1. Write the b-base expansion of n

a. base 2 expansion of 34

34 mod 2 = 0

34 div 2 = 17

17 mod 2 = 1

17 div 2 = 8

8 mod 2 = 0

8 div 2 = 4

4 mod 2 = 0

4 div 2 = 2

2 mod 2 = 0

2 div 2 = 1

1 mod 2 = 1

1 div 2 = $\boxed{0}$ $(100010)_2$

b. base 5 expansion of 125

125 mod 5 = 0

125 div 5 = 25

25 mod 5 = 0

25 div 5 = 5

5 mod 5 = 0

5 div 5 = 1

1 mod 5 = 1

1 div 5 = $\boxed{0}$ $(1000)_5$

c. base 16 expansion of 645

645 mod 16 = 5

645 div 16 = 40

40 mod 16 = 8

40 div 16 = 2

2 mod 16 = 2

2 div 16 = $\boxed{0}$ $(2\,8\,5)_{16}$

2. State the divisibility rule for 9 and prove it

$n = n_k \cdot n_{k-1} \cdot n_{k-2} \cdots n_1 n_0$

$n = n_k \cdot 10^k + n_{k-1} 10^{k-1} + \ldots + n_1 10^1 + n_0 10^0$ where $n_0 = 9$

and $10^x \equiv 1 \pmod 9$ for any integer x

$\therefore n \equiv n_k + n_{k-1} + \ldots + n_1 + n_0 \pmod 9$

Rule: A number $n \in \mathbb{N}$ is divisible by 9 if and only if the sum of its digits is also a multiple of 9

3. Using Fast Exponentiation, find the last digit of 7^{2021}

Fast Integer Exponentiation:

Expansion: base 2 expansion of 2021

2021 mod 2 = 1 ⟵ will be the last exponent:

2021 div 2 = 1010

1010 mod 2 = 0

1010 div 2 = 505

505 mod 2 = 1

505 div 2 = 252

$2021 = (\ldots + 1\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0)$

$\therefore 7^{2021} = 7^{(\ldots + 1\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^0)}$

$= \ldots 7^{2^2} + 7^1$

$\therefore$ Last digit is 7

4. $4 \mid (5^n - 1)$ where n is a non-negative integer

Base case: $n = 1$

$5^1 - 1$ is divisible by 4

4 is divisible by 4

Inductive hypothesis

Assume that for all $k \geq 0$ that:

$4 \mid (5^k - 1)$

Inductive Step:

want to prove that $4 \mid (5^{k+1} - 1)$ for any $k \geq 0$

$5^{k+1} - 1 = 5 \cdot 5^k - 1$

$= 1\cdot(5^k - 1) + 4\cdot 5^k$

and since $5^k - 1$ is divisible by 4 by the inductive hypothesis

and 4 is divisible by 4 so $4 \cdot 5^k$ is divisible by 4

then $(5^k - 1) + 4\cdot 5^k$ is divisible by 4

$\therefore (5^{k+1} - 1)$ is divisible by 4 : $4 \mid (5^{k+1} - 1)$

$\therefore (5^k - 1)$ is divisible by 4: $4 \mid (5^k - 1)$

by induction

5. $5 \mid (2^n + 3^n)$ for all odd positive integers n

Base Case: $n = 1$

$2^1 + 3^1$ is divisible by 5

5 is divisible by 5 ✓

Inductive hypothesis:

assume for all k, such that it is an odd positive integer, that $2^k + 3^k$ is divisible by 5

Inductive step:

want to prove that $5 \mid (2^{k+2} + 3^{k+2})$ since k is an odd positive integer and trying to prove for k+2

$2^{k+2} + 3^{k+2} = 5x$ where x is an integer since proving $2^{k+2} + 3^{k+2}$ is divisible by 5

$2^k \cdot 4 + 3^k \cdot 9 = 5x$

$4(2^k + 3^k) + 5 \cdot 3^k = 5x$

and since 5 is a multiple of 5, then $(5 \cdot 3^k)$ is divisible by 5

and since $(2^k + 3^k)$ is a multiple of 5 by the inductive hypothesis, then $4(2^k + 3^k)$ is divisible by 5

$\therefore$ $(2^{k+2} + 3^{k+2})$ is divisible by 5 : $5 \mid (2^{k+2} + 3^{k+2})$

$\therefore$ $5 \mid (2^n + 3^n)$ for any odd, positive integer n

6. $7 \mid (2021^{6n} - 1)$ for all non negative integers n

$2021^0 = 1$

$2021^1 \equiv 5 \pmod 7$

$2021^2 \equiv 4 \pmod 7$

$2021^3 \equiv 6 \pmod 7$

$2021^4 \equiv 2 \pmod 7$

$2021^5 \equiv 3 \pmod 7$

$2021^6 \equiv 1 \pmod 7$

$\therefore$ $1 \pmod 7 \equiv 2021^{6x}$ where x is a non-negative integer

so $0 \pmod 7 \equiv 2021^{6x} - 1$

and $2021^{6x} - 1$ is divisible by 7 for any non-negative integer x

$\therefore$ $7 \mid (2021^{6n} - 1)$ for any non-negative integer, n