MAL100: Mathematics I — Tutorial Sheet 7 (Continuity) Solutions

Notation: All proofs use standard ε–δ definitions. A function f is *uniformly continuous* on a set S if

$$\forall\varepsilon > 0\ \exists\delta > 0 \text{ such that } \ |x-y| < \delta,\ x, y \in S \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Problem 1. Check whether the following functions are uniformly continuous or not.

(a) $f(x) = \dfrac{1}{x}$ for $x \in (0, \infty)$.

Solution By Steps

Step 1: Idea. Try sequences whose distance tends to 0 but function values do not.

Step 2: Construct sequences. Let $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{1}{n+1}$. Then $|x_n - y_n| = \dfrac{1}{n(n+1)} \to 0$ as $n \to \infty$.

Step 3: Evaluate function difference.

$$|f(x_n) - f(y_n)| = |\ n - (n+1)\ | = 1 \not\to 0.$$

Thus f is *not* uniformly continuous on $(0, \infty)$.

(b) $f(x) = \dfrac{1}{x}$ for $x \in [\alpha, \infty)$ for some $\alpha > 0$.

Solution By Steps

Step 1: Use algebraic estimate. For $x, y \in [\alpha, \infty)$,

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y-x|}{|xy|} \leq \frac{|x-y|}{\alpha^2}.$$

Step 2: Conclude Lipschitz property. The last inequality shows f is Lipschitz with constant $1/\alpha^2$.

Hence f is Lipschitz and therefore uniformly continuous on $[\alpha, \infty)$.

(c) $f(x) = x^2$ for $x \in [a, b]$ with $b > a \geq 0$.

Solution By Steps

Step 1: Use compactness / derivative bound. On $[a, b]$ the derivative $f'(x) = 2x$ is bounded by $2b$.

Step 2: Lipschitz from Mean Value Theorem. For $x, y \in [a, b]$ there exists c between x, y with

$$|x^2 - y^2| = |2c||x-y| \leq 2b|x-y|.$$

Final Answer: f is uniformly continuous on $[a, b]$.

(d) $f(x) = x^2$ for $x \in [a, \infty)$ with $a \geq 0$.

Solution By Steps

Step 1: Seek counterexample sequence. Let $x_n = n$ and $y_n = n + \dfrac{1}{n}$ for integers $n \geq 1$.

Step 2: Distances tend to 0. $|x_n - y_n| = \dfrac{1}{n} \to 0$.

Step 3: Function differences do not tend to 0.

$$|x_n^2 - y_n^2| = |n^2 - (n + \tfrac{1}{n})^2| = \left|n^2 - \left(n^2 + 2 + \tfrac{1}{n^2}\right)\right| = 2 + \tfrac{1}{n^2} \to 2 \neq 0.$$

This shows that f is *not* uniformly continuous on $[a, \infty)$.

(e) $f(x) = \sin(x \sin x)$ for $x \in [0, \infty)$.

Solution By Steps

Step 1: Use Lipschitz property of sine. For all real u, v, $|\sin u - \sin v| \leq |u - v|$. Thus

$$|f(x) - f(y)| = |\sin(x \sin x) - \sin(y \sin y)| \leq |x \sin x - y \sin y|.$$

So uniform continuity of f would follow from uniform continuity of $g(x) := x \sin x$. We will show g is *not* uniformly continuous on $[0, \infty)$.

Step 2: Construct sequences making g blow up in difference. Let

$$x_n = 2n\pi, \qquad y_n = 2n\pi + \frac{1}{4n}.$$

Then $|x_n - y_n| = \dfrac{1}{4n} \to 0$.

Step 3: Compute limits of g.

$$g(x_n) = x_n \sin x_n = 2n\pi \cdot 0 = 0.$$

For y_n,

$$g(y_n) = \left(2n\pi + \frac{1}{4n}\right) \sin\left(\frac{1}{4n}\right) \approx \left(2n\pi + \frac{1}{4n}\right) \cdot \frac{1}{4n} \to 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}.$$

Hence $g(y_n) \to \pi/2$, so $|g(y_n) - g(x_n)| \to \pi/2$.

Step 4: Transfer to f. Using the Lipschitz bound of sin,

$$|f(y_n) - f(x_n)| \geq \big|\,|\sin(g(y_n))| - |\sin(g(x_n))|\,\big| \to |\sin(\tfrac{\pi}{2}) - 0| = 1.$$

Thus $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \not\to 0$.

It follows from here that f is *not* uniformly continuous on $[0, \infty)$.

Problem 2. Does a uniformly continuous function map Cauchy sequences to Cauchy sequences?

Solution By Steps

Step 1: Let f be uniformly continuous and (x_n) a Cauchy sequence in domain D.

Step 2: Use definition of uniform continuity. For given $\varepsilon > 0$ choose $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Step 3: Use Cauchy property of (x_n). Since (x_n) is Cauchy, $\exists N$ with $m, n \geq N \Rightarrow |x_m - x_n| < \delta$. Then for $m, n \geq N$,

$$|f(x_m) - f(x_n)| < \varepsilon.$$

Therefor the answer is yes! A uniformly continuous function maps Cauchy sequences to Cauchy sequences.

Problem 3. Let $f : [a, b] \to \mathbb{R}$ be continuous and satisfy $|f(x)-f(y)| \leq L|x-y|$ for all $x, y \in [a, b]$. Prove f is uniformly continuous.

Solution By Steps

Step 1: Recognize Lipschitz property. The hypothesis is precisely the Lipschitz condition with constant L.

Step 2: Show uniform continuity. Given $\varepsilon > 0$ choose $\delta = \varepsilon/L$ (if $L = 0$ any $\delta > 0$ works). Then for $|x - y| < \delta$,

$$|f(x) - f(y)| \leq L|x - y| < L \cdot \frac{\varepsilon}{L} = \varepsilon.$$

Therefore f is uniformly continuous (Lipschitz functions are uniformly continuous).

Problem 4. Let I be an interval and $f : I \to \mathbb{R}$ continuous with $f(x) \neq 0$ for all $x \in I$. Prove either $f > 0$ on I or $f < 0$ on I. Using this, show that if f, g are continuous on I with $f(x) \neq g(x)$ for all $x \in I$, then either $f > g$ or $f < g$ on I.

Solution By Steps

Step 1: Suppose contrary there are $u, v \in I$ with $f(u) > 0$ and $f(v) < 0$.

Step 2: Apply Intermediate Value Theorem (IVT). Since I is an interval and f is continuous, by IVT there exists c between u and v with $f(c) = 0$, contradicting the hypothesis.

Conclusion (first part): f has constant sign on I: either $f(x) > 0$ for all $x \in I$ or $f(x) < 0$ for all $x \in I$.

Step 3: Second part via $h := f - g$. The function h is continuous and $h(x) \neq 0$ for all $x \in I$. By the first part, h has constant sign, hence either $f - g > 0$ on I (i.e. $f > g$) or $f - g < 0$ on I (i.e. $f < g$).

Problem 5. Let $f : [a, b] \to \mathbb{R}$ be non-constant continuous. Show $f([a, b])$ is an interval. Using this solve the following:

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that

(a) $f(\mathbb{R}) \subset (0, 1) \cup [2, 100)$,

(b) $f(10) = e$.

Explain where $f([1, 2])$ is contained.

Solution By Steps

Step 1: Image of interval is interval. This is standard: by the Intermediate Value Theorem the continuous image of a connected set (an interval) is connected, hence an interval.

Step 2: Use connectedness of $\mathbb{R}$. The set $f(\mathbb{R})$ is a connected subset of $(0, 1) \cup [2, 100)$ (a union of two disjoint open/half-open pieces). Since $f(10) = e \in [2, 100)$, the connected image $f(\mathbb{R})$ cannot lie in the other disjoint component $(0, 1)$. Therefore

$$f(\mathbb{R}) \subset [2, 100).$$

Step 3: Restrict to $[1, 2]$. Since $[1, 2] \subset \mathbb{R}$, $f([1, 2]) \subset f(\mathbb{R}) \subset [2, 100)$ and also $f([1, 2])$ must be an interval (connected). Thus

Thus it follows that $f([1,2])$ is an interval contained in $[2,100)$.

Problem 6. Let $f : I \to \mathbb{R}$ be continuous and $f(c) > 0$ for some $c \in I$. Prove there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (c-\delta, c+\delta)$.

Solution By Steps

Step 1: Use continuity at c. Since $f(c) > 0$, take $\varepsilon = f(c)/2 > 0$. By continuity, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$.

Step 2: Conclude positivity. For such x,

$$f(x) \geq f(c) - |f(x) - f(c)| > f(c) - \frac{f(c)}{2} = \frac{f(c)}{2} > 0.$$

Final Answer: There exists $\delta > 0$ with $f(x) > 0$ for all $x \in (c-\delta, c+\delta) \cap I$.

Problem 7. Prove or Disprove:

(a) There exists a continuous onto function $f : [a,b] \to \mathbb{R}$.

Solution By Steps

Step 1: The continuous image of a closed and bounded interval $[a,b]$ is a closed and bounded interval.

Step 2: Observe $\mathbb{R}$ is not compact. Thus no continuous function from $[a,b]$ can be onto $\mathbb{R}$ (since $\mathbb{R}$ is not compact).

The answer is *No*, such a continuous onto map does not exist.

(b) Let $f : [a,b] \to [0,\infty)$ continuous with $f(x) > 0$ for all $x \in [a,b]$. Then there exists $c > 0$ such that $f(x) > c$ for all x.

Solution By Steps

Step 1: Use compactness again. Since f is continuous on compact $[a,b]$, it attains its minimum $m := \min_{[a,b]} f(x)$.

Step 2: Positivity gives $m > 0$. Because $f(x) > 0$ everywhere, $m > 0$.

Hence the statement if true. Take $c = m$; then $f(x) \geq m > 0$ for all $x \in [a,b]$.

Problem 8. Prove that if $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and f is continuous on $\mathbb{R}$, then f is uniformly continuous.

Solution By Steps

Step 1: Standard fact: additive continuous functions are linear. Continuity at any point (equivalently at 0) implies there exists $k \in \mathbb{R}$ with $f(x) = kx$ for all x. (Sketch: continuity at 0 gives boundedness on a neighborhood of 0, then show $f(q) = qf(1)$ for rational q, then extend by continuity to all reals.)

Step 2: Show linear function is Lipschitz. If $f(x) = kx$ then for all x, y,

$$|f(x) - f(y)| = |k||x - y|.$$

Thus f is Lipschitz (constant $|k|$) and therefore uniformly continuous.

Therefore f is uniformly continuous on $\mathbb{R}$.
