

MAL100: Mathematics I

Tutorial Sheet 4: Cauchy Sequence and Infinite Series

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1. Show that the sequence $x_n = \sqrt{n}$ satisfies $\lim_{n \rightarrow \infty} |x_{n+1} - x_n| = 0$ but it is not a Cauchy sequence.
2. Let $\{x_n\}$ be a sequence of integers. Can it be a Cauchy sequence? If not, why? Describe.
3. Let $\{x_n\}$ be a sequence of real numbers such that $|x_{k+1} - x_k| < a^{-k}$ for some $a > 1$. Prove that $\{x_n\}$ is Cauchy.
4. Let a and b be any two distinct real numbers. Let $x_1 = a$ and $x_2 = b$. Define $x_{n+2} = \frac{x_{n+1} + x_n}{2}$. Prove that $\{x_n\}$ is convergent.
5. Prove that the sequence $\{x_n\}$ of real numbers satisfying the condition $|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$ and for some $c \in (0, 1)$ is convergent.
6. Using Cauchy's principle of convergence prove that $\{\frac{n}{n+1}\}$ is convergent.
7. Prove that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences then so are $\{x_n + y_n\}$ and $\{x_n y_n\}$.
8. Prove or Disprove:
 - (a) If $\sum a_n$ is convergent for $a_n > 0$ then so is $\sum a_n^2$.
 - (b) If $\sum a_n$ is convergent for $a_n > 0$ then so is $\sum \sqrt{a_n}$.
9. If $\sum a_n$ with $a_n > 0$ is convergent, and if $b_n = (a_1 + \dots + a_n)/n$ for $n \in \mathbb{N}$, then show that $\sum b_n$ is always divergent.
10. For which values of x is the series $\sum_{n \geq 0} \frac{(n+1)^n}{n!} x^n$ convergent?
11. Prove that if $\sum_{n \geq 1} n^2 a_n$ is convergent for $a_n > 0$, then prove that $\sum_{n \geq 1} a_n$ is convergent.
12. If $\sum_{n \geq 1} a_n$ be se series of positive terms. Then discuss the convergence/divergence of the series $\sum_{n \geq 1} \frac{a_n}{n^p}$ for $p \geq 0$.