# Note del corso di Calcolabilità e Linguaggi Formali - Lezione 12

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## 1 Nondeterministic tree automata

**Definition 1.** *A* nondeterministic parity tree automaton *(NPTA) for $\Sigma$-labelled full binary trees is a tuple $A = (Q, \Sigma, q_I, \delta, \Omega)$ where $Q$ is a finite set of states, $q_I \in Q$ is the initial state, $\delta : Q \times \Sigma \to \mathcal{P}(Q^2)$ is the transition function and $\Omega : Q \to \{1, \dots, k\}$ is the priority function.*

A *run-tree* of an automaton $A$ over a tree $t$ is a map $\rho : \{0,1\}^* \to Q$ such that

1. $\rho(\varepsilon) = q_I$ and
2. for all $u \in \{0,1\}^*$ we have $(\rho(u0), \rho(u1)) \in \delta(\rho(u), t(u))$.

A run-tree $\rho$ of $A$ over $t$ is *accepting* iff in every path $\pi$ of $\rho$ the least priority that occurs infinitely often is even. An automaton $A$ *accepts* a tree $t$ just if there exists an accepting run-tree of $A$ over $t$.

**Definition 2.** *Given an NPTA $A = (Q, \Sigma, q_I, \delta, \Omega)$ and a tree $t$ the* acceptance parity game *$\mathcal{G}_{A,t} = (V_V, V_R, E, v_I, \Omega')$ where*

- $V_V = \{0,1\}^* \times Q$
- $V_R = \{0,1\}^* \times Q^2$
- $v_I = (\varepsilon, q_I)$
- $E \subseteq (V_V \times V_R) \cup (V_R \times V_V)$ *is the smallest relation such that*
  - *if* $(v, q) \in V_V$ *and* $(q_0, q_1) \in \delta(q, a)$ *and* $t(v) = a$*, then* $((v, q), (v, (q_0, q_1))) \in E$
  - *if* $(v, (q_0, q_1)) \in V_R$*, then* $((v, (q_0, q_1)), (v0, q_0)) \in E$ *and* $((v, (q_0, q_1)), (v1, q_1)) \in E$
- $\Omega'(v, q) = \Omega(q)$ *and* $\Omega'(v, (q_0, q_1)) = \max\{\Omega(q_0), \Omega(q_1)\}$

The acceptance parity game of Definition 2 can be read as follows:

- Verifier can exploit the nondeterminism of $A$ by choosing the run-tree
- Refuter chooses the path

Their combined action is headed towards checking that there exists ($\exists$) a run-tree all ($\forall$) of whose paths are accepting. Again Verifier represents ($\exists$) and Refuter represents ($\forall$).

**Lemma 1.** *Verifier has a winning strategy in $\mathcal{G}_{A,t}$ from vertex $v_0$ iff $t \in L(A)$.*

*Proof.* ($\Rightarrow$) Suppose Verifier has a winning strategy $\sigma$ in $\mathcal{G}_{A,t}$ from $v_0$. Define the map $\rho : \{0,1\}^* \to Q$ inductively as

- $\rho(\varepsilon) = q_I$ ;
- $\rho(v0) = q_0$ if $\rho(v) = q$ and $\sigma(v, q) = (v, (q_0, q_1))$ ;
- $\rho(v1) = q_1$ if $\rho(v) = q$ and $\sigma(v, q) = (v, (q_0, q_1))$ .

Clearly $\rho$ is a run-tree of $A$ over $t$. A path $\pi$ of $\rho$ has vertices
$\ldots, (v, q), (v, (q_0, q_1)), (v0, q_0), (v0, (q'_0, q'_1)), (v01, q'_1), \ldots$
and therefore has priorities
$\ldots, \Omega(q), \max\{\Omega(q_0), \Omega(q_1)\}, \Omega(q_0), \max\{\Omega(q'_0), \Omega(q'_1)\}, \Omega(q'_1), \ldots$.
The least priority occurring infinitely often is even but since each priority appearing at an even position is $\geq$ than its immediate successor, the subsequence obtained by removing all priorities at even positions is such that the least priority occurring infinitely often is still even. Such a subsequence witnesses an accepting path of $\rho$ and all paths of $\rho$ are obtained this way. Hence $\rho$ is accepting.
($\Leftarrow$) Suppose $\rho$ is an accepting run-tree fo $A$ over $t$. Then define the map $\sigma : (V_V \times V_R^*) V_V \to V_R$ as

- $\sigma(\alpha(v, q)) = (v, (q_0, q_1))$ iff $\rho(v0) = q_0$ and $\rho(v1) = q_1$.

Clearly $\sigma$ is a strategy for Verifier in $\mathcal{G}_{A,t}$ from $v_0$. A play $\pi$ respecting $\sigma$ has vertices
$\ldots, (v, q), (v, (q_0, q_1)), (v0, q_0), (v0, (q'_0, q'_1)), (v01, q'_1), \ldots$
and therefore has priorities
$\ldots, \Omega(q), \max\{\Omega(q_0), \Omega(q_1)\}, \Omega(q_0), \max\{\Omega(q'_0), \Omega(q'_1)\}, \Omega(q'_1), \ldots$.
The subsequence obtained by removing all priorities at even positions is the run-tree $\rho$ and the least priority occurring infinitely often in it is even. Since each priority appearing at an even position is $\geq$ than its immediate successor, the least priority occurring infinitely often in $\pi$ is still even. Such a play is winning for Verifier.