Supplementary Materials for

A confounding bridge approach for double negative control inference on causal effects

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This supplementary material includes discussion on the connection between the negative control and the instrumental variable, additional simulation with a continuous exposure, proof of Propositions 3.1/5.1 and Theorems 3.3/3.4/A.1, details for examples, and analysis results for the effect of air pollution in New York and Boston.

A Repairing an invalid instrumental variable with a negative control outcome

The instrumental variable (IV) approach is an influential method to address unmeasured confounding or endogeneity in observational studies. An instrumental variable Z satisfies three core conditions (Wright, 1928; Goldberger, 1972; Angrist et al., 1996).

Assumption A.1 (Instrumental variable). (i) Exclusion restriction, $Z \perp\!\!\!\perp Y \mid (X, U)$; (ii) independence of the confounder, $Z \perp\!\!\!\perp U$; (iii) correlation with the primary exposure, $Z \not\perp\!\!\!\perp X$.

In addition to the three core conditions, the IV approach requires one additional assumption for point identification of a causal effect. Here we consider a structural model that encodes the average causal effect. To ground ideas, we focus on a linear model,

$$E(Y \mid X, U) \quad = \quad \beta X + U, \tag{S.1}$$

where β is the causal parameter of interest. Given Model (S.1), a conventional IV estimator is $\widehat{\beta}_{\text{iv}} = \widehat{\sigma}_{zy}/\widehat{\sigma}_{xz}$ with $\widehat{\sigma}_{zy}$ the sample covariance of Z and Y, and $\widehat{\sigma}_{xz}$ analogously defined. The IV estimator can also be obtained by two stage least square: X is regressed on Z to obtain the fitted values $\widehat{X}$ and then Y is regressed on $\widehat{X}$ (Wooldridge, 2010, chapter 5).

The exclusion restriction is also made in the negative control exposure assumption. Conditions (ii)–(iii) for the IV are not made in the negative control exposure setting, but they are essential for consistency of $\widehat{\beta}_{\text{iv}}$. If either (ii) or (iii) is violated, then $\widehat{\beta}_{\text{iv}}$ is no longer consistent and can be severely biased. Condition (ii) cannot be ensured in application unless the instrumental variable is physically randomized, while violation of (iii) can occur in settings such as Mendelian randomization (Didelez and Sheehan, 2007) where the effects of genetic variants (defining the IV) on the exposure are small.

These problems can be mitigated by incorporating a negative control outcome W. Using

$$b(W, X; \gamma) = \gamma_0 + \gamma_1 X + \gamma_2 W, \quad q(X, Z) = (1, X, Z)^\top, \tag{S.2}$$

and the identity weight matrix for the GMM, leads to the negative control estimator

$$\widehat{\beta}_{\text{nc}} = \widehat{\gamma}_1 = \frac{\widehat{\sigma}_{xw}\widehat{\sigma}_{zy} - \widehat{\sigma}_{xy}\widehat{\sigma}_{zw}}{\widehat{\sigma}_{xw}\widehat{\sigma}_{xz} - \widehat{\sigma}_{xx}\widehat{\sigma}_{zw}}.$$

The estimator can also be obtained by a modified two stage least square: in the first stage W is regressed on (X, Z) to obtain the fitted values $\widehat{W}$ and in the second stage Y is regressed on $(X, \widehat{W})$, then $\widehat{\beta}_{\text{nc}}$ is equal to the coefficient of X in the second stage. A nonzero regression coefficient of Z in the first stage is equivalent to a nonzero denominator in the above expression of $\widehat{\beta}_{\text{nc}}$.

Theorem A.1. Assuming $E(Y \mid U, X) = \beta X + U$, $Z \perp\!\!\!\perp Y \mid (U, X)$, $W \perp\!\!\!\perp (Z, X) \mid U$, $\sigma_{xw} \neq 0$, and given the regularity condition in Hansen (1982) and Hall (2005), then $\widehat{\beta}_{\text{nc}}$ is consistent if either of the following conditions holds, but not necessarily both.

(i) $b(W, X; \gamma)$ in (S.2) is correct in the sense that (2) holds, and $\sigma_{xw}\sigma_{xz} - \sigma_{xx}\sigma_{zw} \neq 0$;

(ii) $Z \perp\!\!\!\perp U$, and $\sigma_{xz} \neq 0$.

These two conditions correspond to the outcome confounding bridge and the IV assumptions, respectively. Given a correct outcome confounding bridge, the negative control estimator is consistent even if IV conditions (ii) and (iii) are not met. In this view, the negative control outcome offers a powerful tool to correct the bias caused by an invalid IV. Although there remains concern about potential bias due to misspecification of the outcome confounding bridge, $\widehat{\beta}_{\text{nc}}$ is strikingly robust if Z is a valid IV. This can be checked by verifying that for a valid IV and a negative control outcome, $\widehat{\sigma}_{zw}$ converges to zero in probability and thus $\widehat{\beta}_{\text{nc}}$ is consistent even if $b(W, X; \gamma)$ is incorrect. Therefore, $\widehat{\beta}_{\text{nc}}$ doubles one's chances to remove confounding bias in the sense that it is consistent if either Z is a valid IV satisfying Assumption A.1, or (Z, W) are a valid negative control pair satisfying Assumptions 2–4. In a measurement error problem, an analogue to $\widehat{\beta}_{\text{nc}}$ was previously derived by Kuroki and Pearl (2014) and Miao et al. (2018). However, they additionally required normality assumptions and both failed to subsequently establish consistency of the estimator under somewhat milder assumptions as in Theorem A.1 and did not recognize the double robustness property and close relationship with two stage least square.

In situations where recorded covariates may not account for all potential sources of confounding, and negative control variables are available, there are several scenarios beyond average treatment effect estimation.

- When the conditional average treatment effect is of interest, Sverdrup and Cui (2023) explored learning conditional average treatment effects.
- For policy evaluation with panel data, Shi et al. (2021) proposed a novel synthetic control method by using the core ideas originated from negative control variables.
- Besides causal inference, such ideas have also inspired other settings, such as survival analysis (Ying et al., 2022, 2023) and off-policy evaluation in the context of partially observed Markov decision processes, as demonstrated by Bennett and Kallus (2021).

These examples highlight the versatility and applicability of negative control variables in various research scenarios and fields.

B Simulations with a continuous exposure

We generate i.i.d. data according to

$$
\begin{aligned}
&V, U \sim N(0,1), \quad \sigma_{uv} = 0.5, \quad Z = 0.5 + 1.5V + \eta U + \varepsilon_1, \\
&X = 0.5 + Z + 0.5V + 0.5V^2 + 1.5U + \varepsilon_2, \quad W = 1 - V + \xi V^2 + 1.5U + \varepsilon_3, \\
&Y = 1 + 0.5X + V + U + 2\varepsilon_3, \quad \varepsilon_1, \varepsilon_2, \varepsilon_3 \sim N(0,1),
\end{aligned}
$$

under multiple parameter settings: $\eta = 0, 0.3, 0.5$ and $\xi = 0, 0.4, 0.6$. We focus on the coefficient of X in the outcome model. We analyze data with the negative control approach (NC), ordinary least square (OLS), and instrumental variable estimation (IV).

Table B1: Coverage probability of 95% negative control confidence interval for the structural parameter.

		$\eta = 0$		0.3		0.5	
	0	0.960	0.946	0.948	0.953	0.941	0.942
$\xi =$	0.4	0.956	0.942	0.971	0.855	0.964	0.712
	0.6	0.962	0.955	0.930	0.763	0.877	0.473

Note: For each setting of η, the first column is for sample size 500 and the second 1500.

For each parameter setting, we replicate 1000 simulations at sample size 500 and 1500, respectively. Figure B1 presents boxplots of three estimators. The negative control estimator has small bias whenever the outcome confounding bridge is correctly specified ($\xi = 0$). When the outcome confounding bridge is incorrect ($\xi = 0.4, 0.6$), although the negative control estimator could be biased, the bias is much smaller than the other two estimators and reduces to zero as the association between Z and U becomes weak ($\eta = 0, 0.3$). This confirms the double robustness property of the proposed negative control estimator of Section 5. From Table B1, the 95% negative control confidence intervals have coverage probability approximating 0.95 if either the outcome confounding bridge is correct or Z is a valid instrumental variable. But when both conditions are violated, the coverage probability is below the nominal level. When Z is a valid instrumental variable ($\eta = 0$), the instrumental variable estimator also performs well with small bias, but is less efficient than the negative control estimator under the settings considered here, and can be severely biased when Z and U are correlated ($\eta = 0.3, 0.5$). The ordinary least square estimator is biased under all settings, due to confounding. Therefore, when a structural model is of interest, we recommend the negative control approach to reduce possible bias caused by confounding or an invalid instrumental variable.

We further simulate and examine situations when the models are misspecified. The data gen-

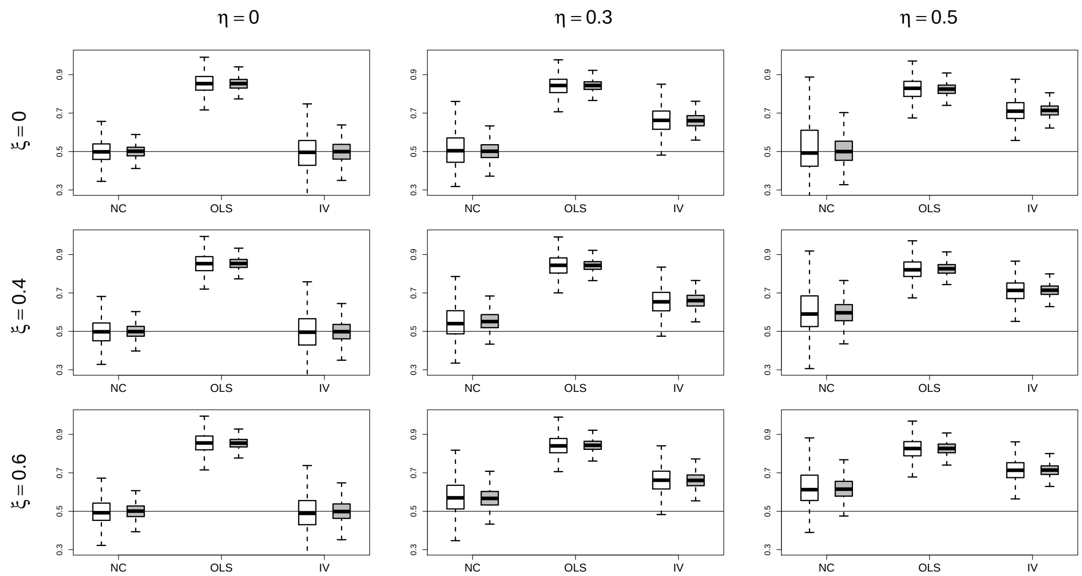

Figure B1: Boxplots for estimators of the structural parameter.

Note: For NC, $b = (1, X, V, W)\gamma$ and $q = (1, X, V, Z - \widehat{Z})^\top$ are used for the GMM with $\widehat{Z}$ obtained from a linear regression of Z on V; for IV, two stage least square is used; for OLS, a linear model is used. White boxes are for sample size 500 and gray ones 1500; the horizontal line marks the true value of the parameter.

erating process is as follow: $V, U \sim N(0, 1)$ with correlation $\sigma_{uv} = 0.5$.

$$Z = 0.5 + 0.5V + U^2 + \varepsilon_1,$$

$$W = 1 - V^2 + \xi U + \varepsilon_2,$$

$$\text{logit}\{p(X = 1 \mid Z, V, U)\} = -0.5 + Z + 0.5V + \eta U^2,$$

$$Y(x) = 1 + 0.5x + 2V^2 + U + 1.5xU + 2\varepsilon_2,$$

with $\varepsilon_1, \varepsilon_2 \sim N(0, 1)$ and η encoding the magnitude of confounding and ξ the association between the negative control outcome and the confounder.

For each choice of $\eta = 0, 0.3, 0.5$ and $\xi = 0.2, 0.4, 0.6$, we replicate 1000 simulations at sample size 500 and 1500, respectively, and summarize results as boxplots in Figure B2. When the confounding bridge function is misspecified, the negative control estimator is biased and has a larger variance than ordinary least square and inverse probability weighted estimators. When there is no unmeasured confounder ($\eta = 0$), ordinary least square and inverse probability weighted estimators are unbiased, but the negative control estimators can be biased even if unconfoundedness holds. Therefore, one needs to carefully sort out negative controls and choice of confounding bridge function especially when domain knowledge is lacking.

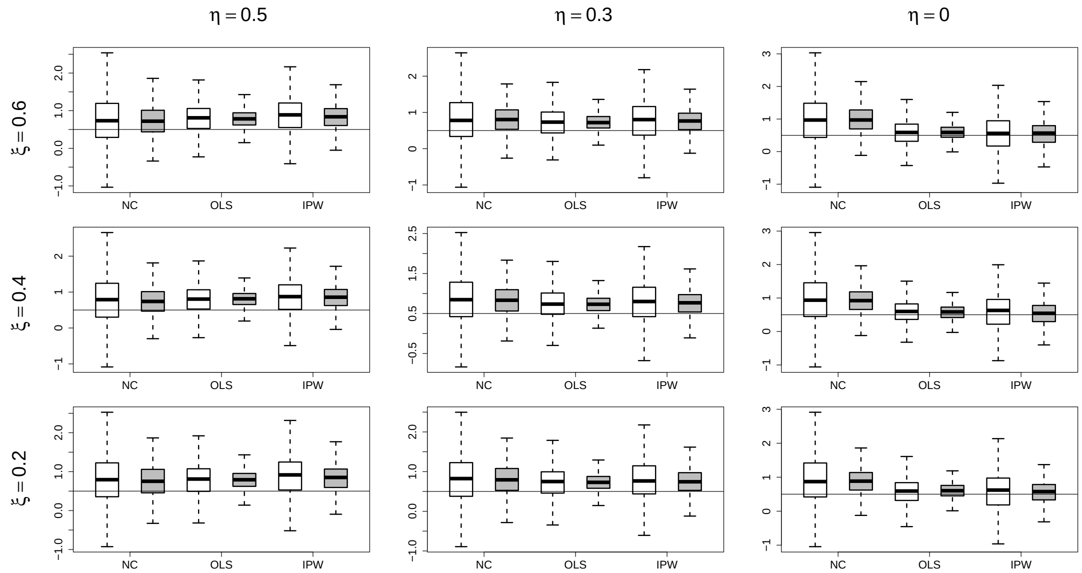

Figure B2: Boxplots for estimators of the average causal effect.

Note: For NC, $b = (1, X, V, W, XV, XW)\gamma$ and $q = (1, X, V, Z, XV, XZ)^{\top}$ are used for the GMM; for IPW, a logistic model for $p(X = 1 \mid V)$ is used; for OLS, a linear model is used.

C Proofs

Proof of Propositions 3.1 and 5.1. Given the outcome confounding bridge Assumption 3, we take expectation over U on both sides of (2) and obtain that for all x,

$$E\{E(Y \mid U, X = x)\} = E\{E(b(W, x) \mid U, X = x)\}.$$

Under the latent ignorability Assumption 1, we have $E\{E(Y \mid U, X = x)\} = E\{E(Y(x) \mid U)\} = E\{Y(x)\}$.

1. Under the negative control outcome Assumption 2, we have $E\{E(b(W, x) \mid U, X = x)\} = E\{E(b(W, x) \mid U)\} = E\{b(W, x)\}$. Therefore, under Assumptions 1–3, we have $E\{Y(x)\} = E\{b(W, x)\}$, completing the proof of Proposition 3.1.

2. Under the positive control outcome Assumption 6, we have $E\{E(b(W, x) \mid U, X = x)\} = E\{E(b(W(x), x) \mid U)\} = E\{b(W(x), x)\}$. Therefore, under Assumptions 1, 3, and 6, we have $E\{Y(x)\} = E\{b(W(x), x)\}$, completing the proof of Proposition 5.1.

$\square$

Proof of Theorems 3.3 and 3.4. Proposition 3.1 implies that under Assumptions 1–3, for all x

$$E\{Y(x)\} = E\{b(W, x)\}, \tag{S.3}$$

which establishes the relationship between the potential outcome mean and the negative control outcome distribution via the outcome confounding bridge. Under Assumptions 2–4, we have that for all x,

$$\begin{aligned}
E(Y \mid Z, X = x) &= E\{E(Y \mid U, Z, X = x) \mid Z, X = x\} \\
&= E\{E(Y \mid U, X = x) \mid Z, X = x\} \\
&= E\{E(b(W, x) \mid U, X = x) \mid Z, X = x\} \\
&= E\{E(b(W, x) \mid U, Z, X = x) \mid Z, X = x\} \\
&= E\{b(W, x) \mid Z, X = x\},
\end{aligned}$$

where the first and fifth equalities are due to the law of iterated expectation, the second and forth are obtained due to the negative control exposure Assumption 4, and the third is implied by the outcome confounding bridge Assumption 3. Therefore, we have that for all x,

$$E\{Y - b(W, x) \mid Z, X = x\} = 0. \tag{S.4}$$

1. If there is no parametric or semiparametric restrictions imposed on the outcome confounding bridge $b(W, X)$, we need completeness of $p(W \mid Z, X)$ for identification of $b(W, X)$. Given Assumption 5, we show uniqueness of the solution to (S.4). Suppose both $b(W, X)$ and $b'(W, X)$ satisfy (S.4), and then we must have that for all x and almost all z,

   $$E\{b(W, x) - b'(W, x) \mid Z = z, X = x\} = 0.$$

   However, Assumption 5 implies that for all x, $b(W, x)$ must equal $b'(W, x)$ almost surely. Thus, the solution to (S.4) is unique, and therefore, the results of Theorem 3.3 hold, i.e., under Assumptions 1–5, the outcome confounding bridge $b(W, X)$ is identified from (S.4), and the potential outcome mean is identified by (S.3).

2. If a parametric or semiparametric model $b(W, X; \gamma)$ is specified for the outcome confounding bridge with a finite or infinite dimensional parameter γ, we only need a weakened version of completeness. Suppose that both $b(W, X; \gamma)$ and $b(W, X; \gamma')$ satisfy (S.4) but $\gamma \neq \gamma'$, and then we must have that for all x and almost all z, $E\{b(W, x; \gamma) - b'(W, x; \gamma') \mid Z = z, X = x\} = 0$, which leads to a contradiction with the condition in Theorem 3.4. Therefore, given Assumptions 1–4 and the weakened completeness condition of Theorem 3.4, the outcome confounding bridge is identified and so is the potential outcome mean.

□

Proof of Theorem A.1. We maintain the following regularity condition for Theorem A.1,

$$\begin{pmatrix} \widehat{\sigma}_{xx} & \widehat{\sigma}_{xw} \\ \widehat{\sigma}_{xz} & \widehat{\sigma}_{zw} \end{pmatrix} \to \begin{pmatrix} \sigma_{xx} & \sigma_{xw} \\ \sigma_{xz} & \sigma_{zw} \end{pmatrix} \text{ in probability,} \tag{S.5}$$

which states consistency of the empirical cross-covariance matrix between (X, Z) and (X, W).

Given that $E(Y \mid U, X) = \beta X + U$, $Z \perp\!\!\!\perp Y \mid (U, X)$, $W \perp\!\!\!\perp (Z, X) \mid U$, then W is a negative control outcome for X and Z is a negative control exposure for W and Y. We apply the GMM with $b(W, X; \gamma) = \gamma_0 + \gamma_1 X + \gamma_2 W$, $q(X, Z) = (1, X, Z)^\top$, and Ω the identity weight matrix. It is equivalent to solving

$$\frac{1}{n} \sum_{i=1}^{n} (1, X_i, Z_i)^\top \{Y_i - (1, X_i, W_i)\gamma\} = 0, \tag{S.6}$$

and leads to the GMM estimator

$$\widehat{\gamma} = \left\{ \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} 1 & X_i & W_i \\ X_i & X_i^2 & X_i W_i \\ Z_i & X_i Z_i & Z_i W_i \end{pmatrix} \right\}^{-1} \left\{ \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} Y_i \\ X_i Y_i \\ Z_i Y_i \end{pmatrix} \right\}.$$

After some algebra, the second component of $\widehat{\gamma}$ can be represented as

$$\widehat{\gamma}_1 = \frac{\widehat{\sigma}_{xw}\widehat{\sigma}_{zy} - \widehat{\sigma}_{xy}\widehat{\sigma}_{zw}}{\widehat{\sigma}_{xw}\widehat{\sigma}_{xz} - \widehat{\sigma}_{xx}\widehat{\sigma}_{zw}}.$$

Assuming the regularity condition (S.5) and $\sigma_{xw}\sigma_{xz} - \sigma_{xx}\sigma_{zw} \neq 0$, then $\widehat{\gamma}_1$ converges in probability to

$$\frac{\sigma_{xw}\sigma_{zy} - \sigma_{xy}\sigma_{zw}}{\sigma_{xw}\sigma_{xz} - \sigma_{xx}\sigma_{zw}}. \tag{S.7}$$

(i) If $b(W, X; \gamma)$ is correct so that $E(Y \mid U, X) = E\{b(W, X; \gamma) \mid U, X\} = E\{\gamma_0 + \gamma_1 X + \gamma_2 W \mid U, X\}$, then we have $\gamma_1 = \beta$ and $E(W \mid U) = (-\gamma_0 + U)/\gamma_2$. Thus, we have $\sigma_{zy} = \beta\sigma_{xz} + \sigma_{zu}$, $\sigma_{zw} = 1/\gamma_2\sigma_{zu}$, $\sigma_{xw} = 1/\gamma_2\sigma_{xu}$, and $\sigma_{xy} = \beta\sigma_{xx} + \sigma_{xu}$; by such substitution, the quantity in (S.7) is in fact equal to β. Therefore, $\widehat{\gamma}_1$ converges in probability to β.

(ii) Given that $W \perp\!\!\!\perp (Z, X) \mid U$, if $Z \perp\!\!\!\perp U$ and $\sigma_{xz} \neq 0$, i.e., Z is a valid instrumental variable, then we have $\sigma_{zw} = 0$. As a result, the quantity in (S.7) is equal to σ_{zy}/σ_{xz}, and thus equal to β. Therefore, $\widehat{\gamma}_1 \to \beta$ in probability.

In summary, $\widehat{\gamma}_1$ is consistent if either condition (i) or (ii) of Theorem A.1 holds, but not necessarily both. □

Equivalence to two stage least square. Solving (S.6) is equivalent to solving

$$\frac{1}{n}\sum_{i=1}^{n}(1, X_i, Z_i)^\top\{Y_i - (1, X_i, \widehat{W}_i)\gamma + \gamma_2(\widehat{W}_i - W_i)\} = 0, \tag{S.8}$$

with $\widehat{W} = (1, X, Z)\widehat{\alpha}$ and $\widehat{\alpha}$ solving the first stage least square,

$$\frac{1}{n}\sum_{i=1}^{n}(1, X_i, Z_i)^\top\{W - (1, X, Z)\alpha\} = 0.$$

In particular, the coefficient of Z obtained in the first stage least square is

$$\frac{\widehat{\sigma}_{xw}\widehat{\sigma}_{xz} - \widehat{\sigma}_{xx}\widehat{\sigma}_{zw}}{\widehat{\sigma}_{xz}^2 - \widehat{\sigma}_{xx}\widehat{\sigma}_{zz}},$$

which can be used to test how far away the denominator in (S.7) is from zero. As a result, (S.6) is equivalent to

$$\frac{1}{n}\sum_{i=1}^{n}(1, X_i, Z_i)^\top\{Y_i - (1, X_i, \widehat{W}_i)\gamma\} = 0,$$

and also equivalent to

$$\frac{1}{n}\sum_{i=1}^{n}(1, X_i, \widehat{W}_i)^\top\{Y_i - (1, X_i, \widehat{W}_i)\gamma\} = 0,$$

because $\widehat{W}_i$ is a linear combination of X_i and Z_i. Therefore, the negative control estimator $\widehat{\beta}_{\mathrm{nc}}$ is equivalent to the two stage least square estimator.

D Details for Examples

A parametric example of single negative control. Consider the data generating process with β encoding the average causal effect:

$$U \sim N(0,1), \qquad W = \alpha_1 U + \sigma_1 \varepsilon_2,$$
$$X = \alpha_2 U + \sigma_2 \varepsilon_1, \qquad Y = \beta X + \alpha_3 U + \sigma_3 \varepsilon_3, \quad \varepsilon_1, \varepsilon_2, \varepsilon_3 \sim N(0,1).$$

Table D2: Two distinct parameter settings with identical observed data distribution.

β	α_1	α_2	α_3	σ_1^2	σ_2^2	σ_3^2
1	1	1	1	1	1	4
-1	$\sqrt{3/5}$	$\sqrt{5/3}$	$\sqrt{15}$	7/5	1/3	2

For the following two parameter settings in Table D2, these two parameter settings with distinct values of β result in identical distribution of (X, Y, W), which is a joint normal distribution with mean zero and covariance matrix:

$$\begin{pmatrix} 2 & 3 & 1 \\ 3 & 9 & 2 \\ 1 & 2 & 2 \end{pmatrix}.$$

Therefore, given the distribution of (X, Y, W), β encoding the average causal effect is not identified. Despite specification of a fully parametric model in the example, the sign of β cannot be inferred from observed data, and the situation does not improve even if the confounder distribution is known.

Details for Example 3.5. We first describe a general result for the relationship between the average causal effect and crude effects. For a outcome confounding bridge function $b(W, X)$, because $E\{Y(x)\} = E\{b(W, x)\}$ and $E\{Y \mid Z, X\} = E\{b(W, X) \mid Z, X\}$, we have that for any two values

x_1, x_0 in the support of X,

$$
\begin{aligned}
& E\{Y(x_1)\} - E\{Y(x_0)\} \\
= \; & \int_w b(w, x_1) p(w) \mathrm{d}w - \int_w b(w, x_0) p(w) \mathrm{d}w \\
= \; & \int_{w,x,z} b(w, x_1) p(w \mid z, x) p(z, x) \mathrm{d}z \mathrm{d}x \mathrm{d}w - \int_{w,x,z} b(w, x_0) p(w \mid z, x) p(z, x) \mathrm{d}z \mathrm{d}x \mathrm{d}w \\
= \; & \int_{w,x,z} b(w, x_1) p(w \mid z, x_1) p(z, x) \mathrm{d}z \mathrm{d}x \mathrm{d}w - \int_{w,x,z} b(w, x_0) p(w \mid z, x_0) p(z, x) \mathrm{d}z \mathrm{d}x \mathrm{d}w \\
& - \int_{w,x,z} b(w, x_1) \{p(w \mid z, x_1) - p(w \mid z, x_0)\} p(z, x) \mathrm{d}z \mathrm{d}x \mathrm{d}w \\
& + \int_{w,x,z} \{b(w, x_1) - b(w, x_0)\} \{p(w \mid z, x) - p(w \mid z, x_0)\} p(z, x) \mathrm{d}z \mathrm{d}x \mathrm{d}w \\
= \; & E\{E(Y \mid Z, x_1) - E(Y \mid Z, x_0)\} \\
& - \int_{w,z} b(w, x_1) \{p(w \mid z, x_1) - p(w \mid z, x_0)\} p(z) \mathrm{d}z \mathrm{d}w \\
& + \int_{w,x,z} \{b(w, x_1) - b(w, x_0)\} \{p(w \mid z, x) - p(w \mid z, x_0)\} p(z, x) \mathrm{d}z \mathrm{d}x \mathrm{d}w.
\end{aligned}
$$

If the outcome confounding bridge has the form $b(W, X) = b_1(X) + b_2(X) b_0(W)$, the last equality reduces to

$$
\begin{aligned}
E\{Y(x_1)\} - E\{Y(x_0)\} \; = \; & E\{E(Y \mid Z, x_1) - E(Y \mid Z, x_0)\} \\
& - b_2(x_1) E\{E(b_0(W) \mid Z, x_1) - E(b_0(W) \mid Z, x_0)\} \\
& + \{b_2(x_1) - b_2(x_0)\} \\
& \times \int_{x,z} \{E(b_0(W) \mid Z = z, x) - E(b_0(W) \mid Z = z, x_0)\} p(z, x) \mathrm{d}z \mathrm{d}x.
\end{aligned}
$$

Next, we consider the setting of Example 3.5 with binary (X, Z) and $b(W, X; \gamma) = \gamma_0 + \gamma_1 X + \gamma_2 W + \gamma_3 X W$, in which case, $b_1(X) = \gamma_0 + \gamma_1 X$, $b_2(X) = \gamma_2 + \gamma_3 X$, $b_0(W) = W$. Then we obtain that

$$
\begin{aligned}
E\{Y(1)\} - E\{Y(0)\} \; = \; & E\{E(Y \mid Z, X = 1) - E(Y \mid Z, X = 0)\} \\
& - (\gamma_2 + \gamma_3) E\{E(W \mid Z, X = 1) - E(W \mid Z, X = 0)\} \\
& + \gamma_3 \sum_{z=0}^{1} \{E(W \mid Z = z, X = 1) - E(W \mid Z = z, X = 0)\} p(Z = z, X = 1).
\end{aligned}
$$

The unknown parameters γ are identified by solving $E(Y \mid Z, X) = E\{b(W, X; \gamma) \mid Z, X\}$:

$$
\gamma_2 = \frac{E(Y \mid Z = 1, X = 0) - E(Y \mid Z = 0, X = 0)}{E(W \mid Z = 1, X = 0) - E(W \mid Z = 0, X = 0)},
$$
$$
\gamma_2 + \gamma_3 = \frac{E(Y \mid Z = 1, X = 1) - E(Y \mid Z = 0, X = 1)}{E(W \mid Z = 1, X = 1) - E(W \mid Z = 0, X = 1)}.
$$

If $\gamma_3 = 0$, then

$$\gamma_2 = \frac{E\{E(Y \mid Z=1, X) - E(Y \mid Z=0, X)\}}{E\{E(W \mid Z=1, X) - E(W \mid Z=0, X)\}}.$$

E Results for the effect of air pollution in New York and Boston

The results for the effect of air pollution in New York and Boston are summarized in Table E3 and E4, respectively. Point estimates and 95% confidence intervals (in brackets) in the table are multiplied by 10000. Confidence intervals and p-values are obtained from a normal approximation and the Newey and West (1987) variance estimator is used to account for serial correlation.

Table E3: Estimates of the effect of air pollution in New York.

	Number of lagged exposures controlled					
	One day		Two days		Three days	
	Estimate	p-value	Estimate	p-value	Estimate	p-value
Ordinary least square						
β_1	37 (1, 72)	0.0410	30 (-6, 66)	0.1016	32 (-3, 68)	0.0742
Confounding test						
α_1	-5 (-39, 29)	0.7662	-3 (-36, 30)	0.8792	-1 (-33, 32)	0.9758
α_2	25 (-7, 57)	0.1188	24 (-7, 54)	0.1327	24 (-7, 54)	0.1328
Negative control estimation						
β_1	-8 (-43, 28)	0.6678	-7 (-45, 30)	0.7024	-7 (-46, 32)	0.7370

Table E4: Estimates of the effect of air pollution in Boston.

	Number of lagged exposures controlled					
	One day		Two days		Three days	
	Estimate	p-value	Estimate	p-value	Estimate	p-value
Ordinary least square						
β_1	1 (-37, 39)	0.9685	-3 (-42, 35)	0.8580	-5 (-43, 34)	0.8160
Confounding test						
α_1	10 (-28, 48)	0.6084	12 (-25, 49)	0.5222	12 (-25, 49)	0.5208
α_2	-7 (-41, 27)	0.6758	-7 (-41, 27)	0.6945	-8 (-42, 26)	0.6596
Negative control estimation						
β_1	-26 (-71, 19)	0.2643	-25 (-71, 21)	0.2813	-25 (-73, 23)	0.3064

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