$$\text{Crab initial positions: } C = \{x_0, x_1, ..., x_N\}\,; N > 0$$

$$\text{Final position: } x$$

$$x_i, x \in \mathbb{R}$$

$$\text{Fuel used: } F(x) = \sum_{i=1}^{N} \frac{|x - x_i|^2 + |x - x_i|}{2}$$

Given $|x - x_i|^2 = (x - x_i)^2$

$$F(x) = \sum_{i=1}^{N} \frac{(x - x_i)^2 + |x - x_i|}{2}$$

$$F_1(x) = \sum_{i=1}^{N} \frac{(x - x_i)^2}{2}; F_2(x) = \sum_{i=1}^{N} \frac{|x - x_i|}{2}; F(x) = F_1(x) + F_2(x)$$

Find the minimum for $F_1(x)$

$$\frac{dF_1(x)}{dx} = \sum_{i=1}^{N}(x - x_i) = Nx - \sum_{1}^{N} x_i$$

$$\frac{d^2F_1(x)}{dx^2} = N > 0 \Rightarrow \frac{dF_1(x)}{dx} = 0 \text{ is a minimum}$$

$$\frac{dF_1(x)}{dx} = 0 \Rightarrow Nx - \sum_{i=1}^{N} x_i = 0 \Rightarrow Nx = \sum_{i=1}^{N} x_i \Rightarrow x = \frac{\sum_{i=1}^{N} x_i}{N} \Rightarrow x = \overline{x_i}$$

Prove that the minimum for $F_1(x)$ is also the minimum for $F_2(x)$

$$\forall x_i \in C; x_j, x_k \in \mathbb{R}$$

$$(x_j - x_i)^2 > (x_k - x_i)^2 \Rightarrow |x_j - x_i| > |x_k - x_i|$$

$$(x_j - x_i)^2 > (x_k - x_i)^2 \Rightarrow (x_j - x_i)^2 + |x_j - x_i| > (x_k - x_i)^2 + |x_k - x_i|$$

$$\sum_{i=1}^{N}(x_j - x_i)^2 > \sum_{i=1}^{N}(x_k - x_i)^2 \Rightarrow \sum_{i=1}^{N} \left[(x_j - x_i)^2 + |x_j - x_i|\right] > \sum_{i=1}^{N} \left[(x_k - x_i)^2 + |x_k - x_i|\right]$$

So the minimum for $F_1(x)$ is also the minimum for $F(x)$

Then, the minimum of $F(x)$ is $x = \overline{x_i}$

Since we can only have integer positions, the final point with the minimum fuel consumption should be x if $x \in \mathbb{Z}$

For $x \notin \mathbb{Z}$, it's obvious that any value greater than $\lceil x \rceil$ or smaller than $\lfloor x \rfloor$ won't be the final point with the minimum fuel consumption, but we can't know which between $\lceil x \rceil$ and $\lfloor x \rfloor$ is the point with the minimum fuel consumption, so we need to check both results and pick the smallest one