CS 362, Lecture 24

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Today's Outline

- Reduction Wrapup
- Approximation algorithms for NP-Hard Problems

Hamiltonian Cycle

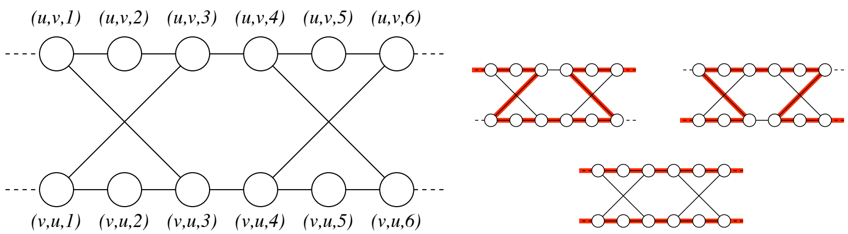
- A *Hamiltonian Cycle* in a graph is a cycle that visits every vertex exactly once (note that this is very different from an *Eulerian cycle* which visits every *edge* exactly once)
- The Hamiltonian Cycle problem is to determine if a given graph G has a Hamiltonian Cycle
- We will show that this problem is NP-Hard by a reduction from the vertex cover problem.

The Reduction

- To do the reduction, we need to show that we can solve Vertex Cover in polynomial time if we have a polynomial time solution to Hamiltonian Cycle.
- Given a graph G and an integer k, we will create another graph G' such that G' has a Hamiltonian cycle iff G has a vertex cover of size k
- As for the last reduction, our transformation will consist of putting together several "gadgets"

Edge Gadget and Cover Vertices

- For each edge (u, v) in G, we have an *edge gadget* in G' consisting of twelve vertices and fourteen edges, as shown below



An edge gadget for (u, v) and the only possible Hamiltonian paths through it.

Edge Gadget

- The four corner vertices $(u, v, 1)$, $(u, v, 6)$, $(v, u, 1)$, and $(v, u, 6)$ each have an edge leaving the gadget
- A Hamiltonian cycle can only pass through an edge gadget in one of the three ways shown in the figure
- These paths through the edge gadget will correspond to one or both of the vertices u and v being in the vertex cover.

Cover Vertices

- G' also contains k *cover vertices*, simply numbered 1 through k

Vertex Chains

- For each vertex u in G, we string together all the edge gadgets for edges (u, v) into a single *vertex chain* and then connect the ends of the chain to all the cover vertices
- Specifically, suppose u has d neighbors $v_1, v_2, \ldots, v_d$. Then G' has the following edges:
  - $d - 1$ edges between $(u, v_i, 6)$ and $(u, v_{i+1}, 1)$ (for all i between 1 and $d - 1$)
  - k edges between the cover vertices and $(u, v_1, 1)$
  - k edges between the cover vertices and $(u, v_d, 6)$

The Reduction

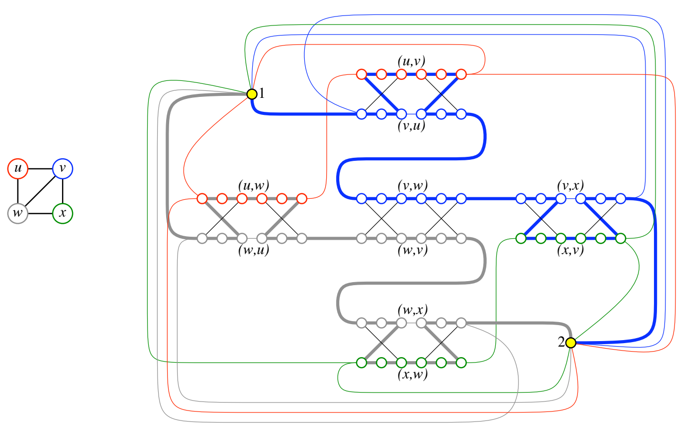
- It's not hard to prove that if $\{v_1, v_2, \ldots, v_k\}$ is a vertex cover of G, then G' has a Hamiltonian cycle
- To get this Hamiltonian cycle, we start at cover vertex 1, traverse through the vertex chain for v_1, then visit cover vertex 2, then traverse the vertex chain for v_2 and so forth, until we eventually return to cover vertex 1
- Conversely, one can prove that any Hamiltonian cycle in G' alternates between cover vertices and vertex chains, and that the vertex chains correspond to the k vertices in a vertex cover of G

Thus, G has a vertex cover of size k iff G' has a Hamiltonian cycle

The Reduction

- The transformation from G to G' takes at most $O(|V|^2)$ time, so the Hamiltonian cycle problem is NP-Hard
- Moreover we can easily verify a Hamiltonian cycle in linear time, thus Hamiltonian cycle is also in NP
- Thus Hamiltonian Cycle is NP-Complete

Example



The original graph G with vertex cover $\{v, w\}$, and the transformed graph G' with a corresponding Hamiltonian cycle (bold edges).
Vertex chains are colored to match their corresponding vertices.

The Reduction

$$\begin{tikzcd}
\boxed{\text{graph } G = (V, E),\ k} \arrow[r, "O(|V|^2)"] & \boxed{\text{graph } G'} \arrow[d, Rightarrow, "\text{Hamiltonian Cycle}"] \\
\boxed{\text{True or False}} & \boxed{\text{True or False}} \arrow[l, "O(1)"']
\end{tikzcd}$$