MAL100: Mathematics I

Tutorial Sheet 4: Cauchy Sequence and Infinite Series

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1. Show that the sequence $x_n = \sqrt{n}$ satisfies $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$ but it is not a Cauchy sequence.

2. Let $\{x_n\}$ be a sequence of integers. Can it be a Cauchy sequence? If not, why? Describe.

3. Let $\{x_n\}$ be a sequence of real numbers such that $|x_{k+1} - x_k| < a^{-k}$ for some $a > 1$. Prove that $\{x_n\}$ is Cauchy.

4. Let a and b be any two distinct real numbers. Let $x_1 = a$ and $x_2 = b$. Define $x_{n+2} = \frac{x_{n+1}+x_n}{2}$. Prove that $\{x_n\}$ is convergent.

5. Prove that the sequence $\{x_n\}$ of real numbers satisfying the condition $|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$ and for some $c \in (0, 1)$ is convergent.

6. Using Cauchy's principle of convergence prove that $\{\frac{n}{n+1}\}$ is convergent.

7. Prove or Disprove:

   (a) If $\sum a_n$ is convergent for $a_n > 0$ then so is $\sum a_n^2$.

   (b) If $\sum a_n$ is convergent for $a_n > 0$ then so is $\sum \sqrt{a_n}$.

8. If $\sum a_n$ an with $a_n > 0$ is convergent, and if $b_n = (a_1 + \cdots + a_n)/n$ for $n \in \mathbb{N}$, then show that $\sum b_n$ is always divergent.