MAL100: Mathematics I

Tutorial Sheet 5: Infinite Series and Limit

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1. Discuss the convergence (absolute/conditional) of the following series:
   (a) $\sum_{n\geqslant 1} \frac{(-1)^n}{(2n-1)\cdot n^2}$.
   (b) $\sum_{n\geqslant 1} \frac{(-1)^n}{n^3\cdot \log(n+1)}$.
   (c) $\sum_{n\geqslant 2} \frac{1}{(\log n)^p}$ for any p.
2. Suppose $\{a_n\}$ be a sequence satisfying $\lim_{n\to\infty}(a_1+1)(a_2+1)\ldots(a_n+1) = a$ for some $a \in (0,\infty)$. Prove that the series $\sum \frac{a_n}{(a_1+1)(a_2+1)\ldots(a_n+1)}$ is convergent and find the sum of the series.
3. Show that if the sequence $\{na_n\}$ and the series $\sum n(a_n - a_{n+1})$ both converge then $\sum a_n$ also does.
4. Give an example of a conditionally convergent series $\sum a_n$ and a bounded sequence $\{b_n\}$ such that $\sum a_n b_n$ is divergent.
5. Let $\sum a_n$ be an absolutely convergent series and $1 + a_n \neq 0$ for any $n \in \mathbb{N}$. Prove that $\sum \frac{a_n}{1+a_n}$ is also absolutely convergent.
6. Prove that $\sum_{n\geqslant 1}(-1)^n \sin\frac{1}{n}$ is convergent but not absolutely convergent.
7. Let $a_n = n^3 \sin\frac{1}{n} - n^2 + \frac{1}{6}$. Then show that $\sum a_n$ is absolutely convergent.
8. Show that $\lim_{x\to 0} e^{1/x}$ does not exist where $x \in \mathbb{R}$ and $x \neq 0$.
9. Using the $\epsilon - \delta$ definition of limit, prove that $\lim_{x\to 2}\sqrt{4x - x^2} = 2$.
10. (a) Let I be an interval in $\mathbb{R}$, let $f : I \to \mathbb{R}$, and let $c \in I$. Suppose there exist constants K and L such that $|f(x) - L| \leqslant K|x - c|$ for $x \in I$. Show that $\lim_{x\to c} f(x) = L$.
    (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x\to 0} f(x) = L$. If $g : \mathbb{R} \to \mathbb{R}$ be such a function that $g(x) = f(ax)$ for some $a \in \mathbb{R}$, then show that $\lim_{x\to 0} g(x) = L$.