README ON SHINTANI DOMAINS FOR TOTALLY COMPLEX QUARTIC FIELDS

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1. Introduction

§2 below is a short user's guide to the PARI/GP program ShintaniTorK41.gp which produces a Shintani domain $\mathcal{F}$ for the action on $\mathbb{C}^* \times \mathbb{C}^*$ of an infinite subgroup $G \subset \mathcal{O}_k^*$ of the units of a totally complex quartic field k. The algorithm is based on our preprint *Shintani fundamental domains for quartic number fields with complex roots of unity*, cited below as {C/E/F}, which in turn relies on our paper *Attractor-repeller construction of Shintani domains for totally complex quartic fields*, J. Number Th. **258** (2024) 146–172, cited below as [CEF].

§3 below describes files with the output of some sample runs of the algorithm.

2. Using the program

2.1. **The Input p, m_1 and m_2.** The user must input an irreducible monic quartic polynomial $p \in \mathbb{Z}[x]$ corresponding to the totally complex quartic number field k, and two integers m_1 and m_2 describing the subgroup $G \subset \mathcal{O}_k^*$. Here $m_1 \geq 1$ is the number of roots of unity in G and m_2 is the index $m_2 := [\mathcal{O}_k^* : GW_k] \geq 1$, where W_k is the torsion subgroup of $\mathcal{O}_k^*$. These integers do not characterize G uniquely in general, but they suffice to produce a Shintani domain.

2.2. **How the algorithm works.** (Go to §2.3 if you just wish to launch the algorithm.) We first ignore the input m_1 and compute a Shintani domain $\widetilde{\mathcal{F}}$ for the subgroup $W_k E_0^{m_2} = W_k G \subset \mathcal{O}_k^*$, where E_0 is a fundamental unit of k, *i. e.* any generator of $\mathcal{O}_k^*/W_k$. The Shintani domain $\mathcal{F}$ for G is then the disjoint union $\mathcal{F} := \bigcup_\zeta \zeta \cdot \widetilde{\mathcal{F}}$, where ζ runs over representatives of $W_k/(W_k \cap G) \cong (W_k G)/G$. Explicitly, $\zeta = \Theta^j$ for $0 \leq j < |W_k|/m_1$, where Θ is a generator of W_k.

The computation of $\widetilde{\mathcal{F}}$ is carried out somewhat differently depending on the number of roots of unity $|W_k|$ in k. Namely,

- If $|W_k| = 2$, we implement the proof of Corollary 17 in [CEF], with $E := E_0^{m_2}$.
- If $|W_k| = 4$ or 6, we implement the proof of Proposition 2 in {C/E/F}, with $E := E_0^{m_2}$.
- If $|W_k| = 8, 10$ or 12, so k is a cyclotomic quartic field, we take $E := E_0^{m_2}$ in the Main Theorem of {C/E/F} and implement its proof for the cyclotomic case (§3 of {C/E/F}).

2.3. **Launching the algorithm.** In PARI/GP upload the file ShintaniTorK41.gp (hosted in Github ShintaniTorK41) by typing \r ShintaniTorK41.gp. Then type

$$\mathsf{F} = \mathsf{torFDK41}(\mathsf{p}, [\mathsf{m}_1, \mathsf{m}_2]);$$

where your input p, m_1 and m_2 describes k and G (see §2.1 above). The output F encoding the Shintani fundamental domain $\mathcal{F}$ for G and some additional data is described next.

2.4. **The output F.** F is a list $\mathsf{F} := [\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3]$ interpreted as follows.

• F_1 (i.e. $\mathsf{F}[1]$ in PARI/GP) has 7 entries $[\mathsf{time}, \mathsf{p}, \mathsf{reg}, \mathsf{disc}, \mathsf{E_0}, \mathsf{N}, \mathsf{v}]$, where
* time = real computation time for F in milliseconds.
* p = is the polynomial in the input. Thus PARI/GP encodes elements of $k := \mathbb{Q}[X]/(\mathsf{p})$ as polymods with respect to p.
* reg = Regulator of k to 19 decimals.
* disc = Discriminant of k.
* $E_0 \in \mathbb{Q}[X]/(\mathsf{p})$ is the fundamental unit of k that was used. Its embedding $\big(E_0^{(1)}, E_0^{(2)}\big)$ in $\mathbb{C}\times\mathbb{C}$ satisfies $|E_0^{(1)}| < 1$.
* N = total number of semi-closed pointed cones in the Shintani domain $\mathcal{F}$.
* v = [#(four-cones), #(three-cones), #(two-cones), #(one-cones)], a vector giving the number of j-dimensional cones in the Shintani domain $\mathcal{F}$ for $4 \geq j \geq 1$.

• F_2 (i.e. $\mathsf{F}[2]$) has the form $[\mathsf{C}_1, \mathsf{C}_2, ..., \mathsf{C}_\mathsf{N}]$, *i. e.* is a list of the N (semi-closed) cones, where $\mathsf{N} = \mathsf{F}[1][6]$ was described above, whose (disjoint) union is $\mathcal{F}$. Each cone C_j is given by ℓ linear inequalities (ℓ depending on the cone) giving ℓ closed or open half-spaces of $\mathbb{C}\times\mathbb{C}$ whose intersection is C_j. Thus, each C_j has the form $[\mathsf{v}_1, \mathsf{v}_2, ..., \mathsf{v}_\ell]$ where $\mathsf{v}_\mathsf{i} = [\mathsf{w}, 1]$ or $[\mathsf{w}, -1]$ and w is an element of k (*i. e.* a polymod depending on i and j). If w is followed by 1, then the corresponding (closed) half-space is the set of $x \in \mathbb{C}\times\mathbb{C}$ satisfying $\mathrm{Tr}_\mathsf{w}(x) \geq 0$. Here Tr_w is the continuous extension to $x \in \mathbb{C}\times\mathbb{C}$ of the map which for $x \in k$ (regarded as embedded in $\mathbb{C}\times\mathbb{C}$) is the trace to $\mathbb{Q}$ of $x\mathsf{w}$. If w is followed by -1, then the corresponding (open) half-space is given by $\mathrm{Tr}_\mathsf{w}(x) > 0$.

• F_3 (i.e. $\mathsf{F}[3]$) has the form $[\overline{\mathsf{C}}_1, \overline{\mathsf{C}}_2, ..., \overline{\mathsf{C}}_\mathsf{N}]$, where $\overline{\mathsf{C}}_\mathsf{j}$ is the closure in $\mathbb{R}^4$ of the cone C_j in F_2. These closed cones $\overline{\mathsf{C}}_\mathsf{j}$ are given by its list of generators (polymods in k).

3. EXAMPLES

In Github ShintaniTorK41, we give Shintani domains for G being the full group of units for 20 quartic fields.
* File ExamplesShK41-M.txt. In PARI/GP type `\r ExamplesShK41-M.txt;`
* File ExamplesShK41-ML.sage. In SAGE-Math type `load('ExamplesShK41-ML.sage')`

Either file contains $\mathsf{examples} := [\mathsf{Ex}_1, \ldots, \mathsf{Ex}_{20}]$, a list of length 20 where each Ex_j has the structure of F described in §2.4.