CSL 101- Discrete Mathematics
Indian Institute of Technology Bhilai
Tutorial Sheet 3

1. Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove by mathematical indution that the arithmetic mean of these numbers is greater or equals to the geometric mean.

2. Show that for any given $n \geq 1$, the numbers $1, 2, ..., n$ can be arranged in a way that the average of any two of these numbers never appears between them. Assume n is a power of 2 and then use induction to prove this fact.

3. Prove that $\sqrt{2}$ is irrational using mathematical induction.

4. Prove that $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$, for all positive integers n, using mathematical induction.(Hints: In case applying inductive steps is a problem, think about a stronger statement)

5. Use mathematical induction to show that a rectangular checkerboard with an even number of cells and two squares missing, one white and one black, can be covered by dominoes.

6. Consider the following model and problem:
   Model: There are 102 coins on a table, 98 are showing heads, and 4 are showing tails. There are two legal moves:

   1. flip over any ten coins, or,
   2. if n is the current number of heads showing, you can place n+1 additional coins on the table, all showing tails.

   Problem: Choose a sequence of moves so that eventually there is exactly one coin showing heads.
   Your task: Prove that there is no sequence of moves that will solve the problem.

7. **Model:** The numbers 1,2,3,4,5 are written on a sheet of paper. In one step, the algorithm picks any two written numbers x and y, writes the value of $|x - y|$ on the paper, and erases x and y. Problem: Choose a sequence of steps so that eventually only 0's are written on the paper. To prove on this quiz: It is impossible for any algorithm to solve this problem. (Hint: Use an invariant involving sum.)

8. Let $x_1, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y1, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

9. Let R be a square in the Euclidean Plane.
   (a) If we select any 5 points inside R, then there must exists two points whose distance is at most $\frac{\sqrt{2}}{2}$.
   (b) If we select any 8 points inside R, then there must exists two points whose distance is at most $\frac{\sqrt{5}}{4}$.

10. A coin collector collects at least 1 coin each day for 100 consecutive days collecting a total of 150 coins. Show that for each value of n with $1 \leq n < 50$, there is a period of consecutive days during which he collected a total of exactly n coins.

11. Assuming that no one has more than 1,000,000 hairs on the head of any person and that the population of New York City was 8,008,278 in 2010, show there had to be at least nine people in New York City in 2010 with the same number of hairs on their heads.

12. Let x be an irrational number. Show that for some positive integer j not exceeding the positive integer n, the absolute value of the difference between jx and the nearest integer to jx is less than 1/n.

13. n couples arrived at a party and were greeted by the host and the hostess at the door. After several rounds of handshaking the host asked the guests as well as the hostess (his wife) to indicate the number of hands each one of them had shaken. He got $2n + 1$ different answers. Given that no one shook hands with his or her own spouse, how many hands had the hostess shaken?

14. Suppose you have sixteen numbers from 29,292,929 to 92,929,292. Show that there are at least [8] two of them such that either their sum or their difference is divisible by 29.