Tutorial 1 (IC 200)

Q→1: Prove that the motion of a particle/body in a central force field remains in a plane.

Ans: For a central force field:

Force, $\vec{F} = f(r)\hat{r}$

then torque, $\vec{\tau} = \vec{r} \times \vec{F}$

$= \vec{r} \times f(r)\hat{r} = \vec{0}$

$\vec{J} = m(\vec{r} \times \vec{v})$

we know that $\vec{\tau} = \frac{d\vec{J}}{dt}$; $\vec{J}$ = angular moment (um)

$\Rightarrow \frac{d\vec{J}}{dt} = \vec{0} \Rightarrow \boxed{\vec{J} = \text{constant}}$ — ①

since, $\vec{J}.\vec{r} = m(\vec{r} \times \vec{v}).\vec{r}$

$= 0$

or $|\vec{J}||\vec{r}| \cos\theta = 0$

if $|\vec{J}| \neq 0$, then $\theta = 90°$

or $\boxed{\vec{J} \perp \vec{r}}$ — ②

Using statement ① and ②, we can say that $\vec{r}$ always remains in a plane which has normal parallel to $\vec{J}$.

Things for student →

① prove that if $\vec{J} = m(\vec{r} \times \vec{v})$ & $\vec{\tau} = \vec{r} \times \vec{F}$, then $\vec{\tau} = \frac{d\vec{J}}{dt}$

② Prove that if $|\vec{J}| = 0$, then particle follow straight line.

Q→2: Show that problem of the dynamics of two bodies interacting with each other via gravitational force can be transformed to the problem of the dynamics of one body moving in central force field.

Ans: ~~Ist method (Newtonian mechanics)~~

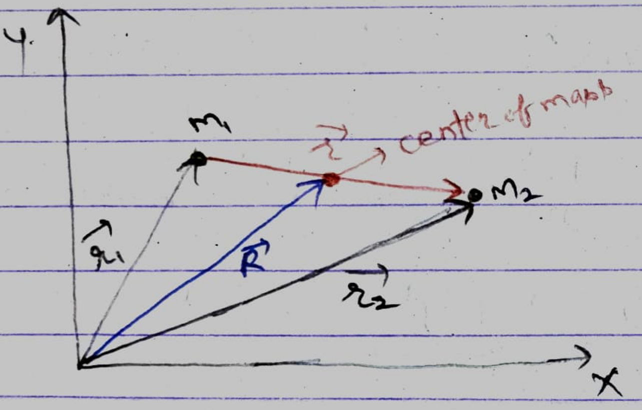

fig (1)

we know →

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad (1)$$

and

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \qquad (2)$$

Solving (1) & (2) for $\vec{r}_1$ and $\vec{r}_2$ →

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1+m_2}\vec{r} \qquad (3)$$

& $$\vec{r}_2 = \vec{R} + \frac{m_1}{m_1+m_2}\vec{r} \qquad (4)$$

from eq (3) & (4), we can say that if we know $\vec{r}$ & $\vec{R}$, we can tell the positions of $\vec{r}_1$ & $\vec{r}_2$ and so we can tell the trajectories of bodies.

But $\ddot{\vec{R}} = \vec{0}$ ⟶ since $\vec{F}_1 = -\vec{F}_2$

$$\ddot{\vec{R}} = \frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{(m_1+m_2)}$$

$$= \frac{\vec{F}_1 + \vec{F}_2}{m_1+m_2} = \vec{0}$$

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Then $\ddot{\vec{R}} = \vec{0} \Rightarrow \dot{\vec{R}} = \text{constant}$

or simply we can take

$$\dot{\vec{R}} = \vec{0} \rightarrow \vec{R} = \text{constant}$$

★ Transforming two body problem to one body central :-

Method - I (Newtonian method) :-

From fig (1) we have →

$$\cancel{m_1} \frac{d^2\vec{r}_1}{dt^2} = + \frac{G\cancel{m_1} m_2}{|\vec{r}|^3} \vec{r} \quad \text{—— (5)}$$

&

$$\cancel{m_2} \frac{d^2\vec{r}_2}{dt^2} = - \frac{G m_1 \cancel{m_2}}{|\vec{r}|^3} \vec{r} \quad \text{—— (6)}$$

$$\frac{d^2\vec{r}_1}{dt^2} - \frac{d^2\vec{r}_2}{dt^2} = \frac{G}{|\vec{r}|^3} (m_1 + m_2) \vec{r} \qquad \text{eq. (5) - (6)}$$

$$\Rightarrow \frac{m_1 m_2}{(m_1 + m_2)} \frac{d^2\vec{r}}{dt^2} = - \frac{G m_1 m_2}{|\vec{r}|^3} \vec{r}$$

can

$$\Rightarrow \boxed{\mu \frac{d^2\vec{r}}{dt^2} = - f(r) \vec{r}} \quad \text{—— (7)}$$

where $\mu = \left(\frac{m_1 m_2}{m_1 + m_2}\right)$

Eq. (7) has only one variable $\vec{r}$ and also follow central force rule.

For student: can ~~3d~~ 3 body problem transformed to 2 body problem ?

Method 3: Lagrangian Method :-

Langrangian → $L = T - V$

$T =$ kinetic energy

$V =$ potential Energy

$$\Rightarrow \quad L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - \frac{G m_1 m_2}{|\vec{r}_2 - \vec{r}_1|}$$

But we ~~know~~ can $\vec{r}_1$ and $\vec{r}_2$ into $\vec{r}$ and $\vec{R}$ using (3) & (4) →

Then
$$L = \frac{1}{2}\left[m_1 \left(\dot{\vec{R}}^2 + \frac{m_2^2}{(m_1+m_2)^2} \dot{\vec{r}}^2 - \frac{2 m_2}{(m_1+m_2)} \dot{\vec{R}} \cdot \dot{\vec{r}} \right) + m_2 \left(\dot{\vec{R}}^2 + \frac{m_1^2}{(m_1+m_2)^2} \dot{\vec{r}}^2 + \frac{2 m_1}{(m_1+m_2)} \dot{\vec{R}} \cdot \dot{\vec{r}} \right) \right] - \frac{G m_1 m_2}{|\vec{r}|}$$

assuming $\dot{\vec{R}} = \vec{0}$

$$\Rightarrow \quad L = \frac{1}{2}\left[(m_1+m_2) \dot{\vec{R}}^2 + \frac{m_1 m_2 (m_1+m_2)}{(m_1+m_2)^2} \dot{\vec{r}}^2 \right] - \frac{G m_1 m_2}{r}$$

$$\Rightarrow \quad \boxed{L = \frac{1}{2} \mu \dot{\vec{r}}^2 - \frac{k}{r}}$$

$\mu = \frac{m_1 m_2}{m_1 + m_2}$

$k = G m_1 m_2$

Lagrangian is only function of one variable r, so it can be ~~assum~~ dealt as one body problem.

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Q→③: ~~Prove Kepler's second law that a planet going around the Sun travels equal amount of area in a given constant time interval no matter which part of~~ of the orbit it is ~~covered~~ covering.

Ans:- if δt is very small then $\delta\theta$ & δr are also small →

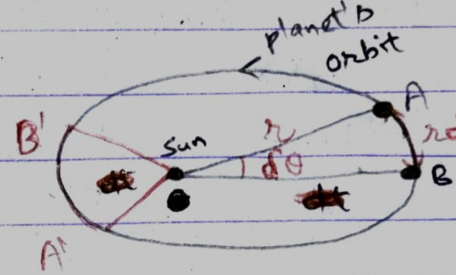

fig ②

Taking $\delta t \to 0$

$\frown OAB \sim \Delta OAB$

$$\Rightarrow \underset{\to 0}{\text{area}(OAB)} \cong \underset{\to 0}{\text{area}(\Delta OAB)}$$

$$\Rightarrow \text{area}(OAB) = \frac{1}{2} r \cdot r\delta\theta = \frac{r^2 \delta\theta}{2}$$

Then area covered per unit time

$$\left. \frac{\delta(\text{area})}{\delta t} \right|_{\delta t \to 0} = \left. \frac{1}{2} r^2 \frac{\delta\theta}{\delta t} \right|_{\delta t \to 0}$$

$$\Rightarrow \boxed{\text{area swept per unit time} = \frac{1}{2}\frac{m}{m} r^2 \frac{d\theta}{dt} = \frac{|\vec{L}|}{2m}}$$

→ constant as $\vec{L}$ = Constan

For Student: If you notice above proof is also valid for unbound orbit. then use 2nd law to explain why parti with nonzero angular momentum will not fall into the centre.

Eccentricity

Q→4 If $\frac{1}{r} = \frac{GMm^2}{L^2}(1+\epsilon\cos\theta)$, prove Kepler's third law $T^2 \propto a^3$; a is semi major axis.

Ans:-

We know from 2nd law of Kepler →

$$\frac{d\,\text{Area}}{dt} = \frac{L}{2m}$$

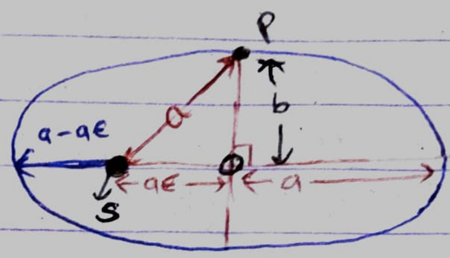

fig ③

$\Rightarrow$ Total Area in one cycle $= \frac{L}{2m}T$, T = time period

$$\Rightarrow \boxed{\pi ab = \frac{L}{2m}T} \quad \text{——} \ (1)$$

To get relation between a & T, we should remove b & L →

(a) $$\frac{1}{r_{min}} = \frac{GMm^2}{L^2}(1+\epsilon\cos\theta)\Big|_{max}$$

$$\Rightarrow \frac{1}{a(1-\epsilon)} = \frac{GMm^2}{L^2}(1+\epsilon)$$

using fig ③ & $\cos\theta|_{max} = 1$

$$\Rightarrow \frac{L^2}{(1-\epsilon^2)} = a.GMm^2 \Rightarrow L^2 = a(1-\epsilon^2)GMm^2$$

(b) $\Delta OSP \rightarrow$ $a^2 - a^2\epsilon^2 = b^2$

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Putting these values in eq ① →

$$\cancel{\pi a(a} \quad \pi a(a\sqrt{1-e^2}) = \frac{T}{2m}\sqrt{a(1-e^2)}\sqrt{GMm^2}$$

$$\Rightarrow 2\cancel{m}\pi a^{2-\frac{1}{2}} = \sqrt{GM\cancel{m^2}}\, T$$

$$\Rightarrow a^3 = \frac{GM}{(2\pi)^2} T^2$$

$$\Rightarrow \boxed{a^3 \propto T^2}$$ and proportional constant $= \frac{GM}{(2\pi)^2}$

only depends on Sun's mass, so valid for all planets. independent to their mass.

Q→5: Derive the orbits (and also try plotting) for the potential energy $V = \alpha r^2$ and also find out for $V = \frac{\alpha}{r^2}$ where α is a constant.

Ans:- (A) $V = \alpha r^2$

~~we have~~ For plotting the orbit we need r & θ relation in polar system, but we have Energy conservation and angular momentum conservation, these two constraint on three parameter → r, θ and t. So we can always find a relation between any of these 2 quantities.

Think

Let us work out mathematics :-

Energy conservation →

$$\underset{\text{constant}}{E} = \underbrace{\frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\theta}^2}_{K.E} + \underbrace{\alpha r^2}_{V} \qquad (1)$$

Angular momentum conservation :-

$$m r^2 \dot{\theta} = L \rightarrow \text{constant}$$

$$\Rightarrow \left(\frac{d\theta}{dt}\right)^2 = \dot{\theta}^2 = \frac{L^2}{m^2 r^4} \quad \text{--- (2)}$$

putting (2) Into (1) →

$$\left(\frac{dr}{dt}\right)^2 = \dot{r}^2 = \frac{2}{m}\left(E - \frac{m}{2}\frac{L^2 r^2}{m^2 r^4} - \alpha r^2\right) \quad \text{--- (3)}$$

To get r & θ relation we should eliminate t →

So ~~(2) ÷ (3)~~ (3) ÷ (2) →

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^6}{r^2}\left[\frac{2mE}{L^2} - \frac{1}{r^2} - \frac{2\alpha m r^2}{L^2}\right]$$

$$\Rightarrow \left(\frac{1}{r^3}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2 r^2} - \frac{1}{r^4} - \frac{2\alpha m}{L^2}$$

$$\Rightarrow \left[\frac{1}{(-2)}\frac{d\left(\frac{1}{r^2}\right)}{d\theta}\right]^2 = \frac{2mE}{L^2}\frac{1}{r^2} - \frac{1}{(r^2)^2} - \frac{2\alpha m}{L^2}$$

If we put $y = \frac{1}{r^2}$, then

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$$\left(\frac{1}{2}\frac{dy}{d\theta}\right)^2 = -\left[y^2 - 2\cdot\frac{mE}{L^2}\cdot y + \left(\frac{mE}{L^2}\right)^2 - \left(\frac{mE}{L^2}\right)^2\right] - \frac{2\alpha m}{L^2}$$

$$\Rightarrow \left(\frac{1}{2}\frac{dy}{d\theta}\right)^2 = -\left(y - \frac{mE}{L^2}\right)^2 - \frac{2m\alpha}{L^2} + \left(\frac{mE}{L^2}\right)^2$$

$$\Rightarrow \left(\frac{1}{2}\frac{d\left(y-\frac{mE}{L^2}\right)}{d\theta}\right)^2 = -\left(y-\frac{mE}{L^2}\right)^2 \underbrace{- \frac{2m\alpha}{L^2} + \left(\frac{mE}{L^2}\right)^2}_{c^2}$$

$\Rightarrow$ [put $z = y - \frac{mE}{L^2}$

$$\left(\frac{1}{2}\frac{dz}{d\theta}\right)^2 = c^2 - z^2 \Rightarrow \boxed{\left(\frac{dz}{d\theta}\right)^2 = 4(c^2 - z^2)}$$

if $c^2 = -ve$, $\left(\frac{dz}{d\theta}\right)^2 = -ve$
not possible

Taking only ~~+ve~~ $-ve$ solution $\rightarrow$

$$\frac{dz}{d\theta} = -2\sqrt{c^2 - z^2}$$

$$\Rightarrow -\int \frac{dz}{\sqrt{c^2-z^2}} = \int 2\,d\theta$$

$$\Rightarrow -\int \frac{dz/c}{\sqrt{1-\frac{z^2}{c^2}}} = 2\theta - 2\theta_0$$

$$\Rightarrow \cos^{-1}\left(\frac{z}{c}\right) = 2(\theta - \theta_0)$$

$$\Rightarrow \boxed{z = c\cos 2(\theta - \theta_0)}$$

for our choice we can take $\theta_0 = 0 \rightarrow$

~~z = c cos θ~~ $\quad z = c \cos 2\theta$

$$\Rightarrow \frac{1}{r^2} - \frac{mE}{L^2} = \sqrt{\frac{m^2E^2}{L^4} - \frac{2m\alpha}{L^2}} \cos 2\theta$$

$$\Rightarrow \frac{1}{r^2} = \frac{mE}{L^2}\left(1 + \underbrace{\frac{L^2}{mE}\sqrt{\frac{m^2E^2}{L^4} - \frac{2m\alpha}{L^2}}}_{\varepsilon} \cdot \cos 2\theta\right)$$

$$\Rightarrow \boxed{\frac{1}{r^2} = \frac{mE}{L^2}\left(1 + \varepsilon \cos 2\theta\right)} \text{ --- } (4)$$

or $\varepsilon = \sqrt{1 - \frac{2\alpha L^2}{mE^2}}$

ellipse

(1) put $r^2 = x^2 + y^2$ and $r\cos\theta = x$, and prove that eq. (4) tells the eq. of ellipse.

For students (2) if we put $\varepsilon = 0$ then

$$\frac{1}{r^2} = \frac{mE}{L^2} \Rightarrow r = \text{constant}$$

= circle equation.

which is similar if eccentricity is zero, then orbit is circular but ε is not eccentricity in equation (4). Then find out formula for eccentricity

(B) $V = \alpha / r^2$

→ As the previous case, we will get

$$\dot{r}^2 = \frac{2}{m}\left(E - \frac{L^2}{2mr^2} - \alpha/r^2\right) \quad \text{—— (1)}$$

and $$\dot{\theta}^2 = \frac{L^2}{m^2 r^4} \quad \text{—— (2)}$$

(1) ÷ (2) → $$\left(\frac{dr}{d\theta}\right)^2 = r^4\left[\frac{2mE}{L^2} - \frac{1}{r^2} - \frac{2\alpha m}{L^2 r^2}\right]$$

$$\Rightarrow \left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{1}{r^2}\underbrace{\left(1 + \frac{2\alpha m}{L^2}\right)}_{C^2}$$

$$\Rightarrow \left(\frac{d\left(\frac{1}{r}\right)}{d\theta}\right)^2 = \frac{2mE}{L^2} - \frac{C^2}{r^2}$$

$$\Rightarrow \boxed{\left(\frac{dy}{d\theta}\right)^2 + C^2 y^2 = \frac{2mE}{L^2}} \quad \text{—— (3)}$$

Now → (A) if $C^2 = 0$, then

$$(dy/d\theta)^2 = \frac{2mE}{L^2}$$

$$y = \theta\sqrt{\frac{2mE}{L^2}} + \text{Const}$$

if $\theta \to 0$ at $r \to \infty$ then const $= 0$.

spiral motion

$$\Rightarrow \boxed{r = \frac{1}{\theta}\sqrt{\frac{2mE}{L^2}}} \quad \text{—— (4)}$$

~~For students~~

(B) if $c^2 > 0 \rightarrow$ $\xrightarrow{}$ unbound orbit

$$\boxed{\frac{1}{r} = \frac{1}{c}\sqrt{\frac{2mE}{L^2}}\, \sin c\theta} \quad \text{--- (5)}$$

(C) if $c^2 < 0 \rightarrow$ $\xrightarrow{}$ Spiral coming towards $r \to 0$ as $\theta \to \infty$

$$\boxed{\frac{1}{r} = \frac{1}{c'}\sqrt{\frac{2mE}{L^2}}\, \sinh c'\theta}, \text{ where } c'^2 = -c^2 \quad \text{--- (6)}$$

For students:

(1) eq. (4), (5) and (6) orbit shapes are mentioned, figure it out why.

(2) In spiral motion, particle will take infinite spirals to reach the center as angular momentum conservation fails ~~after~~ at $r=0$, can you prove this mathematically that it will take infinite time to achieve $r=0$.

~~use~~ $\frac{dr}{dt}$

$\downarrow$

Solve eq (1) to get r and t relation.

(3) If you can notice $c^2 = 0$ & $c^2 < 0$, only possible if $\alpha = -ve$, and we get ~~[illegible]~~ attractive force. Can we use this logic to explain the nature of orbits.

(4) Use the above logic to explain why eq (5) tells an unbound orbit.

Q→6: What is the eccentricity, ε, for the Earth's orbit around the Sun. What would you say: the orbit is close to circular or more like elliptical?

Ans:

Looking at Fig (4), we can say →

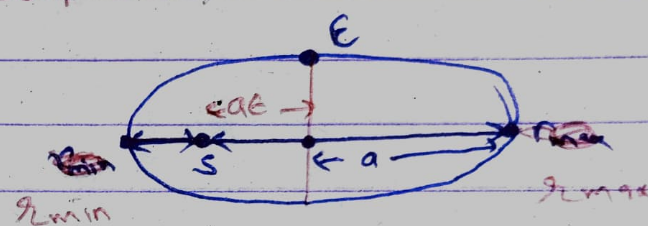

Fig (4)

$$r_{min} = a - a\varepsilon = a(1-\varepsilon)$$

& $$r_{max} = a(1+\varepsilon)$$

then $$\frac{r_{min}}{r_{max}} = \frac{1-\varepsilon}{1+\varepsilon} \Rightarrow \boxed{\frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \varepsilon}$$

We can use data from any website to get r_{min} & r_{max}.

and ε is very close to zero so almost circular.