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Reference: Theorems of Regularity and Singularity of Energy Minimizing Map, Leon Simon.

Definition 1. Let $\Omega \subseteq \mathbb{R}^n$ is an open subset. N compact manifold isometrically embedded in $\mathbb{R}^P$.

we say $u \in W^{1,2}_{loc}(\Omega; N)$ iff $u \in W^{1,2}_{loc}(\Omega; \mathbb{R}^P)$ and $u(x) \in N$ a.e.

the sobolev space

Energy $\mathcal{E}_{B_\rho(y)}(u) := \int_{B_\rho(y)} |Du|^2$

we call a map u is energy minimizing iff for $\forall$ $B_\rho(y) \subset \Omega$. $u \in W^{1,2}(B_\rho(y); N)$

$\mathcal{E}_{B_\rho(y)}(u) \leq \mathcal{E}_{B_\rho(y)}(w)$ for every $w \in W^{1,2}(B_\rho(y), N)$ & $w = u$ in a neighbor of $\partial B_\rho(y)$

Variational Equations.

If we make $\{u_s\}_{s \in (-\delta,\delta)}$ a 1-parameter family of maps from $B_\rho(y)$ to N.

$u_s \equiv u_0$ in $\partial B_\rho(y)$. $u_0 = u$ is energy minimizing.

Then $\mathcal{E}_{B_\rho(y)}(u)$ takes minimum at $s=0$ $\Rightarrow$ $\left.\frac{d\mathcal{E}_{B_\rho(y)}(u_s)}{ds}\right|_{s=0} = 0$

let Π be the nearest point projection to N. for $\forall$ $v \in C_c^\infty(B_\rho(y))$

$u_s = \Pi \circ (u + s \cdot v)$ is in $W^{1,2}(B_\rho(y); N)$ with the same boundary condition.

$u_s = u + s\, d\Pi_u(v) + o(s)$ $\Rightarrow$ $Du_s = Du + s\, d\Pi_u(Dv) + s\, \mathrm{Hess}(\Pi)_u(Du, v) + o(s)$.

Thus $\left.\frac{d\mathcal{E}_{B_\rho(y)}(u_s)}{ds}\right|_{s=0} = 0$ $\Leftrightarrow$ $\int_{B_\rho(y)} d\Pi_u(Dv) + \mathrm{Hess}\,\Pi_u(Du, v) = 0$. $\forall$ $v \in C_c^\infty(B_\rho(y))$

$\Pi: \mathbb{R}^P \to N$. s.t. $\Pi|_N = id_N$. Thus $d\Pi_u(Dv) = \sum_i d\Pi_u(D_i v) = \sum_{i=1}^n D_i u \cdot D_i v$

$\mathrm{Hess}\,\Pi_u = -A_u$ is the second fundamental form

$\mathrm{Hess}\,\Pi_u(Du, v) = \sum_i \mathrm{Hess}\,\Pi_u(D_i u, v) = -v \cdot \sum_{i=1}^n A_u(D_i u, D_i u)$

$\Rightarrow$ $\int_{B_\rho(y)} \sum_{i=1}^n \left(D_i u \cdot D_i v - v \cdot A_u(D_i u, D_i u)\right) = 0$

$\Leftrightarrow$ $\Delta u + \sum_{i=1}^n A_u(D_i u, D_i u) = 0$ (in distribution sense). if u is C^2 we can say that

This is weakly harmonic map.

Description of ε- Regularity Thm.

If 1° $u : \Omega \to N$ is an energy Minimizing map. $u \in W^{1,2}(\Omega)$. $\Omega \subseteq \mathbb{R}^n$

N is a compact analytic manifold isometrically embedded into $\mathbb{R}^p$, $\dim N = q$

2° for $B_R(x_0) \subset \Omega$, Λ, θ are given constant > 0

$$R^{2-n} \int_{B_R(x_0)} |Du|^2 dx < \Lambda$$

Then there exists ε depends on n, N, Λ, θ, s.t.

If 3° $R^{-n} \int_{B_R(x_0)} |u - \lambda_{x_0,R}|^2 dx < \varepsilon$. $\lambda_{x_0,R} = \frac{1}{R^n \cdot \omega_n} \int_{B_R(x_0)} u(x)\, dx$.

Then i) $u \in C^\infty(B_{\theta R}(x_0))$

ii) $R^j \cdot \sup\limits_{B_{\theta R}(x_0)} |D^j u| \le C \cdot \left(R^{-n} \int_{B_R(x_0)} |u - \lambda_{x_0,R}|^2 \right)^{1/2}$. C depends on n, N, Λ, θ and j

Firstly. By technical of scaling, we can always suppose $R = 1$

$u(x)$ satisfied the inequality in $B_R(x_0)$ $\Leftrightarrow$ $v(Rx) = u(x)$ satisfies the inequality in $B_1(x_0)$.

$v(Rx) := u(x)$. $\lambda_{x_0,R} = R^{-n} \int_{B_R(x_0)} u(x)\, dx = R^{-n} \int_{B_1(x_0)} v(Rx)\, dRx$

$v(x) := u(x/R)$ $\qquad d(Rx) = R^n dx$

$$= \int_{B_1(x_0)} v(Rx)$$

α is a multi-index.

$\alpha = (\alpha_1, \cdots, \alpha_n)$

$|\alpha| = \Sigma \alpha_i$, $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$

Secondly. we just need to prove ε- Regular Thm holds for some $0 < \theta < 1$. Then Thm 1 holds for every $0 < \theta < 1$.

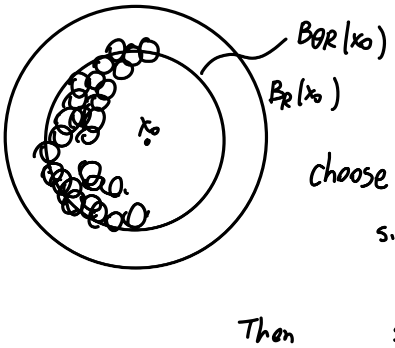
If $\theta_0 = 1/8$ is ok. for another $\theta \in (0,1)$.

$$\forall y \in B_{\theta R}(x_0) \quad B_{(1-\theta)R}(y) \subset B_R(x_0) \subset \Omega. \qquad \int_{B_{(1-\theta)R}(y_0)} |u - \lambda_{x_0,y}|^2$$

$$\sup_{B_{(1-\theta)R/8}(y)} D^j u \le C \left((1-\theta)^{-n} R^{-n} \int_{B_{(1-\theta)R}(y_0)} (u - \lambda_{(1-\theta)R,y})^2 \right)^{1/2}$$

by choosing y appropriately

$$= \underbrace{\left(C \cdot (1-\theta)^{-n/2} \right)}_{C} \cdot \left(R^{-n} \int_{B_R(x_0)} |u - \lambda|^2 \right)^{1/2}.$$



choose finite set $y_1, \cdots, y_Q$

s.t. $B_{\theta R}(x_0) \subset \bigcup_{i=1}^{Q} B_{(1-\theta)R/8}(y_i)$

Then $\sup\limits_{B_{\theta R}} \le \max\limits_{i=1,2,\cdots,Q} \left| \sup\limits_{B_{(1-\theta)R/8}} D^j u \right| \le C \cdot \left(R^{-n} \int_{B_R(x_0)} |u - \lambda|^2 \right)^{1/2}$.

$R=1$. $\theta = 1/8$ Set $B = B_R(x_0)$, $\lambda = \lambda_{x_0,R}$.

u is minimizing map $\Rightarrow$ $\Delta u = -\sum_{j=1}^{n} A_u(D_j u, D_j u)$

where A_u is the second foundamental form of N.

since N is analytic, A_u is analytic w.r.t. u.

A_u = Hess Π_u. Π_u is smooth, even analytic actually.

$$\beta = \sup_{\substack{u\in N \\ \tau \in T_u N,\ |\tau|=1}} A_u(\tau,\tau) \qquad T_uN \cong \mathbb{R}^q$$

$|\tau|=1$, $\{\tau \in T_uN, |\tau|=1\} \cong S^{q-1}$ is compact.

$$= \sup_{N\times S^{q-1}\times S^{q-1}} A_u(x,y)$$, A is analytic, thus continuous, $\sup A$ on a compact set exists.

1. $\|u\|_{C^{1,\alpha}(B_{1/4}(x_0))} \le C\cdot \left(\int_B |u-\lambda|^2\right)^{1/2}$ Then

We use Technical Lemma, $\Delta u = f$ weakly and $u \in W^{1,2}(B)$. $\beta > 0$ fixed cons

condition (4) $|F(x)|^2 \le \beta \cdot |Du|^2$

(5) "Reversed Poincaré inequality"

$$(\rho/2)^{2-n}\int_{B_{\rho/2}(y)} |Du|^2 \le \beta\cdot \rho^{-n}\int_{B_\rho(y)} |u-\lambda_{y,\rho}|^2$$ for $\forall B_\rho(y) \subset B_R(x_0)$. It's not needed in the proof.

For $\forall \alpha \in (0,1)$

Then exists $\delta_0 > 0$ depends on β, n, α.

If (6) $R^{-n}\int_{B_R(x_0)} |u-\lambda_{R,x_0}|^2 \le \delta_0^2$

Then (iii) $u \in C^{1,\alpha}(\overline{B_{R/4}(x_0)})$

(iv) $\|u\|_{C^{1,\alpha}(B_{1/4}(x_0))} \le C\cdot \left(R^{-n}\int_{B_R(x_0)} |u-\lambda_{x_0,R}|^2\right)^{1/2}$.

C depends on n, α, β.

$$|\cdot|_{k,\Omega} := \|\cdot\|_{C^k(\Omega)} = \sum_{j=0}^{k} \sup_\Omega |D^j(\cdot)|$$

$$[u]_{k,\alpha,\Omega} = \sup_\Omega \left|\frac{D^j u(x) - D^j u(y)}{(x-y)^\alpha}\right|$$

$$\|u\|_{C^{k,\alpha}(\Omega)} = |\cdot|_{k,\Omega} + [\cdot]_{k,\alpha,\Omega}.$$

Section 2.8 Reversed Poincaré inequality, u is minimizing map.

If (7) $R^{2-n}\int_{B_R(x_0)} |Du|^2 \le \Lambda$ for a given $\Lambda > 0$.

Then (v) $$\rho^{2-n}\int_{B_{\rho/2}(y)} |Du|^2 \le C\cdot\rho^{-n}\int_{B_\rho(y)} |u-\lambda_{y,\rho}|^2$$ for each $y \in B_{R/2}(x_0)$, $\rho < R/4$.

C depends on n, N, Λ.

we can immediately deduce 1.

From 1. $|Du|_{0,B_{1/4}}$ $[Du]_{\alpha, B_{1/4}}$ bounded by $C\cdot\ell$. and $u \in C^{1,\alpha}(\overline{B}_{1/4})$

Purpose: $|D^j u|_{0,B_{1/8}} \le C(j)\cdot \ell$ for $j = 1, 2, \cdots$.

$u \in C^\infty(B_{1/8})$.

Lemma 3. from Section 1.7

Referrence: Elliptic PDE of 2nd Order
Gilbarg & Trudinger

If $\Delta u = f$ weakly in $B_R(x_0)$. $u \in W^{1,2}(B_R(x_0))$.

case 1. f is bounded. then $u \in C^{1,\alpha}(B_R(x_0))$ for every $\alpha \in [0,1)$.

given $0<\theta<1$, $R^{1+\alpha}[Du]_{\alpha, B_{\theta R}} \le C\cdot(|u|_{0,B_R(x_0)} + R^2 \cdot |f|_{0,B_R(x_0)})$. (vi)

case 2. $f \in C^{k,\alpha}(B_R(x_0))$, $\alpha \in (0,1)$. $k \in \mathbb{N}$. Then $u \in C^{k+2,\alpha}(B_R(x_0))$, and.

$$\sum_{j=1}^{k+2} R^j \cdot |D^j u|_{0,B_{\theta R}} + R^{k+2+\alpha} \cdot [D^{k+2} u]_{\alpha, B_{\theta R}} \le C\cdot\left(|u|_{0,B_R(x_0)} + R^2 |f|_{0;B_R(x_0)} + R^{2+k+\alpha} \cdot [D^k f]_{\alpha, B_R}\right) \quad \text{(vii)}.$$

$u \in C^\infty(B_{1/8})$, we have $u \in C^{1,\alpha}(B_{1/4}) \Rightarrow Du \in C^{0,\alpha}(B_{1/4})$

claim $f = -A_u(D_j u, D_j u)$ α-hölder continuous. in $B_{1/4}$.

$$\sup \left|\frac{f(x)-f(y)}{|x-y|^\alpha}\right| = \Big[A_{u(x)}(D_j u(x), D_j u(x)) - A_{u(x)}(D_j u(x), D_j u(y)) + A_{u(x)}(D_j u(x), D_j u(y)) - A_{u(x)}(D_j u(y), D_j u(y))$$

$$+ A_{u(x)}(D_j u(y), D_j u(y)) - A_{u(y)}(D_j u(y), D_j u(y))\Big] \cdot \frac{1}{|x-y|^\alpha}$$

$$\le 2A_u([Du]_{\alpha, B_{1/4}}, |Du|) + |Du|^2 \cdot \sup_{|\tau|=1} \frac{A_{u(x)}(\tau,\tau) - A_{u(y)}(\tau,\tau)}{|x-y|^\alpha}$$

$$\le 2\beta\cdot[Du]_{\alpha, B_{1/4}} \cdot |Du| + |Du|^2 \cdot \sup_{\substack{|\tau|=1 \\ u\in N}} |DA_u(\tau,\tau)|\, |x-y|^{(1-\alpha)}$$

$$\le C\cdot \ell^2 < C\ell$$

$[f]_{\alpha, B_{1/4}} \le C\ell$. $\quad |f|_{0,B_{1/4}} \le \beta\cdot|Du|^2 \le C^2\beta\cdot\ell^2 < C\ell$.

$|D^2 u|_{0, B_{\theta\cdot 1/4}} \le C\cdot(|u|_{0,B_{1/4}} + |f|_{0,B_{1/4}} + [f]_{\alpha; B_{1/4}}) < C\ell$ $\quad$ choose $1/8 < \theta\cdot 1/4 <$

$[D^2 u]_{\alpha, B_{\theta \cdot 1/4}} \le C\ell$.

$\Rightarrow u \in C^{2,\alpha}(B_{\theta\cdot 1/4})$ $\quad f \in C^{1,\alpha}(B_{\theta\cdot 1/4})$

$\theta\cdot 1/4$ $\quad [f]_{1,\alpha} \le C\ell$. $\quad \theta_j\cdot\left(1/8 + \frac{1}{j}\right) = 1/8 + \frac{1}{j+1}$

$\Rightarrow |D^3 u|_{0, B_{\theta_2\theta_1\cdot 1/4}}$

$\theta_j(1/8 + 1/j) = 1/8 + 1/(j+1)$

$$\sup_{B_{1/8 + 1/(j+1)}} |D^{j+1} u| < C\ell.$$

Application, Section 2.10

1. $y \in$ regular set.
2. $\mathcal{H}^{n-2}(\text{sing } u) = 0$

A more powerful version 2.12.4.

if $n=2$ $\quad \int_{B_R(y)} |Du|^2 < \Lambda$.

Monotonocity formula

u is energy minimizing, for $0<\sigma<\rho<\rho_0$. $\overline{B_{\rho_0}(y)}\subset\Omega$. $R=|x-y|$

we have $\rho^{2-n}\int_{B_\rho(y)}|Du|^2-\sigma^{2-n}\int_{B_\sigma(y)}|Du|^2 = 2\int_{B_\rho(y)\setminus B_\sigma(y)} R^{2-n}\cdot\left|\frac{\partial u}{\partial R}\right|^2$

Proof: A famous lemma is that if a_j are integrable function $j=1,2,\cdots,n$

$a=(a_1,\cdots,a_j)$ is weakly divergence free if $\int_{B_{\rho_0}(y)}\sum_{j=1}^n a_j\cdot D_j\xi=0$

for $\forall\ \xi\in C_c^\infty(B_{\rho_0}(y))$

$\Rightarrow$ $\rho<\rho_0$. $\int_{B_\rho(y)}\sum_{j=1}^n a_j\cdot D_j\xi=\int_{\partial B_\rho(y)}\eta\cdot a\xi$ (Integral by parts). η is outward unit normal.

Then our minimizing map happens the weak divergence free condition

$$\Rightarrow\ \int_{B_\rho(y)}\sum_{i,j=1}^n\left(|Du|^2\cdot\delta_{ij}-2D_iu\cdot D_ju\right)D_i\xi^j=\int_{\partial B_\rho(y)}\sum_{i,j=1}^n\left(|Du|^2\delta_{ij}-2D_iu\cdot D_ju\right)\cdot\rho^{-1}(x^i-y^i)\cdot\xi^j$$

Let $\xi^j=(x^j-y^j)$,
$$(n-2)\cdot\int_{B_\rho(y)}|Du|^2=\int_{\partial B_\rho(y)}\rho^{-1}\cdot\sum_{i=1}^n|Du|^2\cdot(x^i-y^i)^2-\sum_{i,j}\int_{\partial B_\rho(y)}\rho^{-1}\cdot 2D_iu\cdot D_ju\cdot(x^i-y^i)(x^j-y^j)$$
$$=\rho\cdot\int_{\partial B_\rho(y)}|Du|^2-2\rho\cdot\int_{\partial B_\rho(y)}\left|\frac{\partial u}{\partial R}\right|^2$$

Because
$$\left|\frac{\partial u}{\partial R}\right|^2=\sum_{k=1}^p\left(\sum_{i=1}^n D_iu^k\cdot\rho^{-1}(x^i-y^i)\right)^2=\sum_{k=1}^p\left(\sum_{i,j}D_iu^k\cdot D_ju^k\cdot\rho^{-2}(x^i-y^i)(x^j-y^j)\right)$$
$$=\sum_{i,j}D_iu\cdot D_ju\cdot\rho^{-2}\cdot(x^i-y^i)(x^j-y^j)$$

So $(n-2)\int_{B_\rho(y)}|Du|^2=\rho\cdot\int_{\partial B_\rho(y)}\left(|Du|^2-2\left|\frac{\partial u}{\partial R}\right|^2\right)$. Notice $\frac{d}{d\rho}\int_{B_\rho}f=\int_{\partial B_\rho}f$

$$\Rightarrow\ \rho^{1-n}\cdot(n-2)\int_{B_\rho(y)}|Du|^2=\rho^{2-n}\cdot\int_{\partial B_\rho(y)}\left(|Du|^2-2\left|\frac{\partial u}{\partial R}\right|^2\right)$$

$\Rightarrow$ $\frac{d}{d\rho}\rho^{2-n}\int_{B_\rho(y)}|Du|^2=\rho^{2-n}\int_{\partial B_\rho(y)}2\left|\frac{\partial u}{\partial R}\right|^2$, Thus $\rho^{2-n}\int_{B_\rho(y)}|Du|^2$ is decreasing as $\rho\downarrow 0$

And let $\Theta_u(y)$ denote $\lim_{\rho\to 0}\rho^{2-n}\int_{B_\rho(y)}|Du|^2$. $\rho^{2-n}\int_{B_\rho(y)}|Du|^2-\Theta_u(y)=2\int_{B_\rho(y)}R^{2-n}\left|\frac{\partial u}{\partial R}\right|^2$

It is easy to check that $\Theta_u(y)$ is upper semi continuous.

This is called density function.

Lemma of Luckhaus

Definition $v \in W^{1,2}(S^{n-1}; \mathbb{R}^p)$ if $\tilde{v}(rw) = v(w)$, $w \in S^{n-1}$, $r>0$ is in $W^{1,2}$ in some neighbor of S^{n-1}.

$v \in W^{1,2}(S^{n-1}; N)$ if $v \in W^{1,2}(S^{n-1}, \mathbb{R}^p)$ and $v(S^{n-1}) \subset N$.

$v \in W^{1,2}(S^{n-1} \times [a,b]; N)$ if $\tilde{v}(rw,t) = v(w,t)$, $r>0$ is in $W^{1,2}(U \times [a,b])$

where U is a neighbor of S^{n-1}

Lemma 1. N is an arbitrary compact subset of $\mathbb{R}^p$. $u, v \in W^{1,2}(S^{n-1}, N)$, for $\forall\, \varepsilon \in (0,1)$, $\exists\; w \in W^{1,2}(S^{n-1} \times [0,\varepsilon], \mathbb{R}^p)$ s.t. $w = u$ in a neighbor of $S^{n-1} \times \{0\}$.
$w = v$ in a neighbor of $S^{n-1} \times \{\varepsilon\}$

with inequality:

$$\int_{S^{n-1} \times [0,\varepsilon]} |\bar{\nabla} w|^2 \le C \cdot \varepsilon \int_{S^{n-1}} (|\nabla u|^2 + |\nabla v|^2) + C^{-1} \varepsilon^{-1} \int_{S^{n-1}} |u-v|^2$$

$$\mathrm{dist}^2(w(x,s), N) \le C \cdot \varepsilon^{1-n} \left(\int_{S^{n-1}} |\nabla u|^2 + |\nabla v|^2\right)^{1/2} \left(\int_{S^{n-1}} |u-v|^2\right)^{1/2} + C \cdot \varepsilon^{-n} \int_{S^{n-1}} |u-v|^2$$

for a.e. $(x,s) \in S^{n-1} \times [0,\varepsilon]$. ∇ is the gradient on S^{n-1}, $\bar{\nabla}$ is the gradient on $S^{n-1} \times [0,\varepsilon]$.

Remark: If $g \ge 0$ and Integrable on $B_\rho(y)$. Then $\int_{B_\rho(y) \setminus B_{\rho/2}(y)} g\, dx = \int_{\rho/2}^{\rho} \int_{\partial B_\sigma(y)} g\, d\sigma$

This establishes $\int_{\partial B_\sigma} g \le (\theta \cdot \rho/2)^{-1} \int_{B_\rho(y) \setminus B_{\rho/2}(y)} g\, dx$ for a mesure more than $(1-\theta)\rho/2$, $\theta \in (0,1)$.

i.e. $\int_{\partial B_\sigma} g > (\theta \cdot \rho/2)^{-1} \int_{B_\rho(y) \setminus B_{\rho/2}(y)} g\, dx$ for a measure no more than $\theta \cdot \rho/2$.

$w_\sigma(\omega) := w(y + \sigma\omega)$, $\omega \in S^{n-1}$. Then $w_{(\sigma)} \in W^{1,2}(S^{n-1}, \mathbb{R}^p)$

$$\int_{S^{n-1}} |D_\omega w|^2 d\omega \le \sigma^{3-n} \int_{\partial B_\sigma} |Dw|^2 \le 2\theta^{-1} \rho^{2-n} \int_{B_\rho \setminus B_{\rho/2}} |Dw|^2 \quad \text{for } \sigma \text{ with mesure} \ge (1-\theta)\rho/2$$
$\sigma \in (\rho/2, \rho)$.

Corollary 1. N is a compact smooth manifold embedded in $\mathbb{R}^p$. $\Lambda > 0$ a given const.
There are $\delta_0 = \delta_0(n, N, \Lambda)$ and $C = C(n, N, \Lambda)$ s.t.

(1) $u \in W^{1,2}(B_\rho(y), N)$ with $\rho^{2-n} \int_{B_\rho(y)} |\nabla u|^2 \le \Lambda$, $\exists\, \varepsilon$ s.t. $\varepsilon^{-2n} \cdot \rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2 \le \delta_0^2$, then there is $\sigma \in (\frac{3\rho}{4}, \rho)$ such that there is a function $w = w_\varepsilon \in W^{1,2}(B_\sigma(y), N)$ which agrees with u in a neighbor of $\partial B_\sigma(y)$ and

$$\sigma^{2-n} \int_{B_\sigma(y)} |Dw|^2 \le \varepsilon \rho^{2-n} \int_{B_\rho(y)} |Du|^2 + \varepsilon^{-1} C \rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2.$$

(2) If $\varepsilon \in (0, \delta_0]$, and if $u, v \in W^{1,2}(B_{(1+\varepsilon)\rho}(y) \setminus B_\rho(y); N)$ satisfies:

$$\rho^{2-n} \int_{B_{\rho(1+\varepsilon)}(y) \setminus B_\rho(y)} (|Du|^2 + |Dv|^2) \le \Lambda \quad \text{and} \quad \varepsilon^{-2n} \rho^{-n} \int_{B_{\rho(1+\varepsilon)}(y) \setminus B_\rho(y)} |u-v|^2 < \delta_0^2 .$$

Then there is $w \in W^{1,2}(B_{(1+\varepsilon)\rho}(y) \setminus B_\rho(y); N)$ s.t. $w = u$ in neighbor of $\partial B_\rho(y)$

$w = v$ in neighbor of $\partial B_{(1+\varepsilon)\rho}(y)$

And $$\rho^{2-n} \int_{B_{\rho(1+\varepsilon)}(y) \setminus B_\rho(y)} |Dw|^2 \le C \cdot \rho^{2-n} \int_{B_{(1+\varepsilon)\rho}(y) \setminus B_\rho(y)} (|Du|^2 + |Dv|^2) + C \cdot \varepsilon^{-2} \rho^{-n} \int_{B_{(1+\varepsilon)\rho}(y) \setminus B_\rho(y)} |u-v|^2$$

Reverse Poincaré inequality

If u is energy minimizing, $R^{2-n} \int_{B_R(x_0)} |Du|^2 < \Lambda$ for $B_R(x_0) \subset\subset \Omega$, and $\Lambda > 0$ a const

Then $(\rho/2)^{2-n} \int_{B_{\rho/2}(y)} |Du|^2 \le C \cdot \rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2$ for each $y \in B_{R/2}(x_0)$, $\rho < R/4$

$C = C(n, N, \Lambda)$

Proof:
$$\rho^{2-n} \int_{B_\rho(y)} |Du|^2 \le (R/2)^{2-n} \int_{B_{R/2}(y)} |Du|^2 \le 2^{n-2} R^{2-n} \int_{B_R(x_0)} |Du|^2 \le 2^n \Lambda .$$

$(\rho/2)^{2-n} \int_{B_{\rho/2}(y)} |Du|^2 \le C \cdot \rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2$, fix $\varepsilon_0 > 0$, assume $\rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2 < \varepsilon_0$

or we can let $C = 2^n \Lambda / \varepsilon$. Then $(\rho/2)^{2-n} \int_{B_{\rho/2}(y)} |Du|^2 < 2^n \Lambda \le C \cdot \rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2$

consider the case $\rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2 < \varepsilon_0$, choose ε_0 appropriately to use Coro 1.

Then there exists $\sigma \in (3\rho_0/4, \rho_0)$. $w \in W^{1,2}(B_\rho(y); N)$ s.t.

$$\sigma^{2-n} \int_{B_\sigma(y)} |Dw|^2 \le \delta \cdot \rho_0^{2-n} \int_{B_\rho(y)} |Du|^2 + \delta^{-1} C \cdot \rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2 .$$

$$\boxed{\delta^{-2n} \rho_0^{-n} \int_{B_{\rho_0}(y)} |u - \lambda_{y_0,\rho_0}|^2 \le \delta_0^2}$$

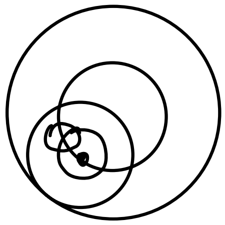
$w = u$ in a neighbor of $\partial B_\sigma(y)$

Then $$\sigma^{2-n} \int_{B_\sigma(y_0)} |Du|^2 \le \sigma^{2-n} \int_{B_\sigma(y_0)} |Dw|^2 \le \delta \cdot \rho_0^{2-n} \int_{B_{\rho_0}(y_0)} |Du|^2 + \delta^{-1} C \cdot \rho_0^{-n} \int_{B_{\rho_0}(y_0)} |u - \lambda_{y_0,\rho_0}|^2$$
$$\le \delta \cdot \Lambda + C \cdot \delta^{2n-1} \delta_0^2 < C\delta .$$

for $\sigma \in (3\rho_0/4, \rho_0)$

choose $y \in B_{\rho_0/2}(y)$. $\rho < \rho_0/4$. $\rho^{2-n} \int_{B_\rho(y)} |Du|^2 \le (\rho_0/2)^{2-n} \int_{B_{\rho_0/2}(y)} |Du|^2 \le (\tfrac{3}{2})^{n-2} (\tfrac{3\rho_0}{4})^{2-n} \int_{B_{3\rho_0/4}(y_0)} |Du|^2$

by Poincare inequality. $\rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2 \le C \cdot \rho^{2-n} \int_{B_\rho(y)} |Du|^2 < C\delta$ for $\rho < \rho_0/4$, $y \in B_{\rho_0/2}(y)$.



$C\delta \le \min\{ \varepsilon^{2n} \delta_0^2, \Lambda \}$.

$\varepsilon_0 < \delta^n \cdot \delta_0$

$$\rho^{2-n} \int_{B_{\rho/2}(y)} |Du|^2 \le C \cdot \varepsilon \cdot \rho^{2-n} \int_{B_\rho(y)} |Du|^2 + C \cdot \varepsilon^{-1} \rho^{-n} \int_{B_\rho(y)} |u - \lambda_{y,\rho}|^2$$

Monotonicity Formula

$$\rho^{2-n}\int_{B_\rho(y)}|Du|^2 - \sigma^{2-n}\int_{B_\sigma(y)}|Du|^2 = 2\int_{B_\rho(y)} |x-y|^{2-n}\left|\frac{\partial u}{\partial \vec{n}}\right|^2 > 0$$

Density Function

$\rho^{2-n}\int_{B_\rho(y)}|Du|^2 \downarrow$ as $\rho \downarrow$ with the same center. Let $\Theta_u(y) = \lim_{\rho\to 0^+}\rho^{2-n}\int_{B_\rho(y)}|Du|^2$

Lemma: Let $B_\rho(y)$ be any ball in $\mathbb{R}^n$, $\varphi : \{\text{convex set in } B_\rho(y)\} \longrightarrow [0,+\infty)$.

subadditive s.t. $\varphi(A) \le \sum_{i=1}^{n}\varphi(A_i)$ for $A \subseteq \bigcup_{i=1}^{n} A_i$

$\exists\ \varepsilon_1$ depends on n, k s.t.

$$\sigma^k\varphi(B_{\sigma/2}(z)) \le \varepsilon_1\sigma^k\varphi(B_\sigma(z)) + \gamma, \text{ where } B_{2\sigma}(z)\subset B_\rho(y)$$

Then $\rho^k\varphi(B_{\rho/2}(y)) \le C\gamma$

if ε is small enough, then the equality holds

By this Lemma, Reverse Poincare Inequa holds.

Compactness Theorem

Lemma 1. If $\{u_j\}$ is a sequence of energy minimizing maps in $W^{1,2}(\Omega;N)$, with $\sup_j |Du_j|^2 < \infty$ for each ball $B_\rho(Y)$ with $\overline{B_\rho}(Y)\subset\Omega$.

then there exists a subseq $\{u_{j'}\}$ and a minimizing map $u\in W^{1,2}(\Omega,N)$

s.t. $u_{j'} \to u$ in $W^{1,2}(B_\rho(y);\mathbb{R}^P)$ for each ball $\overline{B_\rho(y)}\subset\Omega$.

By Rellich Compactness Thm,

there exists $u_{j'} \xrightarrow{L^2_{loc}} u$ and $Du_{j'} \xrightarrow{L^2_{loc}} Du$ weakly.

The main point of Section 2.9 is to prove the strong convergence of Du_j in L^2 norm.

Some Claims.

1. Density Function is upper semi-continuous.

$$\limsup_{y\to y_0} \Theta_u(y) \le \Theta_u(y_0)$$

Define: $\operatorname{reg} u := \{x \in \Omega : u \text{ is } C^\infty \text{ at } x\}$.

$\operatorname{sing} u := \{x \in \Omega : x \notin \operatorname{reg} u\}$.

By ε-Regularity, $\exists\, \varepsilon > 0$, if $\rho^{2-n} \int_{B_\rho(y)} |Du|^2 < \varepsilon$ for some $B_\rho(y) \subset \Omega$.
(ε depends on $n.N.\Lambda$.) Then $y \in \operatorname{reg} u$ and $\sup_{B_{\rho/2}(y)} \rho^j \cdot |D^j u| \le C(n,j,N)$

Coro: $y \in \operatorname{reg} u \iff \Theta_u(y) = 0$

Claim: $\mathcal{H}^{n-2}(\operatorname{sing} u) = 0$

Let K be a compact subset of Ω, choose $\delta < \operatorname{dist}(K, \partial\Omega)$, for

For $y \in K$: $\int_{B_\rho(y)} |Du|^2 \ge \varepsilon \cdot \rho^{n-2}$ by virtue of Coro. $(\Theta_u(y) > \varepsilon)$

Now we choose a maximal pairw'e disjoint collection of balls $B_{\delta/2}(y_i)$ s.t. $y_i \in K \cap \operatorname{sing} u$. $B_{\delta/2}(y_i)$ are pairwise disjoint, $i = 1, 2, \cdots, J$. J is the maximal integer s.t. the disjoint balls exist.

Now we extend the radius, we claim that $\bigcup_{i=1}^{J} B_\delta(y_i) \supset K \cap \operatorname{sing} u$. Otherwise J is not maximal.

So $J\, \varepsilon \cdot (\delta/2)^{n-2} \le \int_{\bigcup_{i=1}^J B_{\delta/2}(y_i)} |Du|^2 \le \int_{\bigcup_{i=1}^J B_\delta(y_i)} |Du|^2 \le \int_{Q_\delta} |Du|^2$.

$Q_\delta := \{x : \operatorname{dist}(x, K \cap \operatorname{sing} u) < \delta\}$. (Obviously $B_\delta(y_i) \subset Q_\delta$)

J is bounded, as $J \cdot |B_{\delta/2}(y_i)| \le |K|$, let $\delta \downarrow 0$. we could see $\bigcup_{i=1}^J B_\delta(y_i)$ has Lebesgue measure 0.

Now by Dominated Convergence Thm, $\int_{Q_\delta} |Du|^2 \to 0$, Thus $J \cdot \delta^{n-2} \to 0$.

$\mathcal{H}^{n-2}(\operatorname{sing} u \cap K) = 0$, let $K \uparrow$ to Ω. we have $\mathcal{H}^{n-2}(\operatorname{sing} u) = 0$

Chapter 3.

$u: \Omega \to N$ energy minimizing map.

Tangent map. Let $\overline{B_{\rho_0}(y)} \subset \Omega$. $\rho > 0$

$u_{y,\rho}(x) := u(y+\rho x)$ for $|x| < \frac{\rho_0}{\rho}$. choose σ, let $\rho \cdot \sigma \le \rho_0$

$D u_{y,\rho}(x) = \rho\, Du(y+\rho x) \Rightarrow \sigma^{2-n} \int_{B_\sigma(0)} |Du_{y,\rho}|^2 = (\sigma\rho)^{2-n} \int_{B_{\sigma\rho}(y)} |Du|^2 \le \rho_0^{2-n} \int_{B_{\rho_0}(y)} |Du|^2$

let σ be fixed. $\rho \to 0$, then $(\sigma\rho)^{2-n} \int_{B_{\sigma\rho}(y)} |Du|^2 \longrightarrow \Theta_u(y)$

By the compactness theorem that $u_{y,\rho_j} \xrightarrow{W^{1,2}} \varphi$ as $\rho_j \to 0$, φ is energy minimizing.

Open question: uniqueness of φ by choosing different seq of ρ_j provided N is analytic.

call φ the tangent map. Then $\sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2 = \Theta_u(y)$ nomatter σ is.

let $\sigma \to 0^+$.

$$\Theta_u(y) = \Theta_\varphi(0) = \sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2, \quad \sigma > 0.$$

By monotonicity formula $0 = \rho^{2-n} \int_{B_\rho(0)} |D\varphi|^2 - \sigma^{2-n} \int_{B_\sigma(0)} |D\varphi|^2 = \int_{B_\rho(0) \setminus B_\sigma(0)} R^{2-n} \left|\frac{\partial \varphi}{\partial R}\right|^2$

$\Rightarrow \varphi(\lambda x) = \varphi(x). \quad \lambda > 0$

Claim 1. $y \in \operatorname{reg} u \Leftrightarrow \exists$ a constant tangent map φ of u at y.

Notice that $y \in \operatorname{reg} u \Leftrightarrow \Theta_u(y) = 0 \Leftrightarrow \varphi \equiv \text{const}$.

Homogeneous Degree Zero Minimizers.

if $\varphi(\lambda x) \equiv \varphi(x)$, $\lambda > 0$, $x \in \mathbb{R}^n$, we have several properties:

1. $\Theta_\varphi(y) \le \Theta_\varphi(0)$

Since $\int_{B_\rho(y)} R^{2-n} \left|\frac{\partial \varphi}{\partial R_y}\right| + \Theta_\varphi(y) = \rho^{2-n} \int_{B_\rho(y)} |D\varphi|^2 \le \rho^{2-n} (\rho+|y|)^{n-2} \cdot (\rho+|y|)^{2-n} \int_{B_{\rho+|y|}(0)} |D\varphi|^2$

$$= \frac{\rho^{2-n}}{(\rho+|y|)^{2-n}} \cdot \Theta_\varphi(0)$$

Let $\rho \to +\infty \Rightarrow \Theta_\varphi(y) \le \Theta_\varphi(0)$ for all $y \in \mathbb{R}^n$

2. $S(\varphi) = \{y \in \mathbb{R}^n : \Theta_\varphi(y) = \Theta_\varphi(0)\}$ is a linear subspace of $\mathbb{R}^n$.

and $\varphi(x) = \varphi(x+y)$ for $\forall\ x \in \mathbb{R}^n$. $y \in S(\varphi)$

$\Theta_\varphi(y) = \Theta_\varphi(0) \Rightarrow \frac{\partial \varphi}{\partial R_y} = 0$ a.e. in $\mathbb{R}^n$. $\Rightarrow \varphi(y+\lambda x) = \varphi(y)$, by $\varphi(\lambda x) = \varphi(x)$, $\lambda > 0$

$\Rightarrow \varphi(x) = \varphi(\lambda x - y + y) = \varphi(\lambda(y + \lambda^{-1}(\lambda x - y))) = \varphi(x + (\lambda - \lambda^{-1})y)$, $\lambda - \lambda^{-1} \in \mathbb{R}$.

Thus $\varphi(\lambda y) = \varphi(0)$ for $\forall\ \lambda > 0$. $\Rightarrow \Theta_\varphi(\lambda y) = \Theta_\varphi(0)$, thus $S(\varphi)$ is a linear subspace.

3. Now $S(\varphi)$ is linear $\subseteq \mathbb{R}^n$. $S(\varphi) = \mathbb{R}^n \iff \varphi \equiv$ const.

if φ is not const, $S(\varphi) \subsetneqq \mathbb{R}^n$. and 0 is obviously a singular point.

and $\varphi(z) = \varphi(0)$ for $z \in S(\varphi)$. the neighbor of z is the same as 0.

$\Rightarrow$ z is aslo a singular point $\Rightarrow$ $S(\varphi) \subset \operatorname{sing} \varphi$.

for all non-constant homogeneous degree zero minimizer φ.

Use Claim 1.

$y \in \operatorname{sing} u$. $\iff$ $\dim S(\varphi) \le n-1$ for every tangent map of u at y

define $S_j := \{ y \in \operatorname{sing} u : \dim S(\varphi) \le j$ for all tangent map φ of u at $y \}$.

Thus $S_0 \subset S_1 \cdots \subset S_{n-3} = S_{n-2} = S_{n-1} = \operatorname{sing} u$. Since $\mathcal{H}^{n-2}(\operatorname{sing} \varphi) = 0$ (Lemma 10, 2.10)

$\dim(S(\varphi)) \le n-3$.

Lemma 1. For each $j = 0, 1, \cdots, n-3$. $\dim S_j \le j$

For each $\alpha > 0$. $S_0 \cap \{x : \Theta_u(x) = \alpha\}$ is a discrete set.

Here dimension refers to Hausdorff dimension. $\dim S_j \le j$ means $\mathcal{H}^{j+\varepsilon}(S_j) = 0$ for $\varepsilon > 0$.

Corollary 1.

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$\dim \operatorname{sing} u \le n-3$, and if N is a 2-dimensional surface of genus ≥ 1.

then $\dim \operatorname{sing} u \le n-4$.

More generally, if all tangent maps $\varphi \in W^{1,2}_{loc}(\mathbb{R}^m; N)$ of u satisfy $\dim S(\varphi) \le m$

then $\dim \operatorname{sing} u \le m$.

We give proof of Lemma 1 here:

firstly, $S_0 \cap \{x : \Theta_u(x) = \alpha\}$ is discrete

Otherwise we have some $\alpha > 0$, $\{y_j\} \subset S_0 \cap \{x : \Theta_u(x) = \alpha\}$. $y_j \to y$.

Thus $\rho_j := |y_j - y|$. $u_{y,\rho_{j'}} \to \varphi$ for a subseq of ρ_j. $\Theta_\varphi(0) = \Theta_u(y) = \alpha$.

Let $\xi_j = |y_j - y|^{-1} (y_j - y)$. $\xi_{j'} \to \xi$ for subseq of ξ_j.

Then $\Theta_{u_{y,\rho_j}}(\xi_j) = \Theta_u(y_j) = \alpha \Rightarrow \lim\limits_{j' \to \infty} \Theta_{y,\rho_j}(\xi_j) = \Theta_\varphi(\xi) \ge \alpha$.

By $\Theta_\varphi(\xi) \le \Theta_\varphi(0) = \alpha \Rightarrow \Theta_\varphi(\xi) = 0 = \Theta_\varphi(0)$. Contradiction to $S(\varphi)$ is 0-dimension.

Secondly, we show $\dim S_j \le j$.

introduce $\eta_{y,\rho}(x) = \rho^{-1}(x-y)$, and Lemma 2:

Lemma 2. For each $y \in S_j$, $\delta > 0$ there is an $\varepsilon > 0$ (depending on u, y, δ).
s.t. for all $\rho \in (0, \varepsilon]$, $\eta_{y,\rho}\{x \in B_\rho(y) : \Theta_u(x) \geq \Theta_u(y) - \varepsilon\} \subset$ δ-neighbor of $L_{y,\rho}$.
where $L_{y,\rho}$ is a j-dimensional subspace of $\mathbb{R}^n$. (depending on y, ρ).

Proof of Lemma 2: If not, there exists $\delta > 0$, $y \in S_j$, $\rho_k \to 0_+$, $\varepsilon_k \to 0_+$
s.t. $\eta_{y,\rho_k}\{x \in B_{\rho_k}(y) : \Theta_u(x) \geq \Theta_u(y) - \varepsilon_k\} \not\subset$ δ-neighbor of L for every $L \subset \mathbb{R}^n$ j-dimension $(*)$

But then $u_{y,\rho_{k'}} \to \varphi$ a tangent map of u.
and $\Theta_u(y) = \Theta_\varphi(0)$, since $y \in S_j$, i.e. $\dim S(\varphi) \leq j$.
there is a j-D subspace $L_0 \supset S(\varphi)$ and $\alpha > 0$ s.t.

$$\Theta_\varphi(x) < \Theta_\varphi(0) - \alpha \text{ for all } x \in \bar{B}_1(0) \text{ and } \operatorname{dist}(x, L_0) \geq \delta.$$

Then $\{\Theta_{u_{y,\rho_k}}(x) \geq \Theta_\varphi(0) - \alpha\} \subset \{x \in \bar{B}_1(0), \operatorname{dist}(x, L_0) < \delta\}$. this is contradiction to $(*)$.

Now come back to the proof of Lemma 1.

Define $S_{j,i}$, $i \in \{1, 2, \cdots\}$ to be the set of y in S_j s.t. Lemma 2 holds for $\varepsilon = i^{-1}$.

Then, by lemma 2, $S_j = \bigcup_{i=1}^{\infty} S_{j,i}$. $S_{j,i,q} = \{x \in S_{j,i} : \Theta_u(x) \in (\frac{q-1}{i}, \frac{q}{i})\}$.

For $y \in S_{j,i,q}$, we have $S_{j,i,q} \subset \{x : \Theta_u(x) > \Theta_u(y) - \frac{1}{i}\}$. Hence By lemma 2,

$(*)$ $\eta_{y,\rho}(S_{j,i,q} \cap B_\rho(y)) \subset$ the δ-neighbor of $L_{y,\rho}$ for some j-dimensional $L_{y,\rho}$ of $\mathbb{R}^n$.

In view of arbitrariness of δ, the proof is completed with Lemma 3:

Lemma 3: There is a function $\beta : (0, +\infty) \to (0, +\infty)$ with $\lim_{t \to 0} \beta(t) = 0$
s.t. if $\delta > 0$ and A has property $(*)$, Then $\mathcal{H}^{j+\beta(\delta)}(A) = 0$

Top dimensional part of the singular set.

Define $\mathrm{sing}_* u = \{ y \in \mathrm{sing}\, u \mid$ exists tangent map φ of u at y s.t. $S(\varphi) = n-3 \}$

Thus $\mathrm{sing}\, u \setminus \mathrm{sing}_* u \subset S_{n-4}$, $\dim(\mathrm{sing}\, u \setminus \mathrm{sing}_* u) \le n-4$

To study this, the key point is homogeneous degree zero minimizer φ with $S(\varphi) = n-3$

without loss of generality. $S(\varphi) = \{0\} \times \mathbb{R}^{n-3}$, $\varphi(x,y) = \varphi_0(x)$. $x \in \mathbb{R}^3$, $y \in \mathbb{R}^{n-3}$

Thus $\mathrm{sing}\, \varphi_0 = \{0\}$ in $\mathbb{R}^3$. hence $\varphi_0|_{S^2}$ is C^∞. so that $\varphi_0|_{S^2} : S^2 \to N$ is a harmonic map.

Thm 3.6:

Let $\varphi^{(j)}$ be a seq of homogeneous degree 0 minimizers.

$\varphi^{(j)}(x,y) \equiv \varphi_0^{(j)}(x)$ for each j. and if $\limsup_{j\to\infty} \int_{B_1(0)} |D\varphi^{(j)}|^2 < \infty$

Then $\limsup_{j\to\infty} \sup_{S^2} |D^\ell \varphi_0^{(j)}| < \infty$ for each $\ell \ge 0$.

By Compact Theorem, there exists subseq $\varphi^{(j')} \to \varphi$. φ is a homogeneous degree zero minimizer

$\varphi(x,y) = \varphi_0(x)$. Thus $\varphi_0|_{S^2}$ is C^∞, for $z \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^{n-3}$, $\sigma < \mathrm{dist}(z, \{0\} \times \mathbb{R}^{n-3})$

we have $|\varphi(x) - \varphi(z)| < C\sigma$. $\Rightarrow \int_{B_\sigma(z)} |\varphi^{(j')} - \varphi(z)|^2 \le \int_{B_\sigma(z)} |\varphi^{(j')} - \varphi|^2 + C\sigma^2$

Apply ε-Regularity here $\Rightarrow$ $\varphi^{(j')} \to \varphi$ in C^k norm for $\forall k$ on a compact subset.

(σ is small, j' is large).

$\Rightarrow$ if $\int_{S^2} |D\varphi|^2 \le \Lambda$. then $\sup_{S^2} |D^\ell \varphi_0| \le C(\ell, N, \Lambda)$

Geometric Picture Near points of $\mathrm{sing}_* u$

$\varphi(x,y) \equiv \varphi_0(x)$, $x \in \mathbb{R}^3$. By definition, $\lim_{j\to\infty} \rho_j^{-n} \int_{B_{\rho_j}(z)} |u - \varphi^{(z)}|^2 = 0$ for some $\rho_j \to 0$

where $\varphi^{(z)} = \varphi((x,y) - z)$.

We claim:

For any homogeneous degree zero minimizing maps: $\varphi : \mathbb{R}^n \to N$, $\dim S(\varphi) = n-3$

$\bar{B}_{\rho_0}(z) \subset \Omega$. we have estimate:

$$\mathrm{sing}\, u \cap B_{\rho/2}(z) \subset \{x : \mathrm{dist}(x, (z + \{0\} \times \mathbb{R}^{n-3})) < \delta(\rho)\cdot\rho\}. \quad \forall \rho < \rho_0$$

$\delta(\rho) = C \cdot \left(\rho^{-n} \int_{B_\rho(z)} |u - \varphi^{(z)}|^2\right)^{1/n}$, C depends on $n. N. \Lambda \ge \rho_0^{2-n} \int_{B_{\rho_0}(z)} |Du|^2$.

Proof: Let $\rho < \rho_0$. $w = (\xi, \eta) \in \mathrm{sing}_* u \cap B_{\rho/2}(0)$. Take $\sigma = \beta_0 |\xi|$. $\beta_0 \le 1/2$ to be chosen

By ε-Regularity, there exists $\varepsilon_0(n, N, \Lambda) > 0$ s.t.

$$\varepsilon_0 \le \sigma^{-n} \int_{B_\sigma(w)} |u - \varphi(w)|^2 \le 2\sigma^{-n} \int_{B_\sigma(w)} |u - \varphi|^2 + 2\sigma^{-n} \int_{B_\sigma(w)} |\varphi - \varphi(w)|^2$$

From Thm 3.6 we know $|D\varphi_0(x)| \le C|x|^{-1}$, C depends only on N, Λ.

$\Rightarrow$ $|\varphi(w) - \varphi(x)| \le C \cdot |\xi|^{-1} \cdot \sigma \le C\beta_0$ for $x \in B_\sigma(w)$

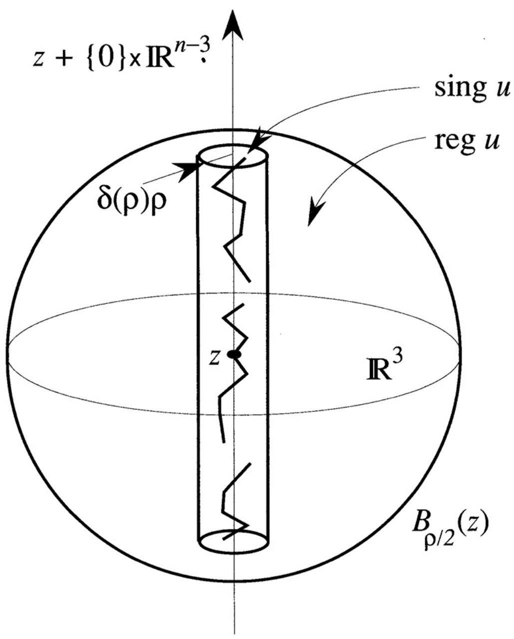
Hence, $\varepsilon_0 \le 2 \cdot \beta_0^{-n} \cdot |\xi|^{-n} \cdot \int_{B_\rho(0)} |u-\varphi|^2 + C \cdot \beta_0^2$, choose β_0 s.t. $C\beta_0^2 \le \frac{1}{2}\varepsilon_0$

we have $|\xi|^n \le C(\rho^{-n} \int_{B_\rho(0)} |u-\varphi|^2) \cdot \rho^n$. $\Rightarrow$ $|\xi| \le \delta(\rho) \cdot \rho$ $\square$

Now we show that if tangent map is unique. the geometric picture shows good information about $\mathrm{sing}_* u$.

Purpose: $\mathrm{sing}_* u$ is contained in a countable union of $(n-3)$-d Lipchitz graph

The Geometric Picture consequence is showed:



where $\delta(\rho) \to 0$ as $\rho \to 0$

3.8 Lemma 1. Let $j \in \{1, \cdots, n-1\}$. Suppose $\delta \in (0,1)$ and A is a subset of $\mathbb{R}^n$ s.t. $\forall y \in A$ there exists a j-D subspace L_y of $\mathbb{R}^n$ and $\rho_y > 0$

s.t. (i) $A \cap B_\rho(y) \subset \{x : \mathrm{dist}(x, y + L_y) \le \delta\rho\}$ for $\forall \rho < \rho_y$.

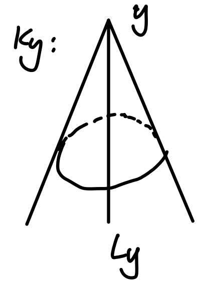
Then $A \subset \bigcup_{i=1}^{\infty} \Sigma_i$, where each Σ_i is the graph of a Lipschitz function over some j-d subspace i.e. there is an open subset U_i of some j-D subspace $L_i \subset \mathbb{R}^n$ and $f_i : L_i \to L_i^\perp$ is Lipschitz function, $\Sigma_i = \{x + f(x)\}$.

In Standard Terminology, A is j-rectifiable.

Proof: Decompose $A = \bigcup_{i=1}^{\infty} A_i$, $A_i := \{y \in A : \text{(i) holds for } \rho_y = \frac{1}{i}\}$, $A_i \subset A_{i+1}$

And A_i satisfies a uniform cone condition:

K_y is the cone s.t. $K_y = \{x : dist(x, y+L_y) < \delta|x-y|\}$.



Then $A_i \cap B_{i+1}(y) \subset K_y$. $\forall\, y \in A_i$

Now select j-dimensional subspaces $L_1, \cdots, L_Q$ of $\mathbb{R}^n$ s.t.

$\forall$ j-D subspace $L \subset \mathbb{R}^n$. $\exists\, L_j$ s.t. $\|L - L_j\| < \delta$.

i.e. $\forall\, y_j \in L_j$. $dist(y, L) < \delta|y|$. Then we decompose A_i:

$A_i = \bigcup_{j=1}^{Q} A_{ij}$. $A_{ij} := \{y \in A_i : \|L_y - L_j\| < \delta\}$. Then $A_{ij} \cap B_{i+1}(y) \subset y + K_j$.

where $K_j = \{x : dist(x, L_j) < 2\delta|x|\}$. $\Rightarrow A_{ij} \cap B_{i+1}(y)$ is contained in a graph of Lip-fun

(δ is small sufficiently) with domain $B_{i+1}(y') \cap L_j$.

(y' is projection of y to L_j)

That completes the proof. $\square$

Our main resulets are:

3.8 Coro 1: If A is a set satisfied the same hypothesis in 3.8 Lemma 1.

And there exists $\rho_y \equiv \rho_0 > 0$ is independent of $y \in A$, then $A \cap B_{\rho}(y)$ is contained in Finite Union of j-dimensional Lipschitz graphs for each $y \in A$.

If $L_y = y + L_0$ for some fixed subspace L_0, then $A \cap B_{\rho_0}(y)$ is contained in graph of Lipschitz func (independent of y) on L_0.

Now we come back to $sing_* u$, we get:

3.8 Coro 2:

There is a $\delta = \delta(n, N, K, \Lambda) > 0$. $K \subset\subset \Omega$. $\Lambda = \sup_{B_\rho(y) \subset K} \rho^{2-n} \int_{B_\rho(y)} |Du|^2$

s.t. For each $y \in sing_* u \cap K$, there exists φ s.t.

$$\rho^{-n} \int_{B_\rho(y)} |u - \varphi|^2 < \delta \text{ for } \forall\, \rho < \rho(\delta, K), \text{ then}$$

$sing_* u \cap B_{\rho(\delta,K)}(y)$ contained in an $(n-3)$ dimensional Lipschitz graph for each $y \in sing_* u \cap K$.