MAL100: Mathematics I - Tutorial Sheet 5 Solutions

Department of Mathematics, Indian Institute of Technology Bhilai

1. Discuss the convergence (absolute/conditional) of the following series:

Conditional convergence : If a series is convergent but not absolutely then it is called conditional convergen

Note : Every absolutely convergence series is convergence.

Limit comparison Test :
If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l \neq 0$$

And $a_n \geq 0$ and $b_n \geq 0$ then $\sum a_n$ and $\sum b_n$ both convergence or divergence together.

(a) $\sum_{n\geq 1} \frac{(-1)^n}{(2n-1)\cdot n^2}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{(2n-1)n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{(2n-1)n^2}$$

Let $a_n = \frac{1}{(2n-1)n^2}$.
$b_n = \frac{1}{n^3}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(2n-1)n^2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{(2n-1)n^2} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2}$$

As $\sum b_n$ converges, the series $\sum a_n$ also converges. Therefore, the original series is **absolutely convergent**.

(b) $\sum_{n\geq 1} \frac{(-1)^n}{n^3\cdot\log(n+1)}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3 \log(n+1)} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3 \log(n+1)}$$

. For $n \geq 2$, we have $\log(n+1) > \log(3) > 1$. This implies $\frac{1}{\log(n+1)} < 1$. So, for $n \geq 2$, we have:

$$\frac{1}{n^3 \log(n+1)} < \frac{1}{n^3}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent series . $\sum_{n=1}^{\infty} \frac{1}{n^3 \log(n+1)}$ converges. Thus, the original series is **absolutely convergent**.

(c) $\sum_{n\geq 2} \frac{1}{(\log n)^p}$ for any p.

Let $a_n = \frac{1}{(\log n)^p}$.
Case 1: $p \leq 0$.
Let $p = -q$ where $q \geq 0$. Then $a_n = (\log n)^q$. Since $\lim_{n\to\infty} \log n = \infty$, if $q > 0$, $\lim_{n\to\infty} a_n = \infty$. If $q = 0$ (i.e., $p = 0$), $\lim_{n\to\infty} a_n = 1$. In either case, the terms do not approach 0. By the n-th term test for divergence, the series diverges for $p \leq 0$.

Case 2: $p > 0$.

The **Cauchy condensation test** says that

$$\sum_{n\geq 2} a_n \quad \text{converges} \quad \iff \quad \sum_{k\geq 1} 2^k a_{2^k} \text{ converges.}$$

Where

$$a_n \geq 0 \quad \forall n \in \mathbb{N}$$

$$a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}$$

Apply the **Cauchy condensation test** (valid because

$$a_n := \frac{1}{(\log n)^p}$$

is eventually positive and nonincreasing).

Now compute

$$2^k a_{2^k} = 2^k \cdot \frac{1}{(\log(2^k))^p} = 2^k \cdot \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \frac{2^k}{k^p}.$$

$$\lim_{k\to\infty} \frac{2^k}{k^p} \neq 0$$

so

$$\sum_{k\geq 1} \frac{2^k}{k^p} \quad \textit{diverges.}$$

Hence $\sum_k 2^k a_{2^k}$ diverges, and by the condensation test the original series

$$\sum_{n\geq 2} \frac{1}{(\log n)^p}$$

also diverges.

Conclusion: The series **diverges for all** $p \in \mathbb{R}$.

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Let

$$P_n := \prod_{k=1}^{n} (a_k + 1)$$

given

$$\lim_{n\to\infty} P_n = a \in (0, \infty).$$

Note that

$$\frac{1}{P_{n-1}} - \frac{1}{P_n} = \frac{1}{p_{n-1}}(1 - \frac{1}{a_n + 1}) = \frac{a_n}{P_n}$$

Hence

$$\sum_{n=1}^{N} \frac{a_n}{P_n} = \sum_{n=1}^{N} \Big(\frac{1}{P_{n-1}} - \frac{1}{P_n}\Big) = 1 - \frac{1}{P_N}.$$

Letting $N \to \infty$ gives

$$\sum_{n=1}^{\infty} \frac{a_n}{(a_1+1)\cdots(a_n+1)} = 1 - \frac{1}{a}.$$

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3. Show that if $\{na_n\}$ and $\sum n(a_n - a_{n+1})$ converge, then $\sum a_n$ converges.

For $N \geq 1$ compute the finite sum

$$\sum_{k=1}^{N} k(a_k - a_{k+1}) = \sum_{k=1}^{N} ka_k - \sum_{k=1}^{N} ka_{k+1}.$$

Change index in the second sum:

$$\sum_{k=1}^{N} ka_{k+1} = \sum_{j=2}^{N+1} (j-1)a_j.$$

Hence

$$\sum_{k=1}^{N} k(a_k - a_{k+1}) = (a_1 + 2a_2 + \cdots + Na_N) - (a_2 + 2a_3 + \cdots + Na_{N+1}).$$

Cancel like terms to obtain the telescoping identity

$$\sum_{k=1}^{N} k(a_k - a_{k+1}) = \sum_{n=1}^{N} a_n - Na_{N+1}.$$

Rearranging gives

$$\sum_{n=1}^{N} a_n = Na_{N+1} + \sum_{k=1}^{N} k(a_k - a_{k+1}). \qquad (\star)$$

By hypothesis $\{na_n\}$ converges; therefore the subsequence $(N+1)a_{N+1}$ converges to the same limit L. In particular na_n is bounded, hence

$$a_n = \frac{na_n}{n} \to 0.$$

Thus

$$Na_{N+1} = (N+1)a_{N+1} - a_{N+1} \;\longrightarrow\; L - 0 = L.$$

By hypothesis the series $\sum_{k=1}^{\infty} k(a_k - a_{k+1})$ converges, so its partial sums

$$\sum_{k=1}^{N} k(a_k - a_{k+1})$$

converge to some finite limit M as $N \to \infty$.

Passing to the limit in $(\star)$ as $N \to \infty$ gives

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} N a_{N+1} + \lim_{N\to\infty} \sum_{k=1}^{N} k(a_k - a_{k+1}) = L + M,$$

which is finite. Hence the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

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4. Example of a conditionally convergent series $\sum a_n$ and a bounded $\{b_n\}$ such that $\sum a_n b_n$ is divergent.

$$\sum a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

$$b_n = (-1)^{n+1}$$

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5. Let $\sum a_n$ be absolutely convergent. Prove that $\sum \frac{a_n}{1+a_n}$ is also absolutely convergent.

Given that $\sum |a_n|$ converges.

$$\implies \lim_{n\to\infty} a_n = 0$$

. We want to show that $\sum \left|\frac{a_n}{1+a_n}\right|$ converges. Let

$$b_n = \left|\frac{a_n}{1+a_n}\right|$$

$$c_n = |a_n|$$

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{\left|\frac{a_n}{1+a_n}\right|}{|a_n|} = \lim_{n\to\infty} \frac{1}{|1+a_n|}$$

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \frac{1}{1} = 1$$

Since the limit is a finite, positive number, and the series $\sum c_n = \sum |a_n|$ converges by hypothesis, the series $\sum b_n = \sum \left|\frac{a_n}{1+a_n}\right|$ must also converge by the Limit Comparison Test. Therefore, the series $\sum \frac{a_n}{1+a_n}$ is **absolutely convergent**.

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6. Prove $\sum_{n\geq 1}(-1)^n \sin\frac{1}{n}$ is conditionally convergent.

The **Leibniz Test** says If

1. $a_n \geq 0$ for all n.
2. The sequence $\{a_n\}$ is monotonically decreasing ($a_{n+1} \leq a_n$).
3. $\lim_{n\to\infty} a_n = 0$.

then

$$\sum_{n=1}^{\infty}(-1)^n a_n$$

is convergent.
The series has the form

$$\sum_{n=1}^{\infty}(-1)^n a_n$$

where

$$a_n = \sin\left(\frac{1}{n}\right)$$

Let's verify these conditions for $a_n = \sin\left(\frac{1}{n}\right)$:

1. $a_n \geq 0$ For $n \geq 1$,
$$\frac{1}{n} \in (0, 1]$$
. Thus, $\sin\left(\frac{1}{n}\right) \geq 0$ for all $n \geq 1$.
2. **Monotonically decreasing**: For
$$n \geq 1$$
$$\implies n + 1 > n$$
,
$$\implies \frac{1}{n+1} < \frac{1}{n}$$
.
$$\sin\left(\frac{1}{n+1}\right) < \sin\left(\frac{1}{n}\right)$$
. The sequence is decreasing.
3. **Limit is zero**:
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$$
.

Since all three conditions of the Leibniz Test are satisfied, the series $\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{1}{n}\right)$ is **convergent**.

Now to show not absolutely convergent .

$$\sum_{n=1}^{\infty}\left|(-1)^n \sin\left(\frac{1}{n}\right)\right|$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$$

Let $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$.

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n}$$
$$= 1 \neq 0$$

So it is not absolutely convergent by limit comparison test,since $\sum b_n$is divergent.

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7. Let $a_n = n^3 \sin\frac{1}{n} - n^2 + \frac{1}{6}$. Show $\sum a_n$ is absolutely convergent.

We use the Taylor series expansion for $\sin x$ around $x = 0$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Substitute $x = 1/n$:

$$\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{6n^3} + \frac{1}{120n^5} - \frac{1}{5040n^7} + \dots$$

Now substitute this into the expression for a_n:

$$a_n = n^3\left(\frac{1}{n} - \frac{1}{6n^3} + \frac{1}{120n^5} - \dots\right) - n^2 + \frac{1}{6}$$
$$= \left(n^2 - \frac{1}{6} + \frac{1}{120n^2} - \dots\right) - n^2 + \frac{1}{6}$$
$$= \frac{1}{120n^2} - \frac{1}{5040n^4} + \dots$$

Let

$$\sum b_n = \sum \frac{1}{n^2}$$
$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{\left|\frac{1}{120n^2} - \frac{1}{5040n^4} + \dots\right|}{\frac{1}{n^2}} = \frac{1}{120}$$

As $\sum \frac{1}{n^2}$ converges, the series $\sum |a_n|$ also converges. Therefore, $\sum a_n$ is **absolutely convergent**.

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8. Show that $\lim_{x\to 0} e^{1/x}$ does not exist.

Let

$$f(x) = e^{1/x}$$
$$a_n = \frac{1}{\log(n+1)} \quad b_n = \frac{-1}{\log(n+1)}$$

then both a_n, b_n converges to zero but its functional value converges to different limit so limit does not exist.

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9. Using $\epsilon - \delta$, prove that $\lim_{x\to 2}\sqrt{4x - x^2} = 2$.

Let $\epsilon > 0$ be given. We choose a $\delta > 0$ such that for all x satisfying $0 < |x-2| < \delta$, we have

$$\left|\sqrt{4x - x^2} - 2\right| < \epsilon.$$

Note that $4x - x^2 = 4 - (x-2)^2$, so

$$\sqrt{4x - x^2} = \sqrt{4 - (x-2)^2}.$$

Since the square root is defined only when $4-(x-2)^2 \geq 0$, we require $|x-2| \leq 2$. Thus, we will restrict $\delta \leq 2$ to ensure the function is defined.

Now observe that for $|x-2| \leq 2$,

$$0 \leq \sqrt{4 - (x-2)^2} \leq 2,$$

so

$$\left|\sqrt{4 - (x-2)^2} - 2\right| = 2 - \sqrt{4 - (x-2)^2}.$$

We want

$$2 - \sqrt{4 - (x-2)^2} < \epsilon.$$

Rearranging:

$$\sqrt{4 - (x-2)^2} > 2 - \epsilon.$$

Since both sides are nonnegative (for $\epsilon \leq 2$), squaring both sides gives:

$$4 - (x-2)^2 > (2-\epsilon)^2,$$

which simplifies to

$$(x-2)^2 < 4 - (2-\epsilon)^2 = 4\epsilon - \epsilon^2.$$

Thus,

$$|x-2| < \sqrt{4\epsilon - \epsilon^2}.$$

We now choose δ as follows:

- If $0 < \epsilon < 2$, let $\delta = \sqrt{4\epsilon - \epsilon^2}$.
- If $\epsilon \geq 2$, let $\delta = 2$.

We verify that this choice works:

Case 1: $0 < \epsilon < 2$.
Let $\delta = \sqrt{4\epsilon - \epsilon^2}$. Suppose $0 < |x-2| < \delta$. Then

$$|x-2| < \sqrt{4\epsilon - \epsilon^2} \quad \Rightarrow \quad (x-2)^2 < 4\epsilon - \epsilon^2.$$

Hence,

$$4 - (x-2)^2 > 4 - (4\epsilon - \epsilon^2) = 4 - 4\epsilon + \epsilon^2 = (2-\epsilon)^2.$$

Taking square roots (which is valid since both sides are positive):

$$\sqrt{4 - (x-2)^2} > 2 - \epsilon,$$

so

$$2 - \sqrt{4 - (x-2)^2} < \epsilon,$$

i.e.,

$$\left|\sqrt{4x - x^2} - 2\right| < \epsilon.$$

Case 2: $\epsilon \geq 2$.
Let $\delta = 2$. Suppose $0 < |x-2| < \delta$. Then $|x-2| < 2$, so $x \in (0,4) \setminus \{2\}$, and

$$\sqrt{4-(x-2)^2} \leq 2.$$

Therefore,

$$\left|\sqrt{4x-x^2} - 2\right| = 2 - \sqrt{4-(x-2)^2} \leq 2 \leq \epsilon.$$

In both cases, the desired inequality holds.
Hence, by the ϵ-δ definition,

$$\lim_{x\to 2} \sqrt{4x-x^2} = 2. \quad \square$$

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10. (a) If $|f(x)-L| \leq K|x-c|$, show that $\lim_{x\to c} f(x) = L$.

We want to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in I$ with $0 < |x-c| < \delta$, we have $|f(x) - L| < \epsilon$.

Given: $|f(x) - L| \leq K|x-c|$ for all $x \in I$.

Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{K}$ (if $K > 0$). If $K = 0$, then $|f(x) - L| \leq 0$ implies $f(x) = L$ for all x, so the limit is trivially L.

Assume $K > 0$. Then, if $0 < |x-c| < \delta = \frac{\epsilon}{K}$, we have:

$$|f(x) - L| \leq K|x-c| < K \cdot \frac{\epsilon}{K} = \epsilon.$$

Thus, for every $\epsilon > 0$, there exists $\delta = \frac{\epsilon}{K} > 0$ such that:

$$0 < |x-c| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Therefore, by the ϵ-δ definition, $\lim_{x\to c} f(x) = L$.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $\lim_{x\to 0} f(x) = L$. Let $g : \mathbb{R} \to \mathbb{R}$ be such that $g(x) = f(ax)$ for some $a \in \mathbb{R}$. We show that $\lim_{x\to 0} g(x) = L$.

We consider two cases:

Case 1: $a \neq 0$

Given $\lim_{x\to 0} f(x) = L$, by the ϵ-δ definition, for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$0 < |x| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

Now, set $\delta' = \frac{\delta}{|a|}$. Then for all x satisfying $0 < |x| < \delta'$, we have:

$$|ax| = |a||x| < |a| \cdot \frac{\delta}{|a|} = \delta,$$

and since $a \neq 0$ and $x \neq 0$, we have $ax \neq 0$. Thus, $0 < |ax| < \delta$, so:

$$|g(x) - L| = |f(ax) - L| < \epsilon.$$

Hence, for every $\epsilon > 0$, there exists $\delta' > 0$ such that:

$$0 < |x| < \delta' \quad \Rightarrow \quad |g(x) - L| < \epsilon,$$

which proves $\lim_{x\to 0} g(x) = L$.

Case 2: $a = 0$

Then $g(x) = f(0)$ for all $x \in \mathbb{R}$. The limit $\lim_{x\to 0} g(x) = f(0)$. For this to equal L, we require $f(0) = L$. However, the given condition $\lim_{x\to 0} f(x) = L$ does not necessarily imply $f(0) = L$ (unless f is continuous at 0). Therefore, the result holds for $a = 0$ only if $f(0) = L$.

$\boxed{\textit{Conclusion}}$

If $a \neq 0$, then $\lim_{x\to 0} g(x) = L$ unconditionally. If $a = 0$, then $\lim_{x\to 0} g(x) = L$ if and only if $f(0) = L$.