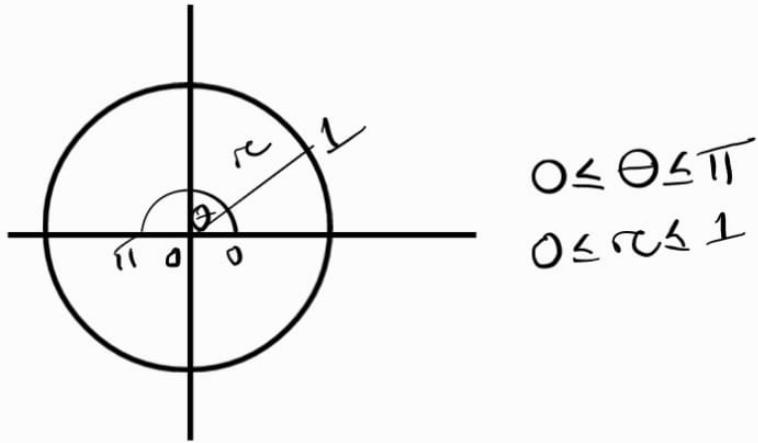


MAL100: Mathematics I  
 Tutorial Sheet 12 Solutions  
 VECTOR CALCULUS

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1. Evaluate  $\iint_R e^{x^2+y^2} dy dx$  where  $R$  is the semicircular region bounded by the  $x$ -axis and  $y = \sqrt{1 - x^2}$ .



**Solution:**

The region  $R$  is the upper half of the unit disk:  $x \in [-1, 1]$ ,  $y \in [0, \sqrt{1 - x^2}]$ .

In polar coordinates:  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,  $dy dx = r dr d\theta$ ,  $x^2 + y^2 = r^2$ .

Region in polar coordinates:  $r \in [0, 1]$ ,  $\theta \in [0, \pi]$ .

$$\iint_R e^{x^2+y^2} dy dx = \int_{\theta=0}^{\pi} \int_{r=0}^1 e^{r^2} r dr d\theta$$

Let  $u = r^2$ ,  $du = 2r dr \Rightarrow r dr = \frac{du}{2}$ .

$$\int_{r=0}^1 e^{r^2} r dr = \frac{1}{2} \int_{u=0}^1 e^u du = \frac{1}{2}(e - 1)$$

$$\int_{\theta=0}^{\pi} \frac{1}{2}(e - 1) d\theta = \frac{1}{2}(e - 1) \cdot \pi$$

$$\boxed{\frac{\pi}{2}(e-1)}$$

**2(a). Calculate the outward flux of  $\vec{F}(x, y) = x\hat{i} + y^2\hat{j}$  across the square bounded by  $x = \pm 1$  and  $y = \pm 1$ .**

**Solution:**

Using divergence theorem:

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2) = 1 + 2y$$

Square:  $x \in [-1, 1]$ ,  $y \in [-1, 1]$ .

$$\text{Flux} = \int_{x=-1}^1 \int_{y=-1}^1 (1 + 2y) dy dx$$

First integrate w.r.t  $y$ :

$$\int_{y=-1}^1 (1 + 2y) dy = [y + y^2]_{-1}^1 = (1 + 1) - (-1 + 1) = 2 - 0 = 2$$

Then integrate w.r.t  $x$ :

$$\int_{x=-1}^1 2 dx = 2 \cdot 2 = 4$$

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**2(b). Evaluate  $\int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{y-x} dx dy$**

**Solution:**

#### Change of Variable

Let

$$x = \phi(u, v), \quad y = \psi(u, v)$$

then

$$\iint_D f(x, y) dx dy = \iint_R f(\phi(u, v), \psi(u, v)) |J| du dv,$$

where the Jacobian is

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

$$\int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{y-x} dx dy.$$

The region satisfies

$$0 \leq y \leq \frac{2}{3}, \quad y \leq x \leq 2 - 2y.$$

Let

$$u = x - y, \quad v = y.$$

Then

$$x = u + v, \quad y = v.$$

Jacobian:

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1, \quad |J| = 1.$$

— Given

$$0 \leq y \leq \frac{2}{3} \Rightarrow 0 \leq v \leq \frac{2}{3},$$

$$\begin{aligned} x = y &\Rightarrow u = 0, \\ x = 2 - 2y &\Rightarrow u + 3v = 2 \Rightarrow u = 2 - 3v. \end{aligned}$$

Thus the region in the  $uv$ -plane is

$$G = \{(u, v) : 0 \leq v \leq \frac{2}{3}, 0 \leq u \leq 2 - 3v\}.$$

—

$$(x + 2y)e^{y-x} = (u + 3v)e^{-u}.$$

Thus

$$\begin{aligned} &\iint_G (u + 3v)e^{-u}|J| du dv \\ &= \int_{v=0}^{2/3} \int_{u=0}^{2-3v} (u + 3v)e^{-u} du dv. \end{aligned}$$

—

$$\begin{aligned} \int (u + 3v)e^{-u} du &= \int ue^{-u} du + 3v \int e^{-u} du \\ &= -(u + 1)e^{-u} - 3ve^{-u}. \end{aligned}$$

Apply limits:

$$\begin{aligned} \int_0^{2-3v} (u + 3v)e^{-u} du &= -[(2 - 3v + 1)e^{-(2-3v)} - 1 + 3ve^{-(2-3v)} - 3v]. \\ &= 2e^{3v-2} + 3ve^{3v-2} - e^{3v-2} + 1 - 3v. \\ &= (3v + 1) - 3e^{3v-2}. \end{aligned}$$

Thus

$$\int_0^{2/3} (3v + 1 - 3e^{3v-2}) dv.$$

—

$$\begin{aligned} \int_0^{2/3} (3v + 1) dv &= \left( \frac{3}{2}v^2 + v \right)_0^{2/3} = \frac{2}{3} + \frac{2}{3}. \\ \int_0^{2/3} 3e^{3v-2} dv &= [e^{3v-2}]_0^{2/3} = e^0 - e^{-2} = 1 - e^{-2}. \end{aligned}$$

Total:

$$\frac{2}{3} + \frac{2}{3} - (1 - e^{-2}) = \frac{1}{3} + e^{-2}.$$

—

$$\boxed{\frac{1}{3} + \frac{1}{e^2}}$$

**3. Evaluate**  $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) dy dx dz$

**Solution:**

Inner integral over  $y$ :

$$\int_{y=x-z}^{x+z} (x+y+z) dy = \left[ (x+z)y + \frac{1}{2}y^2 \right]_{y=x-z}^{x+z}$$

$$\text{At } y = x+z: (x+z)(x+z) + \frac{1}{2}(x+z)^2 = (x+z)^2 + \frac{1}{2}(x+z)^2 = \frac{3}{2}(x+z)^2$$

$$\text{At } y = x-z: (x+z)(x-z) + \frac{1}{2}(x-z)^2 = (x^2 - z^2) + \frac{1}{2}(x^2 - 2xz + z^2)$$

$$\text{Simplify: } x^2 - z^2 + \frac{1}{2}x^2 - xz + \frac{1}{2}z^2 = \frac{3}{2}x^2 - xz - \frac{1}{2}z^2$$

$$\text{Subtract: } \frac{3}{2}(x+z)^2 - \left[ \frac{3}{2}x^2 - xz - \frac{1}{2}z^2 \right]$$

$$\text{First: } \frac{3}{2}(x^2 + 2xz + z^2) = \frac{3}{2}x^2 + 3xz + \frac{3}{2}z^2$$

$$\text{Subtract second: } (\frac{3}{2}x^2 + 3xz + \frac{3}{2}z^2) - (\frac{3}{2}x^2 - xz - \frac{1}{2}z^2) = 3xz + \frac{3}{2}z^2 + xz + \frac{1}{2}z^2 = 4xz + 2z^2$$

$$\text{So inner} = 4xz + 2z^2$$

Now integrate over  $x$  from 0 to  $z$ :

$$\int_{x=0}^z (4xz + 2z^2) dx = [2x^2 z + 2z^2 x]_0^z = 2z^3 + 2z^3 = 4z^3$$

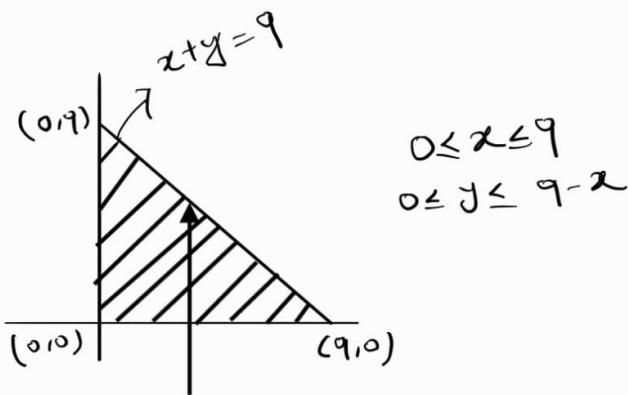
Now integrate over  $z$  from  $-1$  to  $1$ :

$$\int_{z=-1}^1 4z^3 dz = 0 \quad (\text{odd function})$$

0

**4. Evaluate**  $\iint_R (x^2 + y^2) dx dy$  where  $R$  is bounded by  $x = 0, y = 0$  and  $x + y = 9$

**Solution:** Region:  $x \geq 0, y \geq 0, x + y \leq 9$



$$\int_{x=0}^9 \int_{y=0}^{9-x} (x^2 + y^2) dy dx$$

Inner integral over  $y$ :

$$\int_0^{9-x} (x^2 + y^2) dy = x^2(9-x) + \frac{(9-x)^3}{3}$$

So:

$$I = \int_0^9 \left[ x^2(9-x) + \frac{(9-x)^3}{3} \right] dx$$

First term:  $\int_0^9 (9x^2 - x^3) dx = \left[ 3x^3 - \frac{x^4}{4} \right]_0^9 = 3 \cdot 729 - \frac{6561}{4} = 2187 - 1640.25 = 546.75$

Second term: Let  $u = 9-x$ ,  $du = -dx$ , when  $x = 0$ ,  $u = 9$ ; when  $x = 9$ ,  $u = 0$ :

$$\int_0^9 \frac{(9-x)^3}{3} dx = \frac{1}{3} \int_{u=9}^0 u^3 (-du) = \frac{1}{3} \int_0^9 u^3 du = \frac{1}{3} \cdot \frac{9^4}{4} = \frac{6561}{12} = 546.75$$

Total:  $546.75 + 546.75 = 1093.5 = \frac{2187}{2}$

$$\boxed{\frac{2187}{2}}$$

### 5(a). Surface area from rotating $y = \sin(2x)$ , $0 \leq x \leq \pi/8$ about $x$ -axis

**Solution:**

Surface area formula:

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$y = \sin(2x), \frac{dy}{dx} = 2\cos(2x)$$

$$S = 2\pi \int_0^{\pi/8} \sin(2x) \sqrt{1 + 4\cos^2(2x)} dx$$

Let  $u = \cos(2x)$ ,  $du = -2\sin(2x)dx$ , so  $\sin(2x)dx = -\frac{du}{2}$

When  $x = 0$ ,  $u = 1$ ; when  $x = \pi/8$ ,  $u = \cos(\pi/4) = \frac{\sqrt{2}}{2}$

$$S = 2\pi \int_{u=1}^{\sqrt{2}/2} \sqrt{1 + 4u^2} \cdot \left(-\frac{du}{2}\right) = \pi \int_{\sqrt{2}/2}^1 \sqrt{1 + 4u^2} du$$

Let  $2u = \tan \theta$ ,  $2du = \sec^2 \theta d\theta$ ,  $1 + 4u^2 = \sec^2 \theta$

When  $u = \sqrt{2}/2$ ,  $2u = \sqrt{2}$ ,  $\tan \theta = \sqrt{2} \Rightarrow \theta = \arctan(\sqrt{2})$

When  $u = 1$ ,  $2u = 2$ ,  $\theta = \arctan(2)$

$$\sqrt{1 + 4u^2} = \sec \theta, \quad du = \frac{\sec^2 \theta}{2} d\theta$$

So:

$$\pi \int_{\theta=\arctan(\sqrt{2})}^{\arctan(2)} \sec \theta \cdot \frac{\sec^2 \theta}{2} d\theta = \frac{\pi}{2} \int \sec^3 \theta d\theta$$

Recall  $\int \sec^3 \theta d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| + C$

Thus:

$$S = \frac{\pi}{4} [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|]_{\arctan(\sqrt{2})}^{\arctan(2)}$$

At  $\theta = \arctan(2)$ :  $\tan \theta = 2$ ,  $\sec \theta = \sqrt{5}$ , so term =  $2\sqrt{5} + \ln(2 + \sqrt{5})$

At  $\theta = \arctan(\sqrt{2})$ :  $\tan \theta = \sqrt{2}$ ,  $\sec \theta = \sqrt{3}$ , so term =  $\sqrt{6} + \ln(\sqrt{3} + \sqrt{2})$

Difference:  $2\sqrt{5} - \sqrt{6} + \ln \left( \frac{2+\sqrt{5}}{\sqrt{3}+\sqrt{2}} \right)$

Multiply by  $\pi/4$ :

$$\boxed{\frac{\pi}{4} \left[ 2\sqrt{5} - \sqrt{6} + \ln \left( \frac{2+\sqrt{5}}{\sqrt{3}+\sqrt{2}} \right) \right]}$$

**5(b). Surface area from rotating  $y = \sqrt[3]{x}$ ,  $1 \leq y \leq 2$ , about  $y$ -axis**

**Solution:**

$$x = y^3, dx/dy = 3y^2$$

Surface area about  $y$ -axis:

$$S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

So:

$$S = 2\pi \int_1^2 y^3 \sqrt{1 + 9y^4} dy$$

$$\text{Let } u = 1 + 9y^4, du = 36y^3 dy, \text{ so } y^3 dy = du/36$$

$$\text{When } y = 1, u = 10; \text{ when } y = 2, u = 1 + 9 \cdot 16 = 145$$

$$\begin{aligned} S &= 2\pi \int_{u=10}^{145} \sqrt{u} \cdot \frac{du}{36} = \frac{\pi}{18} \int_{10}^{145} u^{1/2} du = \frac{\pi}{18} \cdot \frac{2}{3} \left[ u^{3/2} \right]_{10}^{145} \\ &= \frac{\pi}{27} \left( 145^{3/2} - 10^{3/2} \right) \end{aligned}$$

$$\boxed{\frac{\pi}{27} \left( 145\sqrt{145} - 10\sqrt{10} \right)}$$

**6. Let  $D = [0, 1] \times [0, 1]$  and  $f(x, y)$  as given. Compute the iterated integrals and check Fubini.**

**Solution:**

First,  $\int_0^1 \int_0^1 f(x, y) dy dx$ :

For fixed  $x, y$  from 0 to 1:

- If  $y < x$  and  $y > 0$ :  $f = 1/x^2$
- If  $y > x$ :  $f = -1/y^2$
- If  $y = x$ : 0

So:

$$\int_0^1 f(x, y) dy = \int_0^x \frac{1}{x^2} dy + \int_x^1 \left( -\frac{1}{y^2} \right) dy = \frac{1}{x^2} \cdot x + \left[ \frac{1}{y} \right]_x^1 = \frac{1}{x} + (1 - \frac{1}{x}) = 1$$

Thus:

$$\int_0^1 \int_0^1 f(x, y) dy dx = \int_0^1 1 dx = 1$$

Now  $\int_0^1 \int_0^1 f(x, y) dx dy$ :

For fixed  $y, x$  from 0 to 1:

- If  $x > y$ :  $f = 1/x^2$
- If  $x < y$ :  $f = -1/y^2$

So:

$$\begin{aligned}\int_0^1 f(x, y) dx &= \int_0^y \left( -\frac{1}{y^2} \right) dx + \int_y^1 \frac{1}{x^2} dx = -\frac{1}{y^2} \cdot y + \left[ -\frac{1}{x} \right]_y^1 \\ &= -\frac{1}{y} + \left[ -1 + \frac{1}{y} \right] = -1\end{aligned}$$

Thus:

$$\int_0^1 \int_0^1 f(x, y) dx dy = \int_0^1 (-1) dy = -1$$

The two iterated integrals differ ( $1 \neq -1$ ), so Fubini's theorem does not apply.

1, -1, No

## 7. Prove $|\iint_D f(x, y) dx dy| \leq \alpha(b-a)(d-c)$ given $|f(x, y)| \leq \alpha$

**Solution:**

Since  $-\alpha \leq f(x, y) \leq \alpha$ , by monotonicity of the integral:

$$\iint_D -\alpha dx dy \leq \iint_D f(x, y) dx dy \leq \iint_D \alpha dx dy$$

But  $\iint_D \alpha dx dy = \alpha(b-a)(d-c)$ , and similarly for  $-\alpha$

So:

$$-\alpha(b-a)(d-c) \leq \iint_D f dx dy \leq \alpha(b-a)(d-c)$$

This implies:

$$\left| \iint_D f dx dy \right| \leq \alpha(b-a)(d-c)$$

## 8. Which vector fields are conservative?

(a)  $\vec{F}(x, y) = (x-y)\hat{i} + (x-2)\hat{j}$

Check if  $\nabla \times \vec{F} = 0$  in 2D:

$$\frac{\partial}{\partial x}(x-2) - \frac{\partial}{\partial y}(x-y) = 1 - (-1) = 2 \neq 0$$

Not conservative.

No

(b)  $\vec{F}(x, y) = (3+2xy)\hat{i} + (x^2-3y^2)\hat{j}$

$$\frac{\partial}{\partial x}(x^2-3y^2) - \frac{\partial}{\partial y}(3+2xy) = 2x - 2x = 0$$

Conservative.

Yes

(c)  $\vec{F}(x, y, z) = (2x-3)\hat{i} + z\hat{j} + \cos z\hat{k}$

Check  $\nabla \times \vec{F}$ :

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-3 & z & \cos z \end{vmatrix}$$

*i*-component:  $\frac{\partial}{\partial y}(\cos z) - \frac{\partial}{\partial z}(z) = 0 - 1 = -1 \neq 0$   
 Not conservative.

No

**9. Ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  revolves about *x*-axis. Find volume.**

**Solution:**

Using disk method: For fixed *x*, radius  $y = b\sqrt{1-x^2/a^2}$

Volume:

$$V = \pi \int_{-a}^a [y(x)]^2 dx = \pi \int_{-a}^a b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

Even function, so:

$$V = 2\pi b^2 \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx = 2\pi b^2 \left[x - \frac{x^3}{3a^2}\right]_0^a = 2\pi b^2 \left(a - \frac{a}{3}\right) = 2\pi b^2 \cdot \frac{2a}{3}$$

$\boxed{\frac{4}{3}\pi ab^2}$

**10. Volume of ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$**

**Solution:**

For fixed *z*, cross-section is ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2}$

Semi-axes:  $a\sqrt{1-z^2/c^2}$ ,  $b\sqrt{1-z^2/c^2}$

Area of ellipse =  $\pi \times$  (semi-axis1)  $\times$  (semi-axis2) =  $\pi ab(1 - z^2/c^2)$

Volume:

$$\begin{aligned} V &= \int_{z=-c}^c \pi ab \left(1 - \frac{z^2}{c^2}\right) dz = 2\pi ab \int_0^c \left(1 - \frac{z^2}{c^2}\right) dz \\ &= 2\pi ab \left[z - \frac{z^3}{3c^2}\right]_0^c = 2\pi ab \left(c - \frac{c}{3}\right) = 2\pi ab \cdot \frac{2c}{3} \end{aligned}$$

$\boxed{\frac{4}{3}\pi abc}$

ALTERNATIVE METHOD

We want to compute the volume of the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The volume is given by:

$$V = \iiint_{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1} dx dy dz.$$

### Change of variables to spherical coordinates

Let:

$$x = a\rho \sin \theta \cos \phi, \quad y = b\rho \sin \theta \sin \phi, \quad z = c\rho \cos \theta,$$

where:

$$0 \leq \rho \leq 1, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi.$$

## Jacobian determinant

The Jacobian matrix is:

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{bmatrix} a \sin \theta \cos \phi & a\rho \cos \theta \cos \phi & -a\rho \sin \theta \sin \phi \\ b \sin \theta \sin \phi & b\rho \cos \theta \sin \phi & b\rho \sin \theta \cos \phi \\ c \cos \theta & -c\rho \sin \theta & 0 \end{bmatrix}.$$

$$|J| = abc \cdot \rho^2 \sin \theta$$

The volume becomes:

$$V = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{\rho=0}^1 abc \cdot \rho^2 \sin \theta \, d\rho \, d\theta \, d\phi.$$

Separate the integrals:

$$V = abc \cdot \int_0^{2\pi} d\phi \cdot \int_0^\pi \sin \theta \, d\theta \cdot \int_0^1 \rho^2 \, d\rho.$$

$$\int_0^{2\pi} d\phi = 2\pi, \quad \int_0^\pi \sin \theta \, d\theta = [-\cos \theta]_0^\pi = 2, \quad \int_0^1 \rho^2 \, d\rho = \frac{1}{3}.$$

$$V = abc \cdot 2\pi \cdot 2 \cdot \frac{1}{3} = \frac{4}{3}\pi abc.$$

**Final answer:**

$$\boxed{\frac{4}{3}\pi abc}$$

## 11. Derive Green's theorem from Stokes' theorem

**Solution:**

Stokes' theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$$

Take  $S$  as a planar region in  $xy$ -plane,  $\hat{n} = \hat{k}$

Let  $\vec{F} = P\hat{i} + Q\hat{j} + 0\hat{k}$

Then:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

So  $(\nabla \times \vec{F}) \cdot \hat{n} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

Also  $dS = dx \, dy$

Thus:

$$\oint_C (P \, dx + Q \, dy) = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy$$

which is Green's theorem.

**12. Evaluate**  $\iint_S \vec{F} \cdot \hat{n} d\sigma$  **over**  $S : x^2 + y^2 + z^2 = 4$

**Solution:**

Use divergence theorem:

$$\iint_S \vec{F} \cdot \hat{n} d\sigma = \iiint_V \nabla \cdot \vec{F} dV$$

$$\vec{F} = (2x^3, 2y^3 - xyz, 2z^3 + \frac{xz^2}{2})$$

Compute divergence:

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x^3) + \frac{\partial}{\partial y}(2y^3 - xyz) + \frac{\partial}{\partial z}\left(2z^3 + \frac{xz^2}{2}\right)$$

$$= 6x^2 + (6y^2 - xz) + (6z^2 + xz) = 6x^2 + 6y^2 + 6z^2 = 6(x^2 + y^2 + z^2)$$

On sphere:  $x^2 + y^2 + z^2 = r^2$ ,  $dV = r^2 \sin \phi dr d\phi d\theta$ ,  $r \in [0, 2]$ ,  $\phi \in [0, \pi]$ ,  $\theta \in [0, 2\pi]$

Integrand:  $6r^2 \cdot r^2 \sin \phi = 6r^4 \sin \phi$

So:

$$\begin{aligned} \iiint 6r^4 \sin \phi dr d\phi d\theta &= 6 \int_0^{2\pi} d\theta \int_0^\pi \sin \phi d\phi \int_0^2 r^4 dr \\ &= 6 \cdot (2\pi) \cdot [-\cos \phi]_0^\pi \cdot \left[\frac{r^5}{5}\right]_0^2 = 12\pi \cdot (1 - (-1)) \cdot \frac{32}{5} = 12\pi \cdot 2 \cdot \frac{32}{5} = \frac{768\pi}{5} \end{aligned}$$

$\frac{768\pi}{5}$