Indian Institute of Technology Bhilai

Department of Mathematics

MAL100 : Mid Semester Exam

2 hours | Max mark: 35

Instructions : There are 5 questions each carrying 7 marks. Make sure to answer all the parts of a question together.

1. Answer the following questions

   (i) Find the infimum of the set $A = \{\frac{m}{n} + \frac{4n}{m} : m, n \in \mathbb{N}\}$.

   (ii) Prove that a monotone decreasing sequence which is bounded below is convergent and converges to its infimum.

   (iii) Let $\{a_n\}$ be a sequence defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$. Find $\lim_{n\to\infty} a_n$. (You can apply any method as you wish).

   $(2+3+2)$

2. Answer the following questions

   (i) Test the convergence of the following series

   (a) $\sum_{n=4}^{\infty} \frac{1}{n\log(\log n)}$

   (b) $\frac{1+2}{2^{10}} + \frac{1+2+3}{3^{10}} + \frac{1+2+3+4}{4^{10}} + \frac{1+2+3+4+5}{5^{10}} + \cdots$.

   (ii) Define subsequence of a sequence. Write down a subsequence of $\{\cos\frac{1}{\sqrt{n}}\}$ (You need write a subsequence, no need to explain)

   $((3+2)+(1+1))$

3. Answer the following questions

   (i) Using the definition of limit ($\epsilon - \delta$ method) prove that $\lim_{x\to 2}\sqrt{4x - x^2} = 2$.

   (ii) Using the definition of continuity ($\epsilon - \delta$ method) prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \sin x & \text{if } x \text{ is rational.} \end{cases}$$

   is continuous at the points $n\pi$ for integer n.

   (iii) Define the norm of a partition of an interval $[a, b]$.

   $(3+3+1)$

4. Answer the following questions

   (i) Let $f : [0, 2] \to \mathbb{R}$ is differentiable and f' is continuous on $[0, 2]$ with $f(0) = 0$, $f(1) = 2$, $f(2) = 1$. Prove that $f'(c) = 0$ for some $c \in (0, 2)$.

(ii) Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3x^2 \cos \frac{1}{x^3} + \frac{3}{x} \sin \frac{1}{x^3} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find an antiderivative of f. Can we apply Fundamental Theorem of Calculus (2nd part) to evaluate $\int_{-1}^{1} f(x)dx$? Justify your answer.

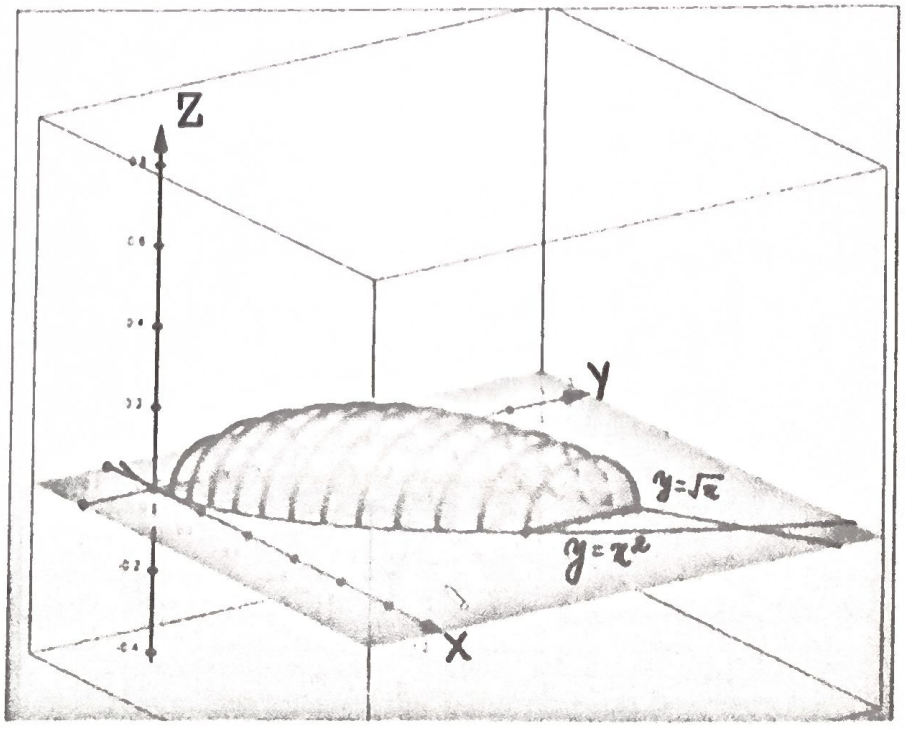
(ii) Define local maximum of a function.

$(3 + (1 + 2) + 1)$

5. Answer the following questions

(i) Find the area of the region bounded by $y = x^2 - 4$ and $y = 2x - 1$. (Draw a rough figure and show all the steps of the calculation).

(ii) Consider the solid whose base lies on the first quadrant of XY plane and bounded by the curves $y = x^2$ and $y = \sqrt{x}$. The cross-section of the solid with a plane perpendicular to X axis is a semi-circle (A semi circle is half of a circle). First find the area of a semi-circle of cross-section and then find the volume of the solid (Refer to the following figure).



$(3 + 4)$