MAL100: Mathematics I

Tutorial Sheet 8: Differentiation

Department of Mathematics
Indian Institute of Technology Bhilai

1. Let $f(x) = |x|^3$, then show that $f^{'}(0), f^{''}(0)$ exists but $f^{'''}(0)$ does not exists.

2. Prove that the function $f : (0, \infty) \to (0, \infty)$ defined by $f(x) = x^{1/n}$ for $n \in \mathbb{N}$ is differentiable.

3. Show that the function
$$f(x) = \begin{cases} x^2, & \text{for } x \in \mathbb{Q} \\ 0, & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$
is continuous and differentiable at 0 and $f^{'}(0) = 0$.

4. Discuss the differentiability of the function
$$f(x) = \begin{cases} x^n, & \text{for } x \geqslant 0 \\ x^m, & \text{for } x < 0. \end{cases}$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} e^{1/x^2} \sin \frac{1}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$$
Show that $f^{'}$ is continuous at 0.

6. Let p be a polynomial of degree bigger than 1 and $k \in \mathbb{R}$. Prove that between any two roots of $p(x) = 0$ there exists a root of $p^{'}(x) + kp(x) = 0$.

7. Prove or Disprove:
   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and satisfying the condition
   $$|f(x) - f(y)| \leqslant |x - y|^{1+\epsilon} \text{ for all } x, y \in \mathbb{R} \text{ and some } \epsilon > 0.$$
   Then f must is constant.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \leqslant (x - y)^2$ for all $x, y \in \mathbb{R}$. Then f is differentiable.
   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable function. If f is even then $f^{'}$ is even and if f is odd then $f^{'}$ is odd.
   (d) Let $f : [2, 5] \to \mathbb{R}$ be continuous and differentiable on $(2, 5)$. Assume that $f(x) = f(x)^2 + \pi$ for all $x \in (2, 5)$. Then $f(5) - f(2) = 3$.

8. Let I be an interval and $f : I \to \mathbb{R}$ be differentiable. Then prove the following:
   (a) If $f^{'}(x) > 0$ for all $x \in I$ then f is strictly increasing.
   (b) If $|f^{'}(x)| \leqslant M$ for all $x \in I$ and for some fixed $M > 0$ then f is uniformly continuous.

9. (Cauchy's Mean Value Theorem). Let $f, g : [a, b] \to \mathbb{R}$ be differentiable such that $g^{'}(x) \neq 0$ for all $x \in [a, b]$. Then there exists $c \in (a, b)$ such that

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f^{'}(c)}{g^{'}(c)}.$$

10. (Darboux's Theorem). Let $f : [a, b] \to \mathbb{R}$ be differentiable. Then $f^{'}$ possesses intermediate value property; i.e., for any y in between $f^{'}(a)$ and $f^{'}(b)$ there exists $x \in [a, b]$ such that $f^{'}(x) = y$. Now using this answer the following question:
Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 0$ for $x \in [-1, 0]$ and $f(x) = 1$ for $x \in (0, 1]$. Does there exists g such that $g^{'} = f$?