Mid-Sem solu^n

(1) Given that $(\Omega, \mathcal{F}, P)$ be a prob space.

$\mathcal{G} = \{A \in \mathcal{F} : P(A) = 0 \text{ or } 1\}$.

(i) $A \in \mathcal{G}$ so $P(A) = 0$ or 1. We know $P(A^c) = 1 - P(A) = 0$ or 1 provided $P(A) = 1$ or 0 respectively. So $A^c \in \mathcal{G}$ $[\frac{1}{2}]$

(ii) $P(\Omega) = 1$ so $\Omega \in \mathcal{G}$ $[\frac{1}{2}]$

(iii) Let $A_1, A_2, \ldots$ be sequ^n of sets in $\mathcal{G}$ so that $P(A_i) = 0$ or 1 for $i \in \mathbb{N}$. Let $P(A_i) = 0$ $\forall i$, we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i) = 0$$

$$\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{G}.$$

$[1]$

Suppose $\exists$ at least one A_j s.t. $P(A_j) = 1$. We know

$$A_j \subseteq \bigcup_{i=1}^{\infty} A_i \Rightarrow P(A_j) \leq P\left(\bigcup_{i=1}^{\infty} A_i\right)$$

$$\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) \geq 1 \Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = 1$$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{Y}.$$ [1].

Q2 Let A = the event that Ram throw 6

B = the event Hari throw 7.

$$P(A) = \frac{5}{36}, \quad P(B) = \frac{6}{36} = \frac{1}{6}.$$ [1]

Ram plays in the 1^{st}, third, fifth .... trials

Therefor Ram will win, if he throws 6 in first trial or third trial or subsequent odd trials

$P(\text{Ram win the game})$

$= P(A) + P(A^c)\,P(B^c)\,P(A) +$

$P(A^c)\,P(B^c)\,P(A^c)\,P(B^c)\,P(A) + \cdots$ [1]

$$= \frac{5}{36} + \frac{31}{36}\cdot\frac{5}{6}\cdot\frac{5}{36} + \frac{31}{36}\,\frac{5}{6}\cdot\frac{31}{36}\cdot\frac{5}{6}\cdot 5/36$$

$+ \cdots$

$$= \frac{5}{36} + \left(\frac{31}{36}\,\frac{5}{6}\right)\frac{5}{36} + \left(\frac{31}{36}\,\frac{5}{6}\right)^2 \frac{5}{36} + \cdots$$

$$= \frac{5}{36}\left[1 + \left(\frac{31}{36}\,\frac{5}{6}\right) + \left(\frac{31}{36}\,\frac{5}{6}\right)^2 + \cdots\right]$$

$$= \frac{5}{36}\,\frac{1}{1 - \frac{155}{216}} = \frac{30}{61}$$ [1]

Q3 (i) Let X denote the absolute difference of number shown on the dice.

$$X \rightarrow 0, 1, 2, 3, 4, 5$$

$$P(X=0) = \sum_{i=1}^{6} P(\{i,i\}) = 6 \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{6}$$

$$P(X=1) = 2 \cdot \sum_{i=1}^{5} P(\{i, i+1\}) = 2 \cdot 5 \left(\frac{1}{6}\right)^2 = \frac{5}{18}$$

$$P(X=2) = 2 \cdot \sum_{i=1}^{4} P(\{i, i+2\}) = 2 \cdot 4 \left(\frac{1}{6}\right)^2 = \frac{2}{9}$$

$$P(X=3) = 2 \sum_{i=1}^{3} P(\{i, i+3\}) = 2 \cdot 3 \left(\frac{1}{6}\right)^2 = \frac{1}{6}$$

$$P(X=4) = 2 \sum_{i=1}^{2} P(\{i, i+4\}) = 2 \cdot 2 \left(\frac{1}{6}\right)^2 = \frac{1}{9}$$

$$P(X=5) = 2 P\{(1,6)\} = \frac{2}{36} = \frac{1}{18}$$

[3M]

(ii) $E(X) = 0\cdot\frac{1}{6} + 1\cdot\frac{5}{18} + 2\cdot\frac{2}{9} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{9}$

$+ 5\cdot\frac{1}{18} = \frac{35}{18} = 1.94$ [1]

$E(x^2) = 0\cdot\frac{1}{6} + 1^2\cdot\frac{5}{18} + 4\cdot\frac{2}{9} + \frac{9}{6} + \frac{16}{9} + \frac{25}{18}$

$= \frac{35}{6} = 5.83$

$Var(X) = \frac{665}{324} = 2.0525$ [1]

(iii) $P(X>3) = \frac{1}{6}$ [1]

Q4 Given $X \sim \exp(2)$. So the density of X is given as

$$f_X(x) = \begin{cases} 2e^{-2x}, & x>0 \\ 0, & o/w \end{cases}$$

$$P(Y=1) = P(X<1) = \int_0^1 2e^{-2x}\,dx$$

$$= [1-e^{-2}]$$ [1]

$$P(Y=2) = P(X=1) = 0$$

$$P(Y=3) = P(1<X<3) = \int_1^3 2e^{-2x}\,dx$$ [1]

$$= (e^{-2} - e^{-6})$$

$$P(Y=4) = P(X=3) = 0$$

$$P(Y=5) = P(X>3) = \int_3^\infty 2e^{-2x}\,dx$$ [1]

$$= e^{-6}$$

So $P(Y=1) + P(Y=3) + P(Y=5) = 1$

So Y is a discrete r.v. [1]

Support of Y = $S_Y = \{1, 3, 5\}$

$$E(Y) = 1\cdot[1-e^{-2}] + 3[e^{-2}-e^{-6}] + 5[e^{-6}]$$

$$= 1 - e^{-2} + 3e^{-2} - 3e^{-6} + 5e^{-6}$$

$$= 1 + 2e^{-2} + 2e^{-6}$$ [2]

Q5 Let Y denote the score on a shot.

Then $Y \to 0, 2, 3, 4.$

$$P(Y=0) = P(X > \sqrt{3}) = \int_{\sqrt{3}}^{\infty} \frac{2}{\pi(1+x^2)}\,dx$$

$$= \frac{2}{\pi} \tan^{-1} x \Big]_{\sqrt{3}}^{\infty} = \frac{1}{3}.$$

$$P(Y=2) = P(1 < X < \sqrt{3}) = \int_{1}^{\sqrt{3}} \frac{2}{\pi(1+x^2)}\,dx = \frac{1}{6}$$

$$P(Y=3) = P\left(\frac{1}{\sqrt{3}} < x < 1\right) = \int_{1/\sqrt{3}}^{1} \frac{2}{\pi(1+x^2)}\,dx = \frac{1}{6}$$

$$P(Y=4) = P\left(x < \frac{1}{\sqrt{3}}\right) = \int_{0}^{1/\sqrt{3}} \frac{2}{\pi(1+x^2)}\,dx = \frac{1}{3}$$

[2]

$E(Y)$ = expected score in a single shot

$$= 0\cdot\frac{1}{3} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{3} = \frac{13}{6}. \quad [1]$$

Q6 Let X be a r.v. with p.m.f.

$$p_X(x) = \begin{cases} \frac{b_i}{\sum b_i}, & \text{if } x = a_i,\ i=1,2,\ldots n. \\ 0, & \text{o/w.} \end{cases} \quad [1]$$

Then $p_X(x)$ is a proper pmf with $S_X = \{a_i : i=1,2,\ldots n\}$. Also $P(X>0)=1$. By Jensen's inequality with $\psi(x) = x^r$, $x>0$ is a convex fun provided $r \geq 1$

So $E(\psi(X)) \geq \psi(E(X))$ [1]

$$\Rightarrow E(X^r) \geq (E(X))^r$$

$$\Rightarrow \sum_{i=1}^{n} a_i^{r} \frac{b_i}{\Sigma b_j} \geq \left(\sum_{i=1}^{n} \frac{a_i b_i}{\Sigma b_j} \right)^{r}$$

$$\Rightarrow \left(\sum_{i=1}^{n} a_i^{r} b_i \right) \left(\sum_{i=1}^{n} b_i \right)^{r-1} \geq \left(\sum_{i=1}^{n} a_i b_i \right)^{r}$$

[1]

Q7 Given equ^n is $x^2+2x+q=0$

where $q \sim U(0,2)$ so the pdf of q

is given as

$$f(q) = \begin{cases} \frac{1}{2}, & 0<q<2 \\ 0, & o/w \end{cases}$$

[1]

The larger root of the given equ^n is

$$y = \frac{-2 + \sqrt{4+4q}}{2}$$

$$= -1 + \sqrt{1+q}$$

$q \in (0,2) \Rightarrow y \in (0, \sqrt{3}-1)$ [1]

Inverse transformation is $q = (1+y)^2 - 1$

$\frac{dq}{dy} = 2(1+y)$. So the density of

Y is

$$f_Y(y) = \begin{cases} 1+y, & 0 < y < \sqrt{3}-1 \\ 0, & o/w. \end{cases}$$

[1]

Q8 (a) $X \sim Exp(\lambda)$

Memoryless property $P(X > a+b \mid X > a) = P(X > b)$

[1]

We have $P(X > a) = e^{-\lambda a}$

So $P(X > a+b \mid X > a) = \frac{P(X > a+b)}{P(X > a)}$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}}$$

$$= e^{-\lambda b} = P(X > b).$$

[1]

(b) Let X denote the number ball is required to gets out

Then $X \sim Geo(0.4)$ [1]

The required prob is

$$P(X > 18 \mid X > 12) = P(X > 6)$$

$$= (0.6)^6.$$ [1]

(c)

$$E(|X|) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-x^2/2}$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-x^2/2} dx = \sqrt{\frac{2}{\pi}}$$ [2]

(d)

X has m.g.f $M_X(t) = e^{2t(1+t)}$

So $M_X(t) = e^{2t + 2t^2}$

$= e^{2t + \frac{1}{2}\cdot 4 \cdot t^2}$

Then $X \sim N(2, 4)$. [1]

$P(X \leq 2) = P\left(\frac{X-2}{2} \leq \frac{2-2}{2}\right)$

$= P(Z \leq 0) = \Phi(0) = \frac{1}{2}$ [1]

(e) Given that $np = 12$, $npq = 4$

$\Rightarrow q = 1/3 \quad p = 2/3, \; n = 18$ $\left[\frac{1}{2}\right]$

$P(X \geq 17) = P(X = 17) + P(X = 18)$

$= 10 \times \left(\frac{2}{3}\right)^{18} = 0.0068$ $\left[\frac{1}{2}\right]$