MAL100: Mathematics I
Tutorial Sheet 11: Differentiation in Several Variables

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Question 1

Check the existence of limits of the following functions at the prescribed points:

(a)

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Check at $(0,0)$.

Solution

Let us take $y = mx$:

$$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x(mx)}{x^2+(mx)^2} = \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}.$$

Limit depends on m, so for different value of m we get different limit.
$\implies$ limit does not exist.

(b)

$$f(x,y) = \frac{(y-x)(1+x)}{(y+x)(1+y)}, \quad x+y \neq 0, \quad -1 < x, y < 1$$

Check at $(0,0)$.

Solution

Let us take $y = mx$:

$$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{(mx-x)(1+x)}{(mx+x)(1+mx)} = \frac{m-1}{m+1}.$$

Limit depends on m, so for different value of m we get different limit.
$\implies$ limit does not exist.

Question 2

Check the differentiability of the following functions at $(0,0)$ using the definition:

(a)

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Solution

Step 1: Continuity at $(0,0)$ Given $\epsilon > 0$, choose $\delta = 2\epsilon$. If $\sqrt{x^2+y^2} < \delta$, then

$$|f(x,y) - f(0,0)| = \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{\frac{x^2+y^2}{2}}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2+y^2}}{2} < \frac{\delta}{2} = \epsilon.$$

So f is continuous at $(0,0)$.

Step 2: Partial derivatives at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

So $a = f_x(0,0) = 0$, $b = f_y(0,0) = 0$.

Step 3: Differentiability limit

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - 0 \cdot h - 0 \cdot k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\frac{hk}{\sqrt{h^2+k^2}}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2}.$$

Let $h = r\cos\theta$, $k = r\sin\theta$:

$$\frac{hk}{h^2+k^2} = \frac{r^2 \cos\theta \sin\theta}{r^2} = \cos\theta \sin\theta.$$

This depends on θ and does not tend to 0 as $r \to 0$. Hence the limit does not exist $\Rightarrow$ **not differentiable**.

(b)

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Solution

Continuity at $(0,0)$**:** Given $\epsilon > 0$, choose $\delta = \epsilon$. If $\sqrt{x^2+y^2} < \delta$, then $|x| \leq \sqrt{x^2+y^2} < \epsilon$.

$$|f(x,y) - f(0,0)| = \frac{|x|^3}{x^2+y^2} \leq |x| \cdot \frac{x^2}{x^2+y^2} \leq |x| < \epsilon.$$

So f is continuous at $(0,0)$.

Step 2: Partial derivatives at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-0}{h} = \lim_{h\to 0} \frac{h^3/h^2}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-0}{k} = \lim_{k\to 0} \frac{0}{k} = 0.$$

So $a = 1$, $b = 0$.

Step 3: Differentiability limit

$$\frac{f(h,k) - 0 - 1\cdot h - 0 \cdot k}{\sqrt{h^2+k^2}} = \frac{\frac{h^3}{h^2+k^2} - h}{\sqrt{h^2+k^2}}.$$

Simplify numerator:

$$\frac{h^3 - h(h^2+k^2)}{h^2+k^2} = \frac{-hk^2}{h^2+k^2}.$$

So the difference quotient is:

$$\frac{-hk^2}{(h^2+k^2)^{3/2}}.$$

Let $h = r\cos\theta$, $k = r\sin\theta$:

$$= \frac{-r\cos\theta \cdot r^2 \sin^2\theta}{r^3} = -\cos\theta \sin^2\theta.$$

This depends on θ and does not tend to 0 as $r \to 0 \Rightarrow$ **not differentiable**.

(c)

$$f(x,y) = \sqrt{|xy|}, \quad f(0,0) = 0.$$

Solution

Step 1: Continuity at $(0,0)$

$$\sqrt{|xy|} \le \sqrt{\frac{x^2+y^2}{2}} \to 0.$$

So f is continuous at $(0,0)$.

Step 2: Partial derivatives at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{\sqrt{|h\cdot 0|}}{h} = 0.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{\sqrt{|0\cdot k|}}{k} = 0.$$

So $a = 0$, $b = 0$.

Step 3: Differentiability limit

$$\frac{f(h,k) - 0 - 0 - 0}{\sqrt{h^2+k^2}} = \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}.$$

Let $h = r\cos\theta$, $k = r\sin\theta$:

$$= \frac{\sqrt{r^2|\cos\theta\sin\theta|}}{r} = \sqrt{|\cos\theta\sin\theta|}.$$

Limit depends on θ and is not zero $\Rightarrow$ **not differentiable**.

(d)

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Solution

Continuity at $(0,0)$: Given $\epsilon > 0$, choose $\delta = \sqrt{2\epsilon}$. If $\sqrt{x^2+y^2} < \delta$, then

$$|f(x,y) - f(0,0)| \leq |xy| \cdot \frac{|x^2-y^2|}{x^2+y^2} \leq |xy| \leq \frac{x^2+y^2}{2} \leq \epsilon.$$

So f is continuous at $(0,0)$.

Step 2: Partial derivatives at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)}{h} = 0.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)}{k} = 0.$$

So $a = 0$, $b = 0$.

Step 3: Differentiability limit

$$\frac{f(h,k) - 0}{\sqrt{h^2+k^2}} = \frac{xy(x^2-y^2)}{(x^2+y^2)^{3/2}}, \quad (x = h, y = k).$$

In polar coordinates: $h = r\cos\theta$, $k = r\sin\theta$:

$$= \frac{r^4 \cos\theta \sin\theta (\cos^2\theta - \sin^2\theta)}{r^3} = r\cos\theta\sin\theta\cos 2\theta.$$

As $r \to 0$, this tends to 0 for all $\theta \Rightarrow$ limit is $0 \Rightarrow$ **differentiable**.

Question 3

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x}, & \text{if } xy \neq 0, \\ 0, & \text{if } xy = 0. \end{cases}$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists but $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not exist.

Solution

Step 1: Double limit exists

For $xy \neq 0$:

$$|f(x,y)| = \left| x\sin\frac{1}{y} + y\sin\frac{1}{x} \right| \leq |x| + |y|.$$

Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$. If $\sqrt{x^2+y^2} < \delta$, then $|x| < \delta$ and $|y| < \delta$, so

$$|f(x,y)| \leq |x| + |y| < 2\delta = \epsilon.$$

Hence, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Step 2: Iterated limit does not exist

Consider $\lim_{x\to 0}[\lim_{y\to 0} f(x,y)]$.

For fixed $x \neq 0$, as $y \to 0$, $x \sin \frac{1}{y}$ oscillates, so $\lim_{y\to 0} f(x,y)$ does not exist for $x \neq 0$. For $x = 0$, $f(0,y) = 0$ so the inner limit is 0.

Since the inner limit fails to exist for $x \neq 0$, the iterated limit $\lim_{x\to 0} \lim_{y\to 0} f(x,y)$ does not exist.

Double limit exists, iterated limit does not.

Question 4

Find the directional derivative of

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

along the unit vector $\mathbf{u} = (u_1, u_2)$. Also deduce whether f is differentiable at $(0,0)$.

Solution

Directional derivative definition

The directional derivative of f at $\mathbf{a}$ in direction $\mathbf{u}$ (unit vector) is:

$$D_{\mathbf{u}} f(\mathbf{a}) = \lim_{t\to 0^+} \frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t}.$$

Here $\mathbf{a} = (0,0)$, $f(0,0) = 0$.

So:

$$D_{\mathbf{u}} f(0,0) = \lim_{t\to 0^+} \frac{f(tu_1, tu_2) - 0}{t}.$$

For $t \neq 0$:

$$f(tu_1, tu_2) = \frac{(tu_1)^3 (tu_2)}{(tu_1)^4 + (tu_2)^2} = \frac{t^4 u_1^3 u_2}{t^4 u_1^4 + t^2 u_2^2}.$$

Factor t^2 in denominator:

$$f(tu_1, tu_2) = \frac{t^4 u_1^3 u_2}{t^2(t^2 u_1^4 + u_2^2)} = \frac{t^2 u_1^3 u_2}{t^2 u_1^4 + u_2^2}.$$

$$\frac{f(tu_1, tu_2)}{t} = \frac{1}{t} \cdot \frac{t^2 u_1^3 u_2}{t^2 u_1^4 + u_2^2} = \frac{t u_1^3 u_2}{t^2 u_1^4 + u_2^2}.$$

We have:

$$D_{\mathbf{u}} f(0,0) = \lim_{t\to 0^+} \frac{t u_1^3 u_2}{t^2 u_1^4 + u_2^2}.$$

Case 1: $u_2 \neq 0$.

Then denominator $\to u_2^2 \neq 0$, numerator $\to 0$, so limit $= 0$.

Case 2: $u_2 = 0$.

Then $\mathbf{u} = (u_1, 0)$ with $|u_1| = 1$.

Expression becomes $\frac{t u_1^3 \cdot 0}{t^2 u_1^4 + 0} = 0$ for $t \neq 0$, so limit $= 0$.

Thus **for all unit vectors u**:

$$D_{\mathbf{u}} f(0,0) = 0.$$

Check differentiability

We need:

$$\lim_{(h,k)\to(0,0)} \frac{\frac{h^3k}{h^4+k^2}}{\sqrt{h^2+k^2}} = 0.$$

Try $k = mh^2$:

$$f(h, mh^2) = \frac{mh}{1+m^2}.$$

Then

$$\frac{f(h, mh^2)}{\sqrt{h^2+k^2}} \to \frac{m}{1+m^2} \cdot \frac{h}{|h|}.$$

Limit depends on m and sign of $h \Rightarrow$ not zero in general $\Rightarrow$ not differentiable.

Conclusion

Directional derivative exists for all $\mathbf{u}$ and equals 0, but f is **not differentiable** at $(0, 0)$.

Question 5

Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2+y^2}{|x|+|y|}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous at $(0, 0)$ but the partial derivatives do not exist.

Solution

Continuity at $(0,0)$
For $(x, y) \neq (0, 0)$:

$$|f(x,y)| = \frac{x^2+y^2}{|x|+|y|}.$$

Using $|x| + |y| \geq \sqrt{x^2+y^2}$, we get:

$$\frac{x^2+y^2}{|x|+|y|} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}.$$

So $|f(x, y)| \leq \sqrt{x^2+y^2}$.

Given $\epsilon > 0$, choose $\delta = \epsilon$. If $\sqrt{x^2+y^2} < \delta$, then $|f(x, y)| < \epsilon$. Hence $\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0) \Rightarrow f$ is continuous at $(0, 0)$.

Partial derivative $f_x(0, 0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-0}{h}.$$

For $h \neq 0$: $f(h,0) = \frac{h^2}{|h|} = |h|$, so

$$\frac{f(h,0)}{h} = \frac{|h|}{h} = \operatorname{sgn}(h).$$

Limit as $h \to 0$ does not exist (left-hand limit -1, right-hand limit $+1$). So $f_x(0,0)$ does not exist.

Question 6

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Show that at origin $f_{xy} \neq f_{yx}$.

Solution

Direct Definition of Mixed Partial Derivatives

For a function $f(x,y)$, the mixed partial derivatives at (a,b) are defined as:

$$f_{xy}(a,b) = \lim_{k \to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$$

where

$$f_x(a,b+k) = \lim_{h \to 0} \frac{f(a+h,b+k) - f(a,b+k)}{h}$$

and

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Similarly,

$$f_{yx}(a,b) = \lim_{h \to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

where

$$f_y(a+h,b) = \lim_{k \to 0} \frac{f(a+h,b+k) - f(a+h,b)}{k}$$

and

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

Compute $f_{xy}(0,0)$

At $(0,0)$, we have:

$$f_{xy}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

First, compute $f_x(0,0)$:

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

Now compute $f_x(0,k)$ for $k \neq 0$:

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} \frac{f(h,k)}{h}$$

since $f(0,k) = 0$.

Now substitute $f(h,k)$:

$$\frac{f(h,k)}{h} = \frac{1}{h} \cdot \frac{hk(h^2 - k^2)}{h^2 + k^2} = \frac{k(h^2 - k^2)}{h^2 + k^2}$$

Take limit as $h \to 0$:

$$f_x(0,k) = \lim_{h\to 0} \frac{k(h^2 - k^2)}{h^2 + k^2} = \frac{k(-k^2)}{k^2} = -k$$

Therefore:

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{-k - 0}{k} = \lim_{k\to 0}(-1) = -1$$

Compute $f_{yx}(0,0)$

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

First, compute $f_y(0,0)$:

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0 - 0}{k} = 0$$

Now compute $f_y(h,0)$ for $h \neq 0$:

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{f(h,k)}{k}$$

since $f(h,0) = 0$.

Now substitute $f(h,k)$:

$$\frac{f(h,k)}{k} = \frac{1}{k} \cdot \frac{hk(h^2 - k^2)}{h^2 + k^2} = \frac{h(h^2 - k^2)}{h^2 + k^2}$$

Take limit as $k \to 0$:

$$f_y(h,0) = \lim_{k\to 0} \frac{h(h^2 - k^2)}{h^2 + k^2} = \frac{h(h^2)}{h^2} = h$$

Therefore:

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h - 0}{h} = \lim_{h\to 0} 1 = 1$$

Conclusion

We have:

$$f_{xy}(0,0) = -1 \quad \text{and} \quad f_{yx}(0,0) = 1$$

Therefore:

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

$$\boxed{f_{xy}(0,0) = -1, \quad f_{yx}(0,0) = 1}$$

Question 7

Give an example of a function of two variables whose partial derivative exist at a point but the function is not continuous at that point.

Solution

Consider the function:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Check partial derivatives at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$
$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0$$

Both partial derivatives exist and equal 0.

Check continuity at $(0,0)$

Along the line $y = mx$:

$$f(x,mx) = \frac{x(mx)}{x^2+(mx)^2} = \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}$$

This depends on m:

Different paths give different limits, so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Thus f is **not continuous** at $(0,0)$.

The function $f(x,y)$ has both partial derivatives existing at $(0,0)$ but is not continuous there.

Question 8

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = \sqrt{|xy|}$ where $x = u$ and $y = u$. Find $\frac{df}{du}$ at $u = 0$.

Solution

Given $f(x,y) = \sqrt{|xy|}$, and $x = u$, $y = u$, we have:

$$f(u,u) = \sqrt{|u \cdot u|} = \sqrt{u^2} = |u|\,.$$

Then:

$$\frac{df}{du} = \frac{d}{du}|u|\,.$$

At $u = 0$:

$$\frac{df}{du}(0) = \lim_{h\to 0} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{|h|}{h}.$$

$\lim_{h\to 0^+} \frac{h}{h} = 1$, $\lim_{h\to 0^-} \frac{-h}{h} = -1$.

The derivative **does not exist** at $u = 0$.

Question 9

Deduce under which condition on the unit vector $\mathbf{u} = (u_1, u_2)$, the directional derivative of

$$f(x, y) = \sqrt{|xy|}$$

at $(0, 0)$ along $\mathbf{u}$ exists.

Solution

The directional derivative is defined as:

$$D_{\mathbf{u}} f(0,0) = \lim_{t \to 0^+} \frac{f(tu_1, tu_2) - f(0,0)}{t}.$$

Since $f(0, 0) = 0$:

$$D_{\mathbf{u}} f(0,0) = \lim_{t \to 0^+} \frac{\sqrt{|(tu_1)(tu_2)|}}{t}.$$

Simplify:

$$f(tu_1, tu_2) = \sqrt{|t^2 u_1 u_2|} = t\sqrt{|u_1 u_2|}.$$

Thus:

$$\frac{f(tu_1, tu_2)}{t} = \sqrt{|u_1 u_2|}.$$

So:

$$D_{\mathbf{u}} f(0,0) = \sqrt{|u_1 u_2|}.$$

This is finite for all unit vectors $\mathbf{u} = (u_1, u_2)$. Therefore, the directional derivative exists for **all** unit vectors $\mathbf{u}$.

Question 10

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Show that the directional derivative exists for all directions at the point $(0, 0)$.

Solution

Let $\mathbf{u} = (u_1, u_2)$ be a unit vector. The directional derivative is:

$$D_{\mathbf{u}} f(0,0) = \lim_{t \to 0^+} \frac{f(tu_1, tu_2)}{t}.$$

For $t > 0$:

$$f(tu_1, tu_2) = \frac{tu_1 u_2^2}{u_1^2 + t^2 u_2^4}.$$

Thus:

$$\frac{f(tu_1, tu_2)}{t} = \frac{u_1 u_2^2}{u_1^2 + t^2 u_2^4}.$$

Taking the limit as $t \to 0^+$:

$$D_{\mathbf{u}} f(0,0) = \frac{u_1 u_2^2}{u_1^2} = \frac{u_2^2}{u_1} \quad \text{if } u_1 \neq 0,$$

and if $u_1 = 0$, then $f(0, tu_2) = 0$, so the limit is 0.

In all cases, the directional derivative exists and is finite. Therefore, it exists for **all** directions at $(0,0)$.

Question 11

Find the directional derivative for the function

$$f(x,y,z) = x^2 + 2y^2 + 2z^2$$

at $(1,1,0)$ in the direction $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Solution

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2x, 4y, 4z)$$

At $(1,1,0)$:

$$\nabla f(1,1,0) = (2,4,0)$$

Direction vector: $\mathbf{v} = (1,-1,2)$

Magnitude: $\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

Unit vector: $\mathbf{u} = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right)$

$$\begin{aligned} D_{\mathbf{u}} f(1,1,0) = \nabla f(1,1,0) \cdot \mathbf{u} &= (2,4,0) \cdot \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right) \\ &= \frac{2}{\sqrt{6}} - \frac{4}{\sqrt{6}} + 0 = -\frac{2}{\sqrt{6}} = -\frac{\sqrt{6}}{3} \end{aligned}$$

Question 12

(a)

Find the tangent plane to the surface $z = x\cos y - ye^x$.

Solution

Let $F(x,y,z) = x\cos y - ye^x - z = 0$.

Gradient:

$$\nabla F = (\cos y - ye^x, -x\sin y - e^x, -1)$$

At (x_0, y_0, z_0) where $z_0 = x_0 \cos y_0 - y_0 e^{x_0}$:

$$\nabla F(x_0, y_0, z_0) = (\cos y_0 - y_0 e^{x_0}, -x_0 \sin y_0 - e^{x_0}, -1)$$

Tangent plane equation:

$$(\cos y_0 - y_0 e^{x_0})(x - x_0) + (-x_0 \sin y_0 - e^{x_0})(y - y_0) - (z - z_0) = 0$$

Or:

$$z = z_0 + (\cos y_0 - y_0 e^{x_0})(x - x_0) + (-x_0 \sin y_0 - e^{x_0})(y - y_0)$$

(b)

Find the tangent line to the curve of intersection of the surfaces $f(x,y,z) = x^2+y^2-2 = 0$ and $g(x,y,z) = x + z - 4 = 0$ at $(1,1,3)$.

Solution

Gradients:

$$\nabla f = (2x, 2y, 0) \Rightarrow \nabla f(1,1,3) = (2,2,0)$$

$$\nabla g = (1,0,1) \Rightarrow \nabla g(1,1,3) = (1,0,1)$$

Direction vector:

$$\nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (2,-2,-2)$$

Tangent line:

$$(x,y,z) = (1,1,3) + t(2,-2,-2), \quad t \in \mathbb{R}$$

Question 13

(a)

Find the maximum and minimum values of $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution

Using Lagrange multipliers with $g(x,y) = x^2 + y^2 - 1 = 0$:

$$\nabla f = (3,4), \quad \nabla g = (2x, 2y)$$

$$3 = 2\lambda x, \quad 4 = 2\lambda y \Rightarrow x = \frac{3}{2\lambda}, \quad y = \frac{2}{\lambda}$$

Substitute into $x^2 + y^2 = 1$:

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1 \Rightarrow \frac{25}{4\lambda^2} = 1 \Rightarrow \lambda^2 = \frac{25}{4}$$

$$\lambda = \pm\frac{5}{2}$$

For $\lambda = \frac{5}{2}$: $x = \frac{3}{5}, y = \frac{4}{5}$, $f = 5$
For $\lambda = -\frac{5}{2}$: $x = -\frac{3}{5}, y = -\frac{4}{5}$, $f = -5$

$$\boxed{\text{Max} = 5, \quad \text{Min} = -5}$$

(b)

Find absolute max/min of $f(x,y) = 2+2x+2y-x^2-y^2$ on triangular region: $x \geq 0, y \geq 0, y \leq 9 - x$.

Solution

Interior critical point:

$$f_x = 2 - 2x = 0 \Rightarrow x = 1, \quad f_y = 2 - 2y = 0 \Rightarrow y = 1$$

$f(1,1) = 4$
Boundary $x = 0$**:** $f(0,y) = 2 + 2y - y^2$
$f_y = 2 - 2y = 0 \Rightarrow y = 1$, $f(0,1) = 3$
Endpoints: $f(0,0) = 2$, $f(0,9) = -61$
Boundary $y = 0$**:** $f(x,0) = 2 + 2x - x^2$
$f_x = 2 - 2x = 0 \Rightarrow x = 1$, $f(1,0) = 3$
Endpoints: $f(0,0) = 2$, $f(9,0) = -61$
Boundary $y = 9 - x$**:** $f(x, 9 - x) = -2x^2 + 18x - 61$
Derivative: $-4x + 18 = 0 \Rightarrow x = 4.5$, $f(4.5, 4.5) = -20.5$
Endpoints: $f(0,9) = -61$, $f(9,0) = -61$
Comparison:
Max: 4 at $(1,1)$, Min: -61 at $(0,9)$ and $(9,0)$
Abs Max $= 4$, Abs Min $= -61$

Question 14

Find the extreme points of the functions:

(a)

$f(x,y) = x^2 + y^2 + x + y + xy$

Solution

Find critical points:

$$f_x = 2x + 1 + y = 0, \quad f_y = 2y + 1 + x = 0$$

From first equation: $y = -2x - 1$
Substitute into second:

$$2(-2x - 1) + 1 + x = -4x - 2 + 1 + x = -3x - 1 = 0 \Rightarrow x = -\frac{1}{3}$$

Then $y = -2(-\frac{1}{3}) - 1 = \frac{2}{3} - 1 = -\frac{1}{3}$
Second derivative test:

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1$$

$$D = (2)(2) - (1)^2 = 3 > 0, \quad f_{xx} > 0 \Rightarrow \text{local minimum}$$

Local minimum at $\left(-\frac{1}{3}, -\frac{1}{3}\right)$

(b)

$f(x,y) = y^2 - x^3$

Solution

Critical points:

$$f_x = -3x^2 = 0, \quad f_y = 2y = 0 \Rightarrow (0, 0)$$

Second derivatives:

$$f_{xx} = -6x, \quad f_{yy} = 2, \quad f_{xy} = 0$$

At $(0, 0)$: $f_{xx} = 0$, $f_{yy} = 2$, $f_{xy} = 0 \Rightarrow D = 0$ (test inconclusive)
Analyze behavior:
For $y = 0$: $f(x, 0) = -x^3$
For $x > 0$: $f < 0$, for $x < 0$: $f > 0 \Rightarrow$ saddle point

Question 15

Let $z = z(x, y)$ be a differentiable function and $x = r\cos\theta$, $y = r\sin\theta$. Prove that:

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Proof

By the chain rule:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta \quad (1)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = -r\frac{\partial z}{\partial x}\sin\theta + r\frac{\partial z}{\partial y}\cos\theta \quad (2)$$

From (1):

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cos\theta\sin\theta$$

From (2):

$$\left(\frac{\partial z}{\partial \theta}\right)^2 = r^2\left[\left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta\right]$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\sin\theta\cos\theta$$

Adding:

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial z}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Question 16

If $f_x(a, b) = f_y(a, b) = 0$, must f have a local maximum or minimum value at (a, b)? Give reasons for your answer.

Solution

No, f does **not** necessarily have a local maximum or minimum at (a, b) when $f_x(a, b) = f_y(a, b) = 0$.

Reason

The conditions $f_x(a, b) = f_y(a, b) = 0$ mean that (a, b) is a **critical point**. However, a critical point can be:

- Local maximum
- Local minimum
- Saddle point (neither max nor min)

Counterexample

Consider $f(x, y) = x^2 - y^2$ at $(0, 0)$:

$$f_x = 2x \Rightarrow f_x(0, 0) = 0, \quad f_y = -2y \Rightarrow f_y(0, 0) = 0$$

So $(0, 0)$ is a critical point.

But:

- Along x-axis ($y = 0$): $f(x, 0) = x^2 \geq 0$ (minimum at 0)
- Along y-axis ($x = 0$): $f(0, y) = -y^2 \leq 0$ (maximum at 0)

This is a **saddle point**, not a local maximum or minimum.

The given conditions only identify critical points, which include saddle points where the function has neither a local maximum nor minimum.

Question 17

Consider $f(x, y) = x^2 + y^2 + 2xy - x - y + 1$ over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

(a)

Show that f has an absolute minimum along the line segment $2x + 2y = 1$ in this square. What is the absolute minimum value?

Solution

Simplify f:

$$f(x, y) = x^2 + y^2 + 2xy - x - y + 1 = (x + y)^2 - (x + y) + 1$$

Let $t = x + y$, then $f(t) = t^2 - t + 1$.
Domain: $0 \le t \le 2$.
Line $2x + 2y = 1$ means $x + y = \frac{1}{2} \Rightarrow t = \frac{1}{2}$.
Minimize $f(t)$: derivative $f'(t) = 2t - 1 = 0 \Rightarrow t = \frac{1}{2}$.
Value: $f\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$.
Check boundaries: $f(0) = 1$, $f(2) = 3 \Rightarrow$ minimum at $t = \frac{1}{2}$.
Line $x + y = \frac{1}{2}$ lies in square for $0 \le x \le \frac{1}{2}$, $0 \le y \le \frac{1}{2}$.

$$\frac{3}{4}$$

(b)

Find the absolute maximum value of f over the square.

Solution

$f(t) = t^2 - t + 1$ is a parabola opening upwards, so maximum at $t = 2$.
$f(2) = 4 - 2 + 1 = 3$.
$t = 2$ means $x + y = 2 \Rightarrow$ only $(1, 1)$ in square.
Check corners: $(0, 0) : 1$, $(1, 0) : 1$, $(0, 1) : 1$, $(1, 1) : 3$.
So maximum value is 3.

$$3$$

Question 18

Show that if $w = f(u, v)$ satisfies the Laplace equation $f_{uu} + f_{vv} = 0$ and if

$$u = \frac{x^2 - y^2}{2}, \quad v = xy,$$

then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

Proof

First derivatives of w
By the chain rule:

$$w_x = f_u u_x + f_v v_x, \quad w_y = f_u u_y + f_v v_y$$

Compute:

$$u_x = x, \quad u_y = -y, \quad v_x = y, \quad v_y = x$$

So:

$$w_x = f_u \cdot x + f_v \cdot y \quad (1), \quad w_y = f_u \cdot (-y) + f_v \cdot x \quad (2)$$

Second derivative w_{xx}
Differentiate (1) w.r.t. x:

$$w_{xx} = \frac{\partial}{\partial x}(f_u) \cdot x + f_u + \frac{\partial}{\partial x}(f_v) \cdot y$$

$$\frac{\partial}{\partial x}(f_u) = f_{uu}u_x + f_{uv}v_x = f_{uu}x + f_{uv}y$$

$$\frac{\partial}{\partial x}(f_v) = f_{vu}u_x + f_{vv}v_x = f_{vu}x + f_{vv}y$$

Thus:

$$\begin{aligned} w_{xx} &= (f_{uu}x + f_{uv}y)x + f_u + (f_{vu}x + f_{vv}y)y \\ &= f_{uu}x^2 + 2f_{uv}xy + f_{vv}y^2 + f_u \quad (3) \end{aligned}$$

Second derivative w_{yy}
Differentiate (2) w.r.t. y:

$$w_{yy} = \frac{\partial}{\partial y}(f_u) \cdot (-y) - f_u + \frac{\partial}{\partial y}(f_v) \cdot x$$

$$\frac{\partial}{\partial y}(f_u) = f_{uu}u_y + f_{uv}v_y = -f_{uu}y + f_{uv}x$$

$$\frac{\partial}{\partial y}(f_v) = f_{vu}u_y + f_{vv}v_y = -f_{vu}y + f_{vv}x$$

Thus:

$$\begin{aligned} w_{yy} &= (-f_{uu}y + f_{uv}x)(-y) - f_u + (-f_{vu}y + f_{vv}x)x \\ &= f_{uu}y^2 - 2f_{uv}xy + f_{vv}x^2 - f_u \quad (4) \end{aligned}$$

Add $w_{xx} + w_{yy}$
From (3) and (4):

$$w_{xx} + w_{yy} = f_{uu}x^2 + 2f_{uv}xy + f_{vv}y^2 + f_u + f_{uu}y^2 - 2f_{uv}xy + f_{vv}x^2 - f_u$$

$$= f_{uu}(x^2 + y^2) + f_{vv}(x^2 + y^2) = (f_{uu} + f_{vv})(x^2 + y^2)$$

Use given condition
Given $f_{uu} + f_{vv} = 0$, we get:

$$w_{xx} + w_{yy} = 0$$