Lecture 16 2022/10/28

Reflexive spaces

Def: $X \xrightarrow{J_X} X^{**}$ is surjective 自反的定义.

即 $X^{**}=X$ (naturally)

Recall ① X Ref $\Leftrightarrow$ X^* Ref

$\Downarrow$

$x^{**}=x \Rightarrow (X^{**})^* = X^*$

② X Ref $\Rightarrow$ 任意闭子空间 Ref.

Consequently, X^* Ref $\Rightarrow X^{**}$ Ref. 又 $X \hookrightarrow X^{**}$

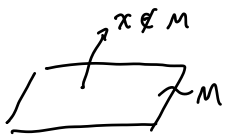
$\Rightarrow X$ Ref by ②

Saturated:

$$\begin{array}{ccc} X & & X^* \\ \cup\!\! \shortmid & & \cup\!\!\shortmid \\ M \text{ closed subspace} & \leftarrow & W \end{array}$$

Then x and M, 可以被 X^* 中某个元素分离.

$\exists\, x^* \in X^*$, $\|x^*\|=1$, $x^*(x)=\|x\|\neq 0$. & $x^*(M)=0$



可以找到 $x^{**}\in X^{**}$, 分离 x^* 和 W.

分离的意思: $x^{**}(x^*)$ 不相交 $x^{**}(W)$. Then automatically $x^*(W)=0$

否则 $(x^{**}, W)\neq \{0\} \Rightarrow (x^{**}, W)=\mathbb{R}$.

Definition:

We say $W \subset X^*$ (closed subsp) is saturated (饱和), 若 $\forall\, x^*\notin W$, x^* 与 W 可以被某个 $x\in X$ 分离.

即 $x^*(x)\neq 0$, $(W,x)=0$

$\Updownarrow$

$\exists\, x\in {}^\circ W$ s.t. $(x, x^*)\neq 0$

$x^* \notin ({}^\circ W)^\circ$

$\Downarrow$

$W=({}^\circ W)^\circ$

$X \xrightarrow{\text{dual}} X^* \xrightarrow{\text{dual}} X^{**}$, X is called a predual of X^*

双对偶空间 bidual

Thm. ① $S\subset X$

$S^\circ \subset X^*$

Then S° is saturated

② $\forall$ 饱和空间均是某个集合的零化子.

③ $W\subset X^*$ 饱和 $\Leftrightarrow$ $W=({}^\circ W)^\circ$ "$\Rightarrow$" pf.

pf of ①: $X^* \supset S^\circ$. 若 $x^* \notin S^\circ$

存在 $s \in S$. $x^*(s) \neq 0$ The $(x^*, s) \neq 0$ but $(S^\circ, s) = 0$ $\Rightarrow$ S° saturated □

Weak topology (拓扑空间中)

若 T_1, T_2 two sets. 若已知 T_2 是 topo. space. $T_1 \longrightarrow T_2^I$.

又有一族映射 $\mathcal{F} = (\varphi_i)_{i\in I}$ $\varphi_i : T_1 \longrightarrow T_2$. $x \longmapsto (\varphi_i(x))_{i\in I}$.

Then T_1 上存在最粗糙的 topology s.t. 所有 φ_i 连续. $(T_1, \sigma(T_1, \mathcal{F}))$

X X^*

net convergence.

$\downarrow$

广义序列.

Def: ① The weak topology on X is the $\sigma(X, X^*)$.

即它是使得 $\forall x^* \in X^*$, $X \xrightarrow{x^*} \mathbb{R}$ 都连续的最小拓扑.

泛函分析结果. $X \xrightarrow{\text{是单射}} \mathbb{R}^{X^*}$

$x \longrightarrow (x^*(x))_{x^* \in X^*}$

$\Leftrightarrow$ $(x_\alpha)_{\alpha\in A}$ generalised sequence (net)

w^* = weak star = weak*.

② The weak* topology on X^* $(\sigma(\underset{\text{底空间}}{X^*}, \underset{\text{测试函数}}{X}))$.

$(x_\alpha^*)_{\alpha\in A}$ $x_\alpha^* \xrightarrow{w^*} x_0^*$ $\overset{\text{definition}}{\Longleftrightarrow}$ $x_\alpha^*(x) \longrightarrow x_0^*(x)$ for any $x \in X$.

也即 $X^* \xhookrightarrow{\text{单射}} \mathbb{R}^X$

$x^* \longmapsto (x^*(x))_{x\in X}$.

$i_0 \in I$. $\prod_{i\in I} X_i \xrightarrow{\pi_{i_0}} X_{i_0}$ 是连续的

induce 乘积空间的拓扑.

Def: (定向集) Directed sets

$A = \{\alpha\}$, 有 $\leq$ pre-order.

(不要求反对称性)

1° $\alpha \leq \alpha$

2° $\alpha_1 \leq \alpha_2$ & $\alpha_2 \leq \alpha_3$ $\Rightarrow$ $\alpha_1 \leq \alpha_3$.

3° $\forall \alpha_1, \alpha_2 \in A$, $\exists \alpha_3$. $\alpha_1 \leq \alpha_3$, $\alpha_2 \leq \alpha_3$.

Def: A net is just a map from a directed set in X.

$A \longrightarrow X$

$\alpha \longmapsto x_\alpha$

我们直接记为 $(x_\alpha)_{\alpha\in A}$.

is also called a generalised sequence

Def: X topological space.

a generalised sequence $(x_\alpha)_{\alpha\in A}$.

is said to converge to $x_0 \in X$.

if $\forall$ neighbourhood U_{x_0} of x_0

$\exists \alpha_0 \in A$. $\forall \alpha \geq \alpha_0$. $x_\alpha \in U_{x_0}$.

Thm 8.5 $\underset{\text{closed subspace}}{W \subset X^*}$ is saturated $\Leftrightarrow$. W is w^*-closed. 弱星闭.

Pf: W saturated $\Rightarrow$ $W = ({}^{\circ}W)^{\circ}$

Lemma: $\forall\ S \subset X$. S° is w^*-closed.

$f_s^{-1}(0)$ is w^*-closed.

$X^* \xrightarrow{f_s} \mathbb{R}$

$x^* \longmapsto (x^*, s)$

w^*-continuous.

Pf: $S^{\circ} = \{x^* \in X^* \mid \underset{\forall s \in S}{(x^*, s) = 0}\} = \bigcap_{s \in S} \{x^* \in X^* \mid (x^*, s) = 0\}$

$\forall\ x^* \notin W$, $X^* \overset{\Phi}{\hookrightarrow} \Phi(X^*) \subseteq \mathbb{R}^X$

$x^* \longmapsto (x^*(x))_{x \in X}$.

$\underset{\text{closed}}{X^* \supset W} \rightsquigarrow \Phi(W)$.

存在 $x_1, \cdots, x_n \in X$.

$x^*(x_i) \subset U_i \underset{\text{open}}{\subseteq} \mathbb{R}$.

$\Phi(W) \cap \{(t_x)_{x \in X} \mid \underset{1 \le i \le n}{t_{x_i} \in U_i}\} = \phi$.

$X^* \xrightarrow{\Phi_{x_1, \cdots, x_n}} \mathbb{R}^n$

$x^* \longmapsto (x^*(x_1), \cdots, x^*(x_n))$

$\Phi_{x_1 \cdots x_n}(W)$ 与 $\Phi_{x_1 \cdots x_n}(x_0^*)$ 可由开集分离.

子空间

$\{z^* \in W \mid (z^*(x_1), \cdots z^*(x_n))\}$ 是 $\mathbb{R}^n$ 中的子空间

$\Rightarrow$ 存在 $z_0^*(x_1, \cdots x_n)$, 使 $x_0^* - z_0^*$ 垂直于子空间. $\forall\ z^* \in W$.

即 $(x_0^* - z_0^*) \cdot (x_1, \cdots x_n) \perp (z^*(x_1), \cdots z^*(x_n))$.

即 $\sum_{i=1}^{n} (x_0^* - z_0)^*(x_i) \cdot z^*(x_i) = 0$.

$\Vert$

$\Big(\underbrace{\sum_{i=1}^{n} (x_0^* - z_0^*)(x_i) \cdot x_i}_{\in {}^{\circ}W}, \ z^* \Big) = 0$

而 $\left(\sum_{i=1}^{n} (x_0^* - z_0^*)(x_i)\, x_i,\ x_0^* - z_0^*\right) = \sum_{i=1}^{n} |(x_0^* - z_0^*)(x_i)|^2 > 0$.

注意 $\left(\sum_{i=1}^{n} (x_0^* - z_0^*)(x_i)\, x_i, z_0^*\right) = 0$

$\Rightarrow \left(\sum_{i=1}^{n} (x_0^* - z_0^*)(x_i)\, x_i,\ x_0^*\right) = \sum_{i=1}^{n} |(x_0^* - z_0^*)(x_i)|^2 > 0$

$\Rightarrow$ x_0^* 与 W 分离. W saturated.

Thm. (Banach–Alaoglu Theorem)

X^* 中的闭单位球是 w^*-compact.

Lecture 17 2022/11/2

下周六下午 2:00~4:00 期中考试

Recall: w^*-topology $X^* \overset{\Phi}{\hookrightarrow} \mathbb{R}^X$

$x^* \overset{\text{单射}}{\longmapsto} (x^*(x))_{x \in X}$.

把 X 与 $\Phi(X)$ 视为集合是 bijection. 把 $\Phi(X)$ 上的乘积拓扑搬到 X 上.

拓扑基: $x_1, \cdots, x_n \in X$, $x_0^* \in X^*$. $\varepsilon_1, \cdots \varepsilon_n > 0$

$\{x^* \in X^* \mid |(x^* - x_0^*)(x_j)| < \varepsilon_j,\ \forall\ 1 \le j \le n\}$ 的有限交,

Tychonov Theorem: if $(X_i)_{i \in I}$ is a family of compact topological space,

Then $\prod_{i \in I} X_i$ (product topology) is compact.

Remark: $X^* \xrightarrow{\Phi} \mathbb{R}^{B_X}$

$x^* \longmapsto (x^*(x))_{x\in B_X}$.

Pf of Banach-Alaoglu Theorem:

$$\bar{B}_{X^*} \xrightarrow{\Phi} \Phi(\bar{B}_{X^*}) \subset [-1,1]^{B_X}$$

$$x^* \longmapsto (x^*(x))_{x\in B_X}.$$

$|x^*(x)| \le \|x^*\|\cdot\|x\| \le 1$.

By Tychonov Thm, $[-1,1]^{B_X}$ is compact

只需证 $\Phi(\bar{B}_{X^*})$ is closed.

$(x_\alpha^*(x))_{x\in B_X} \longrightarrow (y(x))_{x\in B_X}$

a generalised sequence (net).

希望存在 $x_0^* \in \bar{B}_{X^*}$ s.t. $y(x) = x_0^*(x)$.

(即) $(y_x)_{x\in B_X} = (x_0^*(x))_{x\in B_X} \in \Phi(\bar{B}_{X^*})$

$y(x) = \lim_\alpha x_\alpha^*(x)$

先定义 $f: X \longrightarrow \mathbb{R}$

$$f(x) := \begin{cases} \|x\|\cdot y\left(\frac{x}{\|x\|}\right), & x\neq 0 \\ 0 & \text{if } x=0. \end{cases}$$

可以验证 f 是线性的.

下证: $\|f\|\le 1$. 即 $\sup_{\|x\|=1}|f(x)|\le 1$.

$|f(x)| = |y(x)| = |\lim_\alpha x_\alpha^*(x)| \le \lim_\alpha \|x_\alpha^*\|\cdot\|x\| \le 1$.

Proposition: If X is separable, then $(X^*, \sigma(X^*,X))$ is metrisable (可度量化)

弱星拓扑

即存在 X^* 上的一个度量 $d(\cdot,\cdot)$

使得 $(X^*,\sigma(X^*,X))$ 拓扑与 d 诱导的拓扑一样.

Lemma. $\mathbb{R}^{\mathbb{N}}$ is metrisable

Pf. $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} 2^{-n}\cdot \frac{|x_n-y_n|}{|x_n-y_n|+1}$.

claim d is a matric.

① 对称性, 非负性.

② $d((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = 0 \iff (x_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}$.

③ 三角不等式.

验证比较 trivial.

③中可能用到一些常见不等式.

$(\mathbb{R}^{\mathbb{N}}, \text{product}) \xrightarrow[\text{id}]{\text{同胚}} (\mathbb{R}^{\mathbb{N}}, d)$, $(\mathbb{R}^{\mathbb{N}}, d) \xrightarrow{\text{id}} (\mathbb{R}^{\mathbb{N}}, \text{product})$

这个为何连续.

连续.

$\downarrow x_n$ 第 n 个坐标

$\mathbb{R}$

$U_\delta^N(x) = \{(y_n)_{n\in\mathbb{N}} \mid |x_k-y_k|<\varepsilon, k=1,2,\cdots,N\}$ 构成 $(x_n)_{n\in\mathbb{N}}$ 的邻域基.

$\forall \varepsilon>0$, $\exists\ \delta, N$. $\forall y\in U_\delta^N(x)$, $d(x,y)<\varepsilon$.

$(y_n)_{n\in\mathbb{N}}$. $\quad U = \{(y_n)_{n\in\mathbb{N}} \mid |x_k - y_k| < \frac{1}{k}, k\in\mathbb{N}\}$ $\quad U$ 是乘积拓扑下的开集.

U 不包含 U_S^N ?

X^* $\quad X$ separable

$\cup$

S countable dense subset of X.

如果该Lemma 已证,则

Lemma: X^*上的弱星拓扑 $\sigma(X^*, X) = \sigma(X^*, S)$ $\quad X^* \xrightarrow{\text{是同胚}} \mathbb{R}^S$ 可度量.

pf. 只要证明. 若 $\forall s\in S$, $X^* \xrightarrow{s} \mathbb{R}$ 连续.

$\forall x\in X$, $X^* \xrightarrow{x} \mathbb{R}$ 连续.

$(X^*, \sigma(X^*, X)) \xrightarrow{\text{连续}} (X^*, \sigma(X^*, S))$ $\qquad (X^*, \sigma(X^*, S)) \longrightarrow (X^*, \sigma(X^*, X)) \xrightarrow{x} \mathbb{R}$.

$(X^*, \sigma(X^*, S)) \xrightarrow{x} \mathbb{R}$. $\quad \forall \varepsilon > 0$. $y^* \in \{y^* : |(x_0^* - y^*)(s_n)| < \varepsilon\}$

$x_0^* \longrightarrow x_0^*(x)$. $\qquad \forall n = 1, 2, \cdots, N$.

$s_n \xrightarrow{n} x$. $\qquad$ 希望 $|x_0^*(x) - y^*(x)|$ 很小.

Thm. 8.6 X^*中任意 $W \subset X^*$, $\dim W < +\infty$

$\Rightarrow W$ weak*-closed.

pf: $\forall x^* \notin W$. 不妨设 $W = \text{span}\{\underbrace{x_1^*, \cdots x_n^*}_{\text{linear independent}}\}$.

$x^*, x_1^*, x_2^*, \cdots, x_n^*$ 线性独立.

存在dual system. $x, x_1, \cdots, x_n \in X$. $x^*(x) = 1$. $x_i^*(x) = 0$ $\Rightarrow$ W saturated.

Theorem 8.7. $\quad X$ is Reflaxive $\Leftrightarrow$ $\forall\, \underbrace{W \underset{\text{closed}}{\subset} X^*}_{\text{norm topology}}$ is weak*-closed.

pf: X reflaxive $\Rightarrow X^{**} = X$.

$\forall x^* \notin W$. 可以用 x^{**} 分离. $x^{**} \in X = X^{**}$ $\Rightarrow W$ 饱和. $\qquad$ 用 w^* 闭集之交理解.

Lemma: X normed space, $x^* \in X^* \setminus \{0\}$. $\exists\, x_0 \in X$ such that $x_0^*(x_0) \neq 0$.

Then $X = \ker x_0^* \oplus \mathbb{R}x_0$ $\qquad$ pf: ① $\ker x_0^* \cap \mathbb{R}x_0 = \{0\}$.

即
$$\begin{tikzcd} X \arrow[r, two heads] \arrow[dr] & \mathbb{R} \\ & X/\ker x_0^* \arrow[u, "\cong"'] \end{tikzcd}$$

② $\forall x\in X$, $\quad x = x - \lambda x_0 + \lambda x_0$. 令 $x_0^*(x - \lambda x_0) = 0$

得 $\lambda = \dfrac{x_0^*(x)}{x_0^*(x_0)}$

$\forall\, x_0^{**} \in X^{**}\setminus\{0\}$. $\quad X^* \xrightarrow{x_0^{**}} \mathbb{R}$. $\quad \forall\, x^* \in X^*$

$(x_0^*, x_0^{**}) \neq 0$. $\quad X^* = \ker x_0^{**} \oplus \mathbb{R}x_0^*$ $\quad x^* = \lambda x_0^* + \underline{\underline{n^*}}$

$\ker x_0^{**} \underset{\text{closed}}{\subset} X^*$. $\exists\, x_0 \in X$ 分离 $x_0^* \nsubseteq \ker x_0^{**}$. 不妨设 $x_0^*(x_0) = 1$.

$x_0^{**}(x^*) = \lambda(x_0^{**}(x_0^*)) = x_0(x^*)\cdot(x_0^{**}, x_0^*)$ $\Rightarrow$ $x_0(x^*) = \lambda = (\underbrace{(x_0^{**}, x_0^*)x_0}_{\in X}, x^*)$.

Q1: normed vector space X.

单位开球 B_X. 单位闭球 $F_X := \{x\in X \mid \|x\|\le 1\}$. 应有 $\overline{B_X} = F_X$?

$\forall\, x_0 \notin F_X$. $\|x_0\| > 1$. 存在 $0<\varepsilon<1-\|x_0\|$. $\forall\, x\in B(x_0,\varepsilon) := \{x\in X \mid \|x-x_0\|<\varepsilon\}$.

$\|x\| > 1$, 故 $x \notin \overline{B_X}$. $\Rightarrow$ $\overline{B_X} \subset F_X$.

$\forall\, \|x_0\| = 1$. $\forall\, 0<\varepsilon<1$ $\|(1-\frac{\varepsilon}{2})x_0\| = 1-\frac{\varepsilon}{2}$ $\Rightarrow$ $(1-\frac{\varepsilon}{2})x_0 \in B_X$. $\Rightarrow$ $B(x_0,\varepsilon)\cap B_X \neq \emptyset$

$(1-\frac{\varepsilon}{2})x_0 \in B(x_0,\varepsilon)$ $\Rightarrow$ $x_0 \in \overline{B_X}$ $\Rightarrow$ $F_X \subset \overline{B_X}$.

所以 $\overline{B_X} = F_X$.

拓扑基: $x_1,\dots,x_n \in X$. $x_0^* \in X^*$. $\varepsilon_1,\dots\varepsilon_n > 0$

$\{x^*\in X^* \mid |(x^*-x_0^*)(x_j)| < \varepsilon_j.\ \forall\, 1\le j\le n\}$ 的有限交, → 这个拓扑基记为$\mathcal{B}$

这里 $x_1\cdots$ 不是有限个X中元素?

w^* 拓扑是满足 逐连续的最小拓扑. $X^* \xrightarrow{\text{逐}} \mathbb{R}^X$

$\mathbb{R}^X$ 中拓扑是乘积拓扑. 拓扑基应为 $\prod_{x\in X}(U_x)$. U_x 是 $\mathbb{R}$ 中的开集

w^* 是满足 π_x 逐: $X^* \xrightarrow{x} \mathbb{R}$ 连续的最小拓扑.

故 $\{x^*\in X^* \mid |(x^*-x_0^*)(x)| < \varepsilon\}$ 是开集.

w^* 应是 由 $\underset{\varepsilon_x>0}{\forall\, x\in X}$. $\{x^*\in X^* \mid |(x^*-x_0^*)(x)| < \varepsilon_x\}$ 的集合 生成的最小拓扑 (任意并, 有限交).

这么看 $\mathcal{B}$ 应是此拓扑. 根据定义. $\mathbb{R}^X$ 中. $\prod_{x\in X,\ x\text{有限}}(U_x)$. 才是开集. $\prod_{x\in X,\ \text{无限}} U_x$ 不是开集.

因为 $\prod_{x\in X,\ \text{有限}}(U_x)$ 已能满足 π_x 均是连续的. 不需要无限开集乘积.

$\boxed{\textit{Lecture } 18 \quad 2022/11/4}$

Thm. If X is separable, then any bounded set of X^*, w^*-topology is metrizable.

($\overline{B}_{X^*}$ is w^*-compact, if X 还是可分时, $(\overline{B}_{X^*}, \sigma(X^*,X))$是一个可度量的紧集).

Pf: 不妨就考虑 $\overline{B}_{X^*}=\{x^*\in X^* \mid \|x^*\|\le 1\}$ 令 $S\subset X$ countable dense subset.

$$\begin{aligned} X^* &\xrightarrow{\Phi} \mathbb{R}^X \\ x^* &\longmapsto (x^*(x))_{x\in X}. \end{aligned}$$

Then $(X^*, \sigma(X^*,X)) \xrightarrow[\text{homeomorphic}]{\Phi} \Phi(X^*)$.

$\overline{B}_{X^*} \xrightarrow[\text{homeomorphic}]{\Phi} \Phi(\overline{B}_{X^*})$.

≅同胚

定义: $X^* \xrightarrow[\text{单射}]{\Psi} \mathbb{R}^S$ ←可数.

$\mathbb{R}^S$ ⟶ 可度量化.

$x^* \longmapsto (x^*(x))_{x\in S}$.

但不可能是同胚，除非 $\dim X<+\infty$

(任何无穷维X，则X^*上的弱星拓扑不可度量化).

Claim: $\overline{B}_{X^*} \xrightarrow[\cong\text{连续单}]{\Psi} \Psi(\overline{B}_{X^*}) \subset \mathbb{R}^S$. 可度量

从而 $(\overline{B}_{X^*}, \sigma(X^*,X))$ 可度量.

Indeed, 连续性

$$\begin{tikzcd} \overline{B}_{X^*} \arrow[r] \arrow[dr] & \mathbb{R}^S \arrow[d, "\forall s_0"] \\ & \mathbb{R} \end{tikzcd}$$

$x^* \longmapsto x^*(s_0)$

Conversely.

$\Psi(\overline{B}_{X^*})\subset\mathbb{R}^S$上的诱导拓扑

要证: $(\overline{B}_{X^*}, \sigma(X^*,S)) \longrightarrow (\overline{B}_{X^*}, \sigma(X^*,X))$是连续的.

$\Leftrightarrow$ $\forall x_0\in X$. $(\overline{B}_{X^*},\sigma(X^*,S)) \xrightarrow{x_0} \mathbb{R}$ 连续. 而 $S\subset X$ dense. 可以令 $s_n\to x_0$ ($\|s_n-x_0\|\xrightarrow{n\to+\infty}0$)

Claim: $f_{s_n} \overset{n\to+\infty}{\rightrightarrows} f_{x_0}$, $\sup\limits_{x^*\in\overline{B}_{X^*}}|f_{s_n}-f_{x_0}| = \|s_n-x_0\|\longrightarrow 0$

由 $\sigma(X^*,S)$ 定义知 f_{s_n} 连续，连续函数的一致极限连续 $\Rightarrow$ f_{x_0} 连续.

Prop: If $X \xrightarrow{T} Y$ isomorphic, ($T\in B(X,Y)$, T^{-1}存在且$T^{-1}\in B(Y,X)$.).

X, Y Banach

Then X is reflexive $\Leftrightarrow$ Y is reflexive. (同构不变)

Pf: $X\xrightarrow[\cong]{T} Y \Rightarrow Y^*\xrightarrow[\cong]{T^*}X^*$ ($(AB)^*=B^*A^*$).

$TS=I_Y \Rightarrow S^*T^*=I_{Y^*}$

$ST=I_X \Rightarrow T^*S^*=I_{X^*}$.

$X^{**}\xrightarrow[\cong]{T^{**}}Y^{**}$

$$\begin{tikzcd} X^{**} \arrow[r, "T^{**}"] & Y^{**} \\ X \arrow[u, "J_X"] \arrow[ur, phantom, "\circlearrowleft"] & Y \arrow[l, "T^{-1}"'] \arrow[u, "J_Y"'] \end{tikzcd}$$

$\Rightarrow$ J_Y也是满射. Y也是reflexive.

Thm If X^* is separable $\Rightarrow$ X is separable.

(反之不成立). 反例: $l_1^* = l_\infty$. l_1 可分, 而 l_∞ 不可分.

Pf: Let $\{x_n^*\} \underset{\text{subset}}{\overset{\text{dense}}{\subset}} X^*$. Let $\{x_n\}_{n=1}^{\infty} \subset X$, $\|x_n\| = 1$. $|x_n^*(x_n)| \geq \frac{1}{2}\|x_n^*\|$.

Claim: $\overline{\text{span}}\{x_n\} = X$. 则 $\{\sum_{n=1}^{N} r_n x_n \mid N \in \mathbb{N}.\ r_n \in \mathbb{Q}\}$ 在 X 中 dense.

否则 $\overline{\text{span}}\{x_n\} \subsetneq X$. 则存在 $x_0^* \in X^*$. s.t. $\|x_0^*\| = 1$, $x_0^*(\overline{\text{span}}\{x_n\}) = 0$

$\|x_0^* - x_n^*\| \geq |(x_0^* - x_n^*)(x_n)| \geq \frac{1}{2}\|x_n^*\| \Rightarrow \|x_0^*\| \leq \|x_n^*\| + \|x_0^* - x_n^*\| \leq 3\|x_0 - x_n^*\|$

$\Rightarrow \|x_0^* - x_n^*\| \geq \frac{1}{3}\|x_0^*\|$. for $\forall n \in \mathbb{N}_+$.

Cor: X $\boxed{\text{Ref} + \text{Sep}}$ $\Leftrightarrow$ X^* $\boxed{\text{Ref} + \text{Sep}}$

$\Leftarrow$ is trivial. "$\Rightarrow$": X is Ref $\Rightarrow$ $X = X^{**}$ $\Rightarrow$ X^* is sep.

Thm. 6.13 If X sep, then every bdd seq in X^*, has a w^*-convergent subseq.

Proof: ① X sep $\Rightarrow$ Every bdd set in X^* is metrizable in $\sigma(X^*, X)$ topology.

② $(\overline{B_{X^*}}, \sigma(X^*, X))$ is compact (Tychomov Theorem)

③ 紧度量空间中任意序列有收敛子列.

Thm 可分度量空间任一子空间可分.

$X \xrightarrow{\text{嵌入}} \mathbb{R}^{X^*}$ 给出了 X 上的 weak-topology.

Thm. X^* is separable, then $(B_X, \sigma(X, X^*))$ 是可度量的.

Thm. X Reflexive, then $\forall$ bdd seq, has weakly convegent subseq.

Pf: $\{x_n\} \subset X$ bdd seq. $M = \overline{\text{span}}\{x_n\} \subset X$. Then M is Sep + Ref

想要找 $\{x_{n_k}\}$. $x_{n_k} \xrightarrow{\text{weakly}} x_0$ s.t. $\forall x^* \in X^*$, $x^*|_M$ 在 $\{x_n\}$ 上的取值.

$\in M^*$ $\lim (x_{n_k}, x^*) = (x_0, x^*)$

M is Ref + Sep $\Leftrightarrow$ M^* is Ref + Sep. $(M, \sigma(M, M^*)) = (M^{**}, \sigma(M^{**}, M^*))$.

Lemma:

$X \ni x_n \xrightarrow{\sigma(X, X^*)} x \in X$. 则 x_n 有界

Pf: $\forall x^* \in X^*$, $\lim\limits_{n\to\infty} (x_n, x^*)$ 存在. $\Rightarrow \sup\limits_n |(x_n, x^*)| < +\infty$.

把 x_n 看作 $B(X^*, \mathbb{R})$ 中元素. $X^* \xrightarrow{x_n} \mathbb{R}$. 由共鸣定理, $\sup\limits_{x^* \in B_{X^*}} \sup\limits_n |(x_n, x^*)| < +\infty$

术语: convergence in norm 称为强收敛. strong convergence.

Thm. X n.v.s. $\dim X < +\infty$. weak convergence $\Leftrightarrow$ strong convergence. 1.2.4.5.6 9.13.

pf: X 可以直接看作 $\mathbb{R}^n$. $X = \text{span}\{x_1 \cdots x_n\}$ linearly independent.

$X \xrightarrow[\text{isomorphic}]{\Phi} \mathbb{R}^n$ $x_1^*, \cdots x_n^*$.

$\{z_n\}$ 收敛. $z_n = \sum \alpha_i^{(n)} x_i \xrightarrow{w} z_\infty = \sum \alpha_i^{(\infty)} x_i$.

$\sum \alpha_i x_i \longmapsto (\alpha_1, \cdots \alpha_n)$. $\Rightarrow \alpha_i^{(n)} \xrightarrow{n\to\infty} \alpha_i^{(\infty)}$

例 7. ℓ^1 中序列的 weak convergence $\Leftrightarrow$ strong convergence.

pf: 假设不对. 则 $\{x^{(n)}\} \subset \ell^1$, $x^{(n)} \xrightarrow{\sigma(\ell^1, \ell^\infty)} x^{(\infty)}$

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pf: "$\Rightarrow$" 否则存在 $\{x^{(n)}\}_{n=1}^\infty$ in ℓ_1.

$$\begin{cases} x^{(n)} \xrightarrow[n\to\infty]{w} 0 \\ \|x^{(n)}\| \geq 1 \quad \forall n \end{cases}$$

$\|x^{(n)}\| = \sum\limits_j |x_j^{(n)}| = 1$

$\sum\limits_{j=1}^\infty x_j^{(n)} z_j \xrightarrow{n\to\infty} 0 \quad \forall z \in \ell^\infty = \ell_1^*$

$\forall j$. 取 $z = (0, \cdots 0, 1, 0 \cdots)$ 则 $x_j^{(n)} \xrightarrow{n\to\infty} 0$

第j个分量

$\forall \varepsilon > 0$, fix $n_1 \in \mathbb{N}_+$, $\exists m_1 \in \mathbb{N}_+$, s.t. $\sum\limits_{j=m_1}^{+\infty} |x_j^{(n_1)}| < \varepsilon$.

Then $\exists n_2 \in \mathbb{N}_+$. $\sum\limits_{j=1}^{m_1} |x_j^{(n_2)}| < \varepsilon$. $\exists m_2 \in \mathbb{N}_+$. s.t. $\sum\limits_{j=m_2}^{+\infty} |x_j^{(n_2)}| < \varepsilon$.

$\vdots$

$\sum\limits_{j=1}^{m_k} |x_j^{(n_{k+1})}| < \varepsilon$. $\exists m_{k+1} \in \mathbb{N}_+$. s.t $\sum\limits_{j=m_k}^{+\infty} |x_j^{(n_k)}| < \varepsilon$.

Let $z_j x_j^{(n_k)} = |x_j^{(n_k)}|$, $m_{k-1} \leq j \leq m_k$

$$\lim_{n\to\infty} \sum z_j x_j^{(n)} = \lim_{k\to\infty} \left(\left|\sum_{j=1}^{m_k} x^{(n_k)}\right| + \left|\sum_{j=m_k+1}^{m_{k+1}} x^{(n_k)}\right| + \left|\sum_{j=m_{k+1}+1}^{+\infty} x^{(n_k)}\right| \right) \geq (1-4\varepsilon).$$

Chapter 9 Banach Algebra

具体的例子: X Banach 空间, $B(X)=\{X\xrightarrow{T}X \mid$ 有界线性算子$\}$.

- 代数: 线性空间. 乘法.

Banach 代数. 有范数的代数, 使其成为一个 Banach 空间且 $\|ab\|\le\|a\|\cdot\|b\|$

$A\in B(X)$, $\|A\|=\sup\limits_{x\in B_X}\|Ax\|$ $\quad\|AB\|\le\|A\|\cdot\|B\|$

单位元: $I: X\longrightarrow X$, $I(x)=x$. $\quad AI=IA=A$, $\forall\ A\in B(X)$.

如果一个赋范且完备的代数 B (over $\mathbb{R}$ or $\mathbb{C}$).

$\hat{B}:=B\times\mathbb{R}$ (or $\mathbb{C}$). $\quad$ 定义乘法: $(a,\alpha)(b,\beta)=(ab+\alpha b+\beta a,\alpha\beta)$.

(a,α). $\quad$ 定义范数: $\|(a,\alpha)\|=\|a\|+|\alpha|$,

则 $\hat{B}$ is a Banach Algebra, 且 $(0,1)$ is the identity element,

$(a,\alpha)\longleftrightarrow a+\alpha$

$(a+\alpha)(b+\beta)=(ab+a\beta+b\alpha)+\alpha\beta$.

$(a,\alpha)\cdot(b,\beta):=(ab+a\beta+b\alpha,\ \alpha\beta)$.

显然有 $\|(a,\alpha)\cdot(b,\beta)\|\le\|(a,\alpha)\|\cdot\|(b,\beta)\|$.

$a\in B$ is called invertible. if $\exists\ b\in B$. s.t. $ab=ba=e$ (or 1).

$B^{\times}:=\{b\in B\mid b$ is invertible$\}$, b is called the inverse of a. denoted as a^{-1}.

Def: 预解集 resolvent $\quad\rho(a)=\{\lambda\in\mathbb{R}\ (\text{or }\mathbb{C})\mid\lambda-a$ 可逆$\}$. $(\lambda-a)^{-1}$

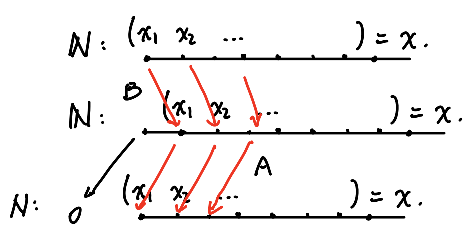
谱 spectrum $\quad\sigma(a)=\{\lambda\in\mathbb{R}\ (\text{or }\mathbb{C})\mid\lambda-a$ 不可逆$\}$ $\qquad\lambda=\lambda e$.

proposition:

if $\mathcal{A}$ is an algebra. $a,b\in\mathcal{A}$. $ab=ba$.

Assume that ab is invertible. Then a,b are invertible

Counter example: if $ab\neq ba$. $\quad\ell^2=\ell^2(\mathbb{N})=\{(a_n)_{n=1}^{\infty}:\sum|a_n|^2<+\infty\}$.



$A\cdot B$ 是恒等映射.

$AB\neq BA$.

A, B 均不可逆.

Fekete Theorem

Thm: Let $\{a_n\}_{n=1}^{\infty}$ a sequence in $\mathbb{R}$.

Assume that $(a_n)_{n=1}^{\infty}$ is sub-additive, i.e. $a_{m+n} \le a_m + a_n$

Then $\lim\limits_{n\to\infty} \frac{a_n}{n} = \inf\limits_{n\ge 1} \frac{a_n}{n} \in [-\infty, +\infty)$.

Pf: 所以只需证: $\limsup\limits_{n\to\infty} \frac{a_n}{n} \le \inf\limits_{n\ge 1} \frac{a_n}{n}$.

即 $\forall\, n_0 \in \mathbb{N}_+$, 都有 $\limsup\limits_{n\to\infty} \frac{a_n}{n} \le \frac{a_{n_0}}{n_0}$, $\forall\, n_0$, 有唯一的 $q \in \{0,1,2,\dots\}$ $r \in \{0,1,\dots,n_0-1\}$.

$n = q n_0 + r$. $\quad \frac{a_n}{n} = \frac{a_{qn_0+r}}{qn_0+r} \le \frac{q a_{n_0} + a_r}{q n_0 + r}$

$$= \frac{a_{n_0} + \frac{a_r}{q}}{n_0 + \frac{r}{q}}, \quad \limsup_{n\to\infty} \frac{a_n}{n} \le \frac{a_{n_0}}{n_0}$$

Corollary:

$(b_n)_{n=1}^{\infty}$, $b_n \ge 0$. 满足 sub-multiplicative

即 $b_{m+n} \le b_m \cdot b_n$

Then $\lim\limits_{n\to\infty} \sqrt[n]{b_n} = \inf\limits_{n\ge 1} \sqrt[n]{b_n}$.

Pf: $\log b_{m+n} \le \log b_m + \log b_n \Rightarrow \log b_n$ 次可加.

$\lim\limits_{n\to\infty} \frac{\log b_n}{n} = \inf\limits_{n\ge 1} \frac{\log b_n}{n} \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{b_n} = \inf\limits_{n\ge 1} \sqrt[n]{b_n}$.

应用: B Banach Algebra.

$\|ab\| \le \|a\|\cdot\|b\|$. 特别 $\|a^{m+n}\| = \|a^m \cdot a^n\| \le \|a^m\| \cdot \|a^n\|$

故 $(\|a^n\|)_{n=1}^{\infty}$ 次可乘. 由 Fekete Thm, $\lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}}$ 存在.

目标: B complex Banach algebra.

Thm: $r(a) = \max\limits_{\lambda\in\sigma(a)} |\lambda|$ (称之为谱半径).

Then $r(a) = \lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}}$

$a \in B$. ← Banach algebra.

Lemma 1: If $\lambda \in \sigma(a) \Rightarrow \lambda^n \in \sigma(a^n)$

即 $a-\lambda$ 不可逆 $\Rightarrow$ $a^n-\lambda^n$ 不可逆.

Pf of Lemma 1: 只需证 $a^n-\lambda^n$ 可逆, 则 $a-\lambda$.

$a^n - \lambda^n = (a-\lambda)(a^{n-1} + a^{n-2}\lambda + \cdots \lambda^{n-1})$.

$= (a^{n-1} + a^{n-2}\lambda + \cdots \lambda^{n-1})\cdot(a-\lambda)$. 可逆.

故 $a-\lambda$ 可逆.

Lemma 2. p is a polynomial. $a \in B$.

Then $p(\sigma(a)) \subset \sigma(p(a))$

Pf: $\forall \lambda \in \sigma(a)$，即 $a-\lambda$ 不可逆. 欲证: $p(\lambda) \in \sigma(p(a))$，即 $p(a)-p(\lambda)$ 不可逆.

反证: 若 $p(a)-p(\lambda)$ 可逆. $Q(x)=p(x)-p(\lambda)=0$ 则 $a-\lambda \mid (p(a)-p(\lambda))$.

$p(x)-p(\lambda)=(x-\lambda)\cdot q(x)=q(x)\cdot(x-\lambda)$. 则 $a-\lambda$ 可逆. 矛盾. $\square$

$\boxed{\textit{Lecture } 20 \quad 2022/11/11}$

$$\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2$$

$$(\alpha_1,\alpha_2) \longmapsto (-\alpha_2,\alpha_1) \longmapsto (-\alpha_1,-\alpha_2)$$

$$A^2=-I$$

$p(\sigma(a)) \subset \sigma(p(a))$

从现在开始考虑 $\mathbb{C}$ 上的 Banach 代数. B

Prop. $p \in \mathbb{C}[z]$. Then $p(\sigma(a))=\sigma(p(a))$, $\forall a \in B$.

Pf. 若 $\lambda \in \sigma(p(a))$. Then $p(a)-\lambda$ 不可逆.

希望 $\lambda \in p(\sigma(a))$. 即 $\lambda = p(\mu)$ for some $\mu \in \sigma(a)$.

$p(z)-\lambda = c(z-z_1)\cdots(z-z_n)$. 从而 $p(a)-\lambda = c\cdot\prod(a-z_n)$

① Case 1: $c=0$. Trivial.

② case 2: $c\neq 0$. 存在 $1\le i\le n$ s.t. $(a-z_i)$ 不可逆. $z_i \in \sigma(a)$, $p(z_i)-\lambda=0$

$\Rightarrow \lambda = p(z_i)$ $\square$

Lemma. $\forall a \in B$, $\sigma(a)\subset\mathbb{C}$ is compact.

Pf: $\Leftrightarrow$ $\rho(a)$ is open & $\sigma(a)$ is bdd.

① $\sigma(a)$ bdd. $\sigma(a) \subset \overline{B(0,\|a\|)} = \{x\in\mathbb{C} \mid |z|\le\|a\|\}$.

只需 $\forall \lambda \in \mathbb{C}$. $|\lambda|>\|a\|$. $\lambda-a$ 可逆. $\lambda - a = \lambda\cdot\underbrace{(1-a/\lambda)}_{\text{可逆. 逆元 } \sum_0^\infty (a/\lambda)^n}$

② if $\lambda \in \rho(a)$. 即 $a-\lambda$ 可逆.

希望 $|z|<\varepsilon$. $\lambda+z-a$ 也可逆.

$(\lambda+z-a) = (\lambda-a)\cdot(1+z/\lambda-a)$，令 $\|z\| < \|\lambda-a\|/2$ 即可.

Prop: $\forall a \in B$, the map is holomorphic

$$\rho(a)=\mathbb{C}\setminus\sigma(a) \xrightarrow{\phi} B.$$

$$z \longmapsto (z-a)^{-1}$$

即 $\forall z_0 \in \rho(a)$. 存在 $B(z_0,r)\subset \rho(a)$

s.t. ϕ 在 $B(z_0,r)$ 上有解析式:

$\phi(z)=\sum_0^{+\infty} a_n(z-z_0)^n$. $a_n \in B$.

Remark: 强解析 $\Leftrightarrow$ 弱解析.

即, $f: U \longrightarrow X$. complex Banach space

$U \subset \mathbb{C}$ is called weakly holomorphic.

$\forall\, l \in X^*$. $\quad U \xrightarrow{f} X \xrightarrow{l} \mathbb{C}$ is holomorphic.

$\quad z \longmapsto f(z) \longmapsto l(f(z))$

Lemma: $\lambda, \mu \in \rho(a)$

$$(\lambda-a)^{-1}-(\mu-a)^{-1} = (\lambda-a)^{-1}\cdot\left[(\mu-a)-(\lambda-a)\right]\cdot(\mu-a)^{-1}$$

$$= (\lambda-a)^{-1}\cdot(\mu-\lambda)\cdot(\mu-a)^{-1} = (\mu-\lambda)\cdot(\lambda-a)^{-1}(\mu-a)^{-1}$$

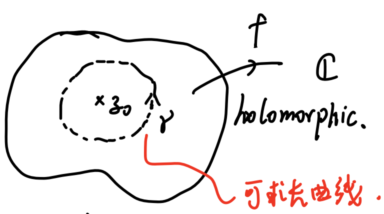
Pf: $z_0 \in \rho(a)$. $\quad B(z_0, r) \subset \rho(a)$. $\quad w \in B(0, r)$.

$$(z_0+w-a)^{-1} = (z_0-a+w)^{-1} = \left[(z_0-a)\cdot\left[1+(z_0-a)^{-1}\cdot w\right]\right]^{-1} = \left(1+(z_0-a)^{-1}w\right)^{-1}\cdot(z_0-a)^{-1}$$

$$= \sum_{n=0}^{+\infty}\left[-(z_0-a)^{-1}w\right]^n\cdot(z_0-a)^{-1}$$

是关于 w 的展开式.

Recall: Cauthy formula.



$n(\gamma, z_0) = 1$.

Then $\quad f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \dfrac{f(z)}{z-z_0}\,dz$.

Lemma: if $|z| > \lim\limits_{n\to\infty}\|a^n\|^{\frac{1}{n}}$ $\quad (\ast)$

Then $(z-a)^{-1} = \sum\limits_{1}^{\infty} z^{-n}a^{n-1}$ 条件 $(\ast)$: 存在 $n \in \mathbb{N}_+$

s.t. $\|a^n\|^{\frac{1}{n}} < |z| \Leftrightarrow \left\|\dfrac{a^n}{z^n}\right\| < 1$.

令 $b = a/z$, $\displaystyle\sum_{k=0}^{+\infty}\|b^k\| = \underbrace{\sum_{r=0}^{n-1}}_{\text{有限个}}\underbrace{\sum_{q=0}^{+\infty}\|b^{qn+r}\|}_{\leq \|b^r\|\cdot\sum_{q=0}^{+\infty}\|b^n\|^q} < +\infty$

在 B 中收敛.

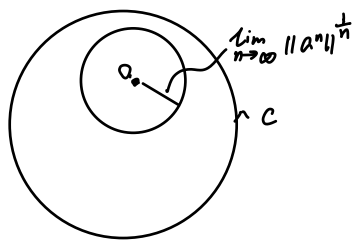
$\Rightarrow \lim\limits_{k\to\infty}\|b^k\| = 0$

$$1-b^k = (1-b)\cdot(1+b+\cdots b^{k-1}) \quad 令\ k\to\infty\ 得\ (1-b)^{-1} = \sum_{k=0}^{\infty} b^k$$

$\square$.

Corollary. $\sigma(a) \subset \{z \in \mathbb{C} \mid |z| \leq \lim\limits_{n\to\infty}\|a^n\|^{\frac{1}{n}}\}$.

$\forall a \in B.$



$z \to z^n$

Lemma: $a^n = \dfrac{1}{2\pi i}\oint_C \dfrac{z^n}{z-a}\, dz.$

$= \dfrac{1}{2\pi i}\oint_C z^n (z-a)^{-1}\, dz.$

$(z-a)^{-1} = \sum\limits_{n=1}^{\infty} z^{-n} a^{n-1}$. 满足 $\sum\limits_{n=1}^{\infty} \|z^n a^{n-1}\| < +\infty$ on C.

$= \dfrac{1}{2\pi i}\oint_C \left(\sum\limits_{k=1}^{\infty} z^{n-k} a^{k-1}\right) dz = \sum\limits_{k=1}^{\infty} \dfrac{1}{2\pi i}\oint_C z^{n-k} a^{k-1}\, dz = \dfrac{1}{2\pi i}\oint_C \dfrac{a^n}{z}\, dz = a^n.$

最终目的:

$a \in B$

解析函数芽

$\mathrm{Hol}(\sigma(a)) = \{ f \mid f \in A(\underbrace{\overline{\sigma(a)}}_{\text{包含}\sigma(a)\text{的某邻域}}) \}.$

$z \mapsto a \qquad 1 \mapsto e$

代数、连续同态 $\phi: \mathrm{Hol}(\sigma(a)) \to B.$

故 $\phi(z^n) = a^n. \quad \phi(P(z)) = P(a)$

ϕ 可以延拓至解析函数.

Theorem: $\max\limits_{\lambda\in\sigma(a)} |\lambda| = \lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}}$, $\sigma(a) \neq \emptyset$. ?

假设 $a \in B$ 有 $\sigma(a) = \emptyset$. 则 $\forall\, l \in B^*$. $l(z-a)^{-1}$ 是 $\mathbb{C} \to \mathbb{C}$ 的有界解析函数.

$(z-a)^{-1} = \sum\limits_{n=1}^{\infty} z^n a^{n-1} \Rightarrow \|(z-a)^{-1}\| \le \sum\limits_{n=1}^{\infty} \dfrac{\|a\|^{n-1}}{|z|^n}$

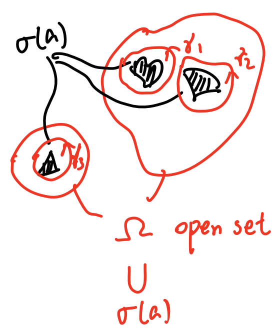
故为常函数.

$(z-a)^{-1}$ 也为常数. 矛盾.

所以 $\sigma(a)$ 是非空紧集.

Back to the Thm. Pf: 令 $m = \max\limits_{\lambda\in\sigma(x)} |\lambda|$, 我们想证 $\lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}} < m + \varepsilon$.

下次再证



可求长

$\Gamma = \bigcup\limits_i \gamma_i.$ ← 每个 γ_i 可求长.

$n(\Gamma, w) = 1. \ \forall\, w \in \sigma(a).$

$(z-a)^{-1}$ 在 Γ 上可定义.

定义 $\tilde{f}(a) := \dfrac{1}{2\pi i}\int_\Gamma f(z)\cdot(z-a)^{-1}\, dz.$

① 是否良定义.

②

Lecture 21 2022/11/16

正规族 $\underset{\text{domain}}{\Omega} \subset \mathbb{C}$

$$\mathcal{O}(\Omega) = hol(\Omega) = \{ f:\Omega \to \mathbb{C} \mid \text{holomorphic} \}$$

拓扑：内闭一致收敛诱导.

Def: $\mathcal{F} \subset \mathcal{O}(\Omega)$ is normal if. 局部一致有界(确认). 即 $\forall K \underset{}{\overset{\text{compact}}{\subset\subset}} \Omega$ $\sup_{f\in\mathcal{F}} \|f\|_{C(K)} < +\infty$.

在内闭一致收敛拓扑下相对紧.

试卷讲解.

Riesz calculus.
复Banach代数(解析演算)

$a \in B$. $\sigma(a) \subset \Omega$ 开集.

可以找 $\Gamma = \gamma_1 \cup \gamma_2 \cdots \cup \gamma_n$ 有限条定向可求长闭曲线.

$n(z,\Gamma) = 1$. $\forall z \in \sigma(a)$

定义: $f \in Hol(\sigma(a)) = \{ f \mid f \text{ hol in an open } U \supset \sigma(a) \}$.

$f(a) := \frac{1}{2\pi i} \oint_\Gamma f(\zeta)/\zeta - a \, d\zeta$ 不依赖 Γ 的选取. Bochner integral.

理解: 可对角化矩阵 $M = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1}$

习题 $f(M) = P \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix} P^{-1}$

Thm. $\forall a \in B$ fixed.

$$Hol(\sigma(a)) \xrightarrow{R_a} B. \qquad f \longmapsto f(a)$$

Then R_a is a homomorphism of algebras.

即 $R_a(fg) = R_a(f) R_a(g)$

$fg(a) = f(a)\cdot g(a) = g(a)\cdot f(a)$

Pf: $\Gamma_1 \longrightarrow f(a)$

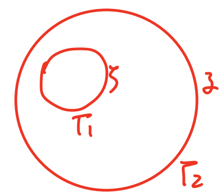
Γ_2

$$f(a)\cdot g(a) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - a} d\zeta \cdot \frac{1}{2\pi i} \oint_\Gamma \frac{g(z)}{z-a} dz.$$

$$= \left(\frac{1}{2\pi i}\right)^2 \cdot \oint_\Gamma \oint_\Gamma f(\zeta) g(z) \cdot (\zeta - a)^{-1} \cdot (z-a)^{-1} d\zeta \, dz$$

Recall: $(\zeta - a)^{-1} - (z-a)^{-1} = (\zeta-a)^{-1} \cdot [z - a - (\zeta - a)] \cdot (z-a)^{-1} = (\zeta - a)^{-1} \cdot (z - \zeta) (z-a)^{-1}$

$$\left(\frac{1}{2\pi i}\right)^2 \cdot \oint_\Gamma \oint_\Gamma f(\zeta)\, g(z) \cdot \frac{(\zeta-a)^{-1} - (z-a)^{-1}}{z-\zeta}$$



$$= \left(\frac{1}{2\pi i}\right)^2 \left(\oint_{\Gamma_1} \left[\oint_{\Gamma_2} \frac{f(\zeta)\, g(z)\, (\zeta-a)^{-1}}{z-\zeta}\, dz \right] d\zeta - \oint_{\Gamma_2} \left[\underbrace{\oint_{\Gamma_1} \frac{f(\zeta)\, g(z)\, (z-a)^{-1}}{z-\zeta}\, d\zeta}_{\text{是解析} \ = 0} \right] dz \right)$$

$$= \frac{1}{2\pi i} \oint_{\Gamma_1} f(\zeta) \cdot (\zeta-a)^{-1} \cdot \left(\frac{1}{2\pi i} \oint_{\Gamma_2} \frac{g(z)}{z-\zeta}\, dz \right) d\zeta. \quad = \frac{1}{2\pi i} \oint_{\Gamma_1} f(\zeta) g(\zeta)\, (\zeta-a)^{-1} d\zeta = \overset{f\cdot g}{h}(a)$$

同理有 $= g \cdot f$. (令 Γ_1 包含 Γ_2).

$$\mathrm{Hol}(\sigma(a)) \xrightarrow{R_a} B,$$
$$f \longmapsto \frac{1}{2\pi i} \oint_\Gamma f(\zeta)(\zeta-a)^{-1} d\zeta$$

if f_n 都在某固定的 $U \supset \sigma(a)$ 上解析.

$f_n \xrightarrow[\text{一致}]{\text{内闭}} f$. 则 $f_n(a) \xrightarrow{n\to\infty} f(a)$.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[rrr, bend right, "g\circ f\ ?"'] & f(a) \arrow[r, "g"] & & g(f(a)) \end{tikzcd}$$
$$a \in B, \quad f(a) \in B, \quad g(f(a)) \in B.$$

Prop. 同态 $\mathrm{Hol}(\sigma(a)) \longrightarrow B$. 加上连续性 唯一确定了 R_a
$$z \longmapsto a$$
$$\mathbb{1} \longmapsto 1.$$

Thm. $f \in \mathrm{Hol}(\sigma(a))$

Then $\sigma(f(a)) = f(\sigma(a))$

Pf: "$\subset$" $\lambda \in \sigma(f(a))$. 即 $\lambda - f(a)$ 不可逆. 我们希望证 $\lambda = f(z)$ 在 $\sigma(a)$ 中有解.

否则 $f(z)-\lambda$ 在 $\sigma(a)$ 上不取 0.

则 $\frac{1}{f(z)-\lambda} \in \mathrm{Hol}(\sigma(a))$.

$$\left(\frac{1}{f(z)-\lambda}\right)(a) \cdot (f(z)-\lambda)(a) = \left(\frac{1}{f(z)-\lambda} \cdot f(z)-\lambda\right) \cdot (a) = \mathbb{1}(a) = 1.$$

$\Rightarrow \lambda - f(a)$ 可逆、矛盾.

"$\supset$" if $\mu = f(\lambda)$, $\lambda \in \sigma(a)$. 想要证. $f(a)-\mu = f(a) - f(\lambda)$ 不可逆.

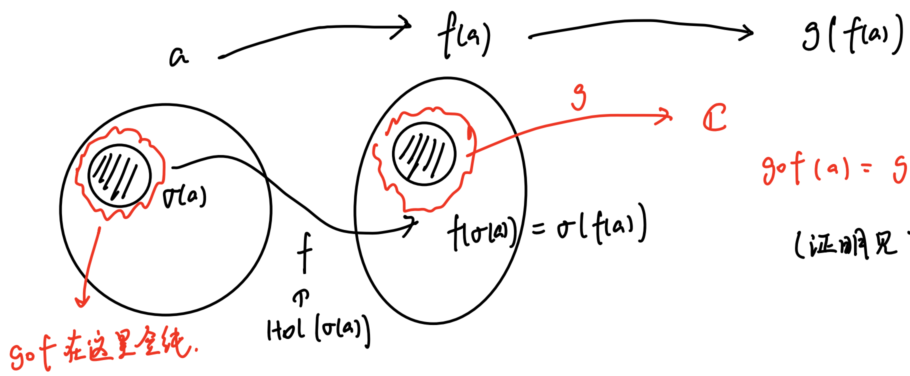
$h(z) = \frac{f(z)-f(\lambda)}{z-\lambda}$ 是解析函数. $h \in \mathrm{Hol}(\sigma(a))$ $f(z)-f(\lambda) = h(z)\cdot(z-\lambda)$.

分别作用到 a: $f(a)-f(\lambda) = h(a)\cdot(a-\lambda)$.

而 $a-\lambda$ 不可逆.

$\Downarrow$

$f(a)-f(\lambda)$ 不可逆.



$g\circ f(a) = g(f(a))$

(证明见76页)

$\boxed{\text{Lecture 22} \quad 2022/11/23}$

$a \in B \longleftarrow$ Banach algebra. 谱半径 $r(a) := \max\limits_{\lambda\in\sigma(a)} |\lambda|$

Theorem: $\boxed{r(a) = \lim\limits_{n\to\infty} \|a^n\|^{\frac{1}{n}}} = \inf\limits_{n} \|a^n\|^{\frac{1}{n}}$ → Fekete theorem. 一直成立.

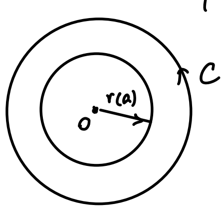
pf: $r(a) \le \underset{n\to\infty}{\underline{\lim}} \|a^n\|^{\frac{1}{n}}$ 极限存在

回忆: $r(a) \le \|a\|$. $\Rightarrow r(a^n) \le \|a^n\|$

谱定理 $\sigma(a^n) = \sigma(a)^n \Rightarrow r(a) \le \|a^n\|^{\frac{1}{n}} \quad \forall n$

反之, 欲证 $\overline{\lim\limits_{n\to\infty}} \|a^n\|^{\frac{1}{n}} \le r(a)$.

Riesz Holomorphic calculus. $\forall$ C a circle centered with radius $r(a)+\varepsilon$



$$a^n = (z^n)\cdot(a) = \frac{1}{2\pi i}\oint_C z^n\cdot(z-a)^{-1}dz.$$

$$\Downarrow$$

$$a^n = \frac{1}{2\pi}\int_0^{2\pi} [(r(a)+\varepsilon)e^{i\theta}]^n \cdot \left((r(a)+\varepsilon)e^{i\theta} - a\right)^{-1} \cdot (r(a)+\varepsilon)\, e^{i\theta}\, d\theta$$

$$\Rightarrow \|a^n\| \le \frac{1}{2\pi}\int_0^{2\pi} [r(a)+\varepsilon]^{n+1} \cdot \underbrace{\| [(r(a)+\varepsilon)e^{i\theta} - a]^{-1} \|}_{\text{关于}\theta\text{连续, 有界. 记上界为}M} d\theta$$

$$\Rightarrow \|a^n\|^{\frac{1}{n}} \le M^{\frac{1}{n}} \cdot (r(a)+\varepsilon)^{\frac{n+1}{n}}. \quad \text{令 } n\to\infty.$$

得 $\overline{\lim\limits_{n\to\infty}} \|a^n\|^{\frac{1}{n}} \le r(a)+\varepsilon$.

Remark: $f(a) = \frac{1}{2\pi i}\int_\Gamma f(z)\cdot(z-a)^{-1}dz. \Rightarrow \quad \|f(a)\| \le \frac{1}{2\pi}\oint_\Gamma |f(z)|\cdot\|(z-a)^{-1}\|\,dz.$

$$\|f(a)\| = \sup_{\substack{\ell\in B^* \\ \|\ell\|=1}} |\ell(f(a))| = \sup_{\substack{\ell\in B^* \\ \|\ell\|=1}} \left|\frac{1}{2\pi i}\int_\Gamma \ell(z-a)^{-1}\cdot f(z)\,dz\right| \le \frac{1}{2\pi}\oint_\Gamma \|(z-a)^{-1}\|\cdot|f(z)|\,dz$$

$$a^n = \frac{1}{2\pi i}\oint_C z^n (z-a)^{-1} dz \quad \Rightarrow \quad \|a^n\| \le \frac{1}{2\pi}\oint_C |z^n|\cdot \|(z-a)^{-1}\| dz$$

$$\le (r(a)+\varepsilon)^{n+1} \cdot \sup_{z\in C} \|(z-a)^{-1}\|$$

特殊的 Banach 代数

Commutative Banach algebra

B Banach 代数. 满足 $ab = ba$. $\forall a, b \in B$

Examples: ① X compact topological space (Hausdorff).

$C(X) = \{ f: X \to \mathbb{C} \mid \text{continuous} \}$. $\|f\|$ = 一致范数. 则 $C(X)$ 为 commutative.

② $L^1(\mathbb{R}) \ni f, g$. $(L^1(\mathbb{R}), *)$ is commutative Banach algebra.

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-y) g(y) dy$$

$$\|f * g\|_{L^1} = \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f(x-y) g(y) dy \right| dx \le \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(x-y) g(y)| dx\, dy = \|f\|_{L^1} \cdot \|g\|_{L^1}.$$

同理. $\mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \} = \{ e^{i2\pi\theta} \mid \theta \in [0,1) \}$

$(L^1(\mathbb{T}), *)$. 卷积是群的操作.

$$(f*g)(e^{i2\pi\theta}) = \int_0^1 f(e^{i2\pi(\theta - z)}) \cdot g(e^{i2\pi z}) dz.$$ 易验证 $\|f*g\|_1 \le \|f\|_1 \cdot \|g\|_1$.

$(\ell^1(\mathbb{Z}), *)$ $\quad (v * w)(n) := \sum_{m\in\mathbb{Z}} v(n-m) w(m)$.

Spectrum of commutative Banach algebra.

||

非 0 可乘线性泛函. 即 $m: B \to \mathbb{C}$. $\quad \mathcal{M}(B) := \{ m: B \to \mathbb{C} \mid \text{multiplicative} \}$.

m 不恒为 0. $m(ab) = m(a)\cdot m(b)$.

multiplicative functional. (不要求 m 连续).

Theorem: $\forall a \in B$. $\lambda \in \sigma(a) \Leftrightarrow \exists\, m \in \mathcal{M}(B)$, s.t. $\lambda = m(a)$.

Def: 理想 ideal. $I \subset B$ subspace. I is called ideal, if I 有吸收性.

即 $\forall a \in I, b \in B$. $ab \in I$.

- 极大理想 ① I is called maximal ideal if $I \subset J$. J is an ideal
  $I \ne B$. Then $I = J$.

② $B = I \oplus \mathbb{C}\cdot e$, $e \in B$ 单位元.

$\forall\ b = a + \lambda e$.

Theorem 9.11 Every ideal $H \neq B$. is contained in a maximal ideal.

Proof. Consider $\mathcal{S} = \{ L \subsetneq B \mid \begin{matrix} H \subset L \\ L \text{ is an ideal} \end{matrix} \}$ 所有比 H 大的真子理想..

$\mathcal{S}$ 有自然的偏序关系.

$\forall\ W \subset \mathcal{S}$ 是一个全序子族. $L_1 \subset L_2$. let $J = \bigcup_{L \in W} L$. 则 J 是一个真子理想.

$\forall\ a \in L_1,\ b \in L_2,\ \lambda a + \mu b \in L_2 \subset J.\ \Rightarrow J$ is a subspace.

J is an ideal. $\forall \begin{matrix} a \in J \\ b \in B. \end{matrix}$ $\begin{matrix} \text{if } a \in L \subset J. \\ ab \in L \subset J. \end{matrix}$

$J \neq B$. 否则 $e \in J$. $\Rightarrow e \in L \subset J$. for some $L \in W$. $\Rightarrow B = L$. Contraction!

故 J is an upper bound of W. By Zorn's Lemma.

$\exists$ maximal element in $\mathcal{S}$. denoted as K. $H \subseteq \underset{\text{ideal}}{\underset{\sim}{K}} \subsetneq B$.

K is maximal element in $\mathcal{S}$. if $H \subset K \subset \underset{\text{idedl}}{\underset{\sim}{K'}} \subsetneq B$. Then $K' = K$. $\Rightarrow$

K is maximal.

Lemma. 任意不可逆元包含在某个极大理想中

pf: $a \notin B^{\times}$, Then $\langle a \rangle = \{ ab \mid b \in B \}$ is an ideal. $e \notin \langle a \rangle \Rightarrow \langle a \rangle \neq B$.

由于 $\langle a \rangle$ 是真子理想. 上述定理表明 $\langle a \rangle$ 包含于某极大理想.

Lemma: $\forall$ $\underset{\sim}{B}$ 中的极大理想 I 一定是闭子空间.

Banach algebra

Thm 9.16 If I is an ideal.

$I \subsetneq B$ is a maximal ideal $\Leftrightarrow B = I + \mathbb{C} \cdot e$.

"$\Leftarrow$" if $B = I + \mathbb{C}e$. J is an ideal, $I \subset J \subsetneq B$.

若 $I \subsetneq J$. $\exists\ b \in J.\ b \notin I$. $b \in B = I + \mathbb{C}e$. $\Rightarrow b = a + \lambda e$ for $a \in I \subset J$.

$\Rightarrow b - a \in J \Rightarrow \lambda e \in J$. 而 $\lambda \neq 0$. $\Rightarrow e \in J \Rightarrow J = B$. 矛盾.

故只能有 $I = J$.

"$\Rightarrow$" Lemma: if $B = I + \mathbb{C}e$. Then I is closed. (I is an ideal)

pf: $\{a_n\} \subset I$. $a_n \longrightarrow a \in B$. 欲证 $a \in I$.

$B = I + \mathbb{C}e$. $a = a_1' + \lambda e$. $\Leftrightarrow$ 证 $\lambda = 0$.

否则 $\lambda \neq 0$. $a_n - a \to 0 = a_n - a_1' - \lambda e \to 0$.

$\Leftrightarrow \ a_n - a_i' \to \lambda e. \quad \Leftrightarrow \ \frac{1}{\lambda}(a_n - a_i') \to e$

可逆元是开集，可逆元的附近也是可逆的。 n 足够大时. $\frac{a_n - a_i'}{\lambda}$ 可逆

而 $\frac{a_n - a_i'}{\lambda} \in I. \Rightarrow I = B.$ I is closed.

回到原定理、if I is a maximal ideal. Then $I \neq B$.

取 $a \notin I$. $L = \langle I, a \rangle. = \{ xa + y \mid x \in B. \ y \in I \}$ 由 I, a 生成的理想.

$I \subsetneq L$. L 是理想 $\Rightarrow L = B$.

B/I. 代数/理想 还是一个代数 (剩余证明见 Lecture 23 中部).

Maximal ideal $I \Leftrightarrow I$ ideal, $I \subsetneq B$. $B = \mathbb{C}e + I$. → 闭理想.

$\left(\begin{array}{l} \mathbb{C}e \oplus I : \quad \lambda e + a = \lambda' e + a' \\ \quad \Rightarrow (\lambda - \lambda')e = a - a' \Rightarrow a - a' = 0, \ \lambda - \lambda' = 0 \end{array} \right)$ 其实 $I \oplus \mathbb{C}e = I + \mathbb{C}e$

Theorem: 极大理想与 multiplicative functional.

$\mathcal{M}(B) \longrightarrow \{\text{极大理想 in } B\}$.

$m \longmapsto \ker m$.

If $m \in \mathcal{M}(B) \Rightarrow \ker m$ 是极大理想. if $a \in \ker m$ $m(ab) = m(a)\cdot m(b) = 0 \quad \forall b \in B$.

$\Rightarrow ab \in \ker m$. $\ker m$ is an ideal.

$$\begin{tikzcd} B \arrow[rr, "m \neq 0"] \arrow[dr] & & \mathbb{C} \\ & B/\ker m \arrow[ur, "\cong"'] & \end{tikzcd}$$

$\Rightarrow B = \ker m \oplus \mathbb{C}e$. $m(e) \neq 0$, 否则 $\ker m = B$.

$\forall b \in B. \quad m(b) = \frac{m(b)}{m(e)} m(e) = m\left(\frac{m(b)}{m(e)} e\right)$

$\Rightarrow m\left(b - \frac{m(b)}{m(e)} e\right) = 0$. $b = \frac{m(b)}{m(e)} e + \underbrace{b - \frac{m(b)}{m(e)} e}_{\ker m.}$

Thus $B = \ker m \oplus \mathbb{C}e$. $\Rightarrow \ker m$ is maximal.

反之，若 I 是真子理想. 且 $I \oplus \mathbb{C}e = B$. $\Rightarrow \exists \, m \in \mathcal{M}(B)$ s.t. $I = \ker m$.

$$\begin{tikzcd} B \arrow[rr, "m"] \arrow[d, equal] & & \mathbb{C}. \\ I + \mathbb{C}e \arrow[dr] & & \\ & B/I \arrow[uur, "\cong"'] & \end{tikzcd}$$

令 $m(e) = 1$. $m(a) = 0 \ \forall a \in I$.

$\forall b \in B$. $b = a + \lambda e$. Then $m(b) = \lambda$.

Verify that m is multiplitive.

$B \xrightarrow{\text{quotient}} B/I$ 是一个代数同态. 从而有可乘性.

I 极大理想, $\Rightarrow I = \ker m$. I is closed $\Rightarrow m$ is bnd linearly functional.

一个 Banach 代数上的任意非 0 可乘泛函是连续泛函.

Thm. 事实上这种泛函的范数 ≤ 1. $L^1(\mathbb{R})$ 是没有单位元的.

proof: $\ker m$, closed. 假设 $|m(a)|=1$. $\|a\|<1$.

Then $|m(a^n)| = |m(a)\cdots m(a)| = 1$.

而 $1 = |m(a^n)| \le \|m\|\cdot\|a^n\| \le \|m\|\cdot\|a\|^n \to 0^+$, 矛盾.

故 $\|m\| \le 1$ 而 $m(e) = m(e\cdot e) = m^2(e) \Rightarrow m(e) = 1$ or $\underset{\sim}{0} \Rightarrow m\equiv 0$

$\Rightarrow \|m\| = 0$ or 1

Thm. $\lambda\in\sigma(a) \iff \exists\, m\in\mathcal{M}(B)$, $\lambda = m(a)$

Pf: 若 $\lambda\notin\sigma(a)$. 即 $a-\lambda$ 可逆. 即 $\exists\, b\in B$, $(a-\lambda)b = e$.

$\Rightarrow \forall\, m\in\mathcal{M}(B)$ $m((a-\lambda)b) = m(e) = 1$.

$$\begin{aligned} &\overset{\shortparallel}{m(a-\lambda)m(b)} = (m(a)-\lambda)m(e)\cdot m(b) = 1 \Rightarrow m(a)\ne\lambda \end{aligned}$$

即 $m(a)=\lambda \Rightarrow \lambda\in\sigma(a)$

反之, $\Leftarrow$: 见 lecture 23 尾部.

Lecture 23 2022/11/25

Holomorphic calculus. Matrix. M $n\times n$ complex matrics, diaognalable

$M = P\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}P^{-1}$. $f(M)$. $f\in\sigma(\{\lambda_1,\lambda_2,\cdots\lambda_n\})$

这个要求是自然的.

$$f(M) = \frac{1}{2\pi i}\oint_\Gamma f(z)\cdot(z-M)^{-1}\,dz$$

$z-M$ 看作单位矩阵 $zI-M$.

$$= P\begin{pmatrix}f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n)\end{pmatrix}\cdot P^{-1}$$

这个等式是自然的.

$z-M = P\begin{pmatrix}z-\lambda_1 & & \\ & \ddots & \\ & & z-\lambda_n\end{pmatrix}\cdot P^{-1} \Rightarrow (z-M)^{-1} = P\begin{pmatrix}\frac{1}{z-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{z-\lambda_n}\end{pmatrix}\cdot P^{-1}$.

$$\oint_\Gamma f(z)(z-M)^{-1}dz = P^{-1}\begin{bmatrix}\frac{1}{2\pi i}\oint_\Gamma f(z)\cdot(z-\lambda_1)^{-1}dz & & \\ & \ddots & \\ & & \frac{1}{2\pi i}\oint_\Gamma f(z)(z-\lambda_n)^{-1}dz\end{bmatrix}P = P^{-1}\begin{bmatrix}f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n)\end{bmatrix}\cdot P.$$

Commutative Banach algebras.

B complex Banach algebra, (有单位元), commutative.

- Ideal 有吸收性的子空间. (抽象代数意义下).

$I\subset B$ (不要求闭) $a\in I$, $b\in B \Rightarrow ab\in I$.

- Maximal Ideal. $I \subsetneq B$ ideal such that

  $\forall\ J \subsetneq B$ ideal $J \supset I$. then $J = I$.

- Theorem: Any maximal ideal in a commutative Banach algebra B is closed.

Proof: Observation 1: I ideal $\Rightarrow \bar{I}$ is ideal. ($\bar{I}$ 是 I 的闭包).

$\forall\ a \in \bar{I},\ b \in B.\quad ab \overset{?}{\in} \bar{I}$

find $a_n \longrightarrow a.\quad a_n \in I.\quad$ Then $a_n b \in I,\quad a_n b \longrightarrow ab$

$\| a_n b - ab \| \le \| a_n - a \| \| b \| \to 0$

Thus $ab \in \bar{I}$. $\bar{I}$ is an ideal.

Observation 2: 真子理想的闭包仍是真子理想.

$B^\times$ is open $\left(a \in B^\times\quad a+b = a\cdot(1+a^{-1}b) \right.$ 可逆只要 $\|a^{-1}b\| < 1$. 即 $\left. \|b\| < \frac{1}{\|a^{-1}\|} \right)$

I 真子理想, 不包含任何可逆元. $I \subset \underbrace{B \backslash B^\times}_{闭集}$ 故 $\bar{I} \subset B \backslash B^\times \neq B$. $\bar{I}$ 仍是真理想.

If I is maximal ideal. Then $I \subset \bar{I} \subsetneq B$. $\Rightarrow I = \bar{I}$. 即 I is closed

- $\forall$ 真理想. $H \subsetneq B$ 都包含于某个极大理想中.

Theorem: An ideal $I \subsetneq B$ is maximal iff $B = I \oplus \mathbb{C}e$. $(**)$

Pf: Recall: 满足 $(**)$ 的理想一定是闭的. 且是极大的.

Indeed, $\forall\ J \subsetneq B$, ideal $I \subset J \subsetneq B$. 假设 $J \neq I$, then $\exists\ b \in J \backslash I.\quad \underset{J \backslash I}{b} = \underset{I}{a} + \underset{\mathbb{C}}{\lambda e}$

$\Rightarrow e = \frac{1}{\lambda}(b-a) \in J$. 矛盾. 故 $I = J$. I maximal.

Conversely. if I is a maximal ideal. I is closed.

Claim: B/I is a Banach algebra. (代数/理想 = 商代数)

$[a]\cdot[b] := [ab]$ well-defined: $[a] = [a']$, $[b] = [b']$. $\Rightarrow [ab] = [a'b']$.

$a - a' \in I\quad b - b' \in I$. Then $ab - a'b' = \underset{\in I}{ab - a'b + a'b - a'b'}$

$\|[ab]\| \le \|[a]\| \cdot \|[b]\|$

$d(ab, I) \le d(a, I)\cdot d(b, I)$. $\quad ab - \underset{I}{(xb - ay + xy)} = (a-x)\cdot(b-y)$.

$d(ab, I) \le \| ab - xb - ay + xy \| \le \| a - x \| \cdot \| b - y \|$

$\Rightarrow d(ab, I) \le d(a, I)\cdot d(b, I)$

Claim 2: B/I, I maximal ideal, then B/I is a field.

(交换代数 + 任意非零元可逆).

if $[a] \neq [0]$. 即 $a \notin I$. if $[a]$ 不可逆. $\Rightarrow [a] \subset J \subsetneq B/I$

极大理想.

J' 由 a 和 I 生成的理想. 则 $J'/I \subset J$

而 $I \subsetneq J'$, 由 I maximal 知 $J' = B$. $\Rightarrow J = B/I$ 矛盾. 故 $[a]$ 可逆.

claim 3: $I + aB = J$ 是一个理想、且是 a, I 生成的最小理想.

由 claim 3 知 $\exists\ c \in I$, $b \in B$ s.t.

$c + ab = e$. $\Rightarrow [ab] = [e] \Rightarrow [a]\cdot[b] = [b]\cdot[a] = [e]$.

即 B/I is a field.

claim 4: B/I Banach algebra, 又是 field.

则 $B/I \cong \mathbb{C}$. $\forall\ [a] \neq [0]$. $\sigma([a]) \neq \emptyset$.

取 $\lambda \in \sigma([a])$、 则 $[a] - \lambda[e]$ 不可逆 $\Rightarrow [a - \lambda e] = [0]$.

Finally, $I + \mathbb{C}e = B$.

Recall: $\mathcal{M}(B) = \{$multiplicative linear functional on $B\}$
(非 0).

$$\mathcal{M}(B) \xrightarrow{\text{Bijection}} \{\text{极大理想 in } B\}.$$

$$m \longrightarrow \ker m.$$

已证: $m \in \mathcal{M}(B)$. $\|m\| < 1$. $\ker m$ 极大理想 $\Rightarrow$ closed $\Rightarrow$ m 有界线性泛函.

Thm 9.10: $\lambda \in \sigma(a) \Leftrightarrow \exists\ m \in \mathcal{M}(B)$. $\lambda = m(a)$.

Pf: $\lambda \in \sigma(a) \Leftrightarrow a - \lambda e$ 不可逆 $\Leftrightarrow a - \lambda \in I \Leftrightarrow I = \ker m$ for some $m \in \mathcal{M}(B)$.

极大理想

Then $m(a - \lambda e) = 0 \Rightarrow m(a) = \lambda m(e)$. $m(e) \neq 0$. $m(e) = m(e)\cdot m(e)$

$\Rightarrow m(a) = \lambda$. $\Rightarrow m(e) = 1$.

作业. (2) (3).

Joint Spectrum:

预解集

Joint resolvent set: $\forall\ (a_1, \cdots, a_n) \in B^n$,

$$\rho(a_1, \cdots a_n) := \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n \mid \sum_1^n b_k(a_k - \lambda_k) = e\}.$$

$$\sigma(a_1, \cdots, a_n) := \mathbb{C}^n \setminus \rho(a_1, \cdots, a_n).$$

Theorem 9.13: $P \in \mathbb{C}[x_1,\cdots,x_n]$ 多元多项式. $x_ix_j = x_jx_i$. (交换的未知量).

$\forall\ (a_1,\cdots,a_n) \in B^n$. $\quad \mu \in \sigma(P(a_1,\cdots a_n)) \iff \exists\ (\lambda_1\cdots\lambda_n) \in \sigma(a_1,\cdots a_n)$

s.t. $\mu = P(\lambda_1,\cdots,\lambda_n)$.

即 $P(\sigma(a_1,\cdots,a_n)) = \sigma(P(a_1,\cdots,a_n))$ (谱映射定理).

Proof: $\sigma(P(a_1,\cdots,a_n)) = \{ m(P(a_1,\cdots a_n)) \mid m \in M(B) \}$

m可乘. $P(x) = \sum_{\substack{\text{finite}\\ \text{sum}}} a_\alpha x^\alpha$. $\alpha \in \mathbb{N}^n$. $\quad m(P(x)) = \sum_{\substack{\text{finite}\\ \text{sum}}} a_\alpha m(x)^\alpha = P(m(x))$

$\sigma(P(a_1,\cdots,a_n)) = \{ P(m(a_1),\cdots,m(a_n)) \mid m \in M(B) \}$

claim: $\{ (m(a_1),\cdots,m(a_n)) \mid m \in M(B) \} = \sigma(a_1,\cdots,a_n)$. (下次再讲).

半群: Semi-group.

$$\begin{cases} u'(t) = Au(t), & t>0 \quad A \in \mathbb{C} \\ u(0) = u_0. & \end{cases}$$ 常微分方程 $\Rightarrow$ 解得 $u = u_0 e^{At}$

$u: [0,+\infty) \longrightarrow \mathbb{C}$. scalar function

如果考虑, $u: [0,+\infty) \longrightarrow X \leftarrow$ Banach 空间. (向量值函数)

Def: · 连续性: ($\forall\ t_0 \in [0,+\infty)$, $\forall\ \varepsilon>0$, $\exists\ \delta>0$, $|t_0-t|<\delta \Rightarrow \|u(t)-u(t_0)\| < \varepsilon$).

· 可微性: We say u is differentiable at $t_0 \in (0,+\infty)$

if $\quad u(t) = u(t_0) + v(t-t_0) + o(t-t_0)$ as $t \to t_0$. $\quad \lim_{t\to t_0} \frac{o(t-t_0)}{t-t_0} = 0$

$v =: u'(t)$

定义了 $u: [0,+\infty) \longrightarrow X$. 在某一点的导数.

令 $A \in B(X)$, $u_0 \in X$. $\quad \begin{cases} u'(t) = Au(t). & t>0 \\ u(0) = u_0. \end{cases}$

形式上. 我们希望最后的解为 $u(t) = e^{tA}u_0$

几件事情: ① 定义 $e^{tA} \in B(X)$.

② 验证 $e^{tA}u_0$ 满足 ODE

③ 验证 解的唯一性.

$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ 在任何紧集上绝对一致收敛. 我们希望定义 $e^{tA} := \sum_{k=0}^{\infty} \frac{t^kA^k}{k!}$, $A^k \in B(X)$

合理性: $\underbrace{\sum_{k=0}^{\infty} \frac{\|t^kA^k\|}{k!}}_{\text{在 } B(X) \text{ 中依范数收敛.}} \le \sum_{k=0}^{+\infty} \frac{|t^k| \cdot \|A\|^k}{k!} = \exp(|t| \cdot \|A\|) < +\infty$.

特别地，$\|e^{tA}\| \le e^{|t|\,\|A\|}$. $e^{tA} \in B(X)$.

定义 $u(t) = e^{tA}u_0$, $t \in [0,+\infty)$. $u(0) = u_0$. $\frac{d}{dt}(e^{tA}u_0) \overset{?}{=} Ae^{tA}u_0$

$$e^{(t+h)A}u_0 - e^{tA}u_0$$

Lemma: if $B, C \in B(X)$. $BC = CB$. Then $e^{B+C} = e^Be^C = e^C \cdot e^B$

Proof: $e^{B+C} = \sum_{k=0}^{\infty} \frac{(B+C)^k}{k!}$. $e^B \cdot e^C = \left(\sum_{m=0}^{\infty} \frac{B^m}{m!}\right)\left(\sum_{n=0}^{+\infty} \frac{C^n}{n!}\right) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{B^m}{m!} \cdot \frac{C^n}{n!}$

$$\sum_{k=0}^{N} \frac{(B+C)^k}{k!} = \sum_{k=0}^{N} \frac{1}{k!} \cdot \sum_{m=0}^{k} \binom{k}{m} B^m C^{k-m} = \sum_{k=0}^{N}\sum_{m=0}^{k} \frac{B^m}{m!} \cdot \frac{C^{k-m}}{(k-m)!} = \sum_{0 \le m+n \le N} \frac{1}{m!\,n!} B^m C^n$$

$$\sum_{m=0}^{N}\sum_{n=0}^{N} \frac{B^m \cdot C^n}{m!\,n!} - \sum_{m+n \le N} \frac{B^m \cdot C^n}{m!\,n!} \le \sum_{\substack{m+n > N \\ m \le N \\ n \le N}} \frac{1}{m!\,n!} \|B\|^m \cdot \|C\|^n = \sum_{m=0}^{N}\sum_{n=0}^{N} \frac{\|B\|^m}{m!} \cdot \frac{\|C\|^n}{n!} - \sum_{m+n \le N} \frac{\|B\|^m}{m!} \cdots$$

$$\to 0 \quad \text{as } N \to +\infty.$$

Lecture 24 2022/11/26

Joint spectrum $\mathcal{B}$ commutative Banach algebra.

$(a_1 \cdots a_n) \in \mathcal{B}^n$. we call $(\lambda_1 \cdots, \lambda_n) \in \mathbb{C}^n$ belongs to $\rho(a_1, \cdots, a_n)$

if $\sum_1^n b_k(a_k - \lambda_k e) = e$. $\Leftrightarrow$ $\sum_1^n b_k(a_k - \lambda_k e)$ 可逆.

存在某个 $b_1 \cdots b_n$

$\mathbb{C}^n \backslash \rho(a_1, \cdots, a_n) = \sigma(a_1, \cdots, a_n)$ 联合谱.

Lemma: $\sigma(a_1 \cdots, a_n) = \{(m(a_1) \cdots, m(a_n)) : m \in \mathcal{M}(\mathcal{B})\}$

与原定理比较: $\sigma(a) = \{m(a) \mid m \in \mathcal{M}(\mathcal{B})\}$.

Proof: $\lambda \in \sigma(a_1, \cdots, a_n)$. $\Leftrightarrow$ $\forall (b_1, \cdots, b_n) \in \mathcal{B}^n$, $\sum_1^n b_k(a_k - \lambda_k)$ in non-invertible.

$$\Leftrightarrow \left\{\sum_1^n b_k(a_k - \lambda_k) \mid (b_1 \cdots b_n) \in \mathcal{B}^n\right\} \subset \mathcal{B} \backslash \mathcal{B}^\times$$

$$\| $$

$$(a_1 - \lambda_1) \cdot \mathcal{B} + (a_2 - \lambda_2)\mathcal{B} \cdots + (a_n - \lambda_n) \cdot \mathcal{B}.$$

claim: $(a_1 - \lambda_1) \cdot \mathcal{B} + (a_2 - \lambda_2)\mathcal{B} \cdots + (a_n - \lambda_n) \cdot \mathcal{B}$ is an ideal. (Trivial).

任意真理想均包含在某个极大理想中. $\Leftrightarrow$ $(a_1 - \lambda_1)\mathcal{B} + \cdots + (a_n - \lambda_n)\mathcal{B} \subset \ker m$.

即 $m((a_1 - \lambda_1)b_1 + \cdots (a_n - \lambda_n)b_n) = 0$ $\forall (b_1, b_2, \cdots, b_n) \in \mathcal{B}^n$ 成立.

$$\| $$

$$\sum_{k=1}^{n} (m(a_k) - \lambda_k) \cdot m(b_k) = 0. \Leftrightarrow m(a_k) - \lambda_k = 0 \quad \forall k = 1, 2 \cdots n.$$

$\Leftrightarrow$ $(\lambda_1 \cdots, \lambda_n) = m(a_1, \cdots, a_n)$. 对某个 $m \in M(\mathcal{B})$ 成立.

半群 semi-group. (one parameter semi-group).

$B, C \in B(X)$. $e^B e^C = e^{B+C}$. $e^A := \sum_{k=0}^{+\infty} \frac{A^k}{k}$, $A^0 \overset{\text{convention}}{=} I$ $\quad [B,C] := BC - CB. = 0$ 可交换.

$u(t) = e^{tA} u_0$, $t \geq 0$, $u_0 \in X$. $A \in B(X)$.

满足方程 $\begin{cases} u' = Au \\ u(0) = u_0 \end{cases}$

$$u(t+h) - u(t) = e^{(t+h)A} u_0 - e^{tA} u_0$$
$$= e^{tA+hA} u_0 - e^{tA} u_0.$$
$$= (e^{hA} - I) \cdot e^{tA} u_0 = \sum_{k=1}^{n} \frac{A^k h^k}{k!} \cdot e^{tA} u_0$$
$$= h\, A \cdot u(t) + o(\|h\|)$$
$$\Rightarrow u'(t) = A \cdot u(t)$$

Uniqueness:

$\begin{cases} u'(t) = Au(t) \\ u(0) = u_0. \end{cases}$ (*) $A \in B(X)$.

has a solution $u'(t) = e^{tA} u_0$.

下证 ODE has a unique solution.

若 u 与 $\tilde{u}$ 都是 (*) 的解. 则 $v(t) = u(t) - \tilde{u}(t)$. 满足 $\begin{cases} v' = Av \\ v(0) = 0 \end{cases}$

$\Leftrightarrow \begin{cases} e^{-tA}(v' - Av) = 0 \\ v(0) = 0 \end{cases} \overset{\text{形式上}}{\Leftrightarrow} \begin{cases} \frac{d}{dt}(e^{-tA} v) = 0 \\ e^{-tA} v(0) = 0 \end{cases}$

严格化: $e^{-(t+h)A} v(t+h) - e^{-tA} v(t) = e^{-tA} \cdot e^{-hA} \cdot (v(t) + h \cdot v'(t) + o(\|h\|)) - e^{-tA} v(t).$

$$= e^{-tA} \cdot (e^{-hA} - I) \cdot v(t) + h \cdot e^{-tA} \cdot e^{-hA} \cdot v'(t) + o(\|h\|)$$
$$= h\, e^{-tA} (v'(t) - Av(t)) + o(h).$$

即 $\frac{d}{dt}(e^{-tA} v) = e^{-tA} \cdot (v' - Av)$.

令 $w(t) = e^{-tA} v(t)$.

$\begin{cases} w' = 0 \\ w(0) = 0 \end{cases}$ 任取 $\ell \in X^*$.

$w_\ell(t) := \ell(w(t))$

则 $w_\ell'(t) = \ell(w'(t))$

$\Rightarrow \begin{cases} w_\ell'(t) = 0 \\ w_\ell(0) = 0 \end{cases} \Rightarrow w_\ell(t) \equiv 0.\ \forall \ell \in X^*. \Rightarrow w(t) \equiv 0.$

故解是唯一的.

$u: [t_0, t_1] \longrightarrow X$. continuous function

we can define Riemann Integral. $\int_{t_0}^{t_1} u(s)\, ds := \lim_{\delta(p) \to 0} \sum_{i=1}^{n} u(s_i') \cdot (s_i - s_{i-1})$.

$P = \{t_0 = s_0 < s_1 \cdots < s_n = t_1\}$

$s_i' \in [s_{i-1}, s_i]$

作为范数意义下收敛.

important

$$\left\| \int_{t_0}^{t_1} u(s) ds \right\| \leq \int_{t_0}^{t_1} \| u(s) \| ds$$

Newton - Lebniz formulas: $\int_{t_0}^{t_1} v'(s)\, ds = v(t_1) - v(t_0)$

Unbounded operators $\rightsquigarrow$ 诱导的半群.

Densely defined closed operators

X Banach spaces.

$$\begin{array}{c} X \\ \text{dense} \ \cup \\ D(A) \xrightarrow{A} X \\ \text{closed operator.} \end{array}$$

Def: $\lambda \in \rho(A) \Leftrightarrow D(A) \xrightarrow[\text{bijection}]{\lambda - A} X$

$(\lambda - A)^{-1} \in B(X)$

$\operatorname{Im}[(\lambda - A)^{-1}] = D(A)$.

Thm 10.1: Condition: A densely defined closed operator on X.

$$\begin{array}{c} X \\ \text{dense} \ \cup \\ D(A) \xrightarrow[\text{closed}]{A} X \end{array}$$

if $\rho(A) \supset [b, +\infty)$ for a fixed $b \geq 0$.

s.t. $\forall \lambda \in [b, +\infty)$. $\|(\lambda - A)^{-1}\| \leq \frac{1}{a + \lambda}$ for a fixed constant $a \in \mathbb{R}$.

Then $\exists \{E_t\}_{t \geq 0} \subset B(X)$ s.t.

(想象中 e^{tA}).

(a) $E_t E_s = E_{t+s}$ $\qquad R_+ \xrightarrow{\text{半群同态}} B(X)$

(b) $E_0 = I$

(c) $\|E_t\| \leq e^{-at}$, $\forall t \geq 0$.

(d) $t \longmapsto E_t x$ is continuous on $[0, +\infty)$ for any $x \in X$.

(strongly continuous)

SOT continuous strong operator topology.

(e) $x \in D(A)$. $t \longmapsto E_t x$ is differential. $\frac{d}{dt} E_t x = A \cdot E_t x$

(f) E_t 与 $(\lambda - A)^{-1}$ 可交换, $\forall \lambda \geq b$, $\forall t \geq 0$

Remark:

我们记上述 E_t 为 e^{tA}

$u(t) = e^{tA} u_0$ if $u_0 \in D(A)$.

$u(t) \in D(A)$

满足 $\begin{cases} u' = Au \\ u(0) = u_0 \end{cases}$

$\frac{d}{dt}(e^{tA} u_0) = A \cdot (e^{tA} u_0)$, $E_t: D(A) \to D(t)$.

Lemma: If $\{B_\lambda\}_{\lambda \geq K} \subset B(X)$.

① $\|B_\lambda\| \leq M < +\infty$, $\forall \lambda \geq K$. for a fixed $M > 0$.

② fix $D \overset{\text{dense}}{\subset} X$. 已知 $\lim_{\lambda \to \infty} B_\lambda x$ 收敛 (依范数).

Then 存在 $B \in B(X)$. s.t. $Bx = \lim_{a \to \infty} B_\lambda x$.

Proof: $\forall x \in X$. $\forall \varepsilon > 0$. $\exists \tilde{x} \in D$. $\|x - \tilde{x}\| < \varepsilon$.

$$\|B_\lambda x - B_\mu x\| \leq \|B_\lambda x - B_\lambda \tilde{x} + B_\lambda \tilde{x} - B_\mu \tilde{x} + B_\mu \tilde{x} - B_\mu x\|$$

$$\leq \|B_\lambda\| \cdot \|x - \tilde{x}\| + \|B_\lambda \tilde{x} - B_\mu \tilde{x}\| + \|B_\mu\| \cdot \|x - \tilde{x}\| \leq 2M\varepsilon + \|B_\lambda \tilde{x} - B_\mu \tilde{x}\|.$$

存在 N 足够大, $\forall \lambda, \mu \geq N$, $\|B_\lambda \tilde{x} - B_\mu \tilde{x}\| < \varepsilon \Rightarrow \|B_\lambda x - B_\mu x\| \leq (2M+1)\varepsilon$.

Pf of Thm: $A \xleftarrow[\lambda\to\infty]{\text{逼近}} A_\lambda = \lambda A(\lambda-A)^{-1}$, $\lambda \geq b$. $[b,+\infty) \subset \rho(A)$.

$\boxed{\text{先假设 } a>0}$

$e^{tA_\lambda} \xrightarrow{\lambda\to\infty} E_t$ 某个有界算子.

验证 E_t 满足我们的条件.

① $\lambda A(\lambda-A)^{-1} \in B(X)$

$\lambda A(\lambda-A)^{-1} = \lambda[(A-\lambda)+\lambda]\cdot(\lambda-A)^{-1} = -\lambda + \lambda^2\cdot\underline{(\lambda-A)^{-1}}$
有界算子.

$$\begin{tikzcd} X \arrow[r, "(\lambda-A)^{-1}"] \arrow[rr, bend right, "I"'] & D(A) \arrow[r, hook, "\lambda-A"] & X \end{tikzcd}$$

② $tA_\lambda = t\lambda A(\lambda-A)^{-1}$. e^{tA_λ} 可以定义 $e^{tA_\lambda} = \sum_{k=0}^{+\infty} \frac{(tA_\lambda)^k}{k!}$

$\|e^{tA_\lambda}\| \leq \exp\left(\frac{-at\lambda}{a+\lambda}\right)$, $\forall\ t\geq 0$, $\lambda \geq b$.

$e^{tA_\lambda} = e^{-t\lambda I + t\lambda^2(\lambda-A)^{-1}} = e^{-t\lambda I}\, e^{t\lambda^2(\lambda-A)^{-1}} = e^{-t\lambda}\cdot e^{t\lambda^2(\lambda-A)^{-1}}$

$\Rightarrow \|e^{tA_\lambda}\| \leq e^{-t\lambda}\exp\left(t\lambda^2\|(\lambda-A)^{-1}\|\right) \leq e^{-t\lambda}\exp\left(t\lambda^2\cdot\frac{1}{\lambda+a}\right) = \exp\left(t\,\frac{-\lambda^2-\lambda a}{\lambda+a} + t\,\frac{\lambda^2}{\lambda+a}\right)$.

$= \exp\left(-\frac{t\lambda a}{\lambda+a}\right)$.

③ $A_\lambda x \xrightarrow{\lambda\to+\infty} Ax$. $\forall\ x\in D(A)$.

$A_\lambda = \lambda A\cdot(\lambda-A)^{-1}$, $\lambda\cdot(\lambda-A)^{-1} = (\lambda-A+A)(\lambda-A)^{-1} = I + A\cdot(\lambda-A)^{-1}$

$\Rightarrow \|A\cdot(\lambda-A)^{-1}\| \leq 1 + \lambda\|(\lambda-A)^{-1}\| \leq 1 + \lambda\cdot\frac{1}{\lambda+a} \leq 2$.

一致有界.线性算子. $x\in D(A)$.

$A\cdot(\lambda-A)^{-1}x \overset{?}{=} (\lambda-A)^{-1}Ax$ $\qquad D(A) \xrightarrow{A-\lambda} X \xrightarrow{(\lambda-A)^{-1}} D(A)$

$(A-\lambda+\lambda)(\lambda-A)^{-1} = -I + \lambda(\lambda-A)^{-1}$

故 $\forall\ x\in D(A)$.

$(\lambda-A)^{-1}\cdot(A-\lambda+\lambda) = -I_{D(A)} + \lambda(\lambda-A)^{-1}$ $\qquad A(A-\lambda)^{-1}x = (A-\lambda)^{-1}Ax$

$\Rightarrow \|A(\lambda-A)^{-1}x\| = \|(\lambda-A)^{-1}Ax\| \leq \|(\lambda-A)^{-1}\|\cdot\|Ax\| \leq \frac{\|Ax\|}{\lambda+a}$

$\Rightarrow A(\lambda-A)^{-1}x \xrightarrow{\lambda\to+\infty} 0$. $\forall\ x\in D(A)$.

即 $-x + \lambda(\lambda-A)^{-1}x \xrightarrow{\lambda\to+\infty} 0$ $\qquad \sup_{\lambda\geq b}\|\lambda(\lambda-A)^{-1}\| < +\infty$

即 $\lambda(\lambda-A)^{-1}x \xrightarrow{\lambda\to+\infty} x$. $x\in \underline{D(A)}$ densely.

特别地, Ax, $x\in D(A)$. $\lambda(\lambda-A)^{-1}Ax \xrightarrow{\lambda\to\infty} Ax$

$\| \ \lambda\cdot A\cdot(\lambda-A)^{-1}x = A_\lambda x$

Lecture 25   2022/11/30

补充说明了解析演算 $\mathrm{Hol}(\sigma(a)) \longrightarrow B.$
$f \longmapsto f(a)$

$f\in \mathrm{Hol}(\sigma(a)) \subset \mathbb{C}$ 是一般的全纯函数于 $\sigma(a)$ 的某邻域 U

注意 $\sigma(a)$ 是紧的.

Proposition: $B \ni a \longmapsto \sigma(a)\subset \mathbb{C}$. 上述映射是关于 Hausdorff distance 连续的.

即 $\forall a\in B,\ \forall \varepsilon>0,\ \exists \delta>0$

$\|a-b\|<\delta \Rightarrow d_H(\sigma(a),\sigma(b))<\varepsilon.$ 即 $\sigma(b)\subset B(\sigma(a),\varepsilon)$,
$\sigma(a)\subset B(\sigma(b),\varepsilon)$.

课上不作证明.

$f\in \mathrm{Hol}(\sigma(a))$ 指 f 在 $\sigma(a)$ 的开邻域上全纯.

当 $b\to a$ 时. $\sigma(b)\subset B(\sigma(a),\varepsilon) \Rightarrow f\in \mathrm{Hol}(\sigma(b))$.

可以定义 $f(a)$. 也可以定义 $f(b)$. That is, $\forall f\in \mathrm{Hol}(\sigma(a))$.

f 可定义于 $B(a,\varepsilon)\subset B$. $f: B(a,\varepsilon)\longrightarrow B$, $\forall b\in B(a,\varepsilon)$.
$b \longrightarrow f(b)$

这里 $f(b)$ 仅仅是记号.

$f(b) := \frac{1}{2\pi i}\oint_\Gamma f(z)\cdot(z-b)^{-1}dz.$

严格来记 $\mathrm{Hol}(\sigma(a)) \longrightarrow B.$
$f \longrightarrow t_a(f).$

为了方便, 我们记为 $f(a)$.

怎样理解:

$(f, a) \longmapsto f(a)$.

看作双变元函数

固定 a 就是解析演算.

固定 f, 就是 $B(a,\varepsilon)\longrightarrow B$ 上一个函数.

对于 $\forall f\in \mathrm{Hol}(\sigma(a))$. 存在 $\varepsilon>0$.

$B(a,\varepsilon) \longrightarrow B.$
$b \longrightarrow t_b(f).$
可以定义.

半群的生成元. X Banach space

Def: ① $\{E_t\}_{t\geq 0} \subset B(X)$ （即 R_+ 参数化的线性算子）为单参数半群.（Semi-group）.

if $\begin{matrix} R_+^{\text{加法}} & \longrightarrow & B(X)^{\text{乘法}} \\ t & \longmapsto & E_t \end{matrix}$ is a homomorphism. 即 $\begin{cases} E_t \cdot E_s = E_{t+s} & s,t \geq 0. \\ E_0 = I. \end{cases}$

Remark: one-parameter semi-group 是交换算子构成的.

② $\{E_t\}_{t\geq 0}$ a semi-group is called strongly continuous (SOT-continuous)

if $\forall x \in X$. the map. $R_+ \ni t \longmapsto E_t x \in X$. is continuous.

③ A densely defined linear operator is called the generator of the semi-group.

If $\boxed{(E_t x)' = A \cdot E_t x \ . \ \forall x \in D(A).}$ 我们记 $E_t = e^{tA}$ 形式的记号.

Thm. If A is a densely defined closed linear operator.

$$\begin{matrix} & X & \\ \text{dense} \cup & & \\ D(A) & \xrightarrow[A]{\text{closed}} & X \end{matrix}$$

s.t. $\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda + a}$

$\forall \lambda \geq b$ for some fixed $b \geq 0$. $a \in \mathbb{R}$.

则 A 是某个 strongly continuous one-parameter semi-group. 的生成元.

且满足 $\|e^{tA}\| \leq e^{-at}$. $t \geq 0$. $[e^{tA}. (\lambda - A)^{-1}] = 0$. (可交换).

proof:

$$\begin{matrix} A_\lambda = \lambda A(\lambda - A)^{-1} & \longrightarrow & e^{tA_\lambda} \\ \Big\downarrow \text{逼近 } \lambda \to \infty. & \longrightarrow & \Big\downarrow \\ A. & & e^{tA} \end{matrix}$$

① $A_\lambda = \lambda A(\lambda - A)^{-1} = \lambda(A - \lambda + \lambda)(\lambda - A)^{-1} = \underbrace{-\lambda + \lambda^2(\lambda - A)^{-1}}$ 有界线性算子.

② $\|e^{tA_\lambda}\| \leq e^{-\lambda t} e^{t\lambda^2 \|(\lambda - A)^{-1}\|} \leq e^{-\lambda t} \exp(t\lambda^2 \frac{1}{\lambda + a}) = \exp(-\frac{at\lambda}{a + \lambda})$.

③ $A_\lambda x \xrightarrow{\lambda \to +\infty} Ax$. $\forall x \in D(A)$.

$\lambda(\lambda - A)^{-1} = (\lambda - A + A)(\lambda - A)^{-1} = I + \underbrace{A(\lambda - A)^{-1}}$ (一致有界).

$\|\lambda(\lambda - A)^{-1}\| \leq \frac{\lambda}{\lambda + a} \leq 1$.

$\Rightarrow$ $A(\lambda-A)^{-1}x = (\lambda-A)^{-1}Ax$. if $x\in D(A)$.

$$\begin{array}{ccc} A(\lambda-A)^{-1}x & & (\lambda-A)^{-1}Ax \\ \| & & \| \\ (\lambda(\lambda-A)^{-1}+I)x & \xrightarrow{\text{相等}} & (\lambda(\lambda-A)^{-1}+I_{D(A)})x. \end{array}$$

$$\|A(\lambda-A)^{-1}x\| = \|(\lambda-A)^{-1}Ax\| \le \frac{\|Ax\|}{\lambda+a} \xrightarrow{\lambda\to+\infty} 0.$$

$x\in D(A)$ 时.

$$\lambda(\lambda-A)^{-1}x = x + \underbrace{A(\lambda-A)^{-1}x} \xrightarrow{\lambda\to\infty} x.$$

而 $\lambda(\lambda-A)^{-1}$ 一致有界. 由引理知 $\lim\limits_{\lambda\to\infty}\lambda(\lambda-A)^{-1}x = x$. $\forall x\in X$.

特别地. $\forall Ax\in X$. $\lim\limits_{\lambda\to\infty}\lambda(\lambda-A)^{-1}Ax = Ax$. 即 $\lim\limits_{\lambda\to\infty} A_\lambda x = Ax$.

④ $e^{tA_\lambda}x$ 收敛到某个极限. $(\lambda\to+\infty)$. $\forall x\in X$.

而 $\|e^{tA_\lambda}\| \le \exp\left(-\frac{at\lambda}{a+\lambda}\right) \le 1$. $\Rightarrow$ e^{tA_λ} 是一致有界的.

只需证 Dense subset 有极限. 下证 $\{e^{tA_\lambda}x\}$ 是 Cauchy 列.

$$e^{tA_\lambda}x - e^{tA_\mu}x = \boxed{e^{stA_\lambda+(1-s)tA_\mu}x}\Big|_{s=0}^{1} = e^{s(tA_\lambda-tA_\mu)+tA_\mu}x\Big|_{s=0}^{1}$$

$v(s)$

$$= \int_0^1 \frac{d}{dt}\left[e^{s(tA_\lambda-tA_\mu)+tA_\mu}x\right]ds.$$

claim: A_μ 与 A_λ 可交换. $\lambda,\mu\ge b$.

由于 $A_\mu = \mu A(\mu-A)^{-1} = -\mu I + \mu^2(\mu-A)^{-1}$ 只需验证 $(\mu-A)^{-1}$ 与 $(\lambda-A)^{-1}$ 可交换.

A_λ — — — $= -\lambda I + \lambda^2(\lambda-A)^{-1}$. $(\mu-A)^{-1} = (\mu-\lambda+\lambda-A)^{-1}$.

Lemma: If C densely defined closed invertible operator. $\forall k\in\mathbb{C}$. C^{-1} 与 $(k+C)^{-1}$ 可交换.

$$\begin{array}{l} X \\ \cup \\ D(C) \xrightarrow[\text{closed}]{C} X. \end{array}$$

先承认. 后续补上证明.

$$e^{tA_\lambda}x - e^{tA_\mu}x = \int_0^1 \frac{d}{ds}\left(e^{s(tA_\lambda-tA_\mu)}e^{tA_\mu}x\right)ds = \int_0^1 (tA_\lambda-tA_\mu)e^{s(tA_\lambda-tA_\mu)}e^{tA_\mu}\,ds.$$

$\|$

$v(1)-v(0)$

$$= t(A_\lambda-A_\mu)\cdot\int_0^1 e^{stA_\lambda+(1-s)tA_\mu}x\,ds = t(A_\lambda-A_\mu)\cdot\int_0^1 V_s(x)\,ds$$

$$= \boxed{t\int_0^1 V_s(A_\lambda-A_\mu)(x)\,ds} \quad (*)$$

$$V_s = e^{stA_\lambda+(1-s)tA_\mu}.$$

$$= \exp\left[st(-\lambda+\lambda^2(\lambda-A)^{-1}] + s(1-t)\cdot[-\mu+\mu^2(\mu-A)^{-1}]\right]$$

$\Rightarrow \|V_s\| \le e^{-st\lambda-(1-s)t\mu} \cdot \exp\left(st\lambda^2 \cdot \frac{1}{\lambda+a}\right) \cdot \exp\left(\frac{(1-s)t\mu^2}{\mu+a}\right),$

$= \exp\left(-\left(\frac{st\lambda a}{\lambda+a} + \frac{(1-s)t\mu a}{\mu+a}\right)\right) \le 1.$

故 $\|e^{tA_\lambda} - e^{tA_\mu} x\| \le t\int_0^1 \|(A_\lambda - A_\mu)x\| ds = t\cdot\|(A_\lambda - A_\mu)\cdot x\|.$

而 $A_\lambda x \xrightarrow[\lambda\to\infty]{x\in D(A)} Ax.$. Thus, $\{A_\lambda x\}_{\lambda\ge b}$ is a cauchy seq $\Rightarrow e^{tA_\lambda}x$ is Cauchy seq. □.

下面记 $E_t x = \lim_{\lambda\to\infty} e^{tA_\lambda}x$ $\|E_t\| \le \lim_{\lambda\to\infty}\|e^{tA_\lambda}\| \le \lim_{\lambda\to\infty}\exp\left(-\frac{at\lambda}{\lambda+a}\right) = e^{-at}$

Claim: $\{E_t\}$ is an one-parameter semi-group generated by A.

1° $E_{s+t} = E_s\cdot E_t$. $E_t x = \lim_{\lambda\to\infty} e^{t\lambda A} x$.

$e^{(s+t)A_\lambda} x = e^{sA_\lambda}\cdot e^{tA_\lambda}x$, 当 $\lambda\to\infty$ 时. 右式 $\xrightarrow{??} E_s\cdot E_t x$.

$\|e^{sA_\lambda}\cdot e^{tA_\lambda}x - E_s\cdot E_t x\| \le \|e^{sA_\lambda}\cdot e^{tA_\lambda}x - e^{sA_\lambda}\cdot E_t x\| + \|e^{sA_\lambda}\cdot E_t x - E_s\cdot E_t x\|.$

$\le \underbrace{\|e^{sA_\lambda}\|}_{\le \exp\left(-\frac{as\lambda}{\lambda+a}\right)\le 1.}\cdot\|e^{tA_\lambda}x - E_t x\| + \|e^{sA_\lambda}(E_t x) - E_s(E_t x)\|.$

$\xrightarrow{\lambda\to\infty} 0.$

2°. $\forall x\in X$ $t \longmapsto E_t x$ 是连续的 (SOT-continuous).

$e^{tA_\lambda}x$ 关于 t 连续 (关于 λ 一致?).

$\|e^{tA_\lambda}x - e^{t_0A_\lambda}x\| \le \int_{t_0}^t \|e^{sA_\lambda}A_\lambda x\| ds \le \int_{t_0}^t \|A_\lambda x\| ds = \|A_\lambda x\|\cdot(t-t_0)$

而 $\|A_\lambda x\| \longrightarrow \|Ax\|$ $\Rightarrow$ $\|A_\lambda x\|$ 关于 λ 一致有界.

所以 $\forall x\in D(A)$, $\{e^{tA_\lambda}x\}_{\lambda\ge b}$ 是关于 t 一致连续.

$\Rightarrow E_t x \xrightarrow{t\to t_0} E_{t_0}x$, $x\in D(A)$. 但是 $\|E_t\| \le e^{-at}$ 一致有界.

$\Rightarrow E_t x \xrightarrow{t\to t_0} E_{t_0}x$. $\forall x\in X$. 所以 $\{E_t\}_{t\ge 0}$ is strongly continuous.

3° claim: E_t 的生成元 (generator) 是 A.

$e^{tA_\lambda}x - e^{t_0A_\lambda}x = \int_{t_0}^t \underbrace{e^{sA_\lambda}\cdot A_\lambda x}_{\downarrow\ E_s A_\lambda x.} ds$ 考虑 $\lim_{\lambda\to\infty}\int_{t_0}^t e^{sA_\lambda}\cdot A_\lambda x\, ds$ 能否交换极限与积分.

$\downarrow \lambda\to\infty$

$E_t x - E_{t_0}x.$

if $x\in D(A)$, $A_\lambda x \xrightarrow{\lambda\to+\infty} Ax$.

$\lim_{\lambda\to\infty} \int_{t_0}^{t} e^{sA_\lambda} A_\lambda x\,ds = \int_{t_0}^{t} E_s A x\,ds. \quad \forall x \in D(A)$. 利用控制收敛定理.

$\| e^{sA_\lambda} A_\lambda x \| \le \underbrace{\| e^{sA_\lambda}\|}_{\le 1} \cdot \underbrace{\| A_\lambda x\|}_{一致有界}$.

故 $E_t x - E_{t_0} x = \int_{t_0}^{t} E_s A x\,ds. \Rightarrow \left.\frac{dE_t x}{dt}\right|_{t=t_0} = \lim_{t\to t_0} \frac{E_t x - E_{t_0} x}{t-t_0} = \lim_{t\to t_0} \frac{\int_{t_0}^{t} E_s A x\,ds}{t-t_0}$

由于 $s \longmapsto E_s A x$ is continuous

希望 $\left.\frac{dE_t x}{dt}\right|_{t=t_0} = A E_{t_0} x$.

Finally: e^{tA_λ} 与 $(\lambda-A)^{-1}$ 可交换. $\lambda \ge b$.

$$\sum_{k=0}^{\infty} \frac{t^k A_\lambda^k}{k!} (\lambda-A)^{-1} \overset{?}{=} (\lambda-A)^{-1} \cdot \sum_{k=0}^{+\infty} \frac{t^k A_\lambda^k}{k!}.$$

只需验证 $A_\lambda^k (\lambda-A)^{-1} = (\lambda-A)^{-1} \cdot A_\lambda^k$. 而 $A_\lambda = -\lambda + \lambda^2(\lambda-A)^{-1}$ $\Downarrow$ 成立.

$e^{tA_\lambda}(\lambda-A)^{-1} = (\lambda-A)^{-1} e^{tA_\lambda}$.

Claim: e^{tA_λ} 与 $(\mu-A)^{-1}$ 可交换.

只需证 A_λ^k 与 $(\mu-A)^{-1}$ 可交换

$\Longleftrightarrow \underbrace{[(\lambda-A)^{-1}, (\mu-A)^{-1}] = 0}_{见预解式引理}$

$$\begin{array}{ccc} e^{tA_\lambda} \cdot (\mu-A)^{-1} x & = & (\mu-A)^{-1} \cdot e^{tA_\lambda} x \\ \downarrow \lambda\to\infty & & \downarrow \lambda\to\infty \\ E_t (\mu-A)^{-1} x & = & (\mu-A)^{-1} E_t x \end{array}$$

Then $E_t A x = A E_t x. \quad \forall x \in D(A)$.

$$A E_t x = b E_t x - (b-A) E_t x = b E_t x - (b-A) \cdot E_t (b-A)^{-1} (b-A) x$$
$$= b E_t x - (b-A) \cdot (b-A)^{-1} E_t (b-A) x = b E_t x - E_t (b-A) x = E_t A x.$$

Lemma: $X \supset_{\text{dense}} D(B) \xrightarrow[\text{closed}]{B} X$. B^{-1} 有界. $(k+B)^{-1}$ 有界

$B^{-1}(k+B)^{-1} x = (k+B)^{-1} \cdot B^{-1} x$.

$X \xrightarrow{B^{-1}} D(B) \xrightarrow{(k+B)^{-1}|_{D(B)}} D(B)$

$X \xrightarrow{(k+B)^{-1}} D(B) \xrightarrow{B^{-1}|_{D(B)}} D(B)$.

事实上.

$$\begin{aligned} (k+B)^{-1} - B^{-1} &= (k+B)^{-1} \cdot [-k I_{D(B)}] \cdot B^{-1} \quad (= (k+B)^{-1} \cdot B^{-1}) \\ &= B^{-1} \cdot [-k \cdot I_{D(B)}] \cdot (k+B)^{-1} \quad (= B^{-1} \cdot (k+B)^{-1}) \end{aligned}$$

Lecture 26 2022/12/2

Thm 10.1 的最后部分.

我们之前假设 $a>0$. 现考虑一般情况. $(a\le 0)$.

$$\|(\lambda-A)^{-1}\|\le \frac{1}{\lambda+a} \quad \text{for} \quad \lambda\in[b,+\infty).$$

define $B = A+a-1$ $D(B)=D(A)$ is densely defined.

A is closed $\Rightarrow$ B is closed

事实上 $\begin{cases} x_n \longrightarrow x \\ \underbrace{Bx_n \rightarrow y} \end{cases} \Leftrightarrow Ax_n+(a-1)x_n \rightarrow y \Rightarrow Ax_n \longrightarrow y+(1-a)x.$

$\Rightarrow$ $x\in D(A)$, $Ax = y+(1-a)x$

$\Rightarrow Bx = y.$ Thus B is closed

$\|(\lambda-A)^{-1}\|\le\frac{1}{\lambda+a}$, $B=A+a-1$. $A=B+1-a$.

$\Updownarrow$

$\|(\lambda-B-1+a)^{-1}\|\le\frac{1}{\lambda+a}$. 令 $\lambda'=\lambda-1+a$. $\lambda=\lambda'+1-a$.

Then $\|(\lambda'-B)^{-1}\|\le\frac{1}{\lambda'+1}$.

由已证结果知 $B \xrightarrow[\text{semigroup}]{\text{one parameter}} F_t$ ("$=e^{tB}$")

$e^{tB} = e^{t(A+a-1)} = e^{tA}\cdot e^{t(a-1)}$ 现在令 $E_t = e^{t(1-a)}F_t$.

则 $\frac{dE_tx}{dt} = \frac{d}{dt}\left(e^{t(1-a)}F_tx\right) = (1-a)e^{t(1-a)}F_tx + e^{t(1-B)}\cdot BF_tx = AE_tx.$

即 E_t has A as a generator.

$E_t\cdot E_s = E_{t+s}$ (显然).

$\|E_t\|\le e^{t(1-a)}\cdot\|F_t\|\le e^{t(1-a)}\cdot e^{-t} = e^{-at}.$

Strongly continuous one-parameter semi-group

$\downarrow$ 称为

C_0 - semigroup. (Hille 的术语).

A generator (满足 Thm 10.1 的条件) $\longrightarrow$ C_0 - semigroup E_t

我们记 $E_t = e^{tA}$

Thm 10.3. $\forall$ C_0 - semigroup 都存在一个 generator (当然是唯一的).

$(E_t x)' = A E_t x$

是唯一的.

Thm. $\forall$ C_0-SG is uniformly determined by its generator.

(所以我们可以记 $A \xrightarrow[\text{生成的}]{\text{记作}} e^{tA}$)

Proof. 假设 $\{E_t\}_{t\geq 0}$, $\{F_t\}_{t\geq 0}$ two C_0-SG s.t.

$$\begin{cases} \dfrac{dE_t x}{dt} = AE_t x = E_t A x. & \forall x \in D(A) \\ \dfrac{dF_t x}{dt} = AF_t x = F_t A x. & \forall x \in D(A) \end{cases}$$

$\forall t>0$, 考虑 $0 \leq s \leq t$. 定义 $w(s) = E_s \cdot F_{t-s}$ $\begin{cases} w(0) = F_t \\ w(t) = E_t. \end{cases}$

故 $\forall x \in D(A)$.

$$\frac{d}{ds}[w(s)x] = \frac{d}{ds}[E_s F_{t-s} x] = E_s A \cdot F_{t-s} x - E_s A F_{t-s} x = 0$$

故 $w(s)x$ = constant vector $0 \leq s \leq t$. $\forall x \in D(A)$.

但由于 $w(s)$ 是 $[0,t]$ 上一致有界的算子 (马上就会证明).

$\Rightarrow$ $w(s)x$ = constant vector, $0 \leq s \leq t$. $\forall x \in X$.

$\Rightarrow$ $w(0)x = w(t)x$ 即 $F_t x = E_t x$, $\forall x \in X$.

Proof of Thm 10.3 : (生成元的存在性). 已知 $\{E_t\}_{t\geq 0}$ C_0-SG.

要证 $E_t = e^{tA}$. A densely defined closed operator.

Define $\forall h>0$, $A_h = \dfrac{E_h - I}{h}$.

我们希望 A_h 在某种意义下的 $(h\to 0^+)$ 极限就是我们想要的 A.

Define $D(A) := \{x \in X : \lim\limits_{h\to 0^+} (\frac{E_h - I}{h})x$ 存在 $\}$.

$A: D(A) \longrightarrow X$. $Ax := \lim_{h\to 0^+} A_h x$.

需要验证: ① $D(A)$ is dense. , A is closed

② A is indeed the generator.

(直观来看. $\forall x\in X$. 作光滑化).

$\forall x\in X$, $s>0$. define $x_s := \frac{1}{s}\int_0^s E_t x\,dt$.

$W := \{ x_s = \frac{1}{s}\int_0^s E_t x\,dt.\ x\in X \}$.

Claim 1: $x_s \xrightarrow{s\to 0^+} x$. Then W is dense.

$\|x_s - x\| \le \frac{1}{s}\int_0^s \underbrace{\|E_t x - x\|}_{\|E_t x - x\| \to 0 \text{ as } t\to 0} dt \xrightarrow{s\to 0^+} 0$

Claim 2: $W \subset D(A) \subset X$.

$A_h x_s = A_s x_h$. $\forall s, h > 0$

Let $h\to 0^+$. Then $x_h \longrightarrow x$.

A_s 有界算子. Then $A_s x_h \xrightarrow{h\to 0^+} A_s x$.

subclaim: $\forall h, s > 0$, $A_h x_s = A_s x_h$

$A_h x_s = \frac{E_h - I}{h}.\ \frac{1}{s}\int_0^s E_t x\,dt = \frac{\int_h^{s+h} E_t x\,dt - \int_0^s E_t x\,dt}{sh}$

$A_s x_h = \left(\int_s^{s+h} E_t x\,dt - \int_0^h E_t x\,dt\right)\cdot\frac{1}{sh}$ //

从而 $\lim_{h\to 0^+} A_h x_s$ 存在. 故 $x_s \in D(A) \Rightarrow W\subset D(A)$. $D(A)$ is dense.

and $Ax_s = A_s x$.

下证 A is a closed operator.

Claim 3: $(E_t x)' = A\cdot E_t x = E_t A x$. $\forall x\in D(A)$.

Indeed, if $h>0$. $\frac{(E_{t+h} - E_t)x}{h} = \frac{E_t\cdot(E_h - I)x}{h} = E_t A_h x \to E_t Ax$.

$= A_h E_t x \Rightarrow \lim_{h\to 0^+} A_h E_t x$ 存在.

根据 $D(A)$ 的定义, $E_t x \in D(A)$. 故 $\lim_{h\to 0^+} A_h E_t x = A E_t x = E_t A x$.

考虑左导数. $\frac{(E_t - E_{t-h})x}{h} = E_{t-h}\cdot\frac{E_h - I}{h}x = E_{t-h}A_h x = A_h E_{t-h} x$.

希望 $\lim_{h\to 0^+} E_{t-h} A_h x = E_t A x$.

$\|E_{t-h}A_h x - E_t A x\| \le \underbrace{\|E_{t-h}(A_h - A)x\|} + \underbrace{\|(E_{t-h} - E_t)Ax\|} \longrightarrow 0$

$\le \underbrace{\|E_{t-h}\|}_{一致有界}\cdot \underbrace{\|A_h x - Ax\|}_{\to 0^+}$

所以. 固定 $T>0$.

uniform bddness principle. $\{E_t x\}_{t\ge 0}$ 连续. $\forall x\in X$. $\{E_t x : 0\le t\le T\}$ 有界.

$\sup_{0\le t\le T}\|E_t x\| < \infty. \Rightarrow \sup_{x\in X}\sup_{0\le t\le T}\|E_t x\| < +\infty.$ 即 $\sup_{0\le t\le T}\|E_t\| < +\infty.$

C_0-SG 在任一有限区间内一致有界.

Claim 4. A is closed operator.

Lemma: $\forall\, z\in D(A).\ A_s z = A z_s = (Az)_s.$

而 $A\cdot \frac{1}{s}\int_0^s E_t z\,dt \underset{(*)}{\overset{?}{=}} \frac{1}{s}\int_0^s E_t A z\,dt. \Rightarrow A z_s = (Az)_s$

Remark: Since A is not necessarily bnd. we don't know $(*)$

$$A_s z_h = A_h z_s = \underbrace{\frac{E_h - I}{h}}_{\text{有界算子}}\cdot \frac{1}{s}\int_0^s E_t z\,dt = \frac{1}{s}\int_0^s E_t A_h z\,dt.$$

$\Big\downarrow h\to 0^+$ 　　　　　　$\Big\downarrow h\to 0^+$ 控制收敛.

$A z_s$ 　　　　　　$\frac{1}{s}\int_0^s E_t A z\,dt = (Az)_s.$

现在任取 $\{x^{(n)}\}\subset D(A)$. $\begin{cases} x^{(n)} \xrightarrow[\text{in } X]{n\to\infty} x. \\ Ax^{(n)} \xrightarrow{n\to\infty} y. \end{cases} \Rightarrow \begin{cases} x^{(n)}{}_s \longrightarrow x_s. \\ (Ax^{(n)})_s \longrightarrow y_s. \end{cases}$

$\forall s,\ \underbrace{x\mapsto x_s}_{\text{有界线性算子}} = \frac{1}{s}\int_0^s E_t x\,dt \Rightarrow \|x_s\| \le \frac{1}{s}\int_0^s \|E_t\|\cdot\|x\|\,dt.$

$\le M\cdot\|x\|.$

$A_s x^{(n)} \longrightarrow A_s x \overset{}{=} y_s$. 而令 $s\to 0^+$. $y = \lim_{s\to 0^+} y_s$. 故 $x\in D(A)$ 时.

$Ax = \lim_{s\to 0^+} A_s x = \lim_{s\to 0^+} y_s = y.$

故 A 是一个闭算子.

由 Claim 1 ~ 4 知, 结论: $\{E_t\}_{t\ge 0}$ C_0-SG.

则 $\frac{E_h - I}{h}x \longrightarrow Ax$. A 即为 $\{E_t\}_{t\ge 0}$ 的生成元

Thm 10.1. $\|(\lambda - A)^{-1}\| \le \frac{1}{\lambda + a}.$

$\boxed{\text{Thm. 10.4.}}$ If $\{E_t\}_{t\ge 0}$ C_0-SG, 假设 $\|E_t\| \le e^{-at}$, $\forall\, t\ge 0$.
$a\in\mathbb{R}$.

则 $\exists\, b\ge 0$. $\|(\lambda - A)^{-1}\| \le \frac{1}{\lambda + a}$, $\lambda \ge b$. (Hilla - Yousida Theorem).

Proof: Assume that $a \ge 0$

Claim 1. $\forall\, s,\lambda > 0$. $\lambda - A_s$ invertible.

且 $\|(\lambda - A_s)^{-1}\| \le \frac{s}{1 + \lambda s - e^{-as}} \le \frac{1}{\lambda}$. (固定的 λ, 关于 $s>0$ 是一致有界算子).

Indeed. $\lambda - A_s = \lambda - \frac{E_s - I}{s} = \frac{\lambda s + 1 - E_s}{s} = \left(\frac{\lambda s+1}{s}\right)\cdot\left[I - \underbrace{\frac{E_s}{\lambda s+1}}\right].$

$\downarrow$

$\left\|\frac{E_s}{\lambda s+1}\right\| < 1.$

且 $(\lambda - A_s)^{-1} = \frac{s}{s\lambda+1}\cdot\sum_{n=0}^{+\infty}\left(\frac{E_s}{\lambda s+1}\right)^n$

$\|E_s\| \leq e^{-as} \Rightarrow \|(\lambda - A_s)^{-1}\| \leq \frac{s}{\lambda+1}\cdot\sum_{n=0}^{+\infty}\frac{e^{-as\cdot n}}{(\lambda s+1)^n} = \frac{s}{\lambda s+1-e^{as}} \leq \frac{s}{\lambda s} = \frac{1}{\lambda}$

$\Rightarrow \|x\| = \|(\lambda-A_s)^{-1}(\lambda-A_s)x\| \leq \frac{s}{\lambda s+1-e^{as}}\|(\lambda-A_s)x\| \quad \forall\, x\in X.$

特别地. $x\in D(A)$. $\|(\lambda-A_s)x\| \geq \frac{\lambda s+1-e^{-as}}{s}\|x\|$

$\Rightarrow \|(\lambda-A)x\| \geq \lim_{s\to0^+}\left(\frac{\lambda s+1-e^{-as}}{s}\right)\cdot\|x\| = (\lambda+a)\|x\|$

故 $D(A) \xrightarrow[\text{closed.}]{\lambda - A} X$ is injective.

$\mathrm{Im}(\lambda-A)$ is closed. ($\lambda - A$ has closed range).

还需证明 $\mathrm{Im}(\lambda-A) \underset{\text{dense}}{\subset} X$.

Claim 2: $\mathrm{Im}(\lambda-A) \supset D(A)$. (Hence dense!!).

$\Updownarrow$. $\forall\, y\in D(A)$. $\exists\, x\in D(A)$. $(\lambda-A)x = y$.

$\forall\, s>0$. $\underbrace{(\lambda-A_s)}_{\text{可逆}}x^{(s)} = y$. $x^{(s)} = (\lambda-A_s)^{-1}y$.
$\exists!\, x^{(s)}\in X$.

sub-claim: $x^{(s)}\in D(A)$. Indeed $[A_s, A_h] = 0$.

故 $\underbrace{(\lambda-A_s)^{-1}}_{\text{有界算子}}\underbrace{A_h}_{\downarrow h\to0^+ \; A.} y = A_h\cdot\underbrace{(\lambda-A_s)^{-1}y}_{x^{(s)}} \Rightarrow x^{(s)}\in D(A)$ 且 $Ax^{(s)} = (\lambda-A_s)^{-1}Ay$.

$\qquad \| \quad A(\lambda-A_s)^{-1}y.$

固定 $\lambda>0$. $(\lambda-A_s)^{-1}$ uniformly bdd.

(可交换).

$(\lambda-A_s)^{-1}y - (\lambda-A_t)^{-1}y = [(\lambda-A_s)^{-1}-(\lambda-A_t)^{-1}]y = (\lambda-A_s)^{-1}\cdot(A_t-A_s)\cdot(\lambda-A_t)^{-1}y$

$= (\lambda-A_s)^{-1}\cdot(\lambda-A_t)^{-1}\cdot\underline{(A_s-A_t)y}$

$\Rightarrow \|(\lambda-A_s)^{-1}y - (\lambda-A_t)^{-1}y\| \leq \frac{1}{\lambda^2}\cdot\|(A_s-A_t)y\|$. 故 $\lim_{s\to0^+}(\lambda-A_s)^{-1}y$ 存在.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \quad \lim_{s\to0^+}x^{(s)}$ 记为 x.

$(\lambda-A_s)x^{(s)} = y.$
$\| \quad \lambda x^{(s)} - A_s x^{(s)}$

$\lim_{s\to0^+}A_s x^{(s)} = \lambda x - y.$

$$(\lambda-A)x^{(s)}-y = (\lambda-A)\cdot(\lambda-A_s)^{-1}y - y = (\lambda-A)(\lambda-A_s)^{-1}y - (\lambda-A_s)\cdot(\lambda-A_s)^{-1}y$$

$$= (A_s-A)\cdot(\lambda-A_s)^{-1}y. \quad = (\lambda-A_s)^{-1}\cdot(A_s-A)y.$$

$$\|(\lambda-A)x^{(s)}-y\| \le \frac{1}{\lambda}\|A_s y - Ay\| \xrightarrow[(y\in D(A))]{s\to 0^+} 0$$

Hence $(\lambda-A)x^{(s)} \longrightarrow y.$ $\quad x^{(s)} \longrightarrow x.$ $\Rightarrow \begin{cases} Ax^{(s)} \longrightarrow \lambda x - y \\ x^{(s)} \longrightarrow x. \end{cases}$

A closed $\Rightarrow$ $Ax = \lambda x - y.$ $x \in D(A).$ 即 $y = (\lambda-A)x$

$\|(\lambda-A)x\| \ge (\lambda+a)\|x\|$

$\lambda - A$ 可逆

$\|(\lambda-A)^{-1}x\| \le \frac{\|x\|}{\lambda+a}$, $x \in Im(\lambda-A)$. is dense in X.

if $a<0$. consider $F_t = e^{at}E_t.$ $\Rightarrow$ 归结到 $a>0$ 的情形.

Lecture 27 2022/12/7

Complex Banach Algebra

$a\in B$, $f\in Hol(\sigma(a))$. $g\in Hol(\sigma(f(a)))$

$g\circ f(a) = g(f(a))$ (☆) . 不妨设 f 非常值.

Thm. (Uniqueness of Riesz calculus).

$a\in B.$ Then $\mathcal{C}_a: Hol(\sigma(a)) \longrightarrow B.$ be such that:
$\qquad\qquad a \longmapsto f(a)$

① $\mathcal{C}_a$ is an algebraic homomorphism (即 $\mathcal{C}$ 线性的. $\mathcal{C}(fg) = \mathcal{C}(f)\cdot\mathcal{C}(g)$).

② $\mathcal{C}(1) = e$. $\mathcal{C}(z) = a$.

③ $\forall\ U\supset\sigma(a)$ is open . If $\{f_n\}_{n=1}^{\infty}\subset Hol(U)$

$f_n \xrightarrow{\text{内闭一致}} f.$ Then $\mathcal{C}(f_n)\longrightarrow\mathcal{C}(f).$

Then $\mathcal{C}(f) = f(a)$

Proof of ☆: 固定 $f\in Hol(\sigma(a))$ 非常值. $\quad Hol(\sigma(f(a))) \longrightarrow B.$
$\qquad g \longmapsto g(f(a))$

比较 $Hol(\sigma(f(a))) \xrightarrow{\mathcal{C}} B.$ 是 Riesz 演算.
$\qquad g \longmapsto g\circ f(a).$

只需验证 ℓ 满足 ①, ②, ③

① $\ell(g_1g_2) = g_1g_2\circ f\ (a) = g_1\circ f(a)\cdot g_2\circ f(a) = \ell(g_1)\cdot\ell(g_2)$

② $\ell(1) = (1\circ f)(a) = 1(a) = e.$

$\ell(z) = (z\circ f)(a) = f(a)$

③ 开集 $U \supset \sigma(f(a))$. $g_n, g: V \to \mathbb{C}$ Hol. $g_n \xrightarrow[\text{in } V]{\text{内闭一致}} g.$

$g_n\circ f \xrightarrow[\text{on } f^{-1}(V)]{\text{内闭一致}} g\circ f.$ $\Rightarrow \|(g_n\circ f)(a) - (g\circ f)(a)\| \to 0.$ 即 $\ell(g_n) \to \ell(g)$

$A \in B(X)$, $E_t = e^{tA}$, $t \longmapsto E_t \in B(X)$ 连续 $\|E_t - E_{t_0}\| \to 0$ as $t \to t_0$

这种 SG is called uniformly continuous semigroup.

Approximate a C_0-SG by a unif continuous S.G. (期末出一题).

Thm 10.5 $\{T_t\}_{t\ge 0}$ C_0-SG in $B(X)$.

Let $A_h = \frac{T_h - I}{h}$, $h>0$. $A_h \in B(X)$.

Then $e^{tA_h}x \xrightarrow{h\to 0^+} T_t x$. $\forall\ t\ge 0$. $x\in X$.

strong opera r topology.

Definition. $B(X) \supset \{D_n\}_{n=1}^{\infty}$, $D\in B(X)$. We say ① $D_n \xrightarrow{SOT} D$ if $\|D_n x - Dx\| \xrightarrow{n\to\infty} 0$ $\forall\ x\in X$.

② $D_n \xrightarrow{WOT} D$. if $D_n x \xrightarrow{\text{weakly}} Dx.$

$\Updownarrow$

$\forall\ x^*\in X^*$, $\in X$.

$(D_n x, x^*) \to (Dx, x^*)$

即 $e^{tA_h} \xrightarrow{SOT} T_t.$

Pf: claim 1. $[A_h, T_t] = 0 = \left[\frac{T_h - I}{h}, T_t\right].$

$\forall\ h>0,\ t>0.$ 而 $[T_h, T_t] = 0$

claim 2. $\forall\ s>0$. $N_s = \sup\limits_{\substack{0\le t\le s\\ 0<h<1}} \|e^{tA_h}\| < +\infty.$

Indeed, $M_s = \sup\limits_{0\le t\le s}\|T_t\| < +\infty$. 事实上, $\forall\ x\in X$. $t\longmapsto T_t x$ 连续

$\Rightarrow \sup\limits_{0\le t\le s}\|T_t x\| < +\infty.$ $\overset{\text{共鸣定理}}{\Longrightarrow} \sup\limits_{0\le t\le s}\|T_t\| < +\infty.$

Hence, $\|T_t\| \le M_1^{t+1}$, $\forall\ t>0.$ $T_t = T_{[t]}\cdot T_{\{t\}}.$

$\Rightarrow \|T_t\| \le \|T_{[t]}\|\cdot\|T_{\{t\}}\| \le \|T_1\|^{[t]}\cdot\|T_{\{t\}}\| \le M_1^{[t]+1} \le M_1^{t+1}$

$$e^{tA_h} = e^{t\cdot\frac{T_h - I}{h}} = e^{t/h\cdot T_h}\, e^{-t/h}$$

$$\Rightarrow \|e^{tA_h}\| = e^{-t/h}\cdot \left\|\sum_{k=0}^{\infty}\frac{(t/h\cdot T_h)^k}{k!}\right\| \le e^{-t/h}\cdot\sum_{k=0}^{\infty}\frac{(t/h)^k}{k!}\cdot\|T_{hk}\| \le e^{-t/h}\cdot\sum_{k=0}^{\infty}\frac{(t/h)^k}{k!}\cdot M_1^{hk+1}$$

$$= \exp\cdot\left(\frac{t}{h}\cdot(M_1^h - 1)\right)\cdot M_1 \le M_1\exp\cdot\left(\frac{M_1^h - 1}{h}\cdot s\right)$$ 令 $\frac{M_1^h-1}{h}$ 取上确界.

$$\Rightarrow \sup_{\substack{0\le t\le s\\ 0<h<1}} \|e^{tA_h}\| < \infty.$$

Claim 3. $A_h x \xrightarrow{h\to 0^+} Ax \quad \forall\, x\in D(A)$.

Now, 令 $\tau = \frac{t}{n}$, $e^{tA_h} - T_t = e^{n\tau A_h} - T_{n\tau} = e^{n\tau A_h} - T_\tau^{\,n} = (e^{\tau A_h} - T_\tau)\cdot\left(\sum_{k=0}^{n-1} e^{k\tau A_h}\cdot T_\tau^{\,n-1-k}\right)$

$\forall\, 0<t\le s,\ 0<h<1$.

$$\Rightarrow \|(e^{tA_h} - T_t)x\| \le \left\|\sum_{k=0}^{n-1} e^{k\tau A_h}\cdot T_{\tau(n-k-1)}\right\|\cdot\|(e^{\tau A_h} - T_\tau)x\|$$

$$\le n N_s\cdot M_s\,\|(e^{\tau A_h} - T_\tau)x\| = \frac{t}{\tau} N_s M_s\,\|(e^{\tau A_h} - T_\tau)x)$$

$$= s N_s M_s\,\left\|\frac{e^{\tau A_h} - T_\tau}{\tau}x\right\|.$$ 令 $\tau\to 0$.

$$\frac{e^{\tau A_h} - T_\tau}{\tau}x = \left(\frac{e^{\tau A_h} - I}{\tau} - \frac{T_\tau - I}{\tau}\right)x \xrightarrow[x\in D(A)]{\tau\to 0^+} A_h x - Ax.$$

Hence, $\forall\, x\in D(A)$. $\|(e^{tA_h} - T_t)x\| \le sN_sM_s\,\|(A_h - A)x\|$.

$$\Rightarrow \lim_{h\to 0^+}\sup_{0\le t\le s}\|(e^{tA_h} - T_t)x\| = 0.\quad x\in D(A).$$ 而 $\|e^{tA_h}\| \le N_s < +\infty$

$\|T_t\| \le M_s < +\infty$.

$\Rightarrow$ $\|e^{tA_h} - T_t\|$ 一致有界. $D(A)$ dense

Thus $\swarrow$ for $\forall\, x\in X$.

Hilbert space : 完备的 Hilbert 空间.

$H\times H \longrightarrow \mathbb{C}$. sesqui-linear. $(u,u) = \|u\|^2 > 0$. if $u\ne 0$.

$u,v \longmapsto (u,v)$ 平行四边形法则.

线性的 anti-linear 反线性的.

Parallelogram.

$(\alpha u, v) = \alpha(u,v)$

$(u,\beta v) = \bar{\beta}(u,v)$

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right).$$

Theorem: 满足 Parallelogram 的范数一定来自于一个内积.

local theory of Banach spaces.

Proof: real case

$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2(u,v)$.

$(u,v) = \frac{1}{2}[\|u+v\|^2 - \|u\|^2 - \|v\|^2]$.

所以定义 $[u,v] := \frac{1}{2}\cdot[\|u+v\|^2 - \|u\|^2 - \|v\|^2]$.

$[u,u] = \frac{1}{2}(4\|u\|^2 - \|u\|^2 - \|u\|^2) = \|u\|^2$.

只需验证 $[u,v]$ 是内积.

(1) $[u,v] = [v,u]$

(2) $[0,u] = 0$.

(3) $[u,v]$ 关于 u 和 v 连续

(4) $[x+y,z] \overset{?}{\longleftrightarrow} [x,z]+[y,z]$.

$2[x+y,z] = \|x+y+z\|^2 - \|x+y\|^2 - \|z\|^2$.

$2[x,z] + 2[y,z] = \|x+z\|^2 - \|x\|^2 - \|z\|^2 + \|y+z\|^2 - \|y\|^2 - \|z\|^2$

$\Longleftrightarrow \quad \|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x+z\|^2 + \|x+y\|^2 + \|y+z\|^2$. (☆)

而 $2\|x+y\|^2 + 2\|y+z\|^2 = \|x+2y+z\|^2 + \|x-z\|^2$. ①

$2\|x\|^2 + 2\|z\|^2 = \|x+z\|^2 + \|x-z\|^2$. ②

$2\|x+y+z\|^2 + 2\|y\|^2 = \|x+z\|^2 + \|x+2y+z\|^2$. ③

$\frac{1}{2}$(③ + ② − ①) 即得 (☆式). 故 (4) 成立.

(4) $\Rightarrow$ $[nx,y] = n[x,y]$ $\Rightarrow$ $[\frac{m}{n}x,y] = \frac{m}{n}\cdot[x,y]$. $\overset{\text{连续性}}{\Rightarrow}$ $[\alpha x,y] = \alpha[x,y]$. $\alpha>0$.

取 $[-\alpha x,y] + [\alpha x,y] = [0,y] = 0 \Rightarrow [-\alpha x,y] = -\alpha[x,y]$. Thus linear.

Complex case.

$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(x,y)$. 记 $\overset{\mathrm{Re}(x,y).}{[x,y]} = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2)$

$(x,y) = \mathrm{Re}(x,y) + \mathrm{Im}(x,y)\cdot i$. 而 $\mathrm{Re}(ix,y) = -\mathrm{Im}(x,y) \Rightarrow \mathrm{Im}(x,y) = -\mathrm{Re}(ix,y)$

定义 $\{x,y\} := [x,y] - i[ix,y]$.

要证 $\{x,y\}$ 是一个复的内积. $\{x,x\} = [x,x] - i\underbrace{[ix,x]}_{=0} = \|x\|^2$.

验证复线性. $\{ix,y\} = i\{x,y\}$. Hermite 型.

$\overline{\{x,y\}} = \{y,x\}$.

Lecture 28 2022/12/9

Thm. (极化恒等式). V complex vector space. H complex Hilbert space.

let $B: V\times V \longrightarrow \mathbb{C}$. sesquilinear.

$B(u,v)$ $B(u,\lambda v) = \bar{\lambda} B(u,v)$.
↑ linear ↑ anti-linear.

Then. $B(u,v) = \frac{1}{4}\sum_{m=0}^{3} i^m B(u+i^m v, u+i^m v)$

Pf: $m=0,1,2,3$. $B(u+i^m v, u+i^m v) = B(u,u) + \overline{i^m} B(u,v) + i^m B(v,u) + B(v,v)$

$\Rightarrow \sum_{m=0}^{3} i^m B(u+i^m v, u+i^m v) = \left(\sum i^m\right)\left(B(u,u)+B(v,v)\right) + 4B(u,v) + \sum_{m=0}^{3} i^{2m} B(v,u)$

$= 4B(u,v)$.

$B(u,v)$ 由对角线 $B(u,u)$ 决定。

Corollary: In a (complex) Hilbert Space

$(u,v) = 1/4 \sum_{m=0}^{3} i^m \|u+i^m v\|^2$

Corollary: if $A, C \in B(H)$. If $(Ax,x) = (Cx,x)$, $\forall x\in H$.

Then $A=C$.

Pf: $B_A(x,y) := (Ax,y)$, $B_C(x,y) := (Cx,y)$.

B_A, B_C are both sesquilinear. 只需证 $B_A(x,x) = B_C(x,x)$. 由极化恒等式 则 $B_A = B_C \Rightarrow A=C$.

这是我们的条件.

$\Downarrow$ $\Uparrow$

$(Ax,y) = (Cx,y)$, $\forall y\in H$

(实上的极化恒等式).
有空再研究.

Def: $A\in B(H)$. we say $B = A^*$. ← Hilbertian adjoint.

If $\langle Ax,y\rangle = \langle x, By\rangle$ $\forall x,y\in H$.

Riesz representation Thm: $H \xrightarrow{\ell^*} \mathbb{C}$.

$\Leftrightarrow \exists\, y\in H$, such that $\ell(x) = (x,y) = \ell_y(x)$. $y \longmapsto \ell_y$ is anti-linear.

$H \xrightarrow{\text{linear}} \overline{H^*}$ $\lambda x := \bar{\lambda} x$.

$y \longmapsto \ell_y$

$\langle Ax,y\rangle = \langle x, A^*y\rangle$

$\langle x, Ay\rangle = \langle A^*x, y\rangle$

Homework: $A: \mathbb{C}^n \longrightarrow \mathbb{C}^n$

$A^* = \bar{A}^T$.

Def: $A\in B(H)$, We say A is

① Hermitian ($\Leftrightarrow$ self-adjoint) iff $A = A^*$.

② normal. iff $[A, A^*] = 0$. $\Leftrightarrow$ $(AA^* = A^*A)$.

③ unitary iff $\bar{A}^T = A^+$ $\Leftrightarrow$ $AA^* = A^*A = I$.

(← 一般用 U, V 来记 $\mathcal{U}(H)$.).
中元素 $U(H)$

Proposition: A normal $\iff \|Ax\| = \|A^*x\| \ \forall x \in H \iff \exists\, \mathcal{U}: H \to H$. unitary s.t. $A = \mathcal{U}A^*$. $(A^* = \mathcal{U}^{-1}A)$.

pf: $\|Ax\| = \|A^*x\| \iff (Ax, Ax) = (A^*x, A^*x) \iff A^*A = AA^*$.

$(Ax, Ax) = (A^*Ax, x)$, $(A^*x, A^*x) = (AA^*x, x)$

$\iff A$ is normal.

If $A = \mathcal{U}A^*$, $A^* = (\mathcal{U}A^*)^* = A\mathcal{U}^* = A\mathcal{U}^{-1} \Rightarrow A^*A = A\mathcal{U}^{-1}\cdot\mathcal{U}A^* = AA^*$.

If A normal, $H \longrightarrow H$, $\overline{\mathrm{Im}A} \xrightarrow{T} \overline{\mathrm{Im}A^*}$, well-defined, $Ax \mapsto A^*x$.

if $Ax = Ay$. Then $\|A^*(x-y)\| = \|A(x-y)\| = 0$.

suppose $H = \overline{\mathrm{Im}A} \oplus (\mathrm{Im}A)^\perp$ $= \overline{\mathrm{Im}A^*} \oplus (\mathrm{Im}A^*)^\perp$. $\Rightarrow H \xrightarrow{\mathcal{U}} H$, $\mathcal{U}|_{\overline{\mathrm{Im}A}} = T$.

$A^*x = TAx = \mathcal{U}Ax$.

Def: $A \in B(H)$ is said to be unitarily diagonalizable. if $\exists \{v_i\}_{i\in I}$ o.n.b. of A. such that $Av_i = \lambda_i v_i$.

eg. $H = \ell^2(\mathbb{N})$. $A \in B(H)$ is unitary diagonalizable. iff $\exists\, \mathcal{U} \in \mathcal{U}(\ell^2)$ such that

$A = \mathcal{U}\begin{pmatrix}\lambda_1 & & \\ & \ddots & \end{pmatrix}\mathcal{U}^{-1}$. $\mathcal{U} = (c_1, c_2, \cdots)$, $\mathcal{U}^* = \begin{pmatrix} c_1^* \\ \vdots \end{pmatrix}$.

$\mathcal{U}^*\mathcal{U} = [\langle c_i, c_j\rangle] = Id \iff \langle c_i, c_j\rangle = \delta_{ij}$. $\overline{\mathrm{span}}\{c_i\}$ is complete.

Thm. $A \in B(H)$. normal & compact. Then A unitarily diag.

Properties of A^*: ① $\|A^*\| = \|A\|$. $\|A^*\| = \sup_{\|x\|=1} \|A^*x\| = \sup_{\substack{\|x\|=1\\ \|y\|=1}} (A^*x, y) = \sup_{\substack{\|x\|=1\\ \|y\|=1}} (x, Ay) = \|A\|$.

significant !

$B(H)$ - C^* algebra. ② $\|A^*A\| = \|A\|^2$

$\|A^*A\| \le \|A^*\|\cdot\|A\| = \|A\|^2$.

$\|A\|^2 = \sup_{\|x\|=1} \|Ax\|^2 = \sup_{\|x\|=1} (Ax, Ax) = \sup_{\|x\|=1} (A^*Ax, x)$

$\le \sup_{\substack{\|x\|=1\\ \|y\|=1}} (A^*Ax, y) = \|A^*A\|$

①, ② 是一般性质
无需 normal.

Thus $\|A^*A\| = \|A\|^2$.

prop. A normal. then $\|A^n\| = \|A\|^n$

pf: $\|A^n\| \le \|A\cdots A\| \le \|A\|\cdots\|A\| = \|A\|^n$.

$\|A^*A^kx\| = \|A\cdot A^kx\|$. $\Rightarrow \sup_{\|x\|=1} \Rightarrow \|A^*A^k\| = \|A^{k+1}\|$

$\|A^k\|^2 = \sup_{\|x\|=1} \|A^kx\|^2 = \sup_{\|x\|=1} (A^kx, A^kx) = \sup_{\|x\|=1} (A^*A^kx, A^{k-1}x) \le \|A^*A^k\|\,\|A^{k-1}\|$

$= \|A^{k+1}\|\cdot\|A^{k-1}\|$

假设 $\|A\|^n = \|A^n\|$. $\forall\ 1 \le n \in k$. 都成立.

$\|A^{k+1}\| \overset{?}{=} \|A\|^{k+1}$. $\|A^k\|^2 \le \|A^{k+1}\| \cdot \|A^{k-1}\| \Rightarrow \|A\|^{2k} \le \|A^{k+1}\| \cdot \|A\|^{k-1}$

$\Rightarrow \|A\|^{k+1} \le \|A^{k+1}\|$.

Thus $\|A^{k+1}\| = \|A\|^{k+1}$. 由归纳法 $\Rightarrow \|A^n\| = \|A\|^n$.

Corollary. A normal.

$r_\sigma(A)$ 谱半径. $= \max\limits_{\lambda \in \sigma(A)} |\lambda| = \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \|A\|$

Proposition: A normal. $\sigma_p(A) := \{\lambda \in \sigma(A) \mid \ker(A-\lambda) \neq 0\}$. ($A$ 的特征根).

point spectrum

$(A-\lambda)v = 0 \Leftrightarrow Av = \lambda v$.

Proposition: A normal & compact.

$\lambda \neq 0$. $\lambda \in \sigma(A)$, 且 $|\lambda| = \|A\|$. Then $\lambda \in \sigma_p(A)$.

Proof: $|\lambda| = \|A\|$, $\lambda \in \sigma(A)$. $\lambda - A = \lambda(I - \frac{1}{\lambda}A)$

By Fredholm alternative & $\lambda - A$ 不可逆 $\Rightarrow$ $\lambda - A$ 不是单射. $\ker(\lambda - A) \neq \{0\}$. 故 $\lambda \in \sigma_p(A)$.

$\dim \ker(\lambda - A) < +\infty$.

(Maybe 期末).

Def. $A \in B(H)$ is called semi-posotive definite if $\langle Ax, x\rangle \ge 0\ \forall x \in H$.

Prop: $A \in B(H)$ is Hermitian $\Leftrightarrow (Ax, x) \in \mathbb{R}\ \forall x \in H$.

$\Updownarrow$

$A = A^*$.

proof: $(Ax, x) \in \mathbb{R}$. $\overline{(Ax, x)} = (Ax, x) \Leftrightarrow (x, Ax) = (Ax, x)$

$\Leftrightarrow (A^*x, x) = (Ax, x)$, $\forall x \in H$.

$\Leftrightarrow A = A^*$.

Borel 演算.

Prop: A Hermitian, then $\|A\| = \underbrace{\sup\limits_{\|x\|=1} |(Ax, x)|}$ 记为 M.

pf: 显然 $M \le \|A\|$. $\|A\| = \sup\limits_{\substack{\|x\|=1 \\ \|y\|=1}} |(Ax, y)|$.

另一边, $\forall x, y \in H$. $\|x\| = \|y\| = 1$.

$(A(x\pm y), x \pm y) = (Ax, x) \pm (Ax, y) \pm \underset{\underset{(y, Ax)}{\|}}{(Ay, x)} + (Ay, y)$.

$= (Ax, x) \pm 2\mathrm{Re}(Ax, y) + (Ay, y)$.

$\Rightarrow$

$4\mathrm{Re}(Ax, y) = (A(x+y), x+y) - (A(x-y), x-y)$.

取绝对值 $\le M(\|x+y\|^2 + \|x-y\|^2)$

平行四边形 $= 2M \cdot (\|x\|^2 + \|y\|^2)$

$= 4M$. $\Rightarrow \mathrm{Re}(Ax, y) \le M$

$\Rightarrow \mathrm{Re}(e^{i\theta}Ax, y) \le M$.

choose θ s.t. $\underset{\overset{\|}{|(Ax, y)|}}{}\ \le M$.

Lecture 29 2022/12/14

Q_1: C_0-semigroup uniquely determined by its generator.

$\{E_t\}_{t\geq 0}$ C_0-SG. Fix $t>0$, $\{F_t\}_{t\geq 0}$. share the same generator.

$W(s) := E_s F_{t-s}$. 对 $x \in D(A)$

$$\frac{d}{ds} E_s(F_{t-s}(x)) = \left(\frac{d}{ds} E_s\right)(F_{t-s}x) + E_s \frac{d}{ds}(F_{t-s}x)$$

用求导的定义验证↑这个东西. $E_t: D(A) \to D(A)$

$F_t: D(A) \to D(A)$.

Q_2: A normal $\iff \|A^*x\| = \|Ax\| \iff \exists\, U \in \mathcal{U}(H)$. $A = U \cdot A^*$.

Lemma 1: $\forall\ T \in B(H)$. $(\mathrm{Im}\,T)^\perp = \ker T^*$

$\ker T = (\mathrm{Im}\,T^*)^\perp$

pf: $y \in (\mathrm{Im}\,T)^\perp \iff y \perp \mathrm{Im}\,T \iff (Tx, y) = 0\ \ \forall\, x \in H$.

$\iff (T^*y, x) = 0,\ \forall\, x \in H. \iff T^*y = 0$

故 $(\mathrm{Im}\,T)^\perp = \ker T^*$.

$T^{**} = T$, 故 $(\mathrm{Im}\,T^*)^\perp = \ker T^{**} = \ker T$

Lemma 2: $A \in B(H)$ normal, then $\overline{\mathrm{Im}\,A} = \overline{\mathrm{Im}\,A^*}$. & $\ker A = \ker A^*$.

pf: ① $\|Ax\| = \|A^*x\| \Rightarrow \|Ax\| = 0 \iff \|A^*x\| = 0$. $\ker A = \ker A^*$.

② $(\mathrm{Im}\,A)^\perp = \ker A^*$ $\Rightarrow \overline{\mathrm{Im}\,A} = (\ker A)^\perp = (\ker A)^\perp = \overline{\mathrm{Im}\,A^*}$

$\quad\quad\quad\quad\quad\quad\quad\ \ \|$

$(\mathrm{Im}\,A^*)^\perp = \ker A$

Lemma 3: $S \overset{\text{subsp}}{\subset} H$. $(S^\perp)^\perp = \bar{S}$.

pf: $\forall\ x \in \bar{S}$. $x \perp S^\perp$? $\forall\ y \in S^\perp$. $(y, x_n) = 0$. $\lim_{n\to\infty} x_n = x$.

$\Rightarrow (y, x) = \lim_{n\to\infty}(y, x_n) = 0 \Rightarrow (S^\perp)^\perp \subset \bar{S}$.

if $x \notin \bar{S}$. $x \in (S^\perp)^\perp$. $\exists$ $x = \underset{\bar{S}^\perp}{\underset{\cap}{x_\perp}} + \underset{\bar{S}}{\underset{\cap}{x_{/\!/}}}$. $x_\perp \neq 0$. $(x, x_\perp) = \|x_\perp\|^2 > 0$.

claim $S^\perp = \bar{S}^\perp$. 而 $x_\perp \in \bar{S}^\perp = S^\perp$. ↕ 矛盾.

$x \in (S^\perp)^\perp \Rightarrow (x, x_\perp) = 0$.

$$H = \left\{ \begin{array}{ccc} Ax & \longrightarrow & A^*x. \\ \overline{\mathrm{Im}\,A} & & \overline{\mathrm{Im}\,A^*} \\ \oplus & & \oplus \\ \ker A^* & \xrightarrow{\mathrm{Id}} & \ker A \end{array} \right.$$

$$\begin{pmatrix} T & 0 \\ 0 & \mathrm{Id} \end{pmatrix} = U \in \mathcal{U}(H).$$

normal 演算： $f(z) = \begin{cases} \bar{z}/z & z \neq 0 \\ 1 & z = 0 \end{cases}$ Borel 函数.

$\begin{cases} f(z)z = zf(z) = \bar{z} \\ f(z)\cdot\overline{f(z)} = \overline{f(z)}\cdot f(z) = 1. \end{cases}$ 则 $\forall A$ normal $\begin{cases} f(A)A = Af(A) = A^* \\ f(A)\cdot f(A)^* = Id. \end{cases}$

正规算子的对角化.

Def: ① 不变子空间 invariant subspace

$S \underset{\substack{\text{closed}\\\text{subsp}}}{\subset} H$. is called an invarient subspace of $T \in B(H)$. if $T(S) \subset S$.

② 约化子空间. reducing subspace

$R \underset{\substack{\text{closed}\\\text{subspace}}}{\subset} H$ is called a reducing subspace of T.

if R and $R^\perp$ are both T-invarient.

S is T-invarient. $S \oplus S^\perp$ 正交直和分解. $T(x) = \underset{\substack{\Uparrow\\ S}}{T_1^S(x)} + \underset{\substack{\Uparrow\\ S^\perp}}{T_2^S(x)}$

$T = \begin{array}{c|cc} & S & S^\perp \\ \hline S & T_{11} & T_{12} \\ S^\perp & T_{21} & T_{22} \end{array}$ so S is T-invariant $\iff T_{12} = 0$

S is T-reducing $\iff T_{12} = T_{21} = 0$ (Block-diagonal)

Prop: A normal. $\forall$ eigenvector of A $(Av = \lambda v)$.

The one-dimensional subspace $\mathbb{C}v = \{\lambda v \mid \lambda \in \mathbb{C}\}$. is A-reducing.

$A = \begin{array}{c|cc} & \mathbb{C}v & v^\perp \\ \hline \mathbb{C}v & * & 0 \\ v^\perp & 0 & * \end{array}$ Moreover, $A|_{(\mathbb{C}v)^\perp}$ is normal.

Pf: ①

1° $\mathbb{C}v$ is A-invariant $A(kv) = kA(v) = k\lambda v \in \mathbb{C}v$.

2° if $w \perp \mathbb{C}v$, we want $Aw \perp \mathbb{C}v$.

$\forall (w \perp \mathbb{C}v)$. $\longrightarrow$ $(Aw, v) = (w, A^*v) = (w, \bar{\lambda}v) = 0$.

claim: if $Av = \lambda v$. Then $A^*v = \bar{\lambda}v$. (A normal) $\Rightarrow$ $A-\lambda$ normal.

$\Downarrow$

$\|(A-\lambda)v\| = 0$. $\|(A^*-\bar{\lambda})v\| = \|(A-\lambda)v\| = 0 \Rightarrow A^*v = \bar{\lambda}v$.

$A = \begin{pmatrix} \lambda & \\ & A_1 \end{pmatrix}$. $A^* = \begin{pmatrix} \bar\lambda & \\ & A_1^* \end{pmatrix}$ $AA^* = A^*A \Rightarrow A_1A_1^* = A_1^*A_1$. 即 A_1 normal.

$A_1 = A|_{(\mathbb{C}v)^\perp}$.

A normal + compact, $\exists\ \lambda \in \sigma(A)$ 且 $|\lambda| = \|A\|$.

step 1: $A = 0$ ok.

step 2: if $A \neq 0$. Then $\exists\ \lambda \in \sigma(A)$, $|\lambda| = \|A\|$.

Fredholm 方法

$\lambda \in \sigma_p(A)$. (即存在特征向量).

(cf. Lecture 28 末的 Prop.)

$\exists\ \|v_0\| = 1$, $v_0 \in H$.

$Av_0 = \lambda v_0$

$$A = \begin{array}{c|cc} & \mathbb{C}v_0 & (\mathbb{C}v_0)^\perp \\ \hline \mathbb{C}v_0 & \lambda_0 & 0 \\ (\mathbb{C}v_0)^\perp & 0 & A_1 \end{array}$$

step 3: if $A_1 = 0$. OK. 否则. repeat step 2. 找 λ_1, v_1.

if never stops, 存在 $|\lambda_0| = \|A\|$, $|\lambda_1| = \|A_1\| \le \|A\| = |\lambda_0|$

$|\lambda_0| \ge |\lambda_1| \ge \cdots \ge |\lambda_n| \ge \cdots > 0$

$v_0 \quad v_1 \quad \cdots \quad v_n$

$Av_n = \lambda_n v_n$.

$$A|_{\overline{\mathrm{span}\{v_i, i\ge 1\}}} = \begin{pmatrix} \lambda_0 & & & & \\ & \lambda_1 & & & \\ & & \ddots & & \\ & & & \lambda_n & \\ & & & & \ddots \end{pmatrix}_{\text{compact}}$$

claim: $\lim\limits_{n\to\infty} |\lambda_n| = 0$. 否则 $|\lambda_n| \ge \delta > 0$. $v_1 \cdots, v_n, \cdots \in$ Unit ball of H.

$Av_n = \lambda_n v_n \Rightarrow Av_n$ 有收敛子列. Av_{n_k}. (互相垂直).

紧算子.

而 $\|Av_{n_k} - Av_{n_l}\| = \|\lambda_{n_k}v_{n_k} - \lambda_{n_l}v_{n_l}\| = \sqrt{\lambda_{n_k}^2 + \lambda_{n_l}^2} \ge \sqrt{2}\delta$

claim: $A|_{(\mathrm{span}\{v_i, i\ge 1\})^\perp} = 0$

Lemma: if $R_1, R_2 \subset H$ reducing subspace & $R_1 \perp R_2 \Rightarrow R_1 \oplus R_2$ is reducing.

if $A|_{(\mathrm{span}\{v_i, i\ge 1\})^\perp} \neq 0$.

$\exists\ \lambda'$, $|\lambda'| = \|A|_{(\mathrm{span}\{v_i, i\ge 1\})^\perp}\| > 0$.

$Av' = \lambda' v'$. 但是 n 充分大时. $|\lambda_n| < \lambda'$.

Then $|\lambda_0| \ge \cdots \ge |\lambda_{n-1}| \ge |\lambda'| > |\lambda_n|$

$\|A|_{(v_0,\cdots,v_n)^\perp}\| = |\lambda_n| < |\lambda'|$. $\dfrac{\|Av'\|}{\|v'\|} = |\lambda'| > \|A|_{(v_0,\cdots,v_n)^\perp}\|$.

Corollary: A normal & compact

则存在 $H_0 \subset H$ closed subspace s.t. $A|_{H_0^\perp} = 0$

separable.

Hilbert 空间中，

有紧算子的起作用的地方一定是一个可分的子空间.

Def: Hilbert-schmidt class H, K Hilbert space.

$T \in S_2(H,K) \longleftarrow$ The Hilbert schmitz class. from H to K.

(if $H = K$, $S_2(H,H)$ 记为 $S_2(H)$)

if $\forall (v_i)_{i\in I}$ orthonormal basis of H. (ONB) $\quad \|T\|_2 := \sqrt{\sum_{i\in I} \|Tv_i\|^2} < +\infty$.

Theorem: $\forall (v_i)_{i\in I}$, $(w_j)_{j\in J}$ two ONB of H.

$$\sum_{i\in I} \|Tv_i\|_H^2 = \sum_{j\in J} \|Tw_j\|_H^2.$$

Proof: 任取 $(h_\alpha)_{\alpha\in A}$ ONB of K. $\quad \sum_{i\in I}\|Tv_i\|_K^2 \underset{\text{勾股定理}}{=} \sum_{i\in I}\left[\sum_{\alpha\in A} |(Tv_i, h_\alpha)|^2\right] = \sum_{\alpha\in A}\|T^* h_\alpha\|_H^2$

同样对 (w_j), $\sum_{j\in J}\|Tw_j\|_K^2 = \sum_{\alpha\in A}\|T^*h_\alpha\|_H^2$

顺便还证明了 $\|T\|_{S_2(H,K)} = \|T^*\|_{S_2(K,H)}$.

Example: (Home work) ① fix an ONB of H, assume that $T: \operatorname{span}\{v_i : i\in I\} \to K$. (not necessarily) linear.

s.t. $\sum_{i\in I}\|Tv_i\|^2 < +\infty$.

show that $T \in B(H,K)$

② 对 $H = \ell_2(\mathbb{N})$, $T = (a_{kl})_{k,l\geq 0}$, Then $\|T\|_2 = \sqrt{\sum_{k,l\geq 0} |a_{kl}|^2}$

Let $(X, \mathcal{B}, \mu)$ be a measure space with finite measure. $\mu(X) < +\infty$.

Let $K: X\times X \to \mathbb{C}$. measurable

$(x,y) \mapsto K(x,y)$.

such that $\displaystyle\int_{X\times X} |K(x,x)|^2 \mu^{\otimes}(dxdy) \overset{\text{Fubini}}{=} \int_{X\times X} |K(x,x)|^2 \mu(dx)\mu(dy) < +\infty$

Then ten integral operator

$$T_K : L^2(X,\mu) \longrightarrow L(X,\mu).$$
$$f \longmapsto T_K f.$$

Hilbert shmitz operator.

$(T_K f)(x) := \int_X K(x,y) f(y) \mu(dy)$. Then $T_K \in S_2(L^2(X,\mu))$.

Lecture 30 2022/12/16

Thm: H Hilbert space $K \in K(H)$. ← 紧算子. Then

K can be approximated by finite rank operators.

That's $\exists\ T_n \in B(H)$. $\mathrm{rank}(T_n) < +\infty$. s.t. $\|K - T_n\| \longrightarrow +\infty$.

Recall: Any operator approximated by finite rank operator is compact.

Hilbert 空间中紧算子 $\Longleftrightarrow$ 有限秩逼近.

Pf: Assume that H is separable, $\dim H = +\infty$. Then H has an ONB $\{v_k\}_{k=1}^{\infty}$

Let P_n be the orthogonal projection from H onto $\mathrm{span}\{v_i\}_{i=1}^{n}$

(事实上 $P_n = \sum_{i=1}^{n} v_i \otimes v_i^*$. $P_n(x) = \sum_{i=1}^{n} v_i \cdot (x, v_i)$.

Let $Q_n = I - P_n$.

Claim 1: $\forall x \in H$. $\lim_{n\to\infty} \|Q_n x\| = 0$. ($\Leftrightarrow \|P_n x - x\| = 0$).

$\|x\|^2 = \sum_{i=1}^{\infty} |(x, v_i)|^2 < +\infty$. $\|Q_n x\|^2 = \sum_{i=n+1}^{+\infty} |(x, v_i)|^2 \longrightarrow 0$

Claim 2: $\forall\ K \in K(H)$, $\|K - P_n \circ K\| \xrightarrow{n\to\infty} 0$

Remark: $\mathrm{rank}(P_n K) = \dim \mathrm{Im}(P_n K) \le \dim(\mathrm{Im} P_n) = n < +\infty$.

Indeed, $\|K - P_n K\| = \|Q_n K\|$. subclaim $\|Q_n K\| \downarrow n \to \infty$ (单调减).

$Q_{n+1} = Q_{n+1} \cdot Q_n \Rightarrow \|Q_{n+1} K\| = \|Q_{n+1} Q_n K\| \le \|Q_{n+1}\| \cdot \|Q_n K\| \le \|Q_n K\|$

假设不收敛到 0. $\exists \delta > 0$. $\|Q_n K\| > \delta$, $\forall n \in \mathbb{N}$.

$\Rightarrow \exists\ u_n \in H$, $\|u_n\| = 1$. $\|Q_n K u_n\| > \delta$. But $K \in K(H)$.

$\Rightarrow \{u_{n_k}\}$ subseq. $K u_{n_k} \xrightarrow{k\to\infty} v \in H$. $\|Q_n v\| \ge \|Q_{n_k} K u_{n_k}\| - \|Q_{n_k}(K u_{n_k} - v)\|$

$\Rightarrow \liminf_{n\to\infty} \|Q_n v\| \ge \delta$. 这与 claim 1 矛盾.

In general, for any Hilbert space H, $K \in K(H)$

$K^*K \in K(H)$ [$K(H)$ is a bilateral ideal closed]

Remark: ($K(H)$ is the unique bilateral closed ideal. Index theory

in orther words, $B(H)/K(H)$ is a simple algebra Caltein algebra.

$(K^*K)^* = K^* K^{**} = K^*K \Rightarrow K^*K$ is Hermitian.

in particular, K^*K is normal.

$\Rightarrow K^*K$ compact & normal $\Rightarrow$ diagonizable

Then $\exists \{v_k\}_{k=0}^{N}$ $N \in \{0,1,2,\cdots\} \cup \{+\infty\}$ st. $(K^*K)v_k = \lambda_k v_k$.

$(K^*K)\cdot((\mathrm{span}\{v_k\})^\perp) = 0$

Lemma. $\forall\ T \in B(H)$. $\ker T = \ker(T^*T)$

Indeed. $\ker T \subset \ker T^*T$ trivially.

conversely. $T^*Tx = 0 \Rightarrow (T^*Tx, x) = 0 \Rightarrow \|Tx\|^2 = 0 \Rightarrow Tx = 0$

$\ker T^*T \subset \ker T$.

$\mathrm{span}\{v_k\}^\perp \subset \ker(K^*K) = \ker K$, Let $H_0 = \overline{\mathrm{span}}(\{v_k\} \cup \mathrm{Im} K) \subset H$. closed subspace

$\mathrm{Im}\, K = K(\underbrace{\overline{\mathrm{span}}\{v_k\}}_{\text{separable}})$ (有界算子把可分子空间映射到可分子空间).

Hence H_0 is separable

Claim 3. H_0 is K-reducing. ($\Leftrightarrow$ H_0 & $H_0^\perp$ is K-invariant.)

$K(H_0) \subset \mathrm{Im} K \subset H_0$, 而 $\mathrm{span}\{v_k\} \subset H_0 \Rightarrow H_0^\perp \subset \mathrm{span}\{v_k\}^\perp \subset \ker(K)$.

$\Rightarrow K(H_0^\perp) = \{0\} \subset H_0^\perp$. $K \xrightarrow[\text{in block}]{\text{written}} \begin{array}{c|cc} & H_0 & H_0^\perp \\ \hline H_0 & K_0 & 0 \\ H_0^\perp & 0 & 0 \end{array}$ $K_0 = K|_{H_0} = P_{H_0} K \cdot P_{H_0}$

由于 K compact $\Rightarrow K_0$ compact. $P_{H_0}: H \longrightarrow H_0$ orthogonal projection.

$K_0 \in K(\underbrace{H_0}_{\text{sparable Hilbert}})$.

Hence K_0 can be approximated by finite rank operators

$T_n \in B(H_0)$. $\|T_n - K_0\| \longrightarrow 0$. $\Rightarrow \left\| \begin{pmatrix} K_0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} T_n & 0 \\ 0 & 0 \end{pmatrix} \right\| \longrightarrow 0$

Hilbert-Schmitz operators: $S_2(H_1, H_2) \ni T$.

$\Leftrightarrow \sum_{i\in I} \|Tv_i\|^2 < +\infty$ $(v_i)_{i\in I}$ ONB of H

$\|T\|_2$ (也记为 $\|T\|_{S_2(H_1,H_2)}$ 或 $\|T\|_{HS}$).

This quantity is independent of the basis of ONB.

$\|T\|_2 = \sqrt{\sum_{i\in I} \|Tv_i\|^2}$

Homework: Prove that $\|\cdot\|_2$ is a norm on $S_2(H_1, H_2)$.

(Remark: Indeed, $S_2(H_1, H_2)$ is a Hilbert space).

$$H_1 \xrightarrow{T_1} H_2 \xrightarrow{T_2^*} H_1.$$
$$H_1 \xrightarrow{T_2} H_2.$$

$Tr(T_2^*, T_1) := \langle T_1, T_2 \rangle_{S_2(H_2, H_1)}$. (Trace 的定义)

$Tr(T_2^*, T_1) := \sum_{i\in I} (T_2^* T_1 v_i, v_i)$.

This is well-defined.

① absolutely convergent

② does not depend on the choice of ONB.

If $H = L^2(X,\mu)$ σ-finite measure. X is polish $\begin{cases} \text{metrizable} \\ \text{complete} \\ \text{separable} \end{cases}$

Then $S_2(H)$ consists of integrable operator of Hilbert Schmitz class.

即 $\forall\, T \in S_2(H)$. $\Longleftrightarrow \exists\, K: X\times X \longrightarrow \mathbb{C}$.

$$\int_{X\times X} |K(x,y)|^2 d\mu(x)\, d\mu(y) < +\infty$$

$$Tf(x) = T_K f(x) := \int_X K(x,y) f(y)\, d\mu(y)$$

Prop: $T_K \in S_2(L^2(X,\mu))$ & $\|T_K\|_{HS} = \left(\int |K(x,y)|^2 d\mu(x) d\mu(y) \right)^{1/2}$

Fubini

Lemma: $L^2(X,\mu)$ is a separable Hilbert space.

Assume that $\{\varphi_k\}_{k=1}^{\infty}$ ONB of $L^2(X,\mu)$. Then $\{\varphi_k \otimes \varphi_l : k, l \geq 1\}$ is an ONB of $L^2(X\times X, \mu\otimes\mu)$

In other words, $L^2(X\times X, \mu\otimes\mu) = L^2(X,\mu) \otimes L^2(X,\mu)$.

Hilbert tensor product.

$$\langle \varphi_k \otimes \varphi_l, \varphi_{k'} \otimes \varphi_{l'} \rangle = \int \varphi_k(x) \varphi_l(y)\, \overline{\varphi_{k'}(x) \varphi_{l'}(y)}\, d\mu(x)\, d\mu(y) = 0$$

Hence. $\{\varphi_k \otimes \varphi_l : k, l \geq 1\}$ is orthogonal. It suffices to show that $\{\varphi_k\otimes\varphi_l, k, l \geq 1\}$ is complete.

即 $\overline{\operatorname{span}}\{\varphi_k \otimes \varphi_l, k, l \geq 1\} = L^2(X\times X, \mu\otimes\mu)$. if $f \perp \operatorname{span}\{\varphi_k \otimes \varphi_l\}$.

即 $\int_{X\times X} f(x,y)\, \overline{\varphi_k(x) \varphi_l(y)}\, d\mu(x)\, d\mu(y) = 0$. $= \int_X \overline{\varphi_l(y)} \left[\int_X f(x,y) \overline{\varphi_k(x)}\, d\mu(x) \right] d\mu(y) = 0$ $\Rightarrow f = 0$ a.e. $\square$

$$\|T_K\|_{HS}^2 = \sum_{k=1}^{\infty} \|T_K\varphi_k\|_{L^2(X,\mu)}^2 = \sum_{k=1}^{\infty}\Big[\sum_{\ell=1}^{\infty} |(T_K\varphi_k, \varphi_\ell)|^2\Big] = \sum_{k,\ell=1}^{\infty}\Big|\int_X\Big[\int_X K(x,y)\varphi_k(y)\,d\mu y\Big]\overline{\varphi_\ell(x)}\,d\mu x\Big|^2$$

$$= \int_{X\times X} |K(x,y)|^2 d\mu(x)\,d\mu(y)$$

极大(其实唯一的) closed bilateral ideal in $B(H)$.

Proposition $S_2(H) \subset \boxed{K(H)}$. (Any Hilbert-Schmitz operator is compact).

(Barry Simon, Trace-ideals).

$S_p(H) \longleftarrow$ Schotten - von Neumann class.

Pf: $\|T\|_{HS}^2 = \sum_{k=1}^{\infty} \|Tv_k\|^2 < +\infty$. 不妨假设 H separable.

$\|T\|_{HS}^2 = \sum_{k=1}^{\infty} \|Tv_k\|^2 < +\infty$. ($\{v_k\}_{k=1}^{\infty}$ ONB of H)

P_n = ortho. prog. onto span$\{v_k\}_{k=1}^{\infty}$ $\|T-TP_n\| \xrightarrow{n\to\infty} 0$

Lemma. A_1. $\|T\| \leq \|T\|_{HS} < +\infty$.

Pf: $\sum_{i\in I} \|Tv_i\|^2 < +\infty$. $\mathcal{D} = \text{span}\{v_k\}_{k=1}^{\infty} \overset{\text{dense}}{\subset} H$.

$\forall x \in \mathcal{D}$, $x = \sum_{k=1}^{\infty} \underbrace{(x,v_k)}_{\text{有限个不为0.}} v_k$ $\|x\|^2 = \sum_{k=1}^{\infty} (x,v_k)^2$

$\|Tx\| \leq \|x\|\cdot\Big(\sum_{i=1}^{+\infty}\|Tv_i\|^2\Big)^{1/2}$. $\Rightarrow \|T\| \leq \|T\|_{HS}$.

Lemma A_2. $T\in S_2(H)$, $B\in B(H)$, Then TB, $BT \in S_2(H)$.

Pf: $\|T\|_2 = \|T^*\|_2$. $T\in S_2(H) \Rightarrow T^* \in S_2(H)$.

$\|BT\|_2^2 = \sum_{i\in I} \|BTv_i\|^2 \leq \|B\|^2\cdot \sum_{i\in I}\|Tv_i\|^2 = \|B\|^2\cdot\|T\|_2^2$

$\Rightarrow \|BT\|_2 \leq \|B\|\cdot\|T\|_2 \Rightarrow BT \in S_2(H)$.

$\|TB\|_2 = \|(TB)^*\|_2 = \|B^*T^*\|_2 \leq \|B^*\|\cdot\|T^*\|_2 = \|B\|\cdot\|T\|_2$

Corollary.

$\underline{\|ATB\|_2 \leq \|A\|\cdot\|T\|_2\cdot\|B\|}$. (Trace ideal inequality).

回到 $T\in S_2(H)$, $\|T\|_2^2 = \sum_{k=1}^{\infty}\|Tv_k\|^2 < +\infty$

P_n = P onto span$\{v_k\}_{k=1}^{n}$ $\|T-TP_n\| \longrightarrow 0$

事实上. $\|T-TP_n\| \leq \|T-TP_n\|_{HS}$

而 $\|T-TP_n\|_{HS}^2 = \sum_{k=1}^{\infty} \|(T-TP_n)v_k\|^2 = \sum_{k=1}^{n} \underbrace{\|Tv_k - TP_nv_k\|^2}_{=0} + \sum_{k=n+1}^{\infty} \|Tv_k - \underbrace{TP_nv_k}_{=0}\|^2$

$= \sum_{k=n+1}^{\infty} \|Tv_k\|^2 \longrightarrow 0$

$\Rightarrow \lim_{n\to\infty} \|T-TP_n\|_{HS} = 0$

$\lim_{n\to\infty} \|T-TP_n\| = 0.$ Since TP_n finite rank.

$\Rightarrow T \in K(H).$

Lecture 31 2022/12/21

Sesqui — 1个半. Sesquilinear form

Def: (Numerical range) H Hilbert space

$A \in B(H)$. $W(A) := \{(Au, u) \mid u \in H, \|u\| = 1\}$. called the numerical range of A.

Remark: $W(A) =$ A 在所有可能的正交基矩阵表示中对角线元素的全体.

Relation between $W(A)$ & $\sigma(A)$.

Thm: $A \in B(H)$. $\sigma(A) \subset \overline{W(A)}$

Proof: if $\lambda \in \sigma(A)$. Then at least one of the following happens.

① $A-\lambda$ not injective. $\ker(A-\lambda) \neq \{0\}$.

② $\overline{\mathrm{Im}(A-\lambda)} \neq H \iff \overline{\mathrm{Im}(A-\lambda)}^{\perp} = \ker(A^*-\bar{\lambda}) \neq \{0\}$

③ $\mathrm{Im}(A-\lambda)$ is not closed

Actually, ① $\exists\ u \in H, \|u\| = 1, Au = \lambda u.$ $\Rightarrow (Au, u) = \lambda.$

② $\exists\ v \in H, \|v\| = 1, A^*v = \bar{\lambda}v.$ $\Rightarrow \bar{\lambda} = (A^*v, v) = (v, Av) = \overline{(Av, v)}$

③ $\nexists\ c > 0$ s.t. $\|(A-\lambda)x\| \geq c\|x\|$

$\iff \exists\ u_k \in H, \|u_k\| = 1, \|(A-\lambda)u_k\| \xrightarrow{k\to\infty} 0$

③ $\Rightarrow \|(Au_k, u_k) - (\lambda u_k, u_k)\| \leq \|Au_k - \lambda u_k\| \cdot \|u_k\| \longrightarrow 0$

Thus $\lambda \in \overline{W(A)}$.

Sesquilinear form: H, is Hilbert. 定义中要求dense.

A SLF on H is a function $D(a) \subset H$ dense, linear subspace

$a: D(a) \times D(a) \longrightarrow \mathbb{C}.$

$a(u, v)$ is linear in u, anti-linear in v.

$\begin{cases} a(\lambda u_1 + \beta u_2, v) = \lambda a(u_1, v) + \beta a(u_2, v) \\ a(u, \lambda v_1 + \beta v_2) = \bar{\lambda} a(u, v_1) + \bar{\beta} a(u, v_2) \end{cases}$

Examples: ① $a: H\times H \longrightarrow \mathbb{C}$, defined by

$a(u,v) = (Au,v) \quad A\in B(H)$.

② A densely defined linear operator on H.

$a(u,v) = (Au,v)$. $\quad D(a) = D(A)$.

Def: a 的伴随算子. Fix a SLF a on H.

we want to define a linear operator A.

$u\in D(A)$ iff $u\in D(a)$ & $\exists\ C = C_u > 0$ s.t. $\underline{|a(u,v)| \le C_u \|v\|.\ v\in D(a)}$. $\overset{H}{\cup}$ dense

$v \longmapsto a(u,v)$. 可以唯一延拓为 $H\longrightarrow \mathbb{C}$ 的有界反线性算子.

In other words, $v\longrightarrow \overline{a(u,v)}$ 可唯一延拓为 $H\rightarrow\mathbb{C}$ 的有界线性型.

由 Riesz representation, $\exists!\ f\in H$.

$\underset{\text{线性}}{a(u,v)} = \underset{\text{反线性}}{\langle f, v\rangle}$. 这时, 记 $f = Au$.

即 $a(u,v) = \langle Au, v\rangle$, 易验证 $A(\lambda u + \beta v) = \lambda Au + \beta Av$.

Thm: Let $a(u,v)$ be densely defined SLF with associated operator A.

$$w(a) = \{a(u,u) \mid u\in H.\ \|u\| = 1\}.$$

Then (a) If $\lambda\notin \overline{w(a)}$, then $A-\lambda$ injection & $C\|(A-\lambda)u\| \ge \|u\|,\ u\in D(A)$.

Remark: If A is closed, , Then $\mathrm{Im}(A-\lambda)$ closed in H.

Proof: $\lambda\notin\overline{w(a)}$, $\exists\ \delta>0$,

$|\lambda - a(u,u)| \ge \delta > 0$, $\forall\ u\in D(a)$, $\|u\| = 1$.

$\forall\ w\in D(a)$, $w\ne 0$, $\frac{w}{\|w\|}\in D(a)$, $\|\frac{w}{\|w\|}\| = 1$.

$\|\lambda - a(\frac{w}{\|w\|}, \frac{w}{\|w\|})\| \ge \delta \Rightarrow |a(w,w) - \lambda\|w\|^2| \ge \delta\|w\|^2$.

$(A-\lambda)u := \tilde{\omega}$, $\quad (\tilde{\omega}, v) = (Au-\lambda u, v) = (Au,v) - \lambda(u,v) = a(u,v) - \lambda(u,v)$.

$\quad\ \ \| $

$Au - \lambda u \qquad v\in D(a)$.

特别地, 令 $v = u$. $(\tilde{\omega}, u) = a(u,u) - \lambda(u,u)$, $|(\tilde{\omega},u)| \ge \delta\|u\|^2$. $\Rightarrow \|\tilde{\omega}\| \ge \delta\|u\|$

即 $\|A-\lambda\| \ge \delta$. $\Rightarrow$ $A-\lambda$ Injection

Convention: We denote $a(u,u)$ as $a(u)$.

A SLF $a(u,v)$ is called Hermitian if $a(u,v) = \overline{a(v,u)}$.

Lemma: Let $a(u,v)$, $b(u,v)$ be two Hermitian SLF.

$|a(u,u)| \le M\, b(u,u)$, $u \in D(a) \cap D(b)$.

Then $|a(u,v)|^2 \le M^2 \cdot b(u,u) \cdot b(v,v)$, $u,v \in D(a) \cap D(b)$.

Proof: Replacing b by Mb, we may assume $M = 1$. $b(u,u) > 0 \in \mathbb{R}$

已知 $|a(u,u)| \le b(u,u)$. $\overset{\text{want}}{\Longrightarrow}$ $|a(u,v)|^2 \le b(u,u) \cdot b(v,v)$.

Assume $u,v \in D(a) \cap D(b)$. $a(u,v) = re^{i\theta} \Rightarrow a(e^{-i\theta}u, v) = r \in \mathbb{R}$.

$\forall t \in \mathbb{R}$, $P(t) = b(v,v)\cdot t^2 + 2a(w,v)t + b(w,w)$.

记 $w = e^{-i\theta}u$. (乘上 $e^{-i\theta}$ 使 $a(u,v)$ 为实数)

$= b(tv,tv) + 2a(w,tv) + b(w,w)$. $a(w,v) = e^{-i\theta}a(u,v)$.

$a(w+tv, w+tv) = a(w,w) + t\,a(t,w) + t\,a(w,v) + a(tv,tv)$.

$a(w-tv, w-tv) = a(w,w) - t\,a(t,w) - t\,a(w,v) + a(tv,tv)$.

$\Rightarrow$ $4a(w,tv) = a(w+tv, w+tv) - a(w-tv, w-tv)$.

$|4a(w,tv)| \le b(w+tv, w+tv) + b(w-tv, w-tv)$. $= 2\,(b(w,w) + b(tv,tv))$

$\Rightarrow$ $b(w,w) + b(tv,tv) - 2|a(w,tv)| \ge 0$

$\Rightarrow$ $b(w,w) + b(tv,tv) + 2a(w,tv) \ge 0$. $\Rightarrow$ $P(t) \ge 0$. $\Delta \ge 0$

即 $|a(u,v)|^2 \le b(u,u)\cdot b(v,v)$

Corollary:

$b(u,v)$ Hermitian, $a(u,u)$ not sesquilinear, $|a(u,u)| \le M \cdot b(u,u)$

Then $|a(u,v)| \le M^2\, b(u,u)\cdot b(v,v)$.

Pf: Hermitian 化操作: $a_1(u,v) = \frac{1}{2}\cdot\left[a(u,v) + \overline{a(v,u)}\right]$.

$a_2(u,v) = \frac{1}{2i}\cdot\left[a(u,v) - \overline{a(v,u)}\right]$

then a_1, a_2 are both SLF.

$a(u,v) = a_1(u,v) + i\,a_2(u,v)$.

$\Rightarrow |a_1(u,u)| \le \frac{1}{2}\left[\,|a(u,u)| + |\overline{a(u,u)}|\,\right] \le M\cdot b(u,u)$. 同理 $|a_2(u,u)| \le M\cdot b(u,u)$

$\Rightarrow \begin{cases} |a_1(u,v)|^2 \le M^2 \cdot b(u,u)\cdot b(v,v) \\ |a_2(u,v)|^2 \le M^2\, b(u,u)\cdot b(v,v). \end{cases}$ $\Rightarrow$ $|a(u,v)|^2 \le 4M^2\, b(u)\, b(v)$.

Corollary. If $b(u,v)$ Hermitian SLF, $b(u) \ge 0$. $u \in D(b)$ semi-positive

$\Rightarrow |b(u,v)|^2 \le b(u)\cdot b(v)$. $\sqrt{b(u+v)} \le \sqrt{b(u)} + \sqrt{b(v)}$.

Thm: TFAE: a is SLF

(1) $a(u,v)$ is Hermitian

(2) $a(u,u) \in \mathbb{R}$, $\forall u \in D(a)$.

(3) $\mathrm{Re}(a(u,v)) = \mathrm{Re}(a(v,u))$ $u,v \in D(a)$. $\quad (a(e^{i\theta}u) = a(u))$.

Pf: (1) $\Rightarrow$ (2). $a(u,u) = \overline{a(u,u)} \in \mathbb{R}$.

(2) $\Rightarrow$ (3). $a(iu+v) = a(iu) + a(v) + a(iu,v) + a(v,iu)$

$= a(u) + a(v) + i\,a(u,v) - i\,a(v,u)$. $\in \mathbb{R}$.

$\Rightarrow \mathrm{Re}(a(u,v)) = \mathrm{Re}(a(v,u))$.

(3) $\Rightarrow$ (1). 希望 $a(u,v) = \overline{a(v,u)}$, 实部相同. 考虑虚部.

而 $\mathrm{Re}\, a(iu,v) = \mathrm{Re}\, a(v,iu)$

$\mathrm{Re}\, a(iu,v) = \mathrm{Re}(i\,a(u,v))$, $\mathrm{Re}\, a(v,iu) = \mathrm{Re}(-i\,a(v,u))$. $\Rightarrow \mathrm{Im}\, a(u,v) = -\mathrm{Im}\, a(v,u)$.

$\Downarrow$

$a(u,v) = \overline{a(v,u)}$.

Numerical range

Theorem 12.9 $a(u,v)$ SLF. Then $W(a)$ is a convex set, $\subset \mathbb{C}$.

Proof: $u,v \in D(a)$, $\|u\| = \|v\| = 1$.

$a(u)$, $a(v) \in W(a)$.

Case 1: $a(u) = a(v) \Rightarrow \forall \theta \in (0,1)$, $(1-\theta)a(u) + \theta a(v) \in W(a)$.

Case 2: $a(u) \neq a(v)$ 固定 $\theta \in (0,1)$. 希望证 $(1-\theta)a(u) + \theta a(v) \in W(a)$.

即存在 $w \in D(a)$, $\|w\| = 1$, $a(w) = (1-\theta)a(u) + \theta a(v)$.

Claim 1: $\exists \gamma \in \mathbb{C}$. $|\gamma| = 1$.

$$\begin{cases} \gamma a(u) = X + iU \\ \gamma a(v) = Y + iU. \end{cases}$$

即 $\gamma a(u)$ 与 $\gamma a(v)$ 有相同虚部

则 $\gamma a(w) = \theta\cdot \gamma a(u) + (1-\theta)\gamma a(v)$

$\Downarrow$

Find $w \in D(a)$, $\|w\| = 1$, $\gamma a(w) = Z + iU$. 即 $\gamma a(u)$, $\gamma a(v)$, $\gamma a(w)$ 有相同虚部.

只需 $Z = (1-\theta)X + \theta Y$. $\Longleftrightarrow a(w) = \theta a(u) + (1-\theta) a(v)$.

对 $\forall\theta$ 都可找到 w.

使 ~ 成立且 $Z = (1-\theta)X + \theta Y$.

$$h(t) := \gamma a\left(\frac{te^{i\varphi}u + (1-t)v}{\|te^{i\varphi}u + (1-t)v\|}\right) - iU.$$

φ 是待定实数.

Claim 2. $\|te^{i\varphi}u + (1-t)v\| \neq 0$

否则 $\|te^{i\varphi}u\| = \|(1-t)v\| \Rightarrow t = 1-t \Rightarrow t = 1/2$

$h(0) = \gamma a(v) - iU = X$

$h(1) = \gamma a(u) - iU = Y$ $\quad \in \mathbb{R}$

$$a(te^{i\varphi}u) = a(tu) = t^2 a(u).$$
$$\| \qquad\qquad\qquad\qquad\qquad \Rightarrow a(u)=a(v). \quad 这与 Case 2 的假设矛盾.$$
$$a(-(1-t)v) = (1-t)^2 a(v)$$

则希望 $h(t)$ 是一个实数. 即 $\gamma a(te^{i\varphi}u+(1-t)v) - i\|t\cdot e^{i\varphi}u+(1-t)v\|^2\cdot U \in \mathbb{R}$.

则利用介值定理, $h(t)$ 在 $(0,1)$ 中取到 $\theta X+(1-\theta)Y$ 的值. ($h(t)$ 显然连续).

从而 $w=\dfrac{te^{i\varphi}u+(1-t)v}{\|te^{i\varphi}u+(1-t)v\|}$ 满足条件. $\gamma a(w)-iU=\theta X+(1-\theta)Y$.

$\alpha=t,\ \beta=(1-t)$

$h(t)$ 是实的 $\Longleftrightarrow$ $\gamma[\alpha^2 a(u)+\beta^2 a(v)+\alpha\beta e^{i\varphi}a(u,v)+\alpha\beta e^{-i\varphi}a(v,u)] - iU[\alpha^2+\beta^2+\alpha\beta e^{i\varphi}(u,v)+\alpha\beta e^{-i\varphi}(v,u)]$

$$= \alpha^2[\underbrace{\gamma a(u)-iU}_{\mathbb{R}}]+\beta^2[\underbrace{\gamma a(v)-iU}_{\mathbb{R}}]+\alpha\beta[\underbrace{\gamma e^{i\varphi}a(u,v)+\gamma e^{-i\varphi}a(v,u)-iUe^{i\varphi}(u,v)-iUe^{-i\varphi}(v,u)}_{与\ \alpha,\beta\ 无关}]$$

即如果选取合适的 φ, $h(t)$ 恒为实数. 只需 $\mathrm{Im}\big[e^{i\varphi}\underbrace{(\gamma a(u,v)-iU(u,v))}_{z_1=r_1e^{i\theta_1}}+e^{-i\varphi}\underbrace{(\gamma a(v,u)-iU(v,u))}_{z_2=r_2e^{i\theta_2}}\big]=0$

$\mathrm{Im}\left(r_1e^{i(\theta_1+\varphi)}+r_2e^{i(\theta_2-\varphi)}\right)=0$ 选取合适的 φ 这是显然能做到的.

Lecture 32 2022/12/23

Closed SLF

Def: A SLF $a(u,v)$ is called closed if $\{u_n\}_{n=1}^{\infty}\subset D(a)$

$$\begin{cases} u_n \xrightarrow{n\to\infty} u \text{ in } H \\ a(u_n-u_m)\xrightarrow{n,m\to\infty} 0\end{cases} \quad \text{we have } u\in D(a) \ \&\ a(u_n-u)\xrightarrow{n\to\infty} 0$$

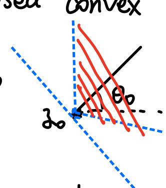
Thm. $a(u,v)$ densely defined closed SLF with associated operator A.

Assume that $\overline{W(a)}$ is NOT the whole plane, a half plane, a strip or a line.

Then A is closed & $\sigma(A)\subset\overline{W(a)}=\overline{W(A)}$

Lemma: W closed convex set, if $W\neq$ plane, half plane, a strip or a line.

Thm: $\exists z_0, \theta_0$



$\leftarrow W$, 即 $\forall z\in W$.

$|\arg(z-z_0)-\theta_0|\le\theta<\frac{\pi}{2}$.

a strip.

即 $|\mathrm{Im}\ e^{-i\theta_0}(z-z_0)|\le\tan\theta;\ \mathrm{Re}(e^{-i\theta_0}(z-z_0))$

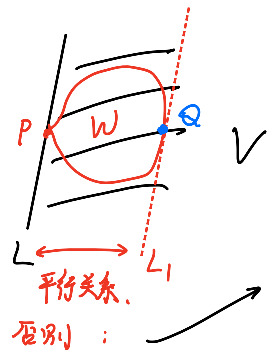
Proof: W closed convex, $W\neq\mathbb{C}$, Hahn-Banach Separation, $\exists$ half-plane V.

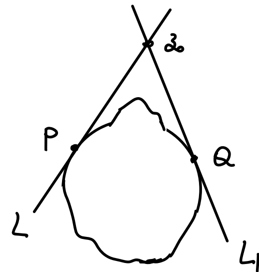
$W\subset V$ such that $L=\partial V$ contains a point $p\in W$.

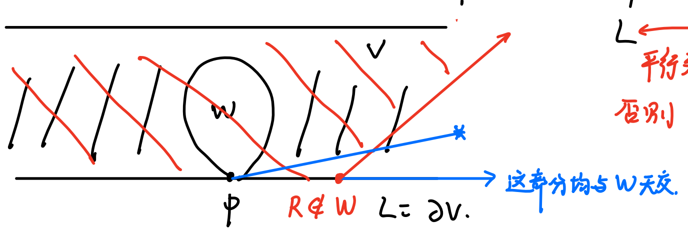
由于 $W \neq$ half-plane, $W \neq V$

$\Rightarrow \exists\, Q \in \partial W \cap \mathring{V}$ $(Q \notin L)$

claim: $L \not\subset W$. 否则 W is a strip.







Then 存在红色射线 L', s.t. $W \subset$ 阴影部分, 否则, 如图中 $\underbrace{\quad}_{\text{p和*的连线}}$ 均 $\subset W$, "不断压低蓝线, 可得 $R \in W$, 矛盾".

不是 $\mathbb{C}$, 不是 half-plane, 不是 strip, 不是 line

$\Updownarrow$

包含在某个锐角确定的扇形中.

Corollary: If $\overline{W(a)}$ is contained in a good domain, then $\exists\, |\gamma| = 1$, $k > 0$, $k_0 \in \mathbb{R}$,

such that: $|a(u)| \le k\left[\mathrm{Re}(\gamma a(u)) + k_0 \|u\|^2\right]$, $\forall\, u \in D(a)$.

Proof: $W(a)$ is convex $\Rightarrow$ $\overline{W(a)}$ is convex.

$\exists\, z_0, \theta_0, \theta$ such that $\left|\mathrm{Im}\, e^{-i\theta_0}(z - z_0)\right| \le \tan\theta \cdot \mathrm{Re}\left(e^{-i\theta_0}(z - z_0)\right)$ $\quad \forall\, z \in \overline{W(a)}$

令 $\gamma = e^{-i\theta_0}$ $\qquad \theta \in (0, \frac{\pi}{2})$. $\qquad z = a(u)$, $\|u\| = 1$, $u \in D(a)$.

$|\mathrm{Im}\, \gamma(a(u) - z_0)| \le \tan\theta \cdot \mathrm{Re}(\gamma a(u) - z_0)$.

$\Rightarrow |\mathrm{Im}\, \gamma a(u)| \le |\mathrm{Im}(\gamma z_0)| + \tan\theta \cdot \mathrm{Re}(\gamma \cdot a(u)) + \overbrace{\tan\theta \cdot \mathrm{Re}(z_0)}^{\text{记为 } \tilde{k} \in \mathbb{R}}$

$\Rightarrow |\mathrm{Im}\, \gamma a(u)| \le \tan\theta \cdot [\mathrm{Re}\, \gamma a(u) + k_0]$. $\qquad k_0 = \tilde{k} + \dfrac{|\mathrm{Im}(\gamma z_0)|}{\tan\theta} \in \mathbb{R}$. $\quad \forall\, u \in D(a)$, 且 $\|u\| = 1$.

$\Rightarrow \left|\mathrm{Im}\, \gamma a(\frac{u}{\|u\|})\right| \le \tan\theta \left[\mathrm{Re}\, \gamma a(\frac{u}{\|u\|}) + k_0\right]$, $\quad \forall\, u \in D(a)$

$\Rightarrow |\mathrm{Im}\, \gamma a(u)| \le \tan\theta \cdot [\mathrm{Re}(\gamma a(u)) + k_0 \|u\|^2]$

Hence, $|a(u)|^2 = |\gamma a(u)|^2 = |\mathrm{Im}(\gamma a(u))|^2 + |\mathrm{Re}(\gamma a(u))|^2 \le (\tan^2\theta + 1) \cdot |\mathrm{Re}(\gamma a(u))|^2 + \cdots$

$\Rightarrow |a(u)| \le k[\mathrm{Re}(\gamma a(u)) + k_0 \|u\|^2]$

Thm 12.13: If $a(u,v)$ such that $W(a)$ is contained in a 好的扇形.

Then $\exists$ Hermitian SLF $b(u,v)$, $D(b) = D(a)$. such that $\exists\, c > 0$

$\frac{1}{c}|a(u)| \le b(u) \le |a(u)| + c\|u\|^2$, $\quad \forall\, u \in D(a) = D(b)$.

Pf: $|a(u)| \le k\cdot[Re(\gamma a(u)) + k_0\|u^2\|]$, Let $b_1(u,v) = \frac{1}{2}[\gamma a(u,v) + \overline{\gamma a(v,u)}]$

$\overline{b_1(v,u)} = \frac{1}{2}[\gamma a(v,u) + \gamma a(u,v)] = b(u,v)$.

即 b is a Hermitian SLF.

Let $b(u,v) := \underbrace{b_1(u,v)}_{\text{Hermitian SLF}} + \underbrace{k_0(u,v)}_{\text{内积, Hermitian SLF}}$. $\Rightarrow$ b is Hermitian SLF.

$|a(u)| \le k[Re(\gamma a(u)) + k_0\|u\|^2]$, $b(u) = b_1(u) + k_0\|u\|^2. = Re(\gamma a(u)) + k\|u\|^2$.

$\Rightarrow$ $b(u) \le |a(u)| + k_0\|u\|^2$, 即 $\frac{1}{C}|a(u)| \le b(u) \le |a(u)| + C\|u\|^2$. □

$a(u,v)$ $\quad W(a) \longrightarrow b$. Hermitian & semi-positive definite.

$\forall \{u_k\} \subset D(a)$, 且 $u \in D(a)$. $\left.\begin{array}{l} u_k \longrightarrow u \\ a(u_k-u) \longrightarrow 0 \end{array}\right\} \Rightarrow a(u_k,v) \longrightarrow a(u,v)$. $\forall\ v \in D(a)$.

$|a(u_k,v) - a(u,v)| = |a(u_k-u,v)|$. $|a(u_k-u,v)| \le 4C^2 b(u_k-u)\cdot b(v)$.

$b(u_k-u) \le |a(u_k-u)| + C\|u_k-u\|^2 \longrightarrow 0$

Semi-positive definite SLF (是可以用来构造新的内积).

If $b(u,u)$ SLF ≥ 0 (半正定), $D(b)$.

Lemma: $S = \{u \in D(b) \mid b(u,u) = 0\}$, is a linear subspace

Pf: $b(u,u) = 0$ & $b(v,v) = 0$ $\Rightarrow$ $b(\alpha u+\beta v, \alpha u+\beta v) = 0$

只需证 $b(u+v, u+v) = 0$.

(Corollary 12.6) $b(u+v,u+v)^{1/2} \le b(u,u)^{1/2} + b(v,v)^{1/2} = 0$

Def: $\Big(\underbrace{D(b)/S}_{\text{商空间}},\ \underbrace{\langle\langle u,v\rangle\rangle := b(u,v)}_{\text{内积}}\Big)$.

即 $\langle\langle u,v\rangle\rangle$ sesquilinear & $\langle\langle u,u\rangle\rangle \ge 0$ & $\langle\langle u,u\rangle\rangle = 0 \Leftrightarrow u \in S$. $[u] = [0]$.

$\overline{D(b)/S}$ 完备化就是一个 Hilbert 空间.

Thm. $a(u,v)$ SLF & $W(a)$ contained in a 好的扇形 (A associated operator).

$\Rightarrow$ A closed, $\sigma(A) \subset \overline{W(a)} = \overline{W(A)}$

Pf: $\{u_k\}_{k=1}^{\infty} \subset D(A)$. 希望: ① $u \in D(A)$ $\underline{a(u,v) \text{ closed}}$.

$\begin{cases} u_k \xrightarrow{k\to\infty} u \\ Au_k \xrightarrow{k\to\infty} f. \end{cases}$ in H.   ② $f = Au$.

$$|a(u_j - u_k)| = |(A(u_j-u_k), u_j-u_k)| \le \|Au_j - Au_k\| \|u_j-u_k\| \xrightarrow{j,k\to\infty} 0.$$

由 a closed 的假设知 $u \in D(a)$

且 $a(u_k-u) \xrightarrow{k\to\infty} 0 \Rightarrow a(u_k, v) \xrightarrow{k\to\infty} a(u,v)$

$$\overset{\shortparallel}{(Au_k, v)} \xrightarrow{k\to\infty} (f,v)$$

Then $a(u,v) = (f,v)$ $\forall v \in H$, 根据 $D(A)$ 的定义及 $u \in D(a) \Rightarrow Au = f$.

故 A is closed. □

$\sigma(A) \subset \overline{W(a)}$, if $\lambda \notin \overline{W(a)}$, 欲证 $A-\lambda$ 可逆.

① 由 Thm 12.3, $A-\lambda$ is injective & $\|u\| \le C\|(A-\lambda)u\|$, $\forall u \in D(A)$.

由 closed Range 定理, $\mathrm{Im}(A-\lambda)$ is closed. 只要证 $\mathrm{Im}(A-\lambda) = H$.

Thm 12.11. $a(u,v)$ closed SLF, $W(a)$ contained in a 好的扇形 且 $0 \notin \overline{W(a)}$

$D(a) \xrightarrow[F]{\text{linear Functional}} \mathbb{C}$. $|F(v)| \le C\sqrt{|a(v)|}$ (变形的 Riesz 表示定理).

则 $\exists!$ $w, u \in D(a)$. $F(v) = a(v,w)$ and $F(v) = \overline{a(u,v)}$. 课上不证.

下证: $\mathrm{Im}(A-\lambda) = H$

$\forall f \in H$, $a_\lambda(u,v) := a(u,v) - \lambda(u,v)$ is SLF.

$D(a) = D(a_\lambda)$ $|(v,f)|^2 \le \|v\|^2 \cdot \|f\|^2 \le C|a_\lambda(v)|^2$

$|a(u) - \lambda\|u\|^2| \ge \delta\|u\|^2$. $\forall u \in D(a)$.

$|a_\lambda(v)| \ge \delta\|v\|^2$.   Thm 12.11

$D(a) \xrightarrow{F} \mathbb{C}$. $|F(v)| \le C|a_\lambda(v)|^{1/2}$ $\Rightarrow$ $\exists$ $u \in D(a_\lambda) = D(a)$. s.t.

$v \longmapsto (v,f)$. $(f,v) = a_\lambda(u,v)$, $\forall v \in D(a)$.

$= a(u,v) - \lambda(u,v)$.

$\Leftrightarrow \underline{(f+\lambda u, v)} = a(u,v) \Rightarrow u \in D(A)$.

$= \langle Au, v \rangle$

$\Updownarrow$

$f + \lambda u = Au$. 即 $f = (A-\lambda)u \in \mathrm{Im}(A-\lambda)$.

Thm 12.18. 12.17.

closable operator.