Soln - Tutorial - 11

① Given

$$X_1, \ldots, X_m \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$$
$$Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$$
indep.

Then
$$\bar{X} \sim N(\mu_1, \sigma_1^2/m)$$
$$\bar{Y} \sim N(\mu_2, \sigma_2^2/n)$$

$$\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}\right)$$

$$\frac{(m-1)S_1^2}{\sigma_1^2} \sim \chi^2_{m-1}, \quad \frac{(n-1)S_2^2}{\sigma_2^2} \sim \chi^2_{n-1}$$

$$\Rightarrow \frac{(m-1)S_1^2}{\sigma_1^2} + \frac{(n-1)S_2^2}{\sigma_2^2} \sim \chi^2_{m+n-2}$$

Hence
$$\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \Bigg/ \sqrt{\frac{(m-1)S_1^2}{\sigma_1^2} + \frac{(n-1)S_2^2}{\sigma_2^2}} \sim t_{m+n-2}$$

$$\Rightarrow \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\frac{(m-1)s_1^2}{\sigma_1^2}+\frac{(n-1)s_2^2}{\sigma_2^2}}} \sqrt{\frac{m+n-2}{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}} \sim t_{m+n-2}$$

② $X_1, \ldots, X_n, X_{n+1} \overset{iid}{\sim} N(\mu, \sigma^2)$

$\bar{X} \sim N(\mu, \sigma^2/n)$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ where $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$

$X_{n+1}-\bar{X} \sim N\left(0, \sigma^2\left(\frac{n+1}{n}\right)\right)$

So $$\frac{X_{n+1}-\bar{X}}{\sqrt{\sigma\left(\frac{n+1}{n}\right)}} \Bigg/ \sqrt{\frac{(n-1)s^2}{\sigma^2}\Big/ n-1} \sim t_{n-1}$$

$$\Rightarrow \left(\frac{X_{n+1}-\bar{X}}{S}\right)\sqrt{\frac{n}{n+1}} \sim t_{n-1}$$

③ $X_1, \ldots, X_m \overset{iid}{\sim} N(\mu_1, \sigma^2)$
$Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu_2, \sigma^2)$ } indep

$$\bar{X} \sim N\left(\mu_1, \frac{\sigma^2}{m}\right), \quad \bar{Y} \sim N\left(\mu_2, \sigma^2/n\right)$$

$$\alpha \bar{X} + \beta \bar{Y} \sim N\left(\alpha\mu_1 + \beta\mu_2 + \sigma^2\left(\frac{\alpha^2}{m} + \frac{\beta^2}{n}\right)\right)$$

$$\Rightarrow \frac{\alpha(\bar{X} - \mu_1) + \beta(\bar{Y} - \mu_2)}{\sigma\sqrt{\frac{\alpha^2}{m} + \frac{\beta^2}{n}}} \sim N(0,1)$$

$$\frac{(m-1)S_1^2}{\sigma^2} + \frac{(n-1)S_2^2}{\sigma^2} \sim \chi^2_{m+n-2}$$

$$\Rightarrow \frac{\alpha(\bar{X} - \mu_1) + \beta(\bar{Y} - \mu_2)}{\sqrt{\frac{(m-1)S_1^2 + (n-1)S_2^2}{m+n-2}}\sqrt{\frac{\alpha^2}{m} + \frac{\beta^2}{n}}} \sim t_{m+n-2}$$

(4) Let $X_1, \ldots, X_5$ denote the life of tires. Then

$$X_1, \ldots, X_5 \overset{iid}{\sim} N(38500, 2500^2)$$

$$\bar{X} \sim N\left(38{,}500, \frac{2500}{5}\right)$$

Required prob is

$$P(\bar{X} < 36{,}000)$$

$$= P\left(\frac{\bar{X} - 38500}{\sqrt{\frac{2500^2}{5}}} < \frac{3600 - 38500}{\sqrt{\frac{2500^2}{5}}}\right)$$

$$= P(Z < -\sqrt{5}) = P(Z < -2.234)$$

$$= 0.0129.$$

(5) Let the population random variable is X. Population mean is μ, population s.d. is σ.

Given $P\left(|\bar{X}-\mu| \leq 25\% \, \sigma\right) = 0.95$

$\Rightarrow P\left(|\bar{x}-\mu| \leq 0.25\sigma\right) = 0.95$

$\Rightarrow P\left(\left|\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right| \leq 0.25\sqrt{n}\right) = 0.95$

As the sample size is sufficiently large

$\Rightarrow P\left(|Z| \leq 0.25\sqrt{n}\right) = 0.95$

$\Rightarrow P\left(-0.25\sqrt{n} \leq Z \leq 0.25\sqrt{n}\right) = 0.95$

i.e. $P(Z \geq 0.25\sqrt{n}) = 0.025$

or $P(Z \leq -0.25\sqrt{n}) = 0.025$

From the table we have

$$0.25\sqrt{n} = 1.96$$

$$\Rightarrow \quad n \approx 61.$$

6 $X_1, \ldots, X_n \overset{iid}{\sim} \operatorname{Exp}(\mu, 1)$

we know the density of $X_{(1)}$ is

$$f_{X_{(1)}}(y) = \begin{cases} n e^{-n(y-\mu)}, & y > \mu \\ 0, & \text{o/w} \end{cases}$$

$$P(X_{(1)} > 3\mu) = \int_{3\mu}^{\infty} n e^{-n(y-\mu)} dy$$

$$= e^{-2n\mu}.$$

7 Given $X \sim N(68, 6.25)$

Let n be the sample size.

$\bar{X} \sim N(68, 6.25/n)$

Required prob is

$$P(\bar{X} - 68 \le 1) \geq 0.95$$

$$\Rightarrow P\left(\frac{\bar{X} - 68}{\sqrt{\frac{6.25}{n}}} \le \frac{\sqrt{n}}{6.25}\right) \geq 0.95$$

$$\Rightarrow P\left(Z \le \frac{\sqrt{n}}{6.25}\right) \geq 0.95$$

From the table value

$$\frac{\sqrt{n}}{6.25} = 1.65$$

$n \simeq 17.$

(8) Let $X_1, \ldots X_{25} \sim N(\mu, 6)$

Required prob is

$$P(s^2 > 9.105)$$

$$= P\left(\frac{24}{6} s^2 > \frac{24 \times 9.105}{6}\right)$$

$$= P(\chi^2_{24} > 36.42) = 0.05$$

From the chisquare table.

(9) Given $n_1 = 5$, $n_2 = 4$.

Required prob is

$$P\left(\frac{s_1^2}{s_2^2} < \frac{1}{5.2} \text{ or } > 6.5\right)$$

$$= 1 - P\left(\frac{1}{5.2} \leq \frac{s_1^2}{s_2^2} \leq 6.5\right)$$

$$= 1 - P\left(\frac{1}{5.2} \leq F_{4,3} \leq 6.25\right)$$

$$\left[\frac{4s_1^2}{\sigma^2} \sim \chi_4^2, \quad \frac{3s_2^2}{\sigma^2} \sim \chi_3^2\right]$$

$$= 1 - P\left(0.1923 < F_{4,3} \leq 6.25\right)$$

$$= 1 - \left[P\left(F_{4,3} \leq 6.25\right) - P\left(F_{4,3} \leq 0.1923\right)\right]$$

This values can be obtained from the table for F – distn

(18) $X_1, \ldots, X_5 \overset{iid}{\sim} N(2.5, 36)$

(a) $P(30 \leq S^2 \leq 40)$

$= P\left(\frac{30 \times 4}{36} \leq \chi^2_4 \leq \frac{40 \times 4}{36}\right) \quad \because \frac{4S^2}{36} \sim \chi^2_4$

$= P(3.3 \leq \chi^2_4 \leq 4.4)$

$= P(\chi^2_4 \geq 3.3) - P(\chi^2_4 \geq 4.4)$

$= 0.5 - 0.35 = 0.15$ (approx)

(From table).

We have $\bar{X} \sim N\left(2.5, \frac{36}{5}\right)$

(b) $P(1.3 \leq \bar{X} \leq 3.5,\ 30 \leq S^2 \leq 44)$

$= P(1.3 \leq \bar{X} \leq 3.5)\, P(30 \leq S^2 \leq 44)$

[$\bar{X}$ & S^2 are indep]

$$= 0.15\, P\left(1.3 \le \bar{X} \le 3.5\right)$$

$$= 0.15\, P\left(\frac{1.3-2.5}{6/\sqrt{5}} \le Z \le \frac{3.5-2.5}{6/\sqrt{5}}\right)$$

$$= 0.15\, P\left(-0.4472 \le Z \le 0.372\right)$$

$$= 0.15\left[\Phi(0.372) - \Phi(-0.4472)\right]$$

$$= 0.15\left[0.64431 - 0.32997\right]$$

$$= 0.15 \times 0.31441.$$

(ii) The joint dist^n of (X_1, X_2) is

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \dfrac{1}{\Gamma(n_1)\Gamma(n_2)\,\theta^{n_1+n_2}}\, x_1^{n_1-1} x_2^{n_2-1} e^{-\frac{x_1+x_2}{\theta}}, & x_1>0,\ x_2>0 \\ 0, & o/w \end{cases}$$

Here $S_{\underline{X}} = (0,\infty)^2$. Let

$$h_1(x_1, x_2) = Y_1 = X_1 + X_2 \quad , \quad h_2(X_1, X_2) = Y_2 = \frac{X_1}{X_1+X_2}$$

$$X_1 = Y_1 Y_2, \qquad X_2 = Y_1(1-Y_2)$$

$$J = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1$$

$y_1 > 0, \; 0 < y_2 < 1.$ So $S_{\underline{y}} = (0,\infty) \times (0,1)$

The joint pdf of (Y_1, Y_2) is

$$f_{\underline{y}}(y_1, y_2) = \frac{(y_1 y_2)^{n_1-1} (y_1(1-y_2))^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\theta^{n_1+n_2}} e^{-\frac{y_1 y_2 + y_1(1-y_2)}{\theta}} \, |-y_1| \times I_{(0,\infty)\times(0,1)}$$

$$= \left\{ \frac{e^{-y_1/\theta} y_1^{n_1+n_2-1}}{\Gamma(n_1+n_2)} I_{(0,\infty)}(y_1) \right\} \times$$

$$\left\{ \frac{1}{B(n_1, n_2)} y_2^{n_1-1} (1-y_2)^{n_2-1} I_{(0,1)}(y_2) \right\}$$

$$= f_{Y_1}(y_1)\, f_{Y_2}(y_2)$$

where $Y_1 \sim \text{Gamma}(n_1+n_2, \theta)$

$Y_2 \sim \text{Beta}(n_1, n_2)$

clearly Y_1 & Y_2 are indep.