MAL100: Mathematics I

Tutorial Sheet 3: Sequence

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1. Discuss the monotonicity of the sequence $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$, for $n \in \mathbb{N}$.

2. Prove that the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is monotonic increasing and bounded above.

3. Let $x_n > 0$. Then $\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} \frac{1}{x_n} = \infty$.

4. Prove that the sequence $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$ is divergent.

5. Discuss the convergence of the following sequences:

   (a) Let $a > 0, x_1 = \sqrt{a}$ and $x_{n+1} = \sqrt{a + x_n}$ for $n \geq 2$,

   (b) Let $x_1 = a > 0$ and $x_{n+1} = x_n + \frac{1}{x_n}$ for $n \geq 2$,

   (c) $x_1 = 1, x_2 = 1, x_{n+2} = x_{n+1} + x_n$ for $n \geq 3$,

   (d) $x_n = \frac{n}{a^n}$ for any $a \in \mathbb{R}$,

   (e) $x_n = \sin(n!\alpha\pi)$ where $\alpha \in \mathbb{R}$,

   and if convergent, find its limit.

6. Prove or Disprove: If $\{x_n\}$ and $\{y_n\}$ are sequences such that $x_n y_n \to 0$. Then one of them converges to 0.

7. Let $0 < a_1 \leq \cdots \leq a_k$. Find the limit of the sequence $(a_1^n + \cdots + a_k^n)^{1/n}$.

8. Prove that a sequence $\{x_n\}$ is unbounded if there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \geq k$ for each $k \in \mathbb{N}$.

9. True/ False:

   (a) For any sequence $\{x_n\}$, the sequence $y_n = \frac{x_n}{1+|x_n|}$ has a convergent subsequence.

   (b) For any sequence $\{x_n\}$, the sequence $y_n = \frac{x_n}{1+x_n}$ has a convergent subsequence.

   (c) $\sin(n)$ does not have any convergent subsequence.

10. Let a be any real number. Let $x_1 = a, x_2 = \frac{1+a}{2}$ and $\{x_n\}$ is defined by $x_{n+1} = \frac{1+x_n}{2}$. Discuss monotonicity and convergence of $\{x_n\}$. If $\{x_n\}$ is convergent, find its limit. For which values of a, $\{x_n\}$ is constant sequence?