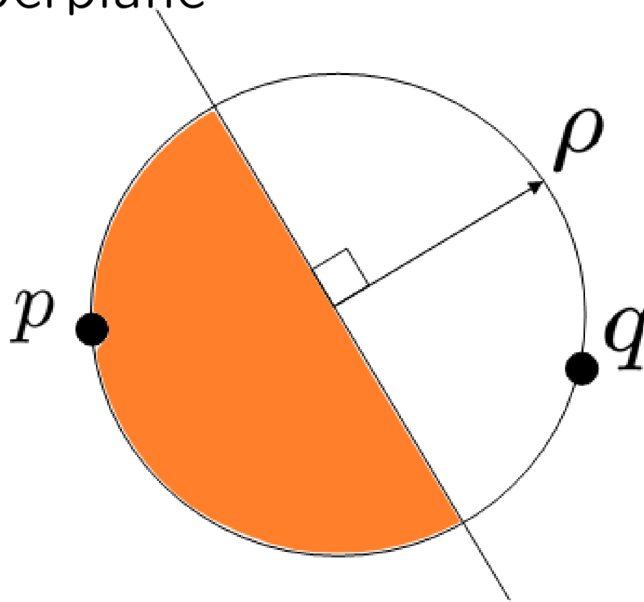
Another LSH: Cosine Similarity

- Cosine Similarity $d_{cos}(p, q) = \cos(p^\wedge q)$
- Random hyper-plane sketch [Charikar02]

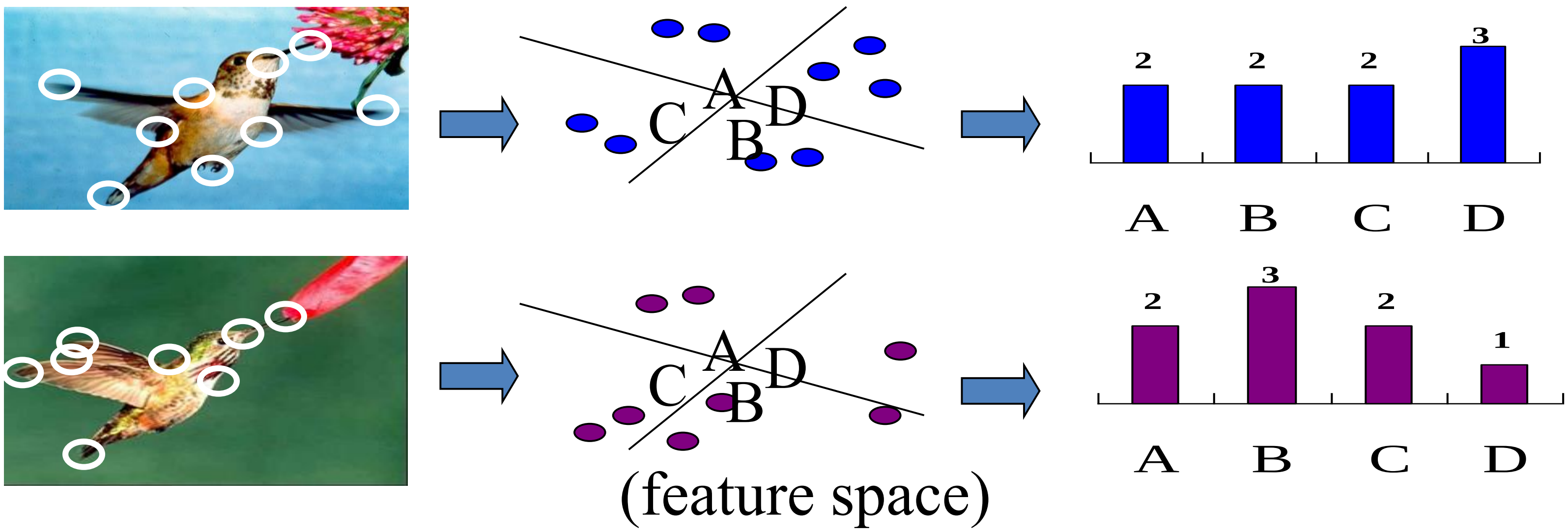
$$h(p) = \begin{cases} 0 & \text{if } \rho \cdot p < 0 \\ 1 & \text{if } \rho \cdot p \geq 0 \end{cases}$$

ρ: normal vector of a random hyperplane

$$E[h(p) \neq h(q)] = \frac{p^\wedge q}{\pi}$$



Recap of Our Scheme



Local feature extraction

Quantization with LSH
- L2 distance
- Cosine similarity
- ……

Histogram matching
- One to one
- One to many

Histogram size = $N2^B$, doesn't explicitly depend on set size!

Evaluation

- Three tasks
  - Object recognition
  - Image similarity search
  - Near duplicate video detection
- Performance measure

$$\text{Accuracy} = \frac{\#\ \text{correctly labeled}}{\text{total number}}$$

$$\text{Average Precision} = \frac{1}{k}\sum_{i=1}^{k}\frac{i}{\text{rank}_i}$$

- Platform: Pentium 4 3.2GHz CPU+ 6GB mem

Recognition: Caltech 101

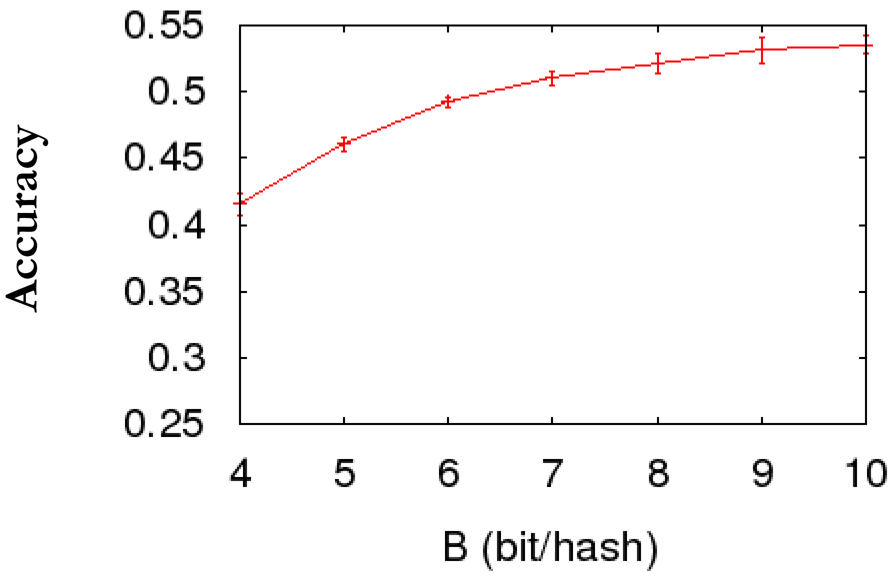
- Benchmark: 101 categories
- Features: 10D PCA SIFT + 2D position feature
- 15 images/category for training, rest for testing

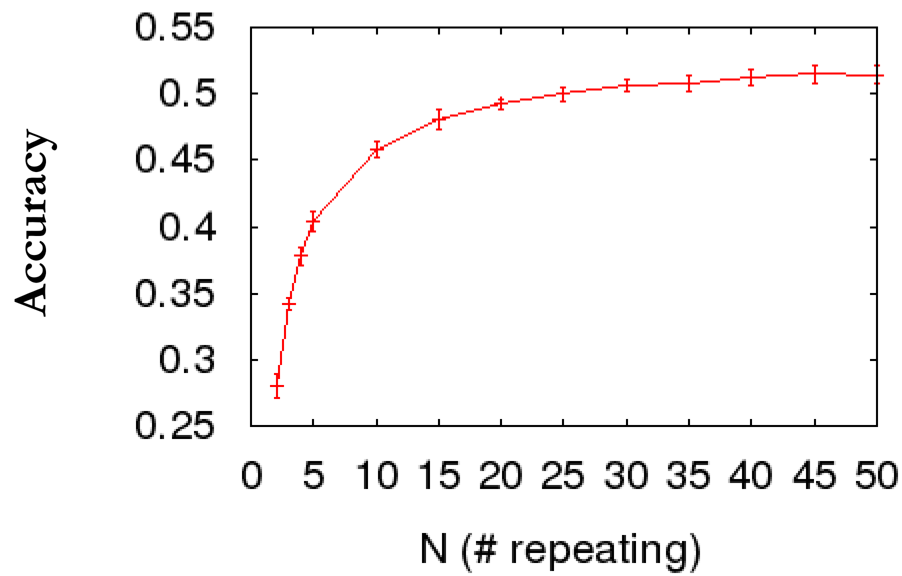
	Grauman05 Pyramid	Zhang07 EMD	Ours A (speed)	Ours B (accuracy)
Accuracy	0.50	0.539	**0.497**	**0.541**
Time/match	100μs	slow	**0.4μs**	**6μs**

All methods use SVM for learning.

Performance vs. Parameters

- Single histogram size: 2^B
- Full histogram size: $N\ 2^B$





Recognition: Graz-01

- Benchmark: bike, person and background
  - High intra-class scale and pose variation
- DoG detector + SIFT feature
- 100 positive + 100 negative for training
- ROC equal error rate

	Opelt04	Zhang07 EMD	Lazebnik06 spatial pyramid	Ours
Bike	0.865	0.920	0.863	**0.883**
Person	0.808	0.880	0.823	**0.805**

Content-Based Image Retrieval

- Benchmark [Lv04]
  - 10K images, 32 sets of manually selected images
- Feature extraction
  - Segmented with JSEG
  - 14-D color & size feature from each segment
- Direct K-NN search

Method	SIMPLIcity EMD like	Lv04 EMD	Ours
Average precision	0.331	0.548	**0.548**
Time/match	N/A	$50\mu s$	$\mathbf{0.3\mu s}$

Near-Duplicate Video Detection

- Benchmark [Wu07]
  - 24 sets of video clips downloaded from Youtube, etc
  - Manually labeled to find near duplicates
- Feature extraction
  - 124 key frames/video clip
  - HSV based color histogram from each key frame

Wu07	SIG_CH	SET_NDK
Average Precision	0.892	0.952
Time/match	"fast"	"minutes"

Ours	SIG_CH	Embedding
Average precision	0.835	0.893
Time/match	0.17ms	4.6ms

Discussion

- Histogram might still be large
  - See paper for two methods of compact representation
  - Further reduction is possible by feature selection
- Precision loss in quantization
  - Vanilla k-NN search is not good enough
  - Can be compensated by machine learning

Conclusion

- Matching sets of features with random histograms
- Efficient
  - fast embedding and matching, good empirical accuracy
  - 20x faster than Pyramid Match (~500 features/image)
  - 150x faster than EMD (~10 features/images)
- Flexible
  - Support various feature and set similarity measures
  - Easy to tradeoff between speed and accuracy