DS100/DSL201: Mathematical Foundations for Data Science

Tutorial 04

06 September 2024

Total points: 20

Question 1 **(2 points)**

(i) Given two vectors $x = (3, 4)$ and $y = (2, 3)$, compute the L_1, L_2, and L_∞ distances between them.

(ii) For vectors $a = (1, 2, 3)$ and $b = (4, 5, 6)$, calculate the L_3 distance.

Question 2 **(4 points)**

Given a data points $P_1 = (0.2, 2)$ and $P_2 = (0.3, 3)$ find the Mahalanobis distance between P_1 and P_2 given that matrix

(i) $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(ii) $\Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

Explain the difference between (i) and (ii).

Question 3 **(2 points)**

For an orthonormal matrix A, $A^T A = I$, if x is vector which is transformed by A i.e. $y = Ax$ than show that L_2 norm of x and L_2 norm of y both are equal.

Question 4 **(5 points)**

Consider a function $d : X \times X \to \mathbb{R}$ defined on a set X. The function d is given by:

$$d(x, y) = |x - y|^{\frac{1}{2}}$$

where $x, y \in X$. Determine whether d is a metric on X.

Question 5 **(4 points)**

Given the following over-determined system of linear equations:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) Find the least-squares solution x to the system $Ax = b$ using the formula for the orthogonal projection. Recall that the least-squares solution x can be found by solving the normal equations $A^T Ax = A^T b$.

(b) Calculate the orthogonal projection of b onto the column space of A.

Question 6 **(3 points)**

Consider a continuous-time signal $x(t) = \sin(2\pi f t)$, where $f = 5$ Hz. The signal is sampled at a rate $f_s = 20$ Hz and then quantized. Answer the following questions:

a. Calculate the sampling interval T_s.

b. Determine the number of samples taken over one signal period.

c. Given the signal amplitude ranges from -1 to 1, and a quantization resolution of 5 bits, calculate the quantization step size Δ.

Question 2 **(4 points)**

Given a data points $P_1 = (0.2, 2)$ and $P_2 = (0.3, 3)$ find the Mahalanobis distance between P_1 and P_2 given that matrix

(i) $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(ii) $\Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

Explain the difference between (i) and (ii).

Solⁿ:- Given $P_1 = (0.2, 2)$

$P_2 = (0.3, 3)$

(i) $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad d_m(a,b) = \sqrt{(a-b)^T M (a-b)}$

$$d_m(P_1, P_2) = \left\{\left(\begin{bmatrix} 0.3 \\ 3 \end{bmatrix} - \begin{bmatrix} 0.2 \\ 2 \end{bmatrix}\right)^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0.3 \\ 3 \end{bmatrix} - \begin{pmatrix} 0.2 \\ 2 \end{pmatrix}\right)\right\}^{1/2}$$

$$= \left\{ \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} \right\}^{1/2}$$

$$= \left\{ \begin{bmatrix} 0.1 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} \right\}^{1/2}$$

$$= (0.01 + 1)^{1/2}$$

$$= (1.01)^{1/2}$$

$$= 1.004 \text{ Ans}$$

ii) $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$d_m = \left\{\left(\begin{bmatrix} 0.3 \\ 3 \end{bmatrix} - \begin{bmatrix} 0.2 \\ 2 \end{bmatrix}\right)^T \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0.3 \\ 3 \end{bmatrix} - \begin{bmatrix} 0.2 \\ 2 \end{bmatrix}\right)\right\}^{1/2}$$

$$d_m = \left\{ \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} \right\}^{1/2} = \left\{ \underbrace{\begin{bmatrix} 0.1 & 1 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 0.1 \\ 1 \end{bmatrix}}_{2\times 1} \right\}^{1/2}$$

$$= \left\{ \begin{bmatrix} 0.1 & 1 \end{bmatrix} \begin{bmatrix} 0.3 \\ 1 \end{bmatrix} \right\}^{1/2} = (0.03 + 1)^{1/2}$$

$$= 1.014 \text{ Ans}$$

$P_1 = (0.2, 2) \quad P_2 = (0.3, 3)$

i) $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad d_m(P_1, P_2) = \sqrt{(P_2 - P_1)^T \Sigma^{-1} (P_2 - P_1)}$

Since inverse of an identity matrix is identity itself, therefore

$d_m = 1.004$ Ans

ii) $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \Sigma^{-1} = \frac{1}{3-0} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$

$$d_m(P_1, P_2) = \left\{ \left(\begin{bmatrix} 0.3 \\ 3 \end{bmatrix} - \begin{bmatrix} 0.2 \\ 2 \end{bmatrix} \right)^T \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0.3 \\ 3 \end{bmatrix} - \begin{bmatrix} 0.2 \\ 2 \end{bmatrix} \right) \right\}^{1/2}$$

$$d_m(P_1, P_2) = \left\{ \begin{bmatrix} 0.1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 \\ 1 \end{bmatrix} \right\}^{1/2}$$

$$= \left\{ \begin{bmatrix} 0.1 & 1 \end{bmatrix} \begin{bmatrix} 0.033 \\ 1 \end{bmatrix} \right\}^{1/2}$$

$$= 1 + 0.0033 = (1.0033)^{1/2}$$

$d_m(P_1, P_2) = 1.0016$ Ans

Question 3 **(2 points)**

For an orthonormal matrix A, $A^T A = I$, if x is vector which is transformed by A i.e. $y = Ax$ than show that L_2 norm of x and L_2 norm of y both are equal.

Sol^n:- Given $A^T A = I$ & $y = Ax$

$$\|y\| = \|Ax\|$$

$$= (Ax)^T (Ax)$$

$$= x^T \underbrace{A^T A}_{I} x$$

$$= x^T x$$

$$\boxed{\|y\| = \|x\|}$$ Ans

Question 6 **(3 points)**

Consider a continuous-time signal $x(t) = \sin(2\pi f t)$, where $f = 5$ Hz. The signal is sampled at a rate $f_s = 20$ Hz and then quantized. Answer the following questions:

a. Calculate the sampling interval T_s.

b. Determine the number of samples taken over one signal period.

c. Given the signal amplitude ranges from -1 to 1, and a quantization resolution of 5 bits, calculate the quantization step size Δ.

Sol^n:- Given Signal freq. $(f) = 5\,Hz$

Sampling freq. $(f_s) = 20\,Hz$

(a) Sampling Interval $f_s = \frac{1}{T_s} = \frac{1}{20} = 0.05$ sec Ans

(b) Time period of signal $(T) = \frac{1}{f} = \frac{1}{5} = 0.2$ sec

No. of samples $(N) = \frac{T}{T_s} = \frac{0.2}{0.05} = 4$ Ans

(c) $\Delta = \frac{V_{max} - V_{min}}{2^n} = \frac{1-(-1)}{2^5} = \frac{2}{2^5} = \frac{2}{32} = \frac{1}{16} = 0.0625$ Ans

Question 4 **(5 points)**

Consider a function $d : X \times X \to \mathbb{R}$ defined on a set X. The function d is given by:

$$d(x, y) = |x - y|^{\frac{1}{2}}$$

where $x, y \in X$. Determine whether d is a metric on X.

Sol^n:- $\because$ $d : X \times X \longrightarrow \mathbb{R}$ It is a metric if it satisfies

i) $d(a, b) \geq 0$ {Non Negativity}

ii) $d(a, b) = 0$ iff $a = b$ {identity}

iii) $d(a, b) = d(b, a)$ {Symmetry}

iv) $d(a, b) \leq d(a, c) + d(c, b)$ {Δ^{el} inequality}

Given $d(x, y) = |x - y|^{1/2}$

i) Since function $d(x, y)$ is always non-negative bcz absolute value is always non-negative.

ii) Identity: $d(x, y) = |x - y|^{1/2}$

$$d(x, y) = 0$$
$$\Rightarrow |x - y|^{1/2} = 0 \text{ iff } x = y$$

Hence identity property is also satisfied.

iii) $d(x, y) = |x - y|^{1/2} = |y - x|^{1/2} = d(y, x)$

Therefore symmetry is also satisfied.

iv) Δ^{el} inequality: $d(x, y) \leq d(x, z) + d(z, y)$ $\forall$ $x, y, z \in X$

$$d(x, y) = |x - y|^{1/2}$$

$$[d(x, y)]^2 = |x - y| \leq |x - z| + |z - y|$$

$$\leq |x - z| + |z - y| + 2\sqrt{|x - z|}\sqrt{|z - y|}$$

$$= \left(\sqrt{|x - z|} + \sqrt{|z - y|}\right)^2$$

$$= \left[d(x,z) + d(z,y)\right]^2$$

$\therefore$ Δ^{el} inequality also satisfied.

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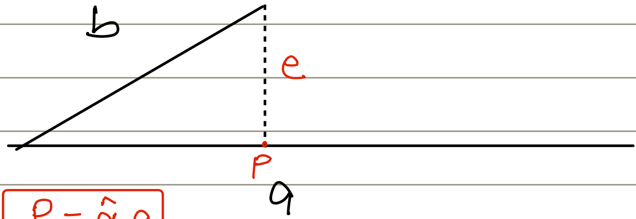
(b) Calculate the orthogonal projection of b onto the column space of A.

Solⁿ:-

$$A^T A x = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b$$



$$\boxed{p = \hat{x} a}$$

$$p = A (A^T A)^{-1} A^T b$$