Some algebraic geometric

k: field (alg closed, character zero)

let $\Gamma_n = k[x_1, \dots, x_n]$

Γ_n is a k-algebra. Suppose A: k-alg

(commutative and unital)

$$k\text{-alg}(\Gamma_n, A) \cong A^n = \{(a_1, \dots, a_n) \mid a_i \in A\}$$

$$\Gamma_n \xrightarrow{\varphi} A \longmapsto \varphi(x_1, \dots, x_n)$$

<u>Lemma.</u> ~~$f = g_1 f_1 + \dots + g_n f_n$~~

~~$g_i \in \Gamma_n$ and $f_i \in \Gamma_n$~~

~~then common zeros of f~~

A : k-alg

$I \subset k[x_1, \dots, x_n]$ ideal

Then define $V_I(A) = \{\vec{x} \in A^n \mid f(\vec{x}) = 0 \ \forall f \in I\}$

Prop. $V_I : k\text{-alg}^{op} \longrightarrow \text{Set}$ is a representable functor and is represented by ~~$k[x]/I$~~ T_n/I

$$V_I(A) \cong k\text{-alg}\left(\frac{k[x_1, \dots, x_n]}{I}, A \right)$$

$$(\varphi(x_1), \dots, \varphi(x_n)) \longleftarrow\!\shortmid \varphi : \frac{k[x_1, \dots, x_n]}{I} \to A$$

for any $f \in I$

$$f(\varphi(x_1), \dots, \varphi(x_n)) =$$

$$\sum c_{(\alpha_1, \dots, \alpha_n)} \varphi(x_1)^{\alpha_1} \cdots \varphi(x_n)^{\alpha_n} =$$

b/c φ: k-alg mor.

$$\varphi\left(\sum c_{(\alpha_1, \dots, \alpha_n)} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \right) =$$

$$\varphi(f(x_1, \dots, x_n)) = 0_A$$

$$\begin{tikzcd} Psh(k\text{-alg}) = [k\text{-alg}^{op}, Set] \\ k\text{-alg}^{op} \arrow[u, "\mathrm{Spec} = \text{yoneda}"] \end{tikzcd}$$

$A : k\text{-alg}$

$$\mathrm{Spec}_A : (k\text{-alg})^{op} \longrightarrow Set$$

$$\mathrm{Spec}_A(\ := \cancel{Hom}\ k\text{-alg}(A, \bigstar)$$

$$\cancel{\mathrm{Spec}_A(k[x_1,\dots,x_n]/I) \cong V_I(A)}$$

$$\mathrm{Spec}_{T_n}(B) = k\text{-alg}(T_n, B) = V_I(B)$$

<u>Defn.</u> Scheme is a presheaf which is locally like Spec.

We can take k to be a comm. unital ring.

Remark A, B, C : (unital) commutative K-algebras.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \arrow[dl, phantom, "\lrcorner"] & b \arrow[d, mapsto] \\ C \arrow[r] & B \otimes_A C & b \otimes 1 \\ c \arrow[r, mapsto] & 1 \otimes c & \end{tikzcd}$$

is a pushout of K-algebras.

Consider B and C as A-modules. Construct

$$B \otimes_A C = \frac{\mathrm{Free}(B \times C)}{\sim} \qquad (f(a) \cdot b, c) \sim (b, g(a) \cdot c)$$

so $(f(a) \cdot b) \otimes c = b \otimes (g(a) \cdot c)$

If

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \arrow[ddr, bend left, "\lambda"] & \\ C \arrow[r] \arrow[drr, bend right, "\gamma"'] & B \otimes_A C \arrow[dr, dashed, "\exists ! u"] & \\ & & D \end{tikzcd}$$

If $\lambda f = \gamma g$
$\exists !\ u : B \otimes_A C \longrightarrow D$
s.t. $u(b \otimes 1) = \lambda(b)$
$u(1 \otimes c) = \gamma(c)$

Construct $u : B \otimes_A C \longrightarrow D$ defined on basic tensors $b \otimes c \longmapsto \lambda(b) \cdot \gamma(c)$

It is well defined b/c $\lambda f = \gamma g$.

$$u(f(a) \cdot b \otimes c) = \lambda(f(a) \cdot b) \cdot \gamma(c) = \lambda f(a) \cdot \lambda(b) \cdot \gamma(c)$$
$$= \lambda(b) \cdot \gamma g(a) \cdot \gamma(c) = u(b \otimes g(a) \cdot c) =$$

Take $f \in A$: k-alg

$$\begin{tikzcd} k[x] \arrow[r, "\text{loc.}"] \arrow[d, "f"'] & k[x,x^{-1}] \cong \dfrac{k[x,y]}{\langle xy-1\rangle} \arrow[d] \\ f \in A \arrow[r, "\text{loc.}"] \arrow[d, "h"'] & A_f \arrow[d] \\ B \arrow[r, "\text{loc.}"] & A_{h(f)} = A_{h\circ f} \end{tikzcd}$$

pushout of k-algebras

$$\sum c_i x^i + \sum d_j y^j \mapsto \sum c_i f^i + \sum \frac{d_j}{f^j}$$

$$\begin{tikzcd} k[x] \arrow[r] \arrow[d] & k[x,x^{-1}] \arrow[d] \\ A \arrow[r] \arrow[d] & A_f \arrow[d] \\ A_g \arrow[r] & A_f \otimes_A A_g \end{tikzcd}$$

(ex)

So $A_f \otimes_A A_g \cong A_{fg}$

Remark. Every commutative ring R is a comm. $\mathbb{Z}$-algebra.

So coproduct and pushout of comm. rings can be given as coproduct & pushout of comm. $\mathbb{Z}$-algebras.

A, B, C : rings

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & B \otimes_A C \end{tikzcd}$$

$\mathbb{Z}$: initial comm. ring

and coproduct

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & B \otimes_{\mathbb{Z}} C \end{tikzcd}$$

JULY					
M		4	11	18	25
T		5	12	19	26
W		6	13	20	27
T		7	14	21	28
F	1	8	15	22	29
S	2	9	16	23	30
S	3	10	17	24	31

Defn. A <u>monoidal monad</u> S on monoidal category $(\mathcal{C}, \otimes, k, a, l, r)$ is a monad (S, η, μ) on category $\mathcal{C}$ plus ~~morphisms~~ natural tranformation

$$\tau_{-,-} : S(- \otimes -) \Rightarrow S(-) \otimes S(-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

compatibility

~~τ_k~~ and a morphism

$$\tau_k : S(k) \longrightarrow k \quad \text{s.t.}$$

$$\begin{tikzcd}
SX \otimes Sk \arrow[d, "1 \otimes \tau_k"'] & S(X \otimes k) \arrow[l, "\tau_{X,k}"'] \arrow[d, "S(r_X)"] \arrow[dl, phantom, "="] \\
SX \otimes k \arrow[r, "r_{S(X)}"'] & S(X)
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
Sk \otimes SX \arrow[d, "\tau_k \otimes 1"'] & S(k \otimes X) \arrow[l, "\tau_{k,X}"'] \arrow[d, "S(l_X)"] \arrow[dl, phantom, "="] \\
k \otimes SX \arrow[r, "l_{SX}"'] & S(X)
\end{tikzcd}$$

$$r_{S(X)} \circ (1 \otimes \tau_k) \circ \tau_{X,k} = S(r_X) \qquad l_{SX} \circ (\tau_k \otimes 1) \circ \tau_{k,X} = S(l_X)$$

and

- Comptability of η ~~and~~ with τ_k:

$$\begin{tikzcd}
k \arrow[d, "\eta"'] \arrow[dr, "\eta"] \\
S(k) \arrow[r, "\tau"'] & k
\end{tikzcd}$$
$=$

- Comptability of η ~~and~~ with $\tau_{X,Y}$:

$$\begin{tikzcd}
X \otimes Y \arrow[d, "\eta_{X\otimes Y}"'] \arrow[dr, "\eta \otimes \eta"] \\
S(X \otimes Y) \arrow[r, "\tau"'] & S(X) \otimes S(Y)
\end{tikzcd}$$
$=$

- Compatibility of μ with τ_k:

$$\begin{tikzcd}
S^2(k) \arrow[r, "S(\tau_k)"] \arrow[d, "\mu_k"'] \arrow[dr, phantom, "="] & S(k) \arrow[d, "\tau_k"] \\
S(k) \arrow[r, "\tau_k"'] & k
\end{tikzcd}$$

- Compatibility of μ with $\tau_{X,Y}$

$$\begin{tikzcd}
S^2(X \otimes Y) \arrow[r, "\mu_{X,Y}"] \arrow[d, "S(\tau_{X,Y})"'] & S(X \otimes Y) \arrow[dd, "\tau_{X,Y}"] \\
S(SX \otimes SY) \arrow[d, "\tau_{SX,SY}"'] \arrow[r, phantom, "="] & \\
S^2X \otimes S^2Y \arrow[r, "\mu_X \otimes \mu_Y"'] & SX \otimes SY
\end{tikzcd}$$

Prop.

Let S be a monoidal monad on a tensor category $(\mathcal{C}, \otimes, k, a, l, r)$.

Then the category $Alg(S)$ of S-algebras is again a tensor category.

Proof. Structure maps of S: $\eta, \mu, \tau_{X,Y}, \tau_k$

where
$$\eta_X : X \longrightarrow S(X)$$
$$\mu_X : S^2(X) \longrightarrow S(X)$$
$$\tau_{X,Y} : S(X \otimes Y) \longrightarrow S(X) \otimes S(Y)$$
$$\tau_k : S(k) \longrightarrow k$$

$\tilde{k} = (k, \tau_k : S(k) \longrightarrow k)$ defines an

S-algebra:

• Compatibility of η with τ_k

$$\begin{tikzcd} k \arrow[d, "\eta_k"'] \arrow[dr, "\eta"] & \\ S(k) \arrow[r, "\tau_k"'] \arrow[ur, phantom, "="] & k \end{tikzcd}$$

and

• Compatibility of η with τ_k

$$\begin{tikzcd} S^2(k) \arrow[r, "S(\tau_k)"] \arrow[d, "\mu_k"'] \arrow[dr, phantom, "="] & S(k) \arrow[d, "\tau_k"] \\ S(k) \arrow[r, "\tau_k"'] & k \end{tikzcd}$$

Tensor product on $Alg(S)$

$\tilde{A} = (A, \alpha : S(A) \longrightarrow A)$

$\tilde{B} = (B, \beta : S(B) \longrightarrow B)$

$$\tilde{A} \otimes \tilde{B} := \left(A \otimes B ,\ \begin{tikzcd} S(A\otimes B) \arrow[rr] \arrow[dr, "\tau_{A,B}"'] & & A \otimes B \\ & SA \otimes SB \arrow[ur, "\alpha \otimes \beta"'] & \end{tikzcd} \right)$$

Check $\bar{A} \otimes \bar{B}$ is indeed an $\mathcal{S}$-algebra.

$$
\begin{tikzcd}
A \otimes B \arrow[d, "\eta_{A\otimes B}"'] \arrow[dr, "= \eta_A \otimes \eta_B"] \arrow[drr, bend left, "1 = 1 \otimes 1"] & & \\
\mathcal{S}(A \otimes B) \arrow[r, "\tau_{A,B}"'] & \mathcal{S}A \otimes \mathcal{S}B \arrow[r, "\alpha \otimes \beta"'] & A \otimes B
\end{tikzcd}
$$

we used $\circ$ compatibility of η with $\tau_{X,Y}$. $\cancel{\tau \circ \eta}$

$(\tau \circ \eta = \eta \otimes \eta)$

$$
\begin{tikzcd}
\mathcal{S}^2(A \otimes B) \arrow[r, "\mathcal{S}(\tau_{A,B})"] \arrow[dd, "\mu_{A\otimes B}"'] & \mathcal{S}(\mathcal{S}A \otimes \mathcal{S}B) \arrow[r, "\mathcal{S}(\alpha \otimes \beta)"] \arrow[d, "\tau_{\mathcal{S}A,\mathcal{S}B}"'] \arrow[dr, phantom, "\text{naturality of } \tau_{X,Y} \; ="] & \mathcal{S}(A \otimes B) \arrow[d, "\tau_{A,B}"] \\
\arrow[r, phantom, "= \text{ Compatibility of } \mu \text{ and } \tau_{X,Y}"] & \mathcal{S}^2A \otimes \mathcal{S}^2B \arrow[r, "\cancel{\mathcal{S}\alpha \otimes \mathcal{S}\beta}"] \arrow[d, "\mu_A \otimes \mu_B"] \arrow[dr, phantom, "\langle \alpha, \beta \rangle \text{: algebras} \; ="] & \mathcal{S}A \otimes \mathcal{S}B \arrow[d, "\alpha \otimes \beta"] \\
\mathcal{S}(A \otimes B) \arrow[r, "\tau_{A,B}"'] & \mathcal{S}A \otimes \mathcal{S}B \arrow[r, "\alpha \otimes \beta"'] & A \otimes B
\end{tikzcd}
$$

$\tilde{k} = (k,\ \mathcal{S}(k) \xrightarrow{\tau_k} k)$ is the unit of tensor in $\mathrm{Alg}(\mathcal{S})$.

~~This corresponds~~

$\tilde{A} = (A,\ \mathcal{S}(A) \xrightarrow{\alpha} A)$

$$\tilde{k} \otimes \tilde{A} = \left(k \otimes A,\ \begin{tikzcd} \mathcal{S}(k\otimes A) \arrow[rr] \arrow[dr, "\tau_{k,A}"'] & {} \arrow[d, phantom, "="] & k \otimes A \\ & \mathcal{S}k \otimes \mathcal{S}A \arrow[ur, "\tau_k \otimes \alpha"'] & \end{tikzcd} \right)$$

$\tilde{k} \otimes \tilde{A} \cong \tilde{A}$

$$\begin{tikzcd} \mathcal{S}(k\otimes A) \arrow[rr, "\mathcal{S}(l_A)"] \arrow[d, "\tau_{k,A}"'] & {} \arrow[d, phantom, "="] & \mathcal{S}(A) \arrow[dd, "\alpha"] \\ \mathcal{S}k \otimes \mathcal{S}A \arrow[r, "\tau_k \otimes 1"] \arrow[d, "\tau_k \otimes \alpha"'] \arrow[dr, phantom, "="] & k \otimes \mathcal{S}A \arrow[ur, "l_{\mathcal{S}(A)}"'] \arrow[dl, "1 \otimes \alpha"] \arrow[r, phantom, "\parallel"] & {} \\ k \otimes A \arrow[rr, "l_A"'] & & A \end{tikzcd}$$

compatibility of $\tau_{X,Y}$, τ_k

$\mathcal{S}$

b/c of naturality of $l_{(-)}$

$$
\begin{array}{ccc}
\mathrm{Alg}(S) & \qquad & \tilde{A} = (A,\ S(A) \xrightarrow{\alpha} A) \\
\Big\downarrow{\scriptstyle U_S} & & \Big\downarrow \\
\mathcal{C} & & A
\end{array}
$$

$$U_S(\tilde{k}) = k.$$

$$U_S(\tilde{A} \otimes \tilde{B}) = U_S(\tilde{A}) \otimes U_S(\tilde{B}) = \cancel{U_S}(A \otimes B$$

U_S is a strict monoidal functor.

$$\left\{ \begin{array}{c} \text{monoidal} \\ \text{monads} \\ \text{on} \\ \mathcal{C} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{monoidal} \\ \text{structures on} \\ \mathrm{Alg}(S) \text{ which} \\ \text{make } U_S \text{ a} \\ \text{strict monoidal} \\ \text{functor.} \end{array} \right\}$$

Prop.

The category CjSlat of complete join semi-lattices is symmetric closed monoidal.

Defn: If M, N, L are sup-lattices ~~and~~ then $f : M \times N \to L$ is a ~~bimorphism~~ bihomomorphism of sup-lattices if f preserves suprema in each variable i.e.

$$f\Big(\bigvee_{i \in I} x_i , y\Big) = \bigvee_{i \in I} f(x_i, y)$$

and

$$f\Big(x , \bigvee_{j \in J} y_j\Big) = \bigvee_{j \in J} f(x, y_j)$$

In CjBLat $M \otimes N$ is the codomain of universal bimorphism

$$\begin{tikzcd} M \times N \arrow[r, "{[\,]}"] \arrow[dr, "f"'] & M \otimes N \arrow[d, dashed, "\exists! \bar{f}"] \\ & L \end{tikzcd}$$

$M \otimes N$ can be obtained as a quotient of free sup-lattice of $M \times N$ (which is $P(M \times N)$) by the equivalence relation generated by

$$\bigvee_{i \in I} x_i \otimes y \sim \left(\bigvee_{i \in I} x_i\right) \otimes y \quad \text{and}$$

$$\bigvee_{j \in J} (x \otimes y_j) \sim x \otimes \left(\bigvee_{j \in J} y_j\right)$$

$$M \otimes N = \frac{P(M \times N)}{\sim}$$

Note: In particular $0 = 0 \otimes y$

$0 = \cancel{x} \otimes 0$

$\forall x, y$.

Unit of tensor on CjSLat

free (complete) sup-lattice on one generator

$P1 = \{\bot \leq \top\}$

CjSLat	AbGrp
$\frac{P(M\times N)}{\sim} = M\otimes N$	$A\otimes B = \frac{Fr_{Ab}(A\times B)}{\sim}$
free sup-lattice on one-generator $= P1$	$\mathbb{Z} =$ free abelian group on one generator
$\bigvee_{i\in I} a_i$	$\sum_{i\in I:\text{finite}} a_i$

$P1\otimes M \cong M$, $M\otimes P1 \cong M$

$$\begin{aligned} P1\otimes M &\xrightarrow{\cong} M \\ \bot\otimes m &\longmapsto 0 \\ \top\otimes m &\longmapsto m \end{aligned}$$

Rmk: M^{op}: opposite poset (category)

$$\mathrm{Hom}(M, (P1)^{op}) \cong \mathrm{Hom}(P1, M^{op})$$

$$\cong M^{op}$$

internal hom : Closed structure.

$$\mathrm{Hom}(M,N) \cong \mathrm{Hom}(N^{op}, M^{op})$$

$$\cong \mathrm{Hom}(N^{op}, \mathrm{Hom}(P1, M^{op}))$$

$$\cong \mathrm{Hom}(N^{op}, \mathrm{Hom}(M, P1^{op}))$$

$$\cong \mathrm{Hom}(N^{op} \otimes M, P1^{op})$$

$$\cong (N^{op} \otimes M)^{op}$$

Also

$$M \otimes N \cong \mathrm{Hom}(M, N^{op})^{op}$$

tensor is given by Hom.

Rmk. G Slat ($\mathcal{S}$) where

$\mathcal{S}$ is any elementary

topos.

In that case

unit is $P1 \cong \Omega$

subobject classifier.

You can do this internal to every topos $\underline{\mathcal{E}}$. $\rightsquigarrow$ K-Z-monad

$1 = * = \{pt\}$

$$(\mathrm{Set}, \times, *) \circlearrowleft P$$

P: power set monad

algebras of P $\cong$ sup-lattices

structure map $\cong$ structure map

$$(X, \alpha : PX \longrightarrow X)$$

α is a join.

$$\begin{tikzcd}
\mathrm{Alg}(P) \arrow[d, "U_P"'] \\
(\mathcal{E}, \times, 1) \circlearrowleft P
\end{tikzcd}$$

Note that U_P is not a

$\underline{\text{Guess}}$ fibers of U_P = propositions. (at best) or contractible.

Symmetric algebra monad:

Idea: for a vector space $V/_k$,

SV = free commutative algebra over V

Construction: $V/_k$: a vector space over a field k.

The symmetric algebra SV is generated by elements of V using operations:

(i) addition and scalar multiplication

(ii) an associative binary operation $\circ$

$V/_k \longrightarrow$ Consider V as a set

$\downarrow$

Consider algebra generated by V

$x \in V,\ y \in V \longmapsto x + y \in \langle V \rangle$

$x \in V,\ \lambda \in k \longmapsto \lambda x \in \langle V \rangle$

$x \in V,\ y \in V \longmapsto x \circ y \in V$ subjet to

+ $(x \circ y) \circ z = x \circ (y \circ z)$
+ $(\lambda x) \circ (\lambda' y) = (\lambda \lambda')(x \circ y)$
+ $v \circ w = w \circ v$

Prop.

SV is a commutative graded algebra

spanned by p-fold products, that is

elements of the form

$$V^{\otimes p} = \{ v_1 \cdots v_p \} \cong \{ v_1 \otimes \cdots \otimes v_p \}$$

$$S_p \times V^{\otimes p} \longrightarrow V_p^{\otimes p}$$

$$(\pi, v_1 \cdots v_p) \longmapsto \prod_{i=1}^{p} v_{\pi(i)}$$

More generally,

Suppose $(\mathcal{C}, \otimes, k)$ a symm. monoidal w/ countable coproducts category.

$V \in \mathcal{C}$

Form tensor powers $V^{\otimes n}$

and their countable coproduct

$$TV = \bigoplus_{n \geq 0} V^{\otimes n} \in \mathcal{C}$$

Q: Is TV a monoid object in $\mathcal{C}$?

Yes, if the tensor product distributes over (these) coproducts

$$k \to TV = \bigoplus_{n \geq 0} V^{\otimes n} \quad \text{is}$$

just embedding of a summand.

$$TV \otimes TV \xrightarrow{m} TV \quad \text{is got by}$$

$$\left(\bigoplus_{m\geq 0} V^{\otimes m} \otimes \bigoplus_{n \geq 0} V^{\otimes n}\right) \xrightarrow{\cong} \bigoplus_{\substack{m\geq 0\\ n\geq 0}} V^{\otimes m} \otimes V^{\otimes n} \xrightarrow{\cong} \bigoplus_{\substack{m\geq 0\\ n\geq 0}} V^{\otimes (m+n)} \to TV$$

Action of symmetric group S_n

$$S_n \times \boxed{V^{\otimes n}} \longrightarrow V^{\otimes n}$$

$$\boxed{S_n} \longrightarrow \boxed{\mathcal{C}(V^{\otimes n}, V^{\otimes n})}$$

$$\pi \longmapsto V^{\otimes n} \longrightarrow V^{\otimes n}$$

$$V \otimes V \otimes \cdots \otimes V \longrightarrow V \otimes \cdots \otimes V$$

$\mathcal{C}$: linear category

$$S_n \longrightarrow \mathcal{C}(V^{\otimes n}, V^{\otimes n})$$
$$\sigma \longmapsto \hat{\sigma}$$

define $\quad P_A : V^{\otimes n} \longrightarrow V^{\otimes n}$

$$P_A = \frac{1}{n!} \sum_{\sigma \in S_n} \hat{\sigma}$$

$$P_A \cdot P_A \in \mathcal{C}(V^{\otimes n}, V^{\otimes n})$$

$$P_A \cdot P_A = \frac{1}{n!} \frac{1}{n!}$$

$$\mathcal{C}(V^{\otimes n}, V^{\otimes n}) \times \mathcal{C}(V^{\otimes n}, V^{\otimes n}) \longrightarrow \mathcal{C}(V^{\otimes n}, V^{\otimes n})$$
$$\hat{\sigma} \qquad\qquad \hat{\sigma} \longmapsto \hat{\sigma}^2 = \hat{\sigma}$$

$$\hat{\sigma}\hat{\tau} = \widehat{\sigma\tau}$$

$$P_A^2 = \frac{1}{n!} \frac{1}{n!} \sum_{\substack{\sigma \in S_n \\ \tau \in S_n}} \widehat{\sigma\tau}$$

I have a group G of size n.

$$\frac{1}{n}\sum_{g\in G} g = \boxed{\operatorname{avg}(G)}$$

$$\frac{1}{n}\frac{1}{n}\sum_{\substack{g\in G\\ h\in G}} gh = \operatorname{avg}(G) \qquad \operatorname{avg}_2(G)$$

$$\frac{1}{n^2}\left(\begin{array}{l} g_0 \cancel{(e+g)} \; (e+g_1 \cdots\cdots g_{n-1}) \\ g_1 \quad (e+g_1+\cdots+g_{n-1}) \\ \qquad \vdots \end{array}\right)$$

$$\cancel{\operatorname{avg}(G)}$$

$$\sum_{g\in G} g \quad \sum_{g\in G} g \quad \cdots \quad \sum_{g\in G} g$$

$$\frac{1}{n^2}\left(n^2 \cdot \operatorname{avg}(G) = \right) = \operatorname{avg}(G)$$

e.g.

$$S_2 \longrightarrow \mathcal{L}(V^{\otimes 2}, V^{\otimes 2})$$

$$1 \longmapsto \hat{1}\,(v_1 \otimes v_2) = v_1 \otimes v_2$$

$$\sigma \longmapsto \hat{\sigma}\,(v_1 \otimes v_2) = v_2 \otimes v_1$$

$$P_A = \frac{1}{2!}(\hat{1} + \hat{\sigma})$$

$$P_A(v_1 \otimes v_2) = \frac{1}{2} v_1 \otimes v_2 + \frac{1}{2} v_2 \otimes v_1 =$$

$$\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$$

$\hat{1}\hat{1} =$

$\hat{\sigma}\hat{\sigma} =$

$\hat{\sigma}\hat{1} =$

$\hat{1}\hat{\sigma} =$

$$P_A\left(\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)\right) =$$

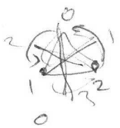
$$\frac{1}{2!}(\hat{1} + \hat{\sigma})\left(\frac{v_1 \otimes v_2 + v_2 \otimes v_1}{2}\right) =$$

$$\frac{1}{4}(\boxed{v_1 \otimes v_2} + v_2 \otimes v_1) + \frac{1}{4}(v_2 \otimes v_1 + v_1 \otimes v_2)$$

$$\frac{1}{2}(v_1 \otimes v_2) + \frac{1}{2}(v_2 \otimes v_1)$$

$$\frac{1}{3!}(\hat{1} + \hat{\mu} + \hat{\mu}^2 + \hat{\rho}_1 + \hat{\rho}_2 + \hat{\rho}_3)$$

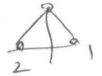
$v_1 \otimes v_2 \otimes v_3$ $\quad$ $v_1 \otimes v_2 \otimes v_3$



$$\frac{1}{6}(v_1 \otimes v_2 \otimes v_3 + v_3 \otimes v_1 \otimes v_2 + v_2 \otimes v_3 \otimes v_1 +$$

$$(v_2 \otimes v_1) \otimes v_3 + v_1 \otimes v_3 \otimes v_2 + v_3 \otimes v_2 \otimes v_1) =$$

$(v_2 \otimes v_1) \otimes v_3$ $\quad$ $v_1 \otimes v_2 \otimes v_3$



9

$$p_A : V^{\otimes n} \longrightarrow V^{\otimes n}$$

$$p_A^2 = p_A$$

If idempotent splits in $\mathcal{C}$, we can form its cokernel:

<u>Remark</u>

$$\begin{tikzcd} A \arrow[r, shift left, "e"] \arrow[r, shift right, "1"'] & A \arrow[r, dashed, two heads, "r"] & B \end{tikzcd}$$

is a coequalizer

$$\begin{tikzcd} A \arrow[r, shift left, "e"] \arrow[r, shift right, "1"'] & A \arrow[r, "r"] \arrow[dr, "f"'] & B \arrow[d, dashed, "fs = x"] \\ & & C \end{tikzcd}$$

$$fe = f$$

$$fsr = f$$

$$xr = f$$

$$\cancel{xr = xrsr = xre = fe = fsr}$$

$$x = xrs = fs$$

$v_1 \otimes \cdots \otimes v_n \longmapsto [v_1 \otimes \cdots \otimes v_n]$

$$V^{\otimes n} \underset{1}{\overset{p_A}{\rightrightarrows}} V^{\otimes n} \overset{\pi}{\dashrightarrow} S^n V$$

and

$$SV = \bigoplus_{n \geq 0} S^n V$$

SV = free commutative monoid object in $(\mathcal{C}, \otimes, k)$

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & SV \arrow[d, dashed] \\ & A \end{tikzcd}$$

A : comm. monoid.

<u>Prop.</u> $S_\bullet$-algebras are exactly k-commutative unital algebras.