Linear Programming: An Overview

- Goal is to ***maximizing profit*** or ***minimizing costs*** and meet ***constraints***.

LP Model Formulation
A Maximization Example

- Product mix problem - Beaver Creek Pottery Company
- How many bowls and mugs should be produced to maximize profits given labor and materials constraints?
- Product resource requirements and unit profit:

	Resource Requirements		
Product	Labor (Hr./Unit)	Clay (Lb./Unit)	Profit ($/Unit)
Bowl	1	4	40
Mug	2	3	50

LP Model Formulation
A Maximization Example

Resource Availability: 40 hrs of labor per day
120 lbs of clay

Decision Variables: x_1 = number of bowls to produce per day
x_2 = number of mugs to produce per day

Objective Function: Maximize $Z = \$40x_1 + \$50x_2$
Where Z = profit per day

Resource Constraints: $1x_1 + 2x_2 \leq 40$ hours of labor
$4x_1 + 3x_2 \leq 120$ pounds of clay

Non-Negativity Constraints: $x_1 \geq 0;\ x_2 \geq 0$

Model Components

- **Objective function** - a linear mathematical relationship describing an objective of the firm, in terms of decision variables - this function is to be maximized or minimized.
- **Constraints** – requirements or restrictions placed on the firm by the operating environment, stated in linear relationships of the decision variables.

LP Model Formulation
A Maximization Example

Complete Linear Programming Model:

Maximize $Z = 40x_1 + 50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Feasible Solutions

A ***feasible solution*** does not violate ***any*** of the constraints:

Example: $x_1 = 5$

$x_2 = 10$

$Z = 40x_1 + 50x_2 = 700$

constraint check: $1(5) + 2(10) = 25 < 40$ hours

constraint check: $4(5) + 3(10) = 50 < 120$

Infeasible Solutions

An ***infeasible solution*** violates ***at least one*** of the constraints:

Example: $x_1 = 10$

$x_2 = 20$

$Z = 40x_1 + 50x_2 = 1400$

Labor constraint check: $1(10) + 2(20) = 50 > 40$ hours

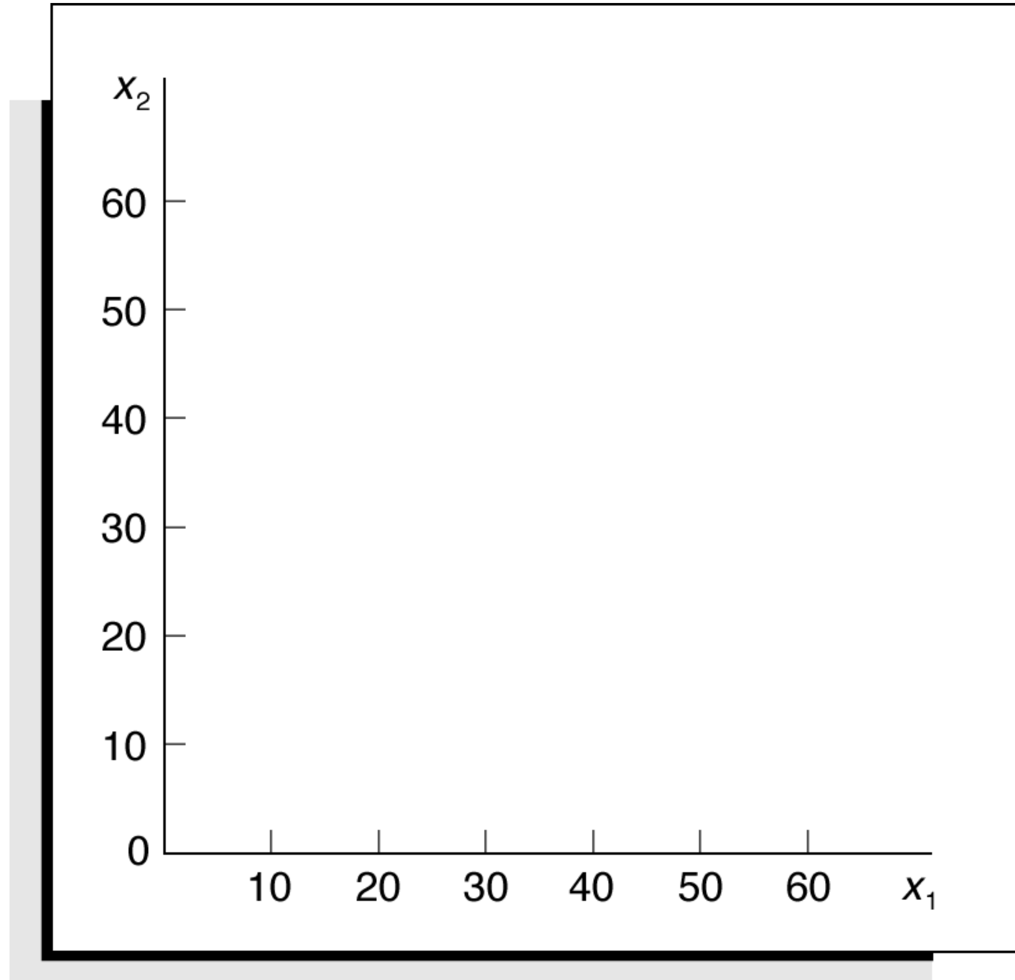
Graphical Solution of Maximization Model

Maximize $Z = 40x_1 + 50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Coordinates for Graphical Analysis

Graphical Solution of Maximization Model

Maximize $Z = 40x_1 + 50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

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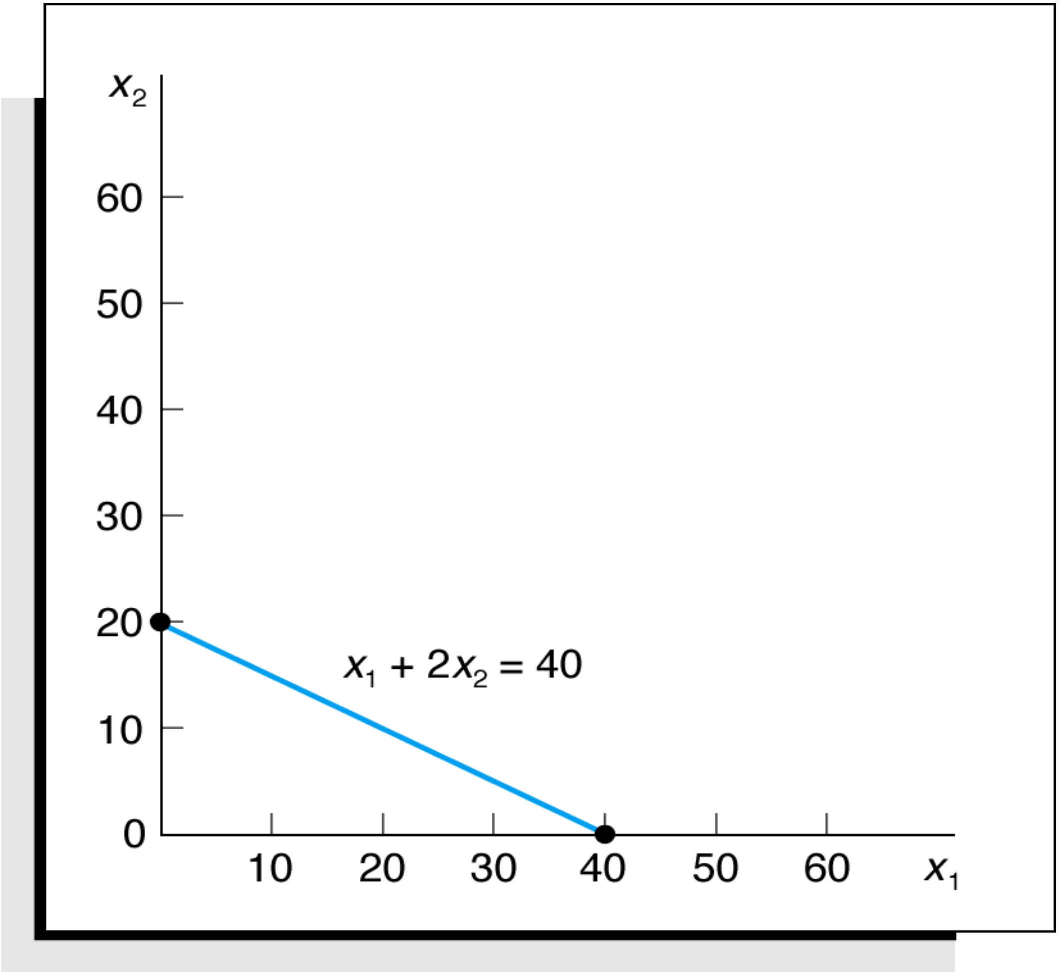

Graphical Solution of Maximization Model

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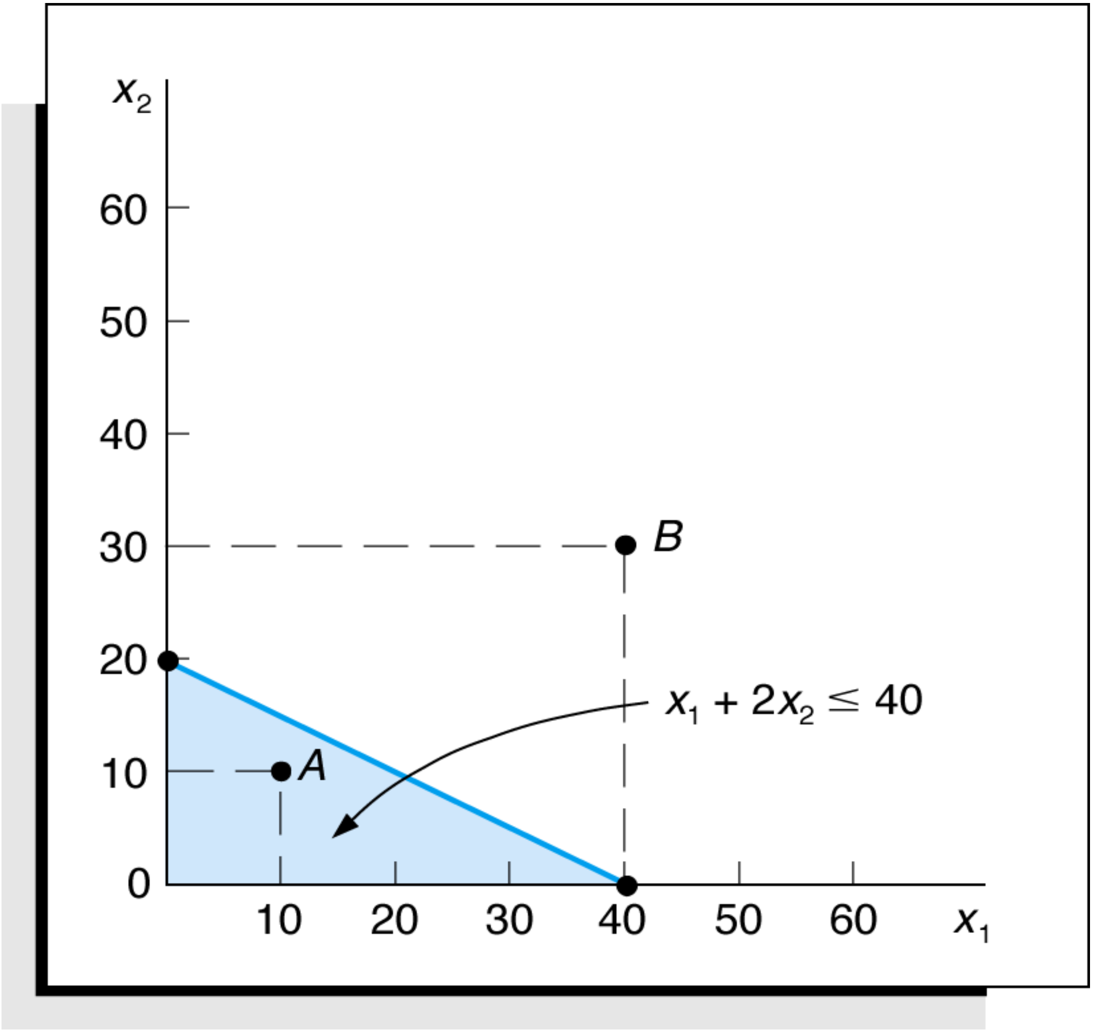

Graphical Solution of Maximization Model

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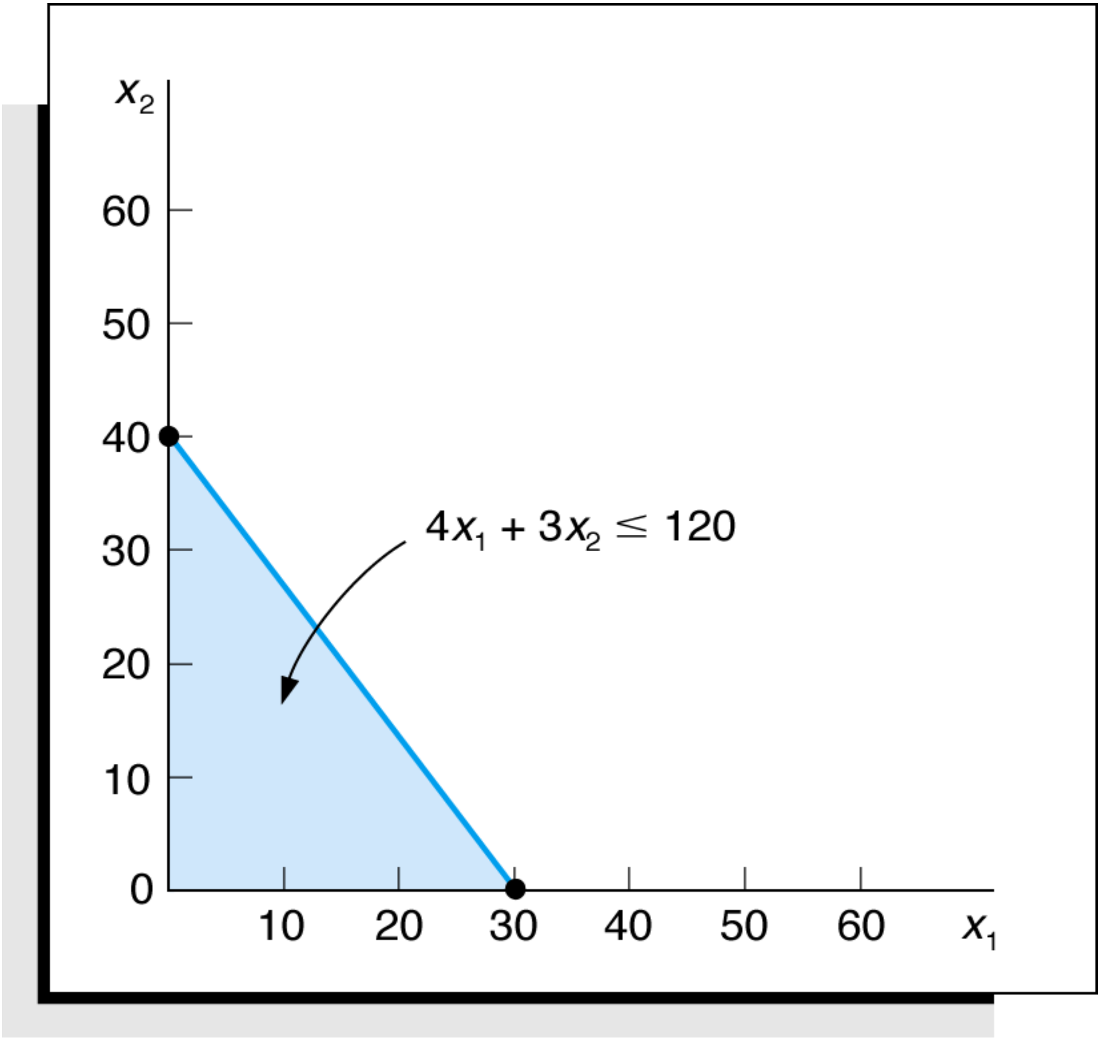

Graphical Solution of Maximization Model

Maximize $Z = 40x_1 + 50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

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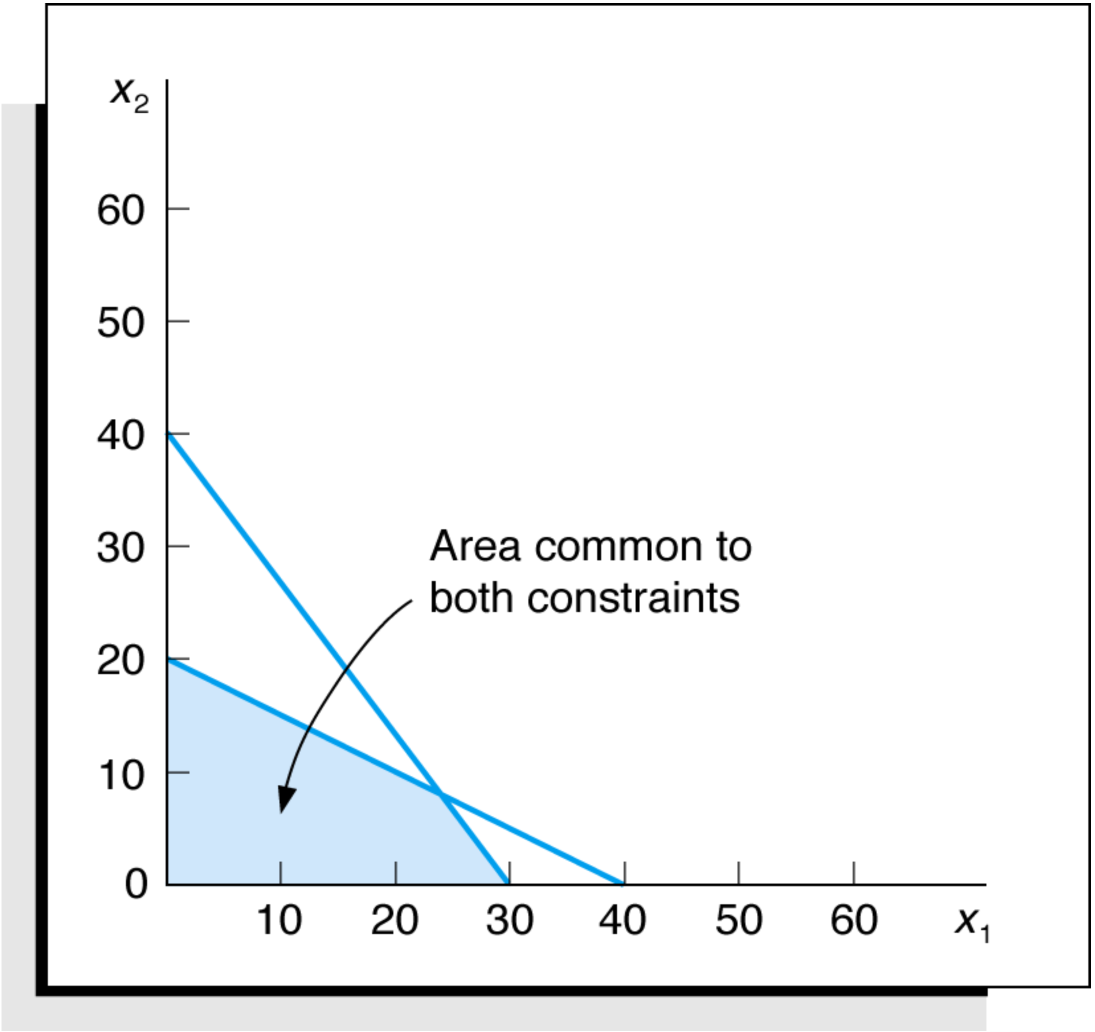

Graphical Solution of Maximization Model

Maximize $Z = 40x_1 + 50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

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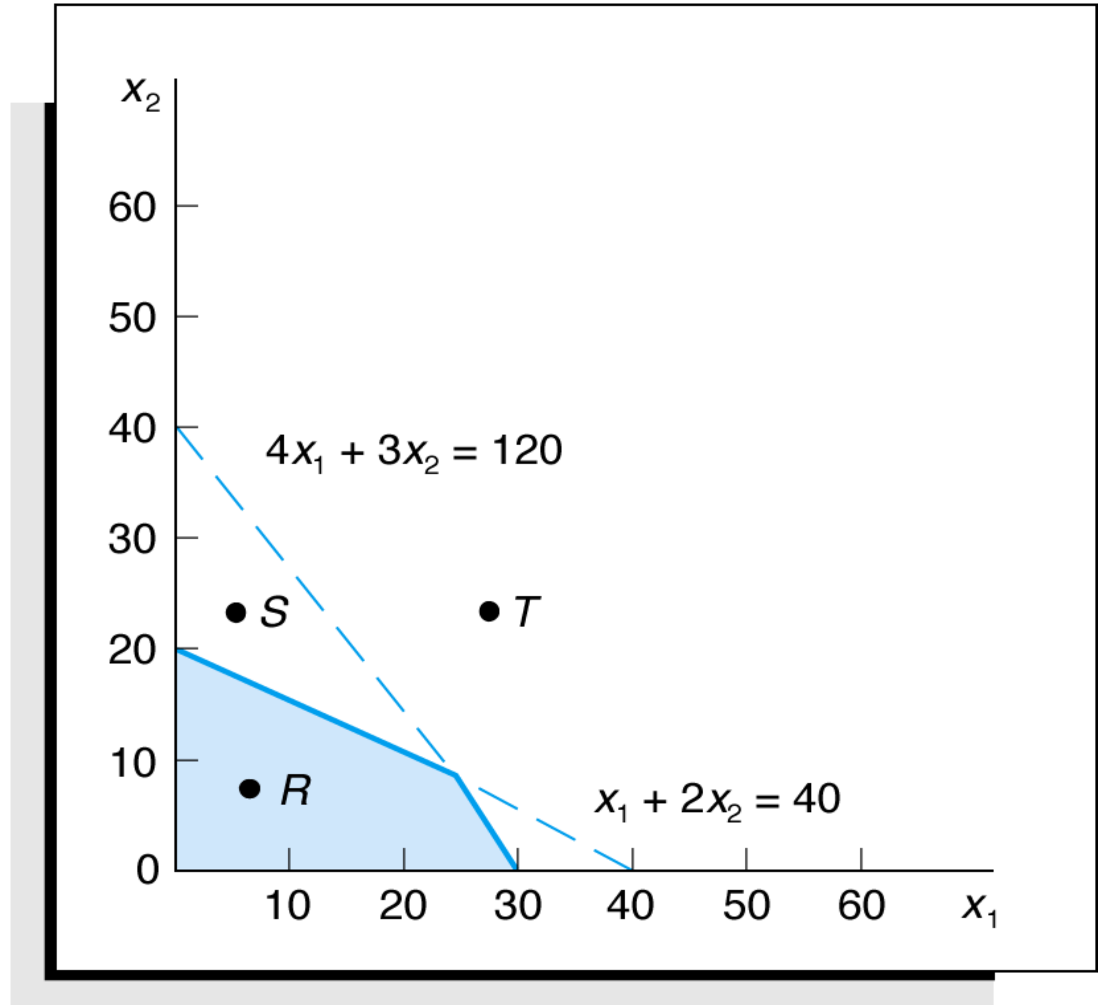


Feasible Solution Area

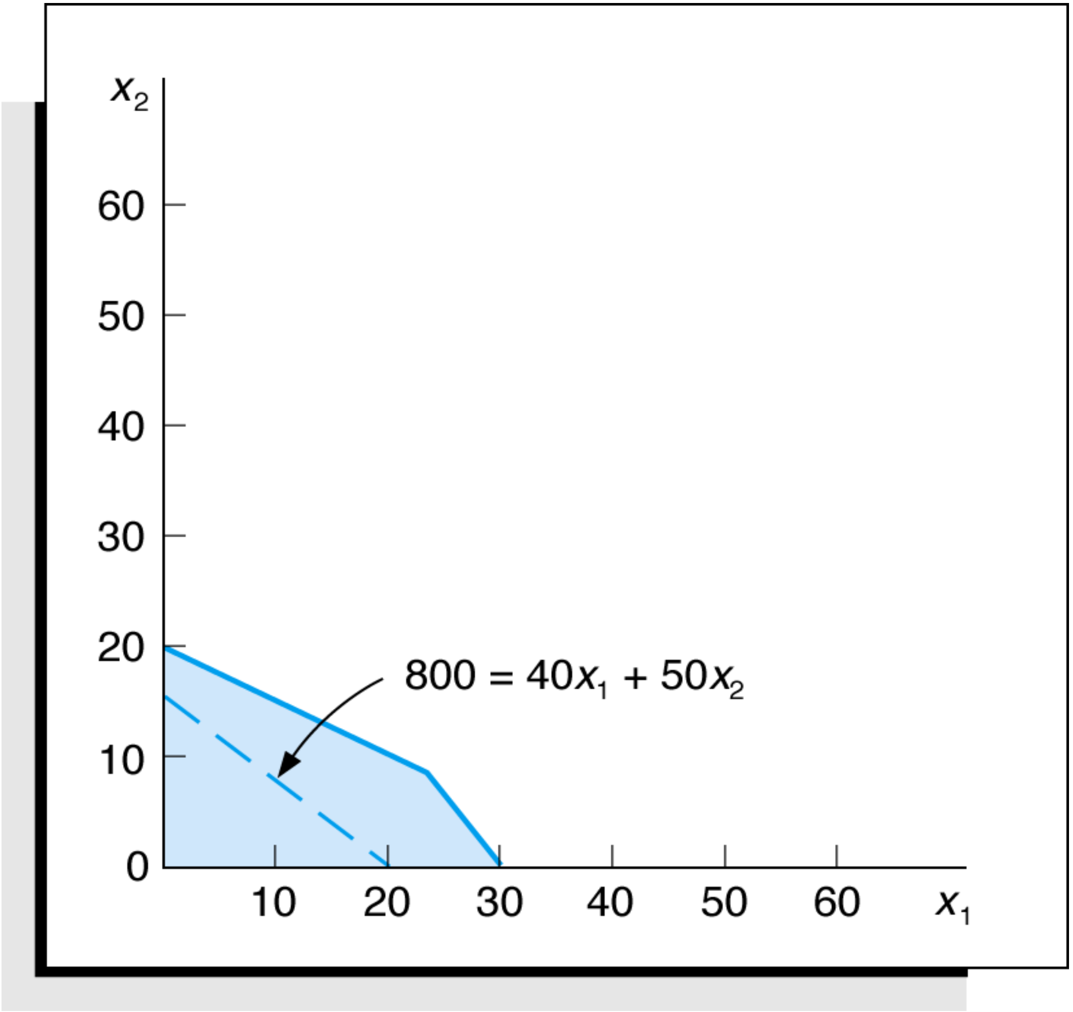
Objective Function Solution = 800
Graphical Solution of Maximization

Maximize $Z = 40x_1 + 50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Objection Function Line for Z = 800

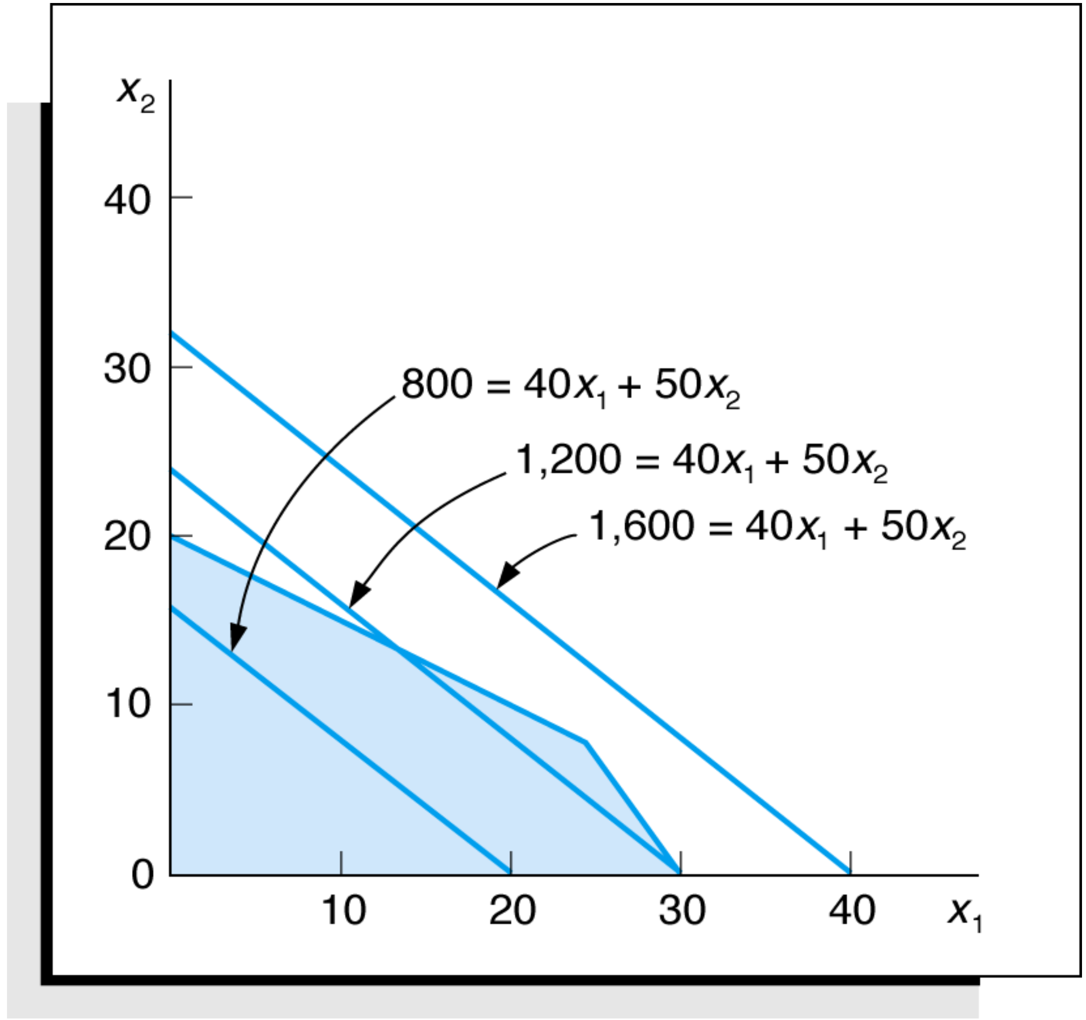
Alternative Objective Function Solution Lines Graphical Solution of Maximization Model

Maximize $Z = 40x_1 + 50x_2$

subject to: $1x_1 + 2x_2 \le 40$

$4x_2 + 3x_2 \le 120$

$x_1, x_2 \ge 0$



Alternative Objective Function Lines

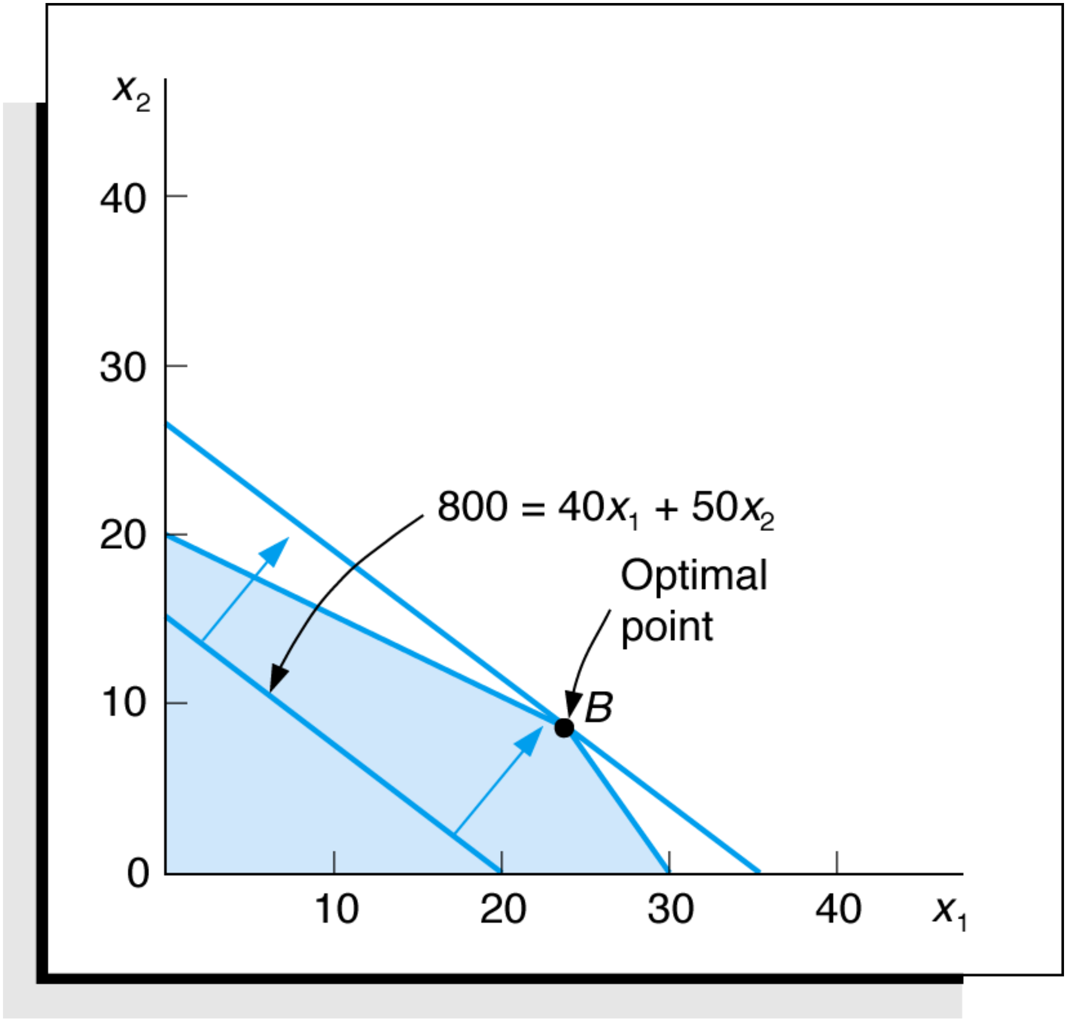
Optimal Solution
Graphical Solution of Maximization

Maximize $Z = 40x_1 + 50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Identification of Optimal Solution Point

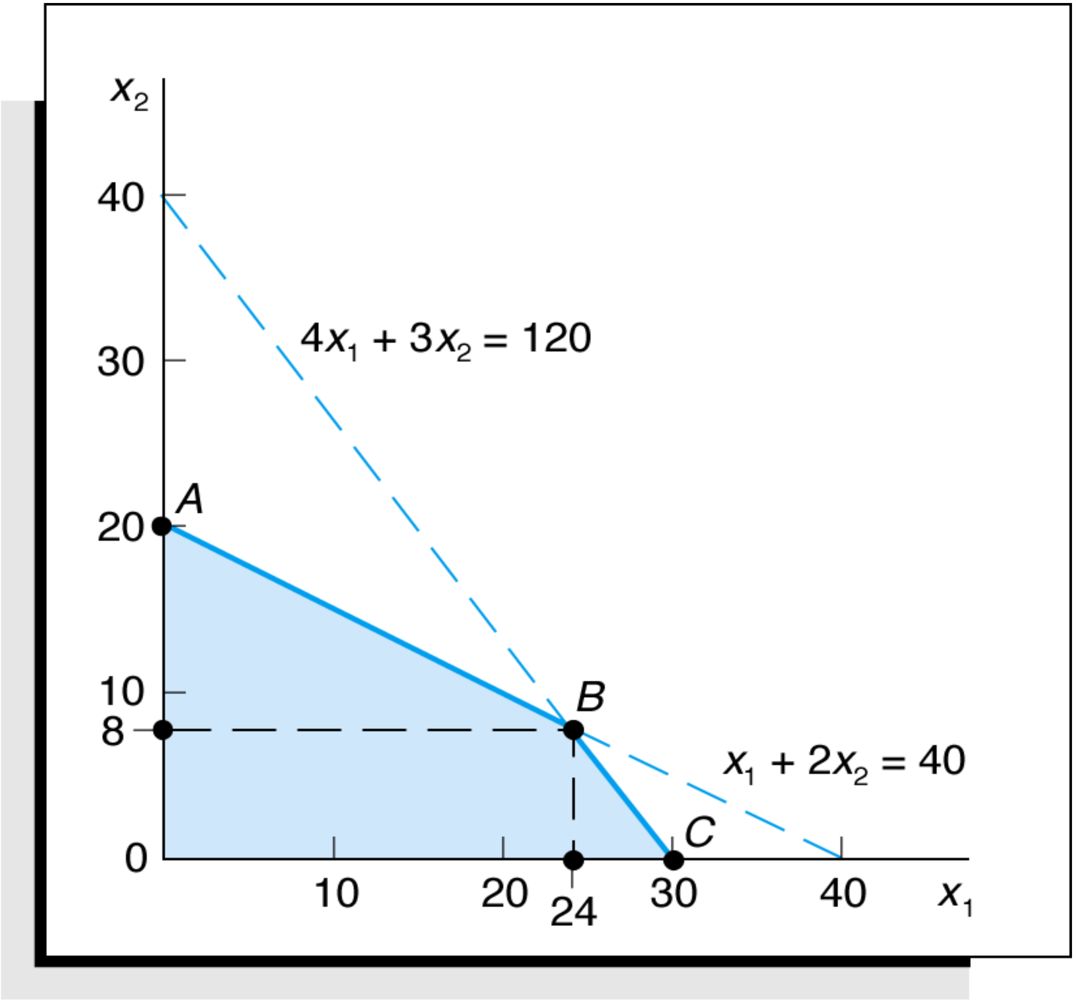
Optimal Solution Coordinates Graphical Solution of Maximization Model

Maximize $Z = 40x_1 + 50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Optimal Solution Coordinates

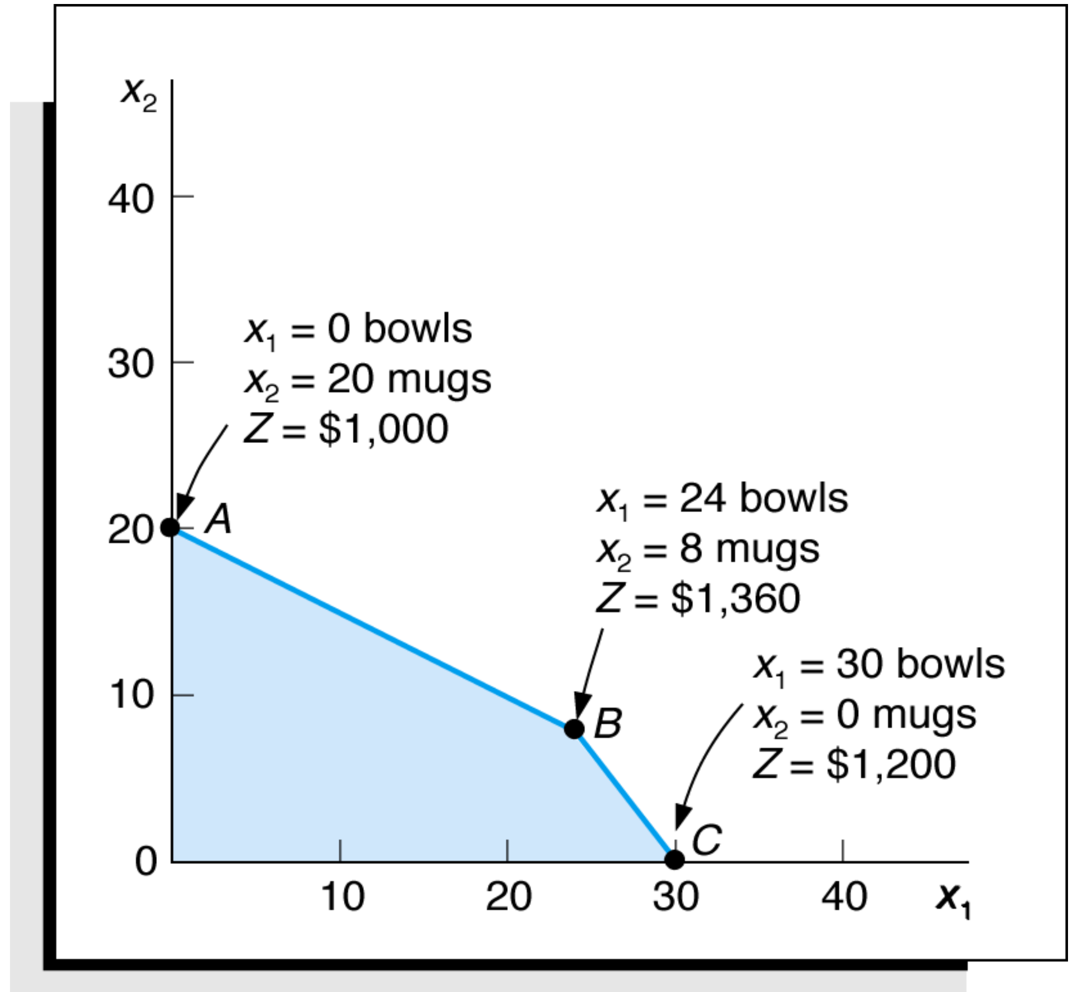
Extreme (Corner) Point Solutions Graphical Solution of Maximization Model

Maximize $Z = 40x_1 + 50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Solutions at All Corner Points

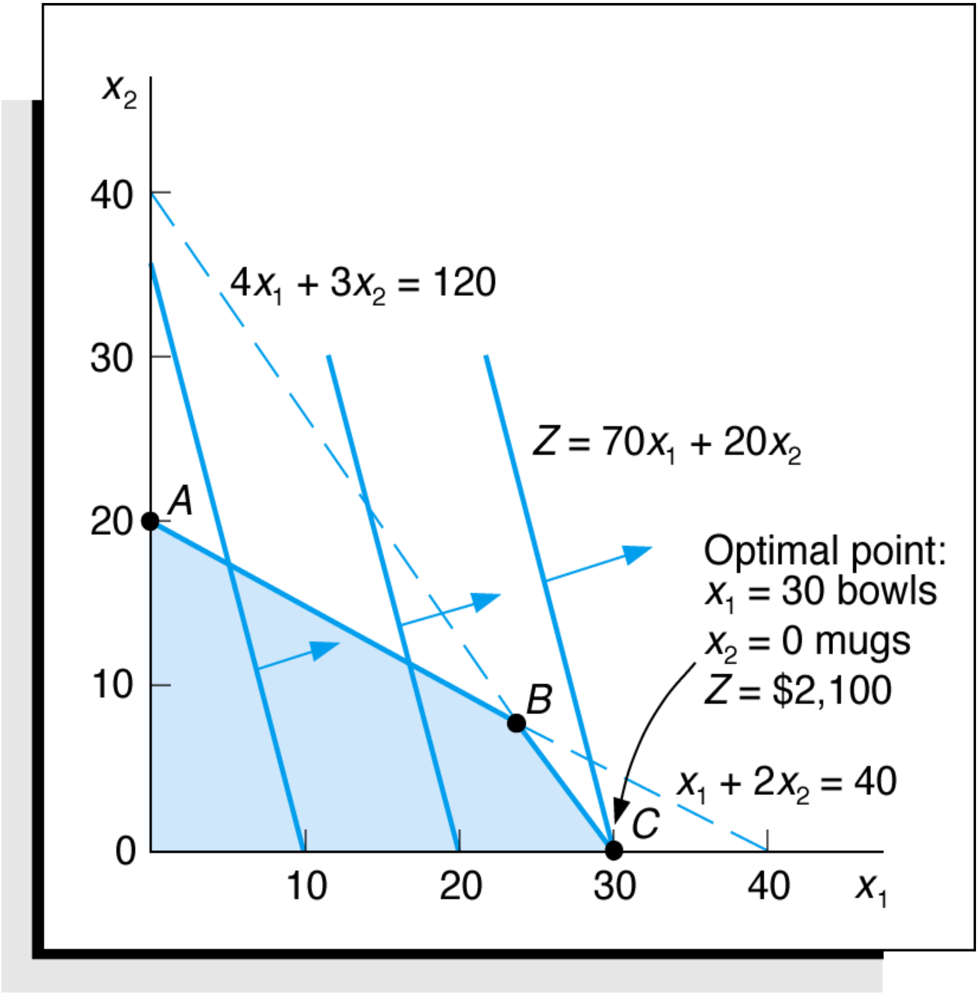
Optimal Solution for New Objective Function
Graphical Solution of Maximization Model

Maximize $Z = 70x_1 + 20x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Optimal Solution with $Z = 70x_1 + 20x_2$

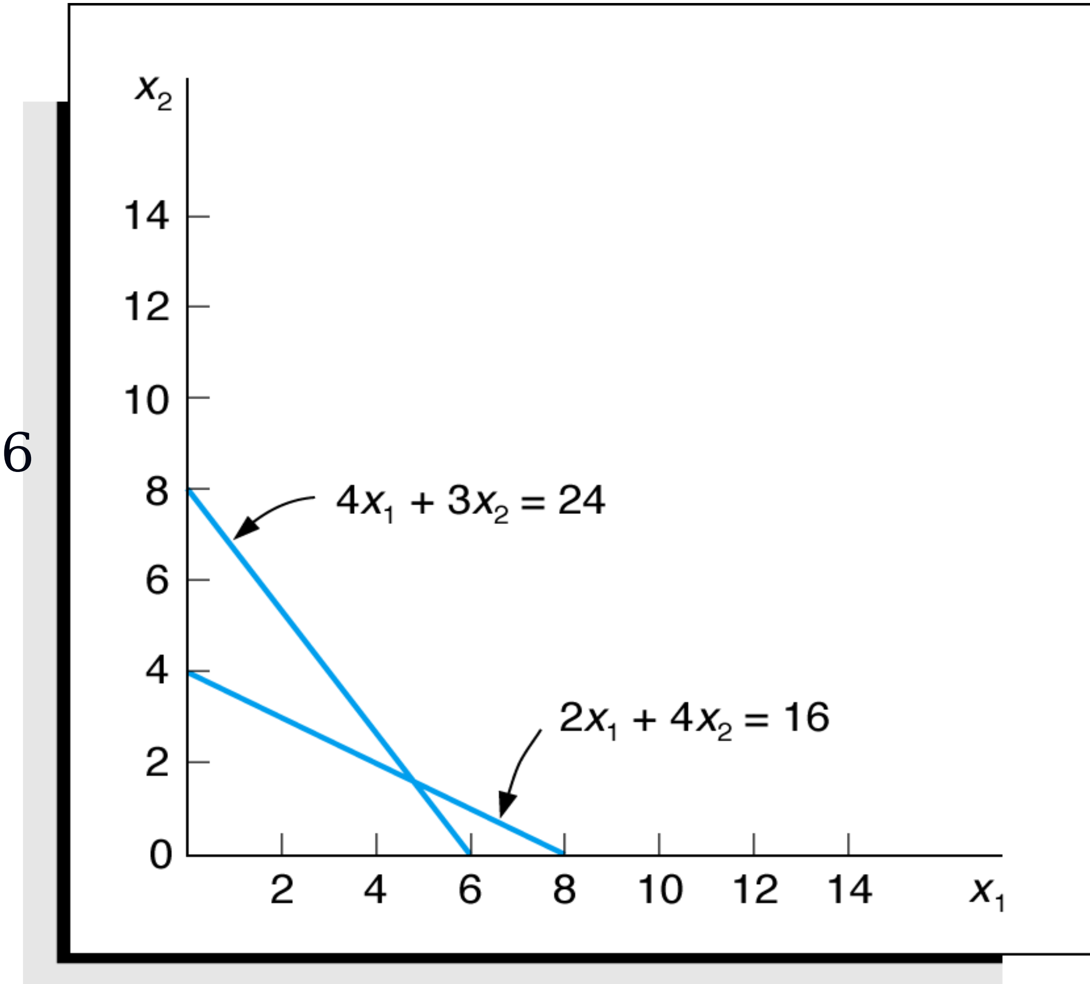
Minimization

Minimize $Z = 6x_1 + 3x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_2 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$



Graph of Both Model Constraints

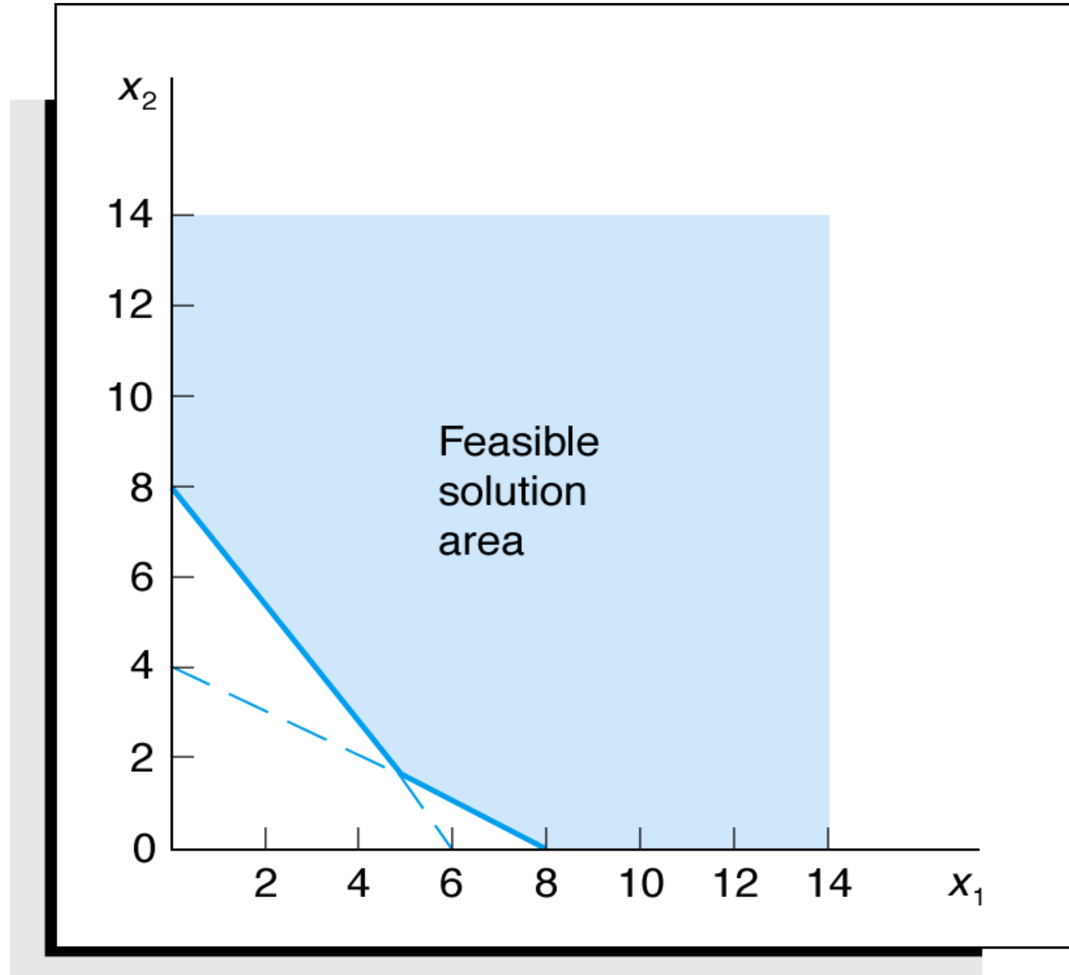
Feasible Region– Minimization

Minimize $Z = 6x_1 + 3x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_2 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$



Feasible Solution Area

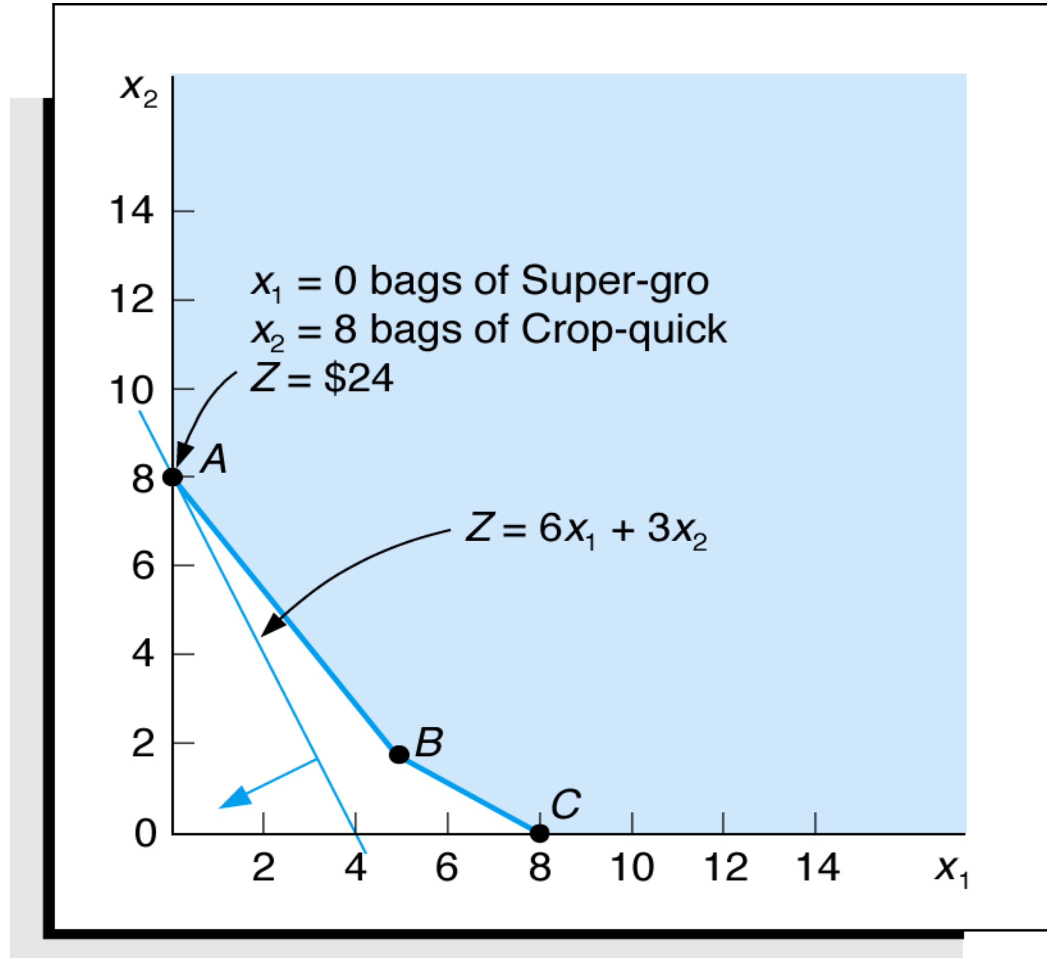
Optimal Solution Point – Minimization

Minimize $Z = 6x_1 + 3x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_2 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$



Optimum Solution Point

Irregular Types of Linear Programming Problems

For some linear programming models, the general rules do not apply.

- Special types of problems include those with:
  - Multiple optimal solutions
  - Infeasible solutions
  - Unbounded solutions

Multiple Optimal Solutions

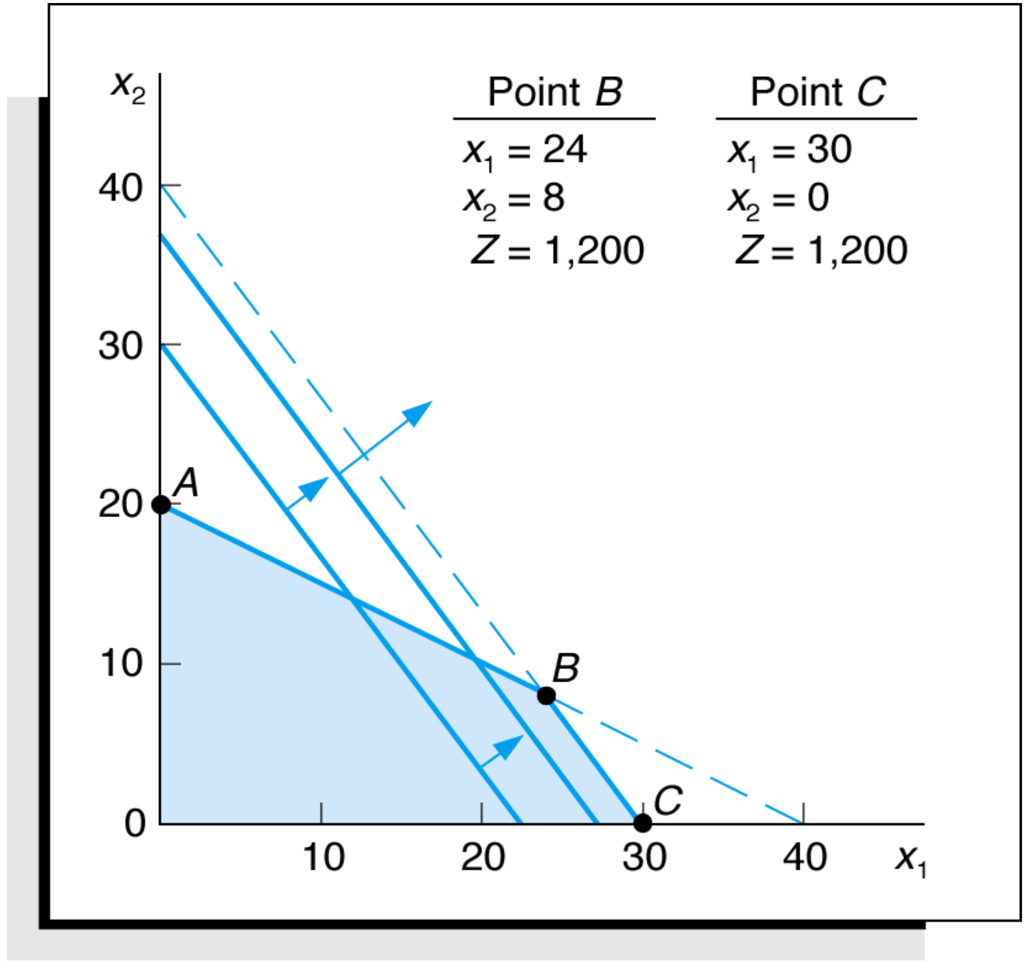
The objective function is **parallel** to a constraint line.

Maximize $Z=40x_1 + 30x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$



Example with Multiple Optimal Solutions

An Infeasible Problem

Every possible solution **violates** at least one constraint:

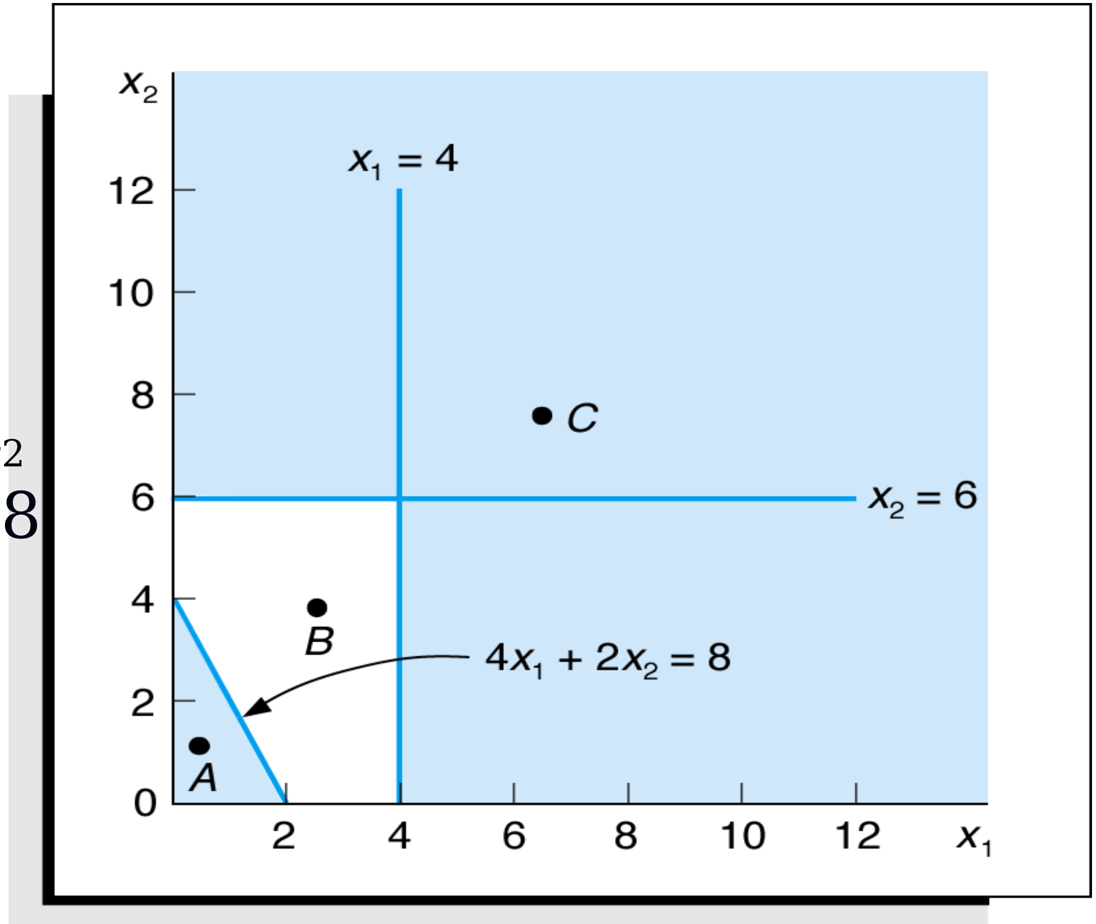
Maximize $Z = 5x_1 + 3x_2$

subject to: $4x_1 + 2x_2 \leq 8$

$x_1 \geq 4$

$x_2 \geq 6$

$x_1, x_2 \geq 0$



Graph of an Infeasible Problem

An Unbounded Problem

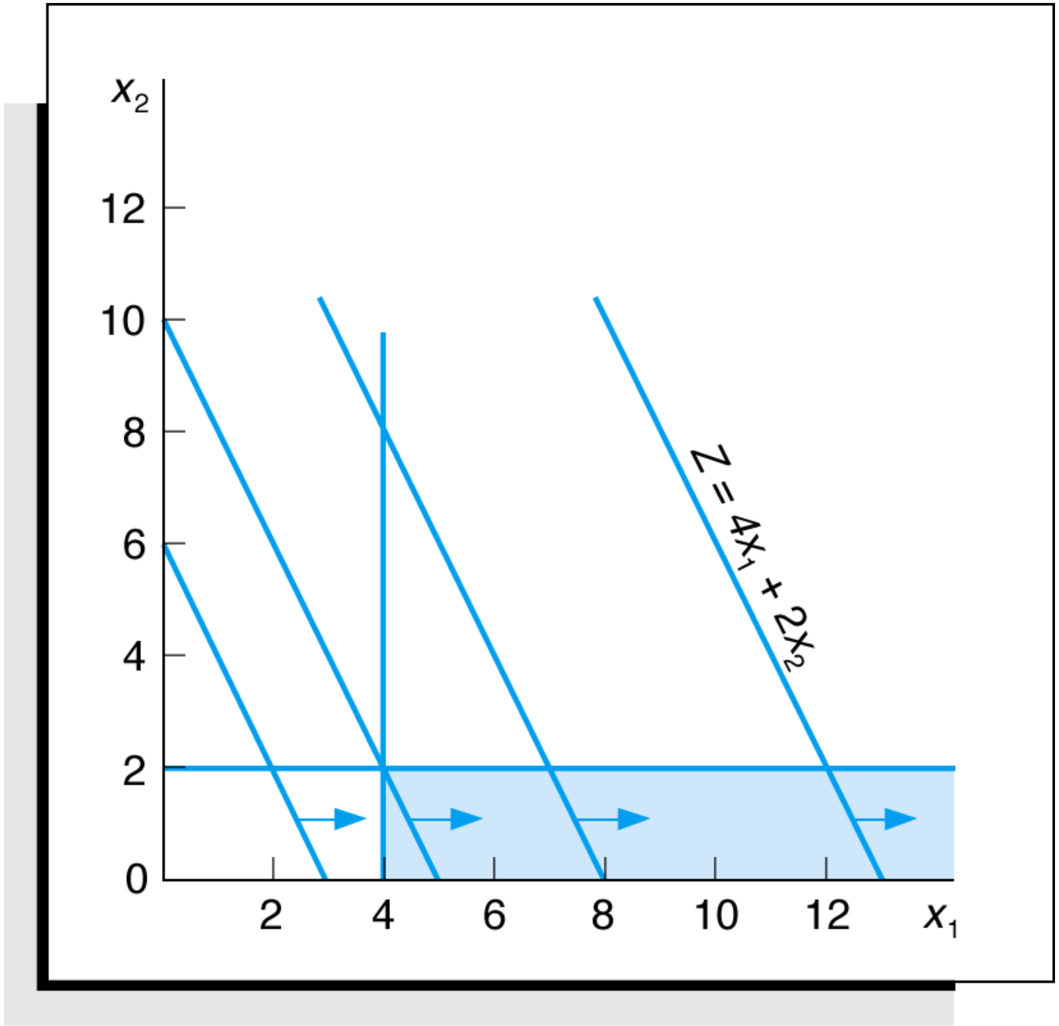
Value of the objective function increases indefinitely:

Maximize $Z = 4x_1 + 2x_2$

subject to: $x_1 \geq 4$

$x_2 \leq 2$

$x_1, x_2 \geq 0$



Graph of an Unbounded Problem

Example Problem

Solve the following model graphically:

Maximize $Z = 4x_1 + 5x_2$

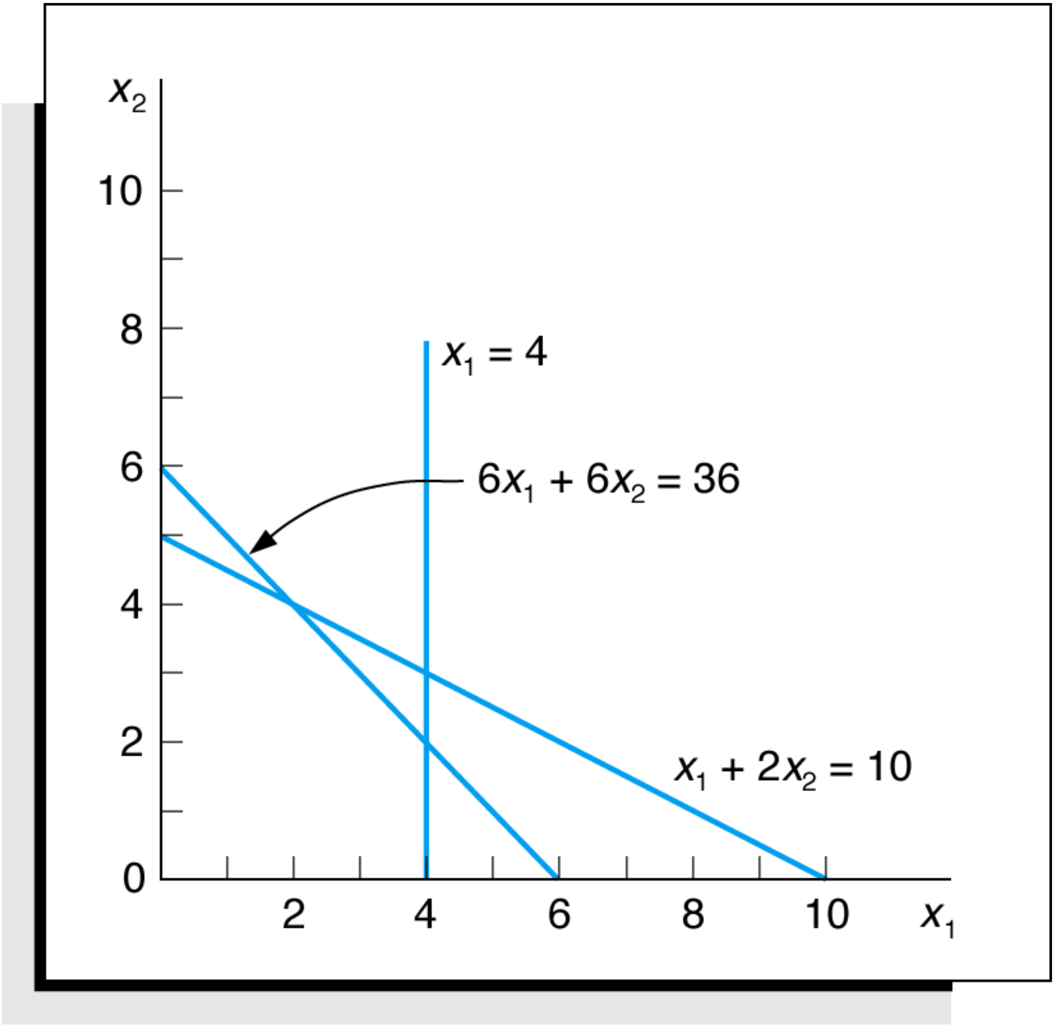
subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 1: Plot the constraints as equations



Constraint Equations

Example Problem

Maximize $Z = 4x_1 + 5x_2$

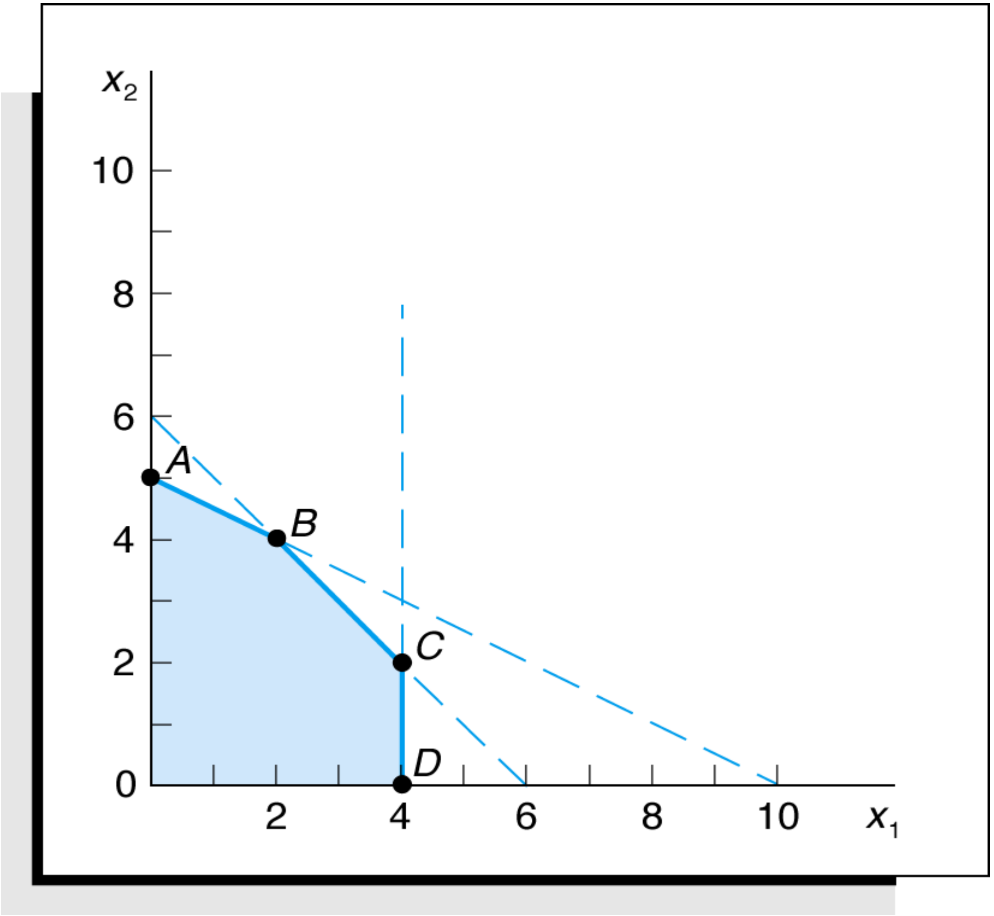
subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 2: Determine the feasible solution space



Feasible Solution Space and Extreme Points

Example Problem

Maximize $Z = 4x_1 + 5x_2$

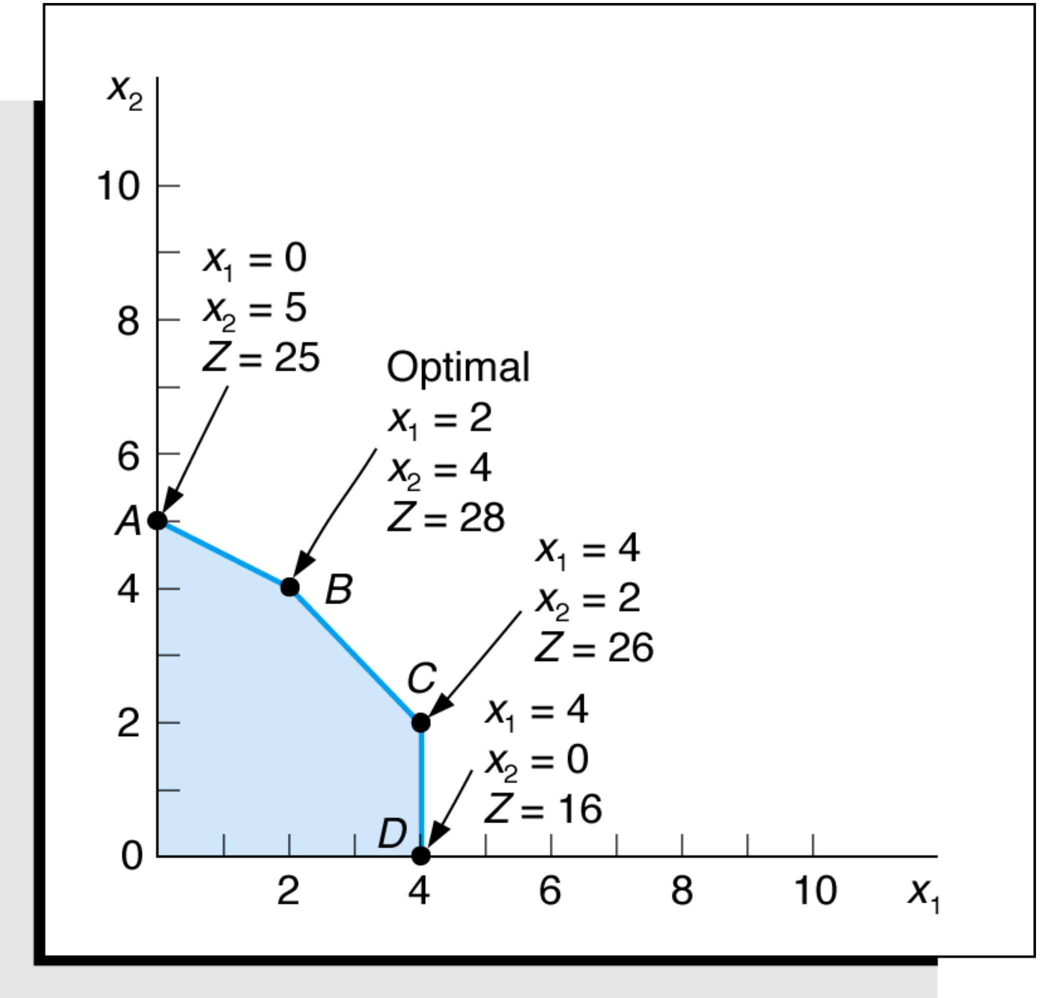
subject to: $x_1 + 2x_2 \leq 10$

$6x_1 + 6x_2 \leq 36$

$x_1 \leq 4$

$x_1, x_2 \geq 0$

Step 3 and 4: Determine the solution points and optimal solution



Optimal Solution Point