PHL 101

(1) Mechanics

(2) Astrophysics

$$m\frac{d^2y}{dt^2} = -mg$$

$$\int d\left(\frac{dy}{dt}\right) = \int -\cancel{m}g\,dt + c$$

$$\frac{dy}{dt} = -gt + c$$

at $t = 0$, $\frac{dy}{dt} = v = v_0$

Method: 2

Equate the K.E. with P.E. at the top

Newton's equations are Vector Equation

$$m\frac{dv_x}{dt} = F_x$$

$$m\frac{dv_y}{dt} = F_y$$

$$m\frac{dv_z}{dt} = F_z$$

$$m\left(\frac{dv_x}{dt}\hat{x} + \frac{dv_y}{dt}\hat{y} + --\right) = \vec{F}$$

K.E $\longrightarrow$ T

P.E $\longrightarrow$ V

$$E = \text{Total Energy} = T + V$$

Another combination of T and V is

$$L = T - V$$

Books:

(1) An introduction to Lagrangian & Hamiltonian Mechanics — Patrick Hamilton

(2) Classical Mechanics: Goldstein, Poole & Safko

(3) Classical Mechanics — Rana & Jog

(4) Feynman's Lectures Vol. I

Lagrangian equations of motion :

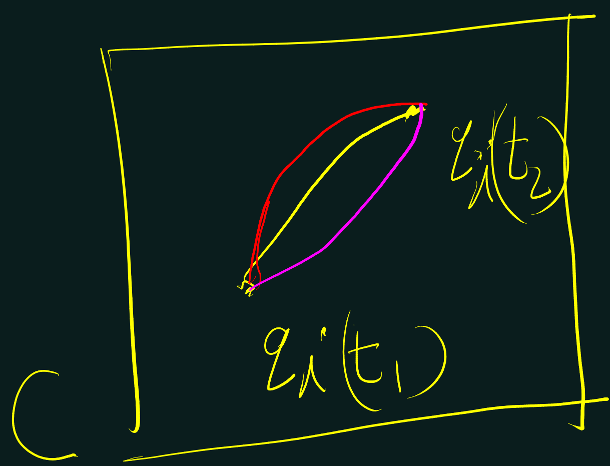
while proceeding from one state at time $t = t_1$ to another final state at time $t = t_2$ the system / Universe / body adopts a path. (which path ?)

The set of coordinates that describe a N dimension system $q_1, q_2, q_3 \ldots$

$\dot{q}_1, \dot{q}_2, \dot{q}_3 \ldots$

Let call them q_i

Think of the following 6N dimensional configuration space, C



The system takes one specific path $q_i(t)$ out of all possible paths (In classical macroscopic case)

[Side Note {In Quantum world it takes all the paths i.e. there is a non-zero probability for each path]

In classical one path is taken which the path of least (extremum) action

what is action:

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$$

where L is the Lagrangian which

$$L = T - V \equiv L(q_i, \dot{q}_i, t)$$

Now, to find the path; we minimize S: $\Rightarrow$ take $\delta S = 0$

$$\delta S = \int \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0$$

But $\displaystyle \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) = \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \delta q$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}} \delta \dot{q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \delta q$$

$$\Rightarrow \delta S = \int_{t_1}^{t_2} \frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\delta q$$

$$= \cancel{\frac{\partial L}{\partial \dot q}\delta q \Big|_{t_1}^{t_2}} + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\delta q\, dt = 0$$

The first term above is having $\delta q(t_1)$ and $\delta q(t_2)$ which are both zero as the initial and end points are fixed/same for all paths

$$\Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

One equation for each coordinate q_i

How many coordinates do we need to describe the system

$3N - s$ where s is the number of constraints

Holonomic constraint: Motion confined to the surface of sphere so

$$x^2+y^2+z^2 = r^2$$

There is an equality therefore it is a holonomic constraint

If there is an inequality then non-holonomic constraints e.g.

Motion can occur inside a sphere

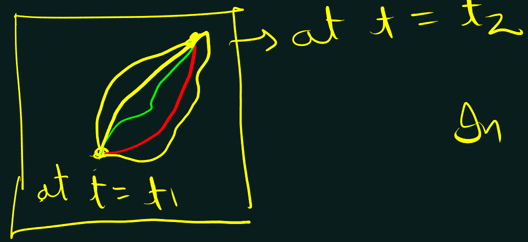
then $x^2 + y^2 + z^2 < r^2$

It is a non-holonomic constraint

Or motion can occur outside the sphere: $x^2 + y^2 + z^2 \geq r^2$

These would be non-holonomic constraints

(Euler) Lagrange equations of motion: $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$



In the intermediate region $q = q(t)$ along the main path

For any other path: $\underset{\text{Actual solution.}}{q(t)} + \underbrace{\delta q}_{\text{variation on top of the actual solution}}$

Free particle: Not in a potential / field or not subject to any force.

In 3D: $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$; $V = 0$

How many coordinates: $(q_i) =$ x, y and z

① $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow$$

$$\frac{d}{dt}\left(m\dot{x}\right) = 0 \Rightarrow m\, x\,(\text{acceleration along } x) =$$

② $$m\ddot{y} = 0$$

③ $$m\ddot{z} = 0$$

Consider a particle in a potential $V = V(x, y, z)$ and $L = T - V$

① $$m\ddot{x} + \frac{\partial V}{\partial x} = 0 \Rightarrow m\ddot{x} = -\frac{\partial V}{\partial x}$$

② $$m\ddot{y} + \frac{\partial V}{\partial y} = 0 \Rightarrow m\ddot{y} = -\frac{\partial V}{\partial y}$$

③ $$m\ddot{z} + \frac{\partial V}{\partial z} = 0 \Rightarrow m\ddot{z} = -\frac{\partial V}{\partial z}$$

$$(m\ddot{x})\,\hat{x} + (m\ddot{y})\,\hat{y} + (m\ddot{z})\,\hat{z} = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)$$

$$\Rightarrow \quad \vec{F} = -\vec{\nabla} V$$

Let's see the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad ; \quad L(q, \dot{q}, t)$$

For free particle $L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

If L does not explicitly depend on q

then $\frac{\partial L}{\partial q} = 0$; q is called a cyclic coordinate

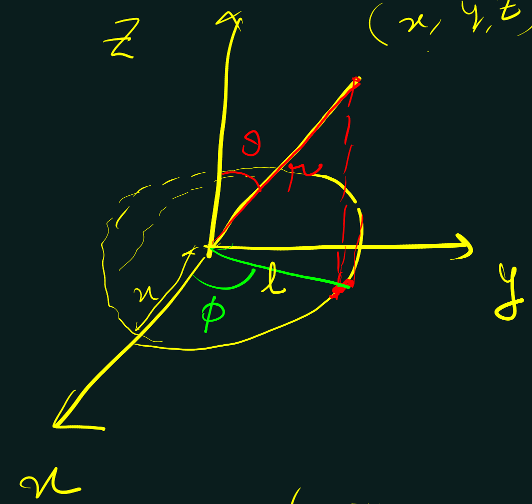
$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}} = \text{constant} \quad \Bigg\} \text{ Conservation Law}$$

$\Rightarrow$ for free particle along x :- $\frac{d}{dt}(m\dot{x}) = 0 \Rightarrow m\dot{x} = \text{constant}$

$m v = \text{constant}$

Conservation of momentum

$(x, y, z) \equiv (r, \theta, \phi)$



$l = r\sin\theta$

$x = l\cos\phi = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$L = T - V$

For a free particle $L = T \cancel{=0} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

L does not depend on ϕ explicitly therefore ϕ is cyclic

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \cancel{\frac{\partial L}{\partial\phi}} = 0$$

$\Rightarrow \frac{\partial L}{\partial\dot{\phi}} = \text{constant} \Rightarrow m r^2\sin^2\theta\,\dot{\phi} = \text{Constant}$

$$\Rightarrow \quad m l^2 \dot{\phi} = \text{constant} = L_z$$ = Conservation of Angular momentum

what if $V \neq 0$ but $V = V_{\text{coulomb}}$

$$T = \frac{1}{2} m \dot{x}^2$$

$$V = \frac{1}{2} k x^2$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

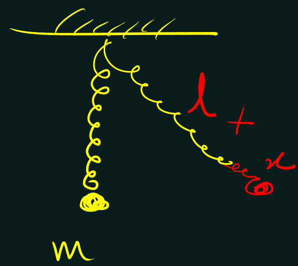
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m \ddot{x} + k x = 0$$

$$\Rightarrow \ddot{x} + \left(\frac{k}{m}\right) x = 0$$

$$\frac{d^2 x}{dt^2} + \omega^2 x = 0$$

$$x = A e^{i\omega t} + B e^{-i\omega t} = C \sin \omega t + D \cos \omega t$$

Spring Pendulum



spring force constant k

mass of blob m

acc. due to gravity g

(1) Non-inertial frames of reference

(2) Dissipative forces: Friction

$\Big\}$ How do we deal with these in Lagrangian Mech.

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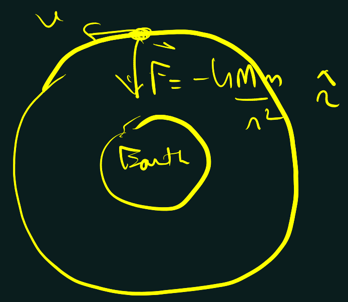
(1) Non-inertial frames



Student A: There is some other pseudo-force

Student B: It is always falling but not falling towards the Earth

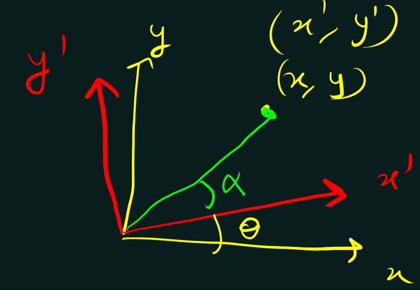
Student C: In the frame of ref. of satellite
Centrifugal (?)

To calculate velocity

$$\underbrace{\frac{mv^2}{R}}_{\text{Centripetal force}} = \frac{GMm}{r^2}$$

Lagragian for a free particle: $L = T - V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - 0$

Rotating frame of reference:



$$x = r\cos(\theta + \alpha)$$
$$y = r\sin(\theta + \alpha)$$

$$x = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha = (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$$
$$y = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha = (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta$$

$$x = x'\cos\theta - y'\sin\theta$$

$$\boxed{\omega = \dot{\theta}}$$

$$y = x'\sin\theta + y'\cos\theta$$

$$z = z$$

$$\dot{x} = \dot{x}'\cos\theta - \dot{y}'\sin\theta - (x'\sin\theta)\omega - (y'\cos\theta)\omega$$
$$\dot{y} = \dot{x}'\sin\theta + \dot{y}'\cos\theta + (x'\cos\theta)\omega - (y'\sin\theta)\omega$$
$$\Rightarrow \quad \dot{x} = (\dot{x}' - y'\omega)\cos\theta - (\dot{y}' + x'\omega)\sin\theta = A\cos\theta - B\sin\theta$$

$$\dot{y} = (\dot{x}' - y'\omega)\sin\theta + (\dot{y}' + x'\omega)\cos\theta = A\sin\theta + B\cos\theta$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (\dot{x}' - y'\omega)^2 + (\dot{y}' + x'\omega)^2 + \dot{z}'^2$$

$$L = \frac{m}{2}\left[(\dot{x}' - y'\omega)^2 + (\dot{y}' + x'\omega)^2 + \dot{z}'^2\right]$$

$\omega = \dot{\theta}$ But ω is a vector $\vec{\omega} = \dot{\theta}\,\hat{z}$

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

Calculate $\vec{\omega} \times \vec{r} = \omega x\,\hat{y} - \omega y\,\hat{x}$

$$\dot{\vec{r}}' + \vec{\omega} \times \vec{r}' = \dot{x}'\hat{x}' + \dot{y}'\hat{y}' + \dot{z}'\hat{z}' + \omega x'\hat{y}' - \omega y'\hat{x}'$$

$$= (\dot{x}' - \omega y')\hat{x}' + (\dot{y}' + \omega x')\hat{y}' + \dot{z}'\hat{z}'$$

$$|\dot{\vec{r}}' + \omega \times \vec{r}'|^2 = (\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 + \dot{z}'^2$$

$$L = \frac{m}{2}\,|\dot{\vec{r}}' + \omega \times \vec{r}'|^2$$

Find out the Equations of motion and realize for yourself that there are two additional forces there : Centrifugal & Coriolis

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2

Dissipative Forces

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

In case of dissipation: the above equations has to be modified.

If dissipative force has the form $F = -\alpha \dot{q}$

then $R = \frac{1}{2}\alpha \dot{q}^2$

And the equations are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial R}{\partial \dot{q}}$$

$F_{restoring} = -kx$

$F_{friction} = -\alpha \dot{x} = -\alpha v$

$T = \frac{1}{2} m \dot{x}^2 \qquad V = \frac{1}{2} k x^2 \qquad R = \frac{1}{2} \alpha \dot{x}^2$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{dR}{d\dot{x}} \Rightarrow -kx - m\ddot{x} = \alpha \dot{x}$$

$$\Rightarrow m\ddot{x} + \alpha \dot{x} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \left(\frac{\alpha}{m}\right)\frac{dx}{dt} + \left(\frac{k}{m}\right)x = 0$$

Let's say $\frac{\alpha}{m} = 2\gamma$ and $\frac{k}{m} = \omega^2$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = 0}$$

For no-friction $\alpha \Rightarrow \gamma = 0$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

we get back S.H.M. equation.