MAL100: Mathematics I

Assignment II

Department of Mathematics
Indian Institute of Technology Bhilai

1. If $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$, where $C_0, C_1, \ldots, C_n$ are real constants, prove that the equation $C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n = 0$ has at least one real root between 0 and 1.

2. Characterize all the differentiable functions $f : [0,1] \to \mathbb{R}$ whose slopes of the tangents are always rationals.

3. Let $f : [0,1] \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} \sqrt{1-x^2}, & \text{if } x \in \mathbb{Q} \\ 1-x, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
Prove that f is not integrable on $[0,1]$.

4. Let $\chi_A : [0,1] \to \mathbb{R}$ for $A \subseteq [0,1]$, be defined by
$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}.$$
Consider $f(x) = \sum_{n=1}^{200} \frac{1}{n^6} X_{[0, \frac{n}{200}]}(x), x \in [0,1]$. Then check whether $f(x)$ is Riemann integrable on $[0,1]$.

5. A function f continuous on $\mathbb{R}$ and $\int_{-x}^{x} f(t)\,dt = 2\int_0^x f(t)\,dt$ for all $x \in \mathbb{R}$. Prove that f is an even function.

6. If f is a real function defined on a convex open set $E \subset \mathbb{R}^n$ such that $(\partial_1 f)(x) = 0$ for every $x \in E$, where $\partial_1 f = \frac{\partial f}{\partial x_1}$, prove that f depends only on $x_2, x_3, \ldots, x_n$.

7. Let $D = [0,2] \times [0,3]$ and define
$$f(x,y) = \begin{cases} 3, & \text{if } x \in \mathbb{Q}, \\ y^2, & \text{if } x \notin \mathbb{Q}. \end{cases}$$
Calculate the iterated integral.

8. The line segment $x = 1 - y$, where $0 \leqslant y \leqslant 1$, is revolved about the y-axis to generate a cone. Find the surface area of the cone (excluding the area of the base). Match the obtained result with the formula of the surface area of a cone in geometry.

9. Let D be a region in $\mathbb{R}^2$ bounded by the curve C oriented counter-clockwise. Then *area of* D is given by
$$\text{Area of } D = \frac{1}{2}\int_C (x\,dy - y\,dx).$$
Then using Green's theorem find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

10. (a) Let S be the surface $x^2 + y^2 + z^2 = 1$, $z \geq 0$. Use Stokes' theorem to evaluate

$$\int_C (2x - y)\, dx - y\, dy - z\, dz$$

where C is the circle $x^2 + y^2 = 1$, $z = 0$ oriented anticlockwise.

(b) Consider the vector field $F = \frac{1}{a^3}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$ on the sphere S of radius a centered at the origin. Show that the flux through S is a constant.