MAL100: Mathematics I

Tutorial Sheet 11: Differentiation in Several Variables

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1. Evaluate
$$\iint_R e^{x^2+y^2}\, dy\, dx$$
where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.

2. (a) Calculate the outward flux of the field
$$F(x,y) = x\hat{\imath} + y^2\hat{\jmath}$$
across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

   (b) Evaluate
$$\int_0^{2/3}\int_y^{2-2y} (x+2y)e^{y-x}\, dx\, dy.$$

3. Evaluate
$$\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz.$$

4. Evaluate
$$\int\int_R (x^2+y^2)\, dx\, dy$$
where R is the region bounded by the lines $x = 0, y = 0$ and $x + y = 9$.

5. (a) Find the surface area of the solid obtained by rotating $y = \sin(2x)$, for $0 \le x \le \pi/8$ about the x-axis.

   (b) Find the surface area of the solid obtained by rotating
$$y = \sqrt[3]{x},\ \text{for } 1 \le y \le 2,$$
about the y-axis.

6. Let $D = [0,1]\times[0,1]$ and define
$$f(x,y) = \begin{cases} \dfrac{1}{x^2}, & 0 < y < x < 1, \\ -\dfrac{1}{y^2}, & 0 < x < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$
Compute $\int_0^1\int_0^1 f(x,y)\, dx\, dy$ and $\int_0^1\int_0^1 f(x,y)\, dy\, dx$. Also check the double integrability of f. Can we apply Fubini's theorem to compute the integral?

7. Let $D = [a,b]\times[c,d]$ and $f : D \to \mathbb{R}$ be a double integrable function. Also let $|f(x,y)| \leqslant \alpha$ holds for all $(x,y) \in D$ and for some $\alpha > 0$. Prove that
$$\int_R f(x,y)\, dx\, dy \leqslant \alpha(b-a)(d-c).$$

8. Which of the following vector fields are conservative?
   (a) $\vec{F}(x,y) = (x-y)\,\hat{\imath} + (x-2)\,\hat{\jmath}$.
   (b) $\vec{F}(x,y) = (3+2xy)\,\hat{\imath} + (x^2-3y^2)\,\hat{\jmath}$.
   (c) $\vec{F}(x,y,z) = (2x-3)\,\hat{\imath} + z\,\hat{\jmath} + \cos z\,\hat{k}$.
9. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolves about the x-axis. Find the volume of the solid of revolution.
10. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
11. Can you derive Green's theorem from Stokes' theorem.
12. Evaluate $\iint_S (2x^3\,\hat{\imath} + (2y^3 - xyz)\,\hat{\jmath} + (2z^3 + \frac{xz^2}{2})\,\hat{k}) \cdot \hat{n}\,d\sigma$ over the surface $S : x^2 + y^2 + z^2 = 4$.