ECO S500 Mathematics for Economists **Ezra Oberfield**
Fall 2018 **Princeton University**
edo@princeton.edu

Practice Problems for Elementary Analysis

In the problems below, $\mathbb{R}^n$ is always endowed with its usual metric.

1. Let $\{s_n\}_{n=1}^{\infty}$ be defined by $s_1 = s_2 = 1$, and $s_{n+1} = s_n + s_{n-1}$ for $n \geq 2$. Find s_{10}.

2. Is the sequence $1, 0, 1, 0, ...$ a subsequence of $1, 2, 3, 4, ...$?

3. Let S be a sequence, prove that every subsequence of a subsequence of S is itself a subsequence of S.

4. If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers with limit L, prove that this limit is unique.

5. If $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $s_n \leq M$ for all n and $\lim_{n\to\infty} s_n = L$, prove that $L \leq M$.

6. If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers such that $s_n \leq t_n$ for all n, and $\lim_n \ s_n = L$ and $\lim_n \ t_n = M$, prove that $L \leq M$. (Hence, inequalities are preserved when taking limits.)

7. Using the property that every subset of reals with an upper bound has a least upper bound, prove that every subset of reals with a lower bound has a greatest lower bound.

8. If $L \leq M + \varepsilon$ for all $\varepsilon > 0$ prove that $L \leq M$.

9. Prove that $\lim_{n\to\infty} \ 2n/(n + 3) = 2$.

10. Decide whether these sequences have limits and find them if they do.

    a) $\left\{\frac{n^2}{n+5}\right\}_{n=1}^{\infty}$.

    b) $\left\{\frac{3n}{n+7n^2}\right\}_{n=1}^{\infty}$.

11. Prove that if $\{|s_n|\}_{n=1}^{\infty}$ converges, then $\{s_n\}_{n=1}^{\infty}$ is bounded. Must $\{s_n\}_{n=1}^{\infty}$ also converge?

12. Suppose that $\lim_{n\to\infty} \ s_n/n = 0$. Must $\{s_n\}_{n=1}^{\infty}$ converge?

13. Let $s_n = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}$. Prove that $\{s_n\}_{n=1}^{\infty}$ is convergent and that $\lim_{n\to\infty} \ s_n \leq \frac{1}{2}$.

14. Let $s_n = \frac{1+2+...+n}{n^2}$. Show that $\{s_n\}_{n=1}^{\infty}$ is monotone and bounded and that $\lim_{n\to\infty} \ s_n = \frac{1}{2}$.

15. Prove $\lim_{n\to\infty} \frac{2n^3+5n}{4n^3+n^2} = \frac{1}{2}$.

16. Prove that every convergent sequence is Cauchy.

17. If $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers having a subsequence that converges to L, prove that $\{s_n\}_{n=1}^{\infty}$ itself converges to L.

18. Prove that every subsequence of a Cauchy sequence is a Cauchy sequence.

19. Prove that $\lim_{x\to -2} x^2 + 3x = -2$ in two ways. First, using the definition of a limit, and second using results about continuous functions.

20. Let $\lfloor x \rfloor$ denote the greatest integer less or equal to x. For example, $\lfloor -4 \rfloor = -4$, $\lfloor -4.3 \rfloor = -5$, $\lfloor 8.1 \rfloor = 8$. Define $f(x) = \lfloor 1 - x^2 \rfloor$ and $g(x) = \lfloor 1 - x \rfloor$. Do the limits $\lim_{x\to 0} f(x)$ and $\lim_{g\to 0} g(x)$ exist? If so, evaluate.

21. Prove that if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$, then $\lim_{x\to a} f(x) = L$.

22. Show that if ρ is a metric for M then so is 2ρ and so is $min(1, \rho)$. (The latter shows that a metric can always be assumed to be bounded, without loss.)

23. Show that if ρ and σ are metrics for M, then so is $\rho + \sigma$.

24. Suppose that ρ_1 and ρ_2 are metrics for $\mathbb{R}$. Define $\rho(x, y) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2)$ for each $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$. Show that ρ is a metric for $\mathbb{R}^2$. Can you generalize this result to higher dimensions of Euclidean space? To arbitrary metric spaces?

25. Recall that for any sequence of real numbers, $\{a_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} a_n < \infty$ means that the limit $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ exists and the limit is denoted by $\sum_{n=1}^{\infty} a_n$. Let l^1 be the set of all real sequences $\{s_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |s_n| < \infty$. Show that if we define the metric $\rho(s, t) = \sum_{n=1}^{\infty} |s_n - t_n|$ for each $s = \{s_n\}_{n=1}^{\infty}$ and $t = \{t_n\}_{n=1}^{\infty}$ in l^1, then ρ is a metric for l^1.

26. Prove that $\{s_n\}_{n=1}^{\infty}$ is bounded in the metric space $< M, \rho >$ if and only if the sequence of real numbers $\{\rho(s_1, s_n)\}_{n=1}^{\infty}$ is bounded.

27. Prove that if $\{s_n\}_{n=1}^{\infty}$ is Cauchy in the metric space $< M, \rho >$, then the sequence of real numbers $\{\rho(s_1, s_n)\}_{n=1}^{\infty}$ is bounded. Conclude that $\{s_n\}_{n=1}^{\infty}$ is bounded.

28. Prove that $\mathbb{R}^n$ is complete.

29. Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous at $x = a$, then so are $\min(f, g)$ and $\max(f, g)$.

30. Prove that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if $f(x) = (f_1(x), ..., f_m(x))$ for some continuous functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, ..., m$.

31. Prove that if G is an open subset of the reals containing x, then G contains a largest open interval containing x.

32. Let f and g be continuous real-valued functions on a metric space M. Prove that $A = \{x \in M : f(x) < g(x)\}$ is open.

33. Let G be an open subset of a metric space M and for each $x \in M$ define $f(x) = 1$ if $x \in G$ and $f(x) = 0$, otherwise. Prove that f is continuous at every point of G. Is f continuous everywhere in M?

34. Show that the union of two disjoint sets, neither of which is open, can be open.

35. If A and B are open (closed) subsets of $\mathbb{R}$, prove that $A \times B$ is an open (closed) subset of $\mathbb{R}^2$.

36. Prove that any finite subset of a metric space is closed.

37. Let A and B be subsets of a metric space, prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Is the same true for intersections?

38. Let f and g be continuous real-valued functions on a metric space M. Prove that $A = \{x \in M : f(x) = 0\}$ is closed.

39. Prove that if f is a non-constant real-valued continuous function on the reals, then the range of f is an interval.

40. The discrete metric assigns distance 1 to any pair of distinct points in $\mathbb{R}$. Prove that $[0, 1]$ is not connected as a subset $\mathbb{R}$ when the metric is the discrete metric.

41. Prove that every finite subset of a metric space is compact.

42. Prove that every closed and bounded subset of $\mathbb{R}^n$ is compact.

43. If A and B are compact subsets of $\mathbb{R}$, prove that $A \times B$ is a compact subset of $\mathbb{R}^2$.

44. If $f : [a, b] \to \mathbb{R}^2$ is continuous, prove that the graph of f, *i.e.*, the set $\{(x, y) \in \mathbb{R}^3 : y = f(x)\}$, is a compact subset of $\mathbb{R}^3$.

45. Give an example of connected subset of a metric space that is not compact.

46. Prove that the maximization problem

$$\max_{(x,y)} \ln x + 7xy - x^2 e^y$$

subject to $(x, y) \in \mathbb{R}^2$ and $x, y \geq 1$ has a solution.

47. Prove that the maximization problem

$$\max_{\{x_1, x_2, \ldots\}} \sum_{n=1}^{\infty} \frac{1}{2^n} (\sin x_n) e^{-x_{n+1}}$$

subject to $x_n \in \mathbb{R}$ and $|x_n| \leq 2$ for every n, has a solution. (Hint: Consider the space M of real sequences $\{x_n\}_{n=1}^{\infty}$ such that $|x_n| \leq 2$ for every n, and consider the metric $\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x_n - y_n|$ for $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$ in M.)