CSL 101- Discrete Mathematics
Indian Institute of Technology Bhilai
Tutorial Sheet 4

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1. A licence plate number consists of a sequence of four uppercase letters followed by three digits. How many licence plate numbers are there?

2. If n and d are positive integers, then d is a divisor of n, if $\frac{n}{d}$ is an integer. Determine the number of divisors of the integer

$$1,170,725,783,076,864 = 2^{17} \times 3^{12} \times 7^{8}.$$

3. Let $f \geq 2$, $m \geq 2$, and $k \geq 2$ be integers such that $k \leq f$ and $k \leq m$. The Carleton Computer Science program has f female students and m male students. The Carleton Computer Science Society has a Board of Directors consisting of k students. At least one of the board members is female and at least one of the board members is male. Determine the number of ways in which a Board of Directors can be chosen.

4. Let m and n be integers with $0 \leq m \leq n$. There are $n+1$ students in Carleton's Computer Science program. The Carleton Computer Science Society has a Board of Directors, consisting of one president and m vice-presidents. The president cannot be a vice president. Prove that

$$(n+1)\binom{n}{m} = (n+1-m)\binom{n+1}{m},$$

by determining, in two different ways, the number of ways to choose a Board of Directors.

5. In how many ways can you paint 200 chairs, if 33 of them must be painted red, 66 of them must be painted blue, and 101 of them must be painted green?

6. How many bitstrings of length 8 are there that contain at least 4 consecutive Os or at least 4 consecutive 1s?

7. A string of letters is called a palindrome if reading the string from left to right gives the same result as reading the string from right to left. For example, *madam* and *racecar* are palindromes. Recall that there are five vowels in the English alphabet: *a, e, i, o,* and *u*.

   In this exercise, we consider strings consisting of 28 characters, with each character being a lowercase letter. Determine the number of such strings that

8. A string of letters is called a palindrome if reading the string from left to right gives the same result as reading the string from right to left. For example, *madam* and *racecar* are palindromes. Recall that there are five vowels in the English alphabet: *a, e, i, o,* and *u*.

   In this exercise, we consider strings consisting of 28 characters, with each character being a lowercase letter. Determine the number of such strings that start and end with the same letter, or are palindromes, or contain vowels only.

9. Let $n \geq 1$ be an integer. Consider a tennis tournament with $2n$ participants. In the first round of this tournament, n games will be played and, thus, the $2n$ people have to be divided into n pairs. What is the number of ways in which this can be done?

10. A password consists of 100 characters, each character being a digit, a lowercase letter, or an uppercase letter. A password must contain at least one digit, at least one lowercase letter, and at least one uppercase letter. How many passwords are there?

    Hint: Recall De Morgan's Law.

$$\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$$

11. Let $n \geq 1$ be an integer.

    - Assume that n is odd. Determine the number of bitstrings of length n that contain more 0's than 1's. Justify your answer in plain English.
    - Assume that n is even.
        - Determine the number of bitstrings of length n in which the number of 0's is equal to the number of 1's.
        - Determine the number of bitstrings of length n that contain strictly more 0's than 1's.
        - Argue that the binomial coefficient

$$\binom{n}{n/2}$$

        is an even integer .

12. Determine the coefficient of x^{11} in the expansion of $(-17x + 714)^{555}$.

13. Let $k \geq 1$ be an integer and consider a sequence $n_1, n_2, \ldots, n_k$ of positive integers. Use a combinatorial proof to show that

$$\binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_k}{2} \leq \binom{n_1 + n_2 + \cdots + n_k}{2}.$$

    Hint: For each i with $1 \leq i \leq k$, consider the complete graph on n_i vertices. How many edges does this graph have?

14. Let $n \geq 1$ be an integer. Prove that

$$\sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} = \binom{2n}{n+1},$$

    by determining, in two different ways, the number of ways to choose $n + 1$ people from a group consisting of n men and n women.

15. Let $n \geq 66$ be an integer and consider the set $S = \{1, 2, \ldots, n\}$.

- Let k be an integer with $66 \leq k \leq n$. How many 66-element subsets of S are there whose largest element is equal to k?
- Use the result in the first part to prove that

$$\sum_{k=66}^{n} \binom{k-1}{66} = \binom{n}{65}.$$

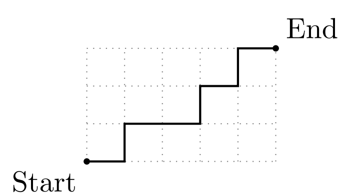
16. Let m and n be integers with $0 \leq m \leq n$, and let S be a set of size n. Prove that

$$\sum_{k=m}^{n} \binom{n}{k} \binom{k}{m} = 2^{n-m} \binom{n}{m},$$

by determining, in two different ways, the number of ordered pairs (A, B) with $A \subseteq S$, $|A| = m$, $B \subseteq S$, and $A \cap B = \emptyset$.

Hint: The size of B can be any of the values $n - m$, $n - (m + 1)$, $n - (m + 2), \ldots, n - n$. What is the number of pairs (A, B) having the properties above and for which $|B| = n - k$?

17. Let $m \geq 1$ and $n \geq 1$ be integers. Consider a rectangle whose horizontal side has length m and whose vertical side has length n. A path from the bottom-left corner to the top-right corner is called valid if, in each step, it either goes one unit to the right or one unit upwards. In the example below, you see a valid path for the case when $m = 5$ and $n = 3$.



How many valid paths are there?