Lecture 1 2022/9/7

主讲老师：郅彦奇、

Final：40% + 60%
平时 期末.
(第九周周五).

教材：《Principles of Functional Analysis》 by Martin Schechter, GSM 36.

$$\begin{cases} f''(x) + f(x) = g(x) \\ f(a) = 1, \quad f'(a) = 0. \end{cases} \quad x \in [a,b].$$

Then the solution is given by

$$f(x) = \cos(x-a) + \int_a^x \sin(x-t) g(t)\,dt.$$

Home work：If $k(x,t)$ is C^1, $a \le x, t \le b$. set $F(x) := \int_a^x k(x,t)\,dt$. 求 $F'(x)$ 给出理由.

$\left(F'(x) = k(x,t) + \int_a^x k'(x,t)\,dt \right)$

$$\begin{cases} f''(x) + f(x) = \sigma(x) \cdot f(x) \\ f(a) = 1, \quad f'(a) = 0 \end{cases}$$

，其中 $\sigma(x)$ 已知.

ODE $\downarrow$ integral equation

$\downarrow$ 它的解一定满足.

$$f(x) = \underbrace{\cos(x-a)}_{u(x)} + \underbrace{\int_a^x \underbrace{\sin(x-a)\sigma(t)}_{k(x,t)} f(t)\,dt}_{Kf}.$$

$f = u + Kf$. (※)

Picard iteration.

for any given continuous function f_0 defined on $[a,b]$.

$f_1 = u + Kf_0$

$f_2 = u + Kf_1$

$f_3 = u + Kf_2$

$\vdots$

$f_{n+1} = u + Kf_n$.

形式上. 若 "$f_n \to f_\infty$"

$f_\infty = u + Kf_\infty$

极限在何种意义下.

$h \in C[a,b]$. $\|h\| := \sup\limits_{a \le t \le b} |h(t)|$.

Fact：$\|f_n - f_m\|$ is small when n, m 足够大. (Cauchy 列)

则 $f_\infty(t) = \lim\limits_{n\to\infty} f_n(t)$ (一致收敛，其极限 $f_\infty \in C[a,b]$)

若 ① $\{f_n\}$ is a Cauchy seq. in $C[a,b]$. 则 $f_n \overset{n\to\infty}{\rightrightarrows} f_\infty \in C[a,b]$

$f = u + \lim\limits_{n\to\infty} Kf_n$

② 希望 $f_n \rightrightarrows f$ 能够推出 $Kf_n \rightrightarrows Kf$.

则 ①+②

$f = u + Kf$

$f_2 = u + Kf_1 = u + K(u + Kf_0) = u + Ku + K^2 f_0$

$f_3 = u + Kf_2 = u + K(u + Kf_1) = u + Ku + K^2 u + K^3 f_0$

$\vdots$

$f_n = u + Ku + K^2 u + \cdots + K^{n-1} u + K^n f_0$

$\|f_m - f_n\| = \| K^n(u - f_0) + K^{n+1} u + \cdots K^{m-1} u + K^m u \|$.

fact：

$\|h_1 + h_2\| \le \|h_1\| + \|h_2\|$

$\|\lambda h\| = |\lambda| \, \|h\|$

Claim A：$\forall h \in C[a,b]$. 都有 $\sum\limits_{n=0}^{\infty} \|K^n h\| < +\infty$

$\Rightarrow \|f_m - f_n\| \le \sum_{n}^{m-1} \|K^l u\| + \|K^n u\| + \|K^m u\|$

Claim B: $\forall\, h \in C[a,b]$. $\|Kh\| \le M\|h\|$

By Claim A. $\{f_n\}$ is a Cauchy Seq, so ① 成立.

And Claim B $\Rightarrow$ ②

Proof of A: $Kf = \int_a^x \sin(x-t)\,\sigma(t) f(t)\,dt$. $\Rightarrow |Kf(x)| \le \int_a^x |\sin(x-t)\,\sigma(t) f(t)|\,dt$

$\le (x-a)\cdot \|\sigma\|\,\|f\|$

$\Rightarrow |K^2 f(x)| \le \|\sigma\| \int_a^x |Kf|\,dt \le \frac{1}{2}(x-a)^2 \cdot \|\sigma\|^2 \cdot \|f\|$

$\vdots$

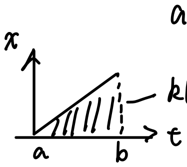
$|K^n f(x)| \le \frac{\|\sigma\|^n}{n!}(x-a)^n \cdot \|f\|$

$\vdots$

so $\sum_{n=0}^{+\infty} \|K^n h\| \le \sum_{n=0}^{\infty} \frac{\|\sigma\|^n (b-a)^n}{n!} \|h\| = e^{\|\sigma\|\cdot(b-a)} \cdot \|h\|$.

$f(x) = u(x) + \int_a^x k(x,t) f(t)\,dt$ — Volterra equation

$a \le t \le x \le b$.



k(x,t) 定义域.

连续.

① Vector space.

(i) addition $+$

(ii) Scalar multiplication $1\cdot v = v$.

② Normed space 赋范空间

$\|\cdot\|: V \to [0,+\infty)$

称为范数 $v \to \|v\|$

semi norm
(i) $\|v\| \ge 0$
(ii) $\|\alpha v\| = |\alpha|\,\|v\|$
(iii) $\|v+w\| \le \|v\| + \|w\|$

(i) $\to$ (i'): $\|v\| \ge 0$

norm: and $(\|v\| = 0 \Leftrightarrow v = 0)$

③ Banach Space (Complete normed Space).

V is a normed space

and all Cauchy seqs are convergent.

Def: 若 $\{v_n\}$ 满足 $v_n \in V$.

$\forall\, \varepsilon > 0$. $\exists\, N \in \mathbb{N}$. $\forall\, n, m \ge N$.

$\|v_n - v_m\| < \varepsilon$.

则存在 $v \in V$. s.t. $\|v_n - v\| \xrightarrow{n\to\infty} 0$

④ Linear operators.

V_1. V_2 normed spaces.

(线性算子).

$K: V_1 \to V_2$ map is called linear operator

if: $K(\lambda v + \mu w) = \lambda K v + \mu K w$.

$\forall\, \lambda, \mu \in \mathbb{R}/\mathbb{C}$, $v, w \in V_1$.

bounded linear operators.

(continuous linear operators).

$\exists\, M > 0$, s.t. $\|Kv\| \le M\|v\|$, $\forall\, v \in V_1$

Thm 1.1. Let X be a Banach Space

$K: X \to X$.

a) $K(v+w) = Kv + Kw$

b) $K(-v) = -Kv$.

c) $\|Kv\| \le M \cdot \|v\|$

d) $\sum_n \|K^n v\| < +\infty$

Then $\forall u \in X$.

$f = u + Kf$.

存在唯一解.

唯一性:

$$\begin{cases} f_1 = u + Kf_1 \\ f_2 = u + Kf_2 \end{cases}$$

$\Rightarrow f_1 - f_2 = K(f_1 - f_2)$

$\Rightarrow f_1 - f_2 = K^n(f_1 - f_2)$

and $\|K^n(f_1-f_2)\| \to 0 \Rightarrow f_1 = f_2$.

Examples: $\mathbb{R}^n$, Euclidean norm

$v = (v^{(1)}, v^{(2)}, \cdots, v^{(n)})$, $\|v\| := \sqrt{\sum_{i=1}^n v_i^2}$

$\{v_n\} \in \mathbb{R}^n$. 在每个分量上.

$|v_n^{(j)} - v_m^{(j)}| \le \|v_n - v_m\|$

$\Rightarrow v_n^{(j)}$ 收敛于 $v^{(j)}$

(根据实数完备性).

Cauchy-Schwartz inequality

(三角不等式).

inner product.

$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$v, w \longmapsto \sum_{i=1}^n v_i w_i$.

denote $(v, w) = \sum_{i=1}^n v_i w_i$

内积诱导范数. $\|v\| = \sqrt{(v,v)}$.

正定双线性型

$V \times V \xrightarrow{B} R$.

$$\begin{cases} B(v,v) \ge 0 \\ B(v,w) = B(w,v). \end{cases}$$

$B(v,w)^2 \le B(v,v) \cdot B(w,w)$.

$\forall v, w \in \mathbb{R}^n$. $\forall \alpha \in \mathbb{R}$.

$(\alpha v + w, \alpha v + w) \ge 0$

$\parallel$

$\alpha^2 (v,v) + (w,w) + 2\alpha(v,w). \ge 0$.

$\Delta = 4(v,w)^2 - 4(v,v)(w,w) \le 0 \Rightarrow (v,w)^2 \le (v,v)(w,w)$.

$\|v+w\|^2 = v^2 + 2(v,w) + w^2 \le v^2 + 2\sqrt{v^2 \cdot w^2} + w^2 = (\|v\| + \|w\|)^2$

Triangle inequality.

习题 2、3、11、22

Hilbert Space

V, Scalar product. (inner product).

$V \times V \to \mathbb{R}$ (v, w) . bilinear.

· 正定

非退化 $(v,v) = 0 \Leftrightarrow v = 0$

完备.

$\|v\| = \sqrt{(v,v)}$.

$(V, \|\cdot\|)$ is a Banach Space.

Lecture 2
2022/9/7

Example 2. ① ℓ_∞ ($\ell_\infty(\mathbb{N})$)

$\ell_\infty(\mathbb{N}) = \{(x_n)_1^\infty \mid x_n \in \mathbb{R},\ \sup |x_n| < \infty\}$

$\|(x_n)_{n=1}^\infty\| = \sup_n |x_n|$.

In general. I index set, $\ell_\infty(I)$: $(x_i)_{i\in I}$.

② $c_0 = \{(x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{R},\ \lim_{n\to\infty} x_n = 0\}$

$\|(x_n)_{n\in\mathbb{N}}\| = \sup_n |x_n|$.

Example 3. ℓ_2. ($\ell_2(\mathbb{N}) \overset{推广}{,}\ \ell_2(I)$).

$\ell_2 = \{(x_n)_{n=1}^\infty \mid x_n \in \mathbb{R},\ \sum_n |x_n|^2 < +\infty\}$.

$\|(x_n)_{n\in\mathbb{N}}\| = \sqrt{\sum_n |x_n|^2}$.

ℓ_2 上有 inner product.

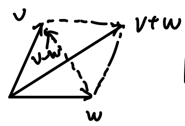
$((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_n x_n y_n$.

↑验证该级数收敛.

Cauthy-schwartz inequality.

$\sum_n^m x_k y_k \le \sqrt{\sum_n^m x_k^2}\cdot\sqrt{\sum_n^m y_k^2} < \varepsilon$. Cauthy sequence

平行四边形法则.



$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$.

Trivial to verify that in Hilbert space. Actually it is sufficient for a Hilbert space.

$\|v+w\|^2 = (v+w, v+w) = \|v\|^2 + \|w\|^2 + 2(v,w)$.

$\|v-w\|^2 = (v-w, v-w) = \|v\|^2 + \|w\|^2 - 2(v,w)$.

Isometric isomorphism
(等距同构)

Def: X, Y two banach spaces are called Isometric isomorphism iff.

若存在有界线性算子 T, S. $X \xrightarrow{T} Y$, $Y \xrightarrow{S} X$.

$\begin{cases} ST = id_X \\ TS = id_Y \end{cases}$

且 $\|x\|_X = \|Tx\|_Y$. $\forall x \in X$.

显然:
$\|y\|_Y = \|Sy\|_X$.
$\forall y \in Y$.

Fourier series.

Consider $f \in C^1[0, 2\pi]$.

$f(0) = f(2\pi)$

级数 uniformly 收敛.

则 $f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx$.

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$.

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$.

$\varphi_0 = \frac{1}{\sqrt{2\pi}}$ $\qquad \alpha_0 = \sqrt{\frac{\pi}{2}}\, a_0$

$\varphi_{2k}(x) = \frac{\cos kx}{\sqrt{\pi}}$ $\qquad \alpha_{2k} = \sqrt{\pi}\, a_k$.

$\varphi_{2k+1}(x) = \frac{\sin kx}{\sqrt{\pi}}$. $\qquad \alpha_{2k+1} = \sqrt{\pi}\, b_k$.

$f(x) = \sum_{n=0}^\infty \alpha_n \varphi_n(x)$

$\int_0^{2\pi} \varphi_m(x)\varphi_n(x)\,dx = \begin{cases} 1 \\ 0 \end{cases}$.

$\begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$

orthonormal
标准正交.

Indeed, $\int_0^{2\pi} \big(\underbrace{\sum_0^{\infty} \alpha_n \varphi_n(x)}_{\text{关于}x\text{一致收敛}}\big) \varphi_j(x)\, dx = \sum_0^{\infty} \int_0^{2\pi} \alpha_n \cdot \varphi_n(x) \cdot \varphi_j(x)\, dx = \alpha_j$.

$$\sum_0^n \alpha_j^2 = \sum_0^n \alpha_j \cdot \int_0^{2\pi} f(x) \varphi_j(x) dx = \int_0^{2\pi} f(x) \cdot \sum_0^n \alpha_j \varphi_j(x)\, dx = \int_0^{2\pi} f(x) \cdot f_n(x)\, dx.$$

$n \to \infty$ (验证 $\int_0^{2\pi} f^2(x) dx = \lim_{n\to\infty} \int_0^{2\pi} f(x) f_n(x)\, dx$).

$$\Rightarrow \sum_0^{\infty} \alpha_j^2 = \int_0^{2\pi} f(x)^2\, dx. \quad \text{Planchel identity}$$

$f \in C^1_{2\pi}[0,2\pi]$. $\|f\| := \sqrt{\int_0^{2\pi} f(x)^2 dx}$ 可以验证是一个范数.

$(f,g) = \int_0^{2\pi} fg\, dx$ is an inner product on $C^1_{2\pi}([0,2\pi])$.

normed space 的完备化.

$(V, \|\cdot\|_V)$ is a normed space.

We say $(W, \|\cdot\|_W)$ is a completion of V.

即 W 是包含 V (等距同构意义下) 的最小 Banach space.

if ① $(W, \|\cdot\|_W)$ is a Banach space.

② $V \xrightarrow[\text{isomorphism}]{\text{Isometric}} W_0 \subset W$. W_0 is dense in W.

Proposition: $\forall$ normed space, $\exists!$ completion

Proof: 存在性: V. 考虑V上所有的 Cauthy 列

$$\hat{V} = \{G = \{g_k\}_{k=1}^{\infty} \mid G \text{ is a Cauthy seq in } V\}.$$

$\hat{V}$ 上加法和数乘 $\alpha G + \beta H = \{\alpha g_k + \beta h_k\}_{k=1}^{\infty}$

$\hat{V} \ni G, H$ 称为等价的 $G \sim H$. 若 $\lim_{k\to\infty} \|g_k - h_k\| = 0$

$\tilde{V} := \hat{V}/\sim$. 加法和数乘仍是 well-defined.

$\tilde{V}$ 上定义范数 $\|\{g_k\}\| := \lim_{k\to\infty} \|g_k\|$

要验证: ① 极限存在 ② 极限不依赖等价类代表元的选取.

不用交的作业.

三角不等式蕴含: $\big|\|x\| - \|y\|\big| \le \|x-y\|$.

故 $\{g_k\}$ Cauthy $\Rightarrow \{\|g_k\|\}$ Cauthy $\Rightarrow \lim_k \|g_k\|$ 存在.

下证 $\tilde{V}$ 是 Banach Space.

$\{G_n\}_{n=1}^{\infty}$ is a Cauthy Seq in $\tilde{V}$, $G_n = \{g_{nk}\}_{k=1}^{\infty}$ a Cauthy seq in V.

$\|G_n - G_m\| < \varepsilon$ for n, m Large enough.

set $g_n = g_{nn}$, Then $G = \{g_n\}$. $\|G_n - G\| \to 0$ as $n \to \infty$.

To make $G \in \tilde{V}$. G must be a Cauthy seq in V.

we set $g_n = g_{nN_n}$. and N_n be the number s.t. $\forall\, k, l > N_n$. $\|g_{nk} - g_{nl}\| < \frac{1}{n}$.

$G = \{g_{nN_n}\}$ is a Cauthy seq. and $\|G_n - G\| \to 0$ as $n \to +\infty$.

$$\|g_{nN_n} - g_{mN_m}\| \le \|g_{nN_n} - \lim_{k\to\infty} g_{nk}\| + \|G_n - G_m\| + \|\lim_{k\to\infty} g_{mk} - g_{mN_m}\|$$
$$\le \frac{1}{n} + \frac{1}{m} + \|G_n - G_m\|.$$

$V \to V_0$

$v \to (v, v, \cdots)$. 是一个等距同构. $\|v\|_V = \|(v, v, \cdots)\|_{\tilde{V}}$

且 V_0 is dense in $\tilde{V}$.

Remark: 度量空间均可完备化.

$$\begin{tikzcd}
 & W \arrow[ddd, leftrightarrow, bend left=70, "\Psi"] \\
V \arrow[r, "\text{同构}", "\phi"'] & W_0 \arrow[u, phantom, "\cup\ \text{dense}"] \arrow[d, leftrightarrow, "\Phi"] \\
V \arrow[r, "\text{同构}", "\varphi"'] & U_0 \arrow[d, phantom, "\cap\ \text{dense}"] \\
 & U
\end{tikzcd}$$

Defin Ψ:

$\forall\, w \in W$. 存在 $\{w_n\} \subset W_0$, $\|w_n - w\| \to 0$ as $n \to \infty$

$\Psi(w) := \lim_{n\to\infty} \Phi(w_n)$ $\Downarrow$ Cauthy seq.

$\Rightarrow$ $\Phi(w_n)$ is also a Cauthy seq in U.

so $\Psi(w)$ 存在. 验证良定义. 双射.

$l_2 \xrightarrow{\text{等距同构}} L^2([0, 2\pi])$

$(\alpha_j)_{j=0}^{\infty} \longmapsto \sum_{j=0}^{\infty} \alpha_j \varphi_j$.

$\|\sum_{j=0}^{\infty} \alpha_j \varphi_j\|_{L^2}^2 = \sum_{j=0}^{\infty} \alpha_j^2$

Lecture 3 2022/9/12

$\ell^2 = \{(\alpha_0, \alpha_1, \cdots) \mid \sum_0^{+\infty} \alpha_i^2 < +\infty\}$ Then $\exists$ one to one corespondence

$L^2 = L^2([0,2\pi])$

$\ell^2 \to L^2 : (\alpha_0, \alpha_1, \cdots) \to f.$

such that

- $\| \sum_0^n \alpha_j \varphi_j - f \|^2 \to 0$ as $n \to +\infty$
- $\|f\|_{L^2}^2 = \sum_0^{+\infty} \alpha_j^2.$ Bessel's identity

$\alpha_j = (f, \varphi_j) = \int_0^{2\pi} f(x) \varphi_j(x)\, dx.$

形式上直接写 $f = \sum_0^{+\infty} \alpha_j \varphi_j$

但是（$f(x) = \sum_0^{+\infty} \alpha_j \varphi_j(x)$ 并未 a.e.）

Carleson result.

Proof: $(\alpha_0, \alpha_1, \cdots) \in \ell_2$.

set $f_n(x) = \sum_0^n \alpha_j \varphi_j(x) \in C_{2\pi}^1$

$\int_0^{2\pi} \varphi_i \varphi_j = \delta_j^i = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$

Claim 1. $\{f_n\}$ Cauthy seq in L^2, $n > m$

$\|f_n - f_m\|_{L^2}^2 = \|\sum_{m+1}^n \alpha_j \varphi_j\|_{L^2}^2 = \sum_{m+1}^n \alpha_j^2 \xrightarrow{\text{OK}}$ Cauthy.

Claim 2. L^2 是完备的. Homework. $\exists f \in L^2$. $\|f_n - f\|_{L^2} \xrightarrow{n\to\infty} 0$

Claim 3. $\alpha_m = (f, \varphi_m)$ $m = 0, 1, 2, \cdots$

$\forall n \geq m$, $(f_n, \varphi_m) = (\sum_0^n \alpha_j \varphi_j, \varphi_m) = \sum_0^n \alpha_j (\varphi_j, \varphi_m) = \alpha_m$

由 Cauthy-Schwartz 不等式 $|(f, \varphi_m) - (f_n, \varphi_m)| \leq \underbrace{\|f - f_n\|}_{\downarrow 0} \cdot \underbrace{\|\varphi_m\|}_{=1}$

$\Rightarrow (f, \varphi_m) = \lim_{n\to\infty} (f_n, \varphi_m) = \alpha_m.$

Claim 4. $\|f\|^2 = \sum_0^\infty \alpha_j^2$, $\|f_n\|^2 = \sum_0^n \alpha_j^2$, $|\,\|f\| - \|f_n\|\,| \leq \|f - f_n\| \to 0$

so $\|f\| = \lim_{n\to\infty} \|f_n\| = \lim_{n\to\infty} \sqrt{\sum_0^n \alpha_j^2} = \sqrt{\sum_0^{+\infty} \alpha_j^2}$

Conversely. $\forall f \in L^2$, $\exists g_n \in C_{2\pi}^1$, $\|f - g_n\|_{L^2} \xrightarrow{n\to+\infty} 0$

$g_n(x) = \sum_0^{+\infty} \alpha_j^{(n)} \varphi_j(x)$

uniformly convergent in $[0, 2\pi]$.

$\{g_n\}$ is a Cauthy seq

$\Rightarrow \{(\alpha_j^{(n)})_{j=0}^{+\infty}\}_{n=1}^{+\infty}$ Cauthy seq in ℓ_2.

Indeed, $\|(\alpha_j^{(n)})_{j=0}^{+\infty} - (\alpha_j^{(m)})_{j=0}^{+\infty}\|_{\ell^2}^2 = \|(\alpha_j^{(n)} - \alpha_j^{(m)})_{j=0}^{+\infty}\|_{\ell^2}^2 = \|g_n - g_m\|_{L^2}^2$

由于 ℓ_2 是完备的，故 $\exists\ (\alpha_1, \alpha_2, \cdots) \in \ell_2$.

s.t. $\sum_{j=0}^{+\infty} (\alpha_j^{(n)} - \alpha_j)^2 \xrightarrow{n\to+\infty} 0$. $\quad \tilde{f} = \sum_0^{+\infty} \alpha_j \varphi_j$. 只需证 $f = \tilde{f}$ in L^2.

$\Leftrightarrow \|f - \sum_0^n \alpha_j \varphi_j\| \xrightarrow{n\to+\infty} 0$, $\quad \|f - \sum_0^{+\infty} \alpha_j\varphi_j\| \le \|f - g_m\| + \|g_m - \sum_0^{+\infty}\alpha_j\varphi_j\|$

$< 2\varepsilon$. when m is large sufficiently.

$\Rightarrow \|f - \sum_0^{+\infty}\alpha_j\varphi_j\| = 0$

H real Hilbert Space

let $\{v_n\}_{n=1}^{\infty} \subset H$. $\quad 1^\circ$ we say $\{v_n\}_{n=1}^{\infty}$ are orthonormal

if $(v_i, v_j) = \delta_j^i = \begin{cases} 0, & \text{if } i \ne j \\ 1, & \text{if } i = j. \end{cases}$

2° We say $\{v_n\}_{n=1}^{\infty}$ is $\boxed{\text{complete}}$ in H.

if $\text{span}\{v_n\}_{n=1}^{\infty} \overset{\text{dense}}{\subset} H$. $\quad$ span：有限线性组合.

Thm. If $\{v_n\} \subset H$ is orthonormal & complet system in H.

Then $\forall\ w \in H$ 均有 $\bullet\ w = \sum_1^{\infty} (w, v_i)\cdot v_i$

$\bullet\ \|w\|^2 = \sum_1^{\infty} (w, v_i)^2$

Proposition if $\{u_i\}_{i\in I}$ orthonormal in H.

Then $\forall\ w \in H$. $\quad \|w\|^2 \ge \sum_{i\in I} (w, v_i)^2 \quad$ Bessel's inequality.

Proof:

$0 \le \|w - \sum_{i=1}^n (w, v_i) v_i\|^2 = (w - \sum_{i=1}^n (w, v_i)v_i,\ w - \sum_{i=1}^n (w, v_i) v_i)$

$= \|w\|^2 - \sum_{i=1}^n (w, v_i)^2$

Now come back to the Thm,

只有有限个非零

Proof: $\forall\ w \in H$. $\exists \{w_n = \sum_{j=1}^{+\infty} \alpha_j^{(n)} v_j\}$ s.t. $\|w_n - w\| \to 0$

$(\alpha_1^{(n)}, \alpha_2^{(n)}, \cdots \alpha_{N_n}^{(n)}, 0, 0, \cdots) \in \ell^2 \Rightarrow \{w_n\}$ Cauchy seq $\Rightarrow \{(\alpha_j^{(n)})_{j=1}^{+\infty}\}_{n=1}^{+\infty}$ Cauchy seq in ℓ^2

$\Rightarrow \exists\, (\alpha_1, \alpha_2, \cdots) \in \ell^2$, s.t. $\sum_{j=1}^{+\infty} (\alpha_j^{(n)} - \alpha_j)^2 \xrightarrow{n \to +\infty} 0$

$$\| w - \sum_{j=1}^{+\infty} \alpha_j v_j \| = \lim_{n\to\infty} \| w_n - \sum_{j=1}^{\infty} \alpha_j v_j \| = \lim_{n\to\infty} \| \sum_{j=1}^{\infty} (\alpha_j^{(n)} - \alpha_j) v_j \| \le \overset{\overset{0}{\shortparallel}}{\lim_{n\to\infty} \sqrt{\sum_{j=1}^{+\infty} (\alpha_j^{(n)} - \alpha_j)^2}}$$

$\Rightarrow \underline{w = \sum_{j=1}^{+\infty} \alpha_j v_j} = \sum_{j=1}^{+\infty} (w, v_j) \cdot v_j$. $\Rightarrow \|w\|^2 = \sum_{j=1}^{+\infty} (w, v_j)^2$ Paserval's identity.

Fourier series (抽象意义下).

Thm. $\{V_i\}_{i=1}^{\infty} \subset H$. orthonormal & complete. Then $\forall\, u, w \in H$.

$$(u, w) = \sum_{i=1}^{\infty} (u, v_i)(w, v_i)$$

Def. V is a vector space, $\ell : V \longrightarrow \mathbb{R}$ (linear map to $\mathbb{R}$ or $\mathbb{C}$) is a linear functional.

if V equipped with a norm $\|\cdot\|$, $\ell : V \to \mathbb{R}$ a linear functional.

We say ℓ is bounded if $\exists\, M > 0$, $|\ell(x)| \le M \cdot \|x\|$. $\forall\, x \in V$.

Example. H is a Hilbert space, fix $v \in H$.

$\ell_v : H \longrightarrow \mathbb{R}$ defined by $\ell_v(w) = (w, v)$ is a linear functional.

又因为 $|\ell_v(w)| = |(w, v)| \le \|w\| \cdot \|v\|$ so $\ell_v(w)$ is a bounded linear functional.

Thm. (Riesz representation Thm).

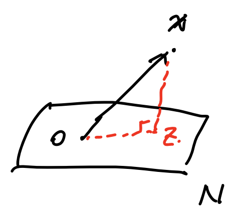
H Hilbert space, 则 $\forall\ \ell \in B(H) : H \to \mathbb{R}$

$\exists!\ w \in H$, s.t. $\ell = \ell_w$.

Lemma A $\quad N \subset H$ closed subspace, $x \notin N$

Then $\exists\ z \in N$, s.t.

$$\|x - z\| = \inf_{w \in N} \|x - w\| := d(x, N)$$



Proof: $\exists\ w_n \in N$, $\|x - w_n\| \xrightarrow{n\to\infty} d$.

平行四边形法则. $\quad 2\|x - w_n\|^2 + 2\|x - w_m\|^2 = \|2x - (w_n + w_m)\|^2 + \|w_n - w_m\|^2$

$$= 4\|x - \frac{(w_n + w_m)}{2}\|^2 + \|w_n - w_m\|^2$$

$$\geq 4d^2 + \|w_n - w_m\|^2.$$

当 n, m 充分大时. $\|x-w_n\|^2 \leq d^2 + \varepsilon/4$, $\|x-w_m\|^2 \leq d^2+\varepsilon/4$ $\Rightarrow$ $\|w_n-w_m\|^2 + 4d^2 \leq 4d^2 + \varepsilon$

$\Rightarrow \|w_n-w_m\|^2 \leq \varepsilon.$

so $\{w_n\}$ is a Cauthy sequence. It has a limit w in N (N is closed),

and $d \leq \|x-w\| \leq \lim\limits_{n\to\infty} \|x-w_n\| + \|w_n - w\| = d$, $w = z$.

<u>Lemma B.</u> The point z in Lemma A is unique & $x-z \perp N$ (即 $(x-z, w) = 0$ for $\forall\, w \in N$).

Proof: $\|x-(z+\alpha y)\|^2$ 在 $\alpha = 0$ 处取到极小值.

$\Rightarrow \|x-z\|^2 - 2\alpha(x-z, y) + \alpha^2\|y\|^2 \Rightarrow (x-z, y) = 0$

if $x-z \perp N$, $x-z' \perp N$. $(x-z, z-z') = 0$, $(x-z', z-z') = 0$ $\Rightarrow (z-z', z-z') = 0 \Rightarrow z = z'$.

Lemma A+B $\Rightarrow \forall\, x \in H$. $x = z + (x-z) = \underset{\in N}{z} + \underset{\in N^\perp}{y}$ 且表达是唯一的.

Pf: $l: H \to \mathbb{R}$

$\ker l = \{y \in H \mid l(y) = 0\} \underset{\text{closed}}{\overset{\text{subspace}}{\subset}} H.$ if $\{y_n\} \in \ker l$. and $y_n \to y$.

Then $l(y) = l(y-y_n) + l(y_n) \leq M\|y-y_n\| \to 0$, so $l(y) = 0$

① 若 $\ker l = H$, 则取 $w = 0$, $l = l_w$

② 若 $\ker l \subsetneq H$, 则存在 $u \neq 0$, $u \perp \ker l$.

$\forall\, v \in H$, $l(v)u - l(u)v \in \ker l$. $\Rightarrow l(v)u - l(u)v \perp u$.

$\Rightarrow 0 = (l(v)\cdot u - l(u)v, u) = l(v)\cdot\|u\|^2 - l(u)(v, u)$

$\Rightarrow l(v) = \dfrac{l(u)}{\|u\|^2}\cdot(v, u) = \left(v, \dfrac{l(u)\cdot u}{\|u\|^2}\right)$, 取 $z = \dfrac{l(u)\cdot u}{\|u\|^2}$.

则 $l(v) = l_z(v)$, 若 $l_{z'} = l_z \Rightarrow (z'-z, v) = 0 \ \forall\, v \in H$

$\Rightarrow z = z'$.

且 $\|w\| = \sup\limits_{\substack{x \in H \\ x \neq 0}} \dfrac{|l_w(x)|}{\|x\|}$, 即 l 的范数.

由于 $|l_w(x)| = |(x, w)| \leq \|x\|\cdot\|w\|$. So $\|w\| \geq \sup \dfrac{|l_w(x)|}{\|x\|}$. Let $x = w$. Then $\dfrac{l_w(w)}{\|w\|} = \|w\|$

Thus equality holds.

有界线性泛函的范数.

$l: V \to \mathbb{R}$. bounded-linear functional

normed space. $\exists M$ st $|l(v)| \le M\|v\|$. $\Rightarrow$ $\frac{|l(v)|}{\|v\|} \le M$. $\forall v \neq 0$.

称最佳的 M 为 l 的范数. 记为 $\|l\|$.

i.e $\|l\| = \sup\limits_{\substack{v\in V \\ v\neq 0}} \frac{|l(v)|}{\|v\|}$

Recall: H. Hibert space $\forall\, l: H \to \mathbb{R}$. b-l-functional

都 $\exists\, u \in H$ $l = l_u$. $l_u(v) = (v,u)$

$\|l_u\| = \sup\limits_{\substack{v\in H \\ v\neq 0}} \frac{|l_u(v)|}{\|v\|}$ 即. $\|l_u\| = \|u\|$.

若 V normed space. 非平凡的 b-l-functional 是否存在.

Thm (Hahn-Banach Extension Theorem).

V vector space. $p: V \to \mathbb{R}$ sublinear
(没有任何其他结构).

$M \subset V$. subspace.

$$\begin{tikzcd} V \arrow[dr, "\exists F \text{ linear functional}"] & \\ M \arrow[u] \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

linear

$f(x) \le p(x)$ $\forall x \in M$.

$F \le p$. $\forall v \in V$

$F|_M = f$.

即 F 为 f 的延拓.

Def (次线性) sublinear

$p: V \to \mathbb{R}$ (vector space.)

$\begin{cases} p(x+y) \le p(x) + p(y). \\ p(\alpha x) = \alpha p(x) \quad \forall \alpha > 0. \end{cases}$

(**).

Recall: Zorn's Lemma

$(X, \preceq)$ partial order. $\begin{cases} ① & x \preceq y,\ y \preceq z \Rightarrow x \preceq z \\ ② & x \preceq x \\ ③ & x \preceq y \text{ 且 } y \preceq x \Rightarrow x = y. \end{cases}$

$\forall x.y$ 未必可比较大小.

if $T \subset X$. x_0 ($\in X$) is called an upper bound of T

if $\forall t \in T$. $t \preceq x_0$.

$S \subset X$ 称为 totally ordered subset

if $\forall s_1, s_2 \in S$ 要么 $s_1 \preceq s_2$, 要么 $s_2 \preceq s_1$.

Zorn's Lemma.

if $T \subset X$. 满足任意全序子集 $S \subset T$ 都有 upper bound in T.

Then T 就有一个极大元.

(极大元. x_0 is T 的极大元 if $x_0 \in T$. 若 $\exists\, t_0 \in T$ st $x_0 \preceq t_0$
则 $x_0 = t_0$).

(**) Proof: 1° $M = V$ OK

2° $M \subsetneq V$. $x_1 \in V \setminus M$.

$$M_1 = M + \mathbb{R}x_1 = \left\{ \underbrace{m+\alpha x_1}_{\text{表达式是唯一的}} \,\middle|\, \begin{array}{l} m\in M \\ \alpha\in\mathbb{R}\end{array} \right\}.$$

$m+\alpha_1 x_1 = m'+\alpha_2 x_1$

$m-m' = \underbrace{(\alpha_2-\alpha_1)x_1}_{=0}$.

由M与x_1张成的最小的线性空间.

一步步地进行延拓.

Claim: $\exists f_1 : M_1 \to \mathbb{R}$ linear

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & \\ \cup & & \searrow \\ M & \xrightarrow{f} & \mathbb{R} \end{array}$$

$f_1(y) \le p(y) \quad \forall y \in M_1$.

Indeed. 这样f_1若存在, 必满足

$$f_1(m+\alpha x_1) = f_1(m) + \alpha f_1(x_1) = f(m) + \alpha \boxed{f_1(x_1)}.$$

仅需定义$f_1(x_1)$即可.

只要找到$f_1(x_1)$ st.

$$f(m)+\alpha f_1(x_1) \le p(m+\alpha x_1) \quad \forall m\in M \ \forall \alpha\in\mathbb{R}$$

则f_1就可以定义.

If $\alpha>0$. $f_1(x_1) \le [p(m+\alpha x_1) - f(m)]/\alpha$

$$= p(\overset{z}{\tfrac{1}{\alpha}m} + x_1) - f(\tfrac{m}{\alpha})$$

if $\alpha<0$. $-f_1(x_1) \le -\frac{1}{\alpha}[p(m+\alpha x_1) - f(m)]$

$$= -f(-\tfrac{1}{\alpha}m) + p(-\tfrac{m}{\alpha} - x_1)$$

$$f_1(x_1) \ge f(\overset{y}{-\tfrac{1}{\alpha}m}) - p(-\tfrac{m}{\alpha} - x_1).$$

若$f_1(x_1)$满足

$f(y) - p(y-x_1) \le f_1(x_1) \le p(z+x_1) - f(z)$ 即可延拓.

$f_1(x_1)$存在 $\Leftrightarrow$ $\sup\limits_{y\in M} \text{LHS} \le \inf\limits_{z\in M} \text{RHS}$.

故只需验证. $f(y) - p(y-x_1) \le p(z+x_1) - f(z)$. $\forall y, z\in M$

$$\Updownarrow$$

$$f(y+z) = f(y)+f(z) \le p(y-x_1) + p(z+x_1).$$

OK!

如此可进行可数次操作. 下面我们证明可比可数做得多.

$$S = \left\{ (D(g), g) \,\middle|\, \begin{array}{c} V \\ \cap \\ D(g) \\ \cap \\ M \end{array} \text{子空间}, \ D(g) \xrightarrow{g} \mathbb{R} \ \begin{array}{l} \text{linear } f \\ g \le p \end{array} \right\}.$$

在S上定义 partial order.

We say $(D(g_1), g_1) \preceq (D(g_2), g_2)$.

if $D(g_1) \subset D(g_2)$. 且 $g_2|_{D(g_1)} = g_1$. (即 g_2 为 g_1 延拓).

证明有极大元.

Claim. $\forall T \subset S$. totally ordered. 则 T 有上界.

$\Downarrow$

由 Zorn's Lemma. S 有极大元. $(D(g_0), g_0)$.

必有 $D(g_0) = V$. g_0 l-f 满足 $g_0 \le p$.

(easy to prove).

Proof of Claim.

$T = \{(D(g_i), g_i),\ i \in I\}$.

$D = \bigcup_{i \in I} D(g_i)$. 显然为一个 linear space.

定义 $g: D \longrightarrow \mathbb{R}$.

$g(v) = g_i(v)$ $v \in D(g_i)$ (良定义).

(D, g) is an upper bound. #

V normed space. $\|\cdot\|: V \longrightarrow \mathbb{R}$. sublinear.

$$\begin{tikzcd} V \arrow[dr, dashed, "\tilde{l}"] & \\ M \arrow[u, phantom, "\cup"] \arrow[r, "l"] & \mathbb{R} \end{tikzcd}$$

$\tilde{l}$. linear functional.

$l(m) \le p(m)$.

l: bounded linear operator.

$|l(m)| \le \underbrace{\|l\|\,\|m\|}_{} = p(m)$.

$\tilde{l}(v) \le \|l\|\,\|v\| = p(v)$

$\tilde{l}(-v) \le \|l\|\,\|-v\| = \|l\|\,\|v\|$

$\Rightarrow |\tilde{l}(v)| \le \|l\|\,\|v\|$.

$\Rightarrow \|\tilde{l}\| \le \|l\|$.

又 $\|\tilde{l}\| \ge \|l\|$

则 $\|\tilde{l}\| = \|l\|$

赋范空间 子空间上的 $\forall$ 有界线性泛函 均可 保范地延拓到大空间上.

V normed $x_0 \in V$. $x_0 \neq 0$.

$\exists F: V \longrightarrow \mathbb{R}$. b-l-f. $\|F\| = 1$. ← a norming functional of x_0.

s.t $F(x_0) = \|x_0\|$.

Proof:

$$\begin{tikzcd} V \arrow[dr, "F"] & \\ \mathbb{R}x_0 \arrow[u, phantom, "\cup"] \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

f: l-f

$\alpha x_0 \mapsto \alpha\|x_0\|$

$f(x_0) = \|x_0\|$

$\|f\| = \sup \dfrac{|f(\alpha x_0)|}{\|\alpha x_0\|} = 1$.

Dual Space. V normed space

对偶空间

记 V' 或 V^* 为

$$V^* = \{ l : V \to \mathbb{R} \mid l \text{ b-l-f} \}.$$

V^* vector space with respect to

$(l_1 + l_2)(v) := l_1(v) + l_2(v)$. $l_1 + l_2 \in V^*$

$(\alpha l)(v) := \alpha l(v)$. $\alpha l \in V^*$.

Thm: V^* is a Banach Space (即使 V 仅是 normed space).

对偶 space 必完备.

Proof: $\{l_n\} \in V^*$. Cauchy seq.

$\big| \|l_n\| - \|l_m\| \big| \le \|l_n - l_m\| \Rightarrow \{\|l_n\|\}$ Cauchy seq.

$\Rightarrow M = \sup\limits_n \|l_n\| < \infty$.

$\forall v \in V$. $|l_n(v) - l_m(v)| \le \|l_n - l_m\| \|v\|$.

$\Rightarrow \{l_n(v)\}$ Cauchy seq.

故 $\lim\limits_{n\to\infty} l_n(v)$ 存在. 记为 $l(v)$.

$|l(v)| = \lim\limits_{n\to\infty} |l_n(v)|$.

$\le \limsup\limits_{n\to\infty} \|l_n\| \|v\| \le M \|v\|$

故 $l \in V^*$.

$\lim\limits_{n\to\infty} \|l_n - l\| = \lim\limits_{n\to\infty} \limsup\limits_{k\to\infty} \|l_n - l_k\| = 0$.

$\Big(|l_n(v) - l(v)| = |l_n(v) - \lim\limits_k l_k(v)|$

$= \lim\limits_{k\to\infty} |l_n(v) - l_k(v)|$

$\le \limsup\limits_{k\to\infty} \|l_n - l_k\| \|v\| \Big)$. #.

Cor. 若 $x_0 \in V$. 满足 $\forall l \in V^*$

$l(x_0) = 0$. 则 $x_0 = 0$.

Pf: Let l be a norming-f. $l(x_0) = \|x_0\| = 0$. $\Rightarrow x_0 = 0$. #.

Thm. Let $M \subset V \leftarrow$ normed space (subspace)

Assume $x_0 \in V$. $d = d(x_0, M) = \inf \|x_0 - m\| > 0$

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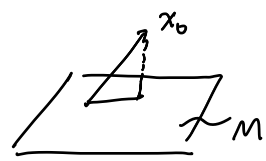
Lecture 5 2022/9/21

Thm 2.9 V normed space
$\cup$
M subspace



$x_0 \in V$. $d(x_0, M) = \inf\limits_{z \in M} \|x - z\| > 0$. Then $\exists\, \ell \in V^*$ s.t.
- $\ell|_M \equiv 0$ $(M \subset \ker \ell)$
- $\|\ell\| = 1$
- $\ell(x_0) = d(x_0, M)$

Remark. If M closed V/M
$\Rightarrow$ $[\ell]$ 是 $[x_0]$ 的 norming functional.

$$\begin{array}{ccc} V & \overset{\ell \,\leftarrow\, \text{Hahn-Banach 保范延拓.}}{\dashrightarrow} & \\ \cup & & \\ M + \mathbb{R}x_0 & \xrightarrow{f} & \mathbb{R} \\ m + \alpha x_0 & \longmapsto & \alpha\, d(x_0, M) \end{array}$$

Pf: 若 $\|f\|_{(M+\mathbb{R}x_0)^*} = 1$，则可结束证明.

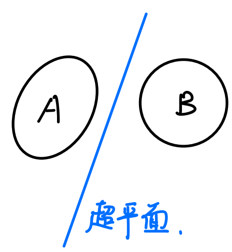
$$\|f\|_{(M+\mathbb{R}x_0)^*} = \sup_{\substack{v \in M + \mathbb{R}x_0 \\ v \neq 0}} \frac{|f(v)|}{\|v\|}$$

显然有 $\|f\| \leq 1$， $\alpha \neq 0$ 时
因为 $|\alpha\, d(x_0, M)| \leq \|m + \alpha x_0\|$
$\Leftrightarrow$ $d(x_0, M) \leq \|x_0 + \frac{1}{\alpha} m\|$ 显然成立.

且有 $\|f\| = 1$： $d(x_0, M) = \lim\limits_n \|x_0 - z_n\|$， $\|f\| \geq \lim\limits_n \dfrac{d(x_0, M)}{\|x_0 - z_n\|} = 1$ $\Rightarrow$ $\|f\| = 1$.

Thm. (Hahn-Banach Separation theorems)

V normed space. $A, B \subset V$ two disjoint convex subset



$\{v \in V \mid \ell(v) = \gamma\}$

① If A is open
Then $\exists\, \ell \in V^*$ (超平面是 closed). $\ell(a) < \gamma \leq \ell(b)$
$\exists\, \gamma \in \mathbb{R}$ $\forall\, a \in A,\ b \in B$.

② A 是紧的, B is closed
$\exists\, \ell \in V^*$, $\exists\, \gamma_1, \gamma_2 \in \mathbb{R}$ $\ell(a) < \gamma_1 < \gamma_2 < \ell(b)$. $\forall\, a \in A$, $b \in B$

预备知识

Minkowski functional (Gauge function / functional).

V vector space (无需其它结构)

Covex set $C \subset V$ is called $\boxed{\text{absorbing}}$ (具有吸收性的)

if $V = \bigcup\limits_{t > 0} tC$ $tC = \{tv \mid v \in C\}$. (换言之, $\forall\, v \in V$, $\exists\, \lambda > 0$, $\lambda v \in C$).

Remark: $0 \in C$.

Lemma V normed space

then any $\underset{\text{convex}}{\ }$ neighbourhood U of 0 is absorbing.

Pf: 

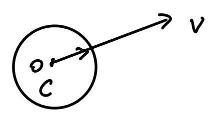
$B(0,\varepsilon)=\{v\in V: \|v\|<\varepsilon\}\subset U$.

$\forall\, w\in V$, $\frac{\varepsilon}{2\|w\|}\cdot w\in B(0,\varepsilon)\subset U$.

Def. (Minkowski functional)

$C\subset V$ absorbing convex subset. The Minkowski functional of C is defined as:

$\mu_C: V\to\mathbb{R}_+$ $\quad \mu_C(v):=\inf\{t>0 \mid \frac{v}{t}\in C\}$.



Corr. C convex absorbing, if $\frac{v}{t}\in C\Rightarrow \frac{v}{t+M}\in C$ $(M>0)$

Indeed, $\frac{v}{t+M}=\frac{v}{t}\cdot\frac{t}{t+M}+0\cdot\frac{M}{t+M}\in C$, $\frac{v}{t}$ 与 0 的凸组合.

Lemma. C convex absorbing, then

μ_C is sublinear (Minkowski functional 是次线性的).

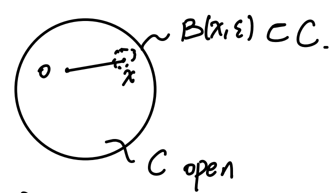
即 $\begin{cases}\mu_C(x+y)\le\mu_C(x)+\mu_C(y)\\ \mu_C(\alpha y)=\alpha\mu_C(y) \quad \forall\,\alpha>0\end{cases}$ Pf: Trivial.

$\begin{cases}\mu_C(x)<t \Rightarrow \frac{x}{t}\in C\\ \frac{x}{t'}\in C\Rightarrow \mu_C(x)\le t'\end{cases}$ (homework, if C open convex absorbing in normed vector space V)

$\mu_C(x)<t\Leftrightarrow\frac{x}{t}\in C$.

Lemma. If C open, convex, absorbing in a normed space V.

then $\mu_C(x)<1$ for all $x\in C$.



figure

Pf. As figure shows. $B(x,\varepsilon)\subset C\Rightarrow \frac{x}{\|x\|}\cdot(\|x\|+\varepsilon/2)\in C$

$\Rightarrow \mu_C(x)\le\frac{\|x\|}{\|x\|+\varepsilon/2}<1$.

Lemma. V normed space $\ell: V\to\mathbb{R}$. linear functional.

If ℓ 在一个开集上有界, 则 $\ell\in V^*$. 可以改成某个含 0 的开集上有界.

Pf: Assume $\ell|_U$ is bdd, U open in V, $\exists\, x_0\in U$, $\varepsilon>0$. $B(x_0,\varepsilon)\subset U$.

$\sup_{z\in B(x_0,\varepsilon)} |l(z)| = M < +\infty$, $\forall\, x\neq 0$, $x_0 + \frac{\varepsilon}{2}\cdot\frac{x}{\|x\|} \in B(x_0,\varepsilon)$.

$\Rightarrow \left| l\left(x_0 + \frac{\varepsilon}{2}\cdot\frac{x}{\|x\|}\right)\right| \le M$

$\left| l(x_0) + \frac{\varepsilon}{2\|x\|} l(x)\right| \le M \Rightarrow \frac{\varepsilon}{2\|x\|} l(x) \le M + |l(x_0)|$

$\Rightarrow l(x) \le \|x\| \cdot \frac{2(M+|l(x_0)|)}{\varepsilon}$

Now we return to Hahn-Banach Separation Thm.

pf of ①: fix $a_0 \in A$ open, $b_0 \in B$. set $C = A - B + \underbrace{b_0 - a_0}_{x_0} = \{a-b+b_0-a_0 \mid a\in A,\ b\in B\}$.

Claim 1: C is an open set containing 0. $\Rightarrow$ C is absorbing.

$C = \bigcup_{b\in B} (\underbrace{A - b + b_0 - a_0}_{\text{开集}}) \Rightarrow C$ open.

Claim 2: C is a convex set. $\alpha(a-b+x_0) + (1-\alpha)(a'-b'+x_0)$

$= [\alpha a + (1-\alpha)a'] - [\alpha b + (1-\alpha)b'] + x_0 \in C$.

$x_0 \notin C$. $\Rightarrow \mu_C(x_0) \ge 1$. $\quad \mathbb{R}x_0 \xrightarrow{f} \mathbb{R}$, $tx_0 \to t$. $\quad f(tx_0) = t \overset{\text{验证}}{\le} \mu_C(tx_0)$.

if $t \le 0$. $\mu_C(tx_0) \ge 0$. OK.

$t > 0$. $t \le t\cdot\mu_C(x_0) = \mu_C(tx_0)$

所以由Hahn Banach 延拓:

$\exists\, \tilde{f}: V \to \mathbb{R}$. $\mathbb{R}x_0 \subset V$ $\quad \tilde{f}|_{\mathbb{R}x_0} = f$. $\quad \tilde{f}(x_0) = 1$.

$\tilde{f}$ linear. $\tilde{f}(x) \le \mu_C(x)\ \forall x \in V$.

$\tilde{f}(x) \le \mu_C(x)\ \forall x\in V$. $\Rightarrow \forall x \in C \Rightarrow \mu_C(x) < 1$. 故 $\forall x\in C$. $\tilde{f}(x) < 1$.

$\Downarrow$

$\forall y \in -C$. $\tilde{f}(y) > -1$.

$\Rightarrow \forall z \in C\cap(-C)$,

$\Rightarrow -1 < \tilde{f}(z) < 1 \Leftrightarrow |\tilde{f}(z)| < 1$. 所以 $\tilde{f} \in V^*$.

$\forall a\in A$, $b\in B$. $a-b+x_0 \in C$. $\tilde{f}(a-b+x_0) \le \mu_C(a-b+x_0) < 1$.

$\Rightarrow \tilde{f}(a) - \tilde{f}(b) + 1 < 1 \Rightarrow \tilde{f}(a) < \tilde{f}(b)\ \forall a\in A, b\in B$.

Lemma. A open convex, $l: V \to \mathbb{R}$ bdd linear functional. (non-constant)

$\Rightarrow l(A) \subset \mathbb{R}$ is an open interval.

pf: A open. $\forall z \in V$. $\forall a\in A$. $\exists B(a,\varepsilon) \subset A$. $a + \alpha\cdot\frac{z}{\|z\|} \in B(a,\varepsilon)$ $\forall \alpha \in (-\varepsilon,\varepsilon)$

$\Rightarrow l(a) + \alpha\cdot\frac{l(z)}{\|z\|} \in l(A)$. $\forall \alpha\in(-\varepsilon,\varepsilon) \Rightarrow l(a)$ 是 $l(A)$ 的内点 $\Rightarrow l(A)$是开集.

而 A 是凸的. 故 $f(A)$ 也是凸的, $\Rightarrow$ $f(A)$ 是连通的 interval.

因此 $\tilde{f}(a) < \tilde{f}(b)$ $\Rightarrow$ $\tilde{f}(A)$ 是一个开区间. 令 $\gamma = \inf_{b\in B} \tilde{f}(b)$.

则 $\tilde{f}(a) < \gamma \le \tilde{f}(b)$ $\forall a, b$.

For ②: A compact. B closed. $A\cap B = \emptyset$

Claim 4: 存在含 A 的凸开集 A', $A'\cap B = \emptyset$.

$\boxed{\textit{Lecture 6} \quad 2022/9/23}$

Examples of dual spaces $\qquad f, g \in L^p(\mu)$

— Minkowski inequality 零测集可测 $\qquad \|f+g\|_p \le \|f\|_p + \|g\|_p$.

$1\le p \le +\infty \qquad (\Omega, \mathcal{B}, \mu)$ complete $\qquad \|f\|_p = \left(\int_\Omega |f(x)|^p d\mu(x)\right)^{1/p}$

σ finite

— Hölder inequality $\quad 1\le p, q \le +\infty$, $\quad p, q$ 称为共轭 (conjugate) $\left(\frac{1}{\infty} = 0\right)$

若 $\frac{1}{p} + \frac{1}{q} = 1$

$f\in L^p(\mu)$, $g \in L^q(\mu)$

$$\left|\int_\Omega fg\, d\mu\right| \le \|f\|_p \cdot \|g\|_q$$

Orlicz space

Recall proof: • $\infty > p \ge 1$ $\quad t \longmapsto t^p$ convex on $[0, +\infty)$

$$\frac{|f+g|}{\|f\|_p + \|g\|_p} \le \frac{|f|}{\|f\|_p} \cdot \frac{\|f\|_p}{\|f\|_p + \|g\|_p} + \frac{|g|}{\|g\|_p} \cdot \frac{\|g\|_p}{\|f\|_p + \|g\|_p}$$

$$\left(\frac{|f+g|}{\|f\|_p + \|g\|_p}\right)^p \le \frac{\|f\|_p}{\|f\|_p + \|g\|_p} \cdot \frac{|f|^p}{\|f\|_p^p} + \frac{|g|^p}{\|g\|_p^p} \cdot \frac{\|g\|_p}{\|f\|_p + \|g\|_p}$$

积分得: $\dfrac{\left(\int_\Omega |f+g|^p d\mu\right)^{1/p}}{\|f\|_p + \|g\|_p} \le 1 \quad \Rightarrow \quad \|f+g\|_p \le \|f\|_p + \|g\|_p$

• $ab = \inf\limits_{t>0} \left\{ \frac{1}{p} t^p a^p + \frac{1}{q} t^{-q} b^q \right\}$, $\quad |fg| \le \frac{1}{p}|tf|^p + \frac{1}{q} |t^{-1} g|^q$

$$\int |fg|\, d\mu = \int \inf_{t>0} \left\{ \frac{1}{p} t^p f^p + \frac{1}{q} t^{-q} g^q \right\} \le \inf_{t>0} \frac{1}{p} t^p \int |f|^p d\mu + \frac{1}{q} t^{-q} \int |g|^q d\mu = \|f\|_p \cdot \|g\|_q$$

$1 \le p < +\infty$ $\quad L^p(\mu)^* = L^q(\mu)$ isometric isomorphism $\quad$ 即 $\quad L^q(\mu) \xrightarrow{i} L^p(\mu)^*$, $\quad g \longmapsto \ell_g$ (isometric linear bijection) $\quad \ell_g : L^p(\mu) \to \mathbb{R}$.

pairing: X normed space, X^* dual space

$X \times X^* \longrightarrow \mathbb{R}$ $\qquad (L^p)^* \xleftrightarrow{\text{identity}} L^q$

$x \in X$, $x^* \in X^*$ bilinear bouded form $\quad ((L^p)^*, L^q) \Longleftrightarrow (L^q, L^q)$ pairing

$(x, x^*) := x^*(x) \le \|x\| \cdot \|x^*\|$

Proof: Assume μ is finite

1° $\ell_g \in L^p(\mu)^*$ & $\|\ell_g\| = \|g\|_{L^q}$.

$|\ell_g(f)| = |\int fg \, d\mu| \overset{\text{Hölder}}{\le} \|f\|_p \cdot \|g\|_q \quad \forall f \in L_p(\mu)$

$\Rightarrow \|\ell_g\| \le \|g\|_q \Rightarrow \ell_g \in L_p^*(\mu)$, 令 $f_0 = |g|^{q-1} \cdot \frac{|g|}{g} \cdot 1(g \ne 0)$.

Then $|\ell_g(f_0)| = \|g\|_q^q$, $\|f_0\|_p = (\int |g|^{pq-p} d\mu)^{1/p} = \|g\|_q^{q-1}$

$\Rightarrow \|\ell_g\| \ge \|g\|_q$

2° i 是 surjective. 即 $\forall \ell : L^p(\mu) \to \mathbb{R}$ bnd linear form

欲证存在 $g \in L^q(\mu)$ $\ell = \ell_g$.

set $\nu : \mathcal{B} \longrightarrow \mathbb{R}_{\ge 0}$, $A \in \mathcal{B}$. $\nu(A) := \ell(1_A)$

验证 ν is a signed measure

(1°) $\nu(\emptyset) = 0$.

(2°) σ-additivity $(A_n)_{n=1}^\infty$ disjoint, $A_n \in \mathcal{B}$ $\quad 1_{\bigcup_{n=1}^\infty A_n} = \sum_{n=1}^\infty 1_{A_n}$

$\| 1_{\bigcup_{n=1}^\infty A_n} - \sum_{n=1}^N 1_{A_n} \|_p \xrightarrow[\text{DCT}]{N \to +\infty} 0$. 故 $\ell(1_{\bigcup_{n=1}^\infty A_n}) = \lim_{N \to \infty} \sum_{n=1}^N \ell(A_n)$

$\Leftrightarrow \nu(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \nu(A_n)$

Hahn decomposition

$\nu = \nu_+ - \nu_-$ (singular) $\qquad \nu_+ = \nu|_{\Omega_+} \quad \Omega_+ \cup \Omega_- = \Omega$, 在 Ω_+ 上. $\mu(A) = 0 \Rightarrow \nu_+(A) = 0$

$\nu_- = -\nu|_{\Omega_-} \quad \Omega_+ \cap \Omega_- = \emptyset$ $\qquad \nu_+ \ll \mu$

ν_+, ν_- 均为正测度. $\qquad$ 同理 $\nu_- \ll \mu$.

Radon - Nikodym Theorem $\Rightarrow$ $\nu_+ = g_1\mu$ $\qquad \nu = (g_1 - g_2)\mu. := g\mu.$

$\nu_- = g_2\mu.$ $\qquad \nu(A) = \int_\Omega 1_A \cdot g \, d\mu.$

$\ell(1_A) = \int_\Omega 1_A g \, d\mu \Rightarrow \ell(f) = \int_\Omega fg \, d\mu \quad \forall f = \mathrm{span}\{1_A, A \in \mathcal{B}\}.$

由于 $\mathrm{span}\{1_A \mid A \in \mathcal{B}\} \overset{dense}{\subset} L^p(\mu)$, $\forall f \in L^p(\mu)$, $\exists f_n$. $\|f_n - f\|_p \to 0$

$\ell(f) = \lim_{n\to\infty} \ell(f_n) = \lim_{n\to\infty} \int_\Omega f_n g \, d\mu.$

下证 $g_1, g_2 \in L^q$. 对于 g_1, 取 $f \in Simp^+(\Omega_+)$ (Ω_+上正的简单函数).

$\left|\int_\Omega f \cdot g_1 \, d\mu\right| = |\ell(f)| \le \underline{\|\ell\| \cdot \|f\|_p} \Rightarrow \|g_1\|_{L^q} \le \|\ell\|$, 再取 $f = \frac{g_1^q}{g_1}$. $f_n \uparrow f$.

$\int f_n g_1 \, d\mu \to \int f g_1 \, d\mu = \int g_1^q \, d\mu.$

The Riesz Representation Theorem.

X ~~locally~~ compact Hausdorff space

(~~$\forall x \in X$, $\exists U$ open, $x \in U$~~ ~~$\bar{U}$ is compact~~) $\quad C_c(X) = \{f: X \to \mathbb{R} \mid$ continous ~~compactly supported~~ $\}$

1° $\forall \ell \in C(X)^*$ $\quad \ell = \ell_+ - \ell_-$, ℓ_+, ℓ_- 均为 positive linear form

$C(X) \xrightarrow{\ell_+} \mathbb{R}$ $\qquad$ 即 $\ell_+(f) \ge 0$, 只要 $f \ge 0$

2° $\forall \ell \in C(X)^*$ positive linear form (自动是 bounded).

$-\|f\|_{C(X)} \cdot \mathbb{1} \le f \le |f| \le \|f\|_{C(X)} \cdot \mathbb{1} \quad \Rightarrow \quad -\|f\|_{C(X)} \cdot \ell(\mathbb{1}) \le \ell(f) \le \|f\|_{C(X)} \cdot \ell(\mathbb{1})$

$\Rightarrow |\ell(f)| \le \|f\|_{C(X)} \cdot \ell(\mathbb{1}) \Rightarrow \ell$ is bounded.

Conclusion: $\exists$ σ-algebra $\mathfrak{M} \supset \mathcal{B}(X) \leftarrow$ X上的 Borel σ-algebra.

$\exists!$ measure μ on $\mathfrak{M}$ s.t. (1) $\boxed{\ell(f) = \int_X f \, d\mu, \ \forall f \in C(X)}$

(2) $(\mathfrak{M}, \mu)$ is complete.

$\forall E \in \mathfrak{M}$. $\mu(E) = \inf\{\mu(\underset{\text{open}}{U}) \mid E \subset U\}$

$\mu(E) = \sup\{\mu(\underset{\text{compact}}{K}) \mid K \subset E\}$ $\quad$ (3) μ is <u>regular</u>. $\qquad$ free ulter-filton.

$L^\infty(\mu) \leftarrow$ abelian Banach algebra

C^*- algebra / Von Neuman algebra. $\qquad$ $\mathbb{N}$ 的 Stone-Čech conpactification.

$\ell^\infty = \{(x_n)_{n=1}^\infty \mid \sup|x_n| < \infty\}$

Proposition: $(\Omega, \mathcal{B}, \mu)$ measure space, $\mu(\Omega) < +\infty$.

If $1 \le p < +\infty$, $\ell \in L^p(\Omega, \mathcal{B}, \mu)^*$

If $\ell(f) = \int g f d\mu$ for all step functions in L^p.

Then $g \in L^q$

Proof: $\boxed{\text{First, we assume } 1 < p < \infty}$ Claim 1. $|g| < +\infty$ μ-a.e.

otherwise, $\exists\ A \in \mathcal{B}$, $\mu(A) > 0$, $g|_A = +\infty$ or $g|_A = -\infty$

then $\ell(\mathbb{1}_A) = \int_A g \, d\mu \in \{-\infty, \infty\}$. impossible.

Now for any $M > 0$, the function $\frac{|g|^q}{g} \mathbb{1}(|g| < M)$ is bdd

Hence $\frac{|g|^q}{g} \mathbb{1}(|g| < M) \in L^p$.

There exist bounded step functions $f_n^+, f_n^- \geq 0$

$$0 \le f_n^+ \uparrow \frac{|g|^q}{g_+} \mathbb{1}(|g| < M) \cdot \mathbb{1}(g_+ > 0)$$

$$0 \le f_n^- \uparrow \frac{|g|}{g_-} \mathbb{1}(|g| < M) \cdot \mathbb{1}(g_- > 0)$$

we have $\frac{|g|^q}{g} \mathbb{1}(|g| < M) = \frac{|g|^q}{g_+} \mathbb{1}(|g| < M) - \frac{|g|^q}{g_-} \mathbb{1}(|g| < M)$

Note that $\begin{cases} f_n^+ > 0 \Longrightarrow g > 0 \\ f_n^- > 0 \Longrightarrow g < 0 \end{cases}$, $\boxed{f_n^+ \cdot f_n^- = 0}$

disjoint support.

Hence, write $f_n = f_n^+ - f_n^-$.

$$0 \le f_n g = f_n^+ g - f_n^- g = f_n^+ g_+ + f_n^- g_-$$

地址：中国湖北武汉市武昌区珞珈山武汉大学数学与统计学院
Address: School of Mathematics and Statistics, Wuhan University
China Zip Code: 430072 Tel: +86-27-68756753 Fax: +86-27-68756753

$$f_n^+ g_+ \uparrow |g|^q \mathbb{1}(|g|<M)\,\mathbb{1}(g>0)$$

$$f_n^- g_- \uparrow |g|^q \mathbb{1}(|g|<M)\,\mathbb{1}(g<0)$$

Hence

$$\lim_{n\to\infty} \int f_n^+ g_+ \, d\mu = \int |g|^q \mathbb{1}(|g|<M)\,\mathbb{1}(g>0)\, d\mu$$

$$\lim_{n\to\infty} \int f_n^- g_- \, d\mu = \int |g|^q \mathbb{1}(|g|<M)\,\mathbb{1}(g<0)\, d\mu$$

Hence

$$\int |g|^q \mathbb{1}(|g|<M)\, d\mu = \lim_{n\to\infty} \left[\int f_n^+ g_+ \, d\mu + \int f_n^- g \, d\mu\right]$$

$$= \lim_{n\to\infty} \int f_n g \, d\mu$$

$$= \lim_{n\to\infty} \cancel{g}\, l(f_n)$$

Since f_n are step functions

Since

$$|l(f_n)| \leq \|l\| \cdot \|f_n\|_p \leq \|l\| \left\| \frac{|g|^q}{g_+} \mathbb{1}(|g|<M)\,\mathbb{1}(g>0) + \frac{|g|^q}{g} \mathbb{1}(|g|<M)\,\mathbb{1}(g<0) \right\|$$

~~But~~ Since

$$|f_n| = f_n^+ + f_n^- \leq \underbrace{\frac{|g|^q}{g_+} \mathbb{1}(|g|<M)\,\mathbb{1}(g>0) + \frac{|g|^q}{g} \mathbb{1}(|g|<M)\,\mathbb{1}(g<0)}$$

$$= |g|^{q-1} \mathbb{1}(|g|<M)$$

Hence $|l(f_n)| \leq \|l\| \cdot \left\| |g|^{q-1} \mathbb{1}(|g|<M) \right\|_p$

$$= \|l\| \left(\int |g|^{(q-1)p} \mathbb{1}(|g|<M)\, d\mu \right)^{1/p}$$

$$= \|l\| \left(\int |g|^q \mathbb{1}(|g|<M)\, d\mu \right)^{1/p}$$

It follows that

$$\int |g|^q \mathbb{1}(|g|<M)\,d\mu \leq \limsup_n |\ell(f_n)| \leq \|\ell\| \left(\int |g|^q \mathbb{1}(|g|<M)\,d\mu\right)^{1/p}$$

Therefore, $$\left(\int |g|^q \mathbb{1}(|g|<M)\,d\mu\right)^{1/q} \leq \|\ell\|$$

Hence $$\left(\int |g|^q d\mu\right)^{1/q} = \lim_{M\to\infty} \left(\int |g|^q \mathbb{1}(|g|<M)\,d\mu\right)^{1/q} \leq \|\ell\|$$

Finally, if $p = 1$, then $q = \infty$.

Assume by contradiction that $\|g\|_\infty = +\infty$.

then $\forall M>0$, there exists $A \in \mathcal{B}$, $\mu(A) > 0$ such that one of the following is satisfied

$$\begin{cases} g(x) \geq M \quad \forall x \in A \quad ① \\ \underline{\text{or}} \quad g(x) \leq -M \quad \forall x \in A \end{cases}$$

Assume for example that ① is satisfied,

then $\ell(\mathbb{1}_A) = \int \mathbb{1}_A g\,d\mu \geq M\mu(A)$

therefore $M\mu(A) \leq \ell(\mathbb{1}_A) \leq \|\ell\| \cdot \|\mathbb{1}_A\|_1 = \|\ell\| \cdot \mu(A)$

$\Rightarrow \|\ell\| \geq M$. But M is arbitrary

$\to \|\ell\| = +\infty$. contradiction

地址：中国湖北武汉市武昌区珞珈山武汉大学数学与统计学院
Address: School of Mathematics and Statistics, Wuhan University
China Zip Code: 430072 Tel: +86-27-68756753 Fax: +86-27-68756753

Linear Operations

本章大部分情况 $D(A)=X$.

X, Y normed space

$X \xrightarrow{A} Y$ is called bounded.

if $\exists M>0$, $\|Ax\| \le M\|x\|$, $\forall x \in X$.

$$\|A\| := \sup_{\substack{x\neq 0 \\ x\in X}} \frac{\|Ax\|}{\|x\|}$$

$\Updownarrow$

$\inf\{M \mid \|Ax\| \le M\cdot\|x\|\}$

$X \supset D(A) \xrightarrow[\text{map}]{A} Y$

$D(A)$: domain of A. A: called operator

若 $\begin{cases} D(A) \subset X \text{ subspace (不一定 closed)} \\ A \text{ is linear} \end{cases}$ is called linear operator.

Lecture 7 2022/9/28

Remark. $L^p(\mu)^* = L^q(\mu)$ (μ: σ-finite), $1 \le p < \infty$, q 是 p 的共轭指标

X normed space, $\forall x \in X$, $\|x\| = \sup_{\substack{x^*\in X^* \\ \|x^*\|\le 1}} |x^*(x)| = \sup_{\substack{x^*\in X^* \\ x^*\neq 0}} \frac{|x^*(x)|}{\|x^*\|}$

$\forall f \in L^p(\mu)$ $\|f\|_{L^p} = \sup_{\substack{g\in L^q \\ \|g\|_{L^q}\le 1}} \left|\int fg\,d\mu\right| = \sup_{\substack{g\in \mathcal{F} \\ g\neq 0}} \frac{\left|\int fg\,d\mu\right|}{\|g\|_{L^q}}$ $\quad \mathcal{F} \subset L^q$ 是给定的稠密集.

$A: X \to Y$, $\|A\| = \sup_{\substack{x\in X \\ x\neq 0}} \frac{\|Ax\|}{\|x\|} = \sup_{\substack{\|x\|\le 1 \\ x\in X}} \|Ax\| = \sup_{\substack{\|x\|=1 \\ x\in X}} \|Ax\|$

Prop. $A: X \to Y$ linear (X, Y normed space)

A is continuous at one pt $x_0 \in X$.

$\Updownarrow$ $\quad \Downarrow$ ②

A is bounded

$\Updownarrow$ $\quad \Downarrow$ ①

A is continuous at all points. (trivial)

Pf. ①: if $\|A\| < +\infty$

$x_n \to x$ $\quad \|x_n - x\| \to 0$

$\|A(x_n - x)\| \le \|A\|\cdot\|x_n - x\| \to 0$

$\Rightarrow Ax_n \to Ax$. #

② 若 $\|A\| = +\infty$. $\exists \{x_n\}_{n=1}^{\infty} \subset X$ $\quad \|Ax_n\| \ge n\|x_n\|$

Let A be continuous at x_0

$x_0 + \frac{x_n}{n\cdot\|x_n\|} \to x_0$. $\left\|A\left(x_0 + \frac{x_n}{n\cdot\|x_n\|}\right) - Ax_0\right\| \ge 1$. 矛盾. $\square$

Thm: $B(X,Y) = \{A: X \to Y \mid$ 有界线性算子$\}$.

线性空间了. $\|\cdot\|$ 是其中的范数. 且 Y is complete $\Leftrightarrow B(X,Y)$ is complete.

Pf: 只验证三角不等式: $A, B \in B(X,Y)$

$$\|A+B\| = \sup_{\substack{x\in X\\ \|x\|\le 1}} \|(A+B)x\| \le \sup_{\substack{x\in X\\ \|x\|\le 1}} \|Ax\| + \|Bx\| \le \sup_{\|x\|\le 1} (\|A\|+\|B\|)\cdot\|x\| = \|A\|+\|B\|$$

if $B(X,Y)$ is Banach, 取 $x_0 \in X$, $\|x_0\|=1$. if $\{y_n\}_{n=1}^{\infty}$ is a Cauthy seq

let x_0^* be the norming functional of x_0: $x_0^* \in X^*$, $\|x_0^*\|=1$. $x_0^*(x_0) = \|x_0\|$.

$\forall\ y \in Y$. 定义 $A^y \in B(X,Y)$, $X \xrightarrow{A^y} Y$ $\quad A^y(x) = x_0^*(x)y$

$x \longrightarrow x_0^*(x)\cdot y$

Then A^{y_n} is Cauthy seq in $B(X,Y)$.

$Y \xrightarrow{\phi \text{ 线性}} B(X,Y)$. Claim ϕ is an isometric embedding, i.e. $\|y\| = \|A^y\|$

$y \longrightarrow A^y$

$$\|A^y\| = \sup_{\substack{x\in X\\ \|x\|=1}} \|A^y x\| = \sup_{\substack{x\in X\\ \|x\|=1}} |x_0^*(x_0)|\cdot\|y\| = \|x_0^*\|\cdot\|y\| = \|y\|$$

A^{y_n} has a limit $A \in B(X,Y)$.

Claim: $\exists\ y \in Y$ s.t. $A = A^y$ $\qquad A^{y_n}x = x_0^*(x)\cdot y_n$, $y_n = A^{y_n}\cdot x_0$

Let $y := Ax_0$. Then proove that $A = A^y$. $\qquad Ax = \lim_{n\to\infty} x_0^*(x)\cdot y_n = x_0^*(x)\cdot \lim_{n\to\infty} y_n$

Lemma: $A_n \to A$ in $B(X,Y)$ $\qquad \Rightarrow Ax_0 = \lim_{n\to\infty} y_n \Rightarrow \lim_{n\to\infty} y_n = y$.

则 $\forall\ x \in X$,

$A_n x \to Ax$. $\quad \|A_n x - Ax\| \le \|A_n - A\|\,\|x\| \to 0$

tensor product.

Grothendieck.

Y Banach $\Rightarrow B(X,Y)$ is Banach

if $\{A_n\}$ Cauthy seq in $B(X,Y) \Rightarrow$ if $x\in X$, $\{A_n x\}$ is a Cauthy seq in Y.

$\|(A_n - A)x\| \le \|A_n - A\|\cdot\|x\|$ 故 $\lim_{n\to\infty} A_n x$ 存在.

记 $Ax := \lim_{n\to\infty} A_n x$. 下证: ① $A \in B(X,Y)$ 线性显然.

② $\|A_n - A\| \to 0$

$$\sup_{\|x\|\le 1} \|Ax\| = \sup_{\|x\|\le 1} \|\lim_{n\to\infty} A_n x\| = \sup_{\|x\|\le 1} \lim_{n\to\infty} \|A_n x\| \le \sup_n \|A_n\| < +\infty.$$

① ✓

$$\sup_{\|x\|\le 1}\|(A-A_n)x\| = \sup_{\|x\|\le 1}\lim_{k\to+\infty}\|(A_k-A_n)\cdot x\| \le \limsup_{k\to\infty}\|A_k-A_n\| < \varepsilon \quad \text{when } n \text{ is large.}$$

#

$Y \overset{\text{iso}}{\hookrightarrow} \underbrace{B(X,Y)}_{\text{normed space}}$ $\quad X^*\otimes Y = \{\sum_{\text{finite}} x_i^*\otimes y_i\}$

$x^*\otimes y$ 可以看作

$X \xrightarrow{x^*\otimes y} Y$

$x \longmapsto x^*(x)y$

$X^*\otimes Y \hookrightarrow B(X,Y).$ $\quad X^*\check{\otimes} Y$

A若为矩阵.

$(A^*x, y) = (x, Ay).$

伴随算子 (adjoint operators)

$X \xrightarrow{A} Y$

$x \longmapsto Ax$

$X^* \xleftarrow{A^*} Y^*$

$A^*y^* \longmapsfrom y^*$

已经定义好的.

$$(\underset{\substack{\cap\\ X^*}}{A^*y^*}, \underset{\substack{\cap\\ X}}{x}) := (\underset{\substack{\cap\\ Y^*}}{y^*}, \underset{\substack{\cap\\ Y.}}{Ax})$$

想要定义的.

要验证:

$x \longmapsto (y^*, Ax)$

① 线性性. (trivial).

② 有界. $y^*(Ax) \le \|y^*\|\cdot\|Ax\|$

$\le \|y^*\|\cdot\|A\|\cdot\|x\|$

Thm: $A\in B(X,Y) \Rightarrow A^*\in B(Y^*,X^*).$

且 $\|A\| = \|A^*\|$

Pf: $$\|A^*\| = \sup_{\substack{y^*\in Y^*\\ \|y^*\|=1}}\|A^*y^*\| = \sup_{\substack{y^*\in Y^*\\ \|y^*\|=1}}\sup_{\substack{x\in X\\ \|x\|=1}}|(A^*y^*, x)| = \sup_{\|x\|=1}\sup_{\|y^*\|=1}|(y^*, Ax)|$$

$$\overset{\text{norming functional 存在性.}}{=} \sup_{\|x\|=1}\|Ax\| = \|A\|$$

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[rr, bend right, "BA"'] & Y \arrow[r, "B"] & Z \end{tikzcd} \qquad \begin{tikzcd} X^* & Y^* \arrow[l, "A^*"'] & Z^* \arrow[l, "B^*"'] \arrow[ll, bend left, "(BA)^*"] \end{tikzcd}$$

$(BA)^* = A^*\cdot B^*$

取对偶 $X \Longmapsto X^*$

算子取对偶 $A\in B(X,Y) \Longmapsto A^*\in B(Y^*,X^*).$

covariant functor

$X \xrightarrow{A} Y$

$x \longmapsto Ax$

$R(A) := \{y\in Y \mid \exists x\in X \text{ s.t. } y = Ax\}$ — 线性空间.

$N(A) := \ker A = \{x\in X \mid Ax = 0\}$ 线性空间.

$$\begin{array}{ccc} X & \xrightarrow{A} & Y. \\ \cup & & \cup \\ \ker(A) & & R(A) \\ \text{closed} & & \text{in general 不知道.} \end{array}$$

$N(A) = A^{-1}\{0\}$ 是闭的.

$$\begin{array}{ccc} X^* & \xleftarrow{A^*} & Y^* \\ \cup & & \cup \\ R(A^*) & & \ker A^* \end{array}$$

$(\ker A,\ R(A^*)) = 0 = (R(A),\ \ker(A^*))$ 这是一个必要条件.

$(x,\ x^*)$ $\qquad$ $(y,\ y^*)$

$y \in R(A) \Rightarrow$
$(y, \ker A^*) = 0$

$\bullet\ (x, x^*) = (x, A^* y^*) = (\underbrace{Ax}_{0} , y^*) = 0$

$\bullet\ (y, y^*) = (Ax, y^*) = (x, \underbrace{A^* y^*}_{0}) = 0$

零化子 (Annihilators).

$\boxed{\textit{Lecture 8} \quad 2022/9/30}$

Annihilator

$$\begin{array}{cc} X & X^* \\ \cup & \cup \\ S & T \end{array}
\qquad
\begin{array}{ccc} X & \xrightarrow{A} & Y \\ \cup & & \cup \\ \ker A & & R(A) \\ & & \\ X^* & \xleftarrow{A^*} & Y^* \\ \cup & & \cup \\ R(A^*) & & \ker A^* \end{array}$$

$\forall$ subset $S \subset X$

$S^\circ := \{x^* \in X^* \mid (x^*, s) = 0\}$

${}^\circ T := \{x \in X \mid (x, T) = 0\}$.

Lemma: S° and ${}^\circ T$ are closed

Pf: $S^\circ = \bigcap_{s \in S} \{x^* \in X^* \mid x^*(s) = 0\}$ 是闭集.

Lemma: S° and ${}^\circ T$ are subspaces.

Pf: $x^*, y^* \in S^\circ$ $\quad (x^*, s) = 0,\ (y^*, s) = 0$

$\forall \alpha, \beta \in \mathbb{R},\ (\alpha x^* + \beta y^*, s) = 0 \Rightarrow \alpha x^* + \beta y^* \in S^\circ$

Lemma: $S \longrightarrow \underbrace{\overline{\text{span}}\, S}_{\text{闭子空间}} = \overline{\left\{ \sum_{\text{finite}} \alpha_i s_i \ \middle|\ \begin{array}{l} \alpha_i \in \mathbb{R} \\ s_i \in S \end{array} \right\}}^{\text{闭包}} \subset X$. Then $S^\circ = (\overline{\text{span}}\, S)^\circ$

Pf: ① if $S_1 \subset S_2 \Rightarrow S_1^\circ \supset S_2^\circ$

故 $S \subset \overline{\text{span}}\, S \Rightarrow S^\circ \supset (\overline{\text{span}}\, S)^\circ$

② $x^* \in S^\circ$ $\quad (x^*, s) = 0 \Rightarrow (x^*, \text{span}\, S) = 0 \overset{\text{连续性}}{\Rightarrow} (x^*, \overline{\text{span}}\, S) = 0$

$\Rightarrow x^* \in (\overline{\text{span}}\, S)^\circ$

$\Rightarrow S^\circ \subset (\overline{\text{span}}\, S)^\circ$

#

Lemma. $W \subset X$ subspace, Then $W = {}^\circ(W^\circ)$

Pf. ① $x \in {}^\circ(W^\circ) \Leftrightarrow (x, W^\circ) = 0$ 由于 $(W, W^\circ) \Rightarrow W \subset {}^\circ(W^\circ)$

② 若 $(x, W^\circ)=0$ 且 $x \notin W$. $\Rightarrow d(x,W)=0$

则存在 $x^* \in X^*$, $\|x^*\|=1$. s.t. $(x^*, W)=0$, $x^*(x)=d(x,W)>0$. 矛盾!

所以 $x \in W$.

bipolar Theorem 拓展知识

$S^\circ = (\overline{\text{span}}\, S)^\circ$ $\qquad \overline{\text{span}}\, S = {}^\circ\big((\overline{\text{span}}\, S)^\circ\big) = {}^\circ(S^\circ)$

$$\begin{array}{ccc} X & \xrightarrow{A} & Y \\ \cup & & \cup \\ \ker(A) & & R(A) \end{array}$$

$R(A) \subset {}^\circ(\ker A^*)$

$$\begin{array}{ccc} X^* & \xleftarrow{A^*} & Y^* \\ \cup & & \cup \\ R(A^*). & & \ker(A^*). \end{array}$$

Theorem.

$R(A) = {}^\circ(\ker A^*)$ iff $R(A)$ is closed.

Pf. "$\Rightarrow$" is trivial.

"$\Leftarrow$", if $R(A)$ is closed

$R(A) = {}^\circ(R(A)^\circ)$ $\quad \forall\, y^* \in R(A)^\circ$

$A^*y^*(x) = y^*(Ax) = 0 \Rightarrow y^* \in \ker(A^*)$.

$\Rightarrow \ker(A^*) \supset R(A)^\circ \Rightarrow {}^\circ\ker(A^*) \subset {}^\circ(R(A)^\circ)$

Actually $R(A)^\circ = \ker A^*$

Closed graph Theorem (Banach)

$$\underset{X}{\text{Banach}} \xrightarrow{A} \underset{Y}{\text{Banach}}$$

(1) $\ker A = 0 \Leftrightarrow A$ injective

(2) $R(A) = Y \Leftrightarrow A$ surjective

故 A is bijection

$A^{-1}: Y \to X$ 是 map A 的逆映射.

$A^{-1}(\alpha y_1 + \beta y_2) = \alpha A^{-1}(y_1) + \beta A^{-1}(y_2)$, so A^{-1} is linear.

A^{-1} is bounded.

Baire's category theorem

Baire 纲定理.

Def: ① 无处稠密集 (nowhere dense sets).

$W \subset X$ is called no where dense

if $\overline{W}$ contains no open subset.

$\Updownarrow$

$\forall\, U \subset X$ open, $\exists\, x \in U$, $x \notin \overline{W}$.

Remark: W nowhere dense $\Leftrightarrow \overline{W}$ nowhere dense

② X is called of the first category 否则，称 X of the second category.

if $X = \bigcup_k W_k$, W_k nowhere dense.
$\uparrow$ 可列并.

(compact metric space is of 2^{nd} category)

Theorem. X Banach space $\Rightarrow$ X is of the second category

Pf: Otherwise, $X = \bigcup_{k=1}^{\infty} W_k$, W_k is nowhere dense

存在 $r_1 < 1$, $x_1 \in X$. $\overline{B(x_1, r_1)} \cap \overline{W_1} = \emptyset$.

$\overline{W_2}$ 存在 $0 < r_2 < \frac{1}{2}$, $x_2 \in B(x_1, r_1)$. $B(x_2, r_2) \cap \overline{W_2} = \emptyset$. $B(x_2, r_2) \subset B(x_1, r_1)$

$\vdots$

$\overline{W_n}$, $\exists\ 0 < r_n < \frac{1}{n}$, $\overline{B(x_n, r_n)} \cap \overline{W_n} = \emptyset$.

$\vdots$

故 $\|x_m - x_n\| < \frac{1}{m}$ for $n > m$. $\{x_n\}$ is a Cauchy Seq, $x_m \xrightarrow{m \to \infty} x \in X$.

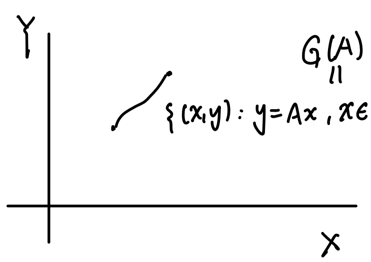
$x \in \bigcap_{n=1}^{\infty} \overline{B(x_n, r_n)}$

Closed operator $D(A) = X$ 不要求

X, Y normed space

$$X \supset D(A) \xrightarrow{A} Y \quad \text{linear operator}$$

is called closed if



$G(A) = \{(x,y) : y = Ax,\ x \in D(A)\} \subset X \times Y$

Graph of A is closed $\Leftrightarrow$ $\underbrace{(x_n, Ax_n)}_{G(A)} \to (x, y) \in X \times Y$. 则 $(x,y) \in G(A)$, 即 $x \in D(A)$ & $y = Ax$.

则只要 $\lim_{n \to \infty} \begin{matrix} x_n \xrightarrow{\text{in } X} x \\ Ax_n \xrightarrow{\text{in } Y} y \end{matrix}$, then $(x, y) \in G(A)$

Thm (closed graph theorem)

if $A: X \to Y$ is closed linear operator

$D(A) = X$.

if X, Y are both Banach spaces, then A is bounded.

Pf: 只需证：存在 $r>0$, $M>0$

$\sup_{\|x\|\le r} \|Ax\| \le M$.

$U_n := \{x\in X \mid \|Ax\|<n\}$

$\Rightarrow X = \bigcup_{n=1}^{\infty} U_n$

Banach space 存在第二纲集. 存在 $k\in\mathbb{N}$. $\overline{U_k}$ 包含开集. $\overline{U_k} \supset B(x_0,t)$, $t>0$

即 WLOG, we can assume $x_0 \in U_k$

$\|Ax_0\|<k$ $B(x_0,t) = x_0 + B(0,t) \subset \overline{U_k}$. $\forall \varepsilon>0$ $\exists z\in U_k$. $\|x_0+x-z_k\|<\varepsilon$. $\|Az\|\le k$

$\|A(x_0-z)\| = \|Ax_0 - Az\| \le \|Az\|+\|Ax_0\| < 2k$. $\Rightarrow x_0 - z \in U_{2k}$

$\forall x\in B(0,t)$. $\forall \varepsilon>0$. $\exists y\in U_{2k}$ s.t. $\|x-y\|<\varepsilon$ 即 $B(0,t)\subset \overline{U_{2k}}$

$\Downarrow$

$B(0,t)\subset \overline{U_{2k\alpha}}$

Lecture 9 2022/10/5

Thm. (Closed Graph Theorem)

$X \xrightarrow{A} Y$ linear operator $\boxed{D(A)=X.}$

Banach

if A is closed, then A is bounded

Recall.

A is closed linear operator from X to Y.

if $X \supset D(A) \xrightarrow{A} Y$ $G(A) \overset{\text{closed}}{\subset} X\times Y$.

$G(A) = \{(x,Ax) \mid x\in D(A)\}$

$\begin{cases} x_n\in D(A) & x_n\to x \\ Ax_n \to y \end{cases} \Rightarrow x\in D(A)$ & $y=Ax$

Pf: $\forall c>0$

$U_c := \{x\in X : \|Ax\|<c\}$

$X = \bigcup_{n=1}^{\infty} U_n$, Banach space $\Rightarrow$ 2^{nd} catagory

$\exists n$, s.t. $\overline{U_n} \supset B(x_0,t)$. $x_0\in X$, $t>0$

Claim 1: $\exists M>0$. s.t.

$B(0,t)\subset \overline{U_M}$

Indeed, $\forall z\in B(0,t)$

$\Rightarrow x_0+z \in B(x_0,t) \subset \overline{U_n}$

故存在 $\{x_k\}_{k=1}^{\infty} \subset U_n$ $(\|Ax_k\|<n)$

$x_k \xrightarrow{k\to+\infty} x_0+z$

$\Updownarrow$

$x_0 - x_k \xrightarrow{k\to+\infty} z$

$\|A(x_0-x_k)\| \le \|Ax_k\|+\|Ax_0\| < \underbrace{n+\|Ax_0\|}_{\text{记为 } M.}$

A 是闭的 $\Rightarrow$ $z\in\overline{U_M}$

Claim 2. $\forall \alpha>0$. $B(0,\alpha t)\subset \overline{U_{\alpha M}}$

$\forall y\in B(0,\alpha t) \Leftrightarrow y/\alpha \in B(0,t)$

$\exists \{y_k\}_{k=1}^{\infty} \subset U_M$. $y_k \xrightarrow{k\to+\infty} y/\alpha \Rightarrow \alpha y_k \xrightarrow{k\to+\infty} y$

即 $\|Ay_k\|<M \Leftrightarrow \|A(\alpha y_k)\| < \alpha M$

$\Leftrightarrow y\in \overline{U_{\alpha M}}$

let $B(0,r)\subset \overline{U_1}$.

Claim 3: $\forall\, \delta \in (0,1)$ $B(0,r) \subset U_{\frac{1}{1-\delta}}$

① $\forall\, z \in B(0,r) \subset \overline{U_1}$

$\exists\, x_1 \in U_1$, $\|z - x_1\| < \delta r$.

即 $z - x_1 \in B(0, \delta r)$.

$\forall\, z \in B(0,r)$, $\exists\, x_1 \in U_1$

$z - x_1 \in B(0,\delta r) \subset \overline{U}_\delta$ (claim 2)

② $\exists\, x_2 \in U_\delta$, $z - x_1 - x_2 \in B(0, \delta^2 r) \subset \overline{U}_{\delta^2}$

③ $\exists\, x_3 \in U_{\delta^2}$, $z - \sum_{i}^{3} x_i \in B(0, \delta^3 r) \subset \overline{U}_{\delta^3}$

$\vdots$

ⓝ $\exists\, x_n \in U_{\delta^{n-1}}$, $z - \sum^n x_i \in B(0, \delta^n r) \subset \overline{U}_{\delta^n}$

$\vdots$

we get $\begin{cases} \|z - \sum_1^n x_i\| < \delta^n r \\ \|A x_n\| < \delta^{n-1} \end{cases} \Rightarrow \begin{cases} \sum_1^n x_i \xrightarrow{\text{in } X} z \\ \boxed{\cdots} \end{cases}$

$\|A x_n\| < \delta^{n-1} \Rightarrow \left\{ \sum_{i=1}^n A_i \right\}_{n=1}^{\infty}$ is a Cauchy Seq in Y.

$\|\sum_m^n A x_i\| \le \sum_m^n \|A x_i\| < \sum_m^n \delta^{i-1} \to 0$

由于 Y is Banach, $A \sum x_i \to y$ in Y.

即 $\begin{cases} \sum^n x_i \xrightarrow{\text{in } X} z \\ A \sum^n x_i \xrightarrow{\text{in } Y} y \end{cases}$ A closed $\Rightarrow y = Az$.

$$\|Az\| = \|y\| = \lim_n \|\sum^n A x_i\| \le \limsup_n \sum^n \|A x_i\| = \sum_{i=1}^{\infty} \|A x_i\| < \frac{1}{1-\delta}.$$

$\forall\, z \in B(0,r)$, $\|Az\| < \frac{1}{1-\delta}$ $\Rightarrow z \in U_{\frac{1}{1-\delta}}$.

$\Rightarrow A$ is bounded. □

Thm. 3.8 X, Y Banach $A \in B(X,Y)$ $\Downarrow$ A closed

$X \xrightarrow{A} Y$

if $\ker A = 0$ $(N(A))$ & $\operatorname{Im} A = Y$ $(R(A))$

then $A^{-1}: Y \to X$ is bounded.

$G(A) = \{(x, Ax) \mid x \in X\} \subset X \times Y$. closed

$G(A^{-1}) = \{(Ax, x) \mid x \in X\} \subset Y \times X$. closed

$\Downarrow$ CGT

A^{-1} is bounded.

$\|A^{-1} y\| \le \|A^{-1}\| \|y\|$

$\frac{\|x\|}{\|A^{-1}\|} \le \|Ax\| \le \|A\| \cdot \|x\|$

$\|x\| = \|A^{-1} A x\| \le \|A^{-1}\| \cdot \|Ax\|$

Def: Two normed space X, Y are called isometric (同构).

$M_1 \|x\| \le \|Tx\| \le M_2 \|x\|$.

if $\exists\, T: X \to Y$, $S: Y \to X$. bounded linear operator

$T \circ S = id_Y$. $S \circ T = id_X$.

Thm. 3.11 $X \leftarrow$ Banach spaces.

$X \supset D(A) \xrightarrow{A} Y$ (Y Banach), A closed operator

Then $A^{-1}: Y \to D(A)$ is bounded

$A^{-1} \in B(Y, X)$

Proof: 在 $D(A)$ 上定义范数 $\|\cdot\|_A$

$\|x\|_A := \|x\| + \|Ax\|$,

A is closed $\Rightarrow (D(A), \|\cdot\|_A)$ is Banach.

$\{x_n\}$ is a Cauchy seq in $D(A)$

Then $\|x_n - x_m\| + \|Ax_n - Ax_m\| < \varepsilon$.

$\Rightarrow \{x_n\}, \{Ax_n\}$ Cauchy seq in X, Y.

$\Rightarrow \lim_n x_n = x$. $\lim_n Ax_n = y$. A closed

$\Rightarrow Ax = y$, $x \in D(A)$ $\|x_n - x\|_A \to 0$ #.

$(D(A), \|\cdot\|_A) \overset{A}{\hookrightarrow\!\!\!\to} Y$.

Thm 3.8 $\Rightarrow A^{+}: Y \to (D(A), \|\cdot\|_A)$

即存在 $M>0$, s.t. $\|A^{+}y\|_A = \|A^{+}y\| + \|y\| \le M\|y\|$, $M>1$.

$\Rightarrow \|A^{+}y\| \le (M-1)\|y\| \quad \forall y \in Y$.

Example Closed but not bnd. operator.

$C_0(\pi) = \left\{ f: \mathbb{R} \to \mathbb{R} \;\middle|\; \begin{matrix} f \text{ continuous} \\ f(0)=0. \; f(x+2\pi) = f(x) \end{matrix} \right\}$. $\|\cdot\|_\infty$

$C_0^1(\pi) = \{ f \in C_0(\pi) \mid f' \text{ continuous} \}$

$C_0(\pi) \leftarrow$ Banach

$\cup$

$C_0^1(\pi) \xrightarrow{D} C(\pi)$

$D: f \to Df$ (微分).

Then D is closed but not bnd.

closedness:

if $\begin{cases} f_n \xrightarrow{C_0(\pi)} f & (一致收敛). \\ Df_n \xrightarrow{C(\pi)} g \end{cases}$

Then $f \in C_0^1(\pi)$. and

$g = Df$ (控制收敛)

but unbounded

$\frac{\|f'\|_\infty}{\|f\|_\infty} \to +\infty$

Closed range operators.

$X \xrightarrow{A} Y$ (Banach) if $\ker A = 0$, $\operatorname{Im} A$ is closed in Y

$\Rightarrow X \hookrightarrow\!\!\!\to \operatorname{Im} A$. 则 $\exists\, c>0$ $\|Ax\| \ge c\|x\| \;\; \forall x \in X$. (☆)

Banach $\quad \cap \quad$ 充要条件.

Y closed.

(☆) $\overset{?}{\Rightarrow} \operatorname{Im} A$ is closed

if $y_n \to y$ for $\{y_n\}_{n=1}^{\infty} \subset \operatorname{Im} A$.

Ax_n is cauchy seq $\Rightarrow \|x_n - x_m\| \le \frac{1}{c} \cdot \|Ax_n - Ax_m\| \Rightarrow \{x_n\}_{n=1}^{\infty}$ is Cauchy seq.

$x_n \to x_\infty$ $\quad \begin{cases} Ax_n \to y \\ x_n \to x_\infty \end{cases} \Rightarrow y = Ax_\infty \in \operatorname{Im} A$

□

Thm 3.12

$X \to$ Banach $\leftarrow Y$

$\cup$

$D(A) \xhookrightarrow{A} Y$

closed linear operator

$\boxed{D(A) = X}$ ← significant !

Then. $\operatorname{Im} A$ closed $\Leftrightarrow \exists\, c>0$. $\|Ax\| \ge c\|x\|$ for all $x \in D(A)$.

Quotient space

X normed space, $M \overset{\text{closed subspace}}{\subset} X$

Define a equivalance relation $\sim_M$ on X

$x, y \in X$ is called equivalant

if $x \sim_M y \overset{\text{definition}}{\Longleftrightarrow} x-y \in M$.

1° $x \sim_M x$ 平凡.

2° $x \sim_M y \Leftrightarrow y \sim_M x$.

3° $x \sim_M y,\ y \sim_M z \Rightarrow x \sim_M z$.

记等价类为 $[x] = x + M$.

$X/_M = \{[x] \mid x \in X\}$.

$\alpha[x] + \beta[y] := [\alpha x + \beta y],\ \forall \alpha, \beta \in \mathbb{R}$.

it is well-defined.

$\|[x]\| := \inf_{z \in [x]} \|z\| = \inf_{y \in M} \|x-y\| = d(x, M)$

Prop: X is normed, $M \subset X$ closed subspace.

$(X/_M, \|\cdot\|)$ is a normed space

1° $\|[x]\| := d(x, M)$ $\quad \|[x]\| = 0 \Leftrightarrow d(x, M) = 0$

M closed $\Rightarrow x \in M \Rightarrow [x] = [0]$

2° $\|[\alpha x]\| = \inf_{y \in M} \|\alpha x - y\| = |\alpha| \inf_{y \in M} \|x - \frac{y}{\alpha}\| = |\alpha| \cdot \|[x]\|$

3° 三角不等式 $\quad \|[x] + [y]\| \le \|[x]\| + \|[y]\|$.

$\Leftrightarrow \inf_{z \in M} \|x+y-z\| \le \inf_{z_1 \in M} \|x - z_1\| + \inf_{z_2 \in M} \|y - z_2\|$.

$\inf_{z \in M} \|x+y-z\| \le \|x+y-z_1-z_2\| \le \|x-z_1\| + \|y-z_2\|$ #

Moreover, if X Banach space

its Quotient space is Banach.

X/M, if $[x_n]$ is a Cauthy seq.

$\|[x_n] - [x_m]\| = \inf \|x_n - x_m - z\| < \varepsilon$.

$\sum_{n=1}^{\infty} \|[x_n]\| < +\infty$

$\Downarrow$

存在 x. $[x] = \sum_{n=1}^{\infty} [x_n]$.

$\|x_n - z_n\| \le \|[x_n]\| + \frac{\varepsilon}{2^n}$.

$\Rightarrow \sum_{n=1}^{\infty} \|x_n - z_n\| < +\infty$,

X Banach $\Rightarrow x = \sum_{n=1}^{+\infty} (x_n - z_n)$ 存在.

$\|[x] - \sum_{n=1}^{N} [x_n]\| \le \|x - \sum_{n=1}^{N} x_n - \sum_{n=1}^{N} z_n\| \xrightarrow{N \to \infty} 0$ #

Thm 3.14.

$$\begin{array}{ccc} X & \leftarrow & \text{Banach} \\ \cup & & \downarrow \\ D(A) & \xrightarrow{A} & Y. \end{array}$$

closed. linear.

Then $\mathrm{Im}(A)$ is closed $\Leftrightarrow \exists\, c > 0$.

$\|Ax\| \ge c\|[x]\|$

Lemma: A is closed $\Rightarrow$ ker A is closed. subspace

$\begin{cases} Ax_n = 0 \\ x_n \to x \end{cases} \Rightarrow x \in D(A)$ & $Ax = 0$ $\Rightarrow x \in \ker A$.

$X/_{\ker A}$ is a Banach. space.

$D(A)/_{\ker A} \xrightarrow{\hat{A}} \mathrm{Im}\,A$. $\quad \hat{A}[x] = Ax$.

$[x] \in X/_{\ker A} \quad \forall x \in D(A)$

$\hat{A}$ is closed: $\begin{cases} [x_n] \to [x]. \\ \hat{A}[x_n] \to y. \end{cases}$

$[x_n] \to [x] \Leftrightarrow x_n + z_n \to x$.

由 A closed

$\Rightarrow \hat{A}$ closed.

$\|Ax\| = \|\hat{A}[x]\|$ $\quad \Rightarrow \|Ax\| \ge c\|[x]\|$

$\Leftrightarrow \|\hat{A}[x]\| \ge c\|[x]\|$

$\Rightarrow \mathrm{Im}\,\hat{A}$ closed.

$\mathrm{Im}\,A = \mathrm{Im}\,\hat{A}$ closed.

Lecture 10　2022/10/7

思考：X 是 Banach，但 Y 不是，找 CGT 反例

① closed Graph Theorem (CGT)

$X \xrightarrow{A \text{ linear}} Y$　$D(A)=X$.

X ← Banach →Y

A closed ⇒ A bnd

② $X \overset{A\in B(X,Y)}{\hookrightarrow} Y$　X Banach, Y Banach

$\ker A = 0$ and $\operatorname{Im} A = Y$

则 $\exists\, C_1, C_2 > 0$

$C_1\|x\| \le \|Ax\| \le C_2\|x\|$

③ X ← Banach → Y

$X \supset D(A) \xrightarrow[\text{closed}]{A} Y$

$\ker A = 0$.　$\operatorname{Im} A = Y$

则 $A^{-1} \in B(Y, X)$

point：$\|x\|_A = \|x\| + \|Ax\|$

A closed ⇒ ($D(A)$ Banach)

④ closed range

$X \xrightarrow{A\in B(X,Y)} Y$,　X ← Banach → Y

Then $\operatorname{Im}(A)$ closed

⇕

$\exists\, c>0$,　$Ax \ge c\|[x]\| = c\, d(x, \ker A)$.

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d] & \operatorname{Im}(A) \subset Y \\ X/\ker(A) \arrow[ur, dashed, "\tilde{A}"] & \end{tikzcd}$$

X Banach，$\tilde{A}$ 线性双射，$X/\ker(A)$ Banach.

⑤ X ← Banach → Y

$X \supset D(A) \xrightarrow{A} Y$,　Then $\operatorname{Im} A$ is closed ⇕

$\ker A = 0$.

$\|Ax\| \ge c\|x\|_A \ge c\|x\|$

⑥ X ← Banach → Y

$X \supset D(A) \xrightarrow{A \text{ closed}} Y$

$X/\ker A \supset D(A)/\ker A \xrightarrow{\hat{A}} Y$

$\operatorname{Im} A$ closed. ⇒ $\|Ax\| \ge c\|[x]\|_A \ge c\|[x]\|$

Thm 3.16

$X \xrightarrow{A} Y$　X ← Banach → Y　if $\operatorname{Im} A$ is closed

⇓ dual

$\operatorname{Im} A^*$ closed.

$X^* \xleftarrow{A^*} Y^*$

$$\begin{tikzcd} X \arrow[rr, bend left, "A"] \arrow[r, "q"] & X/\ker A \arrow[r, "\hat{A}", "\cong"'] & \operatorname{Im} A \arrow[r, hook, "i"] & Y \end{tikzcd}$$

$$\begin{tikzcd} X^* & (X/\ker A)^* \arrow[l, "q^*"', "\text{embedding}"] & (\operatorname{Im} A)^* \arrow[l, "\hat{A}"', "\cong"] & Y^*. \arrow[l, "i^*"'] \end{tikzcd}$$

Pf：

$$\begin{tikzcd} X \arrow[r] \arrow[d, "q"', "\text{quotient}"] & \operatorname{Im} A \arrow[r, hook, "i", "\text{embedding}"'] & Y \\ X/\ker A \arrow[ur, dashed, "\tilde{A}", "\cong"'] & & \end{tikzcd}$$

Banach，Banach

$\operatorname{Im} A^* = \operatorname{Im} \hat{A}^* = (X/\ker A)^* = (\ker A)^\circ$

Lemma：商空间的对偶空间是对偶空间的子空间.

（闭）子空间的对偶空间是对偶空间的商空间.

· $(X/M)^* \subset X^*$　$X \xrightarrow{q} X/M \xrightarrow{l \in (X/M)^*} \mathbb{R}$，$l \circ q \in X^*$.

· $M \subset X$

$M^* = X^*/M^\circ$

$X \supset M$，$\tilde{l}$ Hahn Banach，$M \xrightarrow{l \in M^*} \mathbb{R}$

$l \to l|_M$　$\ker \Phi = M^\circ$

$X^* \overset{\Phi}{\twoheadrightarrow} M^* \Rightarrow X^*/M^\circ \cong M^*$

共鸣定理. (uniform boundness principle)
Banach-steinhaus Theorem.

Thm: $X \longrightarrow Y$
Banach normed space

$W \subset B(X, Y)$.

if $\forall x \in X$, $\sup_{A \in W} \|Ax\| < +\infty$

Then $\sup_{\|x\|=1} \sup_{A \in W} \|Ax\| < +\infty$

$\Leftrightarrow \sup_{A \in W} \|A\| < +\infty$

① Proof: $X = \bigcup_{n=1}^{+\infty} \underbrace{\{x \in X \mid \sup_{A \in W} \|Ax\| \le n\}}_{V_n}$

Clearly. V_n is closed

$V_n = \bigcap \{x : \|Ax\| < n\}$.

X is Banach $\Rightarrow$ X is of 2^{nd} Catagory $\Rightarrow \exists n_0, r > 0$
$V_{n_0} \supset B(x_0, r)$

$\forall z \in B(0, r)$. $x_0 + z \in B(x_0, r)$

$\Rightarrow \begin{cases} \|A(x_0+z)\| \le n_0 \\ \|Ax_0\| \le n_0 \end{cases} \Rightarrow \|Az\| \le 2n_0$
$\forall A \in W$.

$\Rightarrow \sup_{\substack{A \in W \\ \|x\|=1}} \|Ax\| \le \frac{2n_0}{r}$.

② Proof: Replace Y by its completion $\bar{Y}$.

$X \xrightarrow{T} l_\infty(I; \bar{Y})$ $\quad W = \{A_i \mid i \in I\} \subset B(X, Y)$.

$x \longrightarrow Tx = (A_i x)_{i \in I}$.

$l_\infty(I; \bar{Y}) = \{(y_i)_{i \in I} \mid y_i \in \bar{Y} \text{ 且 } \sup_i |y_i| < +\infty\}$

$\|(y_i)_{i \in I}\| := \sup_{i \in I} \|y_i\|$

可以定义的 reason 是 $\sup \|A_i x\| < +\infty$

Claim 1: T is bnd $\longleftarrow$ 只需证 T closed

if $x_n \to x$.
$Tx_n \to (y_i)_{i \in I} \in l_\infty(I; \bar{Y})$.
Then $\|A_i x_n - y_i\| \to 0 \Rightarrow A_i x = y_i$.

Claim 2: $\|T\| = \sup_{i \in I} \|A_i\|$

开映射定理.
(quotient maps are open maps).

Thm 3.18 $\quad$ $X \leftarrow$ Banach
$\cup$
$D(A) \xrightarrow[\text{closed}]{A} Y$ (Banach)

则 $U \in D(A)$ open $\Rightarrow A(U)$ is open

A closed & X is Banach
$\Downarrow$
$(D(A), \|\cdot\|_A)$ is Banach

$$\begin{tikzcd} D(A) \arrow[r, two heads, "A"] \arrow[d, "q"] & Y \\ D(A)/\ker A \arrow[ur, dashed, "\hat{A}"'] & \end{tikzcd}$$
$\hat{A}$ 是同构.

Lemma 1 $\quad$ X Banach, $M \subset X$ closed subsp

Then $X \xrightarrow{q_M} X/M$ is open

Lemma 2 $\quad (D(A), \|\cdot\|_A) \longrightarrow D(A) \subset X$

开集 (in x) $\longrightarrow$ 开集 (in x).

Lecture 11　2022/10/12

习题：$\exists\, f\in l_\infty^*$ s.t. $\liminf\limits_k x_k \le f(x) \le \limsup\limits_k x_k$，$\forall\, x\in l_\infty$

pf: $p(x) := \limsup\limits_{k\to\infty} x_k$

1° $p(x)$ is sublinear.

2° $l_\infty \supset \mathbb{R}\mathbb{1} \xrightarrow{l} \mathbb{R}$

$t\mathbb{1} \longrightarrow t$

$l(x)\le p(x) \Rightarrow \exists\, f(x)\in l_\infty^*$.

$f(x)\le p(x)$.

$f(x)\le \limsup x_k$.

$\Downarrow$

$f(-x)\le -\liminf(x_k)$

$\Downarrow$

$f(x)\ge \liminf(x_k)$.

3° $f(x)\le \limsup\limits_k x_k \le \sup\limits_k |x_k|$

$-f(x)\le \limsup\limits_k(-x_k)\le \sup\limits_k |x_k|$

$\Rightarrow \|f\| < +\infty \Rightarrow f\in l_\infty^*$

Define：$\|\cdot\|$，$|||\cdot|||$ are two norms. on a vector space X are called equivalent

if $\exists\, C_1, C_2 > 0$ s.t. $C_1|||x||| < \|x\| < C_2|||x|||$　$\forall\, x\in X$.

(验证这是等价关系).

Thm 4.2　X vector space，$\dim X<\infty$

Then any two norms on X are equivalent.

pf.

不妨设 $\dim X = n < +\infty$，$v_1, v_2, \cdots, v_n$ 是其中一组基.

$\forall\, x\in X$，$\exists!$ 表示 $x=\sum\limits_{i=1}^n \alpha_i v_i$. $(\ast)$

定义一个范数 $\|x\|_0 = \sqrt{\sum\limits_{i=1}^n |\alpha_i|^2}$ 若$\ast$成立.

$S_1(\|\cdot\|_0) = \{x\in X \mid \|x\|_0 = 1\}$ in X 是紧集.（在 $(X, \|\cdot\|_0)$ 拓扑下）.

令 $\|\cdot\|$为任一给定范数. 下证：$\|\cdot\| \sim \|\cdot\|_0$

1° $\|x\| = \left\|\sum\limits_{i=1}^n \alpha_i v_i\right\| \le \sum\limits_{i=1}^n |\alpha_i|\cdot\|v_i\| \le \sqrt{\sum\limits_{i=1}^n|\alpha_i|^2}\cdot\sqrt{\sum\limits_{i=1}^n\|v_i\|^2} = \|x\|_0\cdot\underbrace{\sqrt{\sum\limits_{i=1}^n\|v_i\|^2}}_{:=C_2}$

2° $(X, \|\cdot\|_0) \xrightarrow{\|\cdot\|} \mathbb{R}_{\ge 0}$ is continuous.　$\big|\|x\|-\|y\|\big| \le \|x-y\| \le C_2\|x-y\|_0$

3° $\underbrace{S_1(\|\cdot\|_0)}_{\text{紧}} \xrightarrow{\|\cdot\|} \mathbb{R}_{\ge 0}$　故 $\|\cdot\|$ 取到紧集上极小值，即 $\exists\, x_0\in S_1(\|\cdot\|_0)$.

$\|x_0\|_0 = 1$.　$\forall\, x\in S_1(\|\cdot\|_0)$.　$\|x\|\ge\|x_0\|>0$.

$\forall\, z\in X$，$z\ne 0$.　$\|z\| \ge \|z\|_0\cdot\|x_0\|$，令 $C_1 = \|x_0\|$.

则 $C_1\|z\|_0 \le \|z\| \le C_2\|z\|_0 \Rightarrow \|\cdot\| \sim \|\cdot\|_0$

Corollary: 1° X n.v.s and $\dim X = n$. (n.v.s = normed vector space).
$\Rightarrow X$ is Banach.

2° Y n.v.s. $M \subset Y$. $\Rightarrow M$ is closed. 等价范数诱导出同一个拓扑.
有限维子空间

Thm 4.6 X normed space, $S_1(X) = \{x \in X \mid \|x\| = 1\}$.

(1) 若 $S_1(X)$ compact $\Rightarrow \dim X < \infty$.

(2) $\bar{B}_1(X) = \{x \in X \mid \|x\| \le 1\}$ compact $\Rightarrow \dim X < \infty$

Lemma. $M \subsetneqq X$. 则 $\forall\, \theta \in (0,1)$, 存在 $x_\theta \in X$.
proper closed subspace $\|x_\theta\| = 1$. $d(x_\theta, M) \ge \theta$.

pf. 存在 $x \in X \setminus M \Rightarrow d(x,M) > 0$ 存在 $m_n \in M$, $\|x - m_n\| \downarrow_{n\to\infty} d(x,M) > 0$
M closed.

$\left\| \frac{x-m_n}{\|x-m_n\|} \right\| = 1$ $d\left(\frac{x-m_n}{\|x-m_n\|}, M\right) = \frac{1}{\|x-m_n\|} d(x,M) \nearrow_{n\to\infty} 1$ 取 n 充分大, $\ge \theta$ 即可.

Pf of Thm 4.6.

1° if $\dim X = \infty$. $S_1(x)$ 不是紧的.

1° $x_1 \in X$, $\|x_1\| = 1$, 令 $\theta = 1/2$

2° $\mathbb{R}x_1 \subsetneqq X$, 存在 $x_2 \in X$, $\|x_2\| = 1$, $d(x_2, \mathbb{R}x_1) \ge \frac{1}{2}$

3° $\mathbb{R}x_1 + \mathbb{R}x_2 \subsetneqq X$, 存在 $x_3 \in X$, $\|x_3\| = 1$, $d(x_3, \mathbb{R}x_1 + \mathbb{R}x_2) \ge \frac{1}{2}$.

⋮

无穷进行下去. 则 $\{x_n\}_{n=1}^\infty \subset S_1$. 且 $\{x_n\}_{n=1}^\infty$ 没有收敛子列!
$\Rightarrow S_1$ 不紧. $(d(x_n, x_m) \ge 1/2)$.

2°
$\bar{B}_1(x) \Rightarrow S_1(x)$
紧 紧.

net convergence.

$l_\infty^* \xrightarrow{f} \mathbb{R}$. Modele theory

Banach limit. ultrafilter

$$\begin{array}{ccc} X & \xrightarrow{y \otimes x^*} & Y \\ & \Downarrow \text{dual} & \\ Y^* & \xrightarrow{x^* \otimes y} & X^* \end{array}$$

验证

有限秩算子. operators of finite rank.

$X \xrightarrow{A} Y$ finite rank $\overset{\text{def}}{\Longleftrightarrow} \dim \operatorname{Im} A < +\infty$.

$\forall\, x^* \in X^*$, $\forall\, y \in Y$
可以定义一个 rank one operator. $X \to Y$. $x \mapsto x^*(x) y$.

我们记这个算子为 $y \otimes x^*$: $X \to Y$.

$Y^* \to X^*$
$y^* \mapsto y^*(y)\, x^*$ 这个算子记为 $x^* \otimes y$.

$\langle (y\otimes x^*)^* y^*, x \rangle = \langle y^*, y\otimes x^*(x) \rangle = x^*(x)\langle y^*, y\rangle = x^*(x)\cdot y^*(y) = \langle y^*(y)x^*, x\rangle$

$\Rightarrow (y\otimes x^*)^* = (x^*\otimes y)$.

$\forall x_1^*, \cdots x_n^* \in X^*$, $\forall y_1, \cdots y_n \in Y$. 定义: $\sum_{i=1}^n y_i\otimes x_i^* : X \longrightarrow Y$.

Proposition: $\forall$ finite rank bnd operator $X\to Y$ 都有上述形式.

if $X \xrightarrow{T} Y$ finite rank bnd operator. $\dim \mathrm{Im}T < \infty$. let $n = \dim \mathrm{Im}T$.

$\mathrm{Im}T = \mathrm{span}\{y_1, \cdots, y_n\}$. $\forall x\in T$. $Tx = \sum_{i=1}^n \alpha_i(x) y_i$. 展开是唯一的

Claim 1: $\alpha_i(x)$ 是线性的.
$$\begin{aligned} T(\lambda x+\mu z) &= \sum_{i=1}^n \alpha_i(\lambda x+\mu z) y_i \\ &\parallel \\ \lambda T(x)+\mu T(z) &= \sum_{i=1}^n (\lambda\alpha_i(x)+\mu\alpha_i(z)) y_i \end{aligned} \Rightarrow \alpha_i(\lambda x+\mu z) = \lambda\alpha_i(x)+\mu\alpha_i(z).$$

Claim 2: α_i 有界. $X \xrightarrow{T} \mathrm{Im}T \subset Y$. $\mathrm{Im}Y$ is closed, Banach, 范数等价.

$Tx = \alpha_1(x)y_1 \cdots + \alpha_n(x)y_n$

$\|T\|\|x\| \geq \|Tx\| \geq C\sum_{i=1}^n |\alpha_i(x)| > C|\alpha_i(x)| \Rightarrow |\alpha_i(x)| < \frac{\|T\|}{C}\cdot\|x\| \Rightarrow \alpha_i(x)$ 有界.

$\Rightarrow \alpha_i \in X^*$. $Tx = \sum \alpha_i(x) y_i = \sum y_i \otimes \alpha_i$

Thm 4.9 X normed space. $K\in F(X) = \{$ finite rand add operators $X\to X\}$.

Then $\begin{cases} \mathrm{Im}(I-K) \text{ is closed} \\ \dim\ker(I-K) = \dim\ker(I-K^*) < +\infty \end{cases}$ $\quad X\xrightarrow{K} X$, $X^* \xleftarrow{K^*} X^*$ $\quad I: X\to X$, $x\to x$ identity map.

Pf: 不妨设 $K = \sum_{i=1}^n x_i\otimes x_i^* : X\to X$.

若 $\dim \mathrm{Im}K = n \Rightarrow \begin{cases} x_1, \cdots, x_n \text{ 线性独立.} \\ x_1^*, \cdots, x_n^* \text{ 线性独立.} \end{cases}$ $\quad$ if $x_n^* = \sum_1^{n-1} \alpha_i x_i^*$. Then $K = \sum_1^{n-1} x_i^*(x_i+\alpha_i x_n)$.

$\Rightarrow \dim \mathrm{Im}K = n$ 矛盾.

$y\in \mathrm{Im}(I-\sum_{i=1}^n x_i\otimes x_i^*)$.

$\Leftrightarrow \exists x\in X$ $y = x - \sum_{i=1}^n x_i^*(x)x_i. = x - (x_1, \cdots x_n)\cdot\begin{bmatrix} x_1^*(x) \\ \vdots \\ x_n^*(x)\end{bmatrix}$ (☆☆).

$$\Rightarrow \begin{bmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y)\end{bmatrix} = \begin{bmatrix} x_1^*(x) \\ \vdots \\ x_n^*(x)\end{bmatrix} - \underbrace{\begin{bmatrix} x_1^*(x_1), \cdots & x_1^*(x_n) \\ \vdots \quad \ddots & \vdots \\ x_n^*(x_1), \cdots & x_n^*(x_n)\end{bmatrix}}_{M = [x_i^*(x_j)]_{i,j}}\cdot\begin{bmatrix} x_1^*(x) \\ \vdots \\ x_n^*(x)\end{bmatrix} \Rightarrow \begin{bmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y)\end{bmatrix} = (I-M)\cdot\begin{bmatrix} x_1^*(x) \\ \vdots \\ x_n^*(x)\end{bmatrix}$$

$M = [x_i^*(x_j)]_{i,j}$. 为常值. (与 K 有关).

$$\begin{bmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{bmatrix} = (I-M)\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \quad ☆$$

未知量.

若 $I-M$ 可逆 $\Rightarrow$ $\forall y\in Y$ ☆ 均有解.

Claim: 固定 $y\in X$. (☆☆) has a solution $\Longleftrightarrow$ (☆) has a solution in $\mathbb{R}^n$

"$\Rightarrow$" has been proved.

反之. 若 $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ 满足☆, 令 $x := y+\sum \alpha_i x_i = y + (x_1 \cdots x_n)\cdot\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

则易验证 x, y 满足(☆☆).

$x_i^*(x) = \alpha_i$.

Lecture 12 2022/10/14

开映照定理

$$\begin{tikzcd} X \arrow[r, phantom, "\leftarrow \text{Banach}"] & \\ D(A) \arrow[u, phantom, "\cup"] \arrow[r, "A"', "\text{closed}"'] & Y \end{tikzcd}$$

(Y ← Banach)

Then A is an open map

$(D(A), \|\cdot\|_A)$ Banach

$\|x\|_A := \|x\| + \|Ax\|$

$(D(A), \|\cdot\|_A) \xrightarrow{\text{连续}} (D(A), \|\cdot\|)$

$x \longmapsto x.$

Bijection

$$D(A)/\ker A \xrightarrow[\hat{A}]{\cong} Y$$

$$\begin{tikzcd} D(A) \arrow[r, "A"] \arrow[d, "q"'] & Y \\ D(A)/\ker A \arrow[ur, "\text{等价}\ \cong"'] & \end{tikzcd}$$

真正的本质:

$X \xrightarrow{q} X/M$

把 $B_1(x)$ 满射映入 $B_1(X/M)$

单位开球. $q(B_1(x)) = B_1(X/M)$

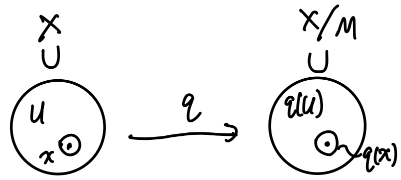
Prop: X is Banach

$M\subset X$ closed subspace

Then $X \xrightarrow{q} X/M$ is an open map

Pf: $U\subset X$ open. Consider $q(U)$

$\forall\ q(x)\in q(U)$, $x\in U$.



若已知 $B(x,\varepsilon)\subset U$.

$\{z \mid \|z-x\|<\varepsilon\} = B(x,\varepsilon)$

$\forall\ y\in B(q(x),\varepsilon)$

设 $q(x_1)=y$.

$\|q(x_1)-q(x)\| < \varepsilon \iff d(x_1-x, M) < \varepsilon$

$\Rightarrow \exists\ m\in M,\ \|x_1-x-m\|<\varepsilon \Rightarrow x_1-m\in B(x,\varepsilon)\subset U.$

$y = q(x_1) = q(x_1-m) \in q(U) \Rightarrow B(q(x),\varepsilon)\subset q(U)$

$\forall\ x\in U$. 故 $q(U)$ 是开集, q是开映射. □

习题22:

$$\begin{tikzcd} X \arrow[r, phantom, "\leftarrow \text{Banach}"] & \\ D(A) \arrow[u, phantom, "\cup"] \arrow[r, "A"] & Y \end{tikzcd}$$

(Y ← Banach)

$$\begin{tikzcd} X & \\ D(B) \arrow[u, phantom, "\cup"] \arrow[r, "B"] & Y \end{tikzcd}$$

$D(B)\supset D(A)$

$D(A+B) = D(A)$ (定义)

$A^{-1}\in B(Y,X)$

$\|BA^{-1}\| < 1.$

$A+B = (I+BA^{-1})A$

若已知 $(I+BA^{-1})^{-1}$ 存在.

则 $(A+B)^{-1} = A^{-1}(I+BA^{-1})^{-1}$

$$\begin{tikzcd} D(A) \arrow[r, "A+B"] \arrow[dr, "A"'] & Y \arrow[l, bend right, "(A+B)^{-1}"'] \arrow[d, bend left, "(I+BA^{-1})^{-1}"] \arrow[d, phantom, "\text{可逆性等价}"'] \\ & Y \arrow[u, "I+BA^{-1}"'] \arrow[ul, bend left, "\text{已知可逆}"] \end{tikzcd}$$

而 $(I+BA^{-1})^{-1} = \sum_{n=0}^{+\infty}(-BA^{-1})^n \in B(X,Y)$.

因为 $\|\sum_{n=0}^{+\infty}(-BA^{-1})^n\| \leq \sum_{n=0}^{+\infty}\|BA^{-1}\|^n < +\infty \Rightarrow \sum_{n=0}^{+\infty}(-BA^{-1})^n$ 收敛

且易验证: $(I+BA^{-1})\cdot(\sum_{n=0}^{+\infty}(-BA^{-1})^n) = I = (\sum_{n=0}^{+\infty}(-BA^{-1})^n)\cdot(I+BA^{-1})$

$\|(I+BA^{-1})^{-1}\| = \|\sum_{n=0}^{+\infty}(-BA^{-1})^n\| \leq \sum_{n=0}^{+\infty}\|BA^{-1}\|^n = \frac{1}{1-\|BA^{-1}\|}$, $\|(A+B)^{-1}\| \leq \|A^{-1}\|\cdot\|(I+BA^{-1})^{-1}\| = \frac{\|A^{-1}\|}{1-\|BA^{-1}\|}$

$\|(A+B)^{-1}-A^{-1}\| \leq \|A^{-1}\|\cdot\|(I+BA^{-1})^{-1}-I\| = \|A^{-1}\|\cdot\|\sum_{n=1}^{+\infty}(-BA^{-1})^n\| \leq \frac{\|A^{-1}\|\cdot\|BA^{-1}\|}{1-\|BA^{-1}\|}$ □

$K = \sum_{i=1}^{n} x_i \otimes x_i^* : X \to X$

若 $\dim \operatorname{Im} K = \operatorname{Rank}(K) = n < +\infty$

则上述表达存在, 且这种表达中:

$x_1, \cdots, x_n$ 线性独立

$x_1^*, \cdots, x_n^*$ 线性独立.

Recall:

Prop: 固定 $y \in X$. (pf cf. Lecture 11)

$(Id - \sum_{i=1}^{n} x_i \otimes x_i^*)\, x = y$ has a solution

$\Leftrightarrow \quad y = x - \sum_{i=1}^{n} x_i \cdot x_i^*(x)$

$\Leftrightarrow \begin{bmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{bmatrix} = (I - \underbrace{[x_k^*(x_l)]_{k,l}}_{M}) \cdot \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

has a solution

($\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ is varieties).

Thm: ① $\operatorname{Im}(I - \sum_{i=1}^{n} x_i \otimes x_i^*)$ is closed.

② $\dim \ker(I - \sum_{i=1}^{n} x_i \otimes x_i^*)$

$\quad \|$

$\dim \ker(I - \sum_{i=1}^{n} x_i^* \otimes x_i) < +\infty$

Pf: Case 1: if $\det(I-M) \neq 0$

事实上有: $\operatorname{Im}(I - \sum_{i=1}^{n} x_i \otimes x_i^*) = X \Leftrightarrow \det(I-M) \neq 0$.

Case 2: if $\det(I-M) = 0$.

Lemma: $x_1^*, \cdots x_n^*$ 相互独立 $\Leftrightarrow$ $X \xrightarrow{L} \mathbb{R}^n$, $x \longrightarrow (x_1^*(x), x_2^*(x), \cdots, x_n^*(x))$ 是满射.

Pf: 若不为满射. 则存在 $\beta = (\beta_1, \cdots, \beta_n) \perp \operatorname{Im} L$, $\beta \neq 0$

$\Leftrightarrow \sum \beta_i x_i^*(x) = 0$ for $\forall x \in X$.

$\Updownarrow$

$\sum \beta_i x_i^* = 0$ 是零算子, 这与 $\beta \neq 0$. x_i^* 线性无关矛盾. 故 L 是满射.

若 L 为满射. $\sum_{i=1}^{n} \lambda_i x_i^* = 0 \Leftrightarrow \sum_{i=1}^{n} \lambda_i x_i^*(x) = 0$ for $\forall x \in X$.

L 是满的. 故 $\exists x \in X$ 使 $x_i^*(x) = \lambda_i \Rightarrow \sum_{i=1}^{n} \lambda_i^2 = 0 \Rightarrow \lambda_i = 0 \Rightarrow x_i^*$ 线性无关 □

Now we return to case 2.

Lemma: T $n\times n$ matrix $T: \mathbb{R}^n \to \mathbb{R}^n$

Then $Tx = y$ has a solution $\Leftrightarrow$ $\forall \alpha \in \mathbb{R}^n$. s.t. $\alpha T = 0$ 都有 $\alpha y = 0$

这是线性代数结论. 证明略.

so $y \in \mathrm{Im}(1 - \sum_1^n x_i \otimes x_i^*)$ $\Leftrightarrow$ $(Id - M) \cdot \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{bmatrix}$ has a solution

$\Leftrightarrow$ $\forall \beta \in \mathbb{R}^n$ s.t. $(\beta_1, \cdots, \beta_n) \cdot [Id - M] = 0$.

Then $(\beta_1, \cdots, \beta_n) \cdot \begin{bmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{bmatrix} = 0$

$\Leftrightarrow$ $(\sum_1^n \beta_i x_i^*, y) = 0$ $\Leftrightarrow$ $y \in {}^\circ\{\sum_1^n \beta_i x_i^* \mid (\beta_1 \cdots \beta_n) \cdot (Id - M) = 0\}$

Then $\mathrm{Im}(1 - \sum_1^n x_i \otimes x_i^*) = {}^\circ\{\sum_1^n \beta_i x_i^* \mid (\beta_1 \cdots \beta_n) \cdot (Id - M) = 0\}$ is closed.

For Thm ②.

claim: $\ker(1 - \sum_1^n x_i \otimes x_i^*) = \{\sum_1^n \alpha_j x_j \mid (Id - M)\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0\}$.

即 $\ker(Id - M) \hookrightarrow\!\!\!\!\rightarrow \ker(1 - \sum_1^n x_i \otimes x_i^*)$

$(\alpha_1, \cdots \alpha_n) \longrightarrow \sum_{j=1}^n \alpha_j x_j$.

$\dim \ker(I - \sum_1^n x_i \otimes x_i^*) = \dim \ker(Id - M)$

Pf: if $z \in \ker(I - \sum_1^n x_i \otimes x_i^*)$

$\Downarrow$

$z = \sum_1^n x_i \cdot x_i^*(z)$

$\Downarrow$

$\begin{bmatrix} x_1^*(z) \\ \vdots \\ x_n^*(z) \end{bmatrix} = \underbrace{[x_k^*(x_l)]_{k,l}}_{M} \cdot \begin{bmatrix} x_1^*(z) \\ \vdots \\ x_n^*(z) \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x_1^*(z) \\ \vdots \\ x_n^*(z) \end{bmatrix} \in \ker(Id - M)$

Conversely, if $z = (x_1, \cdots x_n) \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

and $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = M \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$

Then. $\begin{bmatrix} x_1^*(z) \\ \vdots \\ x_n^*(z) \end{bmatrix} = M \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \Rightarrow x_i^*(z) = \alpha_i$.

so claim is done.

$\dim \ker(1 - \sum_1^n x_i \otimes x_i^*) = \dim \ker(Id - M) \overset{\text{trivial}}{=} \dim \ker(Id - M^T) \overset{✫}{=} \dim \ker(1 - \sum_1^n x_i^* \otimes x_i)$.

explain ✫ : if $y^* \in \ker(1 - \sum_1^n x_i^* \otimes x_i)$, $y^* = \sum_1^n x_i^* \cdot y^*(x_i)$.

$\Downarrow$.

$\begin{bmatrix} y^*(x_1) \\ \vdots \\ y^*(x_n) \end{bmatrix} = \underbrace{[x_l^*(x_k)]_{k,l}}_{M^T} \cdot \begin{bmatrix} y^*(x_1) \\ \vdots \\ y^*(x_n) \end{bmatrix}$.

Through the same discussion

we get ✫.

Theorem （总结）

if X is a normed space, $K \in \underset{\sim}{F(X)}$ 有限秩算子构成的集合.

Then $\mathrm{Im}(I-K)$ closed 且 $\dim\ker(I-K) = \dim\ker(I-K^*) < +\infty$.

4.10 假设 $K \in \mathcal{B}(X)$ 满足 $K_n \in F(X)$, $\|K_n - K\| \to 0$. X Banach.

Then $\mathrm{Im}(I-K)$ closed 且 $\dim\ker(I-K) = \dim\ker(I-K^*) < +\infty$.

Pf: $I-K = I - K_n + K_n - K = \underbrace{I + (K_n - K)}_{\text{Invertible}} - K_n = \underbrace{[I+(K_n-K)]}_{\text{可逆算子}} \cdot [I - \underbrace{(I+K_n-K)^{-1} \cdot K_n}_{\text{finite rank}}]$

$\Rightarrow \mathrm{Im}(I-K)$ closed.

$\dim\ker(I-K) = \dim\ker\left(I - (I+K_n-K)^{-1} K_n\right) < +\infty$

$\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\dim\ker(I-K^*) = \dim\ker\left(I - (I+K_n-K)^{-1} K_n\right)^*$

$\boxed{Lecture\ 13 \quad 2022/10/19}$

Recall: $K \in F(X)$ $\quad X \xrightarrow{K} X$ finite rank.

Then $\begin{cases} \mathrm{Im}(I-K) \text{ closed range} \\ \dim\ker(I-K) = \dim\ker(I-K^*) < +\infty \end{cases}$ $(\bigstar)$

Theorem. if $K_n \in F(X)$ & $K_n \xrightarrow{n\to+\infty} K$ （即 $\|K_n - K\|_{\mathcal{B}(X)} \xrightarrow{n\to+\infty} 0$）

Then $\bigstar$ holds. (pf cf. Lecture 12).

补充: $I - K^* = \{(I+K_n-K)\cdot[I - (I+K_n-K)^{-1}\cdot K_n]\}^* = [I-(I+K_n-K)^{-1}\cdot K_n]^* \cdot (I+K_n-K)^*$.

lemma: if $A \in \mathcal{B}^*(X)$ （可逆算子） $\Rightarrow (A^{-1})^* = (A^*)^{-1}$

Pf. $B = A^{-1}$ 即 $AB = BA = I \Rightarrow B^*A^* = A^*B^* = I^*$. $\Rightarrow (A^*)^{-1} = B^*$

Lemma: G 可逆. $\quad \ker(AG) = G^{-1}\ker(A)$.

$\Rightarrow \dim\ker(I-K)^* = \dim\ker[I - (I+K_n-K)^{-1} K_n]^*$.

Compact operators :

X, Y normed spaces

Define: $X \xrightarrow{K} Y$. $K \in B(X,Y)$ is called compact

if. the image of unitball of X $B_X := \{x \mid \|x\| \le 1\}$

$K(B_X)$ is relatively compact in Y 相对紧 (闭包是紧的).

度量空间中.

紧 $\Leftrightarrow$ 列紧 $\Leftrightarrow$ totally bounded & closed. (完全有界).

relatively compact $\Leftrightarrow$ totally bounded ($\forall \varepsilon>0$、存在有限 ε-net).

$K(X,Y) := \{A \in B(X,Y) \mid A \text{ compact}\}$

Proposition $K \in K(X,Y)$

$\Updownarrow$

$\forall$ bounded subset $M \subset X$.

$K(M)$ relatively compact.

pf: "$\Rightarrow$": 若 $K(M)$ relatively compact.

$M \overset{bdd}{\subset} X \Rightarrow \exists \lambda>0$, $M \subset \lambda \cdot B_X$

$\Rightarrow K(M) \subset K(\lambda B_X) = \lambda K(B_X)$ 相对紧.

Theorem : X normed space , Y Banach space

Then $\underset{\text{closed subspace}}{K(X,Y) \subset B(X,Y)}$

Theorem : X normed space. Y normed space

$K \in K(X,Y)$

$X \xrightarrow[\text{compact}]{K} Y \xrightarrow[\text{bdd}]{A} Z$, Then $AK \in K(X,Z)$.

$W \xrightarrow[\text{bdd}]{B} X \xrightarrow[\text{compact}]{K} Y$, Then $KB \in K(W,Y)$

bdd $\Leftrightarrow$ continuous 会将相对紧映为相对紧. $\underset{相对紧}{A(K(B_X))} \subset A(\underbrace{\overline{K(B_X)}}_{\text{compact}})$ 紧的.

$B(B_W)$ 是有界集 $\rightarrow$ $KB\cdot(B_W)$ 是相对紧.

Hausdorff Distance. Let S be a metric compact space.

$F_1, F_2 \subset S$ two closed subset. (Hence. F_1, F_2 are compact).

Def: $d_H(F_1,F_2) = \inf\{r>0 \mid F_1 \subset B(F_2,r)\ \&\ F_2 \subset B(F_1,r)\}$

$d_H(F_1,F_2) \le d_H(F_1,F_3) + d_H(F_2,F_3)$.

记 $r_1 = d_H(F_1,F_3)+\varepsilon$. $r_2 = d_H(F_2,F_3)+\varepsilon$ $\Rightarrow$ $B(F_1,r_1) \supset F_3$. $B(F_3,r_2) \supset F_2$. $\Rightarrow B(F_1, r_1+r_2) \supset F_2$.

反之，亦有 $B(F_2, r_1+r_2) \supset B(B(F_2, r_2), r_1) \supset B(F_3, r_1) \supset F_1$. $\Rightarrow d_H(F_1, F_2) \le r_1 + r_2$.

令 $\varepsilon \to 0^+$，即有 $d_H(F_1, F_2) \le d_H(F_1, F_3) + d_H(F_2, F_3)$.

若 $d(F_1, F_2) = 0$，则 $F_1 \subset B(F_2, \varepsilon)$，$\forall \varepsilon > 0$. $\Rightarrow F_1 \subset F_2$. vice versa.

$$\bigcap_{\varepsilon > 0} B(F_2, \varepsilon) = \overline{F_2} = F_2$$

so $F_1 = F_2$.

so it is a well-defined distance.

If X is a complete bdd metric space $\Downarrow$ $(\text{closed}(X), d_H)$ is complete. $\|\|$ $\{F \subset X \mid F \text{ closed}\}$.

$X \xrightarrow{T_1} Y$. $d_H(\overline{T_1(B_X)}, \overline{T_2(B_X)}) \le \|T_1 - T_2\|$

$X \xrightarrow{T_2} Y$. $\Leftrightarrow \forall \varepsilon > 0$

$$\begin{cases} \overline{T_1(B_X)} \subset B(\overline{T_2(B_X)}, \|T_1 - T_2\| + \varepsilon). \\ \overline{T_2(B_X)} \subset B(\overline{T_1(B_X)}, \|T_1 - T_2\| + \varepsilon). \end{cases}$$

即 $\forall z \in \overline{T_1(B_X)}$

$$\left.\begin{array}{l} d(z, T_1(B_X)) < \varepsilon/2 \\ \exists x \in B_X \\ \|T_1 x - T_2 x\| \le \|T_1 - T_2\| x \end{array}\right\} \Rightarrow d(z, T_2 x) \le \|T_1 - T_2\| + \varepsilon/2.$$

$\Rightarrow z \in B(\overline{T_2(B_X)}, \|T_1 - T_2\| + \varepsilon)$.

Thm: X normed space, Y Banach space

Then $K(X,Y) \subset B(X,Y)$ closed subspace

Pf: $K_n \in K(X,Y)$, $\|K_n - K\| \xrightarrow{n \to \infty} 0 \Rightarrow \sup_{1 \le n \le \infty} \|K_n\| = M < +\infty$

$\Rightarrow d_H(\overline{K_n(B_X)}, \overline{K(B_X)}) \le \|K_n - K\| \to 0$ $\quad \overline{K_n(B_X)} \subset \underline{\underline{M \cdot B_Y}}$ 有界集 in Y.

Lemma: If F_n are compact subsets in S (metric space).

$d_H(F_n, \underset{\text{closed}}{\underline{F}}) \longrightarrow 0$ Then F is compact.

Pf. 只需证: F is bnd, $\forall \varepsilon > 0$. n large enough. $d(F_n, F) < \varepsilon/2$.

由于 F_n compact. $\forall x \in F$, $\exists x_n \in F_n$

$N_n \subset F_n$

Apply lemma, $\Rightarrow \overline{K(B_X)}$ is compact, $K \in K(X,Y)$.

书本中的 proof： $K(B_X)$，$\{x_n\} \subset B_X$.

K_1 compact. $\exists$ $\{x_n\}$ 的子列 $\{x_n^{(1)}\}$，$K_1 x_n^{(1)}$ converges

K_2 compact. $\exists$ $\{x_n^{(1)}\}$ 的子列 $\{x_n^{(2)}\}$，$K_2 x_n^{(2)}$ converges.

$\vdots$

K_m $\exists$ $\{x_n^{(m-1)}\}$ 的子列 $\{x_n^{(m)}\}$，...

$\vdots$

取 $y_n = x_n^{(n)}$，则 K_m 在 $\{y_n\}$ 中收敛.

$\forall \varepsilon > 0$. 取 N s.t. $\|K_N - K\| < \varepsilon/4$ 设 $\|K_N y_n - K_N y_m\| < \varepsilon/2$

则 $\|K(y_n - y_m)\| < \|K_n - K\| \cdot \|y_n - y_m\| + \|K_N(y_n - y_m)\| < \varepsilon$.

for n, m large enough. $\Rightarrow$ Ky_n is Cauchy Seq.

Remark.

$X \xrightarrow{A} \overset{\text{bdd}}{Y}$ normed space $\Rightarrow$ $\bar{X} \xrightarrow{\bar{A}} \bar{Y}$ 可以唯一延拓到 线性算子.

$\Updownarrow$

$\bar{X} \xrightarrow{\bar{A}} \bar{Y}$ compact.

$\forall \bar{x} \in \bar{X}$ 都存在 $x_n \in X$, $x_n \to \bar{x}$, $Ax_n \to \bar{y}$ well-defined

$\boxed{\textit{Lecture } 14 \quad 2022/10/21}$

$K(X,Y)$ is a linear space

if $A, B \in K(X,Y)$

$(A+B)(B_X) \subset A(B_X) + B(B_X) \subset \overset{\text{相对紧}}{Y}$

$Y \times Y \longrightarrow Y$ continuous

$(y_1, y_2) \longmapsto y_1 + y_2$

$\underbrace{\overline{A(B_X)} \times \overline{B(B_X)}}_{\text{compact}} \Rightarrow \underset{\text{compact.}}{\overline{A(B_X)} + \overline{B(B_X)}}$

$X \xrightarrow{A} Y$，$A \in B(X,Y)$

可唯一延拓为 $\bar{X} \xrightarrow{\bar{A}} \bar{Y}$

$$\begin{tikzcd} \bar{X} \arrow[rr, bend left, "\exists \bar{A} \text{ 延拓}"] & & \\ \cup & & \\ X \arrow[r, "A"] \arrow[rr, bend right, "\text{uniformly convergent}"'] & Y \arrow[r, hook] & \bar{Y} \text{ complete} \end{tikzcd}$$

if $x \in \bar{X}$，$\exists x_n \longrightarrow x \Rightarrow Ax_n \xrightarrow{\text{in } \bar{Y}} y$

则定义 $\bar{A}x = y$ ① well-defined $\left.\begin{matrix} x_n \to x \\ x_n' \to x \end{matrix}\right\} \lim Ax_n = \lim Ax_n'$.

② 线性性

③ $\|\bar{A}\| = \|A\|$

Prop: A is compact $\Longleftrightarrow$ $\bar{A}$ is compact

pf: "$\Leftarrow$":

$$\begin{tikzcd} \bar{X} \arrow[r, "\bar{A}"] & \bar{Y} \\ X \arrow[u, hook] \arrow[r, "A"] & Y \arrow[u, hook] \end{tikzcd} \qquad A = \bar{A}|_X$$

"$\Rightarrow$": $A(B_X)$ relatively compact

$$\overline{B_X}^{\bar{X}} = B_{\bar{X}} = \{\bar{x} \in \bar{X} \mid \|\bar{x}\| \le 1\}$$

$$\bar{A}(B_{\bar{X}}) = \bar{A}(\overline{B_X}) \subset \overline{\bar{A}(B_X)} = \overline{A(B_X)}$$

Theorem: K is compact $\Rightarrow$ K^* is compact. 第八章.

pf: $K(B_X)$ totally bounded, 欲证: $K^*(B_{Y^*})$ totally bounded.

$$(Kx, y^*) = (x, K^*y^*)$$
($x \in B_X$, $y^* \in B_{Y^*}$)

$\forall \varepsilon > 0$, $\exists x_1, \cdots x_n \in B_X$ s.t. $\{Kx_1, \cdots, Kx_n\}$ is a ε-net.

考虑 $Y_n \overset{def}{=} \mathrm{span}\{Kx_1, \cdots Kx_n\}$ finite dimension.

$$\begin{tikzcd} Y \arrow[dr, "y^*"] & \\ Y_n \arrow[u, phantom, "\cup"] \arrow[r, "y^*|_{Y_n}"'] & \mathbb{R} \end{tikzcd}$$

$\{y^*|_{Y_n} : y^* \in B_{Y^*}\} = B_{Y_n^*}$ (有限维空间中的单位球) $\hookrightarrow$ totally bdd

$\exists y_1^*, \cdots, y_m^* \in B_{Y_n^*}$ s.t. $\{y_i^*|_{Y_n}\}_{i=1}^m$ is a ε-net for $B_{Y_n^*}$

Claim: $\{K^*y_1^*, \cdots, K^*y_m^*\}$ is a 3ε-net for $K^*(B_{Y^*})$

$\forall y^* \in B_{Y^*}$, $\exists j: 1 \le j \le m$ $\quad \|y^*|_{Y_n} - y_j|_{Y_n}\| < \varepsilon$

$\forall x \in B_X$, $\exists i: 1 \le i \le n$ $\quad \|Kx - Kx_i\| < \varepsilon$

$$|(x, K^*y^*) - (x, K^*y_j^*)| = |(Kx, y^*) - (Kx, y_j^*)| \le |(K(x - x_i), y^* - y_j^*)| + |(Kx_i, y^* - y_j^*)|$$

$$\le |y^*(Kx - Kx_i)| + |y_j^*(Kx - Kx_i)| + |(Kx_i, y^*|_{Y_n} - y_j^*|_{Y_n})|$$

$$\le 2\|Kx - Kx_i\| + \|Kx_i\| \cdot \|y^*|_{Y_n} - y_j^*|_{Y_n}\| < (2 + \|K\|) \cdot \varepsilon$$

由于 $x \in B_X$ 任意 $\Rightarrow$ $\|K^*(y^* - y_j^*)\| < (2 + \|K\|) \cdot \varepsilon$ $\Rightarrow$ $\{K^*y_j^*\}$ is a $(2 + \|K\|)\varepsilon$ net.

bidual:

$$\begin{tikzcd} X^{**} \arrow[r, "T^{**}"] & Y^{**} \\ X^* & Y^* \arrow[l, "T^*"'] \\ X \arrow[uu, bend left, hook] \arrow[r, "T"] & Y \arrow[uu, bend right, hook, "\text{isometric embedding.}"'] \end{tikzcd}$$

$$\begin{tikzcd} X^{**} \arrow[r, "T^{**}"] & Y^{**} \\ X \arrow[u, hook] \arrow[r, "T"] & Y \arrow[u, hook] \end{tikzcd} \qquad T = T^{**}|_X$$

Definition: $x_1 \cdots x_m \in X$.

$x_1^* \cdots x_m^* \in X^*$

we say $(\{x_i\}_{i=1}^m, \{x_j^*\}_{j=1}^m)$ is a dual system (对偶系)

if $(x_i, x_j^*) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \ne j. \end{cases}$

Theorem: ① $\forall\ x_1,\cdots,x_m \in X$ linearly independent

$\Rightarrow \exists\ x_1^*,\cdots,x_m^* \in X^*$ s.t. $(\{x_i\}_{i=1}^m, \{x_j^*\}_{j=1}^m)$ is a dual system

② $\forall\ y_1^*,\cdots,y_n^* \in X^*$ linearly inde

$\Rightarrow \exists\ y_1,\cdots,y_n \in X$ ——— $(\{y_i\}_{i=1}^n, \{y_i^*\}_{i=1}^n)$ is a dual system

pf: ① $x_1,\cdots x_m$, linearly ind $\Longleftrightarrow$

$$\begin{tikzcd} X^* \arrow[r, "L"] \arrow[d] & \mathbb{R}^m \\ X^*/\ker L \arrow[ur, "\hat{L}"'] & \end{tikzcd} \qquad x^* \longmapsto (x^*(x_1),\cdots x^*(x_m))^T$$

$\hat{L}$ 是 bijection.

取 $\mathbb{R}^m$ 标准正交基 $e_1,\cdots,e_m$

任取 $z_1^*\cdots z_m^*$ s.t. $[z_1^*]\cdots[z_m^*]$ are linearly ind in $X^*/\ker$

$$\hat{L}([z_1^*],\cdots[z_m^*]) = (e_1,\cdots e_m)\cdot\underbrace{[z_j^*(x_i)]_{1\le i,j\le n}}\ \text{(Invertible matrix).}$$

$\exists\ M$, $m\times m$ matrix

$$[z_j^*(x_i)]\cdot M = Id. \quad\Rightarrow\quad \begin{pmatrix} x_1\\ \vdots \\ x_m\end{pmatrix}\cdot \underset{\displaystyle\downarrow \atop \displaystyle (x_1^*,\cdots x_m^*)}{(z_1^*\cdots z_m^*)\cdot M} = Id$$

$(x_1^*,\cdots x_m^*)$ 即为所需

② 同理.

Theorem: (Fredholm alternative)

X Banach, $K\in K(X)$

Then ① $Im(I-K)$ is closed

② $\dim\ker(I-K) = \dim\ker(I-K^*) < +\infty$

Proof: ①
$$\begin{tikzcd} X \arrow[r, "I-K"] \arrow[d] & Im(I-K) \\ X/\ker \arrow[ur, "\widehat{I-K}"'] & \end{tikzcd}$$

$Im(I-K)$ is closed $\Longleftrightarrow \exists\ C>0,\ d(x,\ker(I-K)) \le C\|(I-K)x\|\quad \forall x\in X.$

否则, 存在 $\|(I-K)x_n\| \longrightarrow 0$ s.t. $d(x_n,\ker(I-K)) = 1.$

$$\exists\ z_n\in\ker(I-K) \begin{cases} 1\le \|x_n-z_n\| \le 1+\delta. \\ \|(I-K)\underbrace{(x_n-z_n)}_{\text{记为 } y_n}\| \xrightarrow{n\to+\infty} 0. \end{cases}$$

则 $\{y_n\}$ 有界 $\Rightarrow \{Ky_n\}$ 有收敛子列.

$\|y_{n_j} - Ky_{n_j}\| \xrightarrow{j\to+\infty} 0.$

设 $Ky_{n_j} \xrightarrow{j\to\infty} w.$

而 $d(y_{n_j},\ker(I-K)) = 1$

则 $y_{n_j} \xrightarrow{j\to\infty} w.$

ker 是闭的

$\Downarrow$

$y_{n_j}\to w \Rightarrow \boxed{d(w,\ker(I-K)) = 1} \rightarrow$ 矛盾 $\leftarrow$ $w = Kw \Rightarrow \boxed{w\in\ker(I-K).}$

Proof: ② $\dim \ker(I-K) < +\infty$

$\ker(I-K) \subset X$

只要证: $\forall \ \|x_n\| = 1$, $x_n \in \ker(I-K) \iff x_n = Kx_n$.

都存在 x_{n_k} 收敛子列. 而 $K \in K(X) \Rightarrow \{Kx_n\}$ 有收敛子列.

下证: $\dim \ker(I-K) = \dim \ker(I-K^*)$

Case 1: claim: $\ker(I-K^*) = \{0\} \xRightarrow{\text{then}} \ker(I-K) = \{0\}$.

since $Im(I-K)$ is closed. $Im(I-K) = {}^{\circ}\ker(I-K^*) = X \ \boxed{\Rightarrow} \ \ker(I-K) = \{0\}$.

假设 $\ker(I-K) \neq \{0\}$.

$x_n \xrightarrow{I-K} x_{n-1} \to \cdots \to x_2 \xrightarrow{I-K} \underset{\neq 0}{x_1} \xrightarrow{I-K} 0 \Rightarrow 0 \subsetneqq \ker(I-K) \subsetneqq \ker(I-K)^2 \cdots \subsetneqq \ker(I-K)^n \cdots$

取 $z_n \in \ker(I-K)^n$, $d(z_n, \ker(I-K)^{n-1}) \geq \frac{1}{2}$. 下证 $\{Kz_n\}$ 无收敛子列.

$m<n$, $\|Kz_n - Kz_m\| = \|z_n - \underbrace{(z_m - (I-K)z_m + (I-K)z_n)}_{\in \ker(I-K)^{n-1}}\| \geq \frac{1}{2}$. 故不收敛.

4 5 6 7 11 12 14 18 20

上述证明中有 $\ker(I-K) = \{0\} \iff Im(I-K) = X$. $I-K$ 单射 $\iff$ 满射.

Conversely, similarly argument claims that

$\ker(I-K) = \{0\} \Rightarrow \ker(I-K^*) = \{0\}$

Case 2. $K \in K(X)$. $Im(I-K)$ closed $\dim \ker(I-K) = n \geq 1$. $\dim \ker(I-K^*) = m \geq 1$.

$\ker(I-K) = span\{x_1, \cdots, x_n\}$ 线性独立 dual system: $x_1^*, \cdots, x_n^* \in X^*$ s.t.

$\ker(I-K^*) = span\{y_1^*, \cdots, y_m^*\}$ 线性独立 $x_i^*(x_j) = \delta_{ij}$.

dual system

In particular, $\begin{cases} x_n^*|_{span\{x_1 \cdots x_{n-1}\}} = 0 \\ x_n^*(x_n) = 1 \neq 0 \end{cases}$

$y_1, \cdots, y_m$ $\begin{cases} y_m|_{span\{y_1^*, \cdots, y_{m-1}^*\}} = 0 \\ (y_m, y_m^*) = 1 \neq 0 \end{cases}$

$I - (K + y_m \otimes x_n^*) \xrightarrow{\text{dual}} I - (K^* + x_n^* \otimes y_m)$

Claim: $\ker(I-(K+x_n \otimes y_m^*)) = span\{x_1, \cdots, x_{n-1}\}$ (☆)

$\ker(I-(K^* + y_m^* \otimes x_n)) = span\{y_1^*, \cdots, y_{m-1}^*)$ (☆☆).

Indeed. $1 \leq i \leq n-1$ $(I-(K+y_m \otimes x_n^*))x_i = (I-K)x_i - y_m \cdot (x_n^*, x_i) = 0$

反之 $[I-(K+y_n\otimes x_n^*)](x)=0$. Then $(I-K)x = x_n^*(x)\,y_m$.

Claim $x_n^*(x)=0$. 否则 $y_m=(I-K)\cdot\frac{1}{x_n^*(x)}x \Rightarrow y_m\in Im(I-K)$

$$Im(I-K) = {}^{\circ}\ker(I-K^*).$$

$y_m|_{\ker(I-K^*)}=0$. 这个不成立.

故 $(I-K)x=0 \Rightarrow x\in\ker(I-K)$.

即 $\underline{\mathrm{span}\{x_1,\cdots,x_{n-1}\}}\subset \ker(I-(K+y_m\otimes x_n^*))\subset \underline{\ker(I-K)}$.

只相差一维

而 $(I-(K+y_m\otimes x_n^*))x_n=-y_m\neq 0$.

故 $\ker(I-(K+y_m\otimes x_n^*))=\mathrm{span}\{x_1,\cdots,x_{n-1}\}$.

$\dim\ker(I-K)=n\geq 1 \longrightarrow \dim\ker(I-(K+T_1))=n-1$

$\dim\ker(I-K^*)=m\geq 1 \longrightarrow \dim\ker(I-(K+T_1^*))=m-1$

如果某一步中到0，两者应同时为0

$\Downarrow$

$n=m$

$\square$

Lecture 15 2022/10/26

与第4章课后习题6相关的问题.

Prop. $A,B\in B(X,Y)$, X,Y both Banach spaces.

$\boxed{Im A\subset Im B}\Leftrightarrow$

Example (1) if $A=BC$, $A,B,C\in B(X,Y)$

then $Im A\subset Im B$.

(2)
$$\begin{tikzcd} X \arrow[r, "B"] \arrow[dr, "q_B"'] & Y \\ & X/\ker B \arrow[u, "\hat{B}"'] \end{tikzcd} \qquad Im B = Im \hat{B}$$

If
$$\begin{tikzcd} X \arrow[r, "A"] \arrow[dr, "T"'] & Y \\ & X/\ker B \arrow[u, "\hat{B}"'] \end{tikzcd}$$
if $A=\hat{B}T$

then $Im A\subset Im\hat{B}=Im B$.

Proof: $A,B\in B(X,Y)\Leftrightarrow$

A,B has closed graphs

$$\begin{tikzcd} X \arrow[r, "B"] \arrow[d, "q"'] & Im B \arrow[r, hook] & Y \\ X/\ker B \arrow[ur, "\hat{B}"'] & & \end{tikzcd} \qquad X \xrightarrow{A} Im A \hookrightarrow Y$$

希望改变 $Im B$ 上的范数，使其成为 Banach space.

linear bijection & bnd linear operador

$$\|Bx\|_Y \leq C\|[x]\| = C\,d(x,\ker B) \quad (*).$$

定义 $|||Bx||| := \|[x]\| = d(x,\ker B)$.

这样

由(*)知 $(\mathrm{Im}B, |||\cdot|||) \xrightarrow{\text{bounded}} (\mathrm{Im}B, \|\cdot\|_Y)$.

$Bx \longmapsto Bx$

$X \xrightarrow[\text{bdd}]{A} \mathrm{Im}B \subset Y$. Hence $\mathrm{Graph}(A) = \{(x, Ax) \mid x \in X\} \underset{\text{closed}}{\subset} X \times \mathrm{Im}B \subset X \times Y$

用到了 ImB 上原始的拓扑.

$X \times (\mathrm{Im}B, |||\cdot|||) \xrightarrow{\text{continous}} X \times (\mathrm{Im}B, \|\cdot\|_Y)$

$(x, y) \longmapsto (x, y)$

$\mathrm{Graph}(A) \xleftarrow{\text{原像}} \boxed{\mathrm{Graph}(A)}$ closed

is closed in the new topology （闭集的连续函数原像是闭的）.

但在新的范数下. $(\mathrm{Im}B, |||\cdot|||)$ is Banach. 由闭图像定理，

$$X \xrightarrow{A} (\mathrm{Im}B, |||\cdot|||) \text{ is bounded.}$$

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[dr, "T"'] & (\mathrm{Im}B, |||\cdot|||) \arrow[d, shift left, "u" , "\cong"'] \\ & X/\ker B \arrow[u, bend right, "u^{-1}=\hat{B}"'] \end{tikzcd}$$

T 就是 ② is bounded.

$T := uA$. then $A = u^{-1}T = \hat{B}T$.

关键是分解时. B有界，T有界.

第8章 Reflexive Banach space 自反 Banach 空间.

$X \quad X^*$

$(x, \ x^*)$ pairing $\quad (x, x^*) := x^*(x)$

$\forall x \in X$ defines a bounded linear functional on X^*

$X^* \xrightarrow{J_X x} \mathbb{R}$

$x^* \longmapsto x^*(x) = (x^*, x)$

且 $\|J_X x\|_{X^*} = \sup\limits_{\|x^*\| \le 1} |J_X(x)(x^*)| = \sup\limits_{\|x^*\| \le 1} |(x^*, x)| = \|x\|$

即 $J_X : X \longrightarrow X^{**}$ 等距线性映射.

$x \longmapsto J_X x$

也即 $X \hookrightarrow X^{**}$ 是一个等距嵌入 (isometric embedding). ←永远是 Banach

Definition: X is called reflexive if J_X is surjective.

（即 J_X is an isometric isomorphism) $\quad X \xrightarrow{J_X} X^{**}$

（不仅仅说明 $X \cong X^{**}$，而且是 J_X 诱导的等距同构）.

这个时候可以直接写 $X^{**}=X$.

Example. l_p $1<p<\infty$. $L_p(X,\mu)^* = L_q(X,\mu)$ $\frac{1}{p}+\frac{1}{q}=1$. so $L_p(X,\mu)$ is reflexive.

$(f,g)=\int fg\,d\mu$. $L_p(X,\mu)^{**}=L_p(X,\mu)$.

Theorem 8.1 X 自反 $\Leftrightarrow$ X^* 自反.

Theorem 8.2 自反空间的闭子空间自反.

proof: Assume X is reflexive $Z\subset X$ is a closed subspace

$\forall\, z^{**}\in Z^{**}$ 是否可以找到 $z\in Z$ s.t. $(z^{**}, z^*) = (z^*, z)$ $\forall\, z^*\in Z^*$

$Z\subset X \Rightarrow Z^* \cong X^*/Z^\circ$.

$$\begin{tikzcd} X^* \arrow[r, "\text{surjective}"] & Z^* \arrow[r, "z^{**}", "\text{bnd linear}"'] & \mathbb{R} \\ x^* \arrow[r, mapsto] \arrow[urr, bend right, "\text{bdd linear functional on } X^*"'] & x^*|_Z & \end{tikzcd}$$

即存在 $x_0\in X$, s.t. $\forall\, x^*\in X^*$. $(x^*, x_0) = (x^*|_Z, z^{**})$. 只需证 $x_0\in Z$.

$x_0\in Z$ 则 $(z^*, z^{**}) = (x^*|_Z, z^{**}) = (x^*, x_0) = (x^*|_Z, x_0) = (z^*, x_0)$, $z^{**} = J_Z x_0$

若有 $x_0\notin Z\subset X$.

则存在 x_0^* 满足 $\begin{cases} x_0^*(Z)=0 \\ x_0^*(x_0)\neq 0 \end{cases}$ 则 $0\neq (x_0^*, x_0) = (x_0^*|_Z, z^{**}) = (0, z^{**}) = 0$, 矛盾. □

Proof of Thm 8.1. ① X 自反 $\Rightarrow$ X^* 自反.

X 自反, 即 $X \xrightarrow[\cong]{J_X} X^{**}$, $X^* \xrightarrow{J_{X^*}} X^{***}$ 是否为满射.

$\forall\, x^{***}\in X^{***}$.

$$\begin{tikzcd} X^{**} \arrow[r, "x_0^{***}"] & \mathbb{R} \\ X \arrow[u, "J_X"', "\cong"] \arrow[ur, "\text{bdd linear functional.}"'] & \end{tikzcd}$$

是 X^* 中元素. 记为 x_0^*

是否 $\forall\, x^{**}\in X^{**}$ $(x^{**}, x_0^{***}) \overset{?}{=} (x^{**}, x_0^*)$.

$x_0^* = x_0^{***}\circ J_X$. J_X 是满的,

$\forall\, x^{**}\in X^{**}$, 存在 $x\in X$. $J_X(x) = x^{**}$

$(x^{**}, x_0^*) = (x_0^*, x) = x_0^{***}(J_X(x)) = (x_0^{***}, x^{**})$. □.

② X^* 自反 $\Rightarrow$ X 自反.

Conversely. If X^* reflexive

$X \underset{\text{closed subspace}}{\overset{J_X}{\hookrightarrow}} X^{**}$

即 X 是某个自反空间的闭子空间. 故 X 是 reflexive.