Stars:

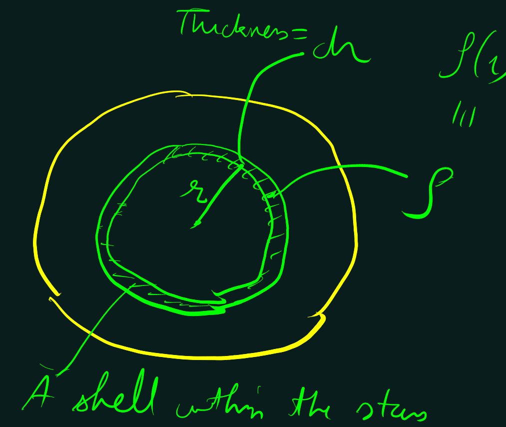
Radius of the Sun $\simeq 7 \times 10^5$ km : Sun is a normal star.

Normal
Classical (1) classical physics is mostly enough to explain the structure, stability and existence of Sun
OR
Main Sequence
star → Main Sequence in the H-R diagram.

(2) Quantum Stars: white dwarf, Neutron stars,
→ Black Holes

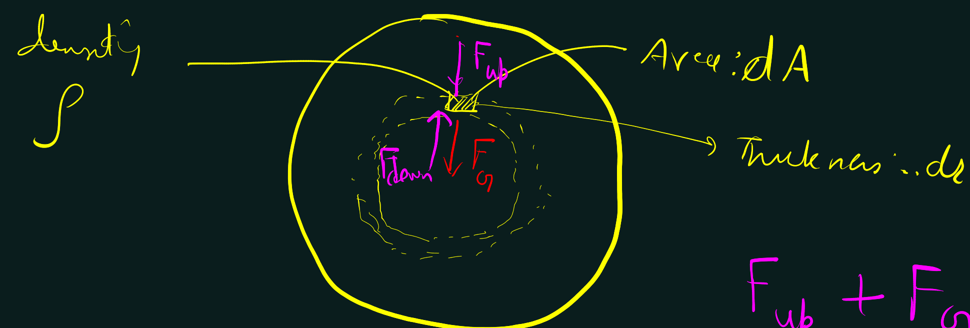
Astrophysics Books:
1. Astronomy: A Physical perspective, M. Kutner
2. Foundations of Astrophysics, B. Ryden
3. Physical Universe, F. Shu
4. An Introduction to Modern Astrophysics, Caroll & Ostlie

Stability of stars: Basic Equation



Mass of the shell: $dM = 4\pi r^2 \rho \, dr$

$$\Rightarrow \boxed{\frac{dM}{dr} = 4\pi r^2 \rho} \qquad (1)$$



$$F_{up} + F_G = F_{down}$$

$$(P + dP)\, dA + \frac{GM \, dm}{r^2} = P \, dA$$

$$\Rightarrow (P + dP)\, \cancel{dA} + \frac{GM \, \rho \, \cancel{dA} \, dr}{r^2} = P \, \cancel{dA}$$

$$\boxed{\frac{dP}{dr} = -\frac{GM}{r^2}\rho} \quad \text{— (2)}$$

$$\boxed{\frac{dM}{dr} = 4\pi r^2 \rho} \quad (1)$$

Mass, Pressure, density

M, P, ρ are unknowns

student propose: Ideal gas law: $P \propto \rho T \Rightarrow P = \frac{\rho k_B T}{m_{proton}}$

But this introduce one more variable which is T

we need to consider some general way

Let's assume $P = K\rho^{\gamma}$

K = constant

(A general power law polytropic relation through index γ)

Let's solve analytically:

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho$$

$$P = K\rho^\gamma$$

Scaling (crude) way

$$\frac{M}{R} = R^2 \frac{M}{R^3} = \frac{M}{R}$$

$$\frac{P}{R} = \frac{M}{R^2}\frac{M}{R^3} \Rightarrow P = \frac{GM^2}{R^4}$$

$$P = K\frac{M^\gamma}{R^{3\gamma}}$$

$$\Rightarrow \quad K\frac{M^\gamma}{R^{3\gamma}} = \frac{GM^2}{R^4} \Rightarrow M^{\gamma-2} = \frac{G}{K} R^{3\gamma-4}$$

for $\gamma = \frac{5}{3}$:

$$M^{\frac{5}{3}-2} = \frac{G}{K} R^{3\cdot\frac{5}{3}-4}$$

$$\Rightarrow M^{-1/3} = \frac{G}{K} R$$

$$\Rightarrow \boxed{R \propto \frac{1}{M^{1/3}}}$$

For $\gamma = \frac{4}{3}$: $\Rightarrow M^{\frac{4}{3}-2} = \frac{G}{K} R^{3\cdot\frac{4}{3}-4}$

$\Rightarrow M^{-2/3} = ? = \frac{G}{K}$

Equation of state:

Boltzmann's ratio = $\frac{N_2}{N_1} = e^{-h\nu/k_B T}$

where $\nu = \nu_2 - \nu_1$

Kinetic Theory

Pressure: $P \propto n\,(K.E.)$; Classical Ideal gas $P \propto n k_B T$

Now what will happen for a quantum gas:

Let's see

Imagine Sun contracted to very small size and it is a plasma of e^- and p^+

Then what is K.E. ?

Let's calculate:

Imagine we have N electrons packed densely in volume V

then (ΔV) volume available to each electron $= \frac{V}{N} = \frac{1}{n}$

The space available to each is $\Delta x \approx (\Delta V)^{1/3} \approx \frac{1}{n^{1/3}}$

$\Rightarrow \quad \Delta x \Delta p \approx \hbar$

Therefore roughly the momentum of such electrons is of the order $p \approx \frac{\hbar}{\Delta x}$

$$\Rightarrow \boxed{p_e \approx \hbar n^{1/3}}$$

Kinetic energy: $K.E. = \frac{p_e^2}{2m_e} = \frac{\hbar^2 n^{2/3}}{2m_e}$

$\Rightarrow$ Pressure $= P = n(K.E.) = \frac{\hbar^2 n^{5/3}}{2m_e}$

$$\Rightarrow \quad \rho = \frac{N m_e + N m_p}{V} \approx n m_p \quad \text{because } m_e << m_p$$

$$\text{And} \quad K.E. = \frac{p_e^2}{2m_e} + \frac{p_p^2}{2m_p} = \frac{\hbar^2 n^{2/3}}{2m_e} + \frac{\hbar^2 n^{2/3}}{2m_p} \approx \frac{\hbar^2 n^{2/3}}{2m_e}$$

$$\Rightarrow n = \rho / m_p$$

$$\Rightarrow P = \frac{\hbar^2}{2m_e m_p^{5/3}} \rho^{5/3} \quad \Rightarrow P = K \rho^{\gamma}$$

$$\text{where } K = \frac{\hbar^2}{2 m_e m_p^{5/3}} \quad \text{and } \gamma = \frac{5}{3}$$

stellar structure:

M, R

$$M^{-1/3} = \frac{G}{K} R \Rightarrow \boxed{R = \frac{\hbar^2}{2 G m_e m_p^{5/3}} M^{-1/3}}$$

we can drop 2
only keeping physics

Find out R for $M = 2 \times 10^{30}$ kg

However $K.E = \frac{p^2}{2m}$ is only valid for non-relativistic particles

what if the particles are relativistic

The $K.E. = \sqrt{p^2c^2 + m_o^2c^4} - m_oc^2$

rest mass energy

Relativistic case $pc >> m_oc^2 \Rightarrow$ $K.E. \approx pc$

Non-relativistic case $pc << m_oc^2 \Rightarrow$ $KE. = m_oc^2\left(1 + \frac{p^2}{m_o^2c^2}\right)^{1/2} - m_oc^2$

$$\Rightarrow KE \approx \cancel{m_oc^2} + m_oc^2\frac{p^2}{2m_o^2c^2} - \cancel{m_oc^2}$$

$$\Rightarrow KE. \approx \frac{p^2}{2m_o}$$

For relativistic case: $K.E. = pc$

$\Rightarrow$ Pressure: $P = n\,(K.E.) = npc$

but $p_e = \hbar\, n^{1/3}$ (electrons) as derived above

$\Rightarrow$ $P = \hbar\, n^{4/3} c = \dfrac{\hbar c\, \rho^{4/3}}{m_p^{4/3}} = K \rho^{\gamma}$

where $K = \dfrac{\hbar c}{m_p^{4/3}}$ and $\gamma = 4/3$

From stellar structure equation: in this case: $M^{-2/3} = \dfrac{G}{K}$

$\Rightarrow$ $M = \left(\dfrac{K}{G}\right)^{3/2} = \left(\dfrac{\hbar c}{G\, m_p^{4/3}}\right)^{3/2}$

$\Rightarrow$ $M = \sqrt{\dfrac{\hbar^3 c^3}{G^3 m_p^4}}$ A constant mass for which radius $R = 0$

$$\Rightarrow \quad M_c = \sqrt{\frac{\hbar^2 c^3}{G^3 m_p^4}}$$

$\hbar = 1.05 \times 10^{-34}$

$c = 3 \times 10^{8}$

$G = 6.67 \times 10^{-11}$

$m_p = 1.67 \times 10^{-24}$

} S.I. units

Remember mass of Sun $= 2 \times 10^{30}$ Kg $= M_\odot$

$M_c \approx 2\, M_\odot$ } we get this but verify

very close to actual $\underline{\underline{M_{ch} \approx 1.4\, M_\odot}}$

This is the Chandrashekhar's limit for W.D. stars.

Let's look at the speed of moving electron (degenerate electrons) in a white dwarf star.

$$\Delta x \, \Delta p \approx \hbar \qquad \text{when} \quad v_e \simeq \frac{p_e}{m_e} \approx \frac{\Delta p_e}{m_e} \approx \frac{\hbar}{\Delta x} \propto \frac{\hbar n^{1/3}}{m_e}$$

ΔV = Volume available to each electron : $\frac{V}{N} = \frac{1}{n}$

$$\Delta x \approx (\Delta V)^{1/3} \approx \frac{1}{n^{1/3}}$$

Therefore $v_e = \frac{\hbar \rho^{1/3}}{m_e m_p^{1/3}} \Rightarrow v_e \propto \rho^{1/3}$

$$\Downarrow$$

$$\rho \propto v_e^3$$

$$v_e = \frac{\hbar}{m_e m_p^{1/3}} \left(\frac{M}{R^3} \right)^{1/3} = \frac{\hbar}{m_e m_p^{1/3}} \left(\frac{(M)^{1/3}}{\frac{K}{G} M^{-1/3}} \right)$$

$$\Rightarrow v_e = \frac{\hbar}{m_e m_p^{1/3}} \frac{G}{K} M^{2/3}$$

At what M ; $v_e = c$

Say at M_{max}

$$\Rightarrow M_{max}^{2/3} = \frac{c\, m_e m_p^{1/3}}{\hbar} \frac{K}{G}$$

$$K = \frac{\hbar^2}{m_e m_p^{5/3}} \Rightarrow M_{max}^{2/3} = \frac{c\, \cancel{m_e} m_p^{1/3}}{\cancel{\hbar}\, G} \frac{\hbar^{\cancel{2}}}{\cancel{m_e} m_p^{5/3}}$$

$$= \frac{\hbar c}{G m_p^{4/3}}$$

$$\Rightarrow M_{max} = \sqrt{\frac{\hbar^3 c^3}{G^3 m_p^4}}$$

Mass of star (now)	Mass of star (at death)	Fate
< 7 Msun	< 1.4 Msun	White Dwarf
> 7 Msun & < 18 Msun	> 1.4 Msun	Neutron Star
> 18 Msun	> 1.4 Msun	Black Hole