Bin Packing

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Here we consider the classical Bin Packing problem: We are given a set $I = \{1, \ldots, n\}$ of *items*, where item $i \in I$ has *size* $s_i \in (0, 1]$ and a set $B = \{1, \ldots, n\}$ of *bins* with *capacity* one. Find an assignment $a : I \to B$ such that the number of non-empty bins is minimal. As a shorthand, we write $s(J) = \sum_{j \in J} s_j$ for any $J \subseteq I$.

8.1 Hardness of Approximation

The Bin Packing problem is NP-complete. More specifically:

Theorem 8.1. *It is* NP*-complete to decide if an instance of* Bin Packing *admits a solution with two bins.*

Proof. We reduce from Partition, which we know is NP-complete. Recall that in the Partition problem, we are given n numbers $c_1, \ldots, c_n \in \mathbb{N}$ and are asked to decide if there is a set $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} c_i = \sum_{i \notin S} c_i$. Given a Partition instance, we create an instance for Bin Packing by setting $s_i = 2c_i / (\sum_{j=1}^{n} c_j) \in (0, 1]$ for $i = 1, \ldots, n$. Obviously two bins suffice if and only if there is a $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} c_i = \sum_{i \notin S} c_i$. $\square$

This allows us to derive a lower bound on the approximabilty of Bin Packing.

Corollary 8.2. *There is no ρ-approximation algorithm with $\rho < 3/2$ for* Bin Packing *unless* $\mathsf{P} = \mathsf{NP}$.

Algorithm for Bin packing

The algorithm is called the First Fit algorithm. It works as follows.

1. **For every item 1 to n**
    a. **If the item cannot be accommodated in the existing bins, then open a new Bin.**

Theorem: The above algorithm uses at most 2.opt number of bins.

Proof: . *Let C be the number of bins required for our algorithm. And let S be the sum of all the items.*

At most one bin can be more than half empty: otherwise the contents of the second half-full bin would be placed in the first.
Therefore, S > (C-1)/2. This implies, C< 2S+1. Since opt>=S, we have C<2.opt+1. Since C is an integer, we have C<=2.opt.

There is a further natural heuristic improvement of First Fit, called First Fit Decreasing: Reorder the items such that $s_1 \geq \cdots \geq s_n$ and apply First Fit. The intuition behind considering large items first is the following: "Large" items do not fit into the same bin anyway, so we already use unavoidable bins and try to place "small" items into the residual space.

Theorem 8.4. First Fit Decreasing *is a* 3/2-*approximation for* Bin Packing. *The algorithm runs in* $O\left(n^2\right)$ *time.*

Proof. Let k be the number of non-empty bins of the assignment a found by First Fit Decreasing and let k^* be the optimal number.

Consider bin number $j = \lceil 2/3k \rceil$. If it contains an item i with $s_i > 1/2$, then each bin $j' < j$ did not have space for item i. Thus j' was assigned an item i' with $i' < i$. As the items are considered in non-increasing order of size we have $s_{i'} \geq s_i > 1/2$. That is, there are at least j items of size larger than $1/2$. These items need to be placed in individual bins. This implies

$$k^* \geq j \geq \frac{2}{3}k.$$

Otherwise, bin j and any bin $j' > j$ does not contain an item with size larger than $1/2$. Hence the bins $j, j+1, \ldots, k$ contain at least $2(k-j)+1$ items, none of which fits into the bins $1, \ldots, j-1$. Thus we have

$$\begin{aligned} s(I) &> \min\{j-1, 2(k-j)+1\} \\ &\geq \min\{\lceil 2/3k \rceil - 1, 2(k-(2/3k+2/3))+1\} \\ &= \lceil 2/3k \rceil - 1 \end{aligned}$$

and $k^* \geq s(I) > \lceil 2/3k \rceil - 1$. This even implies

$$k^* \geq \left\lceil \frac{2}{3}k \right\rceil \geq \frac{2}{3}k$$

and hence the claim. □

Theorem 1 *There is no α-approximation algorithm with $\alpha < \frac{3}{2}$ for Bin Packing unless $P = NP$.*

Proof. Consider the following NP-hard problem.

Partition:

Input: set of items $S = \{1, ..., n\}$ with size $(0 \leq s_i \leq 1) \in Q^+$.

Question: Can we partition S into two parts S' and $S - S'$ such that $\sum_{i \in S} S_i = \sum_{j \in S-S'} S_j$?

Let I be an instance of partition. Scale all S_i's such that $\sum S_i = 2$ and let this instance I' be the input to Bin Packing. If all items of I' fit into 2 bins, since their total sum is 2, both bins must be full and therefore I is a yes instance (the partition is given by the items in 2 bins for I'). On the other hand, if I is a Yes instance, then the corresponding partition implies that the set of items in each part can be fit into one bin for the corresponding instance I'. Therefore, the set of items of I' can be fit into 2 bins if and only if I is a Yes instance. So if we can distinguish between 2 and ≥ 3 for I' then we can decide between Yes and No for I. Therefore, there is no better than $\frac{3}{2}$-approximation for bin packing unless P=NP.

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