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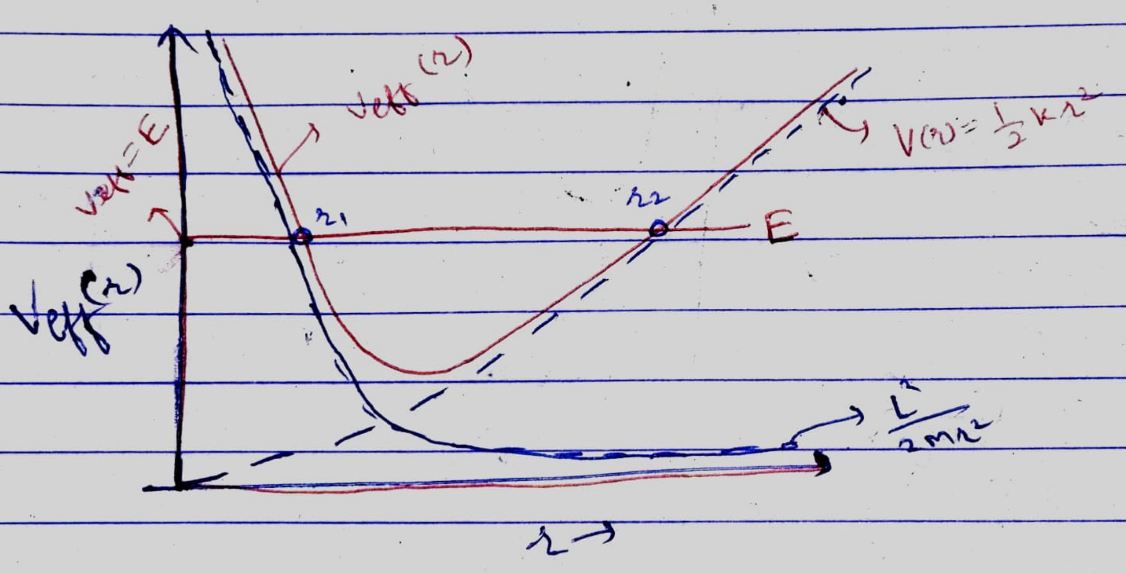
Classical Mechanics

Q① What would be the orbits in the harmonic potential $V = \frac{1}{2}kr^2$ where k is a positive constant and r is the radial distance. Use the method described in the class where you find an effective potential and then analyse it. You can also formally integrate to find the trajectories.

Ans:
$$V(r) = \frac{1}{2}kr^2$$

$$V_{eff} = V(r) + \frac{l^2}{2mr^2} = \frac{1}{2}kr^2 + \frac{1}{2}\frac{l^2}{mr^2}$$



• First, let's calculate some properties of above V_{eff} curve →

ⓐ Minima of the V_{eff} → $\frac{dV_{eff}}{dr} = 0$

Minima as can be seen from the figure. otherwise check $\frac{d^2V}{dr^2}$ value.

$$\Rightarrow kr - \frac{l^2}{mr^3} \Rightarrow r = \left(\frac{l^2}{mk}\right)^{\frac{1}{4}}$$

$$\text{the } (V_{eff})_{min} = \left(\frac{l^2k}{m}\right)^{1/2}$$

ⓑ Values of r_1 & r_2, where $V_{eff} = E$ →

$$\frac{1}{2}kr^2 + \frac{1}{2}\frac{L^2}{mr^2} = E \Rightarrow mkr^4 - 2mEr^2 + L^2 = 0$$

$$\cancel{mkr^4 + L^2 - 2mEr^2 = 0} \Rightarrow r^2 = \frac{2mE \pm \sqrt{4m^2E^2 - 4mkL^2}}{2mk}$$

$$\Rightarrow r^2 = \frac{E \pm \sqrt{E^2 - kL^2/m}}{k} \rightarrow$$ See that $E^2 = kL^2/m$ gives only one real value

then,

$$\boxed{r_1 = \frac{E - \sqrt{E^2 - kL^2/m}}{k}} \quad \& \quad \boxed{r_2 = \frac{E + \sqrt{E^2 - kL^2/m}}{k}}$$

- Now, let us discuss physical meaning of these solⁿ and properties of orbits →

$$\longrightarrow \quad E = V_{eff} + \frac{1}{2}m\dot{r}^2$$

But if $E = V_{eff}(r_1)$ or $E = V_{eff}(r_2)$ then

$$\frac{1}{2}m\dot{r}^2 = 0 \Rightarrow \dot{r} = 0 \Big|_{r=r_1;\ r=r_2}$$

- In an orbit $\dot{r} = 0$ occurs at the orbits boundaries. $\dot{r} = 0$ value at two finite points decides that the orbit is closed.

So, for energy $E > \left(\frac{L^2k}{m}\right)^{1/2}$, all the orbits are closed & elliptical, with r_{min} and r_{max} given above as r_1 & r_2 respectively.

while, for minima, $r = \left(\frac{L^2}{mk}\right)^{\frac{1}{4}} \rightarrow E = \left(\frac{L^2k}{m}\right)^{\frac{1}{2}}$ → orbit is circular as $r_1 = r_2 = r$.

However, from this analysis only, we can't say Elliptical. Only closed orbit can be claimed.