Department of Mathematics
Indian Institute of Technology Bhilai
IC104: Linear Algebra-I
Hints of Tutorial Sheet 3: Linear Transformation

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1. (a) It is easy to verify that $T(c\alpha+\beta) = cT(\alpha)+T(\beta)$ for every $c \in F$ and $\alpha, \beta \in V$. We know that null $(T) = \{x \in F^3 : T(x) = 0\}$. Now $(x+y+z, x-y+z, x+z) = (0,0,0)$ implies that

$$\begin{aligned} x+y+z &= 0 \\ x-y+z &= 0 \\ x+z &= 0, \end{aligned}$$

which can be written as $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then system equivalent to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then $x+z=0$ and $y=0$, z is arbitrary. Let $z=t$, $x = -t$. Then $(x,y,z) = (-t,0,t) = t(-1,0,1)$. Therefore null $(T) = \{t(-1,0,1) : t \in \mathbb{F}\}$. Then basis of null (T) is $\{(-1,0,1)\}$.

Now η be a arbitrary vector in range of T. Then

$$\begin{aligned} \eta &= (x+y+z, x-y+z, x+z) \\ &= x(1,1,1) + y(1,-1,0) + z(1,1,1). \end{aligned}$$

Thus η is a linear combination of the vectors $(1,1,1), (1,-1,0)$. Hence range of T is the subspace spanned by $\{(1,1,1),(1,-1,0)\}$. As $\{(1,1,1),(1,-1,0)\}$ is linealry independent and hence is a basis of range of T.

(b) It is easy to verify that $T(c\alpha+\beta) = cT(\alpha)+T(\beta)$ for every $c \in F$ and $\alpha, \beta \in V$. By following the similar process as in point (a) null $(T) = \{c(-2,4,3) : c \in \mathbb{F}\}$ and basis of null $(T) = \{(-2,4,3)\}$. Again range T is spanned by $\{(-1,1,-2),(2,0,2)\}$. Then the basis of range of T is $\{(-1,1,-2),(2,0,2)\}$.

2. It is given that $T \neq 0$ but $T^2 = 0$. As $T \neq 0$, then there exists a non-zero vector $x^* \in \mathbb{R}^n$ such that $T(x^*) \neq 0$. Now consider a relation $c_1x^* + c_2T(x^*) = 0$. Then $T(c_1x^* + c_2T(x^*)) = T(0)$. As T is linear map, then $c_1T(x^*) + c_2T^2(x^*) = 0$. Again $T^2(x) = 0$, for all $x \in \mathbb{R}^n$, then we get that $c_1 = 0$. Then it is easy to observe that $c_2 = 0$. Therefore $\{x^*, T(x^*)\}$ is linearly independent. In general all $x \in \mathbb{R}^n \setminus N(T)$ will qualify for the solution.

3. Let $\beta = (x, y, z)$ such that $T(\beta) = (9, 3, \alpha)$. Then we have

$$\begin{aligned} 2x + 3y + 4z &= 9 \\ x + y + z &= 3 \\ x + y + 3z &= \alpha \end{aligned}$$

Which can be written as $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ \alpha \end{bmatrix}$. After row operation on the augmented matrix is $\left[\begin{array}{ccc|c} 2 & 3 & 4 & 9 \\ 0 & \frac{-1}{2} & -1 & \frac{-3}{2} \\ 0 & 0 & 2 & \alpha - 3 \end{array}\right]$. This implies that $x = \frac{\alpha-3}{2}$, $y = 6 - \alpha$, $z = \frac{\alpha-3}{2}$. Therefore $\beta = \left(\frac{\alpha-3}{2}, 6 - \alpha, \frac{\alpha-3}{2}\right)$.

4. Yes, beacuse $T(aA_1 + bA_2) = (aA_1 + bA_2)B - B(aA_1 + bA_2) = aT(A_1) + bT(A_2)$. Now, nullity of $T = \{A \mid T(A) = 0\} = \{A \mid AB = BA\}$, i.e set of all matrices that commutes with B. As $M_{2\times 2}(\mathbb{R})$ can be written as the union of sets of commutative and non commutative matrices with B. The range of T contains all those matrices which are images of non commutative matrices with B together with 0 matrix.

5. Since $(1, -1, 1)$ and $(-1, 1, 2)$ are linearly independent, hence extend to basis of $\mathbb{R}^3$ and define $T(0, 1, 0) = (1, 1)$, then a simple calculation gives us the linear transformation as $T(x, y, z) = (\frac{4x+3y+2z}{3}, \frac{7x+3y+2z}{3})$. No, it is not a unique linear transformation of this type.

6. The range and null spaces are identical, hence assume dim(range$T$) = dim(null T) = m. Applying rank-nullity theorem we get $n = 2m$ which is even. Define T as $T(1, 0) = (0, 0)$ and $T(0, 1) = (1, 0)$, then the range space and null space are equal as it is generated by $(1, 0)$.

7. Let $B = \{a_1, a_2, \cdots a_n\}$ be a basis for V. Let $x \in$ range $T \cap$null T then as $\{T(a_1), T(a_2), \cdots T(a_n)\}$ spans range of T, we can write $x = c_1Ta_1 + ... + c_nTa_n$. Now as $x \in N(T)$, we have $0 = T(x) = T(c_1Ta_1 + ... + c_iTa_i) = c_1T^2a_1 + ... + c_iT^2a_i$. Now if $C = \{T^2a_1, T^2a_2, \cdots T^2a_n\}$ is linealry independent set, then $c_k = 0$ for all $k = 1, 2 \cdots n$ which results into $x = 0$. Thus it will be enough to show that nullity of T^2 is zero (then T^2 will be one to one and it will map a linearly independent set B to a linearly independent set C). Now, as $T^2\alpha = T(T\alpha)$ for all $\alpha \in V$, we can apply rank nullity theorem for T^2 on $T(V) = R(T)$ to get nullity of T^2 as 0.

8. Here $A = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & 0 \end{bmatrix}$. We know that the vectors $\{(1, 3, 2), (0, 2, 1), (4, -1, 0)\}$ span

column space and it is easy to see that these vectors are linearly independent. Hence rank of A is 3.

9. In this problem it is given that $m > n$.

   (a) On the contrary, there is a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, which is one-one (injective). Then nullity of $T = 0$. Again by the rank-nullity theorem we know that

   $$\text{nullity of } T + \text{ rank of } T = \dim(\mathbb{R}^m) = m.$$

   Then we have rank of $T = m$, which is not possible as range of T is subset of $\mathbb{R}^n$, that is rank of T is less than equal to n.

   (b) On the contrary, suppose there is a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, which is onto. That is range of $T = \mathbb{R}^m$. Therefore rank of $T = m$. Again by the rank-nullity theorem we got the contradiction.

10. The matrix of T corresponding to the standard basis is $\begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ i & 1 & 0 \end{bmatrix}$. Since $\det(T) = 0$, then T is not invertible.

11. It is easy to check that the matrix of T corresponding to the standard basis is $[T]_\beta = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$. As $\det([T]_\beta) \neq 0$, hence T is invertible and the inverse is $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$.
    Now putting the value of T and I we get that $(T^2 - I)(T - 3I) = 0$ (zero map).

12. Here

    $$\begin{aligned} T(1,0,-1) &= (1,-1) = 5(1,1) + (-2)(2,3) \\ T(1,1,1) &= (2,1) = 4(1,1) + (-1)(2,3) \\ T(1,0,0) &= (1,-1) = 5(1,1) + (-2)(2,3). \end{aligned}$$

    Thus the matrix of T relative to the basis β, β' is $\begin{bmatrix} 5 & 4 & 5 \\ -2 & -1 & -2 \end{bmatrix}$

13. (a) It is easy to check that the matrix of T corresponding to the standard matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

    (b) As $\beta = \{(1,2),(1,-1)\}$ and $T(1,2) = (-2,1) = \frac{-1}{3}(1,2) + \frac{-5}{3}(1,-1)$, $T(1,-1) = (1,1) = \frac{2}{3}(1,2) + \frac{1}{3}(1,-1)$. Therefore the matrix of T corresponding to the matrix is $\begin{bmatrix} \frac{-1}{3} & \frac{2}{3} \\ \frac{-5}{3} & \frac{1}{3} \end{bmatrix}$.

(c) As $(T-cI)(x,y)=(-y-cx,x-cy)$, then the matrix of $T-cI$ corresponding to the standard basis is $\begin{bmatrix}-c & -1\\ 1 & -c\end{bmatrix}$. Therefore $\det(T-cI)=c^2+1\neq 0\ \forall c\in\mathbb{R}$.

(d)

14. (a) Calculating we have $T(1,0,0)=(3,-2-1), T(0,1,0)=(0,1,2), T(0,0,1)=(1,0,4)$. Then the matrix of T corresponding to the standard basis is $\begin{bmatrix}3 & 0 & 1\\ -2 & 1 & 0\\ -1 & 2 & 4\end{bmatrix}$.

(b) Again the matrix of T corresponding to the ordered basis $\{(1,0,1),(-1,2,1),(2,1,1)\}$ is $\begin{bmatrix}\frac{13}{4} & \frac{47}{4} & \frac{11}{2}\\ \frac{-3}{4} & \frac{11}{4} & \frac{-3}{2}\\ \frac{1}{2} & \frac{-11}{2} & 0\end{bmatrix}$.

(c) As determinant of T corresponding to the standard basis is non-zero, then it is invertible. Now inverse of the matrix of T corresponding to the standard basis is $\begin{bmatrix}\frac{4}{9} & \frac{2}{9} & \frac{-1}{9}\\ \frac{8}{9} & \frac{13}{9} & \frac{-2}{9}\\ \frac{-1}{3} & \frac{-2}{3} & \frac{1}{3}\end{bmatrix}$. Then we have

$$T^{-1}(x,y,z)=\left(\frac{4}{9}x+\frac{2}{9}y+\frac{-1}{9}z, \frac{8}{9}x+\frac{13}{9}y+\frac{-2}{9}z, \frac{-1}{3}x+\frac{-2}{3}y+\frac{1}{3}z\right).$$