Department of Mathematics
Indian Institute of Technology Bhilai
IC104: Linear Algebra-I
Tierce Answers

Note: The marking scheme for descriptive questions is in blue color

1. Let V be the vector space of all real polynomials of degree less or equal to 5. Let D be a differentiation operator on V. Find the sum of diagonal elements of matrix of D relative to standard ordered basis of V. [2]

   Ans: The matrix of D relative to standard ordered basis pair of V is

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   which has all the entries in the diagonal as 0. Hence sum of the diagonal entries (also known as trace) is 0.

   (a) 1.5 marks for matrix representation.

   (b) 1 mark for images of basis of $P_5(\mathbb{R})$ in linear combination of basis of $P_5(\mathbb{R})$

   (c) 2 marks if trace is found to be zero by writing a matrix D

   (d) 0 marks otherwise

   (e) 1 mark if only matrix and sum is mentioned

2. Let $P_n(\mathbb{R})$ denotes the vector space of real polynomials in x of degree less or equal to n. Consider $T : P_3(\mathbb{R}) \to P_4(\mathbb{R})$ be a linear transformation given by $p(x) \mapsto x.p(x)$ for every $p(x) \in P_3(\mathbb{R})$. Which of the following belong to range space of T

   (a) 0

   (b) x^4

   (c) $1 + 2x^2$

   (d) $12x(1 + x) + 0.5x^3$

3. Let $T : \mathbb{V} \to W$ be a linear transformations on a vector space V of dimension 4 into a vector space W of dimension 3. If nullity of T be 1, then what will be the rank of T? Your options are:

(a) 1

(b) 2

(c) 3

(d) 4

4. List all pairs of rank and nullity as (rank(T), nullity(T)) which are possible for a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$. [3]

Ans: (0, 3), (1, 2), (2, 1)

(a) 3 marks if (0, 3),(1,2),(2, 1) possibilities are mentioned

(b) 0 if any incorrect possibility is mentioned in the answer or not answered

(c) 2 mark if either of (0, 3), (1, 2), (2, 1) is missing but no other option is mentioned

5. Let V be a finite dimensional vector space over the field F and let $\mathcal{B} = \{\alpha_1, \alpha_2, \cdots \alpha_n\}$ and $\mathcal{B}' = \{\alpha'_1, \alpha'_2, \cdots \alpha'_n\}$ are two ordered bases for V. Let $T : V \to V$ be a linear operator on V then which of the following are correct

(a) $[T]_{\mathcal{B}} = P[T]_{\mathcal{B}'}Q$ for some invertible matrices P and Q.

(b) $[T]_{\mathcal{B}} = P^{-1}[T]_{\mathcal{B}'}P$ where P is an invertible matrix with columns as $P_j = [\alpha_j]_{\mathcal{B}'}$

(c) $[T]_{\mathcal{B}'} = P[T]_{\mathcal{B}}P^{-1}$ where P is an invertible matrix with columns $P_j = [\alpha_j]_{\mathcal{B}}$

(d) $[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P$ where P is an invertible matrix with columns as $P_j = [\alpha'_j]_{\mathcal{B}}$

6. Describe all 2×2 row reduced echelon (RRE) matrices. [4]

Ans: There are four types of 2×2 RRE matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(a) 4 marks if only all the correct types are answered

(b) 2 marks if either of trivial forms are missing

(c) 1 mark if both the trivial types are missing1

(d) 0 mark otherwise which include any wrong type answered

(e) 3 marks if any type is written in a particular form and else are correct

7. For what values of α the following system has unique solutions [2]

$$\alpha^2 x + y = 1$$
$$x + 4y = 2$$

Ans: For the system to have unique solution, $\det \begin{bmatrix} \alpha^2 & 1 \\ 1 & 4 \end{bmatrix} = 4\alpha^2 - 1 \neq 0$. Thus $\alpha \in \mathbb{R} \setminus \{-1/2, 1/2\}$

(a) 1 mark if only values of α are written

(b) 1 mark if correct condition on alpha is written

(c) 2marks if precise values of alpha are mentioned

(d) 0 marks otherwise

8. Let $A = \begin{bmatrix} a & a \\ 0 & b \end{bmatrix}$ be a 2×2 non-zero matrix. Then choose the correct options

(a) If $a \neq 0$ and $b \neq 0$, RRE form of A will never have a zero row

(b) The matrix A is always invertible

(c) If $b = a$ then RRE form of A is an identity matrix

(d) The solution space of the system $Ax = 0$ may have the dimesion 2.

9. For what pairs of (a, b) the following matrices are row equivalent? [4]

$$\begin{bmatrix} 1 & 2 \\ 2 & b \end{bmatrix}, \begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$$

If both the matrices are invertible then their RRE forms are equal (infact will be equal to identity matrix). In this case the choices are $b \neq 4$ and $ab \neq 4$. In case when matrices are not invertible , i.e., $b = 4$ and $ab = 4$, implies $a = 1$ which will correspond to two different RREs of A and B as

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$$

Thus both the matrices are not row equivalent even if $a = 1$ and $b = 4$. Thus matrices will be row equivalent for all pairs of (a, b) lying in $\{(a, b) : ab \neq 4 \ \& \ b \neq 4\}$.

(a) 4 marks for complete answer

(b) 0 marks otherwise

10. Let V be the vector space of real valued functions of one variable on $\mathbb{R}$. Let $S = \{\sin^2 x, \cos^2 x\}$ be a subset of V. Which of the following vectors belong in the the subspace spanned by S

   (a) $f(x) = 1$

   (b) $f(x) = 1 + x$

   (c) $f(x) = \sin 2x$

   (d) $f(x) = \cos 2x$

11. Let V be a vector space over a field F and $\{\alpha, \beta, \gamma\}$ be a linearly independent set. Then which of the following sets are linearly independent are correct.

   (a) $\{\alpha, \alpha + \beta, \alpha + \beta + \gamma\}$

   (b) $\{\alpha + \beta, \beta + \gamma, \gamma + \alpha\}$

   (c) $\{\alpha - \beta, \beta - \gamma, \gamma - \alpha\}$

   (d) $\{\alpha, \alpha + \beta, \alpha + \gamma\}$

12. Which of the following statements are **FALSE**

   (a) An ordered basis of a vector space is unique

   (b) If you remove any vector from basis, it will not remain a linearly independent subset of vector space

   (c) If you remove a vector from basis it will not span the vector space.

   (d) Every basis of a finite dimensional vector space contains the same number of vectors.

13. Let V be the vector space of real valued functions of one variable on $\mathbb{R}$. What is the dimension of subspace spanned by $S = \{1, \sin^2 x, \cos^2 x, \sin 2x, \cos 2x\}$ [3]
   Ans: Observe that $1 = \sin^2 x + \cos^2 x$, $\cos 2x = \cos^2 x - \sin^2 x$ can have a nontrivial linear combination of vectors of S and hence $< S >=< \{\sin^2 x, \cos^2 x, \sin 2x\} >$. It is easy to check that the set $\{\sin^2 x, \cos^2 x, \sin 2x\}$ is linearly independent and hence dimension is 3

   (a) 1 mark if dimension is written correct

   (b) 2 marks if partial argument is given

   (c) 3 marks if answer is completely correct

   (d) 0 otherwise

14. Let A be a matrix of size $m \times n$. Then answer the following questions [4]

(a) What is the upper bound for the row rank of A

(b) What is the the upper bound for the column rank of A

(c) What is the upper bound for rank of A

(d) Is it possible for the matrix A to achieve upper bounds of its column rank and row rank simultaneously.

Ans: (a) m, (b) n, (c) $\min\{m, n\}$, (d) YES it is possible when $m = n$ and A is invertible matrix . Alternatively, answers for (a), (b) and (c) can be $\min\{m, n\}$

(a) 1 mark each for first three answers

(b) 1 mark if last option has correct explanation

(c) 1 mark if either one of them is correct

(d) 2 marks if either two of them are correct

(e) 4 marks if answered correctly

(f) 0 marks otherwise

15. Let V be the vector space of all 2×2 matrices over $\mathbb{R}$. Let

$$W_1 = \left\{ A \in V : A = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix} \right\}$$
$$W_2 = \left\{ A \in V : A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \right\}$$

be subspaces of V. Then find the dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$. [4]

Observe that $W_1 = \left\langle \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$ and $W_2 = \left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\rangle$.

Hence dim(W_1)=2=dim(W_2).

Similarly $W_1 \cap W_2 = \left\{ A \in V : A = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \right\} = \left\langle \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\rangle$.

Thus dim ($W_1 \cap W_2$)=1. Now dim($W_1 + W_2$)=dim(W_1)+dim(W_2)-dim ($W_1 \cap W_2$)=3

(a) 1 mark for each dimension of $W_1, W_2, W_1 + W_2, W_1 \cap W_2$

(b) 4 marks for correct answer

(c) 0 otherwise

(d) 0.5 marks for each dimension if written directly

(e) 2 marks if two dimensions are correct

(f) 0.5 marks each for directly writing two dimensions correct

16. Let $P_1(\mathbb{R})$ be a vector space of real polynomials of degree less or equal to 1. Let $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ be a linear transformation defined as $T(a + bx) = (a + b, b)$. What will be the matrix of T relative to standard ordered basis pair of $P_1(\mathbb{R})$ and $\mathbb{R}^2$. [2]
The standard ordered bases of $P_1(\mathbb{R})$ and $\mathbb{R}^2$ are $\{1, x\}$ and $\{(1, 0), (0, 1)\}$ respectively. Hence $T(1) = (1, 0) = 1(1, 0) + 0(0, 1)$ and $T(x) = (1, 1) = 1(1, 0) + 1(0, 1)$ give rise to the matrix of T relative to standard ordered basis pair of $P_1(\mathbb{R})$ and $\mathbb{R}^2$ as $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) 1 mark of co-ordinate matrix of basis vectors of P_1

(b) 1 mark for matrix

(c) 2 marks if answered completely

(d) 0 otherwise

17. Let $M_{2\times2}(\mathbb{R})$ be the usual vector space of all 2×2 real matrices. Find null space, nullity, range space and rank of the following linear transformation $T : M_{2\times2}(\mathbb{R}) \to \mathbb{R}$ given as [4]

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$$

Ans: $N(T) = \left\{ A \in M_{2\times2}(\mathbb{R}) : A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right\} = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\rangle$. Thus nullity (T)=3. Moreover, from rank nullity theorem, rank (T)=1. Thus T is an onto linear transformation and hence $R(T) = \mathbb{R}$.

(a) 1 mark if either of null space or range space is written

(b) 1 mark each if rank and nullity is written

(c) 3 marks if null space, nullity and rank is written

(d) 0 otherwise

(e) 2 marks if null space and nullity is correct

(f) 2 marks if range space and rank is correct

(g) 4 marks for complete correct answer

18. Let $P_n(\mathbb{R})$ be the vector space of real polynomials in x of degree less or equal to n. Consider a linear transformation $T : P_2(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$ defined as $T(f)(x) = \int_0^x f\ dx$. Find the matrix representation of T with respect to the standard ordered basis of $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$. [4]
It is easy to see that $T(1) = x = 0.1 + 1.x + 0.x^2 + 0.x^3$, $T(x) = x^2/2 = 0.1 + 0.x + 1/2.x^2 + 0.x^3$; $T(x^2) = x^3/3 = 0.1 + 0.x + 0.x^2 + 1/3.x^3$ which gives

$$[T] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

(a) 2 marks if coordinate matrix of image of basis of P_2 is written

(b) 4 If answered correctly

(c) 1 mark if only matrix is written1

(d) 0 otherwise