GITHUB README: COMPUTER VERIFICATION OF CLAIMS MADE FOR THE THREE QUARTIC CYCLOTOMIC FIELDS

This README file describes the computational verification of three claims made in §3 of our preprint *Shintani fundamental domains for quartic number fields with complex roots of unity*, cited below as {C/E/F}.[1] The claims involve unions of cones connected to Shintani domains for the three quartic fields $k = \mathbb{Q}(\Theta) \subset \mathbb{C} \times \mathbb{C}$, where Θ is a root of unity of order $m = 8$, 10 or 12.[2]

We recall from displays (19) and (20) in {C/E/F}¸ the notation

$$\mathcal{C}_0 := \mathbb{R}_{\geq 0} \cdot [1, \Theta, \Theta^2, \Theta^3] \subset \mathbb{C} \times \mathbb{C}, \quad \mathcal{C}_0' := \mathcal{C}_0 - \big(\mathcal{C}_0 \cap (\Theta \cdot \mathcal{C}_0)\big), \quad S := \bigcup_{\ell=0}^{m-1} (\Theta^\ell \cdot \mathcal{C}_0).$$

The first two claims in need of computer verification appear in {C/E/F, display (20)}, while the third one appears on the next to last paragraph of {C/E/F}. Namely,

$$(*) \qquad S = \{0\} \cup \bigcup_{\ell=0}^{m-1} (\Theta^\ell \cdot \mathcal{C}_0') \qquad \text{(disjoint union also claimed)},$$

$$(**) \qquad E \cdot S \subset S \qquad \text{(claimed for any unit } E \in k \text{ of infinite order)},$$

$$(***) \qquad P^{\Delta,\Delta}(c) \subset S \subset P^{\Delta,\Delta}(d) \qquad \text{(existence claimed of such } \Delta, c, d),$$

where $P^{\Delta,\Delta}(\cdot) \subset \mathbb{C} \times \mathbb{C}$ is defined in [CEF, eq. (10)], and we give Δ, c and d below.

To check these claims, we open the software PARI/GP and execute the PARI/GP command

$$\mathsf{bnf} = \mathsf{bnfinit}(\mathsf{p});$$

where p is the cyclotomic polynomial of primitive m-th roots of unity defining k. Thus, $\mathsf{p} = x^4 + 1,\ x^4 - x^3 + x^2 - x + 1$ or $x^4 - x^2 + 1$ for $m = 8, 10$ or 12, respectively. The vector bnf stores basic information on k.

We address (*) first. Since the inclusions $0 \in S$ and $(\Theta^\ell \cdot \mathcal{C}_0') \subset S$ are obvious and $\Theta^m = 1$, it suffices to verify

$$\mathcal{C}_0 - \bigcup_{\ell=0}^{m-1} (\Theta^\ell \cdot \mathcal{C}_0') = \{0\} \qquad \text{and} \qquad \mathcal{C}_0' \cap (\Theta^\ell \cdot \mathcal{C}_0') = \varnothing \quad (1 \leq \ell \leq m-1). \qquad (1)$$

We load the file CycloAlgorithm.gp using the PARI/GP read command

\r CycloAlgorithm.gp;

[1] This should not be confused with the prior article by the same authors, to which we refer as [CEF] below, *Attractor-repeller construction of Shintani domains for totally complex quartic fields*, J. Number Th. **258** (2024) 146–172.

[2] For the purposes of thinking about cones in real vector spaces we regard $k \subset \mathbb{C} \times \mathbb{C}$, embedded by its two complex places, but calculations described below are done in the abstract field $\mathbb{Q}[x]/\mathsf{p}\mathbb{Q}[x]$, where p is the polynomial defining k given below.

In fact, to verify all the statements below the reader may want to inspect this file as it contains the line by line details of the routines described in this README.

The file $\mathsf{CycloAlgorithm.gp}$ contains the routine $\mathsf{DifConeS(bnf)}$, which when bnf is as above returns $[\mathsf{v},\mathsf{w}]$ where v codes the generators of the cones forming $\mathcal{C}_0 - \bigcup_{\ell=0}^{m-1}(\Theta^\ell \cdot \mathcal{C}'_0)$ and $\mathsf{w} = [\varepsilon_1, \ldots, \varepsilon_{m-1}]$, where $\varepsilon_\ell = 0$ if $\mathcal{C}'_0 \cap (\theta^\ell \cdot \mathcal{C}'_0) \neq \varnothing$ and 1 otherwise. As expected, the command

$$\mathsf{DifConeS(bnf)}$$

returns $\big[[\,], [1, \ldots, 1]\big]$, which proves (1) since the empty set of generators $[\,]$ represents $\{0\}$, the only cone without generators. This proves (*).

We now address (**). Since $\Theta \cdot S = S$, to verify (**) for any unit of infinite order $E \in k$, it suffices to do so for $E = u$, where u is the fundamental unit $u = x^2 + x + 1$ for $m = 8$; $u = -x + 1$ for $m = 10$; and $u = -x^3 - x^2$ for $m = 12$ (these u are interpreted modulo $\mathsf{p}\mathbb{Q}[x]$, with p as above defining k). The file $\mathsf{CycloAlgorithm.gp}$ also contains $\mathsf{DiffComplex2(bnf,u)}$. This routine, with bnf and u as above, returns the generators of the cones forming $u \cdot S - S$. As $\mathsf{DiffComplex2(bnf,u)}$ returns $[\,]$, we have proved (**).

We finally address (***). We take $\Delta \subset \mathbb{C}$ to be the closed convex triangle with vertices $1, -\frac{1}{2} + i\frac{2521}{2911}, -\frac{1}{2} - i\frac{2521}{2911}$, where $i := \sqrt{-1} \in \mathbb{C}$. These points approximate the cubic roots of unity. We first try the values (which empirically seem to work)

$$(c', d') := \begin{cases} (1/5, 5) & \text{if } m = 8, \\ (1/6, 4) & \text{if } m = 10, \\ (1/8, 2) & \text{if } m = 12. \end{cases}$$

Unfortunately, the polyhedral complexes $P^{\Delta,\Delta}(c')$ and $P^{\Delta,\Delta}(d')$ are not k-rational (*i. e.* are not a finite union of k-rational cones) as $\Delta \times \Delta \subset \mathbb{C} \times \mathbb{C}$ does not have all vertices in $k \subset \mathbb{C} \times \mathbb{C}$. As our algorithm can only compute with k-rational cones, we cannot directly prove the inclusions (***) above using these parameters (c', d'). We therefore find an ε-deformation (see [CEF, §4.4]) of the identity $f : \mathbb{C}\times\mathbb{C} \to \mathbb{C}\times\mathbb{C}$ with $\varepsilon = 1/150$ such that $\mathsf{A} := f\big(P^{\Delta,\Delta}(c')\big)$ and $\mathsf{R} := f\big(P^{\Delta,\Delta}(d')\big)$ are explicit k-rational polyhedral complexes. To obtain A and R we use the command (again hosted in $\mathsf{CycloAlgorithm.gp}$)

$$[\mathsf{A},\mathsf{R}] = \mathsf{ApproxRComplex}(\mathsf{bnf},\mathsf{c}',\mathsf{d}');$$

with c' and d' as above. Then we check the inclusions $\mathsf{A} \subset S \subset \mathsf{R}$ computing the (k-rational) differences $\mathsf{D1} = \mathsf{A} - \mathsf{S}$ and $\mathsf{D2} = \mathsf{S} - \mathsf{R}$. For this we use the command (hosted in $\mathsf{CycloAlgorithm.gp}$)

$$[\mathsf{D1},\mathsf{D2}] = \mathsf{DiffComplex1}(\mathsf{bnf},\mathsf{A},\mathsf{R});$$

which returns $\mathsf{D1} = [\,]$ and $\mathsf{D2} = [\,]$, *i. e.* $f\big(P^{\Delta,\Delta}(c')\big) \subset S \subset f\big(P^{\Delta,\Delta}(d')\big)$. On the other hand, using [CEF, Lemma 15] (with $\varepsilon = 1/150$) we also find a rational pair (c, d) from (c', d') such that $P^{\Delta,\Delta}(c) \subset f\big(P^{\Delta,\Delta}(c')\big)$ and $f\big(P^{\Delta,\Delta}(d')\big) \subset P^{\Delta,\Delta}(d)$. The pairs (c, d) are obtained using the command (hosted in $\mathsf{CycloAlgorithm.gp}$)

$$[\mathsf{c},\mathsf{d}] = \mathsf{Bounds}([\mathsf{c}',\mathsf{d}']);$$

which returns

$$(c,d) = \begin{cases} \left(\frac{1814222527}{11043058985}, \frac{151154723}{24972421}\right) \approx (0.164, 6.052) & \text{if } (c', d') = (1/5, 5), \\ \left(\frac{3518892479}{26481431049}, \frac{242139697}{51405543}\right) \approx (0.132, 4.710) & \text{if } (c', d') = (1/6, 4), \\ \left(\frac{3299787329}{35272057207}, \frac{121800199}{54326945}\right) \approx (0.093, 2.241) & \text{if } (c', d') = (1/8, 2). \end{cases}$$

Thus we have proved (***).