Tutorial – 8

1) $X = 0, 1, 2, 3$ for white, red, black and blue balls respectively

Y = number in the balls $= 0, 1, 2, 3, 4$.

$f_{X,Y}(i,j) = P(X=i, Y=j)$

marginal of Y.

$Y \backslash X$	0	1	2	3	$f_Y(y)$
0	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{4}{14}$
1	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{4}{14}$
2	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	0	$\frac{3}{14}$
3	$\frac{1}{14}$	$\frac{1}{14}$	0	0	$\frac{2}{14}$
4	$\frac{1}{14}$	0	0	0	$\frac{1}{14}$
$f_X(x)$	$5/14$	$4/14$	$3/14$	$2/14$	1

marginal of X

$$P(X=0, Y=0) = \frac{1}{14}$$ similarly for other

$$P(X=1, Y=0) = 1/14$$

2) (a) $F(x,y) = \begin{cases} 1, & x+2y=1 \\ 0, & x+2y<1 \end{cases}$

$\lim_{x\to\infty} F(x,y) = 1 = G(y)$. $G(y)$ is not a d.f. (marginal) so F is not a distn funn.

(2)

$$F(x,y) = \begin{cases} 0 & \text{if } x<0 \text{ or } x+y<1 \text{ or } y<0 \\ 1 & \text{o/w} \end{cases}$$

For the rectangle $(\frac{1}{4},1] \times (\frac{1}{4},1]$

$$P\left(\tfrac{1}{4}<x\le 1,\ \tfrac{1}{4}<y\le 1\right) = F(1,1) - F(\tfrac{1}{4},1) - F(1,\tfrac{1}{4}) + F(\tfrac{1}{4},\tfrac{1}{4})$$

$$= 1 - 1 - 1 + 0 \quad = -1 < 0$$

$\Rightarrow$ F is not a d.f.

(3) $$p(x,y) = \begin{cases} c(x+2y), & x=1,2, \quad y=1,2 \\ 0, & \text{o/w} \end{cases}$$

(a) $$\sum_x \sum_y p(x,y) = 1 \quad \Rightarrow c\,[3+5+4+6] = 1$$

$$\Rightarrow c = \frac{1}{18}$$

(b) $$p_X(x) = \begin{cases} \displaystyle\sum_{y=1}^{2} \frac{1}{18}(x+2y) = \frac{1}{18}\left[(x+2)+(x+4)\right] \\ \qquad = \frac{1}{18}(2x+6), & x=1,2 \\ 0, & \text{o/w} \end{cases}$$

b) $p_Y(y) = \begin{cases} \sum_{x=1}^{2} \frac{1}{18}(x+2y) = \frac{1}{18}(3+4y), & y=1,2 \\ 0, & o/w. \end{cases}$

(c) $p(x,y) \neq p_X(x)\, p_Y(y) \Rightarrow$ X and Y aree not independent

(d) For $y \in \{1,2\}$

$$p_X(x|Y=y) = \frac{p(x,y)}{p_Y(y)} = \begin{cases} \dfrac{\frac{1}{18}(x+2y)}{\frac{1}{18}(3+4y)}, & x=1,2 \\ 0, & o/w \end{cases}$$

$$p_X(x|Y=2) = \begin{cases} \dfrac{x+4}{11}, & x=1,2 \\ 0, & o/w. \end{cases}$$

$\|^{ly}$ we can calculate $p_Y(y|X=x) = \dfrac{p(x,y)}{p_X(x)}$

$$= \begin{cases} \dfrac{x+2y}{2(x+3)}, & y=1,2 \\ 0, & o/w. \end{cases}$$

(4) The joint pmf is given as

$$p(x,y) = \begin{cases} cxy, & x=1,2,\ y=1,2,\ x \le y \\ 0, & o/w \end{cases}$$

(a) $\displaystyle\sum_{y=1}^{2} \sum_{x=1}^{y} cxy = 1 \Rightarrow c = \frac{1}{7}$

(b) $$p_X(x) = \sum_{y=x}^{2} p(x,y) = \begin{cases} \frac{3}{7}, & x=1 \\ \frac{2\cdot 2}{7} & x=2 \\ 0, & o/w. \end{cases}$$

$$p_Y(y) = \sum_{x=1}^{y} p(x,y) = \begin{cases} \sum_{x=1}^{1} p(x,y) & y=1 \\ \sum_{x=1}^{2} p(x,y) & y=2 \\ 0, & o/w \end{cases} = \begin{cases} \frac{1}{7}, & y=1 \\ \frac{6}{7}, & y=2 \\ 0, & o/w \end{cases}$$

(c) $$p_{Y|X}(y|1) = \begin{cases} \dfrac{p(1,y)}{p_X(1)}, & y=1,2 \\ 0, & o/w \end{cases} = \begin{cases} \frac{y}{3}, & y=1,2 \\ 0, & o/w \end{cases}$$

$$p_{Y|X}(y|2) = \begin{cases} \dfrac{p(2,y)}{p_X(2)}, & y=2 \quad (\text{as } y \ge 2) \\ 0 & o/w \end{cases} = \begin{cases} 1, & y=2 \\ 0, & o/w \end{cases}$$

~~(try yourself)~~

(d) Try yourself

(e) $P(X > Y) = \sum\sum_{x>y} p(x,y) = 0$

$P(X = Y) = \sum\sum_{x=y} p(x,y) = c\,[1\times1 + 2\times2] = 5/7$

$P(X < \frac{2}{3}Y) = P(X < 2/3, Y=1) + P(X < 4/3, Y=2)$

$= 0 + P(X=1, Y=2) = 2/7$

$P(X+Y \geq 3) = 1 - P(X+Y < 3)$

$= 1 - P(X+Y \leq 2) = 1 - P(X=1, Y=1)$

$= 1 - \frac{1}{7} = 6/7.$

(5) The joint p.m.f. is given as

$X \backslash Y$	-1	0	1	$f_X(x)$
0	0	$1/3$	0	$1/3$
1	$1/3$	0	$1/3$	$2/3$
$f_Y(y)$	$1/3$	$1/3$	$1/3$	1

(i) $f_X(x) = \begin{cases} \frac{1}{3}, & x = 0 \\ 2/3, & x = 1 \\ 0, & o/w \end{cases}$ $\quad f_Y(y) = \begin{cases} \frac{1}{3}, & y = -1, 0, 1 \\ 0, & o/w \end{cases}$

(ii) $f_X(0) = 1/3, \quad f_Y(0) = 1/3. \quad f_{X,Y}(0,0) = \frac{1}{3}$

$\neq f_X(0)\, f_Y(0)$

So X and Y are not independent.

(6) (a) $S = \{(x,y) : (x,y) \in \{1,2,3\} \times \{1,2,3,4\},$
$x \leq y\}$

$= \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3)$
$(2,4), (3,3), (3,4)\}$

$\sum\sum_{(x,y) \in S} p(x,y) = 1 \Rightarrow c = \frac{1}{26}.$

(b) $$p_X(x) = \sum_{y=x}^{4} \frac{y}{26} = \begin{cases} 5/13, & x=1 \\ 9/26, & x=2 \\ 7/26, & x=3 \\ 0, & o/w \end{cases}$$

For $y \in \{1,2,3,4\}$

$$f_Y(y) = \sum_{x=1}^{y} \frac{y}{26} = \begin{cases} \frac{1}{26}, & y=1 \\ \frac{2}{13}, & y=2 \\ \frac{9}{26}, & y=3 \\ \frac{6}{13}, & y=4 \\ 0, & o/w \end{cases}$$

(c) $$P(X+Y>4) = P(X=1, Y=4) + P(X=2, Y=3) + P(X=3, Y=3) + P(X=2, Y=4) + P(X=3, Y=4) = \frac{9}{13}.$$

(7) The joint distⁿ fun of (X, Y) is given as

$$F(x,y) = \begin{cases} 0, & x < 0,\ y < 0 \\ \frac{1+xy}{2}, & 0 \le x < 1,\ 0 \le y < 1 \\ \frac{1+x}{2}, & 0 \le x < 1,\ y > 1 \\ \frac{1+y}{2}, & x \ge 1,\ 0 \le y < 1 \\ 1, & x \ge 1,\ y \ge 1 \end{cases}$$

(a) $$F_X(x) = \lim_{y \to \infty} F(x,y) = \begin{cases} 0, & x < 0 \\ \frac{1+x}{2}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

$$F_Y(y) = \lim_{x \to \infty} F(x,y) = \begin{cases} 0, & y < 0 \\ \frac{1+y}{2}, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$

(b) $P\left(\frac{1}{2} \leq X \leq 1,\ \frac{1}{4} < X_2 < 1/2\right)$

$= F\left(1, \frac{1}{2}-\right) - F\left(\frac{1}{2}-, \frac{1}{2}-\right) - F\left(1, \frac{1}{4}\right)$

$+ F\left(\frac{1}{2}-, \frac{1}{4}\right)$

$= \frac{3}{4} - 5/8 - 5/8 + \frac{9}{16} = 1/16$

$P(X=1) = F_X(1) - F_X(1-)$

$= 1-1 = 0.$

$P(X \geq 3/2,\ Y < 1/4) = ?$

$\{X \geq 3/2,\ Y < 1/4\} = \{X \in \mathbb{R},\ Y < 1/4\}$

$- \{X < 3/2,\ Y < 1/4\}$

$P(X \geq 3/2, Y < 1/4) = P(Y < 1/4) - P(X < 3/2, Y < 1/4)$

$= F_Y(1/4-) - F\left(3/2-, \frac{1}{4}-\right) = 5/8 - 5/8 = 0.$