Department of Mathematics
Indian Institute of Technology Bhilai
IC152: Linear Algebra-II
Quiz-III

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1. Which of the following statements are correct

   (i) If α, β are unit vectors orthogonal to each other in an inner product space, then $\|\alpha+\beta\| = 2$.

   (ii) $<(x_1, x_2), (y_1, y_2)> = x_1 y_1 - x_2 y_2$ defines an inner product on $\mathbb{R}^2$.

   (iii) Let V be a n-dimensional inner product space. If W be a m-dimensional subspace of V such that $n = 2m - 1$ then dimension of $W^\perp$ is smaller than dimension of W.

   (iv) Let α, β belong to an inner product space V such that $\|\alpha+\beta\| = \|\alpha-\beta\|$, then α must be orthogonal to β.

   (v) Let T be a linear operator on a finite dimensional inner product space V such that $\|T\alpha\| = \|\alpha\|$ for all $\alpha \in V$ then T is one to one.

   Correct options are (iii) (Using dim V =dim W+dim $W^\perp$) and (v) (For T to be one to one, $T\alpha = 0$ should imply $\alpha = 0$. Let $T\alpha = 0$, then by given relation $0 = \|T\alpha\| = \|\alpha\|$ implies $\alpha = 0$. Hence one to one). Option (i) should have $\|\alpha+\beta\| = \sqrt{2}$. Option (ii) is wrong as $<(1,1),(1,1)> = 0$ but $(1,1) \neq (0,0)$. For option (iv) to be correct, underlying field must be real field.

2. Let α, β be orthogonal vectors in an inner product space $(V, <\cdot,\cdot>)$. Then the vectors $\alpha+\beta$ and $\alpha-\beta$,

   (i) must be orthogonal

   (ii) are orthogonal if and only if $\|\alpha\| = \|\beta\| = 1$

   (iii) are orthogonal if and only if $\|\alpha\| = \|\beta\|$

   (iv) cannot be orthogonal at all

   Correct option is (iii)

3. Let V be an inner product space and W, U are subspaces of V. Prove that if $W \subset U$ then $U^\perp \subset W^\perp$.
   For any $x \in U^\perp$, $<x, y> = 0$ for all $y \in U$. As $W \subset U$, $<x, y> = 0$ for all $y \in W$. Therefore $x \in W^\perp$ leading to $U^\perp \subset W^\perp$.

4. Apply Gram-Schmidt process to the vectors $\alpha_1 = (1, 0, 1), \alpha_2 = (1, 0, -1), \alpha_3 = (0, 3, 4)$ to get an orthogonal basis for $\mathbb{R}^3$ with standard inner product.

   The Gram-Schmidt formula helps to construct, from given linearly independent set $\{\alpha_1, \alpha_2, \alpha_3\}$, an orthogonal set $\{\beta_1 = \alpha_1, \beta_2, \beta_3\}$ in the following way

$$\beta_2 = \alpha_2 - \frac{< \alpha_2, \beta_1 >}{\|\beta_1\|^2}\beta_1, \quad \beta_3 = \alpha_3 - \frac{< \alpha_3, \beta_1 >}{\|\beta_1\|^2}\beta_1 - \frac{< \alpha_3, \beta_2 >}{\|\beta_2\|^2}\beta_2$$

   Applying the above formula, we get $\beta_1 = (1, 0, 1), \beta_2 = (1, 0, -1), \beta_3 = (0, 3, 0)$.

5. Find the matrix of standard inner product $< \cdot, \cdot >$ on $\mathbb{R}^3$ relative to an ordered basis $\mathcal{B} = \{(1, 0, 1), (1, 0, -1), (0, 3, 4)\}$.

   Let M be the matrix of standard inner product on $\mathbb{R}^3$ relative to $\mathcal{B} = \{\alpha_1 = (1, 0, 1), \alpha_2 = (1, 0, -1), \alpha_3 = (0, 3, 4)\}$, then $M_{ij} =< \alpha_j, \alpha_i >$ results into the following matrix

$$M = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & -4 \\ 4 & -4 & 25 \end{bmatrix}$$