GITHUB2: NOTES ON SHINTANI DOMAIN CALCULATIONS IN THE CYCLOTOMIC CASE

This README file describes the computational verification of the three claims made in §3 of the manuscript *Shintani fundamental domains for quartic number fields with many roots of unity* by A. Capuñay, M. Espinoza and E. Friedman. We will refer to that Ms. as [C/E/F]. The claims involve unions of cones connected to Shintani domains for the three quartic cyclotomic fields $k = \mathbb{Q}(\Theta)$, where Θ is a primitive root of unity of order $m = 8$, 10 or 12.

The first two claims in need of computer verification in §3 of [C/E/F] appear in display (20) there, while the third one appears on the next to last paragraph of the paper. Namely,

$$S = \{0\} \cup \bigcup_{\ell=0}^{m-1} (\Theta^\ell \cdot \mathcal{C}_0') \qquad \text{(disjoint union)}, \tag{$*$}$$

$$E \cdot S \subset S, \tag{$**$}$$

$$P^{\Delta,\Delta}(c) \subset S \subset P^{\Delta,\Delta}(d), \tag{$***$}$$

with notation as in display (19) of [C/E/F]. We will give Δ, a and b explicitly.

0.1. Overview of the calculations.

We first address (**) since it is the easiest one. By definition (see display (19) of [C/E/F]) $S := \bigcup_{\ell=0}^{m-1} (\Theta^\ell \cdot \mathcal{C}_0)$ is independent of E and satisfies $\Theta \cdot S = S$. Hence to verify (**) for any unit of infinite order $E \in k$, it suffices to do so for $E = u$ for any generator u of the units of k modulo torsion. This is done using an algorithm giving the difference $E \cdot S - S$ of unions of k-rational cones again as such a difference. SEE FILE XXX This, and other algorithms below, are described in our earlier paper [CEF]*Attractor-repeller construction of Shintani domains for totally complex quartic fields*, J. Number Th. **258** (2024) 146–172.

We now address (*). Since the inclusions $0 \in S$ and $(\Theta^\ell \cdot \mathcal{C}_0') \subset S$ are obvious from the definition of S in §3 of [C/E/F], the equality in (*) is verified by checking that

$$\mathcal{C}_0 - \bigcup_{\ell=0}^{m-1} (\Theta^\ell \cdot \mathcal{C}_0') = \{0\}.$$

This is done using the algorithm IN FILE XXX

The notation $P^{\Delta,\Delta}(c) \subset \mathbb{C} \times \mathbb{C}$ is defined in [CEF,(10)]. For Δ we take the closed convex set (in fact, a triangle) with vertices

$$[1, -\frac{1}{2} + i\frac{2521}{2911}, -\frac{1}{2} - i\frac{2521}{2911}],$$

where $i := \sqrt{-1}$ and $\frac{2521}{2911}$ was chosen as a rational approximation of $\sqrt{3}/2$. We first try the values

$$(c', d') := \begin{cases} (1/5, 5) & \text{if } m = 8, \\ (1/6, 4) & \text{if } m = 10, \\ (1/8, 2) & \text{if } m = 12. \end{cases}$$

Unfortunately, $P^{\Delta,\Delta}(c')$ and $P^{\Delta,\Delta}(d')$ are not k-rational (*i. e.* are not a finite union of k-rational cones). As our algorithm can only compute with k-rational cones, we cannot directly prove the inclusions (**) above. We therefore find an ε-deformation (see [CEF,(19)] with of the identity f with $\varepsilon = 1/150$ such that $f\big(P^{\Delta,\Delta}(c')\big)$ and $f\big(P^{\Delta,\Delta}(d')\big)$ are k-rational. Using FILA YYY we could thus verify $f\big(P^{\Delta,\Delta}(c')\big) \subset S \subset f\big(P^{\Delta,\Delta}(d')\big)$. Using [CEF, Lemma 15] we also find a rational pair (c, d) ESPECIFICAR such that $P^{\Delta,\Delta}(c) \subset f\big(P^{\Delta,\Delta}(c')\big)$ and $f\big(P^{\Delta,\Delta}(d')\big) \subset P^{\Delta,\Delta}(d)$.