MAL100: Mathematics I

Tutorial Sheet 6: Limit and Continuity

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1. Discuss whether the following limits exist or not:
   (a) $\lim_{x\to 0} x^2 \left(1 + 2 + 3 + \cdots + \left[\frac{1}{|x|}\right]\right)$.
   (b) $\lim_{x\to c} f(x)$ for any $c \in \mathbb{R}$, where $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 2 - x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
2. Let $f : (-1, 1) \setminus 0 \to \mathbb{R}$. Show that $\lim_{x\to 0} f(x) = l$ implies that $\lim_{x\to 0} f(\sin x) = l$.
3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{(-1)^n}{n} \sin(\pi x)$ for $x \in [n, n + 1)$ for $n \in \mathbb{Z}$. Then prove that $\lim_{x\to\pm\infty} f(x) = 0$.
4. Discuss the continuity of the following functions:
   (a) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
   (b) $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$
5. Using $\epsilon - \delta$ definition discuss whether the following functions are continuous or not:
   (a) $f(x) = x^n$ for $x \in \mathbb{R}$.
   (b) $f(x) = x^{-1}$ for $x \in (0, \infty)$.
   (c) $f(x) = x^{-1}$ for $x \in (\alpha, \infty), \alpha > 0$.
   (d) $f(x) = x^{1/n}$ for $x \in (0, \infty)$ and $n \in \mathbb{N}$.
6. Using sequential criterion, prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is nowhere continuous.
7. A function $f : J \to \mathbb{R}$ is is called *Lipschitz function* on J if there exists $L > 0$ such that $|f(x) - f(y)| \leqslant L|x - y|$ for all $x, y \in J$. Show that any Lipschitz function is continuous.
8. A function $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Then prove that f is either the *zero function* or if $f(1) = a$ for some $a \in \mathbb{R}$ then $f(x) = ax$.
9. A function $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Then prove that f is either the *zero function* or $f(x) = a^x$ for some $a > 0$.
10. Does a continuous function map a Cauchy sequence to a Cauchy sequence? Describe.