Tutorial - 12, Solu^n

(1) $X_1, \ldots, X_n \overset{iid}{\sim} f_X(x)$ with $f_X(x) = \dfrac{\beta \alpha^\beta}{x^{\beta+1}}$, $x > \alpha$

$\alpha > 0$, $\beta > 2$

$$E(X_1^r) = \int_\alpha^\infty x^r \frac{\beta \alpha^\beta}{x^{\beta+1}} dx = \beta \alpha^\beta \int_\alpha^\infty x^{r-\beta-1} dx$$

$$= -\beta \alpha^\beta \left[\frac{x^{-(\beta-r)}}{\beta - r} \right]_\alpha^\infty = \frac{\beta \alpha^r}{\beta - r} \quad \text{if } \beta > r.$$

$$\mu_1' = E(X_1) = \frac{\beta \alpha}{\beta - 1}, \quad \mu_2' = E(X_1^2) = \frac{\beta \alpha^2}{\beta - 2} \qquad \beta > 2.$$

Now $\dfrac{\beta \alpha}{\beta - 1} = \alpha_1$, $\dfrac{\beta \alpha^2}{\beta - 2} = \alpha_2$ $\qquad \alpha_1 = \frac{1}{n} \sum X_i$, $\alpha_2 = \frac{1}{n} \sum X_i^2$

So $\beta \alpha = \alpha_1 (\beta - 1) \Rightarrow \alpha = \dfrac{\alpha_1 (\beta - 1)}{\beta}$

Again $\beta \left(\dfrac{\alpha_1 (\beta - 1)}{\beta} \right)^2 = \alpha_2 (\beta - 2)$

$$\Rightarrow (\alpha_1^2 - \alpha_2)\beta^2 + \alpha_1^2 - 2\beta(\alpha_1^2 - \alpha_2) = 0.$$

$$\hat{\beta}_{MME} = \frac{2(\alpha_1^2 - \alpha_2) \pm \sqrt{4(\alpha_1^2 - \alpha_2)^2 - 4\alpha_1^2(\alpha_1^2 - \alpha_2)}}{2(\alpha_1^2 - \alpha_2)}$$

$$\hat{\alpha}_{MME} = \frac{\alpha_1 (\hat{\beta}_{MME} - 1)}{\hat{\beta}_{MME}}$$

② $X_1, \ldots, X_n \overset{iid}{\sim} U(-\theta, \theta)$

$E(X_1) = 0, \quad E(X_1^2) = \frac{\theta^2}{3}$. So

$\frac{\theta^2}{3} = \alpha_1 \Rightarrow \hat{\theta}_{MME} = \sqrt{\frac{3}{n}\sum_{i=1}^{n} X_i^2}$

③ $X_1, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\mu, \sigma)$, with density

$$f_X(x) = \begin{cases} \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)}, & x > \mu \\ 0, & \text{ow} \end{cases}$$

$E(X_1) = \mu + \sigma, \quad E(X_1^2) = 2\sigma^2 + 2\mu\sigma + \mu^2.$

Now take

$\alpha_1 = \mu + \sigma, \quad \alpha_2 = 2\sigma^2 + 2\mu\sigma + \mu^2$

$= \sigma^2 + (\mu + \sigma)^2$

$= \sigma^2 + \alpha_1^2$

$\Rightarrow \sigma^2 = \alpha_2 - \alpha_1^2$

So $\hat{\sigma}_{MME} = \sqrt{\alpha_2 - \alpha_1^2}$

$\hat{\mu}_{MME} = \alpha_1 - \sqrt{\alpha_2 - \alpha_1^2}$

(4) Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, with σ is known.

(a)

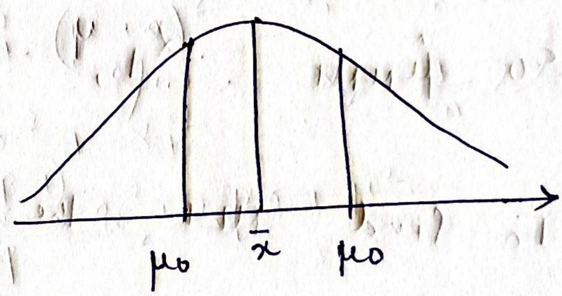
Then the likelihood funⁿ

$$L_{\underline{x}}(\mu) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\Rightarrow \quad l_x(\mu) = \ln L_x(\mu) = -\frac{n}{2}\ln(2\pi) - n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

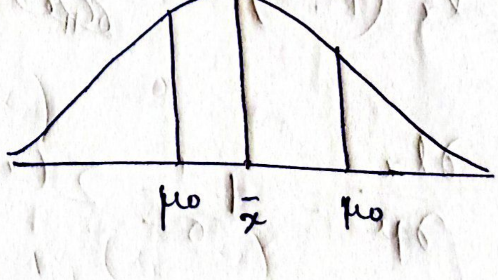
$$\frac{\partial l_x(\mu)}{\partial \mu} = \frac{1}{\sigma^2}\sum(x_i-\mu) \begin{array}{l} >0 \text{ if } \mu < \bar{x} \\ <0 \text{ if } \mu > \bar{x} \end{array}$$

so $\hat{\mu}_{MLE} = \bar{X}$



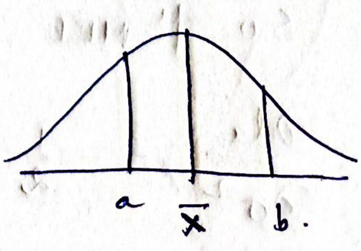
(b) Now $\mu \geq \mu_0$

$$\hat{\mu}_{RML} = \begin{cases} \bar{X} & \text{if } \mu_0 < \bar{X} \\ \mu_0 & \text{if } \mu_0 \geq \bar{X} \end{cases}$$



(c) $\mu \leq \mu_0$

$$\hat{\mu}_{RML} = \begin{cases} \mu_0 & \text{if } \mu \leq \bar{X} \\ \bar{X} & \text{if } \mu_0 > \bar{X} \end{cases}$$



(d) $$\hat{\mu}_{RML} = \begin{cases} b & \text{if } b \leq \bar{X} \\ \bar{X} & \text{if } a < \bar{X} < b \\ a & \text{if } \bar{X} \leq a. \end{cases}$$

⑤ $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, Let $\mu = \mu_0$ known

Similare to 4ⓐ we have

$$\hat{\sigma}_{MLE} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^2}$$

⑥ Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. Then

$$E(X_1) = \mu, \quad E(X_1^2) = \mu^2 + \sigma^2$$

Let $\alpha_1 = \mu$, $\alpha_2 = \mu^2 + \sigma^2 \Rightarrow \sigma^2 = \alpha_2 - \alpha_1^2$

So $\hat{\mu}_{MME} = \alpha_1$, $\hat{\sigma}^2_{MME} = \alpha_2 - \alpha_1^2$

To find MLE the log likelihood fu^n is

$$l_x(\mu, \sigma^2) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\frac{\partial l_x}{\partial \mu} = +\frac{1}{\sigma^2}\sum(x_i - \mu) \begin{cases} > 0 & \text{if } \mu < \bar{x} \\ < 0 & \text{if } \mu > \bar{x} \end{cases}$$

So $\hat{\mu}_{MLE} = \bar{X}$

$$\frac{\partial l_x}{\partial \sigma^2} = -\frac{1}{2}\frac{1}{\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\Rightarrow \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum(X_i - \bar{X})^2$$

(7) $X_1, \dots, X_n \sim \text{Bin}(m, p), \quad 0<p<1$

So the joint p.m.f.

$$p_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} \left[\binom{m}{x_i} p^{x_i} (1-p)^{m-x_i} \right]$$

$$= \left(\prod_{i=1}^{n} \binom{m}{x_i} \right) p^{\sum x_i} (1-p)^{mn-\sum x_i}$$

$$l_{\underline{x}}(p) \ln p_{\underline{X}}(\underline{x}) = \sum_{i=1}^{n} \ln \binom{m}{x_i} + \sum x_i \ln p + \left(mn - \sum x_i\right) \ln(1-p)$$

$$\frac{\partial l_{\underline{x}}(p)}{\partial p} = \frac{\sum x_i}{p} + \frac{mn - \sum x_i}{1-p}(-1)$$

$$= \frac{\sum x_i}{p} - \frac{mn-\sum x_i}{1-p} \quad \begin{cases} >0 \text{ if } p < \frac{\bar{x}}{m} \\ <0 \text{ if } p > \frac{\bar{x}}{m}. \end{cases}$$

$$\Rightarrow \hat{p}_{MML} = \frac{\bar{X}}{m}.$$

So MLE of $p(1-p)$ is $\hat{p}_{MLE}\left(1-\hat{p}_{MLE}\right)$

(8) (a) The joint density is

$$f_{\underline{X}}(\underline{x}) = \begin{cases} \frac{1}{\theta^n}, & 0 < x_{(1)} < x_{(2)} \dots < x_{(n)} < \theta \\ 0, & o/w. \end{cases}$$

$$\hat{\theta}_{MLE} = X_{(n)}$$

(b) Similarly $\hat{\theta}_{MLE} = \frac{X_{(n)}}{2}$.

(c) $\hat{\theta}_{MLE}$ is such that $X_{(n)} - 1 \leq \hat{\theta}_{MLE} \leq X_{(1)} + 1$.

(d) $\hat{\theta}_{MLE}$ is such that $X_{(n)} - 1 \leq \hat{\theta}_{MLE} < X_{(1)}$

(9) $X_1, \ldots, X_n \overset{iid}{\sim} U(-\theta, 2\theta)$.

$$E(X_1) = \frac{1}{3\theta} \int_{-\theta}^{2\theta} x_1 dx_1 = \theta/2.$$

$$E\left(\sum_{i=1}^{n} X_i\right) = \frac{n\theta}{2} \Rightarrow E(2\bar{X}) = \theta.$$

So $\delta(\underline{X}) = 2\bar{X}$ is unbiased for θ.

(10) $X_1, \ldots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.

We know $E(\bar{X}) = \mu$, $E(S^2) = \sigma^2$

Also $\bar{X} \sim N(\mu, \sigma^2)$, $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.

$$E\left(\frac{1}{S}\right) = \frac{\sqrt{n-1}}{\sigma} E\left(\frac{1}{\sqrt{\frac{(n-1)S^2}{\sigma^2}}}\right)$$

$$= \frac{\sqrt{n-1}}{\sigma} E\left(\frac{1}{\sqrt{Y}}\right), \quad Y \sim \chi^2_{n-1}$$

$$E\left(\frac{1}{S}\right) = \frac{\sqrt{n-1}}{\sigma} \int_0^\infty \frac{y^{-1/2}\, y^{\frac{n-1}{2}-1}\, e^{-y/2}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}}\, dy.$$

$$= \frac{\sqrt{n-1}}{\sigma} \int_0^\infty \frac{y^{\frac{n-2}{2}-1}\, e^{-y/2}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}}\, dy$$

Take $t = y/2$ so we get

$$\frac{\sqrt{n-1}}{\sigma} \int_0^\infty \frac{t^{\frac{n-2}{2}-1}\, e^{-t}\, 2^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}}\, dy.$$

$$= \frac{\sqrt{n-1}}{\sigma} \frac{\Gamma\left(\frac{n-2}{2}\right) 2^{\frac{n-2}{2}}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n}{2}-\frac{1}{2}}} = \frac{\sqrt{n-1}}{\sigma} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{n-1}{2}\right)}$$

So $$E\left(\frac{\sqrt{2}\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{n-1}\,\Gamma\left(\frac{n-2}{2}\right)} \frac{1}{S}\right) = \frac{1}{\sigma}$$

Then $T(\underline{X}) = \dfrac{\sqrt{2}\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{n-1}\,\Gamma\left(\frac{n-2}{2}\right)} \dfrac{1}{S}$ is unbiased for $\dfrac{1}{\sigma}$.

$$E\left(\bar{X}\, T(\underline{X})\right) = E(\bar{X})\, E\left(T(\underline{X})\right) = \frac{\mu}{\sigma}$$

($\because$ $\bar{X}$ & S are indep)

So $\dfrac{\sqrt{2}\,\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{n-1}\,\Gamma\left(\frac{n-2}{2}\right)} \dfrac{\bar{X}}{S}$ is unbiased for μ/σ.

For (ii) $(\mu + b\sigma)$ do yourself.

(11) Sample is given $3.3, -0.3, -0.6, -0.9$.

$\sigma = 3$. So σ is known. $\alpha = 0.1$

$n = 4$. $\bar{x} = 0.375$

So 90 % C.I. for μ is

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

We have $z_{0.05} = 1.654$.

$$= \left(0.375 - \frac{1.654 \times 3}{2},\ 0.375 + \frac{1.654 \times 3}{2}\right).$$

(12) $n = 100$, $S = 0.01$, $\alpha = 0.05$. Assuming the sample is coming from a normal population 95 % C.I. (ii) for σ is

$$\left(\sqrt{\frac{99 \times 0.01 \times 0.01}{\chi^2_{0.025,99}}},\ \sqrt{\frac{99 \times (0.01)^2}{\chi^2_{0.975,99}}}\right) \text{———} (*)$$

$\chi^2_{0.025,99} = 128.422$, $\chi^2_{0.975,99} = 73.361$

Substituting all the values in (*) we get CI.

(13)

$n_1 = 36,\quad n_2 = 49,\quad \sigma_1 = 1.2,\quad \sigma_2 = 1.8$

$\bar{x}_1 = 5,\quad \bar{x}_2 = 3.5,\quad \alpha = 0.1$

we know $\quad \bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2,\ \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$

So 90% C.I. for $(\mu_1 - \mu_2)$ is

$$\left(\bar{X}_1 - \bar{X}_2 - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}},\ \bar{X}_1 - \bar{X}_2 + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right) \quad (*)$$

$z_{\alpha/2} = z_{0.05} = 1.654.$

Substituting all the values in (*) we get CI.