Solution of tutorial-1

(1) Given that $\mathcal{F}$ is an algebra. So

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

Let $A_1, A_2, \ldots, A_n \in \mathcal{F}$ then repeated application of (iii) we have $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathcal{F}$.

If $A_i, i = 1(1)n \in \mathcal{F} \Rightarrow A_i^c \in \mathcal{F} \quad i = 1(1)n$.

$\Rightarrow \bigcup_{i=1}^{n} A_i^c \in \mathcal{F} \Rightarrow \left(\bigcup_{i=1}^{n} A_i^c\right)^c \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{n} A_i \in \mathcal{F}$.

(2) $f: X \to Y$, $\mathcal{F}$ is a σ-algebra of subsets of Y.

$f^{-1}(\mathcal{F}) = \{ f^{-1}(A) : A \in \mathcal{F} \}$

Now (i) $f^{-1}(Y) = X \in f^{-1}(\mathcal{F})$

(ii) Let $A \in f^{-1}(\mathcal{F})$ then $\exists\, B \in \mathcal{F}$ s.t $f^{-1}(B) = A$

Now $A^c = \left(f^{-1}(B)\right)^c = f^{-1}(B^c) \in f^{-1}(\mathcal{F})$ $[\because B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}]$.

So $A^c \in f^{-1}(\mathcal{F})$

(iii) Let $\{A_n\}$ be an arbitrary sequence in $f^{-1}(\mathcal{F})$ then $\exists\, \{B_n\}$ in $\mathcal{F}$ s.t $f^{-1}(B_n) = A_n$.

Since $\{B_n\} \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$.

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} f^{-1}(B_n) = f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) \in f^{-1}(\mathcal{F})$$

$\Rightarrow f^{-1}(\mathcal{F})$ is a σ-algebra of subsets of X.

(3) $\mathcal{A}$ is a σ-algebra of subsets of X. $Y \subset X$.

$\mathcal{D} = \{A \cap Y : A \in \mathcal{A}\}$

(i) $X \cap Y \in \mathcal{D}$ $\because X \in \mathcal{A}$. so $Y \in \mathcal{D}$

(ii) Let $B \in \mathcal{D}$ we have to show $(Y - B) \in \mathcal{D}$

$$(Y - B) = (X \cap Y) - (A \cap Y) \quad \because \left[\begin{matrix} B \in \mathcal{D} \text{ so } \exists\, A \in \mathcal{A} \\ \text{s.t } B = A \cap Y \end{matrix}\right]$$

$$= (X - A) \cap Y = A^c \cap Y \in \mathcal{D}$$

$\because A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$.

so $(Y - B) \in \mathcal{D}$.

(iii) Let $\{B_n\} \in \mathcal{D}$ then $\exists \{A_n\} \in \mathcal{A}$ s.t

$B_n = A_n \cap Y$.

$$\bigcup B_n = \bigcup (A_n \cap Y) = \left(\bigcup A_n\right) \cap Y \in \mathcal{D}$$

$(\because \{A_n\} \in \mathcal{A} \Rightarrow \bigcup A_n \in \mathcal{A})$

$\Rightarrow \mathcal{D}$ is a σ-algebra on Y.

(4) Given that $\Omega \in \mathcal{F}$ and $A, B \in \mathcal{F}$ then $A - B = A \cap B^c \in \mathcal{F}$

to prove $\mathcal{F}$ is an algebra we have to prove

(i) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(ii) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

Let $A \in \mathcal{F}$, then $\Omega, A \in \mathcal{F}$ then by the given 1st conditi.

$A^c \cap \Omega \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

Now if $A, B \in \mathcal{F} \Rightarrow A^c, B^c \in \mathcal{F}$

Then by given 2nd condition $A^c \cap B^c \in \mathcal{F}$

Then by (i) $(A^c \cap B^c)^c \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

$\Rightarrow$ $\mathcal{F}$ is an algebra.

(5) Given that $\mathcal{F}$ is an algebra and if $\{A_n\}$ is an increasing sequn of sets in $\mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

since $\mathcal{F}$ is an algebra so (i) $X \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

we want to show $\{A_n\}$ be a arbitrary sequ^n in

$\mathcal{F} \Rightarrow \bigcup A_n \in \mathcal{F}$.

Given a sequ^n $\{B_n\} \in \mathcal{F}$ define.

$$A_n = \bigcup_{k=1}^{n} B_k \quad \text{for all} \quad n \in \mathbb{N}.$$

Then $\{A_n\}$ is increasing sequence

$$\text{and} \quad \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

By the given condition $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$

$\Rightarrow$ $\mathcal{F}$ is a σ-algebra.

(6) $\mathcal{F}$ = set all finite and co-finite subset of Ω.

(i) $\Omega^c = \phi$ is finite $\Rightarrow$ Ω is cofinite $\Rightarrow \Omega \in \mathcal{F}$.

(ii) Suppose $A \in \mathcal{F}$. Then A is finite or co-finite

If A is finite then A^c is co-finite $\Rightarrow A^c \in \mathcal{F}$.

If A is cofinite then A^c is finite $\Rightarrow A^c \in \mathcal{F}$.

(iii) $A, B \in \mathcal{F}$. (a) If A, B is finite then $A \cup B$ is finite
$A \cup B \in \mathcal{F}$

(b) Consider A is finite B is co-finite

Now $(A \cup B)^c = A^c \cap B^c$. Since B is co-finite $\Rightarrow B^c$ is finite $\Rightarrow A^c \cap B^c$ is finite since $A^c \cap B^c \subset B^c \Rightarrow (A \cup B)^c \in \mathcal{F}$

$\Rightarrow$ $A \cup B$ co finite $\Rightarrow$ $A \cup B \in \mathcal{F}$.

(7) $I_n = \{x \in \mathbb{R} : 0 < x < \frac{1}{n}\}$

we want to show that $\bigcap_{n=1}^{\infty} I_n = \phi$.

If $\bigcap_{n=1}^{\infty} I_n \neq \phi$ then $x \in \bigcap_{n=1}^{\infty} I_n$ i.e. $0 < x < \frac{1}{n}$ $\forall n$.

i.e. $\frac{1}{n} > x$ $\forall n \in \mathbb{N}$. ——— $(*)$

Now $x > 0$ then $\exists\, n_0 \in \mathbb{N}$ s.t. $n_0 x > 1 \Rightarrow x > \frac{1}{n_0}$.

Which is contradiction to $(*)$ $\Rightarrow \bigcap_{n=1}^{\infty} I_n = \phi$.

(8) Given $\mathcal{G} \subseteq \mathcal{H}$. By defination of $\sigma(\mathcal{H})$ we have

$\mathcal{H} \subset \sigma(\mathcal{H}) \Rightarrow \mathcal{G} \subseteq \mathcal{H} \subset \sigma(\mathcal{H})$

Again $\sigma(\mathcal{G})$ is the smallest σ-field generated by $\mathcal{G}$

So $\sigma(\mathcal{G}) \subseteq \sigma(\mathcal{H})$.

(9) $\mathcal{F}$ is a σ-algebra. $\sigma(\mathcal{F})$ is the smallest σ-algebra containing $\mathcal{F}$. So $\sigma(\mathcal{F}) \subseteq \mathcal{F}$ ($\sigma(\mathcal{F})$ is smallest).

Again $\mathcal{F}$ contains in $\sigma(\mathcal{F})$ so $\mathcal{F} \subseteq \sigma(\mathcal{F})$

$$\Rightarrow \sigma(\mathcal{F}) = \mathcal{F}.$$

$\mathcal{C}$ be any collection then $\sigma(\mathcal{C})$ is the smallest σ-field contains $\mathcal{C}$. So

$$\sigma(\sigma(\mathcal{C})) = \sigma(\mathcal{C}) \quad (\text{by part 1})$$

(10) and (11) verify the axioms of probability.