THE GENERALIZED PLANCK AREA

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Abstract

Extrapolating the domains of applicability of quantum mechanics and general relativity towards extreme curvatures at the trans-Planckian regime gives rise to a curious realm. In particular, intriguing dynamics is explored beyond the Planck area, which is conventionally derived from the fundamental limits given by the Schwarzschild radius and the Heisenberg uncertainty relation for a single quantum system. We generalize this approach to quantum networks containing several such quantum systems. We show that a fundamental limit on area, our generalized Planck area, is not fixed and depends on the strength of quantum correlations. Thus, somewhat metaphorically, the "pixel size" of reality is governed by quantum correlations, and more broadly, quantum information.

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1. Rethinking uncertainty relations and Planck area

In quantum mechanics one can write down the correlation between measurements. There is a whole field devoted to the consequences and peculiarities of such uniquely quantum correlations [1, 2, 3]. Also in quantum field theory, one is highly interested in certain correlation functions. But the origin of the concept of correlation is statistics. So let us start from there.

The second moment matrix of two random variables X and Y (tightly connected to the covariance or correlation matrix) may be written as (hereinafter we assume for simplicity of notation that the expectation values are 0),

$$\begin{bmatrix} \langle X^2\rangle & \langle XY\rangle \\ \langle YX\rangle & \langle Y^2\rangle \end{bmatrix}, \tag{1}$$

and is positive semi-definite by construction. It turns out to be applicable and highly insightful also in the case of quantum mechanics [4, 5, 6]. Indeed, the second moment matrix of two Hermitian operators, e.g. x and p, may be written as,

$$\begin{bmatrix} \langle x^2\rangle & \langle xp\rangle \\ \langle xp\rangle^\dagger & \langle p^2\rangle \end{bmatrix}. \tag{2}$$

It is also positive semi-definite and so its determinant satisfies:

$$\langle x^2\rangle\langle p^2\rangle - \langle xp\rangle\langle xp\rangle^\dagger \geq 0 \Rightarrow \langle x^2\rangle\langle p^2\rangle \geq |\langle xp\rangle|^2 \geq \frac{|\langle [x,p]\rangle|^2}{4} = \frac{\hbar^2}{4}, \tag{3}$$

wherein $[x,p] = xp - (xp)^\dagger$, and the last inequality is the Cauchy–Schwarz inequality. Overall, we have derived the Heisenberg uncertainty principle:

$$\langle x^2\rangle\langle p^2\rangle \geq \frac{\hbar^2}{4}. \tag{4}$$

At the heart of the above derivation lied the familiar commutation relation of x and p and a specific Lie algebra structure. Once these are relaxed, and deformed commutators are used within deformed algebras [7, 8, 9], one can obtain generalized uncertainty relations of the form:

$$\Delta x \Delta p \geq \frac{\hbar}{2}[1 + \beta(\Delta p)^2 + \gamma], \tag{5}$$

where β and γ are positive. Such uncertainty relations were derived from the calculation of certain scattering amplitudes in string theory at Planckian energies [10, 11, 12, 13] and from various gedanken experiments in the context of black holes and Hawking radiation [14, 15]. When inserting the suitable constants, we obtain from this inequality a gravity-limited resolution of the form of Schwarzschild radius,

$$\langle x^2 \rangle \geq \left(\frac{2G}{c^3}\right)^2 \langle p^2 \rangle \Rightarrow \langle x^2 \rangle^2 \geq \left(\frac{2G}{c^3}\right)^2 \langle p^2 \rangle \langle x^2 \rangle. \tag{6}$$

Using the Heisenberg uncertainty principle of the variances (assuming zero mean variables): $\langle x^2 \rangle \langle p^2 \rangle \geq \frac{|\langle [x,p] \rangle|^2}{4} = \frac{\hbar^2}{4}$:

$$\langle x^2 \rangle^2 \geq \left(\frac{2G}{c^3}\right)^2 \langle p^2 \rangle \langle x^2 \rangle \geq \left(\frac{G|\langle [x,p] \rangle|}{c^3}\right)^2. \tag{7}$$

$$\langle x^2 \rangle \geq \left(\frac{2G}{c^3}\right) \langle p^2 \rangle \ \geq \left(\frac{G|\langle [x,p] \rangle|}{c^3}\right)^2. \tag{8}$$

Overall, we have:

$$\langle x^2 \rangle \geq \frac{G|\langle [x,p] \rangle|}{c^3} = \frac{G\hbar}{c^3} =: \ell_p^2 = \mathcal{A}_p, \tag{9}$$

wherein ℓ_p is the Planck length, and $\mathcal{A}_p$ is the Planck area.

The same line of thought may be somewhat stretched to account for a number of observations of potentially different particles. Consider the position observables, $x_1, \ldots, x_n$, alongside the momentum of one of the particles, p. Omitting again the mean values (they are not of any use to us at the moment), the second moment matrix now reads,

$$\begin{bmatrix} M_x & C_{xp} \\ C_{xp}^{\dagger} & \langle \overline{p}^2 \rangle \end{bmatrix} \succeq 0, \tag{10}$$

where M_x and C_{xp} are, respectively, the moment matrix of the positions and the cross-correlation vector of the positions and momentum:

$$M_x^{(ij)} = \langle x_i x_j \rangle \ , \ C_{xp}^{(i)} = \langle x_i \overline{p} \rangle, \tag{11}$$

where x_i is the position of the ith particle. From Schur complement it follows that:

$$C_{xp}^{\dagger} M_x^{-1} C_{xp} \leq \langle \overline{p}^2 \rangle. \tag{12}$$

The inequalitie (12) may be seen as a generalization of the Heisenberg uncertainty relation. Indeed, by taking just one particle, would yield equation (4). Once this is realized, one may use (12) directly in the derivation of the Planck area. As we shall see the consequences are surprising.

Before we proceed there is one important point to note; The second moment matrix M_x accounts for the positions of many particles. The fundamental area, however, is derived for a single particle, and so one is confronted with either choosing just one of the particles, or, what seems like a more reasonable solution, that of defining a *characteristic position*, say, $x_\star$ (in the same manner, we could have considered in our analysis several momenta using a characteristic momentum). Such a position may be readily defined using the LHS in (12), as,

$$\langle x_\star^2 \rangle^{-1} = \frac{C_{xp}^{\dagger} M_x^{-1} C_{xp}}{\| C_{xp} \|_2^2}. \tag{13}$$

Note that for a single particle, $\langle x_\star^2\rangle = \langle x^2\rangle$. Furthermore, by its very definition,

$$\min_i \langle x_i^2\rangle \le \langle x_\star^2\rangle \le \max_i \langle x_i^2\rangle.$$

Using the above definition of a characteristic position, (12) reads,

$$\langle \overline{p}^2\rangle\langle \overline{x}^2\rangle \ge \|C_{xp}\|_2^2 \tag{14}$$

Reiterating the derivation of the Planck area, while assuming that every particle is localized to within an uncertainty that is greater than the Schwarzschild radius, now yields,

$$\langle x_\star^2\rangle^2 \ge \min_i \langle x_i^2\rangle^2 \ge \left(\frac{2G}{c^3}\right)^2 \langle p^2\rangle\langle x_\star^2\rangle \ge \left(\frac{2G}{c^3}\right)^2 \|C_{xp}\|_2^2 \tag{15}$$

The fundamental bound, the generalized Planck area, is thus given by,

$$\ell_g^2 := \mathcal{A}_g := \frac{2G}{c^3}\|C_{xp}\|_2 = \frac{2G}{c^3}\sqrt{C_{xp}^\dagger C_{xp}}. \tag{16}$$

It is the same as our derivation of the original Planck area, but just replacing position with the characteristic position: The expression of $|\langle [x,p]\rangle|$ in (9) is now replaced with $\sqrt{C_{xp}^\dagger C_{xp}}$, thus yielding a bound on the position:

$$\langle \overline{x}^2\rangle \ge \frac{2G}{c^3}\frac{\|C_{xp}\|_2}{n} =: \mathcal{A}_g =: \ell_g^2. \tag{17}$$

See Figure 1 for further detail.

Do note that an exhaustive (and more symmetric) discussion would require taking every p operator:

$$\begin{bmatrix} M_x & C_{xp} \\ C_{xp}^\dagger & M_p \end{bmatrix} \succeq 0, \tag{18}$$

wherein:

$$M_p^{(ij)} = \langle p_i p_j\rangle,\ C_{xp}^{(ij)} = \langle x_i p_j\rangle, \tag{19}$$

later also involving the characteristic momentum $p_\star$, analogous to $x_\star$.

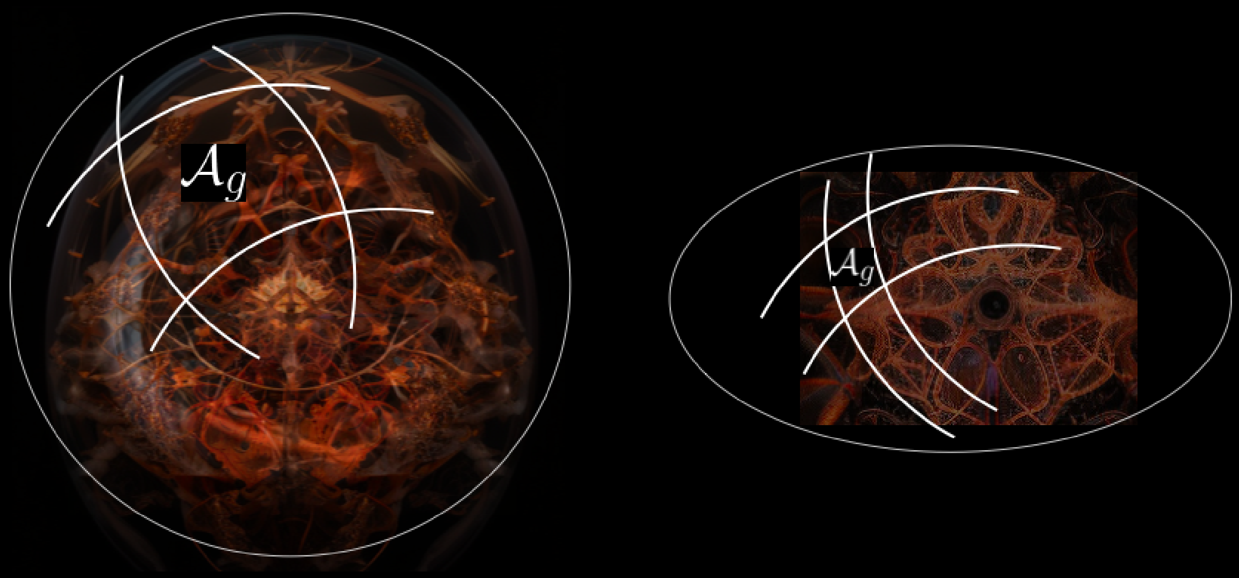

FIGURE 1. Illustration of the generalized Planck area. A main argument in this essay suggests that the fundamental gravitational area, $\mathcal{A}_g$, the one below which geometrical spacetime "gives way", depends on the correlations within a quantum network (the brownish geometrical figures on both sides). Contrary to conventional thinking, the fundamental area limit is not fixed and may be greater or smaller than the currently acceptable Planck area. The more correlated the quantum degrees of freedom are, the greater may $\mathcal{A}_g$ become.

2. Information inflation / deflation

The generalized Planck area thus derived is not fixed. It depends on the correlations between the particles, e.g. between p of one of the particles and the positions of all others in the above analysis. It may be shown that such a quantity is related to the information about p contained in the position variables. Here, information is taken in the classical sense defined by R. A. Fisher just to provide some evidence for this argument to hold, but a more careful treatment should involve the quantum Fisher information. Moreover, we shall illustrate the said relation only in the case of jointly Gaussian positions.

The Fisher information of x about p is given by,

$$I_{x|p} = -E\left[\partial_p^2 \log f(\vec{x}|p)\right], \tag{20}$$

where ∂_p^2 is the second derivative with respect to p, and $f(\vec{x}|p)$ is the conditional density of the position vector given p. Being Gaussian, the conditional density is specified by the first two statistical moments. Assume, without loss of generality, that $\langle\vec{x}\rangle = 0$ and $\langle p\rangle = 0$. The mean and covariance of the conditional density are:

$$\mu = E[\vec{x}|p] = C_{xp}\frac{p}{\langle p^2\rangle}, \quad Q = E[(\vec{x}-\mu)(\vec{x}-\mu)^T|p] = M_x - \frac{C_{xp}C_{xp}^\dagger}{\langle p^2\rangle}. \tag{21}$$

Evaluating (20) with the Gaussian density, $f(\vec{x}|p) \propto \exp\left[-\frac{1}{2}(\vec{x}-\mu)^T Q^{-1}(\vec{x}-\mu)\right]$, yields,

$$I_{x|p} = \frac{C_{xp}^\dagger Q^{-1} C_{xp}}{\langle p^2\rangle^2}. \tag{22}$$

After some more calculations can eventually find the relation:

$$I_{x|p} \geq (C_{xp}^\dagger M_x^{-1} C_{xp})^{-1} = \frac{\langle x_\star^2\rangle}{\|C_{xp}\|_2^2}. \tag{23}$$

which can be used for setting the upper and lower bounds on $\langle x_\star^2\rangle$

$$\mathcal{A}_g \leq \langle x_\star^2\rangle \leq \min\left\{ I_{x|p}\,\|C_{xp}\|_2^2,\ \mathcal{A}_g\sqrt{1+\frac{1}{I_{x|p}-\alpha^{-1}}}\right\} \tag{24}$$

The meaning of (24) is this; The generalized Planck area indicates the amount of information confined within the system (spacetime patch). Once the information $I_{x|p}$ increases, the generalized Planck area $\mathcal{A}_g$ approaches the variance of the characteristic position, $\mathcal{A}_g \to \langle x_\star^2\rangle$. If, on the other hand, the information drops towards $1/\alpha$, then $\mathcal{A}_g$ is of the order of the Fisher information $I_{x|p}$.

2.1. **Discussion.** Let us remind ourselves of the holographic principle [16, 17, 18, 19], which, loosely speaking, implies that a 3D system can be entirely described by a 2D casing. The area of the casing is one Planck area $\mathcal{A}_p$,which may be thought of as a "pixel", per every bit of entropy in the system. Thus, more information would require greater area. The interpretation presented by Equation (16) is somewhat different: it states that the area still grows with an increase of information in a system, but the size of the pixel changes, not the amount of pixels. In an allegory, consider a balloon with a net drawn on it. As we blow air into the balloon (increase the amount of information in the system), the net holes would grow bigger (the size of the pixels increase). Thus, consider a system with N degrees of freedom, the ratio between the two holographic representations should be 1 for a consistent physical description:

$$\mathcal{A}_g = \frac{2G}{c^3}\sqrt{C_{xp}^{\dagger}C_{xp}} = N \cdot \mathcal{A}_p \Rightarrow \sqrt{C_{xp}^{\dagger}C_{xp}} = \frac{N\hbar}{2}, \tag{25}$$

which means that more correlations required more degrees of freedom, exactly as expected.

2.2. **Macroscopic objects and quantum measurement.** A macroscopic object may be thought of as a quantum network having potentially many correlated particles. Consider now a pair of quantum networks (see Figure 2) wherein the positions of some of the particles in one network are correlated with some in the other. The mathematical structure of quantum mechanics dictates that the more correlated the particles are within one of the networks the less they may correlate with their peers in the other network [5]. This observation together with the notion of a generalized Planck area, in particular the term $\| C_{xp} \|_2$, suggest that networks of greater gravitational areas $\mathcal{A}_g$ tend to isolate themselves; large gravitational areas de-correlate. On the other hand, strong cross-correlations between networks require the contraction of the individual networks' $\mathcal{A}_g$.

The above behavior brings to mind the notions of decoherence and apparent collapse in the process of quantum measurement. It may explain why objects of relatively large $\mathcal{A}_g$ cannot possess strong nonlocal correlations such as those characterizing quantum objects.

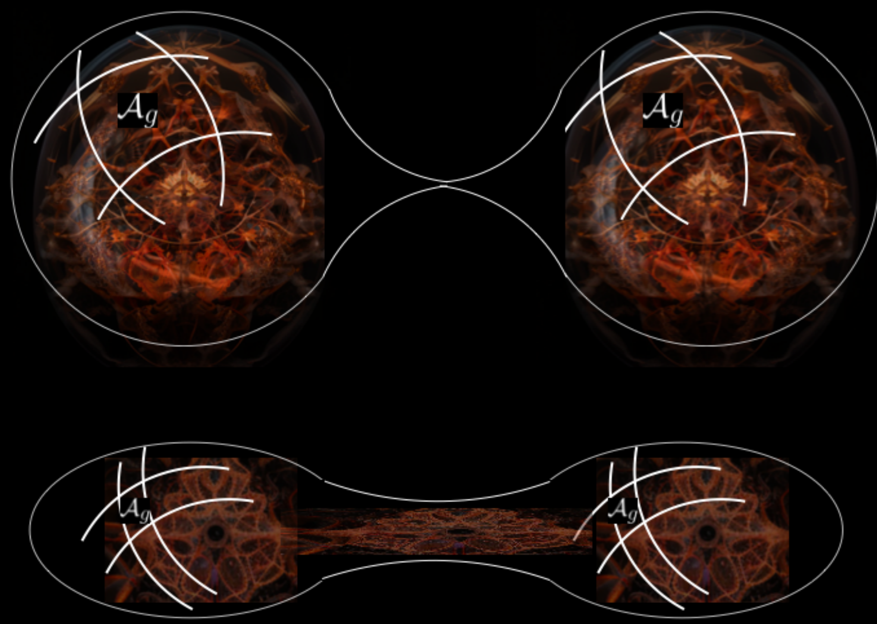

FIGURE 2. Illustration of a quantum-gravitational behavior of two macroscopic bodies. The gravitational areas, $\mathcal{A}_g$, on the left and right depend on the quantum correlations of the respective sub-networks. As such, they admit complementarity relations with the cross-correlations between the two sub-networks: the greater is $\mathcal{A}_g$ of either one of the sub-networks the weaker are their cross-correlations (illustrated by the throat forming in-between the left and right sub-networks), and vice versa.

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