GITHUB README: ON SHINTANI DOMAIN CALCULATIONS IN THE CYCLOTOMIC CASE

This README file describes the computational verification of the three claims made in §3 of the manuscript *Shintani fundamental domains for quartic number fields with many roots of unity* by A. Capuñay, M. Espinoza and E. Friedman. We will refer to that Ms. as [C/E/F]. The claims involve unions of cones connected to Shintani domains for the three quartic cyclotomic fields $k = \mathbb{Q}(\Theta)$, where Θ is a primitive root of unity of order $m = 8$, 10 or 12.

The first two claims in need of computer verification in §3 of [C/E/F] appear in display (20) there, while the third one appears on the next to last paragraph of the paper. Namely,

$$(*) \qquad S \quad = \quad \{0\} \cup \bigcup_{\ell=0}^{m-1} (\Theta^{\ell} \cdot \mathcal{C}_0') \qquad \text{(disjoint union)},$$

$$(**) \qquad E \cdot S \quad \subset \quad S,$$

$$(***) \qquad P^{\Delta,\Delta}(c) \quad \subset S \subset \quad P^{\Delta,\Delta}(d).$$

with notation as in display (19) of [C/E/F]. We will give Δ, c and d explicitly.

0.1. **Overview of the calculations.** We address (*). Since the inclusions $0 \in S$ and $(\Theta^{\ell} \cdot \mathcal{C}_0') \subset S$ are obvious from the definition of S in §3 of [C/E/F], the equality in (*) is verified by checking that

$$L := \mathcal{C}_0 - \bigcup_{\ell=0}^{m-1} (\Theta^{\ell} \cdot \mathcal{C}_0') = \{0\}.$$

To check this, we first open the computational algebraic system PARI/GP and we execute on PARI/GP the following classic command

$$\mathsf{bnf} = \mathsf{bnfinit(p)};$$

where $\mathsf{p} = x^4 + 1,\ x^4 - x^3 + x^2 - x + 1,\ x^4 - x^2 + 1$ defines one of three cyclotomic fields $k = \mathbb{Q}(\Theta)$. And loading the file CycloAlgorithm.gp (*hosted on github*) using the classic command:

\r CycloAlgorithm.gp.

Then to check the difference of cones L given above we use the following command

$$L = \mathsf{DifConeS(bnf)};$$

returning $L = [\ ]$, which is interpreted as the vertex $\{0\}$. This cone difference algorithm, and other algorithms below, are described in our earlier paper [CEF]*Attractor-repeller construction of Shintani domains for totally complex quartic fields*, J. Number Th. **258** (2024) 146–172.

We now address (**). By definition (see display (19) of [C/E/F]) $S := \bigcup_{\ell=0}^{m-1}(\Theta^\ell \cdot \mathcal{C}_0)$ is independent of E and satisfies $\Theta \cdot S = S$. Hence to verify (**) for any unit of infinite order $E \in k$, it suffices to do so for $E = u$ for any generator u of the units of k modulo torsion. This is done by means of a command which calculates difference $E \cdot S - S$ of unions of k-rational cones again as such a difference. More precisely, we apply the following command (hosted in the file: $\mathsf{CycloAlgorithm.gp}$) to calculate $D := u \cdot S - S$,

$$\mathsf{D} = \mathsf{DiffComplex2}(\mathsf{bnf}, \mathsf{u});$$

where we use $u = x^2 + x + 1$ for $m = 8$; $u = -x + 1$ for $m = 10$; and $u = -x^3 - x^2$ for $m = 12$. All of these have with first embedding of module less than 1. These return $D = [\,]$, which means $u \cdot S \subset S$.

We finally address (***). The notation $P^{\Delta,\Delta}(c) \subset \mathbb{C} \times \mathbb{C}$ is defined in [CEF,(10)]. For Δ we take the closed convex set (in fact, a triangle) with vertices

$$\left[1,\ -\frac{1}{2} + i\frac{2521}{2911},\ -\frac{1}{2} - i\frac{2521}{2911}\right],$$

where $i := \sqrt{-1}$ and $\frac{2521}{2911}$ was chosen as a rational approximation of $\sqrt{3}/2$. We first try the values

$$(c', d') := \begin{cases} (1/5, 5) & \text{if } m = 8, \\ (1/6, 4) & \text{if } m = 10, \\ (1/8, 2) & \text{if } m = 12. \end{cases}$$

Unfortunately, the polyhedral complexes $P^{\Delta,\Delta}(c')$ and $P^{\Delta,\Delta}(d')$ are not k-rational (*i. e.* are not a finite union of *k-rational* cones). As our algorithm can only compute with k-rational cones, we cannot directly prove the inclusions (***) above using these parameters (c', d'). We therefore find an ε-deformation (see [CEF,(19)] with of the identity f with $\varepsilon = 1/150$) such that $\mathsf{A} := f\big(P^{\Delta,\Delta}(c')\big)$ and $\mathsf{R} := f\big(P^{\Delta,\Delta}(d')\big)$ are k-rational polyhedral complexes. To obtain these complexes we use the command (again hosted in the file: $\mathsf{CycloAlgorithm.gp}$)

$$[\mathsf{A}, \mathsf{R}] = \mathsf{ApproxRComplex}(\mathsf{bnf}, \mathsf{c}', \mathsf{d}');$$

Then we check the inclusions $\mathsf{A} \subset S \subset \mathsf{R}$ computing the (k-rational) differences $\mathsf{D1} := \mathsf{A} - \mathsf{S}$ and $\mathsf{D2} := \mathsf{S} - \mathsf{R}$, about these we use the following comand

$$[\mathsf{D1}, \mathsf{D2}] = \mathsf{DiffComplex1}(\mathsf{bnf}, \mathsf{A}, \mathsf{R});$$

which returns $\mathsf{D1} = [\,]$ and $\mathsf{D2} = [\,]$, that is, $f\big(P^{\Delta,\Delta}(c')\big) \subset S \subset f\big(P^{\Delta,\Delta}(d')\big)$. On the other hand, using [CEF, Lemma 15] (with $\varepsilon = 1/150$) we also find a rational pair (c, d) from (c', d') such that $P^{\Delta,\Delta}(c) \subset f\big(P^{\Delta,\Delta}(c')\big)$ and $f\big(P^{\Delta,\Delta}(d')\big) \subset P^{\Delta,\Delta}(d)$. Thus we have (***), where such pairs (c, d) are obtained using the command

$$[\mathsf{c}, \mathsf{d}] = \mathsf{Bounds}([\mathsf{c}', \mathsf{d}']);$$

which returns

$$(c, d) = \begin{cases} \left(\frac{1814222527}{11043058985}, \frac{151154723}{24972421}\right) \approx (0.164, 6.052) & \text{if } (c', d') = (1/5, 5), \\[2ex] \left(\frac{3518892479}{26481431049}, \frac{242139697}{51405543}\right) \approx (0.132, 4.710) & \text{if } (c', d') = (1/6, 4), \\[2ex] \left(\frac{3299787329}{35272057207}, \frac{121800199}{54326945}\right) \approx (0.093, 2.241) & \text{if } (c', d') = (1/8, 2). \end{cases}$$