Exercise 2.

Try visualizing for $1, 2, 3$ pegs, and you will get that $T_0 = 0, T_1 = 2, T_2 = 8$, and that the winning strategy is to transfer $n-1$ pegs to peg B via peg C, which takes T_{n-1} moves. Now, transfer the biggest peg to C, which takes one move, and then trasnfer the $n-1$ pegs in B to A, which takes T_{n-1} moves. Transfer biggest from C to B, taking one move, and transfer the other pegs to B, taking again T_{n-1} moves, that is

$$\begin{aligned} T_0 &= 0 \\ T_n &= 3T_{n-1} + 2 \end{aligned}$$

To solve this recurrence, we sum 1 to each side (the idea here is that we want to factor T_{n+1}, and summing 1 will make everything a factor of 3), so we have

$$\begin{aligned} T_0 + 1 &= 1 \\ T_n + 1 &= 3T_{n-1} + 3 \end{aligned}$$

and letting $U_n = T_n + 1$, we get

$$\begin{aligned} U_0 &= 1 \\ U_n &= 3U_{n-1} \end{aligned}$$

so clearly $U_n = 3^n$ and $T_n = 3^n - 1$. To prove that this is indeed the case, let's now use inducction. For $n = 0$, $T_0 = 3^0 - 1 = 0$, and assume $T_{n-1} = 3^{n-1} - 1$, so

$$T_n = 3T_{n-1} + 2 = 3(3^{n-1} - 1) + 2 = 3^n - 3 + 2 = 3^n - 1$$

Exercise 10.

I will describe the optimal strategies, which give rise to the relations between Q_n, the number of turns taken to transfer n disks from A to B, and R_n, the number of turns taken to transfer from the n disks from B back to A, which are

$$\begin{aligned} Q_n &= 2R_{n-1} + 1 \\ R_n &= Q_n + Q_{n-1} + 1 \end{aligned}$$

for $n > 0$. To get Q_n, we need to transfer the top $n-1$ pegs to from A to C, which takes R_{n-1} moves; then we transfr the last one to B, and then we move the top $n-1$ pegs from C to B, taken R_{n-1} turns again, i.e., $Q_n = 2R_{n-1} + 1$.

To get R_n, we first move the $n-1$ smallest from B to A, which takes R_{n-1} turns, then move the biggest peg to C, then move the $n-1$ smallest from A to B, taking Q_{n-1} turns, then move the biggest peg to A, and finally move the smallest $n-1$ pegs from B to A, that is,

$$R_n = R_{n-1} + 1 + Q_{n-1} + 1 + R_{n-1} = 2R_{n-1} + 1 + Q_{n-1} + 1 = Q_n + Q_{n-1} + 1$$

which is what we wanted to show.