$$S_3 = \frac{V_{X_3}[E(Y|X_3)]}{V(Y)} = \frac{25}{67} = 0.37$$
$$S_4 = \frac{V_{X_4}[E(Y|X_4)]}{V(Y)} = \frac{4}{67} = 0.06$$
$$S_5 = \frac{V_{X_5}[E(Y|X_5)]}{V(Y)} = \frac{25}{67} = 0.37.$$

We see that the two most influential factors are X_3 and X_5, each one determining 37% of the output variance.

If we decide to fix those two factors at a given value in their range of variation, we will have only three factors varying, i.e. factors X_1, X_2 and X_4. In such a situation, the model will have a lower variance:

$$V(Y) = \sum_{i=1}^{3} V(X_i) = 4 + 9 + 4 = 17.$$

We conclude that, in this example, by fixing the two most important factors the output variance decreases from 76 to 17, with a reduction of 75%.

Exercise 5

1. Calculate the expansion of f into terms of increasing dimensionality (4.5) for the function (Ishigami and Homma, 1996):

$$f(X_1, X_2, X_3) = \sin X_1 + a \sin^2 X_2 + bX_3^4 \sin X_1. \tag{4.34}$$

The input probability density functions are assumed as follows:[8]

$$p_i(X_i) = \frac{1}{2\pi},$$

when $-\pi \leq X_i \leq \pi$ and

$$p_i(X_i) = 0,$$

when $X_i < -\pi, X_i > \pi$ for $i = 1, 2, 3$.

[8] Note that this does not contradict the assumption that all factors are uniformly distributed within the unit hypercube Ω. It is always possible to map the hypercube to the desired distribution, and the sensitivity measure relative to the hypercube factors is identical to the measure for the transformed factors.

We calculate the decomposition of the function as (4.5) for $k = 3$:

$$
\begin{aligned}
f(X_1, X_2, X_3) =& f_0 + f_1(X_1) + f_2(X_2) + f_3(X_3) + f_{12}(X_1, X_2) \\
&+ f_{13}(X_1, X_3) + f_{23}(X_2, X_3) + f_{123}(X_1, X_2, X_3).
\end{aligned}
$$

Thus

$$
\begin{aligned}
f_0 = E(Y) &= \int\int\int f(X_1, X_2, X_3)p(X_1)p(X_2)p(X_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&= \frac{1}{(2\pi)^3}\int\int\int (\sin X_1 + a\sin^2 X_2 + bX_3^4 \sin X_1)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\
&= \frac{1}{(2\pi)^3}\left[\int \sin X_1\mathrm{d}x_1 + \int a\sin^2 X_2\mathrm{d}x_2 + \int\int bX_3^4\sin X_1\mathrm{d}x_1\mathrm{d}x_3\right] \\
&= \ldots. = \frac{a}{2}.
\end{aligned}
$$

So $f_0 = a/2$.

The $f_i(X_i)$ terms are easily obtained:

$$
\begin{aligned}
f_1(X_1) =& \int\int f(X_1, X_2, X_3)p(X_2)p(X_3)dx_2dx_3 - f_0 \\
=& \frac{1}{(2\pi)^2}\left[(2\pi)^2\sin X_1 + a\int\int \sin^2 X_2\mathrm{d}x_2\mathrm{d}x_3\right. \\
&\left. + b\sin X_1\int\int X_3^4\mathrm{d}x_2\mathrm{d}x_3\right] - \frac{a}{2} \\
=& \ldots = \frac{1}{(2\pi)^2}\left[(2\pi)^2\sin X_1 + 2a\pi^2 + \frac{4}{5}b\sin X_1\pi^6\right] - \frac{a}{2} \\
=& \sin X_1 + \frac{1}{5}b\pi^4\sin X_1 = \left(1 + \frac{1}{5}b\pi^4\right)\sin X_1. \\
f_2(X_2) =& \int\int f(X_1, X_2, X_3)p(X_1)p(X_3)\mathrm{d}x_1\mathrm{d}x_3 - f_0 \\
=& \frac{1}{(2\pi)^2}\left[\int\int \sin X_1\mathrm{d}x_1\mathrm{d}x_3 + (2\pi)^2 a\sin^2 X_2\right. \\
&\left. + b\int\int X_3^4\sin X_1\mathrm{d}x_1\mathrm{d}x_3\right] - \frac{a}{2} \\
=& \ldots = a\sin^2 X_2 - \frac{a}{2}. \\
f_3(X_3) =& \int\int f(X_1, X_2, X_3)p(X_1)p(X_2)\mathrm{d}x_1\mathrm{d}x_2 - f_0 \\
=& \frac{1}{(2\pi)^2}\left[\int\int \sin X_1\mathrm{d}x_1\mathrm{d}x_2 + a\int\int \sin X_2^2\mathrm{d}x_1\mathrm{d}x_2\right.
\end{aligned}
$$

$$+bX_3^4 \int\int \sin X_1 \mathrm{d}x_1 \mathrm{d}x_2 \Big] - \frac{a}{2}$$
$$= \ldots = 0.$$

The $f_{ij}(X_i, X_j)$ terms are computed as

$$f_{12}(X_1, X_2) = \int f(X_1, X_2, X_3) p(X_3) \mathrm{d}x_3 - f_1(X_1) - f_2(X_2) - f_0$$
$$= \sin X_1 + a \sin X_2^2 + b \sin X_1 \frac{1}{2\pi} \int X_3^4 \mathrm{d}x_3 - f_1(X_1) - f_2(X_2)$$
$$- f_0 = 0.$$
$$f_{13}(X_1, X_3) = \int f(X_1, X_2, X_3) p(X_2) \mathrm{d}x_2 - f_1(X_1) - f_3(X_3) - f_0$$
$$= \sin X_1 + a \frac{1}{2\pi} \int \sin X_2^2 \mathrm{d}x_2 + bX_3^4 \sin X_1 - f_1(X_1) - f_3(X_3) - f_0$$
$$= \ldots = \left(bX_3^4 - \frac{1}{5} b\pi^4 \right) \sin X_1.$$
$$f_{23}(X_2, X_3) = \int f(X_1, X_2, X_3) p(X_1) \mathrm{d}x_1 - f_2(X_2) - f_3(X_3) - f_0$$
$$= \frac{1}{2\pi} \int \sin X_1 \mathrm{d}x_1 + a \sin X_2^2 + bX_3^4 \sin X_1 dx_1$$
$$- f_2(X_2) - f_3(X_3) - f_0$$
$$= 0$$

f_{123} is obtained by difference and is equal to zero.

2. Calculate the variances of the terms for the function, according to Equation (4.11).

   First we calculate the unconditional variance of the function:

$$V(f(X)) = \int [f(X_1, X_2, X_3) - E(f(X))]^2 p(X_1) p(X_2) p(X_3) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$
$$= \frac{1}{(2\pi)^3} \int\int\int \big(\sin^2 X_1 + a^2 \sin^4 X_2 + b^2 X_3^8 \sin^2 X_1$$
$$+ 2a \sin X_1 \sin^2 X_2 + 2bX_3^4 \sin^2 X_1$$
$$+ 2abX_3^4 \sin X_1 \sin^2 X_2 + \frac{a}{4} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$$
$$+ a \sin X_1 - a^2 \sin^2 X_2 - abX_3^4 \sin X_1 \big)$$
$$= \frac{1}{(2\pi)^3} \left(\frac{1}{2} + \frac{3}{8} a^2 + \frac{b^2}{18} \pi^8 + \frac{a^2}{4} + \frac{b}{5} \pi^4 - \frac{a^2}{2} \right) (2\pi)^3$$
$$= \frac{1}{2} + \frac{a^2}{8} + \frac{b\pi^4}{5} + \frac{b^2 \pi^8}{18}.$$

We now calculate the variances of Equation (4.11), showing the passages for factor X_1:

$$V_1 = \int f_1^2(X_1)\mathrm{d}x_1 = \int \left(\sin X_1 + \frac{1}{5}b\pi^4 \sin X_1\right)^2 \mathrm{d}x_1$$
$$= \int \left[\sin X_1^2 + \frac{2}{5}b\pi^4 \sin X_1^2 + \frac{1}{25}b^2\pi^8 \sin X_1\right] \mathrm{d}x_1 =$$
$$= \ldots . = \frac{1}{2} + \frac{b\pi^4}{5} + \frac{b^2\pi^8}{50}.$$

The sensitivity index for factor X_1 can be calculated as

$$S_1 = \frac{V_1}{V} = \frac{1/2 + b\pi^4/5 + b^2\pi^8/50}{1/2 + a^2/8 + b\pi^4/5 + b^2\pi^8/18}$$

For the other factors we have

$$V_2 = \frac{a^2}{8}$$
$$V_3 = 0$$
$$V_{12} = 0$$

There is a typo here. It should be, V13=(8b^2*pi^8)/225

$$V_{13} = \frac{b^2\pi^4}{18} - \frac{b^2\pi^8}{50}$$
$$V_{23} = 0$$
$$V_{123} = 0$$

Again the fact that $V_{13} \neq 0$ even if $V_3 = 0$ is of particular interest, as it shows how an apparently noninfluential factor (i.e. with no main effect) may reveal itself to be influential through interacting with other parameters.

3. Show that the terms in the expansion of the function (4.34) are orthogonal.

Let us show, for example, that $f_1(X_1)$ is orthogonal to $f_2(X_2)$:

$$\int\int f_1(X_1)f_2(X_2)\mathrm{d}x_1\mathrm{d}x_2 = \left(1 + \frac{1}{5}b\pi^4\right)\int\int \sin x_1 \left(a\sin^2 x_2 - \frac{a}{2}\right)$$
$$= \left(1 + \frac{1}{5}b\pi^4\right)\int \sin x_1 \int \left(a\sin^2 x_2 - \frac{a}{2}\right)\mathrm{d}x_2,$$

which is equal to zero given that $\int \sin x_1 = 0$. The reader can verify, as a useful exercise, that the same holds for all other pairs of terms.