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Tutorial 06 and 07

1) Sol^n:- For 2x2 matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$, condition for

+ve definitness $a>0$ and $ac-b^2>0$

$$S_1 = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \Rightarrow 9-b^2>0$$

$$9>b^2$$

$$-3<b<3$$

$$S_2 = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix} \Rightarrow 2c-16>0$$

$$c>8 \quad \text{Ans}$$

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2) Sol^n:- $A = \begin{bmatrix} 2 & \sqrt{10} & \sqrt{35} \\ \sqrt{10} & 5 & \sqrt{14} \\ \sqrt{35} & \sqrt{14} & 7 \end{bmatrix}$

$$x^T A x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & \sqrt{10} & \sqrt{35} \\ \sqrt{10} & 5 & \sqrt{14} \\ \sqrt{35} & \sqrt{14} & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= 2x_1^2 + 5x_2^2 + 7x_3^2 + 2\sqrt{10}\, x_1 x_2 + 2\sqrt{35}\, x_1 x_3 + 2\sqrt{14}\, x_2 x_3$$

$$= (\sqrt{2}\, x_1)^2 + (\sqrt{5}\, x_2)^2 + (\sqrt{7}\, x_3)^2 + 2\sqrt{5}\sqrt{2}\, x_1 x_2 + 2\sqrt{7}\sqrt{5}\, x_1 x_3$$

$$+ 2\sqrt{2}\sqrt{7}\, x_2 x_3$$

$$= (\sqrt{2}\, x_1 + \sqrt{5}\, x_2 + \sqrt{7}\, x_3)^2$$

$$x^T A x = (\sqrt{2}\, x_1 + \sqrt{5}\, x_2 + \sqrt{7}\, x_3)^2$$

$x^T A x \geq 0$; $x^T A x = 0$ iff $x_1 = x_2 = x_3 = 0$ ≥ 0

global minima at origin exist.

$x^T A x > 0 \Rightarrow x_1 \neq x_2 = x_3$ global minima does not exist at origin

Ans

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3) Sol^n:- Given: $S, T \rightarrow$ SPDM
$ST \nrightarrow$ Symmetric, $\lambda > 0$

$$STx = \lambda x$$

$$(STx)^T Tx = (\lambda x)^T Tx$$

$$x^T T^T S^T Tx = \lambda x^T Tx$$

R.H.S.: $\lambda x^T Tx > 0$; since T is SPDM $\forall\ x \neq 0$

L.H.S.: let $y = Tx$ OR $x = T^{-1}y$

$$x^T T^T S^T T x = x^T TSTx$$

$$= (T^{-1}y)^T TST(T^{-1}y)$$

$$= y^T \cdot T^{-1} TSTT^{-1} y$$

$$= y^T S y > 0$$

$\therefore$ both L.H.S. & R.H.S. are +ve, it follows that $\lambda > 0$

Ans

For V:

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4) Sol^n:- $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$; $A^TA = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}\right) = 0 \Rightarrow (25-\lambda)^2 - 400 = 0$$

$$25 - \lambda = \pm 20$$

$$\boxed{\lambda_1 = 45 \quad \lambda_2 = 5}$$

singular values $\sigma_1 = \sqrt{45}$; $\sigma_2 = \sqrt{5}$

Eigen vectors corresponding to ~~λ~~ $\lambda_1 = 45$

$$(A^TA - 45I)X_1 = 0 \Rightarrow \begin{bmatrix} 25-45 & 20 \\ 20 & 25-45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -20 & 20 \\ 20 & -20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -20x_1 + 20x_2 &= 0 \\ 20x_1 - 20x_2 &= 0 \end{aligned}$$

$x_1 = x_2 = 1$ (say) $\quad \therefore X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$X_1' = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Eigen vectors corresponding to $\lambda_2 = 5$

$$(A^TA - 5I)X_2 = 0 \Rightarrow \begin{bmatrix} 20 & 20 \\ 20 & 20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$\Rightarrow$ $x_1 = -x_2$ $\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$X_2' = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

Thus: $\boxed{V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$

For U: $AA^T = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 12 & 41 \end{bmatrix}$

$\det\left(\begin{bmatrix} 9-\lambda & 12 \\ 12 & 41-\lambda \end{bmatrix}\right) = 0 \Rightarrow \lambda^2 - 50\lambda + 225 = 0$

$\lambda_1 = 45 \quad \lambda_2 = 5$

E.V. correspond to $\lambda_1 = 45$; $(AA^T - 45I)X_1 = 0$

$\begin{bmatrix} 9-45 & 12 \\ 12 & 41-45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_1 = x_2$ eigenvector $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$X_1' = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$

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For $\lambda_2 = 5$ $\quad (AA^T - 5I)\,x_2 = 0$

$$\begin{bmatrix} 9-5 & 12 \\ 12 & 41-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -3x_2$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \; ; \; X_2' = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

Thus: $\boxed{U = \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}}$

$$\boxed{\Sigma = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}}$$

Ans

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5) Solⁿ:- $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

Right-singular matrix 'V': A^TA

$$A^TA = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det(A^TA - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 5-\lambda & -2 & 1 \\ -2 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right) = [0]$$

$$\Rightarrow \lambda(\lambda^2 - 7\lambda + 6) = 0 \; ; \; \lambda_1 = 6, \; \lambda_2 = 1, \; \lambda_3 = 0$$

Eigen Vector for $\lambda_1 = 6$

$$(A^TA - 6I)X_1 = 0 \Rightarrow \begin{bmatrix} -1 & -2 & 1 \\ -2 & -5 & 0 \\ 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 - 2x_2 + x_3 = 0$$
$$-2x_1 - 5x_2 + 0 \cdot x_3 = 0$$
$$x_1 + 0x_2 - 5x_3 = 0$$

$$x_1 = 5$$
$$x_2 = -2$$
$$x_3 = 1$$

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$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \qquad X_1' = \begin{bmatrix} 5/\sqrt{30} \\ -2/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix}$$

Eigen Vector for $\lambda_2 = 1$

$$(A^TA - I)X_2 = 0 \Rightarrow \begin{bmatrix} 4 & -2 & 1 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 2x_2 + x_3 = 0 \Rightarrow x_3 = 2x_2$$
$$-2x_1 + 0x_2 + 0x_3 = 0$$
$$x_1 + 0x_2 + 0x_3 = 0 \Rightarrow \boxed{x_1 = 0} \quad \boxed{x_2 = 1} \quad \boxed{x_3 = 2}$$

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad X_1' = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

Eigen vector for $\lambda_3 = 0$

$$(A^TA - 0 \cdot I)X_3 = 0 \Rightarrow \begin{bmatrix} 5 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 2x_2 + x_3 = 0$$
$$-2x_1 + x_2 + 0x_3 = 0$$
$$x_1 + 0x_2 + x_3 = 0$$

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad X_3' = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

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Thus: $V = \begin{bmatrix} 5/\sqrt{30} & 0 & 1/\sqrt{6} \\ -2/\sqrt{30} & -1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{30} & 2/\sqrt{5} & -1/\sqrt{6} \end{bmatrix}$ Ans

Singular values Σ: $\sigma_1 = \sqrt{\lambda_1}$ $\sigma_2 = \sqrt{\lambda_2}$ $\sigma_3 = \sqrt{\lambda_3}$

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Ans

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6) Soln :- DFT of sequence $x[n] = \{0, 1, 2, 3\}$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

here $N = 4 \therefore k = 0, 1, 2, 3$

For $k=0$: $X[0] = x[0] + x[1] + x[2] + x[3]$

$= 0 + 1 + 2 + 3$

$= 6$

$k=1$: $X[1] = x[0] + x[1] e^{-j\frac{2\pi \times 1}{4}} + x[2] e^{-j\frac{2\pi \times 2}{4}} + x[3] e^{-j\frac{2\pi \times 3}{4}}$

$= 0 + 1 \cdot e^{-j\frac{\pi}{2}} + 2 \cdot e^{-j\pi} + 3 \cdot e^{-j\frac{3\pi}{2}} = 1j - 2$

$k=2$: $X[3] = x[0] + x[1] e^{-j\frac{2\pi 2}{4}} + x[2] e^{-j\frac{2\pi 4}{4}} + x[3] e^{-j\frac{2\pi 6}{4}}$

$= -2$

$k=3$: $X[3] = x[0] + x[1] e^{-j\frac{2\pi 3}{4}} + x[2] e^{-j\frac{2\pi 6}{4}} + x[3] e^{-j\frac{2\pi 9}{4}}$

$= -2j - 2$

DFT of $x[n]$: $X[k] = \{6, -2+j, -2, -2-2j\}$ Ans

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1) Sol^n:- DFT of $x[n]$:

$$X[K] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

DFT of $x[n-n_0]$

let $y[n] = x[n-n_0]$, so DFT of $y[n]$ is:

$$Y[K] = \sum_{n=0}^{N-1} y[n] e^{-j\frac{2\pi}{N}kn}$$

$$Y[K] = \sum_{n=0}^{N-1} x[n-n_0] e^{-j\frac{2\pi}{N}kn}$$

(say) $m = n - n_0 \Rightarrow n = m + n_0$

$$Y[K] = \sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}k(m+n_0)}$$

$$Y[K] = e^{-j\frac{2\pi}{N}kn_0} \underbrace{\sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}km}}_{X[K]}$$

$$Y[K] = e^{-j\frac{2\pi}{N}kn_0} X[K]$$ Ans

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8) Sol^n:- We know $X[K] = \sum_{n=0}^{N-1} x[n] \exp\left(-j \frac{2\pi}{N} kn\right)$

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[K] \exp\left(j \frac{2\pi}{N} nk\right)$$

$$\Rightarrow x[n] = \frac{1}{N}\left[\sum_{n=0}^{N-1} X[k] e^{j(2\pi/N)kn}\right] = \frac{1}{N}\left[\underbrace{\sum_{n=0}^{N-1} X^*[k] e^{-j\left(\frac{2\pi}{N}\right)nk}}_{\text{DFT of } X^*[K]}\right]^*$$

$\therefore$ Algorithm used to evaluate the DFT can be used to evaluate the IDFT.

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9) Sol^n:- (i) For radix-3 DIT FFT, $x[n]$ is decimated by a factor of 3 to form three sequences of length $\frac{N}{3}$:

$$f_1(n) = x(3n) \qquad n = 0, 1, \ldots \frac{N}{3} - 1$$

$$\cancel{g(n)}\ f_2(n) = x(3n+1)$$

$$f_3(n) = x(3n+2)$$

Expressing N-pt DFT in sequence.

$$X[k] = \sum_{n=0,3,6\ldots} x(n) W_N^{nk} + \sum_{n=1,4,5\ldots} x(n) W_N^{nk} + \sum_{n=2,5,7\ldots} x(n) W_N^{nk}$$

$$= \sum_{l=0}^{\frac{N}{3}-1} f(l) W_N^{3lk} + \sum_{l=0}^{\frac{N}{3}-1} g(l) W_N^{(3l+1)k} + \sum_{l=0}^{\frac{N}{3}-1} h(l) W_N^{(3l+2)k}$$

since $W_N^{3lk} = W_{N/3}^{lk}$, then

$$X[k] = \sum_{l=0}^{\frac{N}{3}-1} f(l) W_{N/3}^{lk} + W_N^{k} \sum_{l=0}^{\frac{N}{3}-1} g(l) W_{N/3}^{lk} + W_N^{2k} \sum_{l=0}^{\frac{N}{3}-1} h(l) W_{\frac{N}{3}}^{lk}$$

$$\therefore X[k] = F[k] + W_N^{k} G[k] + W_N^{2k} H(k)$$

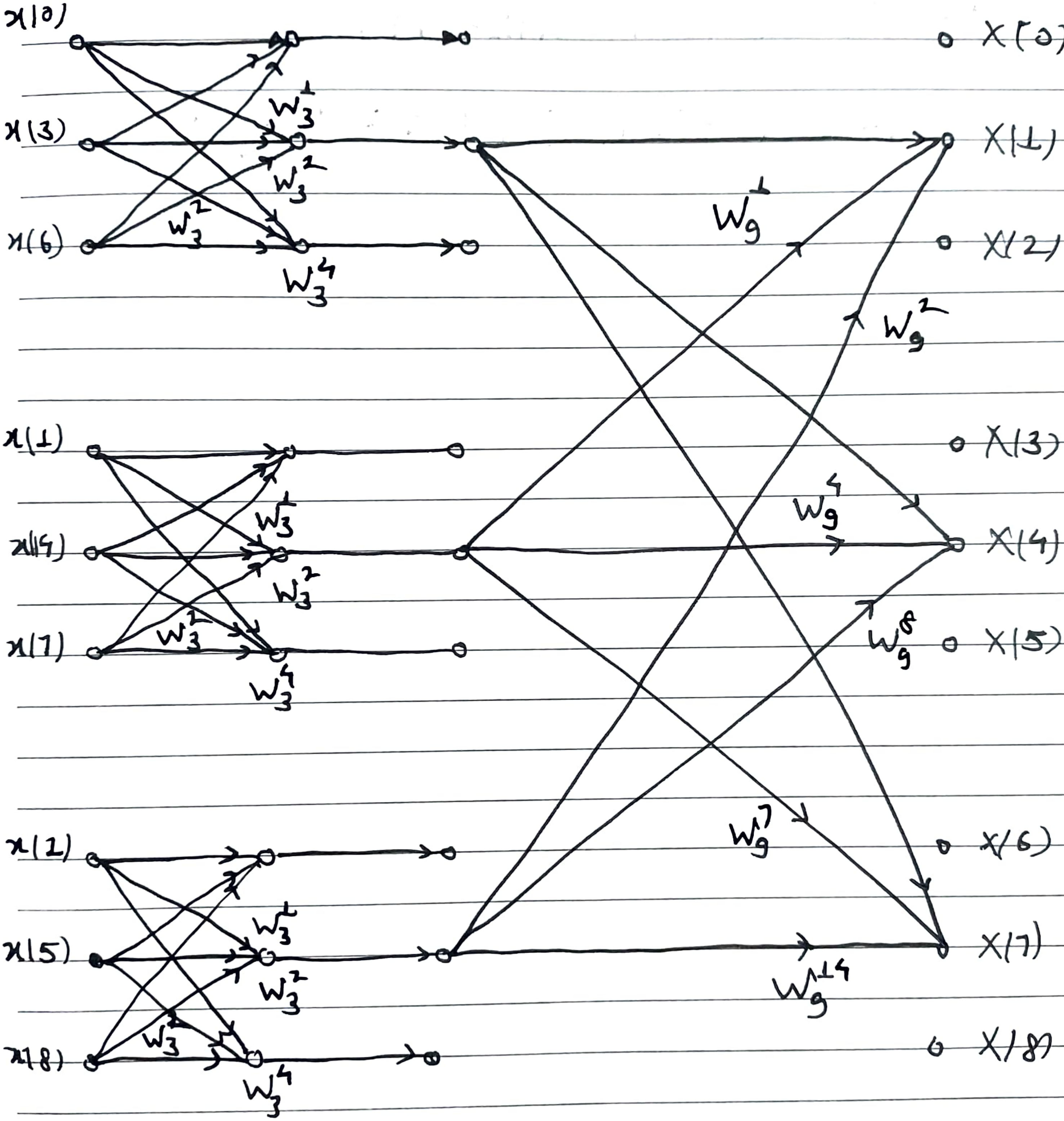

Fig: Flowgraph of 9-point DIT FFT

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ii) • $N = 3^v$: v stages in radix-3 FFT

Total no. of multiplications is : $6N \log_3 N$