System of Linear Equations

Let us consider $F (= \mathbb{R}$ or $\mathbb{C})$ a underlying field. Look at the following scalar linear equation in one variable

$$ax = b, \quad a, b \in \mathbb{R} \quad \text{—— (1)}$$

Let us look at the following three cases

i) The equation (1) has UNIQUE solution if $a \neq 0$

ii) The equation (1) has INFINITLY MANY solutions if $a = 0$ & $b = 0$

iii) The equation has NO solution if $a = 0$ & $b \neq 0$

Let us look at the case of system of two equations (linear) In two variables. We begin with the following examples

$$\left.\begin{aligned} 2x + y &= 3 \\ x + 2y &= 1 \end{aligned}\right\}$$

The system has the solution

$$x = 5/3, \; y = -1/3$$

This solution is UNIQUE.

Consider the case

$$\left.\begin{aligned} 2x + y &= 4 \\ 4x + 2y &= 2 \end{aligned}\right\}$$

The system has NO solutions as the lines are parallel to each other

$$\left(2/4 = 1/2 \right)$$

Finally consider

$$\left.\begin{aligned} 2x + y &= 4 \\ 4x + 2y &= 8 \end{aligned}\right\}$$

The system has infinitly many solutions as the lines are coinciding to each other as

$$\left(2/4 = 1/2 = 4/8 \right)$$

Thus a system of two linear equations has Unique solution, infinitly many solutions or no solutions. The solutions are the intersection points of the lines under consideration.

Let us formulate this discussion for the following general system

$$\left.\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}\right\} \begin{array}{l} a_i, b_i, c_i \in \mathbb{R} \\ (a_i, b_i) \neq (0,0) \\ \text{for } i = 1, 2 \end{array}$$

We may encounter either of the following three cases

I) UNIQUE solution if

$$a_1 b_2 - a_2 b_1 \neq 0$$

II) Infinitly many solutions if

$$a_1 b_2 - b_1 a_2 = 0 \ \& \ b_1 c_2 - b_2 c_1 = 0$$

III) No solution if

$$a_1 b_2 - b_1 a_2 = 0 \ \& \ b_1 c_2 - b_2 c_1 \neq 0$$

Let us consider the general case of a system of m-linear equations in n-variables $x_1, x_2, \ldots x_n$ as

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m \end{array}\right\} .$$

where $a_{ij} \in \mathbb{R}$, $b_i \in \mathbb{R}$ for all $1 \leq i \leq m$ & $1 \leq j \leq n$.

Let us denote by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Then above system can be written as

$Ax = b$, where

$x = (x_1, x_2, \ldots x_n)^T$ and $b = (b_1, b_2, \ldots b_m)^T$

<u>Definition</u> (i) The system $Ax = b$, is homogeneous if $b = 0$ otherwise system is called non-homogeneous.

(ii) The matrix A is called coefficient matrix and the block matrix $[A \ b]$ is called as augmented matrix of the linear system.

Note that $[A \ b]$ is formed by attaching the column vector $b = (b_1, b_2, \ldots b_m)^T$ to the matrix A in $(n+1)^{th}$ column. Thus $[A, b]$ is a matrix of size $m \times (n+1)$.

* One should be carefull while writing the system $Ax = b$, as the unknowns should be arranged in a uniform way.

iii) A solution of the system $Ax = b$ is a vector y of size $n \times 1$ satisfying $Ay = b$.

iv) The collection of all solutions of the system is called solution set of the system.

v) The system $Ax = b$ is called consistent if it has at least one solution otherwise it is inconsistent.
For example $x+y=1$, $2x+2y=3$ is inconsistent while
$x+y=4$, $2x+y=6$ is consistent.

Let us look at the case of homogeneous system, $Ax = 0$
Note the following points
i) $x = 0$ always solves $Ax = 0$, irrespective of choices of A.
ii) If x_1, x_2 are two solutions of $Ax = b$, then $x_1 - x_2$ solves $Ax = 0$.
iii) If x_1 is a solution of $Ax = 0$, then αx is also a solution of $Ax = 0$ $\forall \alpha \in \mathbb{F}$
In general, if $x_1, x_2, \ldots x_k$ are solutions of $Ax = 0$, then any linear combination
$$\sum_{i=1}^{k} \alpha_i x_i \text{ solves } Ax = 0$$
$\forall \alpha_i \in \mathbb{F}$.

Ques: We are discussing the cases of solutions being unique, or no solution or infinitely many solutions.
Is it possible for a system $Ax = b$, $A \in M_{m \times n}(\mathbb{R})$ to have two or finitely many solutions?

Elementary Row Operations

Let us look at the solution process of the following system.

Example:

$$2y+z=2,\quad 2x+3z=5,\quad x+2y+2z=4$$

Step 1: Rewriting the above system into $Ax=b$ form, we get

$$A=\begin{bmatrix}0&2&1\\2&0&3\\1&2&2\end{bmatrix},\ x=\begin{bmatrix}x\\y\\z\end{bmatrix},\ b=\begin{bmatrix}2\\5\\4\end{bmatrix} \text{ and } \begin{bmatrix}0&2&1&2\\2&0&3&5\\1&2&2&4\end{bmatrix}=[A\ b]$$

Step 2. Interchange 1^{st} and 3^{rd} equation to get

$$\begin{aligned}x+2y+2z&=4\\2x+3z&=5\\2y+z&=2\end{aligned},\qquad B_1=\begin{bmatrix}1&2&2&4\\2&0&3&5\\0&2&1&2\end{bmatrix}$$

Step 3 Interchange 2^{nd} and 3^{rd} equation to get

$$\begin{aligned}x+2y+2z&=4\\2y+z&=2\\2x+3z&=5\end{aligned}\qquad B_2=\begin{bmatrix}1&2&2&4\\0&2&1&2\\2&0&3&5\end{bmatrix}$$

Step 4 Multiply equation 2 by $\frac{1}{2}$ to get

$$\begin{aligned}x+2y+2z&=4\\y+\tfrac{1}{2}z&=1\\2x+3z&=5\end{aligned},\qquad B_3=\begin{bmatrix}1&2&2&4\\0&1&\frac{1}{2}&1\\2&0&3&5\end{bmatrix}$$

Step 5: Replace 3^{rd} equation by 3^{rd} equation $-$ 2 times 1^{st} equation

$$\begin{aligned}x+2y+2z&=4\\y+z/2&=1\\-4y-z&=-3\end{aligned}\qquad B_4=\begin{bmatrix}1&2&2&4\\0&1&\frac{1}{2}&1\\0&-4&-1&-3\end{bmatrix}$$

Step 6: Replace equation 3^{rd} by equation 3 plus 4 times equation 2

$$\begin{aligned}x+2y+2z&=4\\y+z/2&=1\\z&=1\end{aligned}\qquad B_5=\begin{bmatrix}1&2&2&4\\0&1&\frac{1}{2}&1\\0&0&1&1\end{bmatrix}$$

Thus the last equation provides
$z=1$. and from equation 2,
on substituting $z=1$, we get $y=1/2$
and from equation 1, we get
$x = 4-2\times 1/2 - 2\times 1 = 1$.
Thus $(1, \frac{1}{2}, 1)$ solves the system uniquely.

Inspired from the above operations applied on the system which resulted into row operations on the augmented matrix, we define the following -

Definition: Let $A \in M_{m\times n}(\mathbb{F})$. Then the elementary row operation are as follows

1) interchanging i^{th} and j^{th} row. $(R_i \leftrightarrow R_j)$
2) multiplying k^{th} row by $0 \neq \lambda \in \mathbb{F}$ $(R_k \to \lambda R_k)$
3) replace i^{th} row by i^{th} row plus λ times j^{th} row for $\lambda \in \mathbb{F}$ $(R_i \to R_i + \lambda R_j)$ $(i \neq j)$

Definition: Two matrices are called row-equivalent if one can be obtained via elementary row operations on other.

Definition: The linear systems $Ax=b$ & $Cx=d$ are row equivalent if $[A\ b]$ is row equivalent to $[C, d]$.

We conclude with the following theorem on row equivalent systems

Theorem: Let $Ax=b$ & $Cx=d$ be two row equivalent linear systems then they have the same solution set.