MAL100: Mathematics I
Assignment II Solutions

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Problem 1 (1). *If $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$, where $C_0, C_1, \ldots, C_n$ are real constants, prove that the equation $C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n = 0$ has at least one real root between 0 and 1.*

Solution 1. *Step 1: Define an auxiliary function*
Let us define the function:

$$F(x) = C_0 x + \frac{C_1}{2} x^2 + \frac{C_2}{3} x^3 + \cdots + \frac{C_n}{n+1} x^{n+1}$$

Step 2: Compute the derivative
Differentiating $F(x)$ term by term:

$$F'(x) = \frac{d}{dx}(C_0 x) + \frac{d}{dx}\left(\frac{C_1}{2} x^2\right) + \frac{d}{dx}\left(\frac{C_2}{3} x^3\right) + \cdots + \frac{d}{dx}\left(\frac{C_n}{n+1} x^{n+1}\right)$$

$$F'(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$$

Step 3: Evaluate at endpoints
At $x = 0$:

$$F(0) = C_0(0) + \frac{C_1}{2}(0)^2 + \frac{C_2}{3}(0)^3 + \cdots + \frac{C_n}{n+1}(0)^{n+1} = 0$$

At $x = 1$:

$$F(1) = C_0(1) + \frac{C_1}{2}(1)^2 + \frac{C_2}{3}(1)^3 + \cdots + \frac{C_n}{n+1}(1)^{n+1}$$

$$F(1) = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1} = 0 \quad \text{(given)}$$

Step 4: Apply Rolle's Theorem
Since:

- *$F(x)$ is a polynomial, hence continuous on $[0,1]$ and differentiable on $(0,1)$*

- *$F(0) = F(1) = 0$*

By Rolle's Theorem, there exists at least one point $c \in (0,1)$ such that:

$$F'(c) = 0$$

But $F'(c) = C_0 + C_1 c + C_2 c^2 + \cdots + C_n c^n$, so:

$$C_0 + C_1 c + C_2 c^2 + \cdots + C_n c^n = 0$$

Step 5: Conclusion

Therefore, the equation $C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n = 0$ has at least one real root c between 0 and 1.

Problem 2 (2). *Characterize all the differentiable functions $f : [0,1] \to \mathbb{R}$ whose slopes of the tangents are always rationals.*

Solution 2. ***Step 1: Interpret the condition***

The condition "slopes of the tangents are always rationals" means:

$$f'(x) \in \mathbb{Q} \quad \text{for all } x \in [0,1]$$

Step 2: Analyze the derivative function

Since f is differentiable on $[0,1]$, f' exists and is a function from $[0,1]$ to $\mathbb{R}$.

Step 3: Use continuity and connectedness

The interval $[0,1]$ is connected. If f' is continuous, then its image $f'([0,1])$ must also be connected.

However, $\mathbb{Q}$ (the set of rational numbers) is totally disconnected in $\mathbb{R}$ (meaning it contains no intervals).

Step 4: Show f' must be constant

The only connected subsets of $\mathbb{Q}$ are single points. Therefore, if f' is continuous and takes only rational values, then $f'([0,1])$ must be a single rational number.

Thus, there exists some $r \in \mathbb{Q}$ such that:

$$f'(x) = r \quad \text{for all } x \in [0,1]$$

Step 5: Integrate to find f

Integrating both sides:

$$f(x) = \int f'(x) dx = \int r\, dx = rx + C$$

where $C \in \mathbb{R}$ is a constant.

Step 6: Conclusion

The functions with the required property are precisely the linear functions:

$$f(x) = rx + C \quad \text{where } r \in \mathbb{Q} \text{ and } C \in \mathbb{R}$$

Problem 3 (3). *Let $f : [0,1] \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} \sqrt{1 - x^2}, & \text{if } x \in \mathbb{Q} \\ 1 - x, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that f is not integrable on $[0,1]$.

Solution 3. ***Step 1: Understand the function behavior***

For $x \in [0,1]$:

- *If $x \in \mathbb{Q}$: $f(x) = \sqrt{1 - x^2}$*

- *If $x \in \mathbb{R} \setminus \mathbb{Q}$: $f(x) = 1 - x$*

Step 2: Compare the two definitions
Let's compare $\sqrt{1 - x^2}$ and $1 - x$ on $[0, 1]$:
For $x \in [0, 1)$:

$$\sqrt{1 - x^2} > 1 - x \quad \text{since } (1 - x^2) > (1 - x)^2 = 1 - 2x + x^2$$

At $x = 0$: $\sqrt{1 - 0} = 1$, $1 - 0 = 1$ At $x = 1$: $\sqrt{1 - 1} = 0$, $1 - 1 = 0$

Step 3: Analyze upper and lower sums
Consider any partition $P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ of $[0, 1]$.
In any subinterval $[x_{i-1}, x_i]$:

- *There exist rational points where $f(x) = \sqrt{1 - x^2}$*

- *There exist irrational points where $f(x) = 1 - x$*

Therefore:

- *Upper sum: $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$, where $M_i \geq \sqrt{1 - x^2}$ for some $x \in [x_{i-1}, x_i]$*

- *Lower sum: $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$, where $m_i \leq 1 - x$ for some $x \in [x_{i-1}, x_i]$*

Step 4: Show the difference doesn't vanish
As the partition gets finer, the upper sum approximates the area under $y = \sqrt{1 - x^2}$ (a quarter circle), while the lower sum approximates the area under $y = 1 - x$ (a straight line).
Area under $y = \sqrt{1 - x^2}$ on $[0, 1] = \frac{\pi}{4} \approx 0.7854$
Area under $y = 1 - x$ on $[0, 1] = \frac{1}{2} = 0.5$
Since these are different, the upper and lower sums cannot converge to the same value.
Step 5: Alternative approach using discontinuity
The function is discontinuous at every point in $(0, 1)$ because:

- *In any neighborhood of any point, there are both rational and irrational numbers*

- *The function takes values from both definitions, which are different in $(0, 1)$*

Since the set of discontinuities is dense in $[0, 1]$, the function is not Riemann integrable.
Step 6: Conclusion
f is not Riemann integrable on $[0, 1]$.

Problem 4 (4). *Let $\chi_A : [0, 1] \to \mathbb{R}$ for $A \subseteq [0, 1]$, be defined by*

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}.$$

Consider $f(x) = \sum_{n=1}^{200} \frac{1}{n^6} X_{[0, \frac{n}{200}]}(x), x \in [0, 1]$. Then check whether $f(x)$ is Riemann integrable on $[0, 1]$.

Solution 4. *Step 1: Understand the function construction*

The function is defined as:

$$f(x) = \sum_{n=1}^{200} \frac{1}{n^6} \chi_{[0, \frac{n}{200}]}(x)$$

where each $\chi_{[0, \frac{n}{200}]}$ is the characteristic function of the interval $[0, \frac{n}{200}]$.

Step 2: Analyze each characteristic function

For each $n = 1, 2, \ldots, 200$:

$$\chi_{[0, \frac{n}{200}]}(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{n}{200} \\ 0 & \text{if } \frac{n}{200} < x \leq 1 \end{cases}$$

This is a step function with one jump discontinuity at $x = \frac{n}{200}$.

Step 3: Properties of step functions

Step functions are Riemann integrable because:

- *They have finitely many discontinuities*

- *The integral equals the sum of areas of rectangles*

Step 4: Analyze the finite sum

Since $f(x)$ is a finite linear combination of step functions:

$$f(x) = \sum_{n=1}^{200} a_n \chi_{[0, \frac{n}{200}]}(x) \quad \text{where } a_n = \frac{1}{n^6}$$

Step 5: Riemann integrability of finite combinations

The set of Riemann integrable functions is closed under:

- *Finite sums*

- *Scalar multiplication*

Since each $\chi_{[0, \frac{n}{200}]}$ is Riemann integrable, and we have a finite sum with constant coefficients, $f(x)$ is Riemann integrable.

Step 6: Explicit description of f

We can describe $f(x)$ explicitly:

For $x \in [0, 1]$, the value $f(x)$ equals the sum of $\frac{1}{n^6}$ over all n such that $\frac{n}{200} \geq x$, i.e., $n \geq 200x$.

So if $200x \leq 1$, then $f(x) = \sum_{n=1}^{200} \frac{1}{n^6}$ (all terms contribute)

If $200x > k$ but $200x \leq k + 1$ for some integer k, then $f(x) = \sum_{n=k+1}^{200} \frac{1}{n^6}$

This shows f is a step function with jump discontinuities at $x = \frac{k}{200}$ for $k = 1, 2, \ldots, 200$.

Step 7: Conclusion

Since f is a step function with finitely many discontinuities, it is Riemann integrable on $[0, 1]$.

Problem 5 (5). *A function f continuous on $\mathbb{R}$ and $\int_{-x}^{x} f(t)dt = 2 \int_0^x f(t)dt$ for all $x \in \mathbb{R}$. Prove that f is an even function.*

Solution 5. ***Step 1: Write the given condition***

We are given that for all $x \in \mathbb{R}$:

$$\int_{-x}^{x} f(t)dt = 2 \int_{0}^{x} f(t)dt$$

Step 2: Split the left-hand side

Using the additive property of definite integrals:

$$\int_{-x}^{x} f(t)dt = \int_{-x}^{0} f(t)dt + \int_{0}^{x} f(t)dt$$

So the equation becomes:

$$\int_{-x}^{0} f(t)dt + \int_{0}^{x} f(t)dt = 2 \int_{0}^{x} f(t)dt$$

Step 3: Simplify

Subtracting $\int_{0}^{x} f(t)dt$ from both sides:

$$\int_{-x}^{0} f(t)dt = \int_{0}^{x} f(t)dt$$

Step 4: Change of variable

Make the substitution $u = -t$ in the left integral:
When $t = -x$, $u = x$; when $t = 0$, $u = 0$
$dt = -du$
So:

$$\int_{-x}^{0} f(t)dt = \int_{x}^{0} f(-u)(-du) = \int_{0}^{x} f(-u)du$$

Step 5: Rewrite the equation

Substituting back:
$$\int_{0}^{x} f(-u)du = \int_{0}^{x} f(t)dt$$

Since this holds for all $x \in \mathbb{R}$, we have:

$$\int_{0}^{x} [f(-u) - f(u)]du = 0 \quad \text{for all } x \in \mathbb{R}$$

Step 6: Differentiate

Differentiate both sides with respect to x using the Fundamental Theorem of Calculus:
Since f is continuous, both sides are differentiable, and:

$$\frac{d}{dx}\left[\int_{0}^{x} [f(-u) - f(u)]du \right] = f(-x) - f(x)$$

But the derivative of the zero function is zero, so:

$$f(-x) - f(x) = 0 \quad \text{for all } x \in \mathbb{R}$$

Step 7: Conclusion

Therefore:
$$f(-x) = f(x) \quad \text{for all } x \in \mathbb{R}$$

which means f is an even function.

Problem 6 (6). *If f is a real function defined on a convex open set $E \subset \mathbb{R}^n$ such that $(\partial_1 f)(x) = 0$ for every $x \in E$, where $\partial_1 f = \frac{\partial f}{\partial x_1}$, prove that f depends only on $x_2, x_3, \ldots, x_n$.*

Solution 6. ***Step 1: Understand the setup***
We have:

- $E \subset \mathbb{R}^n$ *is convex and open*

- $f : E \to \mathbb{R}$ *is a real-valued function*

- $\frac{\partial f}{\partial x_1}(x) = 0$ *for all $x \in E$*

Step 2: Use convexity
Since E is convex, for any two points $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, a_2, \ldots, a_n)$ in E (differing only in the first coordinate), the line segment joining them lies entirely in E.
The line segment can be parameterized as:

$$\gamma(t) = (a_1 + t(b_1 - a_1), a_2, \ldots, a_n), \quad t \in [0, 1]$$

Step 3: Consider the restriction along this line
Define $g(t) = f(\gamma(t)) = f(a_1 + t(b_1 - a_1), a_2, \ldots, a_n)$
Step 4: Compute the derivative
Using the chain rule:

$$g'(t) = \frac{\partial f}{\partial x_1}(\gamma(t)) \cdot (b_1 - a_1) + \sum_{i=2}^{n} \frac{\partial f}{\partial x_i}(\gamma(t)) \cdot 0$$

$$g'(t) = \frac{\partial f}{\partial x_1}(\gamma(t)) \cdot (b_1 - a_1)$$

But we are given that $\frac{\partial f}{\partial x_1}(x) = 0$ for all $x \in E$, so:

$$g'(t) = 0 \quad \text{for all } t \in [0, 1]$$

Step 5: Conclude g is constant
Since $g'(t) = 0$ for all $t \in [0, 1]$, g is constant on $[0, 1]$.
Therefore:
$$g(0) = g(1) \Rightarrow f(a_1, a_2, \ldots, a_n) = f(b_1, a_2, \ldots, a_n)$$

Step 6: General conclusion
This shows that for any fixed $a_2, \ldots, a_n$, the value of f does not depend on x_1. That is, f depends only on $x_2, x_3, \ldots, x_n$.
Step 7: Final statement
We can write:
$$f(x_1, x_2, \ldots, x_n) = h(x_2, x_3, \ldots, x_n)$$

for some function h of $n - 1$ variables.

Problem 7 (7). *Let $D = [0, 2] \times [0, 3]$ and define*

$$f(x, y) = \begin{cases} 3, & \text{if } x \in \mathbb{Q}, \\ y^2, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Calculate the iterated integral.

Solution 7. ***Step 1: Understand the iterated integral***

We need to compute:

$$\int_0^3 \int_0^2 f(x,y)dxdy$$

Step 2: Analyze the inner integral for fixed y

For a fixed $y \in [0,3]$, consider:

$$\int_0^2 f(x,y)dx$$

The function $f(x,y)$ depends on whether x is rational or irrational:

- *If $x \in \mathbb{Q}$: $f(x,y) = 3$*

- *If $x \in \mathbb{R} \setminus \mathbb{Q}$: $f(x,y) = y^2$*

Step 3: Use measure theory facts

In the interval $[0,2]$:

- *The set of rational numbers $\mathbb{Q} \cap [0,2]$ has Lebesgue measure 0*

- *The set of irrational numbers $(\mathbb{R} \setminus \mathbb{Q}) \cap [0,2]$ has full measure (measure 2)*

For Riemann integration, if a function differs from another function only on a set of measure zero, and both are bounded, then they have the same integral.

Step 4: Evaluate the inner integral

For fixed y, the function $x \mapsto f(x,y)$ equals y^2 for almost every $x \in [0,2]$ (except on the rationals, which have measure zero).

Therefore:

$$\int_0^2 f(x,y)dx = \int_0^2 y^2 dx = y^2 \cdot (2-0) = 2y^2$$

Step 5: Evaluate the outer integral

Now compute:

$$\int_0^3 \left[\int_0^2 f(x,y)dx \right] dy = \int_0^3 2y^2 dy$$

$$= 2\int_0^3 y^2 dy = 2 \cdot \left[\frac{y^3}{3} \right]_0^3 = 2 \cdot \frac{27}{3} = 2 \cdot 9 = 18$$

Step 6: Conclusion

The iterated integral is:

$$\int_0^3 \int_0^2 f(x,y)dxdy = 18$$

Note: The other iterated integral $\int_0^2 \int_0^3 f(x,y)dydx$ would be different due to the asymmetry in the definition of f.

Problem 8 (8). *The line segment $x = 1-y$, where $0 \leqslant y \leqslant 1$, is revolved about the y-axis to generate a cone. Find the surface area of the cone (excluding the area of the base). Match the obtained result with the formula of the surface area of a cone in geometry.*

Solution 8. ***Step 1: Understand the geometry***

The line $x = 1 - y$ from $y = 0$ to $y = 1$:

- *At $y = 0$: $x = 1$ (point $(1, 0)$)*

- *At $y = 1$: $x = 0$ (point $(0, 1)$)*

When revolved about the y-axis, this generates a cone.
Step 2: Identify cone parameters
From the line segment:

- *Height $h = 1$ (from $y = 0$ to $y = 1$)*

- *Base radius $r = 1$ (at $y = 0$, $x = 1$)*

- *Slant height $l = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$*

Step 3: Surface area formula from calculus
For a surface generated by revolving $x = g(y)$ about the y-axis from $y = a$ to $y = b$, the surface area is:

$$A = 2\pi \int_a^b x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

Step 4: Apply the formula
Here:

- *$x = 1 - y$*

- *$\frac{dx}{dy} = -1$*

- *$a = 0$, $b = 1$*

So:

$$A = 2\pi \int_0^1 (1-y)\sqrt{1 + (-1)^2} dy = 2\pi \int_0^1 (1-y)\sqrt{2} dy$$

$$= 2\pi\sqrt{2} \int_0^1 (1-y) dy$$

Step 5: Evaluate the integral

$$\int_0^1 (1-y) dy = \left[y - \frac{y^2}{2}\right]_0^1 = \left(1 - \frac{1}{2}\right) - 0 = \frac{1}{2}$$

So:

$$A = 2\pi\sqrt{2} \cdot \frac{1}{2} = \pi\sqrt{2}$$

Step 6: Compare with geometric formula
The lateral surface area of a cone is given by:

$$A_{cone} = \pi r l$$

where r is the base radius and l is the slant height.
Here $r = 1$, $l = \sqrt{2}$, so:

$$A_{cone} = \pi \cdot 1 \cdot \sqrt{2} = \pi\sqrt{2}$$

Step 7: Conclusion
Both methods give the same result: $\pi\sqrt{2}$
The surface area of the cone (excluding the base) is $\pi\sqrt{2}$.

Problem 9 (9). *Let D be a region in $\mathbb{R}^2$ bounded by the curve C oriented counterclockwise. Then area of D is given by*

$$\text{Area of } D = \frac{1}{2}\int_C (xdy - ydx).$$

Then using Green's theorem find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution 9. *Step 1: Recall Green's Theorem*
    Green's Theorem states that for a positively oriented simple closed curve C enclosing region D:

$$\oint_C (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Step 2: Relate to the area formula
    We are given:

$$Area(D) = \frac{1}{2}\oint_C (xdy - ydx)$$

Compare with Green's Theorem:

- $P = -y$

- $Q = x$

Then:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} = 1 - (-1) = 2$$

So by Green's Theorem:

$$\oint_C (-ydx + xdy) = \iint_D 2dA = 2 \cdot Area(D)$$

Therefore:

$$Area(D) = \frac{1}{2}\oint_C (xdy - ydx)$$

which matches the given formula.
    Step 3: Parametrize the ellipse
    The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be parametrized as:

$$x = a\cos t$$
$$y = b\sin t, \quad 0 \le t \le 2\pi$$

Step 4: Compute differentials
    Differentiate the parametrization:

$$dx = -a\sin t \, dt$$
$$dy = b\cos t \, dt$$

Step 5: Compute the integrand
    Compute $xdy - ydx$:

$$xdy - ydx = (a\cos t)(b\cos t \, dt) - (b\sin t)(-a\sin t \, dt)$$

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$$= ab\cos^2 t\,dt + ab\sin^2 t\,dt = ab(\cos^2 t + \sin^2 t)dt = ab\,dt$$

Step 6: Evaluate the line integral

$$\frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{1}{2}\cdot ab \cdot 2\pi = \pi ab$$

Step 7: Conclusion
The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is:

$$Area = \pi ab$$

Problem 10 (10). *(a) Let S be the surface $x^2 + y^2 + z^2 = 1$, $z \geq 0$. Use Stokes' theorem to evaluate*

$$\int_C (2x - y)dx - y\,dy - z\,dz$$

where C is the circle $x^2 + y^2 = 1$, $z = 0$ oriented anticlockwise.
(b) Consider the vector field $F = \frac{1}{a^3}(x_i^2 + y_j^2 + z_k^2)$ on the sphere S of radius a centered at the origin. Show that the flux through S is a constant.

Solution 10. *Part (a)*
Step 1: Identify the vector field
The line integral is:

$$\int_C (2x - y)dx - y\,dy - z\,dz = \int_C \mathbf{F}\cdot d\mathbf{r}$$

where:

$$\mathbf{F} = (2x - y, -y, -z)$$

Step 2: Apply Stokes' Theorem
Stokes' Theorem states:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\,dS$$

where S is any surface bounded by C, and $\mathbf{n}$ is the unit normal consistent with the orientation of C.
We take S to be the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

Step 3: Compute the curl

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -y & -z \end{vmatrix}$$

Compute components:

- **i**-*component:* $\frac{\partial(-z)}{\partial y} - \frac{\partial(-y)}{\partial z} = 0 - (-1) = 1$

- **j**-*component:* $\frac{\partial(2x-y)}{\partial z} - \frac{\partial(-z)}{\partial x} = 0 - 0 = 0$

- **k**-*component:* $\frac{\partial(-y)}{\partial x} - \frac{\partial(2x-y)}{\partial y} = 0 - (-1) = 1$

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So:

$$\nabla \times \mathbf{F} = (1, 0, 1)$$

Step 4: Compute the surface integral

On the sphere $x^2 + y^2 + z^2 = 1$, the outward unit normal is:

$$\mathbf{n} = (x, y, z)$$

So:

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = (1, 0, 1) \cdot (x, y, z) = x + z$$

Thus:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_S (x + z) dS$$

Step 5: Use symmetry

By symmetry:

- $\iint_S x dS = 0$ *(odd function in x, symmetric domain)*

- $\iint_S z dS$ *is not zero*

So:

$$\iint_S (x + z) dS = \iint_S z dS$$

Step 6: Parametrize and compute

Parametrize the upper hemisphere:

$$x = \sin\theta \cos\phi$$
$$y = \sin\theta \sin\phi$$
$$z = \cos\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le 2\pi$$

Surface element: $dS = \sin\theta d\theta d\phi$
Then:

$$\iint_S z dS = \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \cdot \sin\theta d\theta d\phi$$

Step 7: Evaluate the integral

$$\int_0^{2\pi} d\phi = 2\pi$$

$$\int_0^{\pi/2} \cos\theta \sin\theta d\theta = \int_0^{\pi/2} \frac{1}{2} \sin(2\theta) d\theta = \left[-\frac{1}{4} \cos(2\theta) \right]_0^{\pi/2}$$

$$= -\frac{1}{4}(\cos\pi - \cos 0) = -\frac{1}{4}(-1 - 1) = -\frac{1}{4}(-2) = \frac{1}{2}$$

So:

$$\iint_S z dS = 2\pi \cdot \frac{1}{2} = \pi$$

Step 8: Conclusion for part (a)

By Stokes' Theorem:

$$\int_C (2x - y)dx - ydy - zdz = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}dS = \pi$$

Part (b)
Step 1: Identify the vector field
The vector field is:

$$\mathbf{F} = \frac{1}{a^3}(x^2, y^2, z^2)$$

Step 2: Compute the flux
The flux through sphere S of radius a is:

$$\Phi = \iint_S \mathbf{F} \cdot \mathbf{n}dS$$

On the sphere, the outward unit normal is:

$$\mathbf{n} = \frac{1}{a}(x, y, z)$$

Step 3: Compute the dot product

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{a^3}(x^2, y^2, z^2) \cdot \frac{1}{a}(x, y, z) = \frac{1}{a^4}(x^3 + y^3 + z^3)$$

Step 4: Use symmetry
By symmetry of the sphere:

- $\iint_S x^3 dS = 0$ (odd function, symmetric domain)

- $\iint_S y^3 dS = 0$ (odd function, symmetric domain)

- $\iint_S z^3 dS = 0$ (odd function, symmetric domain)

Therefore:

$$\iint_S (x^3 + y^3 + z^3)dS = 0$$

Step 5: Conclusion for part (b)

$$\Phi = \iint_S \mathbf{F} \cdot \mathbf{n}dS = \frac{1}{a^4} \iint_S (x^3 + y^3 + z^3)dS = 0$$

The flux is 0, which is constant (independent of a).