MAL100: Mathematics I
Tutorial Sheet 8:Differentiation
Solutions

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Problem 1

Let $f(x) = |x|^3$. Show that $f'(0), f''(0)$ exist but $f^{(3)}(0)$ does not.

Solution.

$$f(x) = \begin{cases} x^3, & x > 0, \\ (-x)^3 = -x^3, & x < 0. \end{cases}$$

Then,

$$f'(x) = \begin{cases} 3x^2, & x > 0, \\ -3x^2, & x < 0, \end{cases} \qquad f''(x) = \begin{cases} 6x, & x > 0, \\ -6x, & x < 0. \end{cases}$$

At $x = 0$:

$$f'(0) = \lim_{h \to 0} \frac{|h|^3 - 0}{h} = \lim_{h \to 0} |h|^2 = 0.$$

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h \to 0} \frac{f'(h)}{h} = \begin{cases} 3h \to 0, & h > 0, \\ -3h \to 0, & h < 0. \end{cases}$$

Hence $f''(0) = 0$.

For the third derivative:

$$f^{(3)}(0) = \lim_{h \to 0} \frac{f''(h) - f''(0)}{h} = \lim_{h \to 0} \frac{f''(h)}{h} = \begin{cases} 6, & h > 0, \\ -6, & h < 0, \end{cases}$$

which does not exist.

$$\boxed{f'(0) = 0, \quad f''(0) = 0, \quad f^{(3)}(0) \text{ does not exist.}}$$

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Problem 2

We want to prove that $f : (0, \infty) \to (0, \infty)$ defined by $f(x) = x^{1/n}$ for $n \in \mathbb{N}$ is differentiable.

The derivative of f at $x > 0$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Let $a = (x+h)^{1/n}$ and $b = x^{1/n}$. Then $a^n - b^n = (x+h) - x = h$.

We have the factorization:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}).$$

So

$$(x+h)^{1/n} - x^{1/n} = \frac{a^n - b^n}{a^{n-1} + a^{n-2}b + \cdots + b^{n-1}} = \frac{h}{a^{n-1} + a^{n-2}b + \cdots + b^{n-1}}.$$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^{1/n} - x^{1/n}}{h} = \frac{1}{a^{n-1} + a^{n-2}b + \cdots + b^{n-1}},$$

where $a = (x+h)^{1/n}$, $b = x^{1/n}$.

As $h \to 0$, $a \to b$. Each term in the denominator $a^{n-k}b^{k-1}$ (for $k = 1, \ldots, n$) tends to b^{n-1}. There are n such terms, so the denominator tends to $nb^{n-1} = nx^{(n-1)/n}$.

Thus

$$f'(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

Since $x > 0$, the denominator $nx^{(n-1)/n} \neq 0$, so the limit exists for all $x > 0$. Hence f is differentiable on $(0, \infty)$.

$$\boxed{\frac{1}{n}x^{\frac{1}{n}-1}}$$

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Problem 3

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is continuous and differentiable at 0.

Continuity: For any sequence $x_k \to 0$, $|f(x_k)| \leq x_k^2 \to 0$, so f is continuous at 0.

Differentiability:

$$f'(0) = \lim_{h \to 0} \frac{f(h)}{h} = \begin{cases} h \to 0, & h \in \mathbb{Q}, \\ 0, & h \notin \mathbb{Q}. \end{cases}$$

Hence $f'(0) = 0$.

$$\boxed{f \text{ continuous and differentiable at } 0, \quad f'(0) = 0.}$$

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Problem 4

Consider $f(x) = \begin{cases} x^n, & x \geq 0 \\ x^m, & x < 0 \end{cases}$ with $n, m \in \mathbb{N}$.

- For $x > 0$: $f'(x) = nx^{n-1}$

- For $x < 0$: $f'(x) = mx^{m-1}$

- At $x = 0$:

$$f'_+(0) = \lim_{h \to 0^+} \frac{h^n}{h} = \lim_{h \to 0^+} h^{n-1}$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{h^m}{h} = \lim_{h \to 0^-} h^{m-1}$$

Results at $x = 0$:

- If $n = m = 1$: $f'_+(0) = f'_-(0) = 1 \Rightarrow$ differentiable

- If $n > 1$ and $m > 1$: $f'_+(0) = f'_-(0) = 0 \Rightarrow$ differentiable

- If $n = 1, m > 1$: $f'_+(0) = 1, f'_-(0) = 0 \Rightarrow$ not differentiable

- If $n > 1, m = 1$: $f'_+(0) = 0, f'_-(0) = 1 \Rightarrow$ not differentiable

$$\boxed{\text{Differentiable everywhere iff } (n = m = 1) \text{ or } (n > 1 \text{ and } m > 1)}$$

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Problem 5

Let

$$f(x) = \begin{cases} e^{-1/x^2} \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show f' is continuous at 0.

At $x = 0$:

$$f'(0) = \lim_{h \to 0} \frac{e^{-1/h^2} \sin(1/h)}{h} = 0$$

since e^{-1/h^2} decays faster than any power.

For $x \neq 0$:

$$f'(x) = e^{-1/x^2} \left(\frac{2}{x^3} \sin(1/x) - \frac{1}{x^2} \cos(1/x) \right).$$

As $x \to 0$, both terms $\to 0$. Hence $f'(x) \to 0 = f'(0)$.

$$\boxed{f' \text{ continuous at } 0.}$$

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Problem 6

Let p be a polynomial and $k \in \mathbb{R}$. Between any two distinct roots of $p(x) = 0$ there exists a root of $p'(x) + kp(x) = 0$.

Let $a < b$ be consecutive roots of p. Define $h(x) = e^{kx}p(x)$. Then

$$h'(x) = e^{kx}\big(p'(x) + kp(x)\big).$$

Since $h(a) = h(b) = 0$, Rolle's theorem gives $c \in (a, b)$ with $h'(c) = 0$, i.e.

$$p'(c) + kp(c) = 0.$$

$$\boxed{\text{Hence, the claim is true by Rolle's theorem.}}$$

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Problem 7

(a) If differentiable f satisfies $|f(x) - f(y)| \le |x - y|^{1+\varepsilon}$, show f is constant.

$$\left|\frac{f(x + h) - f(x)}{h}\right| \le |h|^\varepsilon \to 0.$$

Thus $f'(x) = 0$ for all x, so f constant.

$$\boxed{\text{True.}}$$

(b) If continuous f satisfies $|f(x) - f(y)| \le (x - y)^2$, then

$$\left|\frac{f(x + h) - f(x)}{h}\right| \le |h| \to 0.$$

So $f'(x) = 0$ everywhere; f differentiable and constant.

$$\boxed{\text{True.}}$$

(c) If f even, f' is **odd**; if f odd, f' is **even**.

$$\boxed{\text{False}}$$

$$\boxed{\text{Derivative of even} = \text{odd, derivative of odd} = \text{even.}}$$

(d) Let $f : [2, 5] \to \mathbb{R}$ be continuous on $[2, 5]$ and differentiable on $(2, 5)$. Assume that

$$f'(x) = [f(x)]^2 + \pi \quad \text{for all } x \in (2, 5).$$

Then $f(5) - f(2) = 3$. ?

By the Mean Value Theorem, there exists $c \in (2, 5)$ such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2}$$

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If $f(5) - f(2) = 3$, then
$$f'(c) = \frac{3}{3} = 1.$$

But from the given differential equation,
$$f'(c) = [f(c)]^2 + \pi.$$

So
$$1 = [f(c)]^2 + \pi \quad \Rightarrow \quad [f(c)]^2 = 1 - \pi.$$

Since $\pi > 1$, we have $1 - \pi < 0$, which implies
$$[f(c)]^2 < 0.$$

This is impossible for a real-valued function f. Hence, the assumption $f(5) - f(2) = 3$ leads to a contradiction.

$$\boxed{\text{False}}$$

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Problem 8

Let f differentiable on interval I.

(a) If $f'(x) > 0$ on I, then by Mean Value Theorem,
$$f(b) - f(a) = f'(c)(b - a) > 0.$$

Hence f strictly increasing.

$$\boxed{\text{True.}}$$

(b) If $|f'(x)| \leq M$, then
$$|f(x) - f(y)| = |f'(\xi)||x - y| \leq M|x - y|.$$

So f is Lipschitz, hence uniformly continuous.

$$\boxed{\text{True.}}$$

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Problem 9 (Cauchy's Mean Value Theorem)

If $f, g : [a, b] \to \mathbb{R}$ are differentiable and $g'(x) \neq 0$, then there exists $c \in (a, b)$ such that

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(c)}{g'(c)}.$$

Proof. Define

$$F(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a)).$$

Then $F(a) = F(b) = 0$, so by Rolle's theorem there exists $c \in (a, b)$ with $F'(c) = 0$. Thus

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)),$$

which gives the result.

$$\boxed{\text{Proved.}}$$

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Problem 10 (Darboux's Theorem)

Let f be differentiable on $[a, b]$. Then f' has the intermediate value property.
Define $g(x) = f(x) - \lambda x$. If $f'(a) < \lambda < f'(b)$, then

$$g'(a) = f'(a) - \lambda < 0, \quad g'(b) = f'(b) - \lambda > 0.$$

By the Mean Value Theorem, there exists $c \in (a, b)$ with $g'(c) = 0 \Rightarrow f'(c) = \lambda$.

$$\boxed{f' \text{ has the intermediate value property.}}$$

Let

$$f(x) = \begin{cases} 0, & \text{for } x \in [-1, 0] \\ 1, & \text{for } x \in (0, 1] \end{cases}$$

Does there exist g with $g' = f$ on $[-1, 1]$?
Answer: No. By Darboux's Theorem, derivatives satisfy the intermediate value property. The function f has a jump discontinuity at 0 (from 0 to 1), so it cannot be a derivative.

$$\boxed{\text{No such } g \text{ exists.}}$$