GITHUB README: ON SHINTANI DOMAINS CALCULATION IN QUARTIC FIELDS WITH MANY ROOTS OF UNITY

In AlgorithmShitaniDomainK41 we given an implementation in PARI/GP to obtain explicit Shinani domains in totally complex quartic number fields k for the action on $\mathbb{C}^* \times \mathbb{C}^*$ of the free group $G = \langle E \rangle$ (torsion-free T), which is based in our earlier paper [CEF]*Attractor-repeller construction of Shintani domains for totally complex quartic fields*, J. Number Th. **258** (2024) 146–172. In this README file we describe an extension of such implementation for the action of the group $G = T \times \langle E \rangle$, where T denotes a subgroup of full torsion group W of k. In this case, the posible orders $|W|$ are 2, 4, 6, 8, 10 and 12. This implementation is supported in the preprint *Shintani fundamental domains for quartic number fields with many roots of unity* by A. Capuñay, M. Espinoza and E. Friedman, which we will refer as [C/E/F]. Now we explain the execution of our algorithm and some examples.

1. Execution

1.1. **On Main Algorithm.** In PARI/GP upload the file ShintaniTorK41.gp (hosted in Github) which contains our implementation to obtain such Shintani domains. Then carry through the command

$$\mathsf{F} = \mathsf{torFDK41}(\mathsf{p}, \mathsf{flag});$$

where p is an irreducible polynomial which defines a totally complex quartic number field k. Here this GP function $\mathsf{torFDK41}(--)$ has one mandatory input p, and an optional one, flag (non-negative integer), whose meaning is as follows:

($\star$) $\mathsf{flag} = 0$ (*default*): you can type $\mathsf{torFDK41}(\mathsf{p})$ or $\mathsf{torFDK41}(\mathsf{p}, 0)$ both return the same result. In this case the data F obtained (*described below*) represents information about of a Shintani domain for the action on $\mathbb{C}^* \times \mathbb{C}^*$ of the group $G = W \times \langle E \rangle$, when W is the *full* torsion group for k, obtained by PARI/GP.

($\star$) $\mathsf{flag} = 1$: if you type $\mathsf{torFDK41}(\mathsf{p}, 1)$ you get the same data (with 4 entries) described in AlgorithmShitaniDomainK41 by the command $\mathsf{FDK41}(\mathsf{p})$, which returns information of a Shintani domain obtained for action of the group $G = \langle E \rangle$ (torsion-free T).

($\star$) $\mathsf{flag} = \mathsf{m} > 1$: if you know a priori the order $t = |W|$ of the full torsion group W of k, then m is a divisor of t. In this case you can type $\mathsf{F} = \mathsf{torFDK41}(\mathsf{p}, \mathsf{m})$; to obtain a data F about a Shintani domain obtained for the action of $G = T \times \langle E \rangle$, where now T represents a subgroup of order m of the full torsion group W of k. Note that $\mathsf{torFDK41}(\mathsf{p}, \mathsf{t}) = \mathsf{torFDK41}(\mathsf{p})$ for t the order of the full torsion group W of k.

1.2. **On data** $\mathsf{F} = \mathsf{torFDK41}(\mathsf{p}, \mathsf{flag})$**.** Leaving aside the case $\mathsf{flag} = 1$, we explain the data obtained in F for the case when $\mathsf{flag} = 0$ or $\mathsf{flag} > 1$. Such F returns a list of three entries of form $\mathsf{F} := [\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3]$ interpreted as follows:
1. The first entry F_1 (i.e., $\mathsf{F}[1]$) has 9 entries of the form

$$[\mathsf{time}, \mathsf{p}, \mathsf{reg}, \mathsf{disc}, \mathsf{tor}, \mathsf{E}, \mathsf{r}, \mathsf{N}, \mathsf{v}]$$

where
$*$ time = real computation time for F in milliseconds.
$*$ p = quartic irreducible polynomial defining a totally complex number field $k :=$ the quotient ring $\mathbb{Q}[X]/(\mathsf{p})$.
$*$ reg = Regulator of k to 19 decimals.
$*$ disc = Discriminant of k.
$*$ tor = vector of two entries of the form $[t_1, t_2]$, where $t_1 = [t, \Theta]$, $t_2 = [m, \Theta^{t/m}]$, such that Θ is a generator of full torsion group W of k of order t, and $\Theta^{t/m}$ is a generator of torsion subgroup T of k of order $m := flag$. Thus m divides t, and $\Theta^{t/m} \in k$.
$*$ E = fundamental unit of k used (given by PARI/GP). The unit E, like all other elements of k below, is given as a polynomial g in $\mathbb{Q}[X]$ of degree at most 3. The associated element of k is the class of g in $\mathbb{Q}[X]/(\mathsf{p})$. Moreover, its embedding $E = (E_1, E_2)$ in $\mathbb{C} \times \mathbb{C}$ satisfy that $|E_1| < 1$.
$*$ r = is a positive integer such that for full torsion of order 2, $\mathsf{r} = 1$ if its regulator $reg(k) > 0.802$, $\mathsf{r} = 3$ otherwise. For the other full torsion of order 4, 6, 8, 10 or 12, $\mathsf{r} = 1$. More details see [C/E/F].
$*$ N = total number of (semi-closed) cones in the Shintani domain constructed.
$*$ $\mathsf{v} = [\#(\mathsf{four-cones}), \#(\mathsf{three-cones}), \#(\mathsf{two-cones}), \#(\mathsf{one-cones})]$ vector which describes information of the number semi-closed $j-$ cones (of dimension $j = 1, 2, 3, 4$) in a Shintani domain obtained by execution of command $\mathsf{torFDK41}(\mathsf{p}, \mathsf{flag})$.

2. The second entry F_2 of F (i.e., $\mathsf{F}[2]$) has the form

$$[\mathsf{C}_1, \mathsf{C}_2, ..., \mathsf{C}_\mathsf{N}]$$

which is a list of the N (semi-closed) cones where $\mathsf{N} = \mathsf{F}[1][8]$ was described above and the union of such cones form a Shintani domain for the action on $\mathbb{C}^* \times \mathbb{C}^*$ of the group $G = T \times \langle E \rangle$, with T subgroup of the full torsion group $W = \langle \Theta \rangle$ (of order $t = |W|$) whose generator is $\mathsf{F}[1][5][2][2] := \Theta^{t/m} \in k$ is of order $\mathsf{F}[1][5][2][1] := m$. Each cone C_j is given by ℓ *linear inequalities* (ℓ depending on the cone) giving ℓ closed or open half-spaces whose intersection is C_j. Thus, each C_j has the form $[\mathsf{v}_1, \mathsf{v}_2, ..., \mathsf{v}_\ell]$ where $\mathsf{v}_\mathsf{i} = [\mathsf{w}, 1]$ or $[\mathsf{w}, -1]$ and w is an element of k (depending on i and j). If w is followed by 1, then the corresponding (closed) half-space is the set of elements x of $\mathbb{R}^4$ with $\text{Trace}(\mathsf{xw}) \geq 0$. If w is followed by -1, then the corresponding (open) half-space is given by $\text{Trace}(\mathsf{xw}) > 0$. Here Trace is the extension to $\mathbb{R}^4$ of the trace map from k to $\mathbb{Q}$.
3. The third entry F_3 of F (i.e., $\mathsf{F}[3]$) has the form

$$[\overline{\mathsf{C}}_1, \overline{\mathsf{C}}_2, ..., \overline{\mathsf{C}}_\mathsf{N}]$$

where $\overline{\mathsf{C}}_\mathsf{j}$ is the closure in $\mathbb{R}^4$ of the cone C_j in F_2. And such closed cones $\overline{\mathsf{C}}_\mathsf{j}$ are given by its list of *generators* in k.

2. EXAMPLES