DS100/DSL201: Mathematical Foundations for Data Science

Tutorial 02 and 03

23 August 2024

Total points: 40

Question 1 **(4 points)**

A bag contains (n+1) coins. It is known that one of these coins has a head on both sides, whereas the remaining coins are fair. One of these coins is selected at random and is tossed. If the probability that the toss results in a head is $\frac{5}{9}$, find the value of n.

Question 2 **(4 points)**

You have one fair coin and one biased coin which lands heads with probability $\frac{3}{4}$. You pick one of the coins at random and flip it three times. It lands heads all three times. Given this information, what is the probability that the coin you picked will be fair?

Question 3 **(4 points)**

An unbiased coin is tossed 100 times. Show that the probability that the number of heads will be between 30 and 70 is greater than 0.933. i.e. $P[30 < X < 70] > 0.933$

Question 4 **(6 points)**

From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75.

(i) Given an upper bound for the probability that a student's test score will exceed 85.

(ii) Suppose, in addition, that the professor knows that the variance of a student's test score is equal to 25. What can be said about the probability that a student will score less than 65?

Question 5 **(6 points)**

Consider a pdf f so that a random variable $X \sim f$ has expected value $E[X] = 5$ and variance $\mathrm{Var}[X] = 100$. Now consider $n = 16$ iid random variables $X_1, X_2, ..., X_{16}$ drawn from f. Let $\bar{X} = \frac{1}{16}\sum_{i=1}^{16} X_i$.

(i) What is $\mathrm{E}[\bar{X}]$?

(ii) What is $\mathrm{Var}[\bar{X}]$?

Assume we know that X is never smaller than 0 and never larger than 20.

(iii) Use the Markov inequality to upper bound $\Pr[\bar{X} > 8]$.

(iv) Use the Chebyshev inequality to upper bound $\Pr[\bar{X} > 8]$.

(v) Use the Chernoff-Hoeffding inequality to upper bound $\Pr[\bar{X} > 8]$.

Question 6 **(4 points)**

Suppose X is a random variable that follows a Binomial Distribution, i.e., $X \sim \mathrm{Bin}(n, p)$ as given below:

$$P[X = x] = \binom{n}{x} p^x (1-p)^{n-x}$$

Find the MLE (Maximum likelihood estimation) for the parameter p.

Question 7 **(4 points)**

Suppose X is a random variable that follows an Exponential Distribution, i.e., $X \sim \mathrm{Exp}(\lambda)$ as given below:

$$f(x;\lambda) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$

Find the MLE (Maximum likelihood estimation) for the parameter λ.

Question 8 **(4 points)**

Prove the following formula for covariance:

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

Question 9 **(4 points)**

Prove that the variance of a scaled random variable is given by:

$$\mathrm{Var}(aX) = a^2 \cdot \mathrm{Var}(X).$$

Question 1 **(4 points)**

A bag contains (n+1) coins. It is known that one of these coins has a head on both sides, whereas the remaining coins are fair. One of these coins is selected at random and is tossed. If the probability that the toss results in a head is $\frac{5}{9}$, find the value of n.

Solⁿ:- Let events

E_1: Coin with two heads is selected

E_2: Fair coin is selected

A: Toss results in heads

Then $P(E_1) = \frac{1}{n+1}$; $P(A|E_1) = 1$

$P(E_2) = \frac{n}{n+1}$; $P(A|E_2) = \frac{1}{2}$

$\therefore$ $P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2)$

$$\frac{5}{9} = \frac{1}{n+1} \times 1 + \frac{n}{n+1} \times \frac{1}{2}$$

$$\Rightarrow \frac{5}{9} = \frac{1}{n+1}\left[1 + \frac{n}{2}\right]$$

$$\Rightarrow 5n + 5 = 9 + \frac{9n}{2}$$

$$\Rightarrow \frac{n}{2} = 4 \Rightarrow \boxed{n = 8}$$ Ans

Question 2 **(4 points)**

You have one fair coin and one biased coin which lands heads with probability $\frac{3}{4}$. You pick one of the coins at random and flip it three times. It lands heads all three times. Given this information, what is the probability that the coin you picked will be fair?

Solⁿ:- Let events

A: Chosen coin lands Heads three times

F: Picked coin is fair

using Baye's rule

$$P(F|A) = \frac{P(A|F)\,P(F)}{P(A)}$$

$$= \frac{P(A|F)\,P(F)}{P(A|F)\,P(F) + P(A|F^c)\,P(F^c)}$$

$$= \frac{(1/2)^3 \times (1/2)}{(1/2)^3 \times (1/2) + (3/4)^3 \times (1/2)}$$

$$\boxed{P(F|A) = \frac{8}{35}}$$ Ans

Question 3 **(4 points)**

An unbiased coin is tossed 100 times. Show that the probability that the number of heads will be between 30 and 70 is greater than 0.933. i.e. $P[30 < X < 70] > 0.933$

Sol^n:- $P[\text{getting heads}] = \frac{1}{2}$

$$E[x] = 100 \times \frac{1}{2} = 50$$

$$Var[x] = 100 \times \frac{1}{2} \times \frac{1}{2} = 25$$

$$\Rightarrow L.H.S = P(30 < X < 70)$$

$$= P(30 - \mu < X - \mu < 70 - \mu)$$

$$= P(30 - 50 < X - \mu < 70 - 50)$$

$$= P(-20 < X - \mu < 20)$$

$$= P[|X - \mu| < 20]$$

using chebyshev's inequality

$$P[|X - \mu| < k] > 1 - \frac{Var(X)}{k^2}$$

$$P[|X - 50| < 20] > 1 - \frac{25}{400} = 1 - \frac{1}{16} = \frac{15}{16}$$

$$\Rightarrow P[|X - 50| < 20] > 0.933$$ Ans

Question 4 **(6 points)**

From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75.

(i) Given an upper bound for the probability that a student's test score will exceed 85.

(ii) Suppose, in addition, that the professor knows that the variance of a student's test score is equal to 25. What can be said about the probability that a student will score less than 65?

Solⁿ :- let X be a random variable which represents test score

Given $E[X] = 75$

(a) By markov's inequality

$$P[X \geq a] \leqslant \frac{E[X]}{a}$$

$$P[X > 85] \leqslant \frac{75}{85} = \frac{15}{17}$$ Ans

(b) Given $\sigma^2 = 25$

By chebyshev's inequality

$$P[|X - E[X]| \geq K] \leqslant \frac{Var[X]}{K^2}$$

$$P[|X-75| \geq 10] \leqslant \frac{25}{(10)^2}$$

$$P[X-75 \geq 10 \text{ OR } X-75 \leqslant -10] \leqslant \frac{1}{4}$$

$$P[\underbrace{X \leqslant 65}_{A} \text{ OR } \underbrace{X \geq 85}_{B}] \leqslant \frac{1}{4} = 0.25$$

Since bound for the probability of A or B is 0.25, which implies both A or B would be individually bounded by 0.25. Therefore

$$P[X \leqslant 65] \leqslant 0.25$$ Ans

Question 5 **(6 points)**

Consider a pdf f so that a random variable $X \sim f$ has expected value $E[X] = 5$ and variance $\text{Var}[X] = 100$. Now consider $n = 16$ iid random variables $X_1, X_2, ..., X_{16}$ drawn from f. Let $\bar{X} = \frac{1}{16}\sum_{i=1}^{16} X_i$.

(i) What is $\text{E}[\bar{X}]$?

(ii) What is $\text{Var}[\bar{X}]$?

Assume we know that X is never smaller than 0 and never larger than 20.

(iii) Use the Markov inequality to upper bound $\Pr[\bar{X} > 8]$.

(iv) Use the Chebyshev inequality to upper bound $\Pr[\bar{X} > 8]$.

(v) Use the Chernoff-Hoeffding inequality to upper bound $\Pr[\bar{X} > 8]$.

Solⁿ:- Given $E[X] = 5$, $\text{Var}[X] = 100$, $n = 16$

$$\bar{X} = \frac{1}{16}\sum_{i=1}^{16} X_i$$

i) $$E[\bar{X}] = E\left[\frac{1}{16}\sum_{i=1}^{16} X_i\right] = \frac{1}{16} \times 16 \times 5 = 5$$ Ans

ii) $$\text{Var}[\bar{X}] = \text{Var}\left[\frac{1}{16}\sum_{i=1}^{16} X_i\right] = \frac{1}{16^2} \times \text{Var}\left(\sum_{i=1}^{16} X_i\right)$$

$$= \frac{1}{16^2} \times 16 \times 100$$

$$= \frac{100}{16} = 6.25$$ Ans

iii) By markov's inequality

$$P[\bar{X} \geq a] \leq \frac{E[\bar{X}]}{a}$$

$$P[\bar{X} > 8] \leq \frac{5}{8}$$ Ans

iv) By chebyshev's inequality

$$P[|\bar{X} - E[\bar{X}]| \geq a] \leq \frac{\text{Var}(\bar{X})}{a^2}$$

$$P[|\bar{X} - 5| \geq 3] \leq \frac{6.25}{3^2}$$

$$P[\bar{X} - 5 \geq 3] \leq 0.694$$

$$P[\bar{X} \geq 8] \leq 0.694$$ Ans

v) Chernoff–Hoeffding inequality

$$P[|\bar{X} - E[\bar{X}]| > \varepsilon] \leq 2\exp\left(\frac{-2\varepsilon^2 n}{\Delta^2}\right)$$

as we know

$$0 \leq X \leq 20$$

$$P[\bar{X} - 5 > 3] \leq 2\exp\left(\frac{-2 \times 3^2 \times 16}{(20-0)^2}\right)$$

$$P[\bar{X} > 8] \leq \exp(-0.72)$$

$$P[\bar{X} > 8] \leq 0.487$$ Ans

Question 6 **(4 points)**

Suppose X is a random variable that follows a Binomial Distribution, i.e., $X \sim \text{Bin}(n, p)$ as given below:

$$P[X = x] = \binom{n}{x} p^x (1-p)^{n-x}$$

Find the MLE (Maximum likelihood estimation) for the parameter p.

Solⁿ:- given $X \sim Bin(n, p)$

$$f(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$L(p) = \log(f) = \log\binom{n}{x} + x \log p + (n-x)\log(1-p)$$

$$\frac{\partial L}{\partial p} = 0$$

$$\Rightarrow \quad 0 + \frac{x}{p} - \frac{(n-x)}{(1-p)} = 0$$

$$\frac{x}{p} = \frac{n-x}{1-p}$$

$$\Rightarrow \quad x(1-p) = (n-x)p$$

$$\Rightarrow \quad x - \cancel{xp} = np - \cancel{xp}$$

$$\boxed{\hat{p} = \frac{x}{n}}$$ Ans

Question 7 **(4 points)**

Suppose X is a random variable that follows an Exponential Distribution, i.e., $X \sim \text{Exp}(\lambda)$ as given below:

$$f(x; \lambda) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0$$

Find the MLE (Maximum likelihood estimation) for the parameter λ.

Sol^n:-

$$f(x, \lambda) = \lambda e^{-\lambda x}$$

$$L(\lambda, x) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$ [Joint density]

$$L(\lambda, x) = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\log L = n \log \lambda - \lambda \sum_{i=1}^{n} x_i$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \quad \frac{n}{\lambda} = \sum_{i=1}^{n} x_i$$

$$\boxed{\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}}$$ Ans

Question 8 **(4 points)**

Prove the following formula for covariance:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

Sol^n:-

$$\text{Cov}(X, Y) = E\left[(X - E[X])(Y - E[Y])\right]$$

$$= E\left[XY - XE[Y] - YE[X] + E[X]E[Y]\right]$$

$$= E[XY] - E[XE[Y]] - E[YE[X]] + E\left[E[X]E[Y]\right]$$

$$= E[XY] - E[X]E[Y] - \cancel{E[X]E[Y]} + \cancel{E[X]E[Y]}$$

$$\boxed{\text{Cov}(X, Y) = E[XY] - E[X]E[Y]}$$ Ans.

Question 9 **(4 points)**

Prove that the variance of a scaled random variable is given by:

$$\mathrm{Var}(aX) = a^2 \cdot \mathrm{Var}(X).$$

Sol^n:-

$$\mathrm{Var}[aX+b] = E[(aX+b)^2] - (E[aX+b])^2$$

$$= E[a^2X^2 + 2aXb + b^2] - [aE[X]+b]^2$$

$$= a^2E[X^2] + \cancel{2abE[X]} + \cancel{b^2} - [a^2(E[X])^2 + \cancel{b^2} + \cancel{2ab}E[X]]$$

$$= a^2[E[X^2] - (E[X])^2]$$

$$= a^2\mathrm{Var}[X]$$ Ans