Question 1

1. (a) (i)

- λ is the arrival rate of the packets / customers / etc. per unit of time
- μ is the service rate of the packets / customers / etc. per unit of time
- $\rho < 1$ to achieve steady state

1. (a) (ii)

$$\sum_{k=0}^{\infty} P_k = 1$$

$$\sum_{k=0}^{m-1} P_0 \left(\frac{(m\rho)^k}{k!} \right) + \sum_{k=m}^{\infty} P_0 \left(\frac{m^m \rho^k}{m!} \right) = 1$$

$$P_0 \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{m^m}{m!} \sum_{k=m}^{\infty} \rho^k \right] = 1$$

$$P_0 \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \sum_{k=m}^{\infty} \rho^{k-m} \right] = 1$$

$$P_0 \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!(1-\rho)} \right] = 1$$

Therefore,

$$P_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!(1-\rho)} \right]^{-1}$$

1. (b) (i)

When $m = 3$,

$$P_k = \begin{cases} P_0 \left(\dfrac{(3\rho)^k}{k!} \right) & k < 3 \\ P_0 \left(\dfrac{3^3 \rho^k}{3!} \right) = P_0 \left(\dfrac{9\rho^k}{2} \right) & k \geq 3 \end{cases} \quad , \quad \rho = \frac{\lambda}{m\mu} < 1$$

$\rho < 1$ for steady state solution to exist

1. (b) (ii)

Substituting $m = 3$ into P_0 obtained in **1. (a) (ii)**,

$$
\begin{aligned}
P_0 &= \left[\sum_{k=0}^{3-1} \frac{(3\rho)^k}{k!} + \frac{(3\rho)^3}{3!(1-\rho)}\right]^{-1} \\
&= \left[\sum_{k=0}^{2} \frac{(3\rho)^k}{k!} + \frac{9\rho^3}{2(1-\rho)}\right]^{-1} \\
&= \left[\sum_{k=0}^{2} \frac{(3\rho)^k}{k!} + \frac{9\rho^3}{2(1-\rho)}\right]^{-1} \\
&= \left[1 + 3\rho + \frac{9\rho^2}{2} + \frac{9\rho^3}{2(1-\rho)}\right]^{-1}
\end{aligned}
$$

1. (b) (iii)

For $m = 3$, the average number of packets in the system is:

$$
\begin{aligned}
E\{k\} &= \sum_{k=0}^{m-1} kP_k + \sum_{k=m}^{\infty} kP_k \\
&= \sum_{k=0}^{2} kP_0 \left(\frac{(3\rho)^k}{k!}\right) + \sum_{k=3}^{\infty} kP_0 \left(\frac{9\rho^k}{2}\right) \\
&= P_0 \sum_{k=0}^{2} \frac{k(3\rho)^k}{k!} + \frac{9P_0}{2} \sum_{k=3}^{\infty} k\rho^k
\end{aligned}
$$

Further simplifying the above expression,

$$
E\{k\} = P_0 \left[\sum_{k=1}^{2} \frac{(3\rho)^k}{(k-1)!} + \frac{9}{2}\left(\sum_{k=0}^{\infty} k\rho^k - \rho - 2\rho^2\right)\right]
$$

$\sum_{k=0}^{\infty} k\rho^k$ can be simplified as

$$
\sum_{k=0}^{\infty} k\rho^k = \rho \sum_{k=0}^{\infty} k\rho^{k-1} = \rho \frac{\partial}{\partial\rho} \sum_{k=0}^{\infty} \rho^k = \rho \frac{\partial}{\partial\rho} \left[\frac{1}{1-\rho}\right] = \frac{\rho}{(1-\rho)^2}
$$

Hence,

$$
\begin{aligned}
E\{k\} &= P_0 \left[3\rho + 9\rho^2 + \frac{9}{2}\left(\frac{\rho}{(1-\rho)^2} - \rho - 2\rho^2\right)\right] \\
&= P_0 \left[3\rho + 9\rho^2 + \frac{9\rho}{2(1-\rho)^2} - \frac{9\rho}{2} - 9\rho^2\right] \\
&= P_0 \left[\frac{9\rho}{2(1-\rho)^2} - \frac{3\rho}{2}\right]
\end{aligned}
$$

Question 2

2. (a) (i)

The formula for the Poisson distribution of k events occurring in a time interval t is:

$$P(k \mid t, \lambda) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t)$$

where λ is the rate at which the events are occurring.

Mean of Poisson distribution:

$$\begin{aligned}
E\{k\} = \sum_{k=0}^{\infty} kP(k \mid t, \lambda) &= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k}{k!} \exp(-\lambda t) \\
&= \exp(-\lambda t) \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \\
&= \lambda t \exp(-\lambda t) \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} \\
&= \lambda t \exp(-\lambda t) \sum_{m=0}^{\infty} \frac{(\lambda t)^m}{m!} \\
&= \lambda t \exp(-\lambda t) \exp(\lambda t) \\
&= \lambda t
\end{aligned}$$

Variance of Poisson distribution:

$$\begin{aligned}
var\{k\} &= E\{k^2\} - (E\{k\})^2 \\
&= E\{k^2\} - (\lambda t)^2
\end{aligned}$$

where

$$\begin{aligned}
E\{k^2\} = \sum_{k=0}^{\infty} k^2 P(k \mid t, \lambda) &= \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k}{k!} \exp(-\lambda t) \\
&= \exp(-\lambda t) \sum_{k=1}^{\infty} \frac{k(\lambda t)^k}{(k-1)!} \\
&= \exp(-\lambda t) \left[\sum_{k=1}^{\infty} \frac{(k-1)(\lambda t)^k}{(k-1)!} + \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \right] \\
&= \exp(-\lambda t) \left[(\lambda t)^2 \sum_{k=2}^{\infty} \frac{(\lambda t)^{k-2}}{(k-2)!} + (\lambda t) \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} \right] \\
&= \exp(-\lambda t) \left[(\lambda t)^2 \exp(\lambda t) + (\lambda t) \exp(\lambda t) \right] \\
&= (\lambda t)^2 + \lambda t
\end{aligned}$$

therefore

$$\begin{aligned}
var\{k\} &= (\lambda t)^2 + \lambda t - (\lambda t)^2 \\
&= \lambda t
\end{aligned}$$

2. (a) (ii)

$$\begin{aligned} P(\text{no arrivals in time interval } T) &= P(k = 0 \mid t = T, \lambda) \\ &= \frac{(\lambda T)^0}{0!} \exp(-\lambda T) \\ &= \exp(-\lambda T) \end{aligned}$$

2. (a) (iii)

$$\begin{aligned} P(\text{at least one arrival in time interval } T) &= 1 - P(\text{no arrivals in time interval } T) \\ &= 1 - \exp(-\lambda T) \end{aligned}$$

2. (b) (i)

The balance equation is:

$$\lambda_{k-1} P_{k-1} + \mu_{k+1} P_{k+1} = \lambda_k P_k + \mu_k P_k$$

The solution of this equation is:

$$\begin{aligned} \lambda_{k-1} P_{k-1} &= \mu_k P_k \\ P_k &= \frac{\lambda}{\mu} \alpha^{k-1} P_{k-1} \end{aligned}$$

Calculating the first few terms,

$$\begin{aligned} P_1 &= \frac{\lambda}{\mu} P_0 \\ P_2 &= \frac{\lambda}{\mu} \alpha P_1 \ = \left(\frac{\lambda}{\mu}\right)^2 \alpha P_0 \\ P_3 &= \frac{\lambda}{\mu} \alpha^2 P_2 = \left(\frac{\lambda}{\mu}\right)^3 \alpha^3 P_0 \\ P_4 &= \frac{\lambda}{\mu} \alpha^3 P_3 = \left(\frac{\lambda}{\mu}\right)^4 \alpha^6 P_0 \end{aligned}$$

Hence, the general solution is:

$$P_k = \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}} P_0 \quad , \quad k \geq 0$$

In order to achieve a steady state solution for all state k, the arrival rate ($\lambda_k = \lambda a^k$) must be smaller than the service rate ($\mu_k = \mu$). If $a \geq 1$, it is possible that $\lambda a^k \geq \mu$ for some λ, μ, or k. Therefore, α must be $0 \leq \alpha < 1$.

2. (b) (ii)

Calculate P_0 using the normalisation condition:

$$\sum_{k=0}^{\infty} P_k = 1$$
$$\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}} P_0 = 1$$
$$P_0 \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}} = 1$$

Therefore,

$$P_0 = \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}}\right]^{-1}$$

$$\begin{aligned} P(\text{two or more people in the system}) &= 1 - P_0 - P_1 \\ &= 1 - P_0 - \frac{\lambda}{\mu} P_0 \\ &= 1 - P_0 \left(1 + \frac{\lambda}{\mu}\right) \\ &= 1 - \left(1 + \frac{\lambda}{\mu}\right) \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}}\right]^{-1} \end{aligned}$$

2. (b) (iii)

$$\begin{aligned} \bar{\lambda} &= \sum_{k=0}^{\infty} \lambda_k P_k \\ &= \sum_{k=0}^{\infty} \lambda \alpha^k \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}} P_0 \\ &= \lambda P_0 \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)+2k}{2}} \\ &= \lambda \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}}\right]^{-1} \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k+1)}{2}}\right] \end{aligned}$$

In **2. (b) (i)**, the value of α is restricted to $0 \leq \alpha < 1$. Consider three cases here:

Case 1: $a = 0$

$$\bar{\lambda} = \lambda \left[1 + \frac{\lambda}{\mu}\right]^{-1}$$

$\bar{\lambda} < \lambda$, thus a steady state solution exists

Case 2: $a = 1$

$$\begin{aligned}\bar{\lambda} &= \lambda \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \right]^{-1} \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \right] \\ &= \lambda\end{aligned}$$

Case 3: $a > 1$

In this case, both infinite sum $\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}}$ and $\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k+1)}{2}}$ will not converge. Therefore, it is possible that the average arrival rate ($\bar{\lambda}$) is greater than the average service rate ($\bar{\mu}$), the queue continues to grow in size and a steady state distrubution does not exist.

N.B. Since the service rate (μ_k) is constant across all states k, thus the average service rate is equal to μ.

2. (b) (iv)

$$P_0 = \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}} \right]^{-1}$$

If $\dfrac{\lambda}{\mu} < 1$, it is known that

$$\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k = \frac{1}{1 - \frac{\lambda}{\mu}}$$

and when $a = 1$,

$$\begin{aligned}\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}} &\equiv \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \\ &= \frac{1}{1 - \frac{\lambda}{\mu}}\end{aligned}$$

Therefore, when $0 \leq \alpha < 1$,

$$\begin{aligned}\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}} &< \frac{1}{1 - \frac{\lambda}{\mu}} \\ \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}} \right]^{-1} &> 1 - \frac{\lambda}{\mu}\end{aligned}$$

and thus

$$P_0 > 1 - \frac{\lambda}{\mu}$$

2. (b) (v)

For $0 \leq \alpha < 1$, by observing at the expression P_0:

$$P_0 = \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}}\right]^{-1}$$

the infinite sum is guaranteed to converge when $\frac{\lambda}{\mu} < 1$ as $k \to \infty$.

Analysis for $\frac{\lambda}{\mu} \geq 1$

- If $\frac{\lambda}{\mu} = 1$, as $k \to \infty$, $\left(\frac{\lambda}{\mu}\right)^k = 1$
- If $\frac{\lambda}{\mu} > 1$, as $k \to \infty$, $\left(\frac{\lambda}{\mu}\right)^k \to \infty$
- For $0 \leq \alpha < 1$, as $k \to \infty$, $\alpha^{\frac{k(k-1)}{2}} \to 0$

Therefore, a steady state solution does not exist for $\frac{\lambda}{\mu} > 1$ as $\infty \times 0$ is undefined. However, a steady state solution does exist for $\frac{\lambda}{\mu} = 1$ as the sum converges.

In summary, the infinite sum will converge when $\frac{\lambda}{\mu} \leq 1$. As a result, the condition $\frac{\lambda}{\mu} < 1$ (or more precisely, $\frac{\lambda}{\mu} \leq 1$) is necessary for a steady state solution to exist.