MAL100: Mathematics I

Tutorial Sheet 2: Sequence

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1. Prove or disprove:
   (a) A set always contains its supremum.
   (b) A set may not contain its infimum.
   (c) If A, B are two subsets of $\mathbb{R}$ and $a \leqslant b$ for all $a \in A, b \in B$, then $\sup A \leqslant \inf B$.
   (d) If $\{x_n\}$ is bounded and $\{x_n y_n\}$ is also bounded then $\{y_n\}$ is bounded.
   (e) For any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$.
2. What can you say about a set A if $\sup A = \inf A$?
3. Find the limit of the following sequences:
   (a) $x_n = \left(1 + \frac{1}{n}\right)^n$.
   (b) $x_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$.
   (c) $x_n = \sum_{k=1}^{n} \frac{1}{n(n+1)}$.
   (d) $x_n = \frac{2^n}{n!}$.
4. Prove that $\lim_{n\to\infty} \frac{a}{n} = 0$ for any $a \in \mathbb{R}$.
5. Give an example of a sequence of rational numbers that converges to an irrational number and a sequence of irrational numbers that converges to a rational number.
6. Let $\{x_n\}$ be a sequence and $y_n = x_{2n-1}, z_n = x_{2n}$. Prove that $\{x_n\}$ is convergent if and only if both $\{y_n\}$ and $\{z_n\}$ converge to the same limit.
7. Let $\{x_n\}$ be a convergent sequence in $\mathbb{R}$ and $\{y_n\}$ be a bounded sequence in $\mathbb{R}$. Discuss the boundedness and convergence of the sequence $\{x_n + y_n\}$.
8. Give examples of two divergent sequences $\{x_n\}$ and $\{y_n\}$ such that
   (a) $\{x_n \pm y_n\}$ is convergent.
   (b) $\{x_n y_n\}$ is convergent.
   (c) $\{\frac{x_n}{y_n}\}$ is convergent.
9. Let $\{a_n\}$ be a sequence defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$. Find $\lim_{n\to\infty} a_n$.
10. Let $a \in \mathbb{R}$ and $x \in (a - \epsilon, a + \epsilon)$ for all $\epsilon > 0$. Then prove that $x = a$.