Tutorial 11: Probability and Statistics (MAL403/IC105)

Indian Institute of Technology Bhilai

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1. Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$, respectively be independent samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$. Then what is the distribution of $\frac{\overline{X}-\overline{Y}-(\mu_1-\mu_2)}{\sqrt{\left[\frac{(m-1)S_1^2}{\sigma_1^2}+\frac{(n-1)S_2^2}{\sigma_2^2}\right]}}\sqrt{\frac{m+n-2}{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}}$.

2. Let $X_1, \ldots, X_n$ are random samples from $N(\mu, \sigma^2)$ and $\overline{X}$ and S^2 respectively, be the sample mean and sample variance. Let $X_{n+1} \sim N(\mu, \sigma^2)$ and assume that $X_1, X_2, \ldots, X_n, X_{n+1}$ are independent. Find the distribution of $\sqrt{\frac{n}{n+1}}\left(\frac{X_{n+1}-\overline{X}}{S}\right)$.

3. Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively. Also let α, β be two fixed real numbers. If $\overline{X}, \overline{Y}$ denote the corresponding sample means, what is the sampling distribution of $\frac{\alpha(\overline{X}-\mu_1)+\beta(\overline{Y}-\mu_2)}{\sqrt{\frac{(m-1)S_1^2+(n-1)S_2^2}{m+n-2}}\sqrt{\frac{\alpha^2}{m}+\frac{\beta^2}{n}}}$

4. A prototype automotive tire has a design life of 38,500 miles with a standard deviation of 2,500 miles. Five such tires are manufactured and tested. On the assumption that the actual population mean is 38,500 miles and the actual population standard deviation is 2,500 miles, find the probability that the sample mean will be less than 36,000 miles. Assume that the distribution of lifetimes of such tires is normal.

5. It is required to estimate the mean of a normal population using a sample sufficiently large, so that the probability will be 0.95 that the sample mean will not differ from population mean by more than 25% of the population standard deviation. How large should be the sample? (Use tables of normal distribution).

6. Let $X_1, X_2, \ldots, X_n$ be random sample taken from a shifted exponential distribution $Exp(\mu, 1)$ with the density function

$$f_Y(y) = \begin{cases} e^{-(x-\mu)}, & x > \mu, \\ 0, & \text{Otherwise} \end{cases}$$

   Find the value of $P(X_{(1)} > 3\mu)$.

7. Let a random sample of size n be drawn from a normal $N(68, 6.25)$ population. What should be the minimum size of the sample such that the difference of the sample mean and the population mean is not more than 1 with probability at least 0.95?

8. Find the probability that a random sample of 25 observations from a normal population with variance $\sigma^2 = 6$ will have sample variance greater than 9.105.

9. Let S_1^2 and S_2^2 be the sample variances from two independent samples of sizes $n_1 = 5$ and $n_2 = 4$ from two normal populations having the same unknown variance . Find (approximately) the probability that $\frac{S_1^2}{S_2^2} < \frac{1}{5.2}$ or > 6.5.

10. A random sample of size 5 is taken from normal population with mean $\mu = 2.5$ and variance $\sigma^2 = 36$.

    (a) Find the probability that the sample variance lies between 30 and 44.

    (b) Find the probability that the sample mean lies between 1.3 and 3.5 and the sample variance lies between 30 and 44.

11. Let X_1 and X_2 be independent random variables with $X_i \sim Gamma(n_i, 1/\theta)$ $n_i > 0, \theta > 0$. Define $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$. Then prove that Y_1 and Y_2 independently distributed with $Y_1 \sim Gamma(n_1 + n_2, \theta)$ and $Y_2 \sim Beta(n_1, n_2)$.