PHYSICS

Relativistic independence bounds nonlocality

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If nature allowed nonlocal correlations other than those predicted by quantum mechanics, would that contradict some physical principle? Various approaches have been put forward in the past two decades in an attempt to single out quantum nonlocality. However, none of them can explain the set of quantum correlations arising in the simplest scenarios. Here, it is shown that generalized uncertainty relations, as well as a specific notion of locality, give rise to both familiar and new characterizations of quantum correlations. In particular, we identify a condition, relativistic independence, which states that uncertainty relations are local in the sense that they cannot be influenced by other experimenters' choices of measuring instruments. We prove that theories with nonlocal correlations stronger than the quantum ones do not satisfy this notion of locality, and therefore, they either violate the underlying generalized uncertainty relations or allow experimenters to nonlocally tamper with the uncertainty relations of their peers.

INTRODUCTION

Quantum mechanics stands out in enabling strong, nonlocal correlations between remote parties. On the one hand, these quantum correlations cannot, in any way, be explained by models of classical physics. On the other hand, quantum theory remains rather elusive about their physical origin (*1*–*3*). If nature allowed nonlocal correlations other than those predicted by quantum mechanics, would that break any known physical principle? This question becomes all more important when the predictions of quantum mechanics are experimentally verified time and again.

Initially, it was speculated that those correlations excluded by quantum mechanics violate relativistic causality—the principle that dictates that experiments can be influenced only by events in their past light cone and can influence events only in their future light cone. However, it was shown that other theories may exist, whose correlations, while not realizable in quantum mechanics, are nevertheless nonsignaling and are hence consistent with relativistic causality (*1*).

Over the past 20 years, many efforts have been invested in a line of research aimed at quantitatively deriving the strength of quantum correlations from basic principles. For example, it was shown that violations of the Bell-CHSH (Clauser-Horne-Shimony-Holt) inequality (*4*) beyond the quantum limit, known as Tsirelson's bound, are inconsistent with the uncertainty principle (*5*). Popescu-Rohrlich–boxes (PR-boxes), the hypothetical models achieving the maximal violation of the Bell-CHSH inequality (*1*), would allow distributed computation to be performed with only one bit of communication (*6*), which looks unlikely but does not violate any known physical law. Similarly, in stronger-than-quantum nonlocal theories, some computations exceed reasonable performance limits (*7*), and there is no sensible measure of mutual information between pairs of systems (*8*). Last, it was shown that superquantum nonlocality does not permit classical physics to emerge in the limit of infinitely many microscopic systems (*9*–*11*), and also violates the exclusiveness of local measurement outcomes in multipartite settings (*12*). However, none of these and other principles that have been proposed (*2*) can explain the set of one- and two-point correlators that fully characterize the quantum probability distributions witnessed in the simplest bipartite two-outcome scenario.

A consequence of relativistic causality within the framework of probabilistic theories is known as the no-signaling condition—the local probability distributions of one experimenter (marginal probabilities) are independent of another experimenter's choices (*1*). While the no-signaling condition is insufficient to single out quantum correlations, it is shown here that an analogous requirement applicable in conjunction with generalized uncertainty relations is satisfied exclusively by quantum mechanical correlations.

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RESULTS

In what follows, we first assume (in the next section) that generalized uncertainty relations are valid within the theory in question. These uncertainty relations broaden the meaning of uncertainty beyond the realm of quantum mechanics and give rise to the Schrödinger-Robertson uncertainty relation when applied to the latter. Then, in the "Independence" section, we additionally assume a certain form of independence—we name relativistic independence (RI)—meaning here that local uncertainty relations cannot be affected at a distance. The above assumptions accord well with experimental observations but generalize the underlying theoretical model beyond the quantum formalism.

Generalized uncertainty relations

Three experimenters—Alice, Bob, and Charlie—perform an experiment, where each of them owns a measuring device. On each such device, a knob determines its mode of operation, either "0" or "1," which allows the measurement of two physical variables: A_0/A_1 on Alice's side, B_0/B_1 on Bob's side, and C_0/C_1 on Charlie's side. Alice and Bob are close to one another, and so, they use the readings from all their devices to empirically evaluate the variances, $\Delta^2_{A_i}$ and $\Delta^2_{B_i}$, and the covariances, $\mathbf{C}(A_i, B_j) \overset{\text{def}}{=} E_{A_iB_j} - E_{A_i}E_{B_j}$, where E_{A_i}, E_{B_j}, and $E_{A_iB_j}$ are the respective one- and two-point correlators. Charlie, on the other hand, is far from them (see Fig. 1).

Assume that measurements of physical variables are generally inflicted with uncertainty. This uncertainty not only affects pairs of local measurements performed by individual experimenters but also governs any number of measurements performed by groups of remote experimenters. In our tripartite setting, for example, the measurements of Alice, Bob, and Charlie are assumed to be jointly governed by the generalized uncertainty relation

$$\Lambda_{ABC} \overset{\text{def}}{=} \begin{bmatrix} \Lambda_C & \mathbf{C}(B,C)^T & \mathbf{C}(A,C)^T \\ \mathbf{C}(B,C) & \Lambda_B & \mathbf{C}(A,B)^T \\ \mathbf{C}(A,C) & \mathbf{C}(A,B) & \Lambda_A \end{bmatrix} \succeq 0 \qquad (1)$$

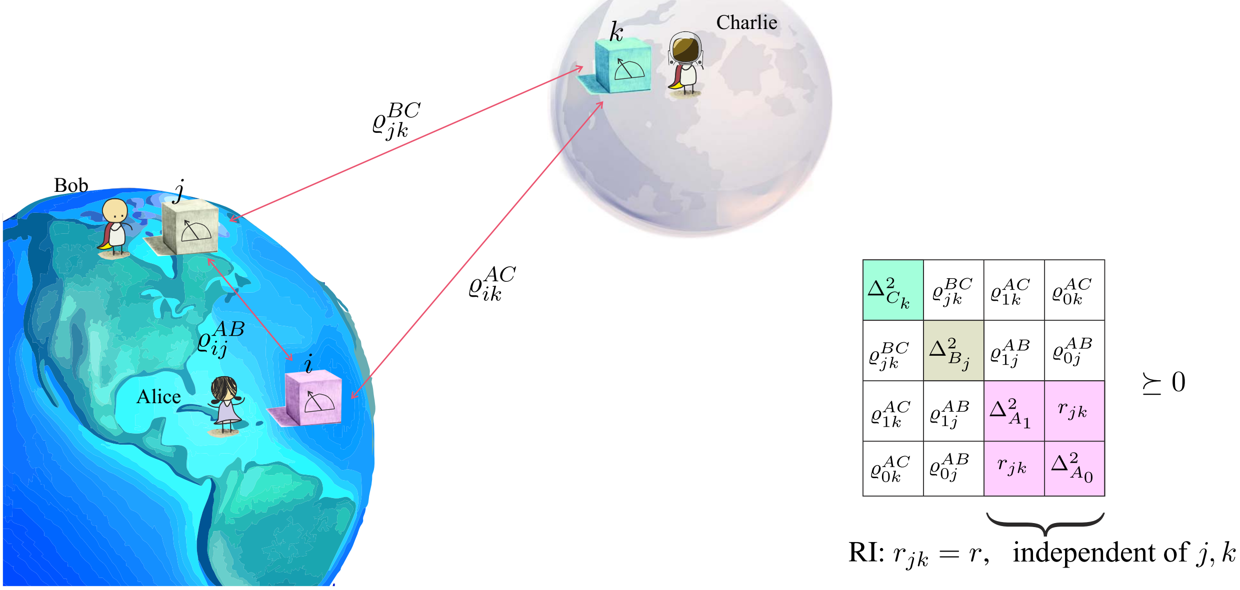

Fig. 1. An illustration of RI in a tripartite scenario. In a theory obeying generalized uncertainty relations (shown in the bottom right corner in the form of a certain positive semidefinite matrix), RI prevents Bob and Charlie from influencing Alice's uncertainty relations, e.g., $\Delta^2_{A_0}\Delta^2_{A_1} \geq r^2_{jk}$, through their choices j and k, i.e., $r_{jk} = r$. Here, $\varrho^{AB}_{ij} = \mathbf{C}(A_i, B_j)$, $\varrho^{AC}_{ik} = \mathbf{C}(A_i, C_k)$, and $\varrho^{BC}_{jk} = \mathbf{C}(B_j, C_k)$ illustrated by the arrows are the covariances of Alice-Bob, Alice-Charlie, and Bob-Charlie measurements, respectively. In the quantum mechanical formalism, a similar matrix inequality gives rise to the Schrödinger-Robertson uncertainty relations of Alice's self-adjoint operators $\hat{A}_0$ and $\hat{A}_1$, as well as between the nonlocal Alice-Bob operators, $\hat{A}_0\hat{B}_j$ and $\hat{A}_1\hat{B}_j$. See Materials and Methods.

which means that Λ_{ABC} is a positive semidefinite matrix. Here, $\mathbf{C}(A, B)$, $\mathbf{C}(A, C)$, and $\mathbf{C}(B, C)$ are the empirical covariance matrices of Alice-Bob, Alice-Charlie, and Bob-Charlie measurements. The diagonal submatrices, e.g., Λ_A, represent the uncertainty relations governing the individual experimenters. Below and in Materials and Methods, Eq. 1 is shown to imply the quantum mechanical Schrödinger-Robertson uncertainty relations (*13*), as well as their multipartite nonquantum generalizations. Moreover, in local hidden-variables theories where all measurement outcomes preexist, Eq. 1 coincides with a covariance matrix, which is, by construction, positive semidefinite and represents the uncertainty of A_i, B_j, and C_k, hence the natural generalization to other theories.

Provided that Bob measured B_j and Charlie measured C_k, the system as a whole is governed by a submatrix of Λ_{ABC}

$$\Lambda^{jk}_{ABC} \stackrel{\text{def}}{=} \begin{bmatrix} \Delta^2_{C_k} & \mathbf{C}(C_k, B_j) & \mathbf{C}(C_k, A_1) & \mathbf{C}(C_k, A_0) \\ \mathbf{C}(C_k, B_j) & \Delta^2_{B_j} & \mathbf{C}(B_j, A_1) & \mathbf{C}(B_j, A_0) \\ \mathbf{C}(C_k, A_1) & \mathbf{C}(B_j, A_1) & \Delta^2_{A_1} & r_{jk} \\ \mathbf{C}(C_k, A_0) & \mathbf{C}(B_j, A_0) & r_{jk} & \Delta^2_{A_0} \end{bmatrix} \succeq 0 \quad (2)$$

Here, r_{jk} is a real number whose value guarantees that $\Lambda^{jk}_{ABC} \succeq 0$. Therefore, it generally depends not only on Alice's choices but also on Bob's j and Charlie's k. The lower 2×2 submatrix in Eq. 2, which is henceforth denoted as the positive semidefinite Λ^{jk}_{ABC}, implies that Alice's measurements satisfy $\Delta^2_{A_0}\Delta^2_{A_1} \geq r^2_{jk}$, as well as other uncertainty relations that depend on r_{jk} rather than r^2_{jk}, i.e., $u^T \Lambda^{jk}_A u \geq 0$, where u is any two-dimensional real-valued vector.

Local hidden-variables theories, quantum mechanics, and nonquantum theories such as the hypothetical PR-boxes (*1*) obey Eq. 2. Moreover, they provide different closed forms for this r_{jk}, which, in general, we are unable to assume. In local hidden-variables theories, where A_0 and A_1 are classical random variables whose joint probability distribution is well defined, Eq. 2 holds for $r_{jk} = \mathbf{C}(A_0, A_1)$, which is independent of j and k. In quantum mechanics, the Schrödinger-Robertson uncertainty relations show that r_{jk} depends exclusively on Alice's self-adjoint operators, in particular their commutator and anticommutator. If Alice and Charlie share a PR-box, then $r_{jk} = (-1)^k$, which, in contrast to the other two theories, depends on k.

Independence

In the above setting, Bob and Charlie may be able to nonlocally tamper with Alice's uncertainty relation, $\Lambda^{jk}_A \succeq 0$, through their j and k. Prohibiting this by requiring that Alice's uncertainty relation as a whole, i.e., the trio Δ_{A_0}, Δ_{A_1}, and r_{jk} would be independent of Bob's j and Charlie's k, leads to the set of quantum mechanical one- and two-point correlators. This condition is named RI.

By RI, the Alice-Bob system, which is governed by the lower 3×3 submatrix of Δ^{jk}_{ABC}, satisfies $\Lambda^{jk}_A \stackrel{\text{def}}{=} \Lambda_A$, for $r_{jk} \stackrel{\text{def}}{=} r$. Swapping the roles of Alice and Bob, where Alice measures A_i, RI similarly implies $\Lambda^{ik}_B \stackrel{\text{def}}{=} \Lambda_B$, for $\bar{r}_{ik} \stackrel{\text{def}}{=} \bar{r}$. That is, RI means

$$\begin{bmatrix} \Delta^2_{B_j} & \mathbf{C}(B_j, A_1) & \mathbf{C}(B_j, A_0) \\ \mathbf{C}(B_j, A_1) & \Delta^2_{A_1} & r \\ \mathbf{C}(B_j, A_0) & r & \Delta^2_{A_0} \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} \Delta^2_{A_i} & \mathbf{C}(A_i, B_1) & \mathbf{C}(A_i, B_0) \\ \mathbf{C}(A_i, B_1) & \Delta^2_{B_1} & \bar{r} \\ \mathbf{C}(A_i, B_0) & \bar{r} & \Delta^2_{B_0} \end{bmatrix} \succeq 0 \quad (3)$$

for $i, j \in \{0, 1\}$. RI (Eq. 3) and the no-signaling condition are distinct and do not follow from one another. The no-signaling condition, for example, dictates that the (marginal) probability distributions of Alice's measurements, and therefore also $\Delta^2_{A_0}$ and $\Delta^2_{A_1}$, are independent of Bob's choices. RI, on the other hand, implies that Λ_A in its entirety must be independent of Bob's choices, which may hold whether or not Alice's marginal probabilities are independent of j. The relationship between the two conditions is discussed in more detail in Materials and Methods.

PR-boxes satisfy the no-signaling condition but violate RI (see Materials and Methods). Moreover, as stated below, RI (Eq. 3) is satisfied exclusively by the quantum mechanical bipartite one- and two-point correlators.

Theorem 1. The conditions (Eq. 3) imply

$$|\varrho_{00}\varrho_{10} - \varrho_{01}\varrho_{11}| \le \sum_{j=0,1}\sqrt{\left(1-\varrho^2_{0j}\right)\left(1-\varrho^2_{1j}\right)}$$
$$|\varrho_{00}\varrho_{01} - \varrho_{10}\varrho_{11}| \le \sum_{i=0,1}\sqrt{\left(1-\varrho^2_{i0}\right)\left(1-\varrho^2_{i1}\right)} \quad (4)$$

where $\varrho_{ij} \stackrel{\text{def}}{=} \mathbf{C}(A_i, B_j)/\left(\Delta_{A_i}\Delta_{B_j}\right)$ is the Pearson correlation coefficient between A_i and B_j.

It is known that any four correlators, $E_{A_iB_i}$, must satisfy Eq. 4 if they are to describe the nonlocality present in a physically realizable quantum mechanical pair of systems (*3*). In addition, all the sets of these correlators permitted by Eq. 4 are possible within quantum mechanics. This result was proven when assuming quantum mechanics and vanishing one-point correlators, $E_{A_i} = E_{B_j} = 0$, independently by Tsirel'son (*14*), Landau (*15*), and Masanes (*16*). More recently, Eq. 4 has been derived for the case of binary measurements from the first level of the Navascues-Pironio-Acin (NPA) hierarchy (*17*). We show without assuming any of these that this bound (in the form of Landau) originates from RI (Eq. 3). Moreover, it is now clear that Eq. 4 must hold not only for binary but also for other, both discrete and continuous, variables. Consequently, Tsirelson's $2\sqrt{2}$ bound (*18*) on the Bell-CHSH parameter (*4*), $\mathcal{B}_{AB} \stackrel{\text{def}}{=} \varrho_{00} + \varrho_{10} + \varrho_{01} - \varrho_{11}$, applies to any type of measurement. For example, Alice's and Bob's measurements may be the position and momentum of some wave function. Quantum theory satisfies the RI condition (Eq. 3) and is therefore subject to Eq. 4. Furthermore, in the case of binary ±1 measurements whose one-point correlators vanish, the first Alice-Bob uncertainty relation in Eq. 3 is given in quantum mechanics by the Schrödinger-Robertson uncertainty relations of $\hat{A}_0\hat{B}_j$ and $\hat{A}_i\hat{B}_j$, where $\hat{A}_1$ and $\hat{B}_j$ are Alice's and Bob's self-adjoint operators. See Materials and Methods for the proof of this theorem and for further details.

Surprisingly, within the quantum formalism Eq. 4 is a special case of another bound with two extra terms.

Theorem 2. In quantum theory, where the Alice and Bob measurements are represented by the self-adjoint operators $\hat{A}_i$ and $\hat{B}_j$, the following holds

$$|\varrho_{00}\varrho_{10} - \varrho_{01}\varrho_{11}| \le \sum_{j=0,1}\sqrt{\left(1-\varrho^2_{0j}\right)\left(1-\varrho^2_{1j}\right) - \eta^2_{\hat{A}}}$$
$$|\varrho_{00}\varrho_{01} - \varrho_{10}\varrho_{11}| \le \sum_{i=0,1}\sqrt{\left(1-\varrho^2_{i0}\right)\left(1-\varrho^2_{i1}\right) - \eta^2_{\hat{B}}} \quad (5)$$

where $\varrho_{ij} \stackrel{\text{def}}{=} (\langle\hat{A}_i\hat{B}_j\rangle - \langle\hat{A}_i\rangle\langle\hat{B}_j\rangle)/(\Delta_{\hat{A}_i}\Delta_{\hat{B}_j})$, and $\eta_{\hat{X}} \stackrel{\text{def}}{=} \frac{1}{2i}\langle[\hat{X}_0, \hat{X}_1]\rangle/(\Delta_{\hat{X}_0}\Delta_{\hat{X}_1})$, with $\hat{X}$ being either $\hat{A}$ or $\hat{B}$. Here, $[\hat{X}_0, \hat{X}_1] \stackrel{\text{def}}{=} \hat{X}_0\hat{X}_1 - \hat{X}_1\hat{X}_0$ is the commutator of $\hat{X}_0$ and $\hat{X}_1$, and $\Delta^2_{\hat{X}} = \langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2$ is the variance of $\hat{X}$. The $\langle\cdot\rangle$ is the quantum mechanical expectation. Note that $\frac{1}{2i}\left[\hat{X}_0, \hat{X}_1\right]$ is self-adjoint and is therefore an observable. Moreover, $|\eta_{\hat{X}}| \le 1$, where $|\eta_{\hat{X}}| = 1$ only if the Robertson uncertainty relation of $\hat{X}_0$ and $\hat{X}_1$ is saturated. The proof of this theorem is given in Materials and Methods.

Local uncertainty relations and nonlocal correlations

The geometry of bipartite RI in Hilbert space is illustrated in Fig. 2. The left picture of Fig. 2 is the geometry underlying the first bound

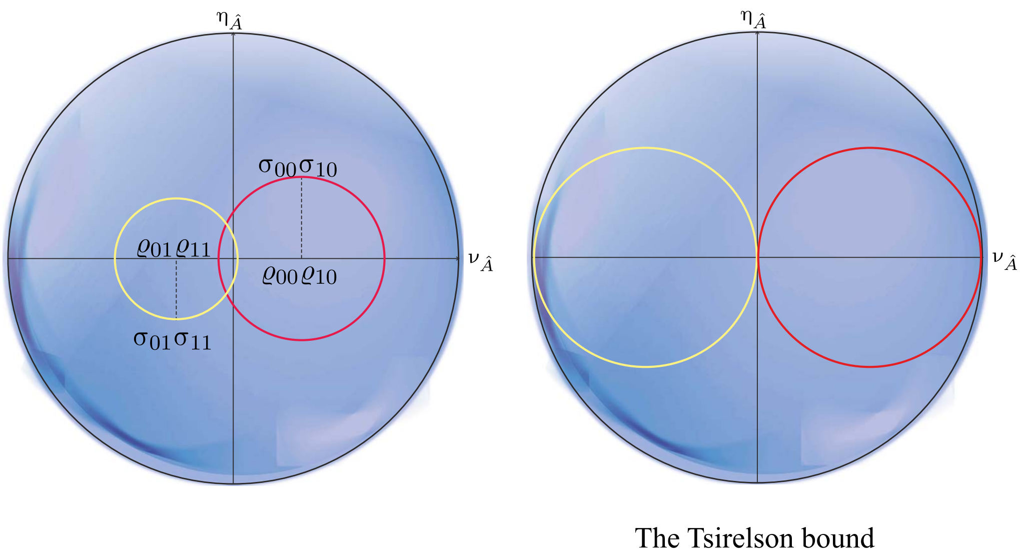

Fig. 2. Geometry of bipartite RI in Hilbert space, the bounds in Eq. 5. The $\eta_{\hat{A}}$ is as defined in Theorem 2, and $\nu_{\hat{A}} \stackrel{\text{def}}{=} \left(\frac{1}{2}\langle\{\hat{A}_0, \hat{A}_1\}\rangle - \langle\hat{A}_0\rangle\langle\hat{A}_1\rangle\right)/\left(\Delta_{\hat{A}_0}\Delta_{\hat{A}_1}\right)$, where $\{\hat{X}, \hat{Y}\}$ is the anti-commutator. Using these definitions, the Schrödinger-Robertson uncertainty relation between Alice's observables is $\nu^2_{\hat{A}} + \eta^2_{\hat{A}} \le 1$, hence the pair of bluish unit discs. Bob's choice, $j = 1$ or $j = 0$, further confines Alice's uncertainty, the $\eta_{\hat{A}}$ and $\nu_{\hat{A}}$, to one of the circles, the yellow or the red, respectively. The extent and location of these circles are determined by the nonlocal covariances, ϱ_{ij}. Quantum mechanics satisfies RI and thus keeps Alice's uncertainty relations independent of Bob's choices, i.e., by allowing only those covariances for which the red and yellow circles intersect. Tsirelson's bound is an extreme configuration where these circles intersect at the origin.

in Eq. 5. This bound arises from the two uncertainty relations (Eq. 3), which, from within quantum mechanics, coincide with the Schrödinger-Robertson uncertainty relations of $\hat{A}_0\hat{B}_j$ and $\hat{A}_1\hat{B}_j$ in the special case of binary measurements. In other cases, Eq. 3 may be viewed as a generalization of the Schrödinger-Robertson uncertainty relations. As shown in Materials and Methods, inside Hilbert space, Eq. 3 describes two circles in the complex plane: one for $j = 0$ (red) and another for $j = 1$ (yellow). The circles are centered at $\varrho_{0j}\varrho_{1j}$, and their respective radii are $\sigma_{0j}\sigma_{1j}$, where $\sigma_{ij}^2 = 1 - \varrho_{ij}^2$. Alice's local uncertainty relations are confined to one or another circle depending on Bob's choice j. Quantum mechanics satisfies RI and thus keeps Alice's uncertainty relations independent of Bob's choice, i.e., by allowing only those covariances ϱ_{ij} for which the red and yellow circles intersect. Tsirelson's bound (the right picture of Fig. 2), for example, is attained when the region of intersection collapses to a single point at the origin.

RI implies that the extent of nonlocality is governed by local uncertainty relations. The interplay between nonlocality as quantified by the Bell-CHSH parameter, $\mathcal{B}$, and Heisenberg uncertainty, where $\hat{A}_0 = \hat{x}$ and $\hat{A}_1 = \hat{p}$ are the position and momentum operators, respectively (see Materials and Methods for the complete derivation), is

$$\left(\frac{\mathcal{B}}{2\sqrt{2}}\right)^2 + \left(\frac{\hbar/2}{\Delta_{\hat{x}}\Delta_{\hat{p}}}\right)^2 \leq 1 \tag{6}$$

It is known that a complete characterization of the set of quantum correlations must follow from inherently multipartite principles (*19*). As shown in Materials and Methods, RI applies to any number of parties with any number of measuring devices. This allows us, for example, to derive a generalization of Eq. 4 for the Alice-Bob, Alice-Charlie, and Bob-Charlie one- and two-point correlators in a tripartite scenario. The property known as monogamy of correlations, the $|\mathcal{B}_{AB}| + |\mathcal{B}_{AC}| \leq 4$, follows as a special case of this inequality. In the same section, it is shown that the correlators in local hidden-variables theories can be similarly bounded by a variant of RI.

DISCUSSION

Within a class of theories obeying generalized uncertainty relations, RI was shown to reproduce the complete quantum mechanical characterization of the bipartite correlations in two-outcome scenarios, and potentially in much more general cases as straightforward corollaries of our approach. To fully characterize the set of quantum correlations would generally require analyzing the uncertainty relation Eq. 1 in an elaborate multipartite setting, accounting for all the parties' cross-correlations and assuming RI (this point, as well as some other technical issues, is discussed in detail in Materials and Methods). All these imply that stronger-than-quantum nonlocal theories may either be incompatible with the uncertainty relations analyzed above or allow experimenters to nonlocally tamper with the uncertainty relations of other experimenters.

MATERIALS AND METHODS

No-signaling condition and RI

A consequence of relativistic causality in probabilistic theories is the no-signaling condition (*1*). Consider the Bell-CHSH setting where a and b are the outcomes of Alice's and Bob's measurements. The joint probability of these outcomes when Alice measured using device i and Bob measured using device j is denoted as $p(a, b \mid i, j)$. The no-signaling condition states that one experimenter's marginal probabilities are independent of another experimenter's choices, namely

$$\begin{aligned}\textstyle\sum_b p(a,b|i,0) = \sum_b p(a,b|i,1) \overset{\text{def}}{=} p(a|i)\\ \textstyle\sum_a p(a,b|0,j) = \sum_a p(a,b|1,j) \overset{\text{def}}{=} p(b|j)\end{aligned} \tag{7}$$

Of course it means that one experimenter's precision is independent of another experimenter's choices

$$\begin{aligned}\Delta_{A_i}^2 = E_{a^2|i,j} - E_{a|i,j}^2 = \textstyle\sum_{a,b} a^2 p(a,b|i,j) -\\ \left(\textstyle\sum_{a,b} ap(a,b|i,j)\right)^2 = \sum_a a^2 p(a|i) - \left(\sum_a ap(a|i)\right)^2\\ \Delta_{B_j}^2 = E_{b^2|i,j} - E_{b|i,j}^2 = \textstyle\sum_{a,b} b^2 p(a,b|i,j) -\\ \left(\textstyle\sum_{a,b} bp(a,b|i,j)\right)^2 = \sum_b b^2 p(b|j) - \left(\sum_b bp(b|j)\right)^2\end{aligned} \tag{8}$$

The no-signaling condition thus implies that the variances of one experimenter in the Alice-Bob uncertainty relations (Eq. 3) are independent of the other experimenter's choices.

RI implies that one experimenter's uncertainty relation is altogether independent of the other experimenter's choices, i.e., that Λ_A as a whole, and therefore also r_j, are independent of j. This does not necessarily imply the no-signaling condition, as there may exist, for example, marginal distributions $p(a \mid i, j)$ that depend on Bob's j whose variances, $\Delta_{A_i}^2$, are nevertheless independent of this j. This shows that RI does not at all require us to assume the no-signaling condition.

PR-boxes violate RI

Consider a tripartite setting where Bob and Charlie are uncorrelated, $\mathbf{C}(B_j, C_k) = 0$, and Alice and Charlie share a PR-box (*1*). The PR-boxes define $E_{A_iC_k} = (-1)^{ik}$, $E_{A_i} = 0$, and $E_{C_k} = 0$. The variances are thus $\Delta_{A_i}^2 = E_{A_i^2} - E_{A_i}^2 = 1$ and $\Delta_{C_k}^2 = E_{C_k^2} - E_{C_k}^2 = 1$, and the covariances are $\mathbf{C}(A_1, C_k) = (-1)^k$ and $\mathbf{C}(A_0, C_k) = 1$. In this case, a permutation of Eq. 2 reads

$$\Lambda_{PR}^{jk} \overset{\text{def}}{=} \begin{bmatrix} \Delta_{B_j}^2 & 0 & \mathbf{C}(A_1,B_j) & \mathbf{C}(A_0,B_j) \\ 0 & 1 & 1 & (-1)^k \\ \mathbf{C}(A_1,B_j) & 1 & 1 & r_{jk} \\ \mathbf{C}(A_0,B_j) & (-1)^k & r_{jk} & 1 \end{bmatrix} \succeq 0 \tag{9}$$

Namely

$$M^{-1}\Lambda_{PR}^{jk}M^{-1} = \begin{bmatrix} 1 & 0 & \varrho_{1j}^{AB} & \varrho_{0j}^{AB} \\ 0 & 1 & 1 & (-1)^k \\ \varrho_{1j}^{AB} & 1 & 1 & r_{jk} \\ \varrho_{0j}^{AB} & (-1)^k & r_{jk} & 1 \end{bmatrix} \succeq 0 \tag{10}$$

where M is a diagonal matrix whose (nonvanishing) terms are all ones but Δ_{B_j}. By the Schur complement condition for positive semidefiniteness, Eq. 10 is equivalent to

$$\begin{bmatrix} 1 & r_{jk} \\ r_{jk} & 1 \end{bmatrix} \succeq \begin{bmatrix} \varrho_{1j}^{AB} & 1 \\ \varrho_{0j}^{AB} & (-1)^k \end{bmatrix} \begin{bmatrix} \varrho_{1j}^{AB} & 1 \\ \varrho_{0j}^{AB} & (-1)^k \end{bmatrix}^T =$$

$$\begin{bmatrix} (\varrho_{1j}^{AB})^2 & \varrho_{1j}^{AB}\varrho_{0j}^{AB} \\ \varrho_{1j}^{AB}\varrho_{0j}^{AB} & (\varrho_{0j}^{AB})^2 \end{bmatrix} + \begin{bmatrix} 1 & (-1)^k \\ (-1)^k & 1 \end{bmatrix} \tag{11}$$

which renders $\varrho_{ij}^{AB} = 0$ (positive semidefiniteness of the matrix obtained by subtracting the right-hand side from the left-hand side implies the nonnegativity of its diagonal entries from which this result follows). The inequality Eq. 11 is equivalent to $-[r_{jk} - (-1)^k]^2 \geq 0$ and only holds for $r_{jk} = (-1)^k$. Such a theory therefore violates RI.

However, the PR-box example teaches us something profound. In this model, complementarity (i.e., the inability to measure both local variables in the same experiment) must be assumed in both Alice's and Charlie's ends; otherwise, Alice, for example, may evaluate

$$\begin{aligned} A_0A_1 &= (A_0C_k)(A_1C_k) = \mathbf{C}(A_0, C_k)\mathbf{C}(A_1, C_k) \\ &= (-1)^0(-1)^k = (-1)^k = r_{jk} \end{aligned} \tag{12}$$

from which she could tell Charlie's choice k. Lack of complementarity immediately leads to signaling in the case of PR-boxes, but as we have seen, the weaker assumption of uncertainty leads to a problem with RI.

Schrödinger-Robertson uncertainty relations and the generalized uncertainty relations Eqs. 1 to 3

Let $\hat{A}_i$ and $\hat{B}_j$ be self-adjoint operators with ±1 eigenvalues and $\langle\hat{A}_i\rangle = \langle\hat{B}_j\rangle = 0$, whose product, $\hat{A}_i\hat{B}_j$, is similarly self-adjoint. The Schrödinger-Robertson uncertainty relations of the corresponding products, $\hat{A}_0\hat{B}_j$ and $\hat{A}_1\hat{B}_j$

$$\Delta^2_{\hat{A}_0\hat{B}_j}\Delta^2_{\hat{A}_1\hat{B}_j} \geq \left(\frac{1}{2}\langle\{\hat{A}_0, \hat{A}_1\}\rangle - \mathbf{C}(\hat{A}_0, \hat{B}_j)\mathbf{C}(\hat{A}_1, \hat{B}_j)\right)^2 + \left(\frac{1}{2i}\langle[\hat{A}_0, \hat{A}_1]\rangle\right)^2 \tag{13}$$

where $\mathbf{C}(\hat{A}_i, \hat{B}_j) = \langle\hat{A}_i\hat{B}_j\rangle$, and the variance, $\Delta^2_{\hat{A}_i\hat{B}_j} = 1 - \mathbf{C}(\hat{A}_i, \hat{B}_j)^2$, can alternatively be written as

$$\begin{bmatrix} 1 & \langle\hat{A}_0\hat{A}_1\rangle \\ \langle\hat{A}_1\hat{A}_0\rangle & 1 \end{bmatrix} \succeq \begin{bmatrix} \mathbf{C}(\hat{A}_1, \hat{B}_j)^2 & \mathbf{C}(\hat{A}_1, \hat{B}_j)\mathbf{C}(\hat{A}_0, \hat{B}_j) \\ \mathbf{C}(\hat{A}_1, \hat{B}_j)\mathbf{C}(\hat{A}_0, \hat{B}_j) & \mathbf{C}(\hat{A}_0, \hat{B}_j)^2 \end{bmatrix} \tag{14}$$

By the Schur complement condition for positive semidefiniteness, this is equivalent to

$$\begin{bmatrix} \Delta^2_{\hat{B}_j} & \mathbf{C}(\hat{A}_1, \hat{B}_j) & \mathbf{C}(\hat{A}_0, \hat{B}_j) \\ \mathbf{C}(\hat{A}_1, \hat{B}_j) & \Delta^2_{\hat{A}_1} & \langle\hat{A}_0\hat{A}_1\rangle \\ \mathbf{C}(\hat{A}_0, \hat{B}_j) & \langle\hat{A}_1\hat{A}_0\rangle & \Delta^2_{\hat{A}_0} \end{bmatrix} \succeq 0 \tag{15}$$

because $\Delta^2_{\hat{B}_j} = \langle\hat{B}_j^2\rangle - \langle\hat{B}_j\rangle^2 = 1$ and $\Delta^2_{\hat{A}_i} = \langle\hat{A}_i^2\rangle - \langle\hat{A}_i\rangle^2 = 1$. This, in turn, implies

$$\begin{bmatrix} \Delta^2_{\hat{B}_j} & \mathbf{C}(\hat{A}_1, \hat{B}_j) & \mathbf{C}(\hat{A}_0, \hat{B}_j) \\ \mathbf{C}(\hat{A}_1, \hat{B}_j) & \Delta^2_{\hat{A}_1} & r \\ \mathbf{C}(\hat{A}_0, \hat{B}_j) & r & \Delta^2_{\hat{A}_0} \end{bmatrix} \succeq 0 \tag{16}$$

with $r = \langle\{\hat{A}_0, \hat{A}_1\}\rangle/2$. The inequalities in Eq. 3 generalize the uncertainty relation Eq. 16 to arbitrary measurements. The inequalities Eqs. 1 and 2 further extend Eq. 16 to include the remaining measurements of Alice, Bob, and Charlie.

Proof of Theorem 1

By the Schur complement condition for positive semidefiniteness, the first condition in Eq. 3 is equivalent to $\Lambda_A \succeq \Delta_{B_j}^{-2}\mathbf{C}(A, B_j)\mathbf{C}(A, B_j)^T$. This can be normalized

$$\begin{aligned} M^{-1}\Lambda_A M^{-1} &= \begin{bmatrix} 1 & r' \\ r' & 1 \end{bmatrix} \succeq \begin{bmatrix} \varrho_{1j}^2 & \varrho_{0j}\varrho_{1j} \\ \varrho_{0j}\varrho_{1j} & \varrho_{0j}^2 \end{bmatrix} \\ &= \Delta_{B_j}^{-2}M^{-1}\begin{bmatrix} \mathbf{C}(A_1, B_j) \\ \mathbf{C}(A_0, B_1) \end{bmatrix}\begin{bmatrix} \mathbf{C}(A_1, B_j)\mathbf{C}(A_0, B_1) \end{bmatrix}M^{-1} \end{aligned} \tag{17}$$

where $r' \stackrel{\text{def}}{=} \frac{r}{\Delta_{A_1}\Delta_{A_0}}$ and M is a diagonal matrix whose (nonvanishing) entries are Δ_{A_1} and Δ_{A_0}. This condition is equivalent to

$$|r' - \varrho_{0j}\varrho_{1j}| \leq \sqrt{(1 - \varrho_{0j}^2)(1 - \varrho_{1j}^2)} \tag{18}$$

which follows from the nonnegative determinant of the matrix obtained by subtracting the right-hand side from the left-hand side in Eq. 17. This, together with the triangle inequality, yield

$$\begin{aligned} &|\varrho_{00}\varrho_{10} - r' + r' - \varrho_{01}\varrho_{11}| \leq |r' - \varrho_{00}\varrho_{10}| + \\ &|r' - \varrho_{01}\varrho_{11}| \leq \sum_{j=0,1}\sqrt{(1 - \varrho_{0j}^2)(1 - \varrho_{1j}^2)} \end{aligned} \tag{19}$$

The second inequality in Eq. 4 is similarly obtained by swapping the roles of Alice and Bob, i.e., from the second RI condition in Eq. 3.

Proof of Theorem 2

In the Hilbert space formulation of quantum mechanics, Alice's measurements are represented by the self-adjoint operators $\hat{A}_0$ and $\hat{A}_1$. Similarly, Bob's measurements are represented by the self-adjoint operators $\hat{B}_j$. The Schrödinger-Robertson uncertainty relations of $\hat{A}_0$ and $\hat{A}_1$ are

$$\Delta^2_{\hat{A}_0}\Delta^2_{\hat{A}_1} \geq \left(\frac{1}{2}\langle\{\hat{A}_0, \hat{A}_1\}\rangle - \langle\hat{A}_0\rangle\langle\hat{A}_1\rangle\right)^2 + \left(\frac{1}{2i}\langle[\hat{A}_0, \hat{A}_1]\rangle\right)^2 \tag{20}$$

where $\Delta^2_{\hat{A}_i} = \left\langle\hat{A}_i^2\right\rangle - \left\langle\hat{A}_i\right\rangle^2$ is the variance of $\hat{A}_i$. This may alternatively be written as

$$\Lambda_{\hat{A}} = \begin{bmatrix} \Delta^2_{\hat{A}_1} & r_Q \\ r^*_Q & \Delta^2_{\hat{A}_0} \end{bmatrix} \succeq 0 \tag{21}$$

where $r_Q \overset{\text{def}}{=} \langle \hat{A}_1 \hat{A}_0 \rangle - \langle \hat{A}_1 \rangle \langle \hat{A}_0 \rangle$ with r^*_Q being its complex conjugate. It can be recognized that this leads to Alice's part in the generalized uncertainty relation in Eq. 2, where $r_{jk} = (r_Q + r^*_Q)/2$ is independent of j and k.

We shall show that the RI condition, the first inequality in Eq. 3, holds in Hilbert space. This condition tells that

$$\Lambda_{\hat{A}\hat{B}} = \begin{bmatrix} \Delta^2_{\hat{B}_j} & \langle \hat{A}_1 \hat{B}_j \rangle - \langle \hat{A}_1 \rangle \langle \hat{B}_j \rangle & \langle \hat{A}_0 \hat{B}_j \rangle - \langle \hat{A}_0 \rangle \langle \hat{B}_j \rangle \\ \langle \hat{A}_1 \hat{B}_j \rangle - \langle \hat{A}_1 \rangle \langle \hat{B}_j \rangle & \Delta^2_{\hat{A}_1} & \langle \hat{A}_1 \hat{A}_0 \rangle - \langle \hat{A}_1 \rangle \langle \hat{A}_0 \rangle \\ \langle \hat{A}_0 \hat{B}_j \rangle - \langle \hat{A}_0 \rangle \langle \hat{B}_j \rangle & \langle \hat{A}_0 \hat{A}_1 \rangle - \langle \hat{A}_1 \rangle \langle \hat{A}_0 \rangle & \Delta^2_{\hat{A}_0} \end{bmatrix}, \quad j = 0, 1 \tag{22}$$

where $\Delta^2_{\hat{B}_j} = \left\langle \hat{B}^2_j \right\rangle - \left\langle \hat{B}_j \right\rangle^2$ is a positive semidefinite matrix. Let $U^* = [u_1, u_2, u_3]$ be any 3×1 complex-valued vector, and denote $|\phi\rangle$ as the underlying state. Note that

$$U^* \Lambda_{\hat{A}\hat{B}} U = V^* V \geq 0 \tag{23}$$

where

$$V \overset{\text{def}}{=} u_1 \left(\hat{B}_j - \langle \hat{B}_j \rangle \right) |\phi\rangle + u_2 \left(\hat{A}_1 - \langle \hat{A}_1 \rangle \right) |\phi\rangle + u_3 \left(\hat{A}_0 - \langle \hat{A}_0 \rangle \right) |\phi\rangle \tag{24}$$

which shows that $\Lambda_{\hat{A}\hat{B}} \succeq 0$, and therefore, Eq. 3 holds.

In what follows, we show that $\Lambda_{\hat{A}\hat{B}} \succeq 0$ implies the first bound in Eq. 5. Note that

$$M^{-1} \Lambda_{\hat{A}\hat{B}} M^{-1} = \begin{bmatrix} 1 & \varrho_{1j} & \varrho_{0j} \\ \varrho_{1j} & 1 & \dfrac{r_Q}{\Delta_{\hat{A}_1} \Delta_{\hat{A}_0}} \\ \varrho_{0j} & \dfrac{r^*_Q}{\Delta_{\hat{A}_1} \Delta_{\hat{A}_0}} & 1 \end{bmatrix} \succeq 0, \quad j = 0, 1 \tag{25}$$

where M is a diagonal matrix whose (nonvanishing) entries are $\Delta_{\hat{B}_j}$, $\Delta_{\hat{A}_1}$, and $\Delta_{\hat{A}_0}$. By the Schur complement condition for positive semidefiniteness, Eq. 25 is equivalent to

$$\begin{bmatrix} 1 - \varrho^2_{1j} & \dfrac{r_Q}{\Delta_{\hat{A}_1} \Delta_{\hat{A}_0}} - \varrho_{1j} \varrho_{0j} \\ \dfrac{r^*_Q}{\Delta_{\hat{A}_1} \Delta_{\hat{A}_0}} - \varrho_{1j} \varrho_{0j} & 1 - \varrho^2_{0j} \end{bmatrix} \succeq 0, \quad j = 0, 1 \tag{26}$$

This, in turn, is equivalent to the requirement that the determinant of this matrix is nonnegative, i.e., that

$$\left(1 - \varrho^2_{1j}\right)\left(1 - \varrho^2_{0j}\right) \geq \left(\frac{\langle \{\hat{A}_0, \hat{A}_1\} \rangle / 2 - \langle \hat{A}_0 \rangle \langle \hat{A}_1 \rangle}{\Delta_{\hat{A}_1} \Delta_{\hat{A}_0}} - \varrho_{0j} \varrho_{1j} \right)^2 + \left(\frac{1}{2i} \frac{\left\langle [\hat{A}_0, \hat{A}_1] \right\rangle}{\Delta_{\hat{A}_1} \Delta_{\hat{A}_0}} \right)^2, \quad j = 0, 1 \tag{27}$$

namely

$$\sqrt{(1 - \varrho^2_{1j})(1 - \varrho^2_{0j}) - \eta^2_{\hat{A}}} \geq \left| \frac{\langle \{\hat{A}_0, \hat{A}_1\} \rangle / 2 - \langle \hat{A}_0 \rangle \langle \hat{A}_1 \rangle}{\Delta_{\hat{A}_1} \Delta_{\hat{A}_0}} - \varrho_{0j} \varrho_{1j} \right|, \quad j = 0, 1 \tag{28}$$

where $\eta_{\hat{A}}$ is as defined in the theorem. This, together with the triangle inequality, implies the first bound in the theorem

$$|\varrho_{00} \varrho_{10} - \varrho_{01} \varrho_{11}| \leq \sum_{j=0,1} \left| \frac{\langle \{\hat{A}_0, \hat{A}_1\} \rangle / 2 - \langle \hat{A}_0 \rangle \langle \hat{A}_1 \rangle}{\Delta_{\hat{A}_1} \Delta_{\hat{A}_0}} - \varrho_{0j} \varrho_{1j} \right| \leq \sum_{j=0,1} \sqrt{(1 - \varrho^2_{1j})(1 - \varrho^2_{0j}) - \eta^2_{\hat{A}}} \tag{29}$$

The remaining bound similarly follows from the second RI condition in Eq. 3. It is was previously noted that for the case where $\hat{A}^2_i = \hat{B}^2_j = I$ and $\langle A_i \rangle = \langle B_j \rangle = 0$, the inequality Eq. 27 coincides with the Schrödinger-Robertson uncertainty relations of $\hat{A}_0 \hat{B}_j$ and $\hat{A}_1 \hat{B}_j$, the inequality Eq. 13.

Nonlocality and Heisenberg uncertainty

An interesting corollary of Theorem 2 is that there is a bound, a generalization of Tsirelson's $2\sqrt{2}$ bound, for different values of $\eta_{\hat{A}}$ and $\eta_{\hat{B}}$. In particular,

$$|\mathscr{B}| \leq 2\sqrt{2} \sqrt{1 - \max\{\eta^2_{\hat{A}}, \eta^2_{\hat{B}}\}} \tag{30}$$

A geometrical view of this bound is given in Fig. 2. Application of Eq. 30 to $\hat{A}_0 = \hat{x}$ and $\hat{A}_1 = \hat{p}$, the position and momentum operators, yields

$$|\mathscr{B}| \leq 2\sqrt{2} \sqrt{1 - \left(\frac{\hbar/2}{\Delta_{\hat{x}} \Delta_{\hat{p}}} \right)^2} \tag{31}$$

which follows from the definition of $\eta_{\hat{A}}$ and the identity $[\hat{x}, \hat{p}] = i\hbar$. This elucidates the interplay between the extent of nonlocality and the Heisenberg uncertainty principle. The greater the uncertainty $\Delta_{\hat{x}} \Delta_{\hat{p}}$, the stronger the nonlocality may get, where Tsirelson's $2\sqrt{2}$ bound corresponds to the limit $\Delta_{\hat{x}} \Delta_{\hat{p}} \to \infty$.

More generally, RI implies a close relationship between nonlocality as quantified by the Bell-CHSH parameter and the uncertainty parameter r in Eq. 3. This is summarized in the next theorem.

Theorem 3

By RI

$$\left(\frac{\mathscr{B}}{2\sqrt{2}} \right)^2 + |r'|^2 \leq 1 \tag{32}$$

where, as before, $r' \stackrel{\text{def}}{=} \frac{r}{\Delta_{A_1}\Delta_{A_0}}$. In quantum mechanics where $r = r_Q$ in Eq. 21, this relation assumes an explicit form

$$\left(\frac{\mathscr{B}}{2\sqrt{2}}\right)^2 + \left|\frac{\langle \hat{A}_0\hat{A}_1\rangle - \langle \hat{A}_0\rangle\langle \hat{A}_1\rangle}{\Delta_{\hat{A}_0}\Delta_{\hat{A}_1}}\right|^2 \le 1 \quad (33)$$

Proof. Assume that $\varrho_{ij} = (-1)^{ij}\varrho$, a configuration underlying the maximal Bell-CHSH parameter, i.e., $\mathscr{B} = 4\varrho$. RI in Eq. 3 implies Eq. 18, which, in this case, yields

$$[r' - (-1)^j \varrho^2]^2 \le (1-\varrho^2)^2 \quad (34)$$

That is

$$|r'|^2 + \left(\frac{\mathscr{B}}{2\sqrt{2}}\right)^2 - 2(-1)^j r'\varrho^2 \le 1 \quad (35)$$

where we have used the identity $\varrho = \mathscr{B}/4$. Averaging Eq. 35 for $j = 0$ and $j = 1$ implies the theorem.

Locality from RI

The preceding sections forged a theory-free notion of nonlocality in the form of correlators that satisfy RI. Can locality (as appearing in classical statistical theories), which is normally defined by means of Bell inequalities, be similarly characterized? We will show that locality is, in some sense, a variant of RI.

The first RI condition in Eq. 3 may alternatively be written as

$$\mathcal{M}^Q \stackrel{\text{def}}{=} \begin{bmatrix} M^{-1}\Lambda_A M^{-1} - \tilde{R}_0\tilde{R}_0^T & 0_{2\times 2} \\ 0_{2\times 2} & M^{-1}\Lambda_A M^{-1} - \tilde{R}_1\tilde{R}_1^T \end{bmatrix} \succeq 0 \quad (36)$$

where M is a diagonal matrix whose (nonvanishing) entries are Δ_{A_1} and Δ_{A_0}, and $\tilde{R}_j^T = [\varrho_{0j}, \varrho_{1j}]$. RI may further restrict the underlying correlators when the off-diagonal blocks do not vanish. Locality is implied, for example, by

$$\mathcal{M}^L \stackrel{\text{def}}{=} \begin{bmatrix} M^{-1}\Lambda_A M^{-1} - \tilde{R}_0\tilde{R}_0^T & \tilde{R}_0\tilde{R}_1^T \\ \tilde{R}_1\tilde{R}_0^T & M^{-1}\Lambda_A M^{-1} - \tilde{R}_1\tilde{R}_1^T \end{bmatrix} \succeq 0 \quad (37)$$

In particular

$$u\mathcal{M}^L u^T = 4 - \mathscr{B}^2 \ge 0 \quad (38)$$

where $u = [1, 1, 1, -1]$ and $\mathscr{B} \stackrel{\text{def}}{=} \varrho_{00} + \varrho_{10} + \varrho_{01} - \varrho_{11}$ is the Bell-CHSH parameter.

The nonvanishing off-diagonal matrices in Eq. 37 essentially render the underlying uncertainty relations of both experimenters ineffective. To see how, note that the matrix in Eq. 37 (but not that in Eq. 36) is the covariance of the four products A_iB_j , $i, j = 0, 1$, where A_i and B_j are Alice's and Bob's measurement outcomes. Therefore, the joint probabilities of A_0 and A_1, and of B_0 and B_1, exist, and the correlators satisfy the Bell-CHSH inequality. As mentioned in the main text, here, the parameter $r = \mathbf{C}(A_0, A_1)$, and $\Delta^2_{A_0}\Delta^2_{A_1} \ge r^2$. However, this form of the uncertainty relation cannot be saturated but for the trivial case of deterministic A_0 and A_1.

RI in general multipartite settings

Suppose that some experimenters are located at spacetime region S and some others at spacetime region T. Each experimenter has an arbitrary number of measuring devices. We shall denote the vectors of measurements in S and T by S_i and T_j, where the indices i and j represent sets of choices of measuring devices in each region. As in the bipartite case, we may write $\Lambda_S(i)$ and $\Lambda_T(j)$ for the uncertainty relations underlying the sets of measurements i in S and j in T. The covariances between S_i and T_j may similarly be expressed by a matrix R.

RI dictates that uncertainty relations in S are independent of choices in T. Therefore, S is independent of whether $j = 0$ or $j = 1$ in T. This is expressed mathematically by

$$\begin{bmatrix} \Lambda_T(0) & R_0^T \\ R_0 & \Lambda_S \end{bmatrix} \succeq 0, \begin{bmatrix} \Lambda_T(1) & R_1^T \\ R_1 & \Lambda_S \end{bmatrix} \succeq 0 \quad (39)$$

But also in the converse direction, uncertainty relations in T are independent of choices in S

$$\begin{bmatrix} \Lambda_T & \bar{R}_0^T \\ \bar{R}_0 & \Lambda_S(0) \end{bmatrix} \succeq 0, \begin{bmatrix} \Lambda_T & \bar{R}_1^T \\ \bar{R}_1 & \Lambda_S(1) \end{bmatrix} \succeq 0 \quad (40)$$

Below, we use these to derive a bound on the quantum mechanical, Alice-Bob, Alice-Charlie, and Bob-Charlie, one- and two-point correlators. The relation thus obtained generalizes Eq. 4 in this tripartite setting.

We note that Eqs. 39 and 40 do not represent the most general approach for characterizing nonlocal correlations. Nevertheless, they facilitate analyses and particularly the derivation of the theorems that follow. A complete characterization of the set of quantum correlations would require analyzing Eq. 1 in a general multipartite setting. In such a case, the cross-correlations between the S and T subsets would have to be accounted for. To some degree, this is practiced in the derivation of Theorem 4, where it is assumed that Bob and Charlie are correlated. Disconnecting them by making their correlations zero leads to the well-known monogamy relation in Theorem 5.

In the tripartite case, where Alice in S measures either A_0 or A_1, and Bob and Charlie in T measure (B_l, C_k) or $(B_{l'}, C_{k'})$, RI in Eq. 39 holds for

$$\Lambda_T(0) \stackrel{\text{def}}{=} \begin{bmatrix} \Delta^2_{C_k} & \mathbf{C}(C_k, B_l) \\ \mathbf{C}(C_k, B_l) & \Delta^2_{B_l} \end{bmatrix},$$

$$\Lambda_T(1) \stackrel{\text{def}}{=} \begin{bmatrix} \Delta^2_{C_{k'}} & \mathbf{C}(C_{k'}, B_{l'}) \\ \mathbf{C}(C_{k'}, B_{l'}) & \Delta^2_{B_{l'}} \end{bmatrix}, \Lambda_S \stackrel{\text{def}}{=} \begin{bmatrix} \Delta^2_{A_1} & r \\ r & \Delta^2_{A_0} \end{bmatrix} \quad (41)$$

where

$$R_0^T = \begin{bmatrix} \mathbf{C}(A_1, C_k) & \mathbf{C}(A_0, C_k) \\ \mathbf{C}(A_1, B_l) & \mathbf{C}(A_0, B_l) \end{bmatrix},$$

$$R_1^T = \begin{bmatrix} \mathbf{C}(A_1, C_{k'}) & \mathbf{C}(A_0, C_{k'}) \\ \mathbf{C}(A_1, B_{l'}) & \mathbf{C}(A_0, B_{l'}) \end{bmatrix} \quad (42)$$

Theorem 4

The RI condition (Eq. 39) with the matrices in Eqs. 41 and 42 implies

$$|\zeta_{01}(l,k)-\zeta_{01}(l',k')|\leq\sqrt{(1-\zeta_{11}(l,k))(1-\zeta_{00}(l,k))}+\sqrt{(1-\zeta_{11}(l',k'))(1-\zeta_{00}(l',k'))} \tag{43}$$

where

$$\zeta_{ij}(l,k)\stackrel{\text{def}}{=}[\varrho^{AC}_{ik}\varrho^{AC}_{jk}-\varrho^{BC}_{lk}\varrho^{AB}_{il}\varrho^{AC}_{jk}-\varrho^{BC}_{lk}\varrho^{AB}_{jl}\varrho^{AC}_{ik}+\varrho^{AB}_{il}\varrho^{AB}_{jl}]/(1-(\varrho^{BC}_{lk})^2) \tag{44}$$

and $\varrho^{XY}_{ij}\stackrel{\text{def}}{=}C(X_i,Y_j)/(\Delta_{X_i}\Delta_{Y_j})$. Note that letting $\varrho_{AC}=\varrho^{BC}=0$ in Eq. 43 recovers the bound on the Alice-Bob correlators, the first inequality in Eq. 4.

Proof. Substituting Eq. 41 into Eq. 39 yields

$$\Lambda_{ABC}\stackrel{\text{def}}{=}\begin{bmatrix}\Delta^2_{C_k} & \mathbf{C}(C_k,B_l) & \mathbf{C}(C_k,A_1) & \mathbf{C}(C_k,A_0)\\ \mathbf{C}(C_k,B_l) & \Delta^2_{B_l} & \mathbf{C}(B_l,A_1) & \mathbf{C}(B_l,A_0)\\ \mathbf{C}(C_k,A_1) & \mathbf{C}(B_l,A_1) & \Delta^2_{A_1} & r\\ \mathbf{C}(C_k,A_0) & \mathbf{C}(B_l,A_0) & r & \Delta^2_{A_0}\end{bmatrix}\succeq 0 \tag{45}$$

and similarly for k' and l'. This is equivalent to

$$M^{-1}\Lambda_{ABC}M^{-1}=\begin{bmatrix}1 & \varrho^{BC}_{lk} & \varrho^{AC}_{1k} & \varrho^{AC}_{0k}\\ \varrho^{BC}_{lk} & 1 & \varrho^{AB}_{1l} & \varrho^{AB}_{0l}\\ \varrho^{AC}_{1k} & \varrho^{AB}_{1l} & 1 & r'\\ \varrho^{AC}_{0k} & \varrho^{AB}_{0l} & r' & 1\end{bmatrix}\succeq 0 \tag{46}$$

where $r'\stackrel{\text{def}}{=}r/(\Delta_{A_1}\Delta_{A_0})$ and M is a diagonal matrix whose (nonvanishing) entries are Δ_{C_k}, Δ_{B_l}, Δ_{A_1}, and Δ_{A_0}. By the Schur complement condition for positive semidefiniteness, Eq. 46 is equivalent to

$$\begin{bmatrix}1 & r'\\ r' & 1\end{bmatrix}\succeq\begin{bmatrix}\varrho^{AC}_{1k} & \varrho^{AC}_{0k}\\ \varrho^{AB}_{1l} & \varrho^{AB}_{0l}\end{bmatrix}^T\begin{bmatrix}1 & \varrho^{BC}_{lk}\\ \varrho^{BC}_{lk} & 1\end{bmatrix}^{-1}\begin{bmatrix}\varrho^{AC}_{1k} & \varrho^{AC}_{0k}\\ \varrho^{AB}_{1l} & \varrho^{AB}_{0l}\end{bmatrix} \tag{47}$$

which holds if and only if the determinant of the matrix obtained by subtracting the right-hand side from the left-hand side in Eq. 47 is nonnegative. Carrying out this calculation for k,l and then for k',l' and invoking the triangle inequality yield Eq. 43.

The next theorem shows that the bound Eq. 43 implies monogamy of correlations. This means that breaking of monogamy necessarily violates RI.

Theorem 5

If Charlie and Bob are uncorrelated, $\mathbf{C}(C_k, B_j) = 0$, then by RI

$$\mathscr{B}^2_{AB}+\mathscr{B}^2_{AC}\leq 8 \tag{48}$$

and therefore also $|\mathscr{B}_{AB}|+|\mathscr{B}_{AC}|\leq 4$, where both Bell-CHSH parameters, $\mathscr{B}_{AB}$ and $\mathscr{B}_{AC}$, are for the same pair, A_0, A_1.

Proof. Substituting $\varrho^{BC}_{jk}=0$ in Eq. 47 implies

$$2(1\pm r')=u^T\begin{bmatrix}1 & r'\\ r' & 1\end{bmatrix}u\geq u^T\begin{bmatrix}\varrho^{AC}_{1k} & \varrho^{AB}_{1j}\\ \varrho^{AC}_{0k} & \varrho^{AB}_{0j}\end{bmatrix}\begin{bmatrix}\varrho^{AC}_{1k} & \varrho^{AB}_{1j}\\ \varrho^{AC}_{0k} & \varrho^{AB}_{0j}\end{bmatrix}^T$$

$$u=[\varrho^{AB}_{0j}\ \pm\varrho^{AB}_{1j}]^2+[\varrho^{AC}_{0k}\pm\varrho^{AC}_{1k}]^2 \tag{49}$$

for $u^T = [1, \pm 1]$. Therefore

$$4\geq\left[\varrho^{AB}_{00}\pm\varrho^{AB}_{10}\right]^2+\left[\varrho^{AC}_{00}\pm\varrho^{AC}_{10}\right]^2+\left[\varrho^{AB}_{01}\pm\varrho^{AB}_{11}\right]^2+\left[\varrho^{AC}_{01}\pm\varrho^{AC}_{11}\right]^2\geq\frac{1}{2}\mathscr{B}^2_{AB}+\frac{1}{2}\mathscr{B}^2_{AC} \tag{50}$$

from which the theorem follows.

Monogamy of correlations in general multipartite settings

The above result is a special case of the more general scenario where any number of experimenters is correlated with Alice but uncorrelated among themselves. Suppose that there are n experimenters whose measurements are uncorrelated, $\mathbf{C}(M^k_i,M^l_j)=0$, where M^k_i stands in for the kth physical variable measured by the ith experimenter. In this case, the generalized uncertainty relations underlying Alice measurements A_0, A_1 and the n other measurements $M^{i_1}_1,\ldots,M^{i_n}_n$ are described by

$$\left[\begin{array}{c|cc} I_{n\times n} & \begin{array}{c}\varrho^1_{0,i_1}\\ \vdots\\ \varrho^n_{0,i_n}\end{array} & \begin{array}{c}\varrho^1_{1,i_1}\\ \vdots\\ \varrho^n_{1,i_n}\end{array}\\ \hline & 1 & r'_{i_1,\ldots,i_n}\\ & & 1\end{array}\right]\succeq 0 \tag{51}$$

where $\varrho^s_{i,k}\stackrel{\text{def}}{=}C(A_i,M^k_s)/(\Delta_{A_i}\Delta_{M^k_s})$. This matrix is obtained as an extension of Eq. 2 following a normalization similar to the one in previous sections. In this case, Alice's uncertainty relations are governed by the parameter $r'_{i_1,\ldots,i_n}$, which may depend on the choices of all of the other experimenters.

Theorem 6

RI implies

$$\sum_{s=1}^{n}|\mathscr{B}_s|\leq\sqrt{2n}(\sqrt{1+r'}+\sqrt{1-r'})\leq 2\sqrt{2n} \tag{52}$$

where $\mathscr{B}_s\stackrel{\text{def}}{=}\varrho^s_{0,i_s}+\varrho^s_{1,i_s}+\varrho^s_{0,j_s}-\varrho^s_{1,j_s}$ is the Bell-CHSH parameter of Alice and the sth experimenter. Tsirelson's bound and the monogamy property of correlations follow from this inequality as special cases for $n = 1$ and $n = 2$, respectively.

Proof. If RI holds, then $r'_{i_1,\ldots,i_n}=r'_{j_1,\ldots,j_n}=r'$. By the Schur complement condition for positive semidefiniteness, Eq. 51 is equivalent to

$$\begin{bmatrix}1 & r'\\ r' & 1\end{bmatrix}\succeq\sum_{s=1}^{n}\begin{bmatrix}\varrho^s_{0,i_s}\\ \varrho^s_{1,i_s}\end{bmatrix}[\varrho^s_{0,i_s}\ \varrho^s_{1,i_s}] \tag{53}$$

and similarly

$$\begin{bmatrix} 1 & r' \\ r' & 1 \end{bmatrix} \succeq \sum_{s=1}^{n} \begin{bmatrix} \varrho^s_{0,j_s} \\ \varrho^s_{1,j_s} \end{bmatrix} [\varrho^s_{0,j_s} \varrho^s_{1,j_s}] \tag{54}$$

Both Eqs. 53 and 54 imply

$$2(1 \pm r') \geq \sum_{s=1}^{n} (\varrho^s_{0,i_s} \pm \varrho^s_{1,i_s})^2, 2(1 \pm r') \geq \sum_{s=1}^{n} (\varrho^s_{0,j_s} \pm \varrho^s_{1,j_s})^2 \tag{55}$$

which are obtained similarly to Eq. 49. By norm equivalence

$$2n(1 \pm r') \geq \left(\sum_{s=1}^{n} |\varrho^s_{0,i_s} \pm \varrho^s_{1,i_s}|\right)^2, 2n(1 \pm r') \geq \left(\sum_{s=1}^{n} |\varrho^s_{0,j_s} \pm \varrho^s_{1,j_s}|\right)^2 \tag{56}$$

Last, invoking the triangle inequality

$$\sum_{s=1}^{n} |\mathscr{B}_s| \leq \sum_{s=1}^{n} \left|\varrho^s_{0,i_s} + \varrho^s_{1,i_s}\right| + \left|\varrho^s_{0,j_s} - \varrho^s_{1,j_s}\right| \leq \sqrt{2n(1+r')} + \sqrt{2n(1-r')} \leq 2\sqrt{2n} \tag{57}$$

Tighter than Schrödinger-Robertson uncertainty relations following from Eq. 3

Alice's uncertainty relations are represented by the 2×2 lower submatrix Λ_A in the generalized uncertainty relation Eq. 3. This shows that Eq. 3 is more stringent than any uncertainty relation derived exclusively from $\Lambda_A \succeq 0$. Consider, for example, a generalized uncertainty relation of the form

$$\begin{bmatrix} \Lambda_D & C \\ C^T & \Lambda_A \end{bmatrix} \succeq 0 \tag{58}$$

where D is an invertible $n \times n$ matrix and C is $n \times 2$ cross-covariance matrix. By the Schur complement condition for positive semidefiniteness, this inequality is equivalent to $\Lambda_A \succeq C^T \Lambda_D^{-1} C$, which, unless C vanishes, is tighter than $\Lambda_A \succeq 0$.

As shown in the preceding sections, from within quantum mechanics, the inequality $\Lambda_A \succeq 0$, which follows from the lower 2×2 submatrix in Eqs. 2 and 3, is equivalent to the Schrödinger-Robertson uncertainty relations underlying Alice's observables $\hat{A}_0$ and $\hat{A}_1$. That quantum mechanics obey generalized uncertainty relations like Eq. 3, and more generally Eq. 58, implies that any uncertainty relation derived from $\Lambda_A \succeq 0$ makes only a small part of the story. There are many more restrictions arising from our approach, all of which are tighter than the Schrödinger-Robertson uncertainty relation that are obeyed by Alice's observables. One such uncertainty relation is given below.

Let $D = \hat{A}_i^m$, where $\hat{A}_i$ is one of Alice's observables, i = 0, 1, and m is an integer, $m > 1$. From within quantum mechanics, the generalized uncertainty Eq. 58 is now given by

$$\begin{bmatrix} \Delta^2_{\hat{A}_i^m} & \mathbf{C}(\hat{A}_i^m, \hat{A}_1) & \mathbf{C}(\hat{A}_i^m, \hat{A}_0) \\ \mathbf{C}(\hat{A}_1, \hat{A}_i^m) & \Delta^2_{\hat{A}_1} & \mathbf{C}(\hat{A}_1, \hat{A}_0) \\ \mathbf{C}(\hat{A}_0, \hat{A}_i^m) & \mathbf{C}(\hat{A}_0, \hat{A}_1) & \Delta^2_{\hat{A}_0} \end{bmatrix} \succeq 0 \tag{59}$$

where $\mathbf{C}(\hat{A}_i, \hat{A}_j) \stackrel{\text{def}}{=} \langle \hat{A}_i \hat{A}_j \rangle - \langle \hat{A}_i \rangle \langle \hat{A}_j \rangle$. The quantities $\Delta^2_{\hat{A}_i^m}$ and $\mathbf{C}(\hat{A}_i^m, \hat{A}_1)$ in Eq. 59 involve higher statistical moments of the underlying observables. The inequality Eq. 59 is equivalent to

$$\Lambda_A = \begin{bmatrix} \Delta^2_{\hat{A}_1} & \mathbf{C}(\hat{A}_1, \hat{A}_0) \\ \mathbf{C}(\hat{A}_0, \hat{A}_1) & \Delta^2_{\hat{A}_0} \end{bmatrix} \succeq \Delta^{-2}_{\hat{A}_i^m} \begin{bmatrix} \mathbf{C}(\hat{A}_1, \hat{A}_i^m) \\ \mathbf{C}(\hat{A}_0, \hat{A}_i^m) \end{bmatrix} [\mathbf{C}(\hat{A}_i^m, \hat{A}_1) \mathbf{C}(\hat{A}_i^m, \hat{A}_0)] \tag{60}$$

by the Schur complement condition for positive semidefiniteness. Let $v^T \stackrel{\text{def}}{=} [1, \pm 1]/\sqrt{2}$ and note that

$$v^T \Lambda_A v = \frac{1}{2}\Delta^2_{\hat{A}_1} + \frac{1}{2}\Delta^2_{\hat{A}_0} \pm \left[\frac{1}{2}\langle\{\hat{A}_1, \hat{A}_0\}\rangle - \langle \hat{A}_1 \rangle \langle \hat{A}_0 \rangle\right] \geq \frac{1}{2\Delta^2_{\hat{A}_i^m}} |\mathbf{C}(\hat{A}_1, \hat{A}_i^m) \pm \mathrm{C}(\hat{A}_0, \hat{A}_i^m)|^2 \tag{61}$$

Therefore

$$\Delta^2_{\hat{A}_1} + \Delta^2_{\hat{A}_0} \geq 2\left|\frac{1}{2}\langle\{\hat{A}_1, \hat{A}_0\}\rangle - \langle \hat{A}_1 \rangle \langle \hat{A}_0 \rangle\right| + \frac{1}{\Delta^2_{\hat{A}_i^m}} |\mathbf{C}(\hat{A}_1, \hat{A}_i^m) \pm \mathbf{C}(\hat{A}_0, \hat{A}_i^m)|^2 \tag{62}$$

This uncertainty relation is to be contrasted with

$$\Delta^2_{\hat{A}_1} + \Delta^2_{\hat{A}_0} \geq 2\left|\frac{1}{2}\langle\{\hat{A}_1, \hat{A}_0\}\rangle - \langle \hat{A}_1 \rangle \langle \hat{A}_0 \rangle\right| \tag{63}$$

which follows from $\Lambda_A \succeq 0$ using similar arguments. Note also that much like the Maccone-Pati uncertainty relations (*20*), these additive inequalities do not become trivial in the case where the state coincides with an eigenvector of one of the observables.

The measurability of r_j in a bipartite setting

In what follows, we examine RI from a different perspective. As mentioned in the main text, this condition may be viewed as the requirement that one experimenter's uncertainty relations are independent of another experimenters' choices. We claim that if it were not so, relativistic causality would have been necessarily violated. Our argument is based on the measurability of r_j in Alice's Λ_A^j.

Lemma 1. There exists an r_{jk} that is independent of j and k such that Eq. 2 holds with $\mathbf{C}(C_k, B_j)$ = 0 if and only if the four intervals $[d_{jk}(-), d_{jk}(+)]$, $j, k \in \{0,1\}$, with the $d_{jk}(-)$ and $d_{jk}(+)$ given below, all intersect.

$$d_{jk}(\pm) \stackrel{\text{def}}{=} \varrho^{AB}_{0j} \varrho^{AB}_{1j} + \varrho^{AC}_{0k} \varrho^{AC}_{1k} \pm \sqrt{[1 - (\varrho^{AB}_{0j})^2 - (\varrho^{AC}_{0k})^2][1 - (\varrho^{AB}_{1j})^2 - (\varrho^{AC}_{1k})^2]} \tag{64}$$

Proof. The inequality Eq. 2 may be written as

$$M^{-1}\Lambda^{jk}_{ABC}M^{-1} = \begin{bmatrix} 1 & \varrho^{BC}_{jk} & \varrho^{AC}_{1k} & \varrho^{AC}_{0k} \\ \varrho^{BC}_{jk} & 1 & \varrho^{AB}_{1j} & \varrho^{AB}_{0j} \\ \varrho^{AC}_{1k} & \varrho^{AB}_{1j} & 1 & r'_{jk} \\ \varrho^{AC}_{0k} & \varrho^{AB}_{0j} & r'_{jk} & 1 \end{bmatrix} \succeq 0 \quad (65)$$

where $r'_{jk} \stackrel{\text{def}}{=} r_{jk}/(\Delta_{A_0}\Delta_{A_1})$ and M is a diagonal matrix whose nonvanishing entries are Δ_{C_k}, Δ_{B_j}, Δ_{A_1}, and Δ_{A_0}. As $\varrho^{BC}_{0j} = \mathbf{C}(C_k, B_j)/(\Delta_{B_j}\Delta_{C_k}) = 0$, the Schur complement condition for positive semidefiniteness implies that Eq. 65 is equivalent to

$$\begin{bmatrix} 1 & r'_{jk} \\ r'_{jk} & 1 \end{bmatrix} - \begin{bmatrix} \varrho^{AC}_{1k} & \varrho^{AB}_{1j} \\ \varrho^{AC}_{0k} & \varrho^{AB}_{0j} \end{bmatrix} \begin{bmatrix} \varrho^{AC}_{1k} & \varrho^{AB}_{1j} \\ \varrho^{AC}_{0k} & \varrho^{AB}_{0j} \end{bmatrix}^T \succeq 0 \quad (66)$$

which holds if and only if the diagonal entries obey, $1 - (\varrho^{AB}_{1j})^2 - (\varrho^{AC}_{1k})^2 \geq 0, i = 0, 1$, and the determinant of this matrix satisfies

$$\left[1 - (\varrho^{AB}_{0j})^2 - (\varrho^{AC}_{0k})^2\right]\left[1 - (\varrho^{AB}_{1j})^2 - (\varrho^{AC}_{1k})^2\right] - (r'_{jk} - \varrho^{AB}_{0j}\varrho^{AB}_{1j} - \varrho^{AC}_{0k}\varrho^{AC}_{1k})^2 \geq 0 \quad (67)$$

Namely, Eq. 66 holds if and only if

$$\left|r'_{jk} - \varrho^{AB}_{0j}\varrho^{AB}_{1j} - \varrho^{AC}_{0k}\varrho^{AC}_{1k}\right| \leq \sqrt{\left[1 - (\varrho^{AB}_{0j})^2 - (\varrho^{AC}_{0k})^2\right]\left[1 - (\varrho^{AB}_{1j})^2 - (\varrho^{AC}_{1k})^2\right]} \quad (68)$$

for $j, k \in \{0, 1\}$. It thus follows that $r'_{jk} \in |d_{jk}(-), d_{jk}(+)|$. If these intervals all intersect, then there is r and $r' \stackrel{\text{def}}{=} r/\Delta_{A_0}\Delta_{A_1}$, which are independent of j, k such that $r'_{jk} = r'$. In particular

$$\max_{j,k} d_{jk}(-) \leq r' \leq \min_{j,k} d_{jk}(+) \quad (69)$$

Conversely, if there is such $r'_{jk} = r'$, then the underlying intervals necessarily intersect.

Lemma 1 shows that in the absence of Charlie, $\varrho^{AC}_{1k} = \varrho^{BC}_{jk} = 0$, the parameter r_j in a bipartite Alice-Bob setting satisfies

$$\varrho_{0j}\varrho_{1j} - \sqrt{(1 - \varrho^2_{0j})(1 - \varrho^2_{1j})} \leq r'_j \leq \varrho_{0j}\varrho_{1j} + \sqrt{(1 - \varrho^2_{0j})(1 - \varrho^2_{1j})} \quad (70)$$

where $\varrho_{1j} = \mathbf{C}(A_i, B_j)/(\Delta_{A_i}\Delta_{B_j})$ and $r'_j \stackrel{\text{def}}{=} r_j/(\Delta_{A_0}\Delta_{A_1})$.

Let D_j be the range of admissible r_j in Eq. 70. Unless $D_0 \cap D_1 \neq \emptyset$, RI cannot be satisfied. We shall show that whenever the two intervals D_0 and D_1 do not intersect, in which case RI fails, signaling takes place. Define

$$\epsilon \stackrel{\text{def}}{=} \min_{w_j \in D_j} |w_0 - w_1| \quad (71)$$

It can be recognized that this ϵ is the smallest of the four possible numbers

$$\epsilon = \left|\varrho_{00}\varrho_{10} - \varrho_{01}\varrho_{11} \pm \sqrt{(1 - \varrho^2_{00})(1 - \varrho^2_{10})} \pm \sqrt{(1 - \varrho^2_{01})(1 - \varrho^2_{11})}\right| \quad (72)$$

Assume now that the intervals D_0 and D_1 do not intersect and thus $\epsilon > 0$. Here is a procedure that Alice may, in principle, follow for detecting a signal from Bob using her local measurements. Let τ be a set of local parameters describing Alice's nontrivial system (for practical reasons, τ can be discretized). The precision is represented for any physical variable A by the variance $\Delta^2_A(\tau)$. This $\Delta^2_A(\tau)$ can be evaluated empirically by measuring A in many trials of an experiment while reproducing time and again the same set τ.

For any real parameter $\theta \in [-\pi, \pi]$, Alice is able to evaluate

$$g(\theta, \tau) \stackrel{\text{def}}{=} \cos(\theta)^2 \frac{\Delta_{A_0}(\tau)}{\Delta_{A_1}(\tau)} + \sin(\theta)^2 \frac{\Delta_{A_1}(\tau)}{\Delta_{A_0}(\tau)} \quad (73)$$

Her uncertainty relation Eq. 1 dictates that this quantity is bounded from below

$$\min_{\tau} g(\theta, \tau) \geq \max\{0, r'_j \sin(2\theta)\} \quad (74)$$

which follows from $[\cos\theta, -\sin\theta]\Lambda^j_A[\cos\theta, -\sin\theta]^T \geq 0$. That Alice may reach r'_j means that for some θ, a subset of parameters τ^* saturating Eq. 74 exists

$$\min_{\tau} g(\theta, \tau) = (\theta, \tau^*) = r'_j \sin(2\theta) \quad (75)$$

which also implies that Λ^j_A is a singular matrix and therefore $\Lambda^2_{A_0}(\tau^*)\Lambda^2_{A_1}(\tau^*) = r^2_j$.

Suppose that Alice and Bob agree in advance to repeat the underlying experiment N times, for a sufficiently large N. Alice may choose a new set τ and a device with which to measure in the beginning of each trail. All this time, Bob uses only one of his devices, say the jth one. Using the measurement outcomes from all these trails, Alice may approximate $g(\theta, \tau)$ for each τ in the domain of these parameters. According to Eq. 75, Alice may then evaluate $\tilde{r}_j'$, an estimate of r'_j, using the approximated minimum of $g(\theta, \tau)$. In practice, her estimate is accurate up to an error term, δ_j, of the order $\mathcal{O}(1/\sqrt{N})$, i.e., $\tilde{r}'_j = r'_j + \delta_j$. It now follows that for sufficiently large N

$$|\tilde{r}'_0 - \tilde{r}'_1| = |r'_0 - r'_1 + \delta_0 - \delta_1| \geq |\epsilon + \mathcal{O}(1/\sqrt{N})| \quad (76)$$

Alice may therefore be able to evaluate a number whose magnitude is as large as ϵ and whose sign tells whether Bob measured first using $j = 0$ and then using $j = 1$ or the opposite. Of course, if independence holds, in which case $\epsilon = 0$, Alice will not detect any signal from Bob via her local uncertainty relations.

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Acknowledgments: We are grateful to N. Gisin and A. Elitzur for helpful discussions and comments that improved the overall presentation of the idea. We especially wish to express our gratitude to Y. Aharonov for many insightful discussions. In addition, we wish to thank anonymous reviewers for very helpful comments and suggestions. **Funding:** A.C. acknowledges support from Israel Science Foundation grant 1723/16. E.C. acknowledges support from the Engineering Faculty in Bar Ilan University. **Author contributions:** Both authors developed the concepts and worked out the mathematical proofs. **Competing interests:** The authors declare that they have no competing interests. **Data and materials availability:** All data needed to evaluate the conclusions in the paper are present in the paper. Additional data related to this paper may be requested from the authors.

Submitted 24 October 2018
Accepted 19 February 2019
Published 12 April 2019
10.1126/sciadv.aav8370

Citation: A. Carmi, E. Cohen, Relativistic independence bounds nonlocality. *Sci. Adv.* **5**, eaav8370 (2019).