MAL100: Mathematics I

Tutorial Sheet 7: Continuity

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1. Check whether the following functions are uniformly continuous or not:
   (a) $f(x) = \frac{1}{x}$ for $x \in (0, \infty)$.
   (b) $f(x) = \frac{1}{x}$ for $x \in [\alpha, \infty)$ for some $\alpha > 0$.
   (c) $f(x) = x^2$ for $x \in [a, b]$ for $b > a \geqslant 0$.
   (d) $f(x) = x^2$ for $x \in [a, \infty)$ for $a \geqslant 0$.
   (e) $f(x) = \sin(x \sin x)$ for $x \in [0, \infty)$.
2. Does uniformly continuous function map Cauchy sequence to a Cauchy sequence?
3. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \leqslant L|x - y|$ for all $x, y \in [a, b]$. Prove that f is uniformly continuous.
4. Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a continuous function such that $f(x) \neq 0$ for any $x \in I$. Prove that either $f > 0$ or $f < 0$ throughout I. Using this result prove that if f and g be two continuous functions such that $f(x) \neq g(x)$ for any $x \in I$ then either $f > g$ or $f < g$.
5. Let $f : [a, b] \to \mathbb{R}$ be a non-constant continuous function. Show that $f([a, b])$ is an interval. Using this solve the following problem: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that
   (a) $f(\mathbb{R}) \subset (0, 1) \cup [2, 100)$, and
   (b) $f(10) = e$.

   Explain where does $f([1, 2])$ contained in.
6. Let $f : I \to \mathbb{R}$ be a continuous function such that $f(c) > 0$ for some $c \in I$. Prove that there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.
7. Prove or Disprove:
   (a) There exists a continuous onto function $f : [a, b] \to \mathbb{R}$.
   (b) Let $f : [a, b] \to [0, \infty)$ be a continuous function such that $f(x) > 0$ for all $x \in [a, b]$. Then there exists $c > 0$ such that $f(x) > c$ for all $x \in [a, b]$.
8. Prove that if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and f is continuous on $\mathbb{R}$ then f is uniformly continuous.