Department of Mathematics
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IC152: Linear Algebra-II
Quiz-IV

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1. Let $\langle \cdot, \cdot \rangle$ be an inner product on V over $\mathbb{R}$. Consider the following statements

   (I) If $\langle u, v\rangle = \langle 2u, -v\rangle$ for all $v \in V$, then $u = 0$

   (II) $|\langle u, v\rangle| \leq \frac{1}{2}(\langle u, u\rangle) + \langle v, v\rangle)$ for all $u, v \in \mathbb{V}$

   Then which of the following options is correct

   (a) both (I) and (II) are correct

   (b) both (I) and (II) are false

   (c) (I) is correct (II) is false

   (d) (I) is false (II) is correct

   Option (a) is correct as both the statements are correct. Note that if $\langle u, v\rangle = \langle 2u, -v\rangle$ for all $v \in V$, then in particular, choose $v = u$, to get $\langle u, u\rangle = -2\langle u, u\rangle \implies \langle u, u\rangle = 0$ or $u = 0$. The second statement follows from Cauchy-Schwartz inequality together with direct application of the inequality $ab < \frac{1}{2}(a^2 + b^2)$.

2. Let $P_2(\mathbb{R})$ be an inner product space of polynomials of degree atmost 2 over a real filed with inner product defined as

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

   Let $\{f_1, f_2, f_3\}$ be an orthogonal set in $P_2(\mathbb{R})$, where $f_1(x) = 1$, $f_2(x) = x + c_1$, $f_3(x) = x^2 + c_2 f_2 + c_3$, then value of $2c_1 + c_2 + 3c_3$ is equal to

   (a) 1

   (b) -2

   (c) -3

   (d) 0

   Correct option is (c). It can be verified by solving a system of 3 equations obtained from $\langle f_1, f_2\rangle = 0$, $\langle f_2, f_3\rangle = 0$ and $\langle f_3, f_1\rangle = 0$.

3. Let V be a finite dimensional vector space over $\mathbb{R}$ and T be a non-zero linear operator on V satisfying $T^2 = \lambda T$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ and $Tx \neq \lambda x$ for some $x \in V$. Then which of the following statement are correct

(a) T is invertible

(b) T is diagonalizable

(c) null space of (T) is non-trivial

(d) λ is the only eigenvalue of T

A annihilating polynomial for T is $x^2 - \lambda x$. The choices for minimal polynomial are $t, t-\lambda$ and $t(t-\lambda)$ but $T \neq 0$ and $T \neq \lambda I$ (as if $T = \lambda I$, then there $Tx = \lambda x$ for all $x \in V$ contradicting the given assumption). Hence the minimal polynomial of T is $t(t-\lambda)$. As 0, is an eigenvalue, T will not be invertible as matrix of T relative to any ordered basis of V will have zero determinant and consequently will have a nontrivial null space. Moreover, as minimal polynomial of T is written in the product of distinct linear factors, T will be diagonalizable. Option (d) is incorrect as 0 is also an eigenvalue other than λ.

4. Let M be a 3×3 real symmetric matrix with eigen values as 1, 0, 3. The eigen vectors corresponding the eigenvalues 1 and 0 are $(1,1,1)^t$ and $(1,-1,0)^t$ respectively. Find the value of M_{33}.

Let $M = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$. Then by trace formula $a + d + f = 4$ and using $M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1. \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0. \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we get $a+b+c = b+d+e = c+e+f = 1$ and $a = b, b = d, c = e$. On solving these equations we get $M_{33} = f = 7/3$.

5. Let M be a 3×3 matrix such that $M \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix}$. Then find out the vector $M^3 \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}$. Observe that -3 is an eigenvalue of M with corresponding eigenvector $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. Now $(-3)^3 = -27$ will be an eigenvalue for M^3 with eigenvector $\begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}$ as well because $\begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix} = -1/2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ which gives $M^3 \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix} = . \begin{bmatrix} -27 \\ 27/2 \\ 0 \end{bmatrix}$