MAL100: Mathematics I

Tutorial Sheet 9: Riemann Integration

Department of Mathematics
Indian Institute of Technology Bhilai

1. Let f be a Riemann integrable function on $[a, b]$. Prove that $|f|$ is integrable and
$$\left| \int_a^b f(x)dx \right| \leqslant \int_a^b |f(x)|dx.$$

2. Check the integrability of the following functions (if integrable, find the integral):
   (a) $f : [0, 1] \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in \mathbb{Q} \\ \frac{1}{3}, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
   (b) $f : [0, 1] \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} 1, & \text{if } x \in [0, 1) \\ 2^{10}, & \text{if } x = 1. \end{cases}$$
   (c) $f : [0, 1] \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} x, & \text{if } x \in [0, 1] \cap \mathbb{Q} \\ 0, & \text{if } x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$
   (d) $f : [0, 1] \to \mathbb{R}$ defined by
   $$f(x) = \begin{cases} x^2, & \text{if } x \in [0, 1] \cap \mathbb{Q} \\ x^3, & \text{if } x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

3. Prove or Disprove:
   (a) If $|f|$ is integrable then so is f.
   (b) If $f : [a, b] \to \mathbb{R}$ be integrable and $f \geqslant 0$ on $[a, b]$ then $\int_a^b f \geqslant 0$.
   (c) If f is continuous on $[a, b]$ and $\int_a^b f(x)dx = 0$ then $f(c) = 0$ for at least one $c \in [a, b]$.
   (d) If f is continuous on $[a, b]$ and $\int_a^b f(x)g(x)dx = 0$ for all integrable function g then $f \equiv 0$.

4. Show by an example that continuity is not necessary for the existence of antiderivative.

5. Let f be continuous and positive on $[a, b]$ and $M = \max\{f(x) : x \in [a, b]\}$. Prove that
$$\lim_{n\to\infty} \left(\int_a^b f(x)^n dx \right)^{1/n} = M.$$

6. (First Mean Value Theorem). Let f, g be continuous on $[a, b]$. Assume that g does not change sign on $[a, b]$. Then there exists $c \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$.

7. Evaluate the following limits:

(a) $\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{1}{2n}\right)$.

(b) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right)$.

(c) $\lim_{n\to\infty}((1+\frac{1^2}{n^2})+(1+\frac{2^2}{n^2})+\cdots+(1+\frac{n^2}{n^2}))^{1/n}$.

8. Prove that the function $f(x)=[x]$ defined on $[0,3]$ is integrable but the integral can not be determined by *Fundamental Theorem of Calculus*.