Tutorial - 5 Solution

1

X be a r.v. with $E(X) = 3 = \mu$.

$E(X^2) = 13 \Rightarrow \sigma^2 = E(X^2) - (E(X))^2 = 13 - 9 = 4$

$Var(X) = 4$.

$P(-2 < X < 8) = P(-2-3 < X-3 < 8-3)$

$= P(|X-3| < 5) = 1 - P(|X-3| \geq 5)$

By chebyshev inequality

$P(|X-3| \geq 5) \leq \frac{4}{25}$

$\Rightarrow 1 - P(|X-3| \geq 5) \geq 1 - \frac{4}{25} = \frac{21}{25}$.

2

X be a r.v. with m.g.f.

$M(t) = \frac{e^{-2t}}{8} + \frac{e^{-t}}{4} + \frac{e^{2t}}{8} + \frac{e^{3t}}{2}$

$= P(X=-2)e^{-2t} + P(X=-1)e^{-1 \cdot t} + P(X=2)e^{2 \cdot t} + P(X=3)e^{3 \cdot t}$

$P(X=-2)=\frac{1}{8}$, $P(X=1)=\frac{1}{4}$, $P(X=2)=\frac{1}{8}$, $P(X=3)=\frac{1}{2}$

$$P(X^2=4) = P(X=-2)+P(X=2)$$

$$= \frac{1}{8}+\frac{1}{8} = \frac{1}{4}.$$

3 If $t>0$, $g(x)=e^{tx}$ is positive, increasing in x.

Hence $$P(X\geq a) = P(e^{tX}\geq e^{ta})$$

$$\leq \frac{E(e^{tX})}{e^{ta}} \quad \text{[By Marckov's inequality]}$$

$$= e^{-at}M(t).$$

If $t<0$, then $h(x)=e^{tx}$ is positive, decreasing and hence

$$P(X\leq a) = P(e^{tX}\geq e^{at}) = e^{-at}M(t)$$

(4)

$S_X = \{-2, -1, 0, 1, 2, 3\}$

(a) $\sum_{x \in S_X} f_X(x) = 1 \Rightarrow 6K + 0.6 = 1$

$\Rightarrow K = \frac{0.4}{6} = \frac{1}{15}$

(b) $P(X < 2) = P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1)$

$= \frac{1}{10} + \frac{1}{15} + \frac{1}{5} + \frac{2}{15} = \frac{1}{2}$

$P(-2 < X < 2) = P(X = -1) + P(X = 0) + P(X = 1)$

$= \frac{1}{15} + \frac{1}{5} + \frac{2}{15} = 2/5$

(c)

$$F_X(x) = \begin{cases} 0, & x < -2 \\ \frac{1}{10}, & -2 \le x < -1 \\ \frac{1}{6}, & -1 \le x < 0 \\ \frac{11}{30}, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ 5/4, & 2 \le x < 3 \\ 1, & x \geq 3. \end{cases}$$

(d) $E(X) = \sum_{x \in S_X} x f_X(x) = -2 \cdot \frac{1}{10} + \frac{(-1)}{15} + 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{15} + 2 \times \frac{3}{10} + 3 \times \frac{1}{\cancel{0}5}$

$= \frac{16}{15}$

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = 4\cdot\frac{1}{10} + \frac{1}{15} + 0 + \frac{2}{15} + 4\times\frac{3}{10} + 9\times\frac{1}{5}$$

$$= \frac{4}{10} + \frac{1}{15} + \frac{2}{15} + \frac{12}{10} + \frac{9}{5} = \frac{18}{5}$$

$$Var(X) = \frac{18}{5} - \left(\frac{16}{15}\right)^2 = \frac{554}{225}$$

(e) $$M_X(t) = E(e^{tX}) = \sum_{x\in S_X} e^{tx} f_X(x)$$

$$= \left[\frac{e^{-2x}}{10} + \frac{e^{-x}}{15} + \frac{1}{15} + \frac{2e^{x}}{15} + \frac{3e^{2x}}{10} + \frac{e^{3x}}{5}\right]$$

5 (a) $$\int_a^b f(x)\,dx = 1$$

$$\Rightarrow p\int_0^1 x\,dx + p\int_1^2 dx + p\int_2^3 (3-x)\,dx = 1$$

$$\Rightarrow p = \frac{1}{2}$$

So the pdf is $$f(x) = \begin{cases} x/2, & 0 < x \le 1 \\ 1/2 & 1 < x \le 2 \\ \frac{3-x}{2}, & 2 < x < 3 \\ 0, & o/w. \end{cases}$$

(b) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 \frac{x^2}{2}\,dx + \int_1^2 \frac{x}{2}\,dx + \int_2^3 \frac{x(3-x)}{2}\,dx$

$= 3/2$

~~$E(x^2) = \frac{8}{3}$, So $Var(x) = \frac{8}{3} - \frac{9}{4}$~~

$E(x^2) = \int_0^1 \frac{x^3}{2}\,dx + \int_1^2 \frac{x^2}{2}\,dx + \int_2^3 \frac{x^2(3-x)}{2}\,dx$

$= 8/3$

$Var(x) = \frac{8}{3} - \frac{9}{4} = 5/12$

$M_X(t) = E(e^{tX}) = \int_0^1 e^{tx}\,\frac{x}{2}\,dx + \int_1^2 \frac{e^{tx}}{2}\,dx + \int_2^3 e^{tx}\,\frac{(3-x)}{2}\,dx$

6 Let $h(c) = E(X-c)^2 = c^2 - 2cE(X) + E(X^2)$

$h'(c) = 2c - 2E(X)$ & $h''(c) = 2 > 0$

~~It~~ follows that $h(c)$ has minimum at

$$C = E(X) = \mu$$

$$\Rightarrow \quad E(X-c)^2 \geq E(X-\mu)^2 \quad \forall\, c \in \mathbb{R}$$

7 Consider $\Delta = E(|X-c|) - E(|X-m|)$

Case-1 $\quad -\infty < c < m.$

$$\Delta = \int_{-\infty}^{c} (c-x)\, f_X(x)\, dx + \int_{c}^{\infty} (x-c)\, f_X(x)\, dx$$

$$- \int_{-\infty}^{m} (m-x) f_X(x)\, dx - \int_{m}^{\infty} (x-m) f_X(x)\, dx$$

$$= 2C F_X(C) - C + 2 \int_{C}^{m} x f_X(x)\, dx \quad \left(\text{using } F_X(m) = \tfrac{1}{2}\right)$$

$$\geq 2C F_X(x) - C + 2C \left[F_X(m) - F_X(C)\right] = 0$$

$$\left[\text{Again using } F_X(m) = \tfrac{1}{2}\right]$$

<u>Case-II</u> $\quad -\infty < m < c < \infty$

$$\Delta = 2C F_X(C) - C - 2 \int_{m}^{C} x f_X(x)\, dx \geq 0$$

8 a

$$E(\psi(X)) = \int_0^\infty \psi(x)\, f_X(x)\, dx$$

$$= \int_0^\infty \int_0^x h(t)\, f_X(x)\, dt\, dx$$

$$= \int_0^\infty \int_t^\infty h(t)\, f_X(x)\, dx\, dt$$ [change of order of integration is allowed as integrand is non-negative]

$$= \int_0^\infty h(t) \int_t^\infty f_X(x)\, dx\, dt = \int_0^\infty h(t)\, P(X > t)\, dt.$$

(b) Take $h(t) = \alpha t^{\alpha-1} \quad t \in (0, \infty)$

$$E(X^\alpha) = \alpha \int_0^\infty t^{\alpha-1} P(X > t)\, dt.$$

(c) $F(t) \geq G(t) \quad \forall t > 0 \Rightarrow$

$$P(Y > t) \geq P(X > t) \quad \forall t > 0.$$

$$\Rightarrow E(Y^K) = K \int_0^\infty t^{K-1} P(Y > t)\, dt$$

$$\geq K \int_0^\infty t^{K-1} P(X > t)\, dt = E(X^K)$$

(Note that $F(0) = G(0) = 0 \Rightarrow S_X \,\&\, S_Y \subseteq (0, \infty)$)

9 Given $f_X(\mu+x) = f_X(\mu-x) \quad \forall x \in \mathbb{R}$

Let $Y_1 = X-\mu, \quad Y_2 = \mu - X$

Then $f_{Y_1}(y) = f_X(y+\mu) \quad \forall y \in \mathbb{R}$

$f_{Y_2}(y) = f_X(\mu-y)$

$\Rightarrow f_{Y_1}(y) = f_{Y_2}(y) \quad \forall y \in \mathbb{R}$

$\Rightarrow Y_1 \overset{d}{=} Y_2$ i.e. $X-\mu \overset{d}{=} \mu - X$

$E(X-\mu) = E(\mu - X) \Rightarrow E(X) = \mu.$

(10) We have $f_X(x) = f_X(-x)$

$\Rightarrow X \stackrel{d}{=} -X \Rightarrow E(X^3) = -E(X^3)$

$\Rightarrow E(X^3) = 0$

The given distn is symmetric about 0. So $E(X) = 0$, & $P(X > 0) = 1/2$.
