Soluⁿ of Tutorial - 4

(1)

(a) Since X is continuous type with p.d.f $f(x)$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-1/2}^{1/2} (K - |x|)\,dx = 1$$

$$\Rightarrow \int_{-1/2}^{0} (K + x)\,dx + \int_{0}^{1/2} (K - x)\,dx = 1 \Rightarrow K = 5/4$$

Also for $K = 5/4$, $f(x) \geq 0 \quad \forall x \in \mathbb{R}$.

(b) $$P(X < 0) = \int_{-\infty}^{0} f(x)\,dx = \int_{-1/2}^{0} \left(\frac{5}{4} + x\right) dx = \frac{1}{2}$$

$$= P(X \leq 0).$$

$\|^{ly}$ do others

(c) for $x < \frac{1}{2}$, $F_X(x) = 0$.

$-\frac{1}{2} \le x < 0$, $F_X(x) = \int_{-1/2}^{x} \left(\frac{5}{4} + t\right) dt = \frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}$

$0 \le x < \frac{1}{2}$, $F_X(x) = \int_{-1/2}^{0} \left(\frac{5}{4} + t\right) dt + \int_{0}^{x} \left(\frac{5}{4} - t\right) dt$

$$= -\frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}$$

$x \geq \frac{1}{2}$, $F_X(x) = \int_{-1/2}^{0} \left(\frac{5}{4} + t\right) dt + \int_{0}^{1/2} \left(\frac{5}{4} - t\right) dt = 1.$

$$F_X(x) = \begin{cases} 0, & x < -1/2 \\ \frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}, & -\frac{1}{2} \le x < 0 \\ -\frac{x^2}{2} + \frac{5x}{4} + \frac{1}{2}, & 0 \le x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$$

$$= \begin{cases} 0, & x < -1/2 \\ -\frac{x|x|}{2} + \frac{5}{4}x + \frac{1}{2}, & -\frac{1}{2} \le x < 1/2 \\ 1 & x \geq 1/2 \end{cases}$$

(3)

(2) is same as (1)

(3) $f(x) = \begin{cases} \frac{1}{\beta}\left(1 - \frac{|x-\alpha|}{\beta}\right), & \alpha-\beta < x < \alpha+\beta \\ 0, & o/w \end{cases}$, $\alpha \in \mathbb{R}, \beta > 0$

$$\int_{\alpha-\beta}^{\alpha+\beta} \frac{1}{\beta}\left\{1 - \frac{|x-\alpha|}{\beta}\right\} dx, \quad y = \frac{x-\alpha}{\beta}$$

$$= \int_{-1}^{1} \left(1-|y|\right) dy = 2\int_{0}^{1} (1-y)\, dy = 1$$

Also $f(x) \geq 0 \quad \forall x \in \mathbb{R}$.

$\Rightarrow$ $f(x)$ is a pdf.

$$F(x) = \int_{\alpha-\beta}^{x} \frac{1}{\beta}\left(1 - \frac{|t-\alpha|}{\beta}\right), \quad \alpha-\beta < t < \alpha+\beta$$

$$= \int_{-1}^{\frac{x-\alpha}{\beta}} \left(1-|y|\right) dy, \quad y = \frac{t-\alpha}{\beta}$$

If $\frac{x-\alpha}{\beta} < 0$ i.e. $x < \alpha$, we have

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$$F(x) = \int_{-1}^{\frac{x-\alpha}{\beta}} (1+y)\, dy = \frac{(1+y)^2}{2} \Big|_{-1}^{\frac{x-\alpha}{\beta}}$$

$$= \frac{1}{2}\left[1 + \frac{x-\alpha}{\beta}\right]^2, \quad \alpha-\beta < x \le \alpha$$

Now if $\frac{x-\alpha}{\beta} > 0$, so for $\alpha \le x < \alpha+\beta$

$$F(x) = \int_{-1}^{0} (1-|y|)\, dy + \int_{0}^{\frac{x-\alpha}{\beta}} (1-|y|)\, dy$$

$$= \frac{1}{2} + \int_{0}^{\frac{x-\alpha}{\beta}} (1-y)\, dy = 1 - \frac{1}{2}\left(1 - \frac{x-\alpha}{\beta}\right)^2$$

$$F(x) = \begin{cases} 0, & x \le \alpha-\beta \\ \frac{1}{2}\left[1+\left(\frac{x-\alpha}{\beta}\right)\right]^2, & \alpha-\beta < x \le \alpha \\ 1 - \frac{1}{2}\left(1 - \frac{x-\alpha}{\beta}\right)^2, & \alpha < x \le \alpha+\beta \\ 1, & x > \alpha+\beta \end{cases}$$

$E(X) = \alpha.$

$$Var(X) = E(X-\alpha)^2 = \int_{\alpha-\beta}^{\alpha+\beta} \frac{(x-\alpha)^2}{\beta}\left\{1 - \frac{|x-\alpha|}{\beta}\right\} dx$$

$$= \beta^2 \int_{-1}^{1} y^2 (1-|y|)\, dy = 2\beta^2 \int_{0}^{1} y^2(1-y)\, dy$$

$$= \beta^2/6$$

4. X be a r.v. with d.f. $F_X(x)$.

$Y_1 = |X|$. $F_{Y_1}(y_1) = P(Y_1 \le y_1) = 0$ if $y_1 < 0$

$= P(-y_1 \le X \le y_1)$, if $y_1 \geq 0$

$= F_X(y_1) - P(X < -y_1)$

$= F_X(y_1) - F_X(-y_1) + P(X = -y_1)$

So the c.d.f. of Y_1 is

$$F_{Y_1}(y_1) = \begin{cases} 0, & y_1 < 0 \\ F_X(y_1) - F_X(-y_1) + P(X = -y_1), & y_1 \geq 0 \end{cases}$$

consider $Y_2 = aX + b$, $a \neq 0$, $b \in \mathbb{R}$.

$$F_{Y_2}(y_2) = P(Y_2 \le y_2) = P(aX + b \le y_2)$$

$$= \begin{cases} P\left(X \le \frac{y_2 - b}{a}\right) & \text{if } a > 0 \\ P\left(X \geq \frac{y_2 - b}{a}\right) & \text{if } a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y_2 - b}{a}\right) & \text{if } a > 0 \\ 1 - F_X\left(\frac{y_2 - b}{a}\right) + P\left(X = \frac{y_2 - b}{a}\right), & \text{if } a < 0. \end{cases}$$

$$Y_3 = \max(x,0) = \begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x \le 0 \end{cases}$$

max (x, 0) is alway grater than equl to zero

$$F_{Y_3}(y_3) = P(Y_3 \le y_3) = \begin{cases} 0, & \text{if } y_3 < 0 \\ P(Y_3 \le 0) = P(Y_3 < 0) + P(Y_3 = 0) & \text{if } y_3 = 0 \\ P(Y_3 \le y_3) = P(Y_3 < 0) + P(Y_3 = 0) + P(0 < Y_3 \le y_3) & \text{if } y_3 > 0 \end{cases}$$

$\max(X,0) = 0$ equivalent to $X \le 0$.

$$= \begin{cases} 0, & \text{if } y_3 < 0 \\ P(X \le 0) & \text{if } y_3 = 0 \\ P(X \le 0) + P(0 \le X \le y_3) & \text{if } y_3 > 0 \end{cases}$$

$$= \begin{cases} 0, & y_3 < 0 \\ F_X(0), & y_3 = 0 \\ F_X(y_3), & y_3 > 0 \end{cases} = \begin{cases} 0, & y_3 < 0 \\ F_X(y_3), & y_3 \ge 0. \end{cases}$$

$$Y_4 = \min(x,0) = \begin{cases} x, & x < 0 \\ 0, & x \ge 0. \end{cases}$$

[min (x, 0) is either 0 or lessthan 0.]

So min (x, 0) is always less than a (+, ve no. or zero.]

$$P(Y_4 \le y_4) = \begin{cases} P(Y_4 \le y_4) = 1 & \text{if } y_4 \ge 0 \\ P(X \le y_4) & \text{if } y_4 < 0. \end{cases}$$

$$P(Y_4 \le y_4) = \begin{cases} 1 & \text{if } y_4 \ge 0 \\ P(X \le y_4) & \text{if } y < 0 \end{cases}$$

(5) The p.m.f. of X is given as $P(X=-2)=\frac{1}{5}$, $P(X=-1)=\frac{1}{6}$

$P(X=0)=\frac{1}{5}$, $P(X=1)=\frac{1}{15}$, $P(X=2)=\frac{11}{30}$

Let $Y=X^2$. Then the ~~set~~ $Y \in \{0,1,4\}$

$$P(Y=y) = \begin{cases} \frac{1}{5}, & y=0 \\ \frac{1}{6}+\frac{1}{15}, & y=1 \\ \frac{1}{5}+\frac{11}{30}, & y=4 \end{cases} = \begin{cases} \frac{1}{5}, & y=0 \\ \frac{7}{30}, & y=1 \\ \frac{17}{30}, & y=2 \end{cases}$$

C.d.f. of Y is given as

$$F_Y(y) = \begin{cases} 0, & y<0 \\ \frac{1}{5}, & 0\le y<1 \\ \frac{1}{5}+\frac{7}{30}, & 1\le y<2 \\ 1 & y\geq 2 \end{cases} = \begin{cases} 0, & y<0 \\ \frac{1}{5}, & 0\le y<1 \\ \frac{13}{30}, & 1\le y<2 \\ 1, & y\geq 2. \end{cases}$$

(6) The p.d.f. of X is given as

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1<x<1 \\ 0, & o/w. \end{cases}$$

Let $Y=\max(X,0)$. Apply question 1 we have

$$P(Y\le y) = \begin{cases} 0, & y<0 \\ F_X(y), & y\geq 0 \end{cases} = \begin{cases} 0, & y<0 \\ \frac{1}{2} & y=0 \\ \frac{1}{2}+\frac{y}{2} & 0<y\le 1 \\ 1, & y>1 \end{cases}$$

7) $f_X(x) = \frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$

$Y = X^2$, So we have $h(x) = x^2$ &

Now $h(x)$ is strictly decreasing in $(-\infty, 0)$ with inverse $h^{-1}(y) = -\sqrt{y}$

Again $h(x)$ is strictly increasing in $(0, \infty)$ with inverse $h^{-1}(y) = \sqrt{y}$.

Also we have $h(-\infty, 0) = h(0, \infty) = (0, \infty)$

Then the density of Y is given as

$$f_Y(y) = f_X(-\sqrt{y}) \left| -\frac{1}{2\sqrt{y}} \right| + f_X(\sqrt{y}) \left| \frac{1}{2\sqrt{y}} \right|, \quad y \in (0, \infty)$$

$$= \frac{1}{2} e^{-\sqrt{y}} \frac{1}{2\sqrt{y}} + \frac{1}{2} e^{\sqrt{y}} \frac{1}{2\sqrt{y}}, \quad y \in (0, \infty)$$

$$= \frac{1}{2\sqrt{y}} e^{-\sqrt{y}}, \quad 0 < y < \infty.$$

8) $f_X(x) = \begin{cases} c(x+1), & -1 \le x \le 2 \\ 0, & o/w. \end{cases}$

$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow c = \frac{2}{9}.$

$$f_X(x) = \begin{cases} \frac{2}{9}(x+1), & -1 \le x \le 2 \\ 0, & o/w. \end{cases}$$

$Y = X^2 = h(X)$

$h(x)$ is strictly decreasing in $(-1,0)$ with inverse

$h^{-1}(y) = -\sqrt{y}$, & $h(-1,0) = (0,1)$

$h(x)$ is strictly increasing in $(0,2)$ with inverse

$h^{-1}(y) = \sqrt{y}$, & $h(0,2) = (0,4)$.

$$f_Y(y) = f_X(-\sqrt{y}) \left|\frac{d}{dy}(-\sqrt{y})\right| I_{(0,1)}(y) + f_X(\sqrt{y}) \left|\frac{d}{dy}(\sqrt{y})\right| I_{(0,4)}(y)$$

$$= \begin{cases} \frac{2}{9}(-\sqrt{y}+1)\frac{1}{2\sqrt{y}} + \frac{2}{9}(\sqrt{y}+1)\frac{1}{2\sqrt{y}}, & 0 < y < 1 \\ \frac{2}{9}(\sqrt{y}+1)\frac{1}{2\sqrt{y}}, & 2 < y < 4 \\ 0, & o/w. \end{cases}$$

$$= \begin{cases} \frac{2}{9\sqrt{y}}, & 0 < y < 1 \\ \frac{\sqrt{y}+1}{9\sqrt{y}}, & 1 < y < 4, \\ 0, & o/w. \end{cases}$$

[Finding c.d.f do yourself.]

(9) $f_X(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0, & o/w. \end{cases}$

$Y = 40(1-X) = h(X)$. $h(x)$ is strictly increasing in $(0,1)$.

$h^{-1}(y) = x = \left(1 - \frac{y}{40}\right)$, $\frac{d h^{-1}(y)}{dy} = -\frac{1}{40}$.

$x \in (0,1)$ then $y \in (0, 40)$.

$$f_Y(y) = \begin{cases} \frac{3}{40}\left(1 - \frac{y}{40}\right)^2, & 0 < y < 40 \\ 0, & o/w. \end{cases}$$

(10) $X =$ number of female applicants among the final 5.

$X = 0, 1, 2, 3, 4, 5$

$$P(X = i) = \frac{\binom{9}{i}\binom{6}{5-i}}{\binom{15}{5}}, \quad i = 0, 1, 2, 3, 4, 5$$

Let $Y =$ number of male applicants $= (5 - X)$

$Y = 0, 1, 2, 3, 4, 5$

$$P(Y=y) = P(5-X=y) = P(X=5-y)$$

$$= \frac{\binom{9}{5-y}\binom{6}{y}}{\binom{15}{5}}, \quad y=0,1,2,3,4,5.$$

is the p.m.f. of Y.