End Sem Solu^n

Q1 (a) $X \sim \mathcal{P}(\lambda_1)$ and $Y \sim \mathcal{P}(\lambda_2)$ and they are independent. Now for non-negative integer $m < n$

$$P(X=m \mid X+Y=n) = \frac{P(X=m, Y=n-m)}{P(X+Y=n)} \quad [1]$$

$$= \frac{e^{-\lambda_1}\frac{\lambda_1^m}{m!}\, \frac{e^{-\lambda_2}\lambda_2^{n-m}}{(n-m)!}}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \binom{n}{m}\frac{\lambda_1^m \lambda_2^{n-m}}{(\lambda_1+\lambda_2)^n} = \binom{n}{m}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m \left(1-\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-m}$$

$m = 0, 1, 2, \dots, n.$

[1]

This is a binomial dist^n.

(b) $X_i \sim \mathcal{E}xp(1)$, $f_{X_i}(x_i) = e^{-x}$, $x > 0$

$$E(2n\bar{X}) = 2E(\Sigma X_i) = 2n \quad [\tfrac{1}{2}]$$

$$Var(2n\bar{X}) = 4\, Var(\sum X_i)$$

$$= 4 \sum_{i=1}^{n} Var(X_i) = 4n \quad [1\tfrac{1}{2}]$$

$$E(Y) + Var(Y) = 2n + 4n = 6n.$$

ⓒ $X_1, X_2, \ldots, X_9 \overset{iid}{\sim} N(-1, 9)$

$Y_1, Y_2, \ldots Y_9 \overset{iid}{\sim} N(1, 16)$

$Z_1, \ldots \quad Z_{25} \overset{iid}{\sim} N(0, 9)$

Now $\quad X_i + Y_i \sim N(0, 25)$

$$W = \sum_{i=1}^{9} \left(\frac{X_i + Y_i}{5}\right)^2 \sim \chi^2_9$$

Now $\quad \dfrac{Z_i^2}{9} \sim \chi^2_1, \quad V = \sum_{i=1}^{25} \dfrac{Z_i^2}{9} \sim \chi^2_{25}$ $[1]$

$$\frac{W}{9} \Big/ \frac{V}{25} = \frac{\dfrac{\sum_{i=1}^{9}(X_i + Y_i)^2}{25} \Big/ \cancel{9}}{\dfrac{\sum_{i=1}^{25} Z_i^2}{\cancel{9}} \Big/ \cancel{25}}$$

$$= \frac{\sum_{i=1}^{9} (X_i + Y_i)^2}{\sum_{i=1}^{25} Z_i^2} \sim F_{9,25} \qquad [1]$$

(d) $W_1 = X_1 + X_2, \quad W_2 = \sum_{i=1}^{8} X_i$

$$Cov(W_1, W_2) = Cov\left(X_1 + X_2, \sum_{i=1}^{8} X_i\right)$$

$$= Cov\left(X_1, \sum_{i=1}^{8} X_i\right) + Cov\left(X_2, \sum_{i=1}^{8} X_i\right)$$

$$= Cov(X_1, X_1) + \sum_{i=2}^{8} Cov(X_1, X_i) + Cov(X_2, X_2) + \sum_{\substack{i=1 \\ i \neq 2}}^{8} Cov(X_1, X_i) \qquad [1]$$

$$= Var(X_1) + 0 + Var(X_2) = 2\sigma^2$$

Now $Var(W_1) = 2\sigma^2, \quad Var(W_2) = 8\sigma^2$

$$Cor(W_1, W_2) = \frac{2\sigma^2}{4\sigma^2} = \frac{1}{2}. \qquad [1]$$

(e) $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ Bernoulli(θ)

So $X_1 + X_2 \sim$ Bin $(2, \theta)$

$$E(T) = \frac{1}{2n} P(X_1 + X_2 = 1)$$ [1]

$$= \frac{1}{2n} \binom{2}{1} \theta (1-\theta)$$

$$= \frac{\theta(1-\theta)}{n}$$

So T is unbiased for $\frac{\theta(1-\theta)}{n}$. [1]

(Q2) The joint density given as

$$f_{X,Y}(x,y) = \begin{cases} cxy(x+y), & 0<x<1,\ 0<y<1 \\ 0, & o/w \end{cases}$$

(i) $\int_0^1 \int_0^1 cxy(x+y)\, dx\, dy = 1 \Rightarrow c = 3$

[2]

(ii) $f_X(x) = 3\int_0^1 xy(x+y)\,dy \quad , 0<x<1$

$= \frac{3}{2}x^2 + x$

$$f_X(x) = \begin{cases} \frac{3}{2}x^2+x, & 0<x<1 \\ 0, & o/w \end{cases} \quad [1]$$

$$f_Y(y) = \begin{cases} 3\int_0^1 xy(x+y)\,dx, & 0<y<1 \\ 0, & o/w \end{cases}$$

$$= \begin{cases} \frac{3}{2}y^2+y, & 0<y<1 \\ 0, & o/w \end{cases} \quad [1]$$

(iii) $P\left(\frac{1}{2} \le X \le \frac{3}{4}, \frac{1}{3} \le Y \le \frac{2}{3}\right)$

$$= 3\int_{1/2}^{3/4}\int_{1/3}^{2/3} xy(x+y)\,dy\,dx = \frac{311}{3456}$$

[2]

(iv) $$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_0^1 x\left(\frac{3}{2}x^2 + x\right) dx = \frac{17}{24}$$

$$E(Y) = \int_0^1 y\left(\frac{3}{2}y^2 + y\right) dy = \frac{17}{24}$$

[1]

$$E(XY) = 3\int_0^1\int_0^1 xy\, xy\,(x+y)\, dx\,dy$$

$$= 3\int_0^1 \left(\frac{x^3}{3} + \frac{x^2}{4}\right) dx = \frac{1}{2}$$

$$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{2} - \left(\frac{17}{24}\right)^2 = -\frac{1}{576}$$

[1]

(v) $$f_{X,Y}(x,y) \neq f_X(x)\, f_Y(y)$$

So X, Y are not dependent. [1]

③ The required prob is

$$P(\min(X_1, X_2) \geq 2) = P(X_1 \geq 2)\, P(X_2 \geq 2)$$ [1]

$$= \left(\int_2^\infty \frac{1}{2} e^{-x_1/2}\, dx_1\right)\left(\int_2^\infty \frac{x_2}{4} e^{-x_2/2}\, dx_2\right)$$ [1]

$$= \left[-e^{-x_1/2}\right]_2^\infty \frac{1}{4}\left(\left[-2x_2 e^{-\frac{x_2}{2}}\right]_2^\infty + \int_2^\infty 2 e^{-x_2/2}\, dx_2\right)$$

$$= \frac{1}{4} e^{-1}\left(4e^{-1} + 4e^{-1}\right) = 2e^{-2}.$$ [1]

Q4 Let $X_i \sim \text{Gamma}(n_i, 1/\theta)$, $n_i > 0$, $\theta > 0$, and X_1, X_2 are indep.

So the joint distⁿ of X_1 & X_2 is

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \dfrac{x_1^{n_1-1} \, x_2^{n_2-1} \, e^{-\frac{x_1+x_2}{\theta}}}{\Gamma(n_1)\Gamma(n_2)\,\theta^{n_1+n_2}}, & x_1 > 0, \; x_2 > 0 \\ 0, & \text{o/w} \end{cases}$$

[1]

Here $S_{\underline{X}} = (0, \infty)^2$. Define

$$Y_1 = X_1 + X_2, \quad Y_2 = \frac{X_1}{X_1 + X_2}$$

$$X_1 = Y_1 Y_2, \quad X_2 = Y_1(1 - Y_2)$$

$$J = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1 \quad \&$$

[1]

$y_1 > 0$, $0 < y_2 < 1$. Hence $S_{\underline{Y}} = (0, \infty) \times (0, 1)$

The joint pdf of (Y_1, Y_2) is

$$f_{\underline{Y}}(y_1, y_2) = \frac{(y_1 y_2)^{n_1-1}\left(y_1(1-y_2)\right)^{n_2-1}}{\Gamma(n_1)\,\Gamma(n_2)\,\theta^{n_1+n_2}} \times$$

$$e^{-\frac{y_1 y_2 + y_1(1-y_2)}{\theta}} \, |-y_1| \times$$

$$I_{(0,\infty)\times(0,1)}$$

$$= \left\{ \frac{e^{-y_1/\theta}\, y_1^{n_1+n_2-1}}{\theta^{n_1+n_2}\,\Gamma(n_1+n_2)} I_{(0,\infty)}(y) \right\} \times$$

$$\left\{ \frac{1}{B(n_1, n_2)}\, y_2^{n_1-1} (1-y_2)^{n_2-1} I_{(0,1)}(y_2) \right\}$$

$$f_{Y_1}(y_1)\, f_{Y_2}(y_2)$$

where $Y_1 \sim \text{Gamma}(n_1+n_2, 1/\theta)$

$Y_2 \sim \text{Beta}(n_1, n_2)$ [1]

Clearly Y_1 & Y_2 are indep.

Q5 Let $X_1, X_2, \ldots, X_n$ be a random Sample from an $\text{Exp}(\mu, \sigma)$

So the likelihood fun is

$$L_{\underline{x}}(\mu,\sigma) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{1}{\sigma}\sum_{i=1}^{n}(x_i-\mu)} \quad [1]$$

$x_i > \mu, \quad i = 1(1)n.$

$$l_{\underline{x}}(\mu,\sigma) = -n\ln\sigma - \frac{1}{\sigma}\sum_{i=1}^{n}(x_i-\mu)$$

$$= -n\ln\sigma - \frac{1}{\sigma}\left[n\bar{x} - n\mu\right] \rightarrow (*)$$

For a fixed σ $(*)$ has maximum if μ takes maximum value so

$$\hat{\mu}_{ML} = X_{(1)} = \min\{X_1, \ldots, X_n\}.$$

Again $\dfrac{\partial l_x(\mu,\sigma)}{\partial\sigma} = -\dfrac{n}{\sigma} + \dfrac{1}{\sigma^2}\sum(x_i-\mu)$

$= 0$

$$\Rightarrow \hat{\sigma}_{ML} = \frac{1}{n} \sum_{i=1}^{n} (X_i - X_{(1)}) \quad [1\tfrac{1}{2}]$$

MLE of $\mu + \sigma$ is $\hat{\mu}_{ML} + \hat{\sigma}_{ML}$ $[\tfrac{1}{2}]$

Q6 Given $X_1, X_2, \ldots, X_m \overset{iid}{\sim} N(\mu_1, \sigma^2)$

$Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} N(\mu_2, \sigma^2)$

Then $\bar{X} \sim N\left(\mu_1, \frac{\sigma^2}{m}\right)$,

$\bar{Y} \sim N\left(\mu_2, \frac{\sigma^2}{n}\right)$

$$2\bar{X} - 5\bar{Y} \sim N\left(2\mu_1 - 5\mu_2, \frac{4\sigma^2}{m} + \frac{25\sigma^2}{n}\right)$$

$\cancel{x}$

$$\frac{(m-1)S_1^2}{\sigma^2} \sim \chi^2_{m-1}$$

$$\frac{(n-1)S_2^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\frac{(m-1)S_1^2}{\sigma^2} + \frac{(n-1)S_2^2}{\sigma^2} \sim \chi^2_{m+n-2} \quad [1]$$

$$\Rightarrow \quad S_p^2 = \frac{(m-1)S_1^2 + (n-1)S_2^2}{m+n-2}$$

So we have $\dfrac{(m+n-2)S_p^2}{\sigma^2} \sim \chi^2_{m+n-2}$

From (*) $$\frac{2\bar{X} - 5\bar{Y} - (2\mu_1 - 5\mu_2)}{\sigma\sqrt{\frac{4}{m} + \frac{25}{n}}} \sim N(0,1) \qquad [1]$$

Hence we have

$$T = \frac{\dfrac{2\bar{X} - 5\bar{Y} - (2\mu_1 - 5\mu_2)}{\cancel{\sigma}\sqrt{\frac{4}{m} + \frac{25}{n}}}}{\sqrt{\dfrac{\cancel{(m+n-2)}S_p^2}{\cancel{\sigma^2}} \Big/ \cancel{(m+n-2)}}}$$

$$\Rightarrow \sqrt{\frac{mn}{4n+25m}} \; \frac{2\bar{X} - 5\bar{Y} - (2\mu_1 + 5\mu_2)}{S_p} \sim t_{m+n-2}.$$

Let $k = \sqrt{\dfrac{mn}{4n+25m}}$

So T is a pivotal quantity.

So 95% CI for η is

$$P\left(-t_{0.025,\, m+n-2} \le T \le t_{0.025,\, m+n-2}\right)$$

$$= 0.95$$

$$\Rightarrow P\left(-t_{0.025,\, m+n-2} \le k\,\frac{2\bar{X} - 5\bar{Y} - \eta}{S_p} \le t_{0.025,\, m+n-2}\right) = 0.95$$

So 95% CI for η is

$$\left(2\bar{X} - 5\bar{Y} - \frac{S_p}{k}\, t_{0.025,\, m+n-2}\,,\; 2\bar{X} - 5\bar{Y} + \frac{S_p}{k}\, t_{0.025,\, m+n-2}\right) \quad [1]$$

Q7 (i) Unbiased Estimator: A estimator $T(\underline{X})$ is said to be an unbiased estimator of θ if $E(T(\underline{X})) = \theta$ [1]

Let $H_0 : \theta \in \Theta_0$
$H_1 : \theta \in \Theta_1$

(ii) Prob of type-I error is defined as P(reject H_0 when H_0 is true) [1]

(iii) Prob of type-II error is defined as P(not reject H_0 when H_0 is false) [1]

(iv) Power of a test is $= 1 -$ Prob of type-II error [1]

Q8 $X \sim U(0,\theta)$

$H_0: \theta = 1$

$H_1: \theta = 2$

The test is reject H_0 if $X \geq \frac{1}{2}$.

$$P(\text{Type I error}) = P_{\theta=1}\left(X \geq \frac{1}{2}\right)$$

$$= \int_{1/2}^{1} 1\, dx = 1 - \frac{1}{2} = \frac{1}{2}$$

[1]

$$P(\text{Type-II error}) = P_{\theta=2}\left(X \leq 1/2\right)$$

$$= \int_{0}^{1/2} \frac{1}{2}\, dx = \left.\frac{x}{2}\right]_0^{1/2} = \frac{1}{4}.$$

[1]

Q9 X having density

$$f(x) = \begin{cases} \frac{\theta}{x}\left(\frac{3}{x}\right)^{\theta}, & x > 3 \\ \end{cases}$$

$H_0 : \theta = 1$

$H_1 : \theta = 2$

$$f_0(x) = \begin{cases} \frac{1}{x}\left(\frac{3}{x}\right), & x > 3 \\ 0, & o/w \end{cases}$$

$$f_1(x) = \begin{cases} \frac{2}{x}\left(\frac{3}{x}\right)^2, & x > 3 \\ 0, & o/w \end{cases}$$

So MP test is reject H_0 if

$$\frac{f_1(x)}{f_0(x)} \geq K$$ [1]

$$\Rightarrow \frac{\frac{2}{\cancel{x}}\left(\frac{3}{x}\right)^{\cancel{2}}}{\frac{1}{\cancel{x}}\cancel{\left(\frac{3}{x}\right)}} \geq K$$

$$\Rightarrow \frac{6}{x} \geq K \Rightarrow x \leq K_1$$

Now $\alpha = 0.1$

$$P_{\theta=1}(X \leq K_1) = \frac{1}{10}$$ [1]

$$\Rightarrow \int_3^{K_1} \frac{1}{x} \cdot \frac{3}{x}\, dx = \frac{1}{10}$$

$$\Rightarrow \left[-\frac{3}{x}\right]_3^{K_1} = \frac{1}{10}$$

$$\Rightarrow \left(1 - \frac{3}{K_1}\right) = \frac{1}{10} \Rightarrow K_1 = \frac{10}{3}$$

So The MP test at level $\alpha = 0.1$ is

Reject H_0 if $X \le \frac{10}{3}$. [1]