Department of Mathematics
Indian Institute of Technology Bhilai
IC104: Linear Algebra-I
Quiz-2 Answers

Note: Text in blue color is the answer of the question. The marking scheme for descriptive questions is mentioned in red color against each question. All the questions except 3, 6 and 7 are of 2 marks each. Questions 3, 6 and 7 are of 4 marks each.

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1. Let us consider a linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ as $T(1,0) = (0,1,-2)$ and $T(0,1) = (0,-2,4)$. Then which of the following options are correct

   (a) T is unique linear transformation of the above type

   (b) T is one to one

   (c) T is onto

   (d) rank of T is 1

2. Let m, n be any arbitrary positive integers. Give an example of a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ which is one to one. Ans: There are two cases. Case 1: When $n > m$. In this case let $T : \mathbb{R}^n \to \mathbb{R}^m$ be one to one ,i.e., nullity of T is equal to 0. Now apply rank nullity theorem, to get rank of T=dimension of $\mathbb{R}^n = n$ which is not possible as rank of T can never be more than m Thus there does not exist any linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ which is one to one. The second case is $n \leq m$. In this case we can have several examples such as, when $n < m$ we can have $T(x_1, x_2, \cdots, x_n) = (x_1, x_2, \cdots x_n, x_{n+1}, \cdots x_m)$ and when $n = m$ we have $T(x, x') = (-x, x')$, where $x \in \mathbb{R}$ and $x' \in \mathbb{R}^{n-1}$
   Marks distribution is as follows:

   (a) 1 mark if it is written that there is no 1-1 linear transformation when $n > m$, using rank nullity theorem

   (b) 1 mark if only $n < m$ case is answered

   (c) 2 marks if answer is completely correct with general values of n and m

   (d) 1 marks if the answer is completely correct with special values of n and m.

   (e) No marks otherwise

3. Give an example of distinct linear trasnformations, U and T such that $N(T) = N(U)$ and $R(T) = R(U)$, where $N(T)$ and $R(T)$ are the standard notations for null and range space of T respectively. One such example is $T, U : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (y,x)$, $U(x,y) = (-x,y)$. Here $N(T) = U(T) = \{0\}$ and $R(T) = R(U) = \mathbb{R}^2$.

2 marks if only example is given and no justification provided and 4 marks for example and proper justification.

4. Let $P_n(\mathbb{R})$ be the vector space of real polynomials in x of degree less or equal to n. Consider $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be defined as $T(f)(x) = f'(x)$ and $U : P_2(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ be defined as $U(f)(x) = \int_0^x f \, dx$. Find the matrix representation of linear operator TU with respect to the standard ordered basis of $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$. Ans: Observe that $TU : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ is an identity operator. As the dimension of $P_2(\mathbb{R})$ is 3, we get matrix of TU a 3×3 identity matrix.

   2 marks if the answer is identity matrix of order 3x3 else 0.

5. Let V, W and Z are vector spaces over a field F and $T : V \to W$ and $U : W \to Z$ be linear transformatons. Then which of the following statements are true

   (a) If UT is one to one then U and T both are one to one

   (b) If U and T are one to one then UT is also one to one

   (c) If UT is onto then both U and T are onto

   (d) If UT is one to one then T is one to one but U may not be one to one.

6. Let $P_1(\mathbb{R})$ be a vector space of real polynomials of degree less or equal to 1. Let $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ be a linear tranformation defined as $T(a + bx) = (a, a + b)$. What will be the matrix of T relative to standard ordered basis pair of $P_1(\mathbb{R})$ and $\mathbb{R}^2$. The standard ordered bases of $P_1(\mathbb{R})$ and $\mathbb{R}^2$ are $\{1, x\}$ and $\{(1,\ 0),\ (0,\ 1)\}$ respectively. Hence $T(1) = (1, 1) = 1(1, 0) + 1(0, 1)$ and $T(x) = (0, 1) = 0(1, 0) + 1(0, 1)$ give rise to the matrix of T relative to standard ordered basis pair of $P_1(\mathbb{R})$ and $\mathbb{R}^2$ as $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

   The marks distribution is as follows

   (a) 2 marks if only the images of basis vectors are mentioned

   (b) 2 marks if only the matrix is written without giving images of basis vectors of $P_1(\mathbb{R})$

   (c) 4 marks if matrix and images both are written correctly

   (d) 1 mark if one of the images of basis vector is correct

   (e) No mark if matrix of T and images of basis vectors both are wrong

7. Let $V = P_n(\mathbb{R})$ be the vector space of real polynomials in x of degree less or equal to n. Consider a linear transformation $D : V \to V$, defined as $D(f)(x) = f'(x)$. Find the null and range space of D. Ans: All the constant polynomials will be in the null space of D. Hence $N(D) = \mathbb{R}$. Also the range space of D will be spanned by $\{1, x, x^2 \cdots x^{n-1}\}$ and hence $R(D) = P_{n-1}(\mathbb{R})$.

   2 marks for each null and range space