Q1) Sol^n:- $f(x,y)=\begin{cases} e^{-k(x+y)} & ; 0\le y<x<\infty \\ 0 & ; o/w. \end{cases}$

(a) $\because \iint f(x,y)\,dx\,dy = 1$

$\Rightarrow \int_0^\infty \int_0^x e^{-ky} e^{-kx}\,dy\,dx = 1$

$\Rightarrow \int_0^\infty e^{-kx}\left[\frac{e^{-ky}}{-k}\right]_0^x dx = 1$

$\Rightarrow \frac{-1}{k}\int_0^\infty e^{-kx}(e^{-kx}-1)\,dx = 1$

$\Rightarrow \frac{-1}{k}\int_0^\infty (e^{-2kx}-e^{-kx})\,dx = 1$

$\Rightarrow \frac{-1}{k}\left[\frac{e^{-2kx}}{-2k}+\frac{e^{-kx}}{k}\right]_0^\infty = 1$

$\Rightarrow \frac{-1}{k}\left[(0+0)-\left(\frac{-1}{2k}+\frac{1}{k}\right)\right] = 1$

$\Rightarrow \left(\frac{-1}{k}\right)\left[\frac{1}{k}\left(\frac{1}{2}-1\right)\right] = 1$

$\Rightarrow \frac{1}{2k^2} = 1$

$\Rightarrow \boxed{k = \frac{1}{\sqrt{2}}}$ Ans

(b) Marginal dist. of X

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x e^{-\frac{1}{\sqrt{2}}(x+y)}\,dy$

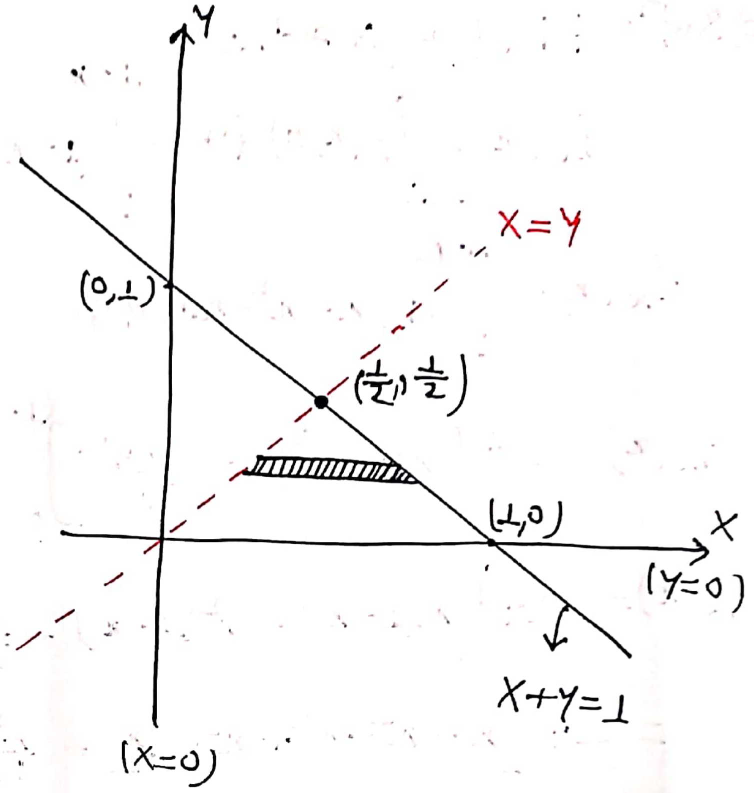
$f_X(x) = \int_0^x e^{-x/\sqrt{2}} \cdot e^{-y/\sqrt{2}}\,dy = \cancel{x}$

$\Rightarrow f_X(x) = e^{-x/\sqrt{2}}\int_0^x e^{-y/\sqrt{2}}\,dy = e^{-x/\sqrt{2}}\left[\frac{e^{-y/\sqrt{2}}}{-1/\sqrt{2}}\right]_0^x = e^{-x/\sqrt{2}}(-\sqrt{2})\left[e^{-x/\sqrt{2}}-1\right]$

$\Rightarrow \boxed{f_X(x) = \sqrt{2}e^{-x/\sqrt{2}}\left(1-e^{-x/\sqrt{2}}\right)}$ Ans

Q2) Sol^n:

$$f_{X,Y}(x,y)=\begin{cases}24x(1-x-y) & x\ge 0,\ y\ge 0,\ x+y\le 1\\ 0 & \text{: otherwise}\end{cases}$$



$$P(X>Y)=P(X-Y>0)$$

OR

$$P(Y-X<0)$$

$$=\int_{y=0}^{y=1/2}\int_{x=y}^{1-y} f(x,y)\,dx\,dy$$

$$=\int_{y=0}^{1/2}\int_{x=y}^{x=1-y} 24x(1-x-y)\,dx\,dy$$

$$=24\int_{y=0}^{1/2}\int_{x=y}^{x=1-y}[x-x^2-xy]\,dx\,dy$$

$$=24\int_{y=0}^{1/2}\left[\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^2y}{2}\right]_y^{1-y}dy$$

$$=24\int_{y=0}^{1/2}\left[\frac{(1-y)^2}{2}-\frac{(1-y)^3}{3}-\frac{y(1-y)^2}{2}-\left(\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^3}{2}\right)\right]dy$$

$$=24\int_{y=0}^{1/2}\left[\frac{(1-y)^2}{2}-\frac{(1-y)^3}{3}-\frac{y+y^3-2y^2}{2}-\frac{y^2}{2}+\frac{5}{6}y^3\right]dy$$

$$=24\int_{y=0}^{1/2}\left[\frac{(1-y)^2}{2}-\frac{(1-y)^3}{3}-\frac{y}{2}+\frac{y^3}{3}+\frac{y^2}{2}\right]dy$$

$$=24\left[\frac{(1-y)^3}{(-6)}-\frac{(1-y)^4}{(-12)}-\frac{y^2}{4}+\frac{y^4}{12}+\frac{y^3}{6}\right]_0^{1/2}$$

$$=\frac{3}{4}$$ Ans

(b) Solⁿ :- Find $P(X > 1/2)$

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{1-x} (1-x-y)\,24x\,dy$$

$$f_x(x) = \int_0^{(1-x)} 24(x - x^2 - xy)\,dy$$

$$f_x(x) = 24\left[xy - x^2y - \frac{xy^2}{2}\right]_0^{(1-x)}$$

$$f_x(x) = 24\left[x(1-x) - x^2(1-x) - \frac{1}{2}x(1-x)^2\right]$$

$$= 24\left[x - x^2 - x^2 + x^3 - \frac{x}{2} - \frac{x^3}{2} + x^2\right]$$

$$= 24\left[\frac{x^3}{2} - x^2 + \frac{x}{2}\right] \#$$

$$\boxed{f_x(x) = 12(x^3 - 2x^2 + x)}$$

$$P(X > 1/2) = \int_{1/2}^{1} f_x(x)\,dx$$

$$= \int_{1/2}^{1} 12(x^3 - 2x^2 + x)\,dx = 12\left[\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{x^2}{2}\right]_{1/2}^{1}$$

$$\boxed{P(X > \tfrac{1}{2}) = 5/16}$$ Ans

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Q3) Sol^n:- F_y $F_Y(y) = P(Y \le y) = P(X^2 \le y)$

$= P(X \le \pm\sqrt{y})$

$= P(-\sqrt{y} \le x \le \sqrt{y})$

$= \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,dx$

$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} e^{-|x|}\,dx$

$= \frac{1}{2}\int_{-\sqrt{y}}^{\sqrt{y}} e^{-|x|}\,dx = \frac{1}{\cancel{2}} \times \cancel{2} \int_0^{\sqrt{y}} e^{-x}\,dx$

$= \left[\frac{e^{-x}}{-1}\right]_0^{\sqrt{y}} = \left[\frac{e^{-\sqrt{y}}}{-1} + 1\right]$

$\boxed{F_Y(y) = 1 - e^{-\sqrt{y}}}$ Ans

$F_Y'(y) = 0 - \frac{e^{-\sqrt{y}}}{-2\sqrt{y}}$

$\boxed{f_Y(y) = \frac{e^{-\sqrt{y}}}{2\sqrt{y}}}$ Ans

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Q4) Solⁿ:- $f_T(t) = 2e^{-2t}$; $0 \le t < \infty$

[I]

$$R = \frac{1}{T} \quad (T \neq 0)$$

$$\Rightarrow T = \frac{1}{R}$$

$$\boxed{t = \frac{1}{r}} \Rightarrow \boxed{dt = \frac{-1}{r^2}}$$

$$f_R(r) = \left|\frac{dt}{dr}\right| f_T(t)$$

$$= \left|\frac{-1}{r^2}\right| 2e^{-2t}$$

$$= \frac{1}{r^2} 2e^{-2/r}$$

$0 \le t < \infty$

$0 < r < \infty$

$$\boxed{f_R(r) = \frac{2}{r^2} e^{-2/r}}$$ Ans

[II] $F_R(r) = P(R \le r) = P\left(\frac{1}{T} < r\right) = P\left(\frac{1}{r} < T\right)$

$$= 1 - P\left(\frac{1}{r} > T\right)$$

$$F_R(r) = 1 - F_T\left(\frac{1}{r}\right) \quad \text{——— } (*)$$

Now,

$$F_T(t) = P(T \le t)$$

$$= \int_0^t 2e^{-2t} = 2\left[\frac{e^{-2t}}{-2}\right]_0^t = -\left(e^{-2t} - 1\right)$$

$$= 1 - e^{-2t}$$

From eqⁿ (*)

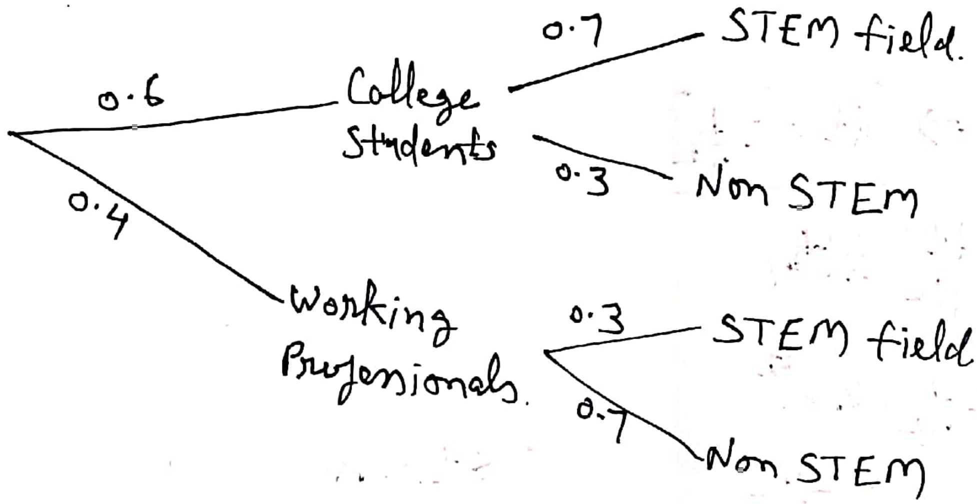
$$F_R(r) = 1 - \left(1 - e^{-2t}\right) = e^{-2t}$$

$$F_R(r) = e^{-2/r} \qquad [\because t = 1/r]$$

$$\frac{d}{dr} F_R(r) = e^{-2/r} \frac{2}{r^2}$$

$$\boxed{f_R(r) = \frac{2}{r^2} e^{-2/r}}$$ Ans ; $0 < r < \infty$

Q5) Sol^n:-



Prob (Participants involved in STEM)

$= 0.6 \times 0.7 + 0.4 \times 0.3$

$= 0.42 + 0.12$

$= 0.54$ Ans

Q6) Sol^n:- $X \sim U(\alpha, \beta)$

$$f_X(x) = \frac{1}{\beta-\alpha} \; ; \; \alpha < x < \beta$$

(a) $$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha}\, dx$$

$$= \left(\frac{1}{\beta-\alpha}\right) \int_{\alpha}^{\beta} x\, dx$$

$$= \left(\frac{1}{\beta-\alpha}\right) \left[\frac{x^2}{2}\right]_{\alpha}^{\beta}$$

$$= \left(\frac{1}{\beta-\alpha}\right) \frac{1}{2} (\beta^2 - \alpha^2)$$

$$\boxed{E[X] = \frac{\alpha+\beta}{2}}$$ Ans

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_{\alpha}^{\beta} x^2 \times \frac{1}{\beta-\alpha}\, dx = \frac{1}{\beta-\alpha} \left[\frac{x^3}{3}\right]_{\alpha}^{\beta}$$

$$E[X^2] = \frac{1}{3(\beta-\alpha)} (\beta^3 - \alpha^3)$$

$$E[X^2] = \frac{1}{3\cancel{(\beta-\alpha)}} \cancel{(\beta-\alpha)} (\beta^2 + \alpha\beta + \alpha^2)$$

$$\boxed{E[X^2] = \frac{\alpha^2 + \alpha\beta + \beta^2}{3}}$$

$$Var(X) = \frac{\alpha^2 + \alpha\beta + \beta^2}{3} - \left(\frac{\alpha+\beta}{2}\right)^2$$

$$\boxed{Var(X) = \frac{(\beta-\alpha)^2}{12}}$$ Ans

Q7) Sol^n:- $f_X(x) = \frac{x}{9} e^{-x/3}$, $x > 0$

$$F_X(x) = P(X<x) = \int_0^x f_X(x)\,dx$$

$$= \int_0^x \frac{x}{9} e^{-x/3}\,dx$$

$$= \frac{1}{9}\int_0^x x e^{-x/3}\,dx = \frac{1}{9}\left[x e^{-x/3}(-3) - \int 1 \times e^{-x/3}(-3)\,dx\right]_0^x$$

$$= \frac{1}{9}\left[-3x e^{-x/3} + 3(-3) e^{-x/3}\right]_0^x$$

$$= -\frac{1}{9}\left[3x e^{-x/3} + 9 e^{-x/3} - (0+9)\right]$$

$$\boxed{F_X(x) = 1 - e^{-x/3} - \frac{1}{3} x e^{-x/3}}$$

$$E[X] = \int_0^\infty x \frac{x}{9} e^{-x/3}\,dx = \frac{1}{9}\int_0^\infty x^2 e^{-x/3}\,dx$$

$$= \frac{1}{9}\int_0^\infty x^2 e^{-x/3}\,dx$$

$$= \frac{1}{9}\left[x^2 e^{-x/3}(-3) - \int 2x e^{-x/3}(-3)\,dx\right]_0^\infty$$

$$= \frac{1}{9}\left[-3x^2 e^{-x/3} + 6\left[x e^{-x/3}(-3) - \int 1 \times e^{-x/3}(-3)\,dx\right]\right]_0^\infty$$

$$= \frac{1}{9}\left[-3x^2 e^{-x/3} + 6\left[-3x e^{-x/3} + 3 e^{-x/3}(-3)\right]\right]_0^\infty$$

$$= \frac{1}{9}\left[-3x^2 e^{-x/3} - 18 x e^{-x/3} - 54 e^{-x/3}\right]_0^\infty$$

$$= \frac{1}{9}\left[(0-0-0) - (0-0-54)\right] = \frac{54}{9} = 6 \text{ Ans}$$

$Var[X] = E[X^2] - (E[X])^2$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2 \cdot \frac{x}{9} e^{-x/3}\,dx = \frac{1}{9}\int_0^{\infty} x^3 e^{-x/3}\,dx$

$= \frac{1}{9}\left[x^3 e^{-x/3}(-3) + 9\int x^2 e^{-x/3}\,dx\right]_0^{\infty}$

$= \frac{1}{9}\left[-3x^3 e^{-x/3} + 9\left[x^2 e^{-x/3}(-3) + 6\int x e^{-x/3}\,dx\right]\right]_0^{\infty}$

$= \frac{1}{9}\left[-3x^3 e^{-x/3} + 9\left[-3x^2 e^{-x/3} + 6\left[-3x e^{-x/3} - 9e^{-x/3}\right]\right]\right]_0^{\infty}$

$= \frac{1}{9}\left[-3x^3 e^{-x/3} + 9\left[-3x^2 e^{-x/3} - 18x e^{-x/3} - 54 e^{-x/3}\right]\right]_0^{\infty}$

$= \frac{1}{9}\left[-3x^3 e^{-x/3} + (-27)x^2 e^{-x/3} - 162x e^{-x/3} - 486 e^{-x/3}\right]_0^{\infty}$

$= \frac{1}{9}\left[(0-0-0-0) - (0-0-0-486)\right]$

$= \frac{486}{9} = 54$

Var

$Var(X) = (54)^2 - (6)^2$

$= 2916 - 36$

$\boxed{Var(X) = 2880}$

(b) $P(X<6) = \int_0^6 \frac{x}{9} e^{-x/3}\,dx$

$= \frac{1}{9}\left[x e^{-x/3}(-3) + (-9)e^{-x/3}\right]_0^6$

$= \frac{1}{9}\left[6e^{-2}(-3) - 9e^{-2} - (0-9)\right]$

$= \frac{1}{9}\left[-18e^{-2} - 9e^{-2} + 9\right]$

$\boxed{P(X<6) = 1 - e^{-2} - 2e^{-2}}$

$= 0.593$