CSL 252- Design and Analysis of Algorithms
Indian Institute of Technology Bhilai
Tutorial Sheet 1

Asymptotic Notations

1. True or false: (a) $2n = O(n)$, (b) $n^2 = O(n)$, (c) $n^2 = O(n\log^2 n)$, (d) $n\log n = O(n^2)$, (e) $3^n = 2^{O(n)}$, (f) $2^{2^n} = O(2^{2^n})$.

2. * True or false: (a) $n = o(2n)$, (b) $2n = o(n^2)$, (c) $2^n = o(3^n)$, (d) $1 = o(n)$, (e) $n = o(\log n)$, (f) $1 = o(1/n)$.

3. Check for each pair of expressions below, whether $f(n)$ is $O(g(n))$, $\Omega(g(n))$, $\Theta(g(n))$. Assume $k \geq 1$; $\epsilon > 0$; $c > 1$ are all constants.
   (a) $f(n) = \log^k n$, $g(n) = n^\epsilon$,
   (b) $f(n) = n^k$, $g(n) = c^n$,
   (c) $f(n) = \sqrt{n}$, $g(n) = n^{\sin n}$,
   (d) $f(n) = 2^n$, $g(n) = 2^{\frac{n}{2}}$,
   (e) $f(n) = \log(n!)$, $g(n) = \log(n^n)$.

4. Prove that if $f = O(g)$, then $f + g = \Theta(g)$.

5. Let $f(n)$ and $g(n)$ be two asymptotically non-negative functions. Prove that $\max\{f(n), g(n)\} \in \Theta(f(n) + g(n))$.

6. * Prove that $o(g(n)) \cap \omega(g(n))$ is an empty set.

7. * Prove that $n! \in \omega(2^n)$, and $n! \in o(n^n)$.

8. Consider the function $g(n) = f(n) + \frac{a_1}{n} + \frac{a_2}{n^2}$.
   (a) Prove or disprove, $g(n) = O(f(n) + 1)$.
   (b) Prove or disprove, $g(n) = O(f(n))$.

9. Let for $n \geq 1$, $H_n = 1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n}$. Prove that H_n is in $\Theta(\log n)$.

10. Prove that if $f_1(n) = \Omega(g_1(n))$ and $f_2(n) = \Omega(g_2(n))$, then $f_1(n) + f_2(n) = \Omega(\min\{g_1(n), g_2(n)\})$.

11. . Suppose that $f_1(n) = \Theta(g_1(n))$ and $f_2(n) = \Theta(g_2(n))$. Is it true that $f_1(n) + f_2(n) = \Theta(g_1(n) + g_2(n))$? Is it true that $f_1(n) + f_2(n) = \Theta(\max\{g_1(n), g_2(n)\})$? Is it true that$f_1(n) + f_2(n) = \Theta(\min g_1(n), g_2(n))$? Justify your answer.

12. . Prove that if $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n)f_2(n) = O(g_1(n)g_2(n))$.

13. Prove that if $f_1(n) = \Omega(g_1(n))$ and $f_2(n) = \Omega(g_2(n))$, then $f_1(n)f_2(n) = \Omega(g_1(n)g_2(n))$.

14. Prove or disprove: For all functions $f(n)$ and $g(n)$, either $f(n) = O(g(n))$ or $g(n) = O(f(n))$.

15. Prove or disprove: If $f(n) > 0$ and $g(n) > 0$ for all n, then $O(f(n) + g(n)) = f(n) + O(g(n))$.

16. Prove or disprove: $O(f(n)^a) = O(f(n))^a$ for all $a \in R^+$.

17. Prove or disprove: $O((x + y)^2) = O(x^2) + O(y^2)$.

18. Multiply $\log n + 6 + O(\frac{1}{n})$ by $n + O(\sqrt{n})$ and simplify your answer as much as possible.

19. Show that big-O is transitive. That is, if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

20. Prove that if $f(n) = O(g(n))$, then $f(n)^k = O(g(n)^k)$.

21. Prove or disprove: If $f(n) = O(g(n))$, then $2^{f(n)} = O(2^{g(n)})$.

22. Prove or disprove: If $f(n) = O(g(n))$, then $\log(f(n)) = O(\log g(n))$.

23. Suppose $f(n) = \Theta(g(n))$. Prove that $h(n) = O(f(n))$ iff $h(n) = O(g(n))$.

24. Prove or disprove: If $f(n) = O(g(n))$, then $f(n)/h(n) = O(g(n)/h(n))$.