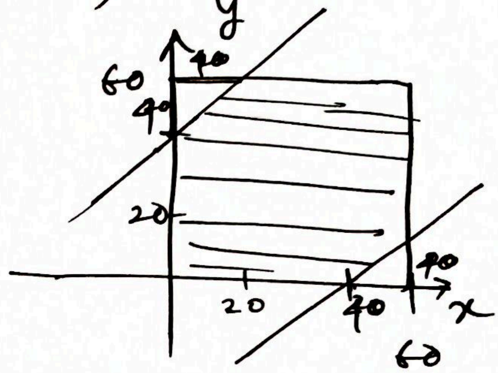
(1) X - denote the time of arrival of boy

$Y \to$ denote the time of arrival of girl.

Then $X \sim U(0, 60)$, $Y \sim U(0, 60)$

$$P(|X-Y| \le 20) = P(-20 < X-Y < 20)$$



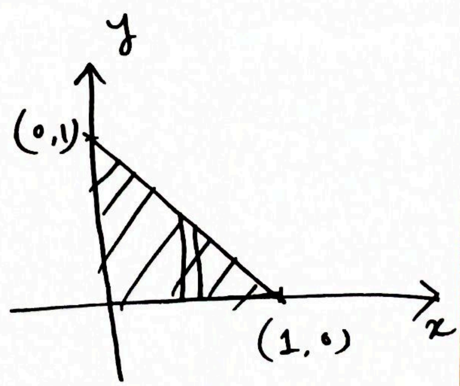
$$= \frac{60 \times 60 - 2 \times \frac{1}{2} \times 40 \times 40}{3600}$$

$$= 5/9.$$

So the required probability is $5/9$.

(2) The joint density is given as

$$f_{X,Y}(x,y) = \begin{cases} K(1-x-y), & x>0,\ y>0,\ x+y<1 \\ 0, & o/w. \end{cases}$$



$$\int_{x=0}^{1} \int_{y=0}^{1-x} f_{X,Y}(x,y)\, dy\, dx = 1$$

$$= \int_0^1 \left[\int_0^{1-x} K(1-x-y)\, dy \right] dx = 1 \quad \Rightarrow K = 6.$$

The marginal pdf of x is

$$f_X(x) = \int_{y=0}^{1-x} 6(1-x-y)\,dy \qquad 0<x<1$$

$$= 3(1-x)^2, \quad 0<x<1$$

$$f_X(x) = \begin{cases} 3(1-x)^2, & 0<x<1 \\ 0, & o/w. \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{x=0}^{1-y} 6(1-x-y)\,dx, & 0<y<1 \\ 0, & o/w \end{cases}$$

$$= \begin{cases} 3(1-y)^2, & 0<y<1 \\ 0, & o/w. \end{cases}$$

$$E(X) = \int_0^1 3x(1-x)^2\,dx = \frac{1}{4}.$$

$$E(Y) = \int_0^1 3y(1-y)^2\,dy = 1/4.$$

$$E(XY) = \int_0^{1}\int_0^{1-x} 6(1-x-y)\cdot xy\,dy\,dx = \frac{1}{20}$$

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{20} - \frac{1}{16} = \frac{-5+4}{80} = -\frac{1}{80}.$$

$$\sigma_X^2 = E(X^2) - E(X)^2 = \frac{1}{10} - \frac{1}{16} = \frac{16-10}{160} = \frac{6}{160} = \frac{3}{80}$$

$$\sigma_Y^2 = \frac{3}{80}$$

$$\rho_{X,Y} = \frac{-\frac{1}{80}}{\sqrt{\frac{3}{80} \times \frac{3}{80}}} = -\frac{1}{80} \times \frac{80}{3} = -\frac{1}{3}.$$

(3) (a) we have $c \sum_{x=0}^{3} \sum_{y=1}^{4} (3x+4y) = 1 \Rightarrow c = \frac{1}{232}$

(b) The marginal p.m.f of X

$$f_X(x) = \sum_{y=1}^{4} \frac{1}{232}(3x+4y), \quad x = 0,1,2,3$$

$$= \frac{1}{232}\left[12x + 4(1+2+3+4)\right]$$

$$= \frac{1}{232}(12x+10) = \frac{1}{58}(3x+10)$$

$$f_X(x) = \begin{cases} \frac{1}{58}(3x+10), & x = 0,1,2,3 \\ 0 & \text{o/w.} \end{cases}$$

u^{ly}

$$f_Y(y) = \begin{cases} \sum_{x=0}^{3} \frac{1}{232}(3x+4y), & y = 1,2,3,4 \\ 0 & \text{o/w.} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{116}(9+8y), & y = 1,2,3,4 \\ 0, & o/w. \end{cases}$$

(c)

$$P(X \geq 2 \mid Y \leq 3) = \frac{P(X \geq 2, Y \leq 3)}{P(Y \leq 3)}$$

$$= \frac{\frac{1}{232} \sum_{x=2}^{3} \sum_{y=1}^{3} (3x+4y)}{\sum_{y=1}^{3} \frac{1}{116}(9+8y)} =$$

$$P(Y=2 \mid X=3) = \frac{P(Y=2, X=3)}{P(X=3)} = \frac{P(X=3, Y=2)}{P(X=3)}$$

$$= \frac{\frac{1}{232}(9+8)}{\frac{1}{58}(9+10)}$$

(4) ⓐ The joint p.m.f is given as

X \ Y	-1	0	1	$f_X(x)$
0	0	1/3	0	1/3
1	1/3	0	1/3	2/3
$f_Y(y)$	1/3	1/3	1/3	1

$$f_X(x) = \begin{cases} \frac{1}{3}, & x = 0 \\ 2/3 & x = 1 \\ 0, & o/w \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{3}, & y = -1,0,1 \\ 0, & o/w. \end{cases}$$

(b) $E(X) = 0\cdot\frac{1}{3} + 1\cdot\frac{2}{3} = 2/3$

$E(Y) = -1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = 0$

$$E(XY) = \sum_{x=0,1}\sum_{y=-1,0,1} xy\, f_{X,Y}(x,y)$$

$$= 0\cdot(-1)\cdot 0 + 0\cdot 0\cdot\frac{1}{3} + 0\cdot(1)\cdot 0 + 1\cdot(-1)\frac{1}{3} + 1\cdot 0\cdot\frac{1}{3} + 1\cdot 1\cdot\frac{1}{3} = 0$$

So $cov(X,Y) = E(XY) - E(X)E(Y)$

$= 0 - 0 = 0.$

$\rho_{XY} = 0.$

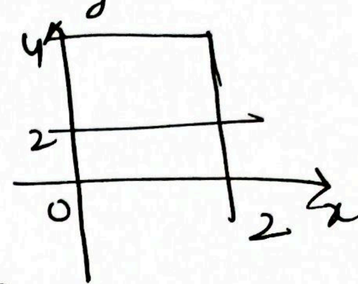
(c) We have $f_X(0) = \frac{1}{3}$, $f_Y(0) = \frac{1}{3}$, $f_{X,Y}(0,0) = \frac{1}{3}$

$\neq f_X(0)\, f_Y(0).$

So X and Y aree not independent.

(5) Do yourself.

(6) $$f_{X,Y}(x,y) = \begin{cases} \dfrac{6-x-y}{8}, & 0<x<2,\ 2<y<4 \\ 0, & o/w. \end{cases}$$



(a) The marginal density of X is

$$f_X(x) = \begin{cases} \displaystyle\int_2^4 \left(\frac{6-x-y}{8}\right) dy, & 0<x<2 \\ 0, & o/w. \end{cases}$$

$$= \begin{cases} \dfrac{6-2x}{8}, & 0<x<2 \\ 0, & o/w. \end{cases}$$

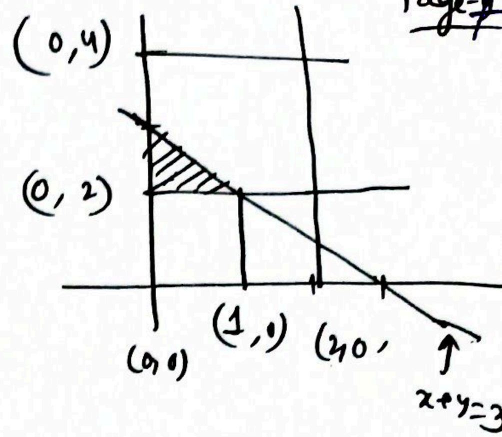
||^y

$$f_Y(y) = \begin{cases} \displaystyle\int_0^2 \frac{6-x-y}{8}\, dx, & 2<y<4 \\ 0, & o/w \end{cases} = \begin{cases} \dfrac{5-y}{4}, & 2<y<4 \\ 0, & o/w. \end{cases}$$

(b) $$P(X<1, Y<3) = \int_{x=0}^{1}\int_{y=2}^{3} \frac{6-x-y}{8}$$

$$= \frac{1}{16}\int_0^1 (7-2x)\, dx = \frac{3}{8}.$$

$$P(X+Y<3) = \int_{x=0}^{1}\int_{y=2}^{3-x}\left(\frac{6-x-y}{8}\right) dy\, dx$$



$$= \frac{1}{16}\int_0^1 (x^2-8x+7)\, dx = 5/24.$$

The conditional p.d.f of X given Y=y is $(2<y<4)$.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{8}(6-x-y)}{\frac{1}{4}(5-y)}, \quad 0<x<2$$

So for $2<y<4$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{6-x-y}{2(5-y)}, & 0<x<2 \\ 0, & o/w. \end{cases}$$

So $$P(X<1|Y=3) = \int_0^1 f_{X|Y}(x|3)\, dx = \int_0^1 \frac{6-x-3}{2(5-3)}\, dx$$

$$= \frac{1}{4}\int_0^1 (3-x)\, dx = 5/8.$$

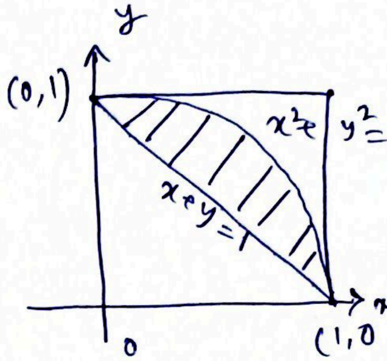
Now $$P(X<1 \mid Y<3) = \frac{P(X<1,\, Y<3)}{P(Y<3)}$$

$$P(Y<3) = \int_2^3 \frac{5-y}{4}\, dy = 5/8$$

So $$P(X<1 \mid Y<3) = \frac{3}{8} \Big/ 5/8 = 3/5.$$

(7) Let X and Y be random variables denoting the two number choosen from the interval (0,1). Then

$X \sim U(0,1)$, $Y \sim U(0,1)$



the joint pdf of (X,Y) is

$$f(x,y) = \begin{cases} 1, & 0<x<1,\ 0<y<1 \\ 0, & o/w. \end{cases}$$

Let $A = \{(x,y) : x+y>1,\ x^2+y^2<1\}$.

we have to find $P((X,Y) \in A)$

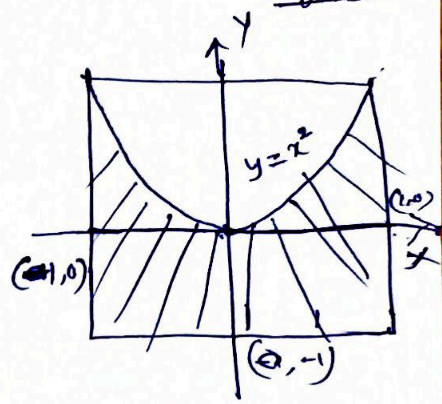
$$P((X,Y)\in A) = \iint_A f(x,y)\,dx\,dy = \int_{x=0}^{1}\int_{y=1-x}^{\sqrt{1-x^2}} dx\,dy = \left(\frac{\pi}{4}-\frac{1}{2}\right)$$

The area of shaded region is $\left(\frac{\pi}{4}-\frac{1}{2}\right)$

$\left(\frac{1}{4}\times \text{area of circle} - \text{area of triangle}\right)$

(8) In this case again $X \sim U(-1,1)$, $Y \sim U(-1,1)$

$$f_{XY}(x,y) = \begin{cases} \frac{1}{4}, & -1<x<1,\ -1<y<1 \\ 0, & o/w. \end{cases}$$



Roots are real iff

$$4x^2 - 4y \geqslant 0 \quad \text{i.e. } x^2 \geqslant y$$

$$A = \{(x,y) \in \mathbb{R}^2 : -1<x<1,\ -1<y<1 \ \& \ x^2 \geqslant y\}.$$

$$\text{Area of } A = 2 + 2\int_0^1 x^2\,dx = \frac{8}{3}.$$

So the required prob is $\dfrac{\frac{8}{3}}{4} = \dfrac{2}{3}$.

9) We have $f_{X,Y}(x,y) = \begin{cases} k(1-x-y), & x>0,\ y>0,\ x+y<1 \\ 0, & o/w \end{cases}$

$k = 6$. The marginal of X & Y are

$$f_X(x) = \begin{cases} 3(1-x)^2, & 0<x<1 \\ 0, & o/w \end{cases}$$

$$f_Y(y) = \begin{cases} 3(1-y)^2, & 0<y<1 \\ 0, & o/w. \end{cases}$$

So ~~to~~ for a given $y \in (0,1)$ the condition of X given $Y=y$

$$f_{X|Y}(x|Y=y) = \begin{cases} \frac{6(1-x-y)}{3(1-y)^2}, & 0<x<1-y \\ 6, & o/w. \end{cases}$$

$$E(X|Y=\tfrac{1}{2}) = \int_0^{\frac{1}{2}} \frac{x\,6(1-x-\frac{1}{2})}{3\cdot(\frac{1}{2})^2}\,dx$$

$$= 4\int_0^{1/2} x(1-2x)\,dx = -\frac{1}{6}.$$

For a given $x \in (0,1)$, the conditional pdf of

~~X~~ ~~X~~ given $X = x$

$$f_{Y|X}(y|X=x) = \begin{cases} \dfrac{6(1-x-y)}{3(1-x)^2}, & 0 < y < 1-x. \\ 0, & o/w \end{cases}$$

$$E(Y|X=x) = \int_0^{1-x} \frac{y\,6(1-x-y)}{3(1-x)^2}\, dy$$

$$E(Y|X=1/2) = \int_0^{1/2} \frac{6y(1-y-\frac{1}{2})}{3(\frac{1}{2})^2}\, dy = -1/6.$$

$$\text{Var}(X|Y=1/2) = \int_0^{1/2} \frac{(x+\frac{1}{6})^2\, 6(1-x-1/2)}{3(\frac{1}{2})^2}\, dx$$

$$\text{var}(Y|X=1/2) = \int_0^{1/2} \frac{(y+\frac{1}{6})^2\, 6(1-1/2-y)}{3(\frac{1}{2})^2}\, dy.$$

10 We have

$$f_{X,Y}(x,y) = P(X=x, Y=y) = \begin{cases} c(3x+4y), & x=0,1,2,3; \\ & y=1,2,3,4 \\ 0, & o/w. \end{cases}$$

Here $c = \frac{1}{232}$

We have

$$f_X(x) = P(X=x) = \begin{cases} \frac{1}{58}(3x+10), & x=0,1,2,3 \\ 0, & o/w \end{cases}$$

$$f_Y(y) = P(Y=y) = \begin{cases} \frac{1}{116}(9+8y), & y=1,2,3,4 \\ 0, & o/w. \end{cases}$$

For a given y,

$$E(X|Y=y) = \sum_{x=0}^{3} x \, P(X=x|Y=y)$$

$$= \sum_{x=0}^{3} x \frac{P(X=x, Y=y)}{P(Y=y)}$$

In particular

$$E(X|Y=1) = \sum_{x=0}^{3} \frac{x P(X=x, Y=1)}{P(Y=1)} = \sum_{x=0}^{3} \frac{\frac{x}{232}(3x+4)}{\frac{17}{116}}$$

$$= \sum_{x=0}^{3} \frac{17}{2} x(3x+4)$$

For a given x

$$E(Y|X=x) = \sum_{y=1}^{4} y \frac{P(Y=y, X=x)}{P(X=x)}$$

$$= \sum_{y=1}^{4} \frac{y \frac{1}{232}(3x+4y)}{\frac{1}{58}(3x+10)}$$

In particular $x=0$

$$E(Y|X=0) = \sum_{y=1}^{4} \frac{y}{4} \cdot \frac{4y}{10} = \sum_{y=1}^{4} \frac{y^2}{10}.$$

(11) The joint density of X & Y is given as

$$f_{X,Y}(x,y) = \begin{cases} c\, e^{-(x+y)}, & y > x > 0 \\ 0, & \text{ow} \end{cases}$$

We know

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$$\Rightarrow C\int_{0}^{\infty}\int_{x}^{\infty} Ce^{-(x+y)}\,dy\,dx = 1$$

$$\Rightarrow C = 2$$

$$f_X(x) = 2\int_{x}^{\infty} e^{(x+y)}\,dy \quad x > 0$$

$$f_X(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{ow} \end{cases}$$

Now for a fixed $x > 0$

$$f_{Y|X}(y|x) = \begin{cases} e^{-(y-x)}, & y > x \\ 0, & \text{ow} \end{cases}$$

So $E(Y|X=2) = \int_2^\infty y e^{-(y-2)} dy$

$= \int_0^\infty (t+2) e^{-t} dt = \Gamma(2)+2\Gamma(1)$
$= 3.$

(12) The joint p.d.f. of X & Y is

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$$

$$= \begin{cases} \alpha y e^{-y(x+\alpha)}, & x>0,\ y>0 \\ 0, & o/w. \end{cases}$$

Marginal of X is

$$f_X(x) = \int_0^\infty \alpha y e^{-y(x+\alpha)} dy$$

$$= \frac{\alpha}{(x+\alpha)^2} \Gamma(2) \quad , x > 0$$

$$f_X(x) = \begin{cases} \frac{\alpha}{(x+\alpha)^2} & , x > 0 \\ 0, & o/w \end{cases}$$

Now for a fixed $x > 0$

$$f_{Y|X}(y|x) = \begin{cases} \frac{\alpha y e^{-y(x+\alpha)}}{\alpha/(x+\alpha)^2} & , y > 0 \\ 0 & o/w \end{cases}$$

$$= \begin{cases} y(x+\alpha)^2 e^{-y(x+\alpha)} & , y > 0 \\ 0, & o/w. \end{cases}$$