Tutorial -10

(1) $X_1 \sim N(200, 8)$, $X_2 \sim N(104, 8)$, $X_3 \sim N(108, 15)$
$X_4 \sim N(120, 15)$, $X_5 \sim N(210, 15)$ and these are indep.

$$\left.\begin{array}{l} U = \dfrac{X_1 + X_2}{2} \sim N(152, 4) \\ V = \dfrac{X_1 + X_2 + X_3}{2} \sim N(146, 5) \end{array}\right\rangle \text{ indep}$$

$$W = U - V \sim N(6, 9)$$

$$P(U > V) = P(W > 0) = P\left(\frac{W-6}{3} > -6/3\right)$$

$$= 1 - \Phi(-2)$$

2 (a) Define the r.v.

$$X_i = \begin{cases} 1 & \text{if the } i\text{th die show even number on its upper face}, \quad i = 1, 2, \dots, 6 \\ 0, & \text{o/w.} \end{cases}$$

$$P(\text{show even number}) = \frac{3}{6} = \frac{1}{2}$$

So $X_i \sim \text{Bernoulli}\left(\frac{1}{2}\right)$, Also X_i's are indep.

$S = \sum X_i \sim \text{Bin}(6, 1/2)$, $E(S) = 6 \cdot \frac{1}{2} = 3$, $Var(S) = npq$
$= 3/2$.

(b) $M_1(t) = \left(\frac{3}{4} + \frac{1}{4}e^t\right)^3$ and $M_2(t) = e^{2(e^t - 1)}$

So

$X_1 \sim \text{Bin}(3, 1/4)$, $X_2 \sim \mathcal{P}(2)$

$$P(X_1 + X_2 = 1) = P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 0)$$
$$= P(X_1 = 0)\, P(X_2 = 1) + P(X_1 = 1)\, P(X_2 = 0)$$
(By indep)

$$= \binom{3}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3 \frac{e^{-2}2^0}{1!} + \binom{3}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^2 e^{-2}$$

$$= \frac{81}{64} e^{-2}$$

(3) $Var(X) = E(X^2) - (E(X))^2 = 2 = Var(Y)$

$$Cov\left(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3}\right) = \frac{2}{9} var(X) + \frac{1}{2} Cov(X,Y) + \frac{1}{9} Cov(X,Y)$$

$$+ \frac{2}{9} Var(Y)$$

$$= \frac{8}{9} + \frac{5}{9} cov(X,Y)$$

$$= \frac{8}{9} + 5/9 \times \frac{1}{3} \times \sqrt{2} \times \sqrt{2} = \frac{34}{27}.$$

(4) $Cov(X_i, X_j) = \sigma_i \sigma_j \rho_{ij} \quad i \neq j$

$$\Rightarrow E(X_i X_j) = \mu_i \mu_j + \rho_{ij} \sigma_i \sigma_j , \quad i \neq j$$

$$Cov(Y,Z) = E(Y - E(Y))(Z - E(Z))$$

$$= E\left[\left(\sum_{i=1}^{n} a_i (X_i - \mu_i)\right)\left(\sum_{j=1}^{n} b_j (X_j - \mu_j)\right)\right]$$

$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j (X_i - \mu_i)(X_j - \mu_j)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j \, cov(X_i, X_j)$$

$$= \sum a_i b_i \sigma_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i \neq j}}^{n} a_i b_j \, cov(X_i, X_j)$$

(5) Let (X,Y) be a r.v s.t.

$$P((X,Y)=(x_i,y_i)) = \frac{1}{n}, \quad i=1,2,\dots,n.$$

Then $E(XY) = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$, $E(X) = \frac{1}{n}\sum x_i = 0 = E(Y)$

$$E(X^2) = \frac{1}{n}\sum x_i^2 = Var(X)$$

$$E(Y^2) = \frac{1}{n}\sum Y_i^2 = var(Y)$$

$$\rho^2(X,Y) \le 1 \Rightarrow cov^2(X,Y) \le Var(X)\,Var(Y).$$

$$\Rightarrow \left(\frac{1}{n}\sum x_i y_i\right)^2 \le \left(\frac{1}{n}\sum x_i^2\right)\left(\frac{1}{n}\sum y_i^2\right)$$

$$\Rightarrow \left(\sum x_i y_i\right)^2 \le \left(\sum x_i^2\right)\left(\sum y_i^2\right)$$

(6) Let $\underline{X} = (x_1, x_2)$, $\underline{Y} = (Y_1, Y_2)$

$$S_{\underline{X}} = \{(0,0), (0,1), (1,1)\}$$

$$S_{\underline{Y}} = \{(0,0), (-1,1), (0,2)\}$$

(a)

$$p_{Y_1,Y_2}(y_1,y_2) = P(X_1 - X_2 = y_1,\ X_1 + X_2 = y_2)$$

$$= P\left(X_1 = \frac{y_1+y_2}{2},\ X_2 = \frac{y_2-y_1}{2}\right)$$

$$= \begin{cases} \left(\frac{2}{3}\right)^{y_2}\left(\frac{1}{3}\right)^{2-y_2}, & \underline{y} \in S_{\underline{Y}} \\ 0, & \text{o/w} \end{cases}$$

(b) $f_{Y_1}(y_1) = \sum_{y_2} p_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{2}{9}, & y_1 = -1 \\ 5/9, & y_1 = 0 \\ 2/9, & y_1 = 1 \\ 0, & o/w \end{cases}$

$p_{Y_2}(y_2) = \sum_{y_1} p_{Y_1,Y_2}(y_1,y_2) = \begin{cases} \frac{1}{9}, & y_2 = 0 \\ 4/9, & y_2 = 1 \\ 4/9, & y_2 = 2 \\ 0, & o/w \end{cases}$

(c) $E(Y_2) = 0\cdot\frac{1}{9} + 1\cdot\frac{4}{9} + 2\cdot\frac{4}{9} = 4/3$

$E(Y_2^2) = \frac{20}{9}$, $Var(Y_2) = 4/9$.

$E(Y_1 Y_2) = 0\cdot\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 + (-1)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{2-1} + 1\cdot\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^{2-1}$

$+ 0 \times \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{2-2} = 0.$

$E(Y_1) = 0$. So $cov(Y_1,Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = 0$

(d) $P(Y_1 = 0, Y_2 = 0) = \frac{1}{9} \neq P(Y = 0)\,P(Y_2 = 0) = 5/9 \times \frac{1}{9}$.

$\Rightarrow$ Y_1 and Y_2 are not indep.

(7) $f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \begin{cases} 2e^{-(x_2+2x_3)}, & 0<x_1<1,\ x_2>0,\ x_3>0 \\ 0, & o/w. \end{cases}$

$$f_{X_1}(x_1) = \int_0^\infty \int_0^\infty f_{X_1,X_2,X_3}(x_1,x_2,x_3)\, dx_2\, dx_3$$

$$= \begin{cases} 1, & 0 < x_1 < 1 \\ 0, & o/w. \end{cases}$$

$||^{ly}$ $f_{X_2}(x_2) = \begin{cases} e^{-x_2}, & x_2 > 0 \\ 0, & o/w \end{cases}$ $\quad f_{X_3}(x_3) = \begin{cases} 2e^{-2x_3}, & x_3 > 0 \\ 0, & o/w. \end{cases}$

(b)

$\Rightarrow$ $f_{X_1,X_2,X_3}(x_1,x_2,x_3) = f_{X_1}(x_1)\, f_{X_2}(x_2)\, f_{X_3}(x_3)$

$\Rightarrow$ X_1, X_2, X_3 are indep.

(c) X_1, X_2, X_3 are indep.

$\Rightarrow$ (X_1, X_2) and X_3 are indep

$\Rightarrow$ $(X_1 + X_2)$ and X_3 are indep.

(d) Since X_1 & X_2 are indep

$f_{X_1|X_2}(x_1 | x_2 = 2) = f_{X_1}(x_1) = \begin{cases} 1, & 0 < x_1 < 1 \\ 0, & o/w. \end{cases}$

(8) (X, Y) have the density

$$f_{X,Y}(x,y) = \frac{1}{\pi\sqrt{3}} \exp\left[-\frac{2}{3}\left(x^2 - xy + y^2\right)\right] \quad x, y \in \mathbb{R}$$

(a) $\Rightarrow$ $\rho = \frac{1}{2}$. $E(X) = 0$, $E(Y) = 0$, $Var(X) = Var(Y) = 1$.

(b)

$$W \equiv X \mid Y=1 \sim N\left(0 + \frac{1}{2}\cdot 1\left(\frac{1-0}{1}\right),\ 1\left(1-\frac{1}{4}\right)\right)$$

$$= N\left(\frac{1}{2}, 3/4\right)$$

$$P(-1 < X < 1 \mid Y=1) = P(-1 < W < 1)$$

$$= P\left(\frac{-1-1/2}{\sqrt{3}/2} < Z < \frac{1-1/2}{\sqrt{3}/2}\right)$$

$$= P\left(\frac{-3/2}{\sqrt{3}/2} < Z < \frac{1}{2}\cdot 2/\sqrt{3}\right)$$

$$= P\left(-\sqrt{3} < Z < \frac{1}{\sqrt{3}}\right) = \Phi\left(\frac{1}{\sqrt{3}}\right) - \Phi(-\sqrt{3}).$$

$$2X+3Y \sim N\left(0,\ 4+9+2\cdot\frac{1}{2}\cdot 2\cdot 3\right) \equiv N(0, 19)$$

$$\mathrm{Var}(2X+3Y) = 19.$$

$$P(-5 < 2X+3Y < 8) = P\left(\frac{-5-0}{\sqrt{19}} < Z < \frac{8-0}{\sqrt{19}}\right)$$

$$\Rightarrow \Phi\left(\frac{8}{\sqrt{19}}\right) - \Phi\left(-5/\sqrt{19}\right).$$