MAL100: Mathematics I - Tutorial Sheet 3 Solutions

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1. Let the sequence
   $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$, for $n \in \mathbb{N}$.
   now
   $$x_{n+1} - x_n - = \frac{1}{n+1} - \log \frac{n+1}{n} < 0,$$
   for $n \in \mathbb{N}$.
   hint
   $$\frac{x}{1+x} < \log(1+x) < x \quad \text{for all } x > 0 \tag{1}$$

2. Now we have to show $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is monotonic increasing and bounded above.
   for monotonic inccreasing
   hint use AM-GM relation on $1, \underbrace{1 + \frac{1}{n} \ldots 1 + \frac{1}{n}}_{n \ times}$
   total n+1 term
   for bounded above
   as $x_n = \left\{\left(1 + \frac{1}{n}\right)^n\right\}$ take both side logarithemic then use equation (1).

3. Let $x_n > 0$. Then
   $$\lim_{n\to\infty} x_n = 0 \iff \lim_{n\to\infty} \frac{1}{x_n} = \infty.$$

($\Rightarrow$) Use $\epsilon = 1/M$ in limit definition.

($\Leftarrow$) Use $M = 1/\epsilon$ in limit definition.

4. We have to show the sequence $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$ is divergent.

$$x_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \ldots$$

We have:

$$x_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\right)$$

Note that:
$$\frac{1}{3} + \frac{1}{4} > \frac{1}{2}, \quad \frac{1}{5} + \cdots + \frac{1}{8} > \frac{1}{2}, \quad \ldots, \quad \frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k} > \frac{1}{2}$$

There are k such groups (each contributing more than $\frac{1}{2}$). Therefore:

$$x_{2^k} > 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + k \cdot \frac{1}{2} = 1 + \frac{k}{2}$$

Since the sequence x_n is increasing, for any $n \geq 2^k$, we have:

$$x_n \geq x_{2^k} > 1 + \frac{k}{2}$$

As $k \to \infty$, $n = 2^k \to \infty$, and $1 + \frac{k}{2} \to \infty$. This means the sequence x_n is not bounded above.

5. (a) Let $a > 0$, $x_1 = \sqrt{a}$ and $x_{n+1} = \sqrt{a + x_n}$ for $n \geq 2$.
   Convergence: Yes.
   Reasoning: The sequence is strictly increasing (by induction) and bounded above. Since increasing so

   $$x_{n+1} > x_n \forall n \in \mathbb{N}$$
   $$\implies x_n^2 - x_n - a > 0$$

   As $a > 0$ given so it has two real solution.By calculation you get one is positive solution and another is negative.So sequence x_n is bounded.
   Limit: Let $L = \lim_{n\to\infty} x_n$. Then $L = \sqrt{a + L} \implies L^2 - L - a = 0$.
   Solving: $L = \frac{1+\sqrt{1+4a}}{2}$ (positive root).

   (b) Let $x_1 = a > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for $n \geq 2$.
   Convergence: No.
   Reasoning: The sequence is strictly increasing. Suppose it converges to L. Then $L = L + \frac{1}{L} \implies \frac{1}{L} = 0$, contradiction.

   (c) $x_1 = 1$, $x_2 = 1$, $x_{n+2} = x_{n+1} + x_n$ for $n \geq 3$ (Fibonacci sequence).
   Convergence: No.
   Reasoning: The sequence is strictly increasing , unbounded above so $x_n \to \infty$.

   (d) $x_n = \frac{n}{a^n}$ for any $a \in \mathbb{R}$.
   Convergence:
   - If $|a| < 1$, then $|a^n| \to 0$ faster than $n \to \infty$, so $x_n \to \infty$ (diverges).
   - If $|a| = 1$, then $x_n = n$ (if $a = 1$) or $x_n = (-1)^n n$ (if $a = -1$), both diverge.
   - If $|a| > 1$, then $x_n \to 0$ (converges to 0). Use ratio test: $\left|\frac{x_{n+1}}{x_n}\right| = \frac{n+1}{n} \cdot \frac{1}{|a|} \to \frac{1}{|a|} < 1$.

   Limit when convergent: 0 for $|a| > 1$.

   (e) $x_n = \sin(n!\alpha\pi)$ where $\alpha \in \mathbb{R}$.
   Convergence:
   - If α is rational, say $\alpha = \frac{p}{q}$, then for $n \geq q$, $n!\alpha$ is an integer, so $x_n = \sin(k\pi) = 0$ for all $n \geq q$. Thus converges to 0.
   - If α is irrational, x_n oscillates in $[-1, 1]$.

   Limit when convergent: 0 (only for rational α).

6. Prove or disprove: If $\{x_n\}$ and $\{y_n\}$ are sequences such that $x_n y_n \to 0$, then at least one of them converges to 0.
   This statement is **false**.
   counterexample:

   - Consider $x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ $\quad y_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$
     Then $x_n y_n = 0$ for all n, so $x_n y_n \to 0$ But neither $\{x_n\}$ nor $\{y_n\}$ converges to 0 (both diverge)

7. Given $0 < a_1 \leq \cdots \leq a_k$.
Let

$$x_n = (a_1^n + \cdots + a_k^n)^{1/n}$$
$$(a_k^n)^{1/n} \leq x_n \leq (a_k^n + \cdots + a_k^n)^{1/n}$$
$$(a_k^n)^{1/n} \leq x_n \leq a_k(k)^{1/n}$$

take limit $n \to \infty$ and use **Sandwich Theorem** you get

$$\lim_{n\to\infty} x_n = a_k$$

8. Given $\{x_n\}$ be a sequence such that there exists a subsequence $\{x_{n_k}\}$

$$x_{n_k} \geq k$$

for each $k \in \mathbb{N}$
we have to show $\{x_n\}$ is unbounded.
On the contrary,suppose $\{x_n\}$ is bounded .
We know a sequence is bounded then all of its subsequnce is also bounded.
But here a subsequence $x_{n_k} \geq k$ for each $k \in \mathbb{N}$
So we get a contradiction.
So sequence $\{x_n\}$ is unbounded.

9. (a) **T** ; Since $\frac{x_n}{1+|x_n|}$ is bounded by 1 then by Bolzano–Weierstrass Theorem, $\{\frac{x_n}{1+|x_n|}\}$ has a convergent subsequence.

   (b) **F**; Take $x_n = -1 + \frac{1}{n}$. Then $\frac{x_n}{1+|x_n|} = -n + 1$ is an unbounded sequence.

   (c) **F** ; Note that $|\sin n| \leqslant 1$. Then $\{x_n\}$ is a bounded sequence. Hence Bolzano–Weierstrass Theorem it has a convergent subsequence.

10. Let a be any real number.
$x_1 = a, x_2 = \frac{1+a}{2}$ and
$\{x_n\}$ is defined by

$$x_{n+1} = \frac{1 + x_n}{2}$$

So successively

$$x_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \frac{a}{2^{n-1}}$$

Now

$$x_{n+1} - x_n = \frac{1}{2^n} + \frac{a}{2^n} - \frac{a}{2^{n-1}}$$
$$= \frac{1 - a}{2^n}$$

So x_n is increasing if 1-a > 0 and decreasing if 1-a <0.
For constant sequence $x_{n+1} = x_n$ for all n in $\mathbb{N}$
so $x_1 = x_2$
it implies

$$a = 1$$