Note del corso di Calcolabilità e Linguaggi Formali - Lezione 12

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1 Nondeterministic tree automata

Definition 1. *A* nondeterministic parity tree automaton *(NPTA) for Σ-labelled full binary trees is a tuple $A = (Q, \Sigma, q_I, \delta, \Omega)$ where Q is a finite set of states, $q_I \in Q$ is the initial state, $\delta : Q \times \Sigma \to \mathcal{P}(Q^2)$ is the transition function and $\Omega : Q \to \{1, \ldots, k\}$ is the priority function.*

A *run-tree* of an automaton A over a tree t is a map $\rho : \{0,1\}^* \to Q$ such that

1. $\rho(\varepsilon) = q_I$ and
2. for all $u \in \{0,1\}^*$ we have $(\rho(u0), \rho(u1)) \in \delta(\rho(u), t(u))$.

A run-tree ρ of A over t is *accepting* iff in every path π of ρ the least priority that occurs infinitely often is even. An automaton A *accepts* a tree t just if there exists an accepting run-tree of A over t.

Definition 2. *Given an NPTA $A = (Q, \Sigma, q_I, \delta, \Omega)$ and a tree t the* acceptance parity game *$\mathcal{G}_{A,t} = (V_V, V_R, E, v_I, \Omega')$ where*

- $V_V = \{0,1\}^* \times Q$
- $V_R = \{0,1\}^* \times Q^2$
- $v_I = (\varepsilon, q_I)$
- $E \subseteq (V_V \times V_R) \cup (V_R \times V_V)$ *is the smallest relation such that*
  - *if* $(v, q) \in V_V$ *and* $(q_0, q_1) \in \delta(q, a)$ *and* $t(v) = a$, *then* $((v, q), (v, (q_0, q_1))) \in E$
  - *if* $(v, (q_0, q_1)) \in V_R$, *then* $((v, (q_0, q_1)), (v0, q_0)) \in E$ *and* $((v, (q_0, q_1)), (v1, q_1)) \in E$
- $\Omega'(v, q) = \Omega(q)$ *and* $\Omega'(v, (q_0, q_1)) = \max\{\Omega(q_0), \Omega(q_1)\}$

The acceptance parity game of Definition 2 can be read as follows:

- Verifier can exploit the nondeterminism of A by choosing the run-tree
- Refuter chooses the path

Their combined action is headed towards checking that there exists ($\exists$) a run-tree all ($\forall$) of whose paths are accepting. Again Verifier represents ($\exists$) and Refuter represents ($\forall$).

Lemma 1. *Verifier has a winning strategy in $\mathcal{G}_{A,t}$ from vertex v_I iff $t \in L(A)$.*

Proof. ($\Rightarrow$) Suppose Verifier has a winning strategy σ in $\mathcal{G}_{A,t}$ from v_I. Define the map $\rho : \{0,1\}^* \to Q$ inductively as

- $\rho(\varepsilon) = q_I$;
- $\rho(v0) = q_0$ if $\rho(v) = q$ and $\sigma(v, q) = (v, (q_0, q_1))$;
- $\rho(v1) = q_1$ if $\rho(v) = q$ and $\sigma(v, q) = (v, (q_0, q_1))$.

Clearly ρ is a run-tree of A over t. A path π of ρ has vertices
$\ldots, (v, q), (v, (q_0, q_1)), (v0, q_0), (v0, (q_0', q_1')), (v01, q_1'), \ldots$
and therefore has priorities
$\ldots, \Omega(q), \max\{\Omega(q_0), \Omega(q_1)\}, \Omega(q_0), \max\{\Omega(q_0'), \Omega(q_1')\}, \Omega(q_1'), \ldots$.
The least priority occurring infinitely often is even but since each priority appearing at an even position is $\geq$ than its immediate successor, the subsequence obtained by removing all priorities at even positions is such that the least priority occurring infinitely often is still even. Such a subsequence witnesses an accepting path of ρ and all paths of ρ are obtained this way. Hence ρ is accepting.
($\Leftarrow$) Suppose ρ is an accepting run-tree fo A over t. Then define the map $\sigma : (V_V \times V_R^*)V_V \to V_R$ as

- $\sigma(\alpha(v, q)) = (v, (q_0, q_1))$ iff $\rho(v0) = q_0$ and $\rho(v1) = q_1$.

Clearly σ is a strategy for Verifier in $\mathcal{G}_{A,t}$ from v_0. A play π respecting σ has vertices
$\ldots, (v, q), (v, (q_0, q_1)), (v0, q_0), (v0, (q_0', q_1')), (v01, q_1'), \ldots$
and therefore has priorities
$\ldots, \Omega(q), \max\{\Omega(q_0), \Omega(q_1)\}, \Omega(q_0), \max\{\Omega(q_0'), \Omega(q_1')\}, \Omega(q_1'), \ldots$.
The subsequence obtained by removing all priorities at even positions is the run-tree ρ and the least priority occurring infinitely often in it is even. Since each priority appearing at an even position is $\geq$ than its immediate successor, the least priority occurring infinitely often in π is still even. Such a play is winning for Verifier.

Definition 3. *Given an NPTA* $A = (Q, \Sigma, q_I, \delta, \Omega)$ *the* non-emptiness parity game $\mathcal{G}_A = (V_V, V_R, E, v_I, \Omega')$ *where*

- $V_V = Q$
- $V_R = \{((q, a), (q_0, q_1)) : (q_0, q_1) \in \delta(q, a)\}$.
- $v_I = q_I$
- $E \subseteq (V_V \times V_R) \cup (V_R \times V_V)$ *is the smallest relation such that*
  - *if* $q \in V_V$ *and* $(q_0, q_1) \in \delta(q, a)$*, then* $(q, ((q, a), (q_0, q_1))) \in E$
  - *if* $v = ((q, a), (q_0, q_1)) \in V_R$*, then* $(v, q_0) \in E$ *and* $(v, q_1) \in E$
- $\Omega'(q) = \Omega(q)$ *and* $\Omega'((q, a), (q_0, q_1)) = \Omega(q)$

Lemma 2. *Verifier has a winning strategy in* $\mathcal{G}_A$ *from vertex* v_I *iff* $L(A) \neq \emptyset$.

Proof. ($\Rightarrow$) Suppose Verifier has a winning strategy σ in $\mathcal{G}_A$ from v_I. Then $\delta(q, a) \neq \emptyset$ for every pair $(q, a) \in Q \times \Sigma$. The map σ induces a function $\hat{\sigma} : Q \to \Sigma \times Q^2$ given by $\hat{\sigma}(q) = (a, q_0, q_1)$ iff $\sigma(q) = ((q, a), (q_0, q_1))$. Therefore

the solution to the fixpoint equation $\tau(q) = (a, \tau(q_0), \tau(q_1))$ with constraints $\hat{\sigma}(q) = (a, q_0, q_1)$ is a tree whose first coordinates define a tree $t \in L(A)$.
($\Leftarrow$) Suppose $t \in L(A)$ is an input tree and let ρ be the associated run-tree.

Let $\bar{t}(\varepsilon) = t(\varepsilon)$ and $\bar{t}(uj) = \bar{t}(u)t(uj)$ for $u \in \{0,1\}^*$ and $j \in \{0,1\}$.

Let also $\bar{\rho}(\varepsilon) = \rho(\varepsilon)$ and $\bar{\rho}(uj) = \bar{\rho}(u)\rho(uj)$ for $u \in \{0,1\}^*$ and $j \in \{0,1\}$.

Similarly Then the map $\sigma : (V_V V_R)^* V_V \to V_R$ given by $\sigma(\alpha q) = ((q, a), (q_0, q_1))$ if α "contains" the string $u = i_1 \dots i_n$ and $t(i_1) \dots t(i_1 \dots i_n) = a$ and $\rho(i_1) \dots \rho(i_1 \dots i_n) = q$.