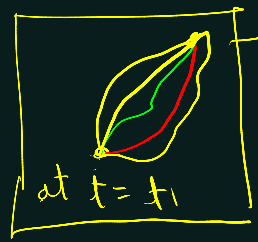
(Euler) Lagrange equations of motion: $\dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) = 0$




In the intermediate region $q = q(t)$ along the main path

For any other path : $\underset{\text{Actual solution.}}{q(t)} + \underbrace{\delta q}_{\text{variation on top of the actual solution}}$

---

Free particle: Not in a potential / field or not subject to any force.

In 3D: $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ ; $V = 0$

How many coordinates : $(q_i) =$ $x$ , $y$ and $z$

(1) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow$$

$$\frac{d}{dt}\left(m\dot{x}\right) = 0 \Rightarrow m \times (\text{acceleration along } x) =$$

(2) $$m\ddot{y} = 0$$

(3) $$m\ddot{z} = 0$$

---

Consider a particle in a potential $V = V(x, y, z)$ and $L = T - V$

(1) $$m\ddot{x} + \frac{\partial V}{\partial x} = 0 \Rightarrow m\ddot{x} = -\frac{\partial V}{\partial x}$$

(2) $$m\ddot{y} + \frac{\partial V}{\partial y} = 0 \Rightarrow m\ddot{y} = -\frac{\partial V}{\partial y}$$

(3) $$m\ddot{z} + \frac{\partial V}{\partial z} = 0 \Rightarrow m\ddot{z} = -\frac{\partial V}{\partial z}$$

$$(m\ddot{x})\hat{x} + (m\ddot{y})\hat{y} + (m\ddot{z})\hat{z} = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)$$

$$\Rightarrow \quad \vec{F} = -\vec{\nabla} V$$

---

Let's see the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad ; \quad L(q, \dot{q}, t)$$

For free particle $L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

If $L$ does not explicitly depend on $q$

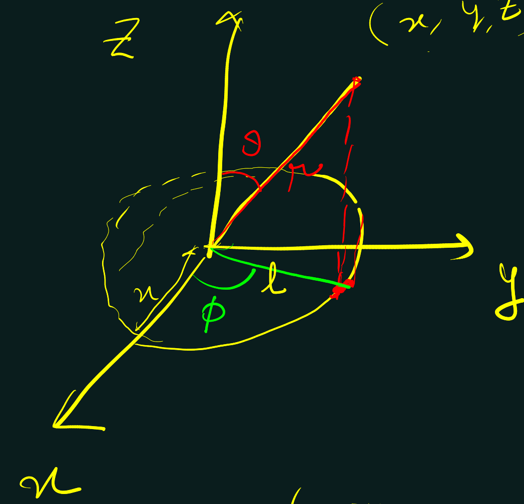
the $\frac{\partial L}{\partial q} = 0$ ; $q$ is called a cyclic coordinate

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}} = \text{constant} \quad \left\{ \begin{array}{l} \text{Conservation} \\ \text{Law} \end{array} \right.$$

$\Rightarrow$ for free particle along $x$ : $\frac{d}{dt}(m\dot{x}) = 0 \Rightarrow m\dot{x} = \text{constant}$

$m v = \text{constant}$

Conservation of momentum




$(x, y, z) \equiv (r, \theta, \phi)$

$l = r \sin\theta$

$x = l \cos\phi = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

$L = T - V$

For a free particle $L = T = 0 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2\right)$$

$L$ does not depend on $\phi$ explicitly therefore $\phi$ is cyclic

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \cancel{\frac{\partial L}{\partial \phi}} = 0$$

$$\Rightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{constant} \Rightarrow m r^2 \sin^2\theta\, \dot{\phi} = \text{constant}$$

$\Rightarrow \quad m l^2 \dot{\phi} = \text{constant} = L_z$ = Conservation of Angular momentum

what if $V \neq 0$ but $V = V_{\text{coulomb}}$

---


frictionless surface

$T = \frac{1}{2} m \dot{x}^2$

$V = \frac{1}{2} k x^2$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m \ddot{x} + k x = 0$$

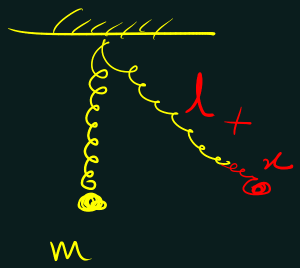
$$\Rightarrow \ddot{x} + \left(\frac{k}{m}\right) x = 0$$

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0$$

$$x = A e^{i\omega t} + B e^{-i\omega t} = C \sin\omega t + D \cos\omega t$$

Spring

Pendulum




spring force constant $k$

mass of blob $m$

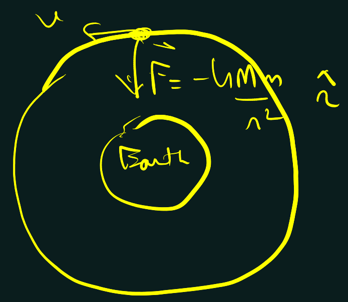
acc. due to gravity $g$

(1) Non-inertial frames of reference
(2) Dissipative forces: Friction

How do we deal with these in Lagrangian Mech.

---

(1) Non-inertial frames




Student A: There is some other pseudo-force

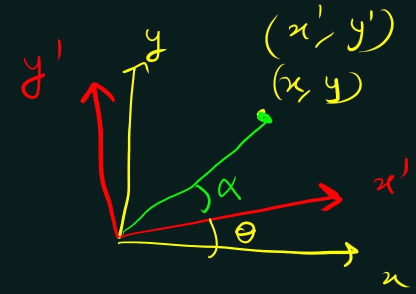
Student B: It is always falling but not falling towards the Earth

Student C: In the frame of ref. of satellite
Centrifugal (?)

To calculate velocity $\underbrace{\frac{mv^2}{R}}_{\text{Centripetal force}} = \frac{GMm}{r^2}$

Lagrangian for a free particle : $L = T - V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - 0$

Rotating frame of reference :




$$x = r \cos(\theta + \alpha)$$
$$y = r \sin(\theta + \alpha)$$

$$x = r \cos\theta \cos\alpha - r \sin\theta \sin\alpha = (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$$
$$y = r \sin\theta \cos\alpha + r \cos\theta \sin\alpha = (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta$$

$$x = x' \cos\theta - y' \sin\theta$$
$$\boxed{\omega = \dot{\theta}}$$
$$y = x' \sin\theta + y' \cos\theta$$

$$z = z$$

$$\dot{x} = \dot{x}' \cos\theta - \dot{y}' \sin\theta - (x' \sin\theta)\omega - (y' \cos\theta)\omega$$
$$\dot{y} = \dot{x}' \sin\theta + \dot{y}' \cos\theta + (x' \cos\theta)\omega - (y' \sin\theta)\omega$$
$$\Rightarrow \quad \dot{x} = (\dot{x}' - y'\omega)\cos\theta - (\dot{y}' + x'\omega)\sin\theta = A\cos\theta - B\sin\theta$$

$$\dot{y} = (\dot{x}' - y'\omega)\sin\theta + (\dot{y}' + x'\omega)\cos\theta = A\sin\theta + B\cos\theta$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (\dot{x}' - y'\omega)^2 + (\dot{y}' + x'\omega)^2 + \dot{z}'^2$$

$$L = \frac{m}{2}\left[(\dot{x}' - y'\omega)^2 + (\dot{y}' + x'\omega)^2 + \dot{z}'^2\right]$$

---

$\omega = \dot{\theta}$ But $\omega$ is a vector $\vec{\omega} = \dot{\theta}\,\hat{z}$

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

Calculate $\vec{\omega} \times \vec{r} = \omega x\,\hat{y} - \omega y\,\hat{x}$

$$\dot{\vec{r}}' + \vec{\omega} \times \vec{r}' = \dot{x}'\hat{x}' + \dot{y}'\hat{y}' + \dot{z}'\hat{z}' + \omega x'\hat{y}' - \omega y'\hat{x}'$$

$$= (\dot{x}' - \omega y')\,\hat{x}' + (\dot{y}' + \omega x')\,\hat{y}' + \dot{z}'\,\hat{z}'$$

$$|\dot{\vec{r}}' + \omega \times \vec{r}'|^2 = (\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 + \dot{z}'^2$$

$$L = \frac{m}{2} \left| \dot{\vec{r}}' + \omega \times \vec{r}' \right|^2$$

Find out the Equations of motion and realize for yourself that there are two additional forces there: Centrifugal & Coriolis

---

2

Dissipative Forces

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

In case of dissipation: the above equations has to be modified.

If dissipative force has the form $F = -\alpha \dot{q}$

then $R = \frac{1}{2} \alpha \dot{q}^2$

And the equations are

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial R}{\partial \dot{q}}$$

$F_{restoring} = -kx$

$F_{friction} = -\alpha \dot{x} = -\alpha v$

$T = \frac{1}{2} m \dot{x}^2 \qquad V = \frac{1}{2} k x^2 \qquad R = \frac{1}{2} \alpha \dot{x}^2$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{dR}{d\dot{x}} \Rightarrow -kx - m\ddot{x} = \alpha \dot{x}$$

$$\Rightarrow m\ddot{x} + \alpha \dot{x} + kx = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \left(\frac{\alpha}{m}\right)\frac{dx}{dt} + \left(\frac{k}{m}\right)x = 0$$

Let's say $\frac{\alpha}{m} = 2\gamma$ and $\frac{k}{m} = \omega^2$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = 0}$$

For no-friction $\alpha \Rightarrow \gamma = 0$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

we get back S.H.M. equation

Let's consider a trial function : $x(t) = A e^{i\omega t}$

$$\dot{x} = \frac{dx}{dt} = i\omega A e^{i\omega t}$$

$$\ddot{x} = \frac{d^2x}{dt^2} = (i\omega)^2 A e^{i\omega t} = -\omega^2 A e^{i\omega t}$$

$$\Rightarrow \quad -\omega^2 A e^{i\omega t} + \omega^2 A e^{i\omega t} = 0$$

---

Try with $x(t) = B e^{-i\omega t}$

This is also a solution (you can verify)

Then a linear combination $x(t) = A e^{i\omega t} + B e^{-i\omega t}$ is also a solution

$$\Rightarrow x(t) = A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t)$$

$$\Rightarrow x(t) = \underbrace{(A+B)}_{D}\cos\omega t + i\underbrace{(A-B)}_{C}\sin\omega t$$

$$x(t) = C\sin\omega t + D\cos\omega t$$

Fix the constants, say $x = 0$ at $t = 0$

---

Let's try one more $x(t) = A\cos(\omega t + \phi)$

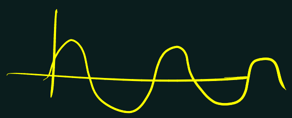
---

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \Rightarrow D^2 + \omega^2 = 0$$

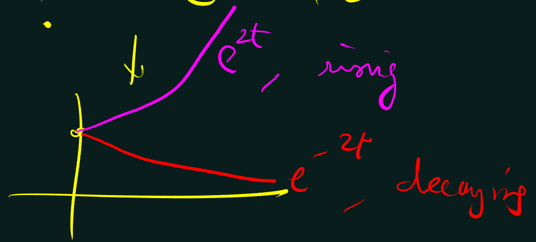
$D$ → $\alpha_1 \to i\omega$, $\alpha_2 \to -i\omega$

$$\Rightarrow x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

$$= Ae^{i\omega t} + Be^{-i\omega t} \rightarrow$$



Say $\frac{d^2x}{dt^2} - 4x = 0 \Rightarrow x(t) = ? = Ae^{2t} + Be^{-2t}$




---

Let's take up the spring problem with friction

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega^2 x = 0 \Rightarrow D^2 + 2\gamma D + \omega^2 = 0$$

Find two roots $\alpha_1$ & $\alpha_2$

$$\alpha_1, \alpha_2 : \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}}{2}$$

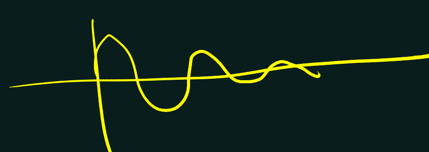
$$\Rightarrow \alpha_1 = -\gamma + \sqrt{\gamma^2 - \omega^2} \quad \text{and} \quad \alpha_2 = -\gamma - \sqrt{\gamma^2 - \omega^2}$$

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t} = Ae^{(-\gamma + \sqrt{\gamma^2 - \omega^2})t} + Be^{(-\gamma - \sqrt{\gamma^2 - \omega^2})t}$$

$$= \underbrace{e^{-\gamma t}}_{\text{Decaying Term}} \Big[ \underbrace{Ae^{t\sqrt{\gamma^2 - \omega^2}} + Be^{-\sqrt{\gamma^2 - \omega^2}\, t}}_{\text{if } \gamma > \omega \text{ then rising or decaying}} \Big]$$

If $\gamma < \omega$ then consider $\gamma^2 - \omega^2 = -K^2$ = which is negative

$$x(t) = e^{-\gamma t}\left[Ae^{iKt} + Be^{-iKt}\right]$$



Oscillator with Dissipation / damping and also forcing

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = F \cos \omega_0 t$$

$$= \frac{F}{2} e^{i\omega_0 t} + \frac{F}{2} e^{-i\omega_0 t}$$

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = \frac{F}{2} e^{i\omega_0 t}$$

Trial solution $x(t) = A e^{i\omega_0 t}$

$$\Rightarrow (i\omega_0)^2 A e^{i\omega_0 t} + 2\gamma (i\omega_0) A e^{i\omega_0 t} + \omega^2 A e^{i\omega_0 t} = \frac{F}{2} e^{i\omega_0 t}$$

$$\Rightarrow A = \frac{F/2}{\omega^2 + 2i\gamma\omega_0 - \omega_0^2}$$

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = \frac{F}{2} e^{-i\omega_0 t}$$

Try with $x(t) = B e^{-i\omega_0 t}$

$$B = \frac{F/2}{\omega^2 - 2i\gamma\omega_0 - \omega_0^2}$$

---

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega^2 x = F \cos\omega_0 t$$

$$x(t) = A e^{i\omega_0 t} + B e^{-i\omega_0 t} = \frac{F/2}{\omega^2 + 2i\gamma\omega_0 - \omega_0^2} e^{i\omega_0 t} + \frac{F/2}{\omega^2 - 2i\gamma\omega_0 - \omega_0^2} e^{-i\omega_0 t}$$

(Real) + (Imaginary)

Homework

$$\frac{F}{2}\left[\frac{\alpha + i\beta}{(\omega^2 + \omega_0^2) + i(2\gamma\omega_0)} + \frac{\alpha - i\beta}{(\omega^2 - \omega_0^2) - i(2\gamma\omega_0)}\right]$$

Astrophysics:

we will use the Lagrangian dynamics to solve the central force problem: Planetary Motion: (Atomic Model)

Gravity

$$F_G = \frac{G m_1 m_2}{r^2}$$

Electrostatic

$$F_E = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$$

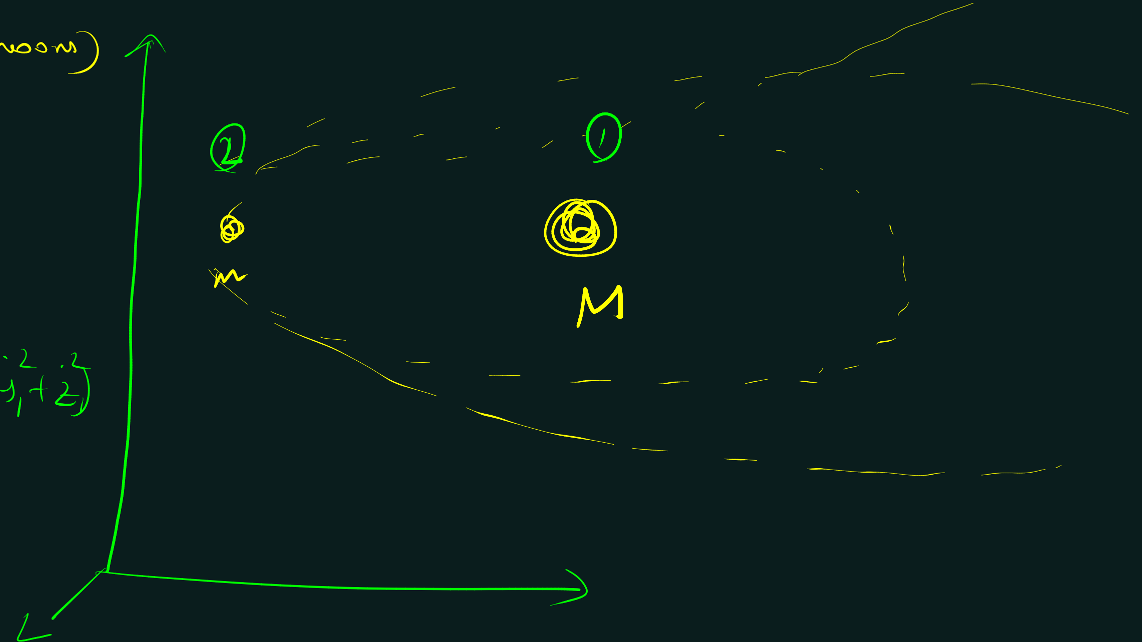
$$\frac{G}{(1/4\pi\epsilon_0)} \approx \frac{10^{-11}}{10^{+9}} \approx 10^{-20}$$

(Gravity is extremely weak)

Then why Gravity moves planets, not the electromagnetic force

Astronomy: Dynamics (planets, moons)

Two Body Problem




$$\mathcal{L} = T_1 + T_2 - V_1 - V_2$$

$$= \frac{m}{2}\left(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right) + \frac{M}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2\right) - V_1 - V_2$$

This problem is equivalent to

motion of one-body moving in central potential / force field

(centred at centre of mass of 2-body system)

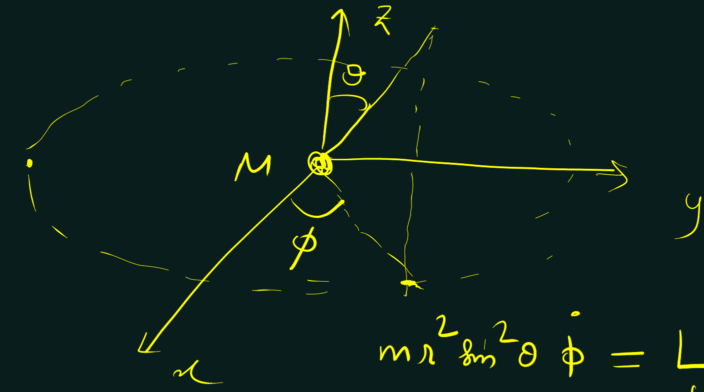
Equivalent to $\downarrow$

1-body in $V_{central}$

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{GM\mu}{r}$$

$$\mu = \frac{mM}{m+M} \approx m \text{ if } m << M$$

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + \frac{GMm}{r}$$




Go to $r, \theta, \phi$ system

$$m r^2 \sin^2\theta\, \dot{\phi} = L$$

Angular momentum along z

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) + \frac{GMm}{r}$$

$\phi$ is cyclic: So, $\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = 0 \Rightarrow \frac{d}{dt}\left(m r^2\sin^2\theta\,\dot{\phi}\right) = 0$

$$\Rightarrow m r^2 \sin^2\theta\,\dot{\phi} = L$$

If $L$ is preserved/conserved then $\theta = \frac{\pi}{2}$; motion is in the plane

$x = r\cos\phi$

$y = r\sin\phi$ $\qquad \theta = \frac{\pi}{2}$

$z = 0$

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + \cancel{r^2\dot{\theta}^2} + r^2\sin^2\theta\,\dot{\phi}^2\right) - \frac{GMm}{r}$$

1

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{GMm}{r}$$

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = 0$$

$$\Rightarrow \quad m r \dot{\phi}^2 - m\ddot{r} = \frac{GMm}{r^2}$$

$$\Rightarrow \quad \ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2}$$

Let's extract all the information about the possible solution physically

$$\mathcal{L} = T - V$$

$$E = T + V = \frac{m\left(\dot{r}^2 + r^2\dot{\phi}^2\right)}{2} - \frac{GMm}{r}$$

But $m r^2 \dot{\phi} = L \Rightarrow \dot{\phi} = \dfrac{L}{m r^2}$

$$\Rightarrow E = \frac{m}{2}\dot{r}^2 + \frac{m}{2} r^2 \frac{L^2}{m^2 r^4} - \frac{GMm}{r}$$

$$= \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

$$E = \frac{1}{2} m \dot{r}^2 + V_{eff} \quad \text{where} \quad V_{eff} = \frac{L^2}{2mr^2} - \frac{GMm}{r} = \frac{A}{r^2} - \frac{B}{r}$$

where $A = \frac{L^2}{2m}$ and $B = GMm$

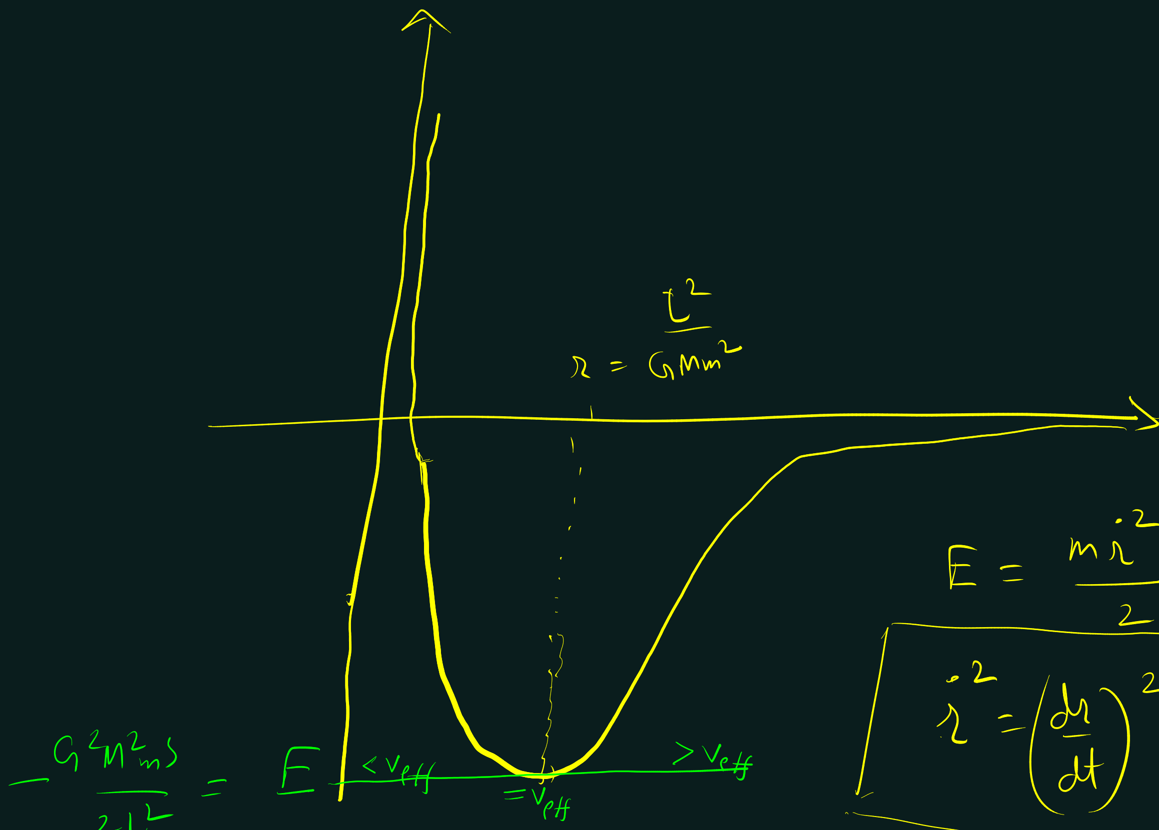
Plot $V_{eff}$ as a function of $r$:

Minima: $\frac{dV_{eff}}{dr} = 0 \Rightarrow -\frac{2A}{r^3} + \frac{B}{r^2} = 0$

$$\Rightarrow r = \frac{2A}{B} = \frac{\cancel{2} L^2}{\cancel{2} G M m^2}$$

Minima at $r = \frac{L^2}{GMm^2}$ and $V_{eff}$ at minimum $= \frac{1}{r}\left[\frac{A GMm^2}{L^2} - B\right]$

$$V_{eff, min} = \frac{GMm^2}{L^2}\left[\frac{L^2}{2m}\frac{GMm^2}{L^2} - GMm\right] = -\frac{GMm^3}{2L^2}$$




$$E = \frac{m\dot{r}^2}{2} + V_{eff}$$

$$\boxed{\dot{r}^2 = \left(\frac{dr}{dt}\right)^2 = \frac{2}{m}\left[E - V_{eff}\right]}$$

$$V_{eff} = \frac{A}{r^2} - \frac{B}{r}$$

$E$ is a constant of motion like $L$

Consider different values of $E$ ; say we consider $E = \frac{GMm^2}{-2L^2} \equiv V_{eff,\,min}$

$$\Rightarrow \quad \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - V_{eff}\right)} = 0 \text{ at min. point}$$

$$E = \frac{m\dot{r}^2}{2} + \frac{m r^2 \dot{\phi}^2}{2} - \frac{GMm}{r} \quad \text{where } \dot{\phi} = \frac{L}{mr^2}$$

Conservation of Angular momentum, $\phi$ is cyclic

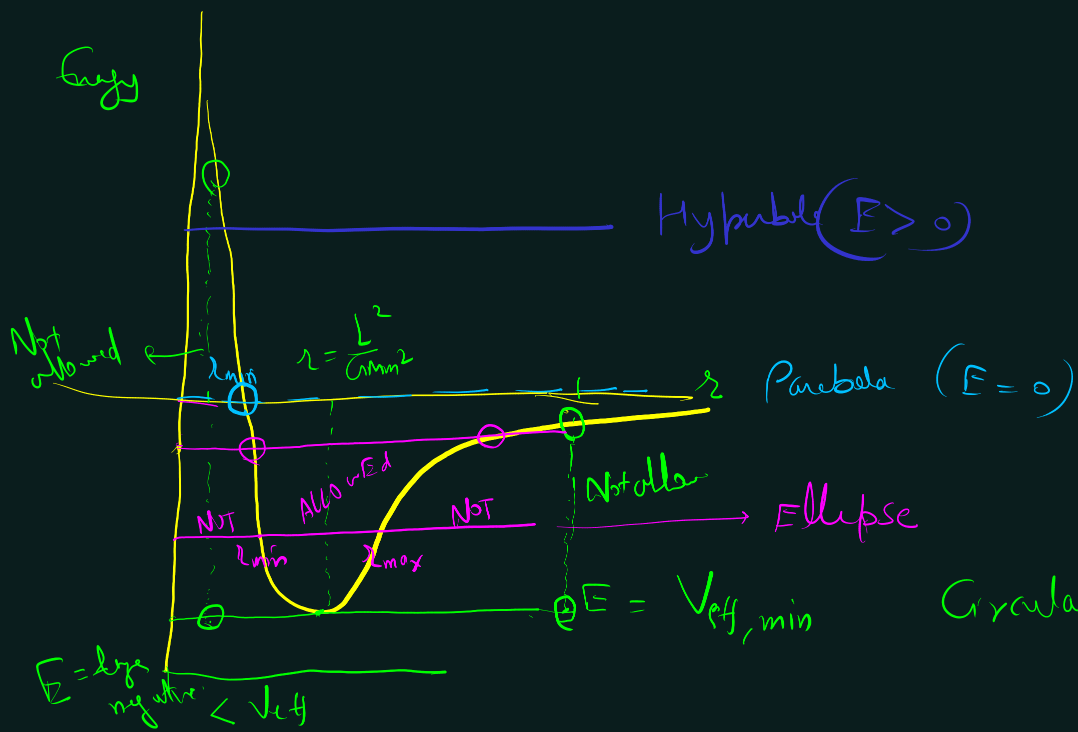
$$E = \frac{m\dot{r}^2}{2} + \underbrace{\frac{L^2}{2mr^2} - \frac{GMm}{r}}_{V_{eff}(r)} \checkmark$$

$$\Rightarrow E = \frac{m\dot{r}^2}{2} + V_{eff}(r)$$

$$\Rightarrow \dot{r} = \sqrt{\frac{2}{m}\left[E - V_{eff}(r)\right]}$$

At $r = \frac{L^2}{GMm^2}$; for $E = V_{eff,min}$

$\dot{r} = 0 \Rightarrow r = \text{constant} = \frac{L^2}{GMm^2}$




Circular Motion / trajectory

What are the $r_{min}$ & $r_{max}$ for Elliptical orbit. Let's calculate below

$$E = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

Ellipse: At $r = r_{min}$ or at $r = r_{max}$ ; $\dot{r} = 0$

$$\Rightarrow E = \frac{L^2}{2m} u^2 - GMm\, u \qquad \text{where we defined } u = \frac{1}{r}$$

$$\Rightarrow u^2 - 2\frac{GMm^2}{L^2} u - \frac{2mE}{L^2} = 0$$

$$\Rightarrow u = \frac{2\frac{GMm^2}{L^2} \pm \sqrt{4\frac{G^2M^2m^4}{L^4} + \frac{8mE}{L^2}}}{2} =$$

$$= \frac{GMm^2}{L^2}\left[1 \pm \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}\right]$$

$$\Rightarrow r = \frac{L^2}{GMm^2}\left[1 \pm \varepsilon\right]^{-1} \quad \text{where } \varepsilon = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}} = \text{Eccentricity}$$

$$r_{min} = \frac{L^2}{GMm^2(1+\varepsilon)}$$

$$r_{max} = \frac{L^2}{GMm^2(1-\varepsilon)}$$

for $E = 0$; $\varepsilon = 1$

$$\Rightarrow r_{min} = \frac{L^2}{2GMm^2} \; ; \; r_{max} = \infty$$

what if eccentricity $= 0 \Rightarrow \varepsilon = 0 \Rightarrow r_{min} = \frac{L^2}{GMm^2}$

$r_{max} = \frac{L^2}{GMm^2}$

which is a circle.

Entire trajectory is understood.

We will solve the Lagrangian Equation Now

Eqn is : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$

$$\Rightarrow \mathcal{L} = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\phi}^2}{2} + \frac{GMm}{r}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \; ; \; \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 - \frac{GMm}{r^2}$$

$$\Rightarrow \quad m\ddot{r} - mr\dot{\phi}^2 + \frac{GMm}{r^2} = 0$$

$$\Rightarrow \quad \ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2}$$

Also from $\phi$ equation

$$mr^2\dot{\phi} = L \Rightarrow \dot{\phi} = \frac{L}{mr^2}$$

$\Rightarrow$

$$\Rightarrow \quad \ddot{r} - \frac{L^2}{m^2 r^3} = -\frac{GM}{r^2}$$

Define $u = \frac{1}{r} \Rightarrow r = \frac{1}{u} \Rightarrow \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt}$

$$\boxed{\dot{\phi} = \frac{L}{mr^2}}$$

$$\ddot{r} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{d\phi}\left[\frac{d\phi}{dt}\frac{dr}{d\phi}\right]\frac{d\phi}{dt} = \frac{d}{d\phi}\left[\dot{\phi}\frac{dr}{d\phi}\right]\dot{\phi}$$

$$\Rightarrow \ddot{r} = \frac{L}{mr^2}\frac{d}{d\phi}\left[\frac{L}{mr^2}\left(-\frac{1}{u^2}\frac{du}{d\phi}\right)\right]$$

$$= -\frac{L^2}{m^2}u^2\frac{d^2u}{d\phi^2}$$

$$\Rightarrow \quad -\frac{L^2}{m^2}u^2\frac{d^2u}{d\phi^2} - \frac{L^2}{m^2}u^3 = -GMu^2$$

$$\Rightarrow \boxed{\frac{d^2u}{d\phi^2} + u = \frac{GMm^2}{L^2}} \quad \text{——— (1)}$$

$$\frac{d^2u}{d\phi^2} + u = 0 + \frac{GMm^2}{L^2}$$

First solve $\frac{d2u}{d\phi^2} + u = 0$ then add to it the sol. of $\frac{d2u}{d\phi^2} + u = \frac{GMm^2}{L^2}$ with trial method

$$\Rightarrow \quad u = A\cos(\phi + \phi_0)$$

Let's say now the solution of (1) is $u = A\cos(\phi + \phi_0) + C$

$$\Rightarrow \quad \frac{d^2u}{d\phi^2} = -A\cos(\phi + \phi_0)$$

$$\Rightarrow \frac{d^2u}{d\phi^2} + u = -A\cos(\phi + \phi_0) + A\cos(\phi + \phi_0) + C = \frac{GMm^2}{L^2}$$

$$\Rightarrow C = \frac{GMm^2}{L^2}$$

Therefore

$$u = A\cos(\phi + \phi_0) + \frac{GMm^2}{L^2}$$