Note del corso di Calcolabilità e Linguaggi Formali - Lezione 12

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1 Parity games

Definition 1. *A* parity game *is a tuple* $\mathcal{G} = (V, R, \lambda, C, \Omega)$

- V *is a finite set of states vertices partitioned into two sets* V_V *(vertices owned by "Verifier") and* V_R *(vertices owned by "Refuter"),*
- $R \subseteq V \times V$ *is the accessibility relation,*
- C *is a finite set of labels and* $\lambda : V \times C$ *is the labelling function,*
- $\Omega : V \to \{1, \dots, k\}$ *is the priority function.*

A *play* is a finite or infinite sequence of vertices. A finite play π is winning for Verifier if its last vertex belongs to V_V. An infinite play $\pi = v_0, v1, \dots$ is winning for Verifier if in the sequence $\Omega(\pi) = \Omega(v_0), \Omega(v1), \dots$ the least priority that occurs infinitely often is even. We let

- $Win = \{\pi : \pi \text{ is a winning play for Verifier}\}$, so that $Win \subseteq V^* \cup V^\omega$.
- $Acc = \{\lambda(\pi) : \pi \in Win\}$, so that $Acc \subseteq C^* \cup C^\omega$.

Example 1. For a graph (V, R) and a finite set $F \subseteq C$. We can build a game such that $Acc \cap C^\omega = F^\omega$ by setting: $\Omega(v) = 0$ for every $v \in F$.

Example 2. For a graph (V, R) and a finite set $F \subseteq C$ we let $Reach_F$ be the set of plays π with $\pi = v_0, v_1, \dots \in V^\omega$ and such that there exists i with $\lambda(v_i) \in F$. We can build a game such that $Acc \cap C^\omega = \lambda(Reach_F)$ by setting:

2 Nondeterministic tree automata

Definition 2. *A* nondeterministic parity tree automaton *(NPTA) for* Σ*-labelled full binary trees is a tuple* $A = (Q, \Sigma, q_I, \delta, \Omega)$ *where*

- Q *is a finite set of states,*
- $q_I \in Q$ *is the initial state,*
- $\delta : Q \times \Sigma \to \mathcal{P}(Q^2)$ *is the transition function and*
- $\Omega : Q \to \{1, \dots, k\}$ *is the priority function.*

Remark 1. Note that the transition function $\delta : Q \times \Sigma \to \mathcal{P}(Q^2)$ is by some authors seen as a function $\delta : Q \times \Sigma \times \{0, 1\} \to \mathcal{P}(Q)$.

A *run-tree* of an automaton A over a tree t is a map $\rho : \{0,1\}^* \to Q$ such that

1. $\rho(\varepsilon) = q_I$ and
2. for all $u \in \{0,1\}^*$ we have $(\rho(u0), \rho(u1)) \in \delta(\rho(u), t(u))$.

A run-tree ρ of A over t is *accepting* iff in every path π of ρ the least priority that occurs infinitely often is even. An automaton A *accepts* a tree t just if there exists an accepting run-tree of A over t.

Definition 3. *Given an NPTA $A = (Q, \Sigma, q_I, \delta, \Omega)$ and a tree t the* acceptance parity game *$\mathcal{G}_{A,t} = (V_V, V_R, E, v_I, \Omega')$ where*

- $V_V = \{0,1\}^* \times Q$
- $V_R = \{0,1\}^* \times Q^2$
- $v_I = (\varepsilon, q_I)$
- $E \subseteq (V_V \times V_R) \cup (V_R \times V_V)$ *is the smallest relation such that*
  - *if $(v, q) \in V_V$ and $(q_0, q_1) \in \delta(q, a)$ and $t(v) = a$, then $((v, q), (v, (q_0, q_1))) \in E$*
  - *if $(v, (q_0, q_1)) \in V_R$, then $((v, (q_0, q_1)), (v0, q_0)) \in E$ and $((v, (q_0, q_1)), (v1, q_1)) \in E$*
- $\Omega'(v, q) = \Omega(q)$ *and* $\Omega'(v, (q_0, q_1)) = \max\{\Omega(q_0), \Omega(q_1)\}$

The acceptance parity game of Definition 3 can be read as follows:

- Verifier can exploit the nondeterminism of A by choosing the run-tree
- Refuter chooses the path

Their combined action is headed towards checking that there exists ($\exists$) a run-tree all ($\forall$) of whose paths are accepting. Again Verifier represents ($\exists$) and Refuter represents ($\forall$).

Lemma 1. *Verifier has a winning strategy in $\mathcal{G}_{A,t}$ from vertex v_I iff $t \in L(A)$.*

Proof. ($\Rightarrow$) Suppose Verifier has a winning strategy σ in $\mathcal{G}_{A,t}$ from v_I. Define the map $\rho : \{0,1\}^* \to Q$ inductively as

- $\rho(\varepsilon) = q_I$;
- $\rho(v0) = q_0$ if $\rho(v) = q$ and $\sigma(v, q) = (v, (q_0, q_1))$;
- $\rho(v1) = q_1$ if $\rho(v) = q$ and $\sigma(v, q) = (v, (q_0, q_1))$.

Clearly ρ is a run-tree of A over t. A path π of ρ has vertices
$\ldots, (v, q), (v, (q_0, q_1)), (v0, q_0), (v0, (q'_0, q'_1)), (v01, q'_1), \ldots$
and therefore has priorities
$\ldots, \Omega(q), \max\{\Omega(q_0), \Omega(q_1)\}, \Omega(q_0), \max\{\Omega(q'_0), \Omega(q'_1)\}, \Omega(q'_1), \ldots$.
The least priority occurring infinitely often is even but since each priority appearing at an even position is $\geq$ than its immediate successor, the subsequence obtained by removing all priorities at even positions is such that the least priority occurring infinitely often is still even. Such a subsequence witnesses an accepting path of ρ and all paths of ρ are obtained this way. Hence ρ is accepting.
($\Leftarrow$) Suppose ρ is an accepting run-tree fo A over t. Then define the map $\sigma : (V_V \times V_R^*) V_V \to V_R$ as

– $\sigma(\alpha(v,q)) = (v,(q_0,q_1))$ iff $\rho(v0) = q_0$ and $\rho(v1) = q_1$.

Clearly σ is a strategy for Verifier in $\mathcal{G}_{A,t}$ from v_0. A play π respecting σ has vertices
$\dots,(v,q),(v,(q_0,q_1)),(v0,q_0),(v0,(q_0',q_1')),(v01,q_1'),\dots$
and therefore has priorities
$\dots,\Omega(q),\max\{\Omega(q_0),\Omega(q_1)\},\Omega(q_0),\max\{\Omega(q_0'),\Omega(q_1')\},\Omega(q_1'),\dots$.
The subsequence obtained by removing all priorities at even positions is the run-tree ρ and the least priority occurring infinitely often in it is even. Since each priority appearing at an even position is $\geq$ than its immediate successor, the least priority occurring infinitely often in π is still even. Such a play is winning for Verifier.

Definition 4. *Given an NPTA* $A = (Q,\Sigma,q_I,\delta,\Omega)$ *the* non-emptiness parity game $\mathcal{G}_A = (V_V,V_R,E,v_I,\Omega')$ *where*

– $V_V = Q$
– $V_R = \{((q,a),(q_0,q_1)) : (q_0,q_1) \in \delta(q,a)\}$.
– $v_I = q_I$
– $E \subseteq (V_V \times V_R) \cup (V_R \times V_V)$ *is the smallest relation such that*
  • *if* $q \in V_V$ *and* $(q_0,q_1) \in \delta(q,a)$, *then* $(q,((q,a),(q_0,q_1))) \in E$
  • *if* $y = ((q,a),(q_0,q_1)) \in V_R$, *then* $(y,q_0) \in E$ *and* $(y,q_1) \in E$
– $\Omega'(q) = \Omega(q)$ *and* $\Omega'((q,a),(q_0,q_1)) = \Omega(q)$

Lemma 2. *Verifier has a winning strategy in* $\mathcal{G}_A$ *from vertex* v_I *iff* $L(A) \neq \emptyset$.

Proof. ($\Rightarrow$) Suppose Verifier has a winning strategy σ in $\mathcal{G}_A$ from v_I. Then $\delta(q,a) \neq \emptyset$ for every pair $(q,a) \in Q \times \Sigma$. The solution to the fixpoint equation $\tau(q) = (a,\tau(q_0),\tau(q_1))$ with constraints $\sigma(q) = ((q,a),(q_0,q_1))$ is a tree whose first coordinates define a tree $t \in L(A)$.

($\Leftarrow$) Let $\alpha \in (V_V V_R)^* V_V$ be an initial segment of a play in $\mathcal{G}_A$ from v_I. The run-trees ρ of the trees $t \in L(A)$. Suppose is an input tree and let ρ be an associated run-tree.

Let $\bar{t}(\varepsilon) = t(\varepsilon)$ and $\bar{t}(uj) = \bar{t}(u)t(uj)$ for $u \in \{0,1\}^*$ and $j \in \{0,1\}$. Let also $\bar{\rho}(\varepsilon) = \rho(\varepsilon)$ and $\bar{\rho}(uj) = \bar{\rho}(u)\rho(uj)$ for $u \in \{0,1\}^*$ and $j \in \{0,1\}$.

Come si fa a definire la strategia vincente? La run-tree su t potrebbe non visitare tutti gli stati e quindi per gli stati ne non compaiono come faccio ad avere la scelta giusta del successore per la strategia?

is a sequence $x_1(y_1,z_1),\dots,x_n(y_n,z_n)$ with $x_i \in Q$, $y_i \in Q \times \Sigma$ and $z_i \in Q^2$. The trace of α is then $y_1,\dots,y_n$. Then the map $\sigma : (V_V V_R)^* V_V \to V_R$ given by $\sigma(\alpha q)$ = component-wise first if α "contains" the string $u = i_1 \dots i_n$ and $t(i_1)\dots t(i_1\dots i_n) = a$ and $\rho(i_1)\dots\rho(i_1\dots i_n) = q$.

3 Finite automata on infinite words

A nondeterminitstic *finite automaton* on Σ-labelled infinite words is a tuple $A = (Q,\Sigma,q_I,\delta)$ where

- Q is a finite set of states;
- $q_I \in Q$ is the initial state;
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is the transition function.

A run of a word $\alpha \in \Sigma^\omega$ is a sequence $\varrho \in Q^\Omega$ such that $\varrho(0) = q_I$ and $\varrho(i+1) \in \delta(\varrho(i), \alpha(i))$ for all $i \geq 0$.

We let $\mathrm{Inf}(\varrho) = \{q \in Q : \varrho^{-1}(q) \text{ is infinite}\}$.

Definition 5. *A* nondeterministic Büchi automaton *on Σ-labelled infinite words is a finite automaton $A = (Q, \Sigma, q_I, \delta, F)$ where $F \subseteq Q$ is the set of final states. The language of A, $L(A)$, is the set of words for which there exists a run ϱ with $Inf(\varrho) \cap F \neq \emptyset$.*

Definition 6. *A* nondeterministic Müller automaton *on Σ-labelled infinite words is a finite automaton $A = (Q, \Sigma, q_I, \delta, \Omega)$ where $\Omega = \{F_1, \ldots, F_k\}$ is a finite set with $F_i \subseteq Q$. The language of A, $L(A)$, is the set of words for which there exists a run ϱ such that $Inf(\varrho) \in \Omega$.*

Definition 7. *A* nondeterministic Rabin automaton *on Σ-labelled infinite words is a finite automaton $A = (Q, \Sigma, q_I, \delta, \Omega)$ where $\Omega = \{(E_1, F_1), \ldots, (E_k, F_k)\}$ is a finite set with $E_i, F_i \subseteq Q$. The language of A, $L(A)$, is the set of words for which there exists a run ϱ such that there exists a pair $(E, F) \in \Omega$ such that $Inf(\varrho) \cap E = \emptyset$ and $Inf(\varrho) \cap F \neq \emptyset$.*

Definition 8. *A* nondeterministic Parity automaton *on Σ-labelled infinite words is a finite automaton $A = (Q, \Sigma, q_I, \delta, \Omega)$ where $\Omega : Q \to \{1, \ldots, k\}$ is the set of priorities (or colors). The language of A, $L(A)$, is the set of words for which there exists a run ϱ such that the minimum element of $Inf(\Omega \circ \varrho)$ is even.*

Theorem 1. *Let $A = (Q, \Sigma, q_I, \delta, F)$ be a nondeterministic Büchi automaton. Then there exists a nondeterministic Müller automaton A' such that $L(A') = L(A)$.*

Proof. Define $\Omega = \{G \in \mathcal{P}(Q) : G \cap F \neq \emptyset\}$. Then the nondeterministic Müller automaton $A' = (Q, \Sigma, q_I, \delta, \Omega)$ has the desired property.

Theorem 2. *Let $A = (Q, \Sigma, q_I, \delta, \Omega)$ be a nondeterministic Müller automaton. Then there exists a nondeterministic Büchi automaton A' such that $L(A') = L(A)$.*

Proof. Define $Q' = Q \cup \{(G, q, X) : G \in \Omega, q \in G, X \subseteq G\}$ and $F = \{(G, q, \emptyset) : G \in \Omega, q \in G\}$.

Let $q \in Q$ and suppose $p \in \delta(q, a)$. Then also $p \in \delta'(q, a)$ and for each $G \in \Omega$ we have

$$\begin{cases} (G, p, \{q\}) \in \delta'(q, a) & \text{if } \{q\} \subset G \text{ and } p \in G \\ (G, p, \emptyset) \in \delta'(q, a) & \text{if } \{q\} = G \text{ and } p \in G \end{cases}$$

and

$$\begin{cases} (G, p, X \cup \{q\}) \in \delta'((G, q, X), a) & \text{if } X \cup \{q\} \subset G \text{ and } p \in G \\ (G, p, \emptyset) \in \delta'((G, q, X), a) & \text{if } X \cup \{q\} = G \text{ and } p \in G \end{cases}$$

The nondeterministic Büchi automaton $A' = (Q', \Sigma, q_I, \delta', F)$ has the desired property.

Theorem 3. *Let $A = (Q, \Sigma, q_I, \delta, \Omega)$ be a nondeterministic Rabin automaton. Then there exists a nondeterministic Müller automaton A' such that $L(A') = L(A)$.*

Proof. Define $\Omega' = \{G \in \mathcal{P}(Q) : \exists (E, F) \in \Omega.\ (G \cap E = \emptyset \wedge G \cap F \neq \emptyset)\}$. Then the nondeterministic Müller automaton $A' = (Q, \Sigma, q_I, \delta, \Omega')$ has the desired property.

Theorem 4. *Let $A = (Q, \Sigma, q_I, \delta, \Omega)$ be a nondeterministic Parity automaton. Then there exists a nondeterministic Rabin automaton A' such that $L(A') = L(A)$.*

Proof. Suppose that $\{0, \ldots, k\}$ is the set of colors for A. Let $r = \lfloor \frac{k}{2} \rfloor$ and $\Omega' = \{(E_0, F_0), \ldots, (E_r, F_r)\}$ where

- $E_i = \{q \in Q : \Omega(q) < 2i\}$
- $F_i = \{q \in Q : \Omega(q) \leq 2i\}$

Then the nondeterministic Rabin automaton $A' = (Q, \Sigma, q_I, \delta, \Omega')$ has the desired property.

Theorem 5. *Let $A = (Q, \Sigma, q_I, \delta, F)$ be a nondeterministic Büchi automaton. Then there exists a nondeterministic Parity automaton A' such that $L(A') = L(A)$.*

Proof. Let $\{0, 1\}$ be the set of colors and define

$$\Omega(q) = \begin{cases} 0 & \text{if } q \in F \\ 1 & \text{if } q \notin F \end{cases}$$

Then the nondeterministic Parity automaton $A' = (Q, \Sigma, q_I, \delta, \Omega)$ has the desired property.

Theorem 6. *Let $A = (Q, \Sigma, q_I, \delta, F)$ be a nondeterministic Büchi automaton. Then there exists a nondeterministic Rabin automaton A' such that $L(A') = L(A)$.*

Proof. Define $\Omega = \{(\emptyset, F)\}$. Then the nondeterministic Rabin automaton $A' = (Q, \Sigma, q_I, \delta, \Omega)$ has the desired property.