MAL100: Mathematics I

Assignment I

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1. Find the supremum and infimum of the sequence $\{x_n\}$, where
$$x_n = \frac{(-1)^n}{n} + \sin\left(\frac{n\pi}{2}\right).$$

2. Let a_0 and b_0 be two positive real numbers. Define the sequences $\{a_n\}$ and $\{b_n\}$ by the recurrence relations
$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n}$$
Prove that both sequences $\{a_n\}$ and $\{b_n\}$ converge.

3. Let x be a real number. Prove that there exists a sequence of rational numbers $\{r_n\}$ and a sequence of irrational numbers $\{i_n\}$ such that $r_n \to x$ and $i_n \to x$.

4. Prove that
$$\lim_{n\to\infty} \frac{1}{n}\{(a+1)(a+2)\dots(a+n)\}^{1/n} = \frac{1}{e} \quad \text{for any} \quad a > 0.$$

5. Show that the series $\sum \log(1 + \frac{1}{n})$ is divergent.

6. If $\{a_n\}$ is a sequence of positive real numbers and bounded above then show that $\sum \frac{a_n}{1+a_n^2}$ is divergent. Give an example of an unbounded sequence of positive real numbers such that $\sum \frac{a_n}{1+a_n^2}$ is convergent.

7. Discuss the convergence of the series
$$\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \dots.$$
If convergent then find the sum of the series.

8. Let $\{a_n\}$ be a sequence of real numbers such that $\sum a_n^2$ is convergent. Prove that $\sum a_n^3$ is also convergent.

9. Prove that the series $\sum_{n\geqslant 1} \frac{(-1)^n \log n}{n \log(\log n)}$ is conditionally convergent?

10. Find the real numbers p for which
    (a) $\sum_{n\geqslant 1} \frac{(-1)^n}{n^p}$ is convergent And for what values of p the series converges absolutely?
    (b) $\sum_{n\geqslant 0} \frac{n^p}{e^n}$ is convergent.
    (c) $\sum_{n\geqslant 0} \sin^n p$ is convergent.
    (d) $\sum_{n\geqslant 1} \frac{1}{n(\log(\log n))^p}$

11. Let $f : [1, \infty) \to [1, \infty)$ be a continuous monotone decreasing function. Let $a_n = f(n)$ and $b_n = \int_1^n f(t)dt$. Prove that

(a) $\sum_{n\geqslant 1} a_n$ converges if $\{b_n\}$ converges,

(b) $\sum_{n\geqslant 1} a_n$ diverges if $\{b_n\}$ diverges.

Using this result, for what value of p, determine whether the series $\sum_{n\geqslant 1} \frac{\log n}{n^p}$ converges?

12. Let $\sum_{n\geqslant 1} a_n$ be convergent series of non-negative and decreasing terms. Prove that $na_n \to 0$. Now using this result, discuss the convergence of the series $\sum_{n\geqslant 1} \frac{1}{an+b}$ for any $a > 0, b \geqslant 0$.