# ADER and DeC: arbitrarily high order (explicit) methods for PDEs and ODEs

IMPLICIT, IMEX

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# Outline

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## Motivation: high order accurate explicit method

Methods used to solve a hyperbolic PDE system for $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}^D$

$$\partial_t u + \nabla_{\mathbf{x}} \mathcal{F}(u) = 0. \tag{1}$$

Or ODE system for $\boldsymbol{u} : \mathbb{R}^+ \to \mathbb{R}^S$

$$\partial_t \boldsymbol{u} = F(\boldsymbol{u}). \tag{2}$$

Applications:

- Fluids/transport
- Chemical/biological processes

How?

- Arbitrarily high order accurate
-

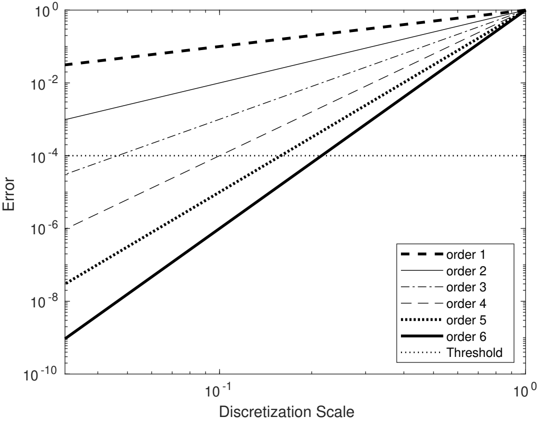
## Motivation: high order accurate explicit method

Methods used to solve a hyperbolic PDE system for $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}^D$

(1)

Or ODE system for $\boldsymbol{u}$ :

(2)

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$$\partial_t \boldsymbol{u} = F(\boldsymbol{u}). \tag{2}$$

Applications:

- Fluids/transport
- Chemical/biological processes

How?

- Arbitrarily high order accurate
- Explicit (if nonstiff problem)

# Classical time integration: Runge–Kutta

$$\boldsymbol{u}^{(1)} := \boldsymbol{u}^n, \tag{3}$$

$$\boldsymbol{u}^{(k)} := \boldsymbol{u}^n + \Delta t \sum_{s=1}^{K} a_{ks} F\left(t^n + c_s \Delta t, \boldsymbol{u}^{(s)}\right), \quad \text{for } k = 2, \dots, K, \tag{4}$$

$$\boldsymbol{u}^{n+1} := \boldsymbol{u}^n + \Delta t \sum_{s=1}^{K} b_s F\left(t^n + c_s \Delta t, \boldsymbol{u}^{(s)}\right). \tag{5}$$

## Classical time integration: Explicit Runge–Kutta

$$\boldsymbol{u}^{(k)} := \boldsymbol{u}^n + \Delta t \sum_{s=1}^{k-1} a_{ks} F\left(t^n + c_s \Delta t, \boldsymbol{u}^{(s)}\right), \quad \text{for } k = 1, \dots, K.$$

- Easy to solve
- High orders involved:
  2. ◦ Order conditions: system of many equations
  1. ◦ Stages $K \geq d$ order of accuracy (e.g. RK44, RK65)

# Classical time integration: Implicit Runge–Kutta

$$\boldsymbol{u}^{(k)} := \boldsymbol{u}^n + \Delta t \sum_{s=1}^{K} a_{ks} F\left(t^n + c_s \Delta t, \boldsymbol{u}^{(s)}\right), \quad \text{for } k = 1, \dots, K.$$

- More complicated to solve for nonlinear systems
- High orders easily done:

  B, C, D

  - Take a high order quadrature rule on $[t^n, t^{n+1}]$
  - Compute the coefficients accordingly, see Gauss–Legendre or Gauss–Lobatto polynomials
  - Order up to $d = 2K$

## ADER and DeC

Two iterative explicit arbitrarily high order accurate methods.

DG SPACE-TIME

- ADER[1] for hyperbolic PDE, after a first analytic more complicated approach.
- Deferred Correction (DeC): introduced for explicit ODE[2], extended to implicit ODE[3] and to hyperbolic PDE[4].

IMEX DeC

[1] M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. Journal of Computational Physics, 227(18):8209–8253, 2008.

[2] A. Dutt, L. Greengard, and V. Rokhlin. Spectral Deferred Correction Methods for Ordinary Differential Equations. BIT Numerical Mathematics, 40(2):241–266, 2000.

[3] M. L. Minion. Semi-implicit spectral deferred correction methods for ordinary differential equations. Commun. Math. Sci., 1(3):471–500, 09 2003.

[4] R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. Journal of Scientific Computing, 73(2):461–494, Dec 2017.

# Outline

## DeC high order time discretization: $\mathcal{L}^2$

2 OPERATORS $\mathcal{L}^1, \mathcal{L}^2$

$$u(t) = \sum_{i=0}^{M} \varphi_i(t) \cdot u^i \qquad t \in [t^n, t^{n+1}]$$

$$\varphi_i(t^j) = \delta_{ij}$$

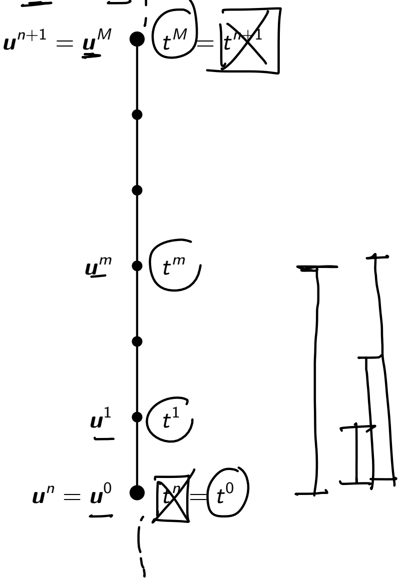
High order in time: we discretize our variable on $[t^n, t^{n+1}]$ in $M$ substeps ($\boldsymbol{u}^m$).

$$\boxed{\partial_t \boldsymbol{u} = F(\boldsymbol{u}(t)).}$$

Thanks to Picard–Lindelöf theorem, we can rewrite

$$\left( \int_{t^0}^{t^m} \partial_t u = u^m - u^0 \right) \qquad \boldsymbol{u}^m = \boldsymbol{u}^0 + \boxed{\int_{t^0}^{t^m} F(\boldsymbol{u}(t))dt.}$$

and if we want to reach order $r+1$ we need $M = r$.

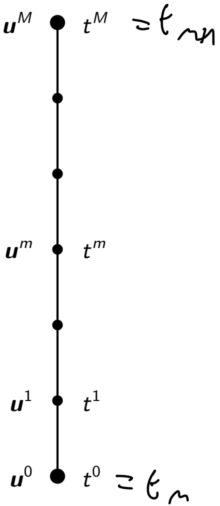


## DeC high order time discretization: $\mathcal{L}^2$

$$\boxed{F(u(t)) \approx \sum_{i=0}^{M} \varphi_i F(u^i)}$$

More precisely, for each $\sigma$ we want to solve $\mathcal{L}^2(\boldsymbol{u}^{n,0},\dots,\boldsymbol{u}^{n,M})=0$, where

$$\mathcal{L}^2(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M)=\begin{pmatrix} \boldsymbol{u}^M-\boldsymbol{u}^0+\sum_{r=0}^M\int_{t^0}^{t^M}\underline{F(\boldsymbol{u}^r)}\varphi_r(s)\mathrm{d}s \\ \vdots \\ \boldsymbol{u}^1-\boldsymbol{u}^0+\sum_{r=0}^M\int_{t^0}^{t^1}F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \end{pmatrix}$$




- $\mathcal{L}^2=0$ is a system of $M\times S$ coupled (non)linear equations
- $\mathcal{L}^2$ is an implicit method (collocation method: Gauss, LobattoIIIA)
- Not easy to solve directly $\mathcal{L}^2(\underline{\boldsymbol{u}}^*)=0$
- High order (equispaced $M+1$, Gauss–Lobatto $2M$), depending on points distribution

# DeC high order time discretization: $\mathcal{L}^2$

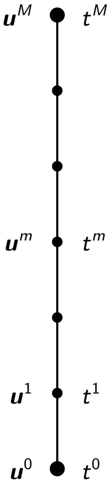
$\Delta t = t^{n+1} - t^n$

$u \in \mathbb{R}^S$

$\theta^m_r := \frac{1}{\Delta t}\int_{t^0}^{t^m} \varphi_r(s)\,ds$

More precisely, for each $\sigma$ we want to solve $\mathcal{L}^2(\boldsymbol{u}^{n,0}, \dots, \boldsymbol{u}^{n,M}) = 0$, where

$$\mathcal{L}^2(\boldsymbol{u}^0, \dots, \boldsymbol{u}^M) = \begin{pmatrix} \boldsymbol{u}^M - \boldsymbol{u}^0 + \Delta t \sum_{r=0}^M \theta_r^M F(\boldsymbol{u}^r) \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 + \Delta t \sum_{r=0}^M \theta_r^1 F(\boldsymbol{u}^r) \end{pmatrix}$$

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NEWTON-COTES FORMULAS

$U^{n+1} = U^M$

## DeC low order time discretization: $\boxed{\mathcal{L}^1}$

(EXPLICIT)

Instead of solving the implicit system directly (difficult), we introduce a first order scheme $\mathcal{L}^1(\boldsymbol{u}^{n,0},\dots,\boldsymbol{u}^{n,M})$:

$$\mathcal{L}^1(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M)=\begin{pmatrix}\boldsymbol{u}^M-\boldsymbol{u}^0+\Delta t\beta^M F(\boldsymbol{u}^0)\\ \vdots \\ \boldsymbol{u}^1-\boldsymbol{u}^0+\Delta t\beta^1 F(\boldsymbol{u}^0)\end{pmatrix}$$

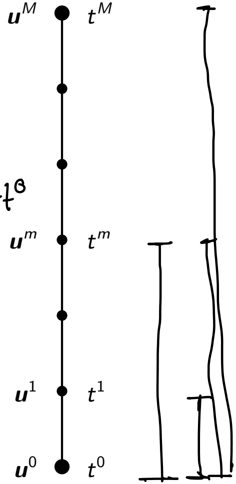
$$\int_{t^0}^{t^m} 1\cdot dt = t^m - t^0$$

- First order approximation
- Explicit Euler
- Easy to solve $\mathcal{L}^1(\underline{\boldsymbol{u}})=0$

$$u^m - u^0 \neq \int_{t^0}^{t^m} \underbrace{F(u(t))}\, dt \approx F(u^0)$$

EXPLICIT

$$\beta^m := \frac{t^m - t^0}{\Delta t}$$

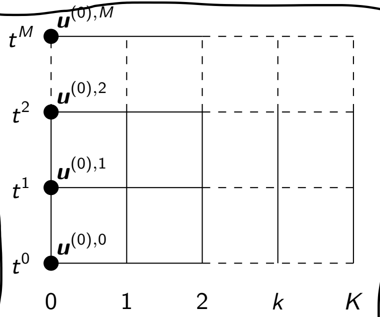


# Deferred Correction[5]

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

- $\mathcal{L}^1(\underline{\boldsymbol{u}}) = 0$, first order accuracy, easily invertible.
- $\boxed{\mathcal{L}^2(\underline{\boldsymbol{u}}) = 0,}$ high order $M+1$. $(\geq n+1)$

$$\boldsymbol{u}^{0,(k)} := \boldsymbol{u}(t^n), \quad k = 0, \dots, K,$$
$$\boldsymbol{u}^{m,(0)} := \boldsymbol{u}(t^n), \quad m = 1, \dots, M$$
$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}) \text{ with } k = 1, \dots, K.$$




### Theorem (Convergence DeC)

- $\mathcal{L}^2(\underline{\boldsymbol{u}}^*) = 0$
- *If $\mathcal{L}^1$ coercive with constant $C_1$*
- *If $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz with constant $C_2\Delta t$*

*Then $\|\underline{\boldsymbol{u}}^{(K)} - \underline{\boldsymbol{u}}^*\| \leq C(\Delta t)^K$*

[5] A. Dutt, L. Greengard, and V. Rokhlin. BIT Numerical Mathematics, 40(2):241–266, 2000.

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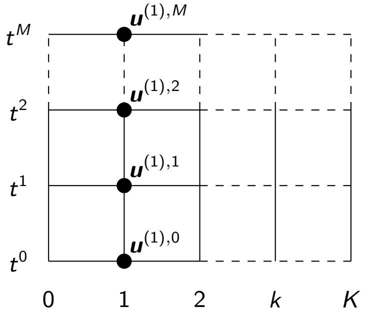


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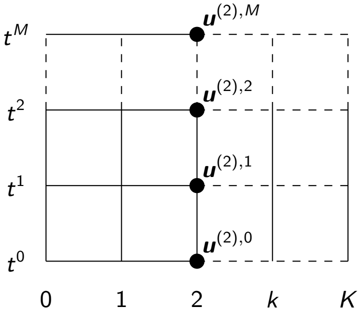


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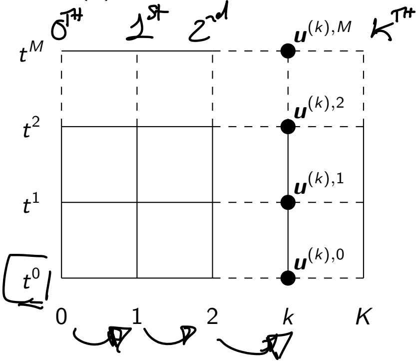


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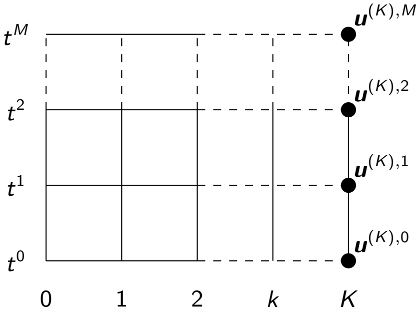


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 $U^{m,(0)} = U(t^m)$ $\quad U^{(K),m} = u(t^m)$ $\quad \mathcal{L}^1(u^{(K)}) = \mathcal{L}^1(u^{(K-1)}) - \mathcal{L}^2(u^{(K-1)})$

**Proof.**

Let $f^*$ be the solution of $\boxed{\mathcal{L}^2(\underline{u}^*) = 0.}$ We know that $\mathcal{L}^1(\underline{u}^*) = \mathcal{L}^1(\underline{u}^*) - \mathcal{L}^2(\underline{u}^*)$, so that

(C) $\mathcal{L}^1$ COERCIVE $\quad \|\mathcal{L}^1(\underline{u}) - \mathcal{L}^1(\underline{v})\| \geq C_1 \|\underline{u} - \underline{v}\|$

(L) $\mathcal{L}^1 - \mathcal{L}^2$ LIPSCHITZ $\quad \|\mathcal{L}^1(\underline{u}) - \mathcal{L}^2(\underline{u}) - (\mathcal{L}^1(\underline{v}) - \mathcal{L}^2(\underline{v}))\| \leq \underline{C_2 \Delta t} \|\underline{u} - \underline{v}\|$

$$\|\underline{u}^{(K)} - \underline{u}^*\| \overset{(C)}{\leq} \frac{1}{C_1} \|\mathcal{L}^1(\underline{u}^{(K)}) - \mathcal{L}^1(\underline{u}^*)\| \overset{DEC}{=} \frac{1}{C_1} \|\mathcal{L}^1(\underline{u}^{(K-1)}) - \mathcal{L}^2(\underline{u}^{(K-1)}) - \mathcal{L}^1(\underline{u}^*) + \underbrace{\mathcal{L}^2(\underline{u}^*)}_{=0}\|$$

$$\overset{(L)}{\leq} \frac{C_2 \Delta t}{C_1} \|\underline{u}^{(K-1)} - \underline{u}^*\| \leq \left(\frac{C_2 \Delta t}{C_1}\right)^K \|\underline{u}^{(0)} - \underline{u}^*\| \quad \boxtimes$$

□

**Proof.**

Let $f^*$ be the solution of $\mathcal{L}^2(\underline{\boldsymbol{u}}^*)=0$. We know that $\mathcal{L}^1(\underline{\boldsymbol{u}}^*)=\mathcal{L}^1(\underline{\boldsymbol{u}}^*)-\mathcal{L}^2(\underline{\boldsymbol{u}}^*)$, so that

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k+1)})-\mathcal{L}^1(\underline{\boldsymbol{u}}^*)=\left(\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)})-\mathcal{L}^2(\underline{\boldsymbol{u}}^{(k)})\right)-\left(\mathcal{L}^1(\underline{\boldsymbol{u}}^*)-\mathcal{L}^2(\underline{\boldsymbol{u}}^*)\right)$$

$$\begin{aligned} C_1||\underline{\boldsymbol{u}}^{(k+1)}-\underline{\boldsymbol{u}}^*|| \leq & ||\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k+1)})-\mathcal{L}^1(\underline{\boldsymbol{u}}^*)||= \\ = & ||\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)})-\mathcal{L}^2(\underline{\boldsymbol{u}}^{(k)})-(\mathcal{L}^1(\underline{\boldsymbol{u}}^*)-\mathcal{L}^2(\underline{\boldsymbol{u}}^*))|| \leq \\ \leq & C_2\Delta||\underline{\boldsymbol{u}}^{(k)}-\underline{\boldsymbol{u}}^*||. \end{aligned}$$

$$||\underline{\boldsymbol{u}}^{(k+1)}-\underline{\boldsymbol{u}}^*|| \leq \left(\frac{C_2}{C_1}\Delta\right)||\underline{\boldsymbol{u}}^{(k)}-\underline{\boldsymbol{u}}^*|| \leq \left(\frac{C_2}{C_1}\Delta\right)^{k+1}||\underline{\boldsymbol{u}}^{(0)}-\underline{\boldsymbol{u}}^*||.$$

After $K$ iteration we have an error at most of $\left(\frac{C_2}{C_1}\Delta\right)^K||\underline{\boldsymbol{u}}^{(0)}-\underline{\boldsymbol{u}}^*||$. □

## DeC: Coercivity and Lipschitz continuity (sketch)

C $\|\mathcal{L}^1(\underline{u}) - \mathcal{L}^1(\underline{v})\| \geq C_1 \|\underline{u} - \underline{v}\|$

$$\left\| \begin{matrix} u^n - \cancel{u^0} + \Delta t \cancel{\beta^n F(u^0)} & - (v^n - \cancel{v^0} + \Delta t \cancel{\beta^n F(v_0)}) \\ \vdots & \\ u^1 - \cancel{u^0} + \Delta t \cancel{\beta^1 F(u^0)} & - (v^1 - \cancel{v^0} + \Delta t \cancel{\beta^1 F(v_0)}) \end{matrix} \right\| = \|\underline{u} - \underline{v}\| \qquad C_1 = 1$$

$\|F(u) - F(v)\|_2 \leq \underline{C} \|u - v\|_2$

$\underline{F(u)} \approx u \partial_x u$

$\underline{u^0, v^0}$ $\qquad v^0 = u^0 = u(t^n)$

L $\mathcal{L}^1(u) = 0$ 1st order approx of $u^{ex}$ $\qquad \mathcal{L}^2(u) = 0$ p-th order approx of $u^{ex}$

$\mathcal{L}^1(u) \approx \mathcal{L}^1(u^{ex}) + O(\Delta t)$ $\qquad \mathcal{L}^2(u) = \mathcal{L}^2(u^{ex}) + O(\Delta t^p)$

$F(u) = \partial_x \frac{u^2}{2}$

$\|\mathcal{L}^1(u) - \mathcal{L}^2(u)\| \lesssim \|\mathcal{L}^1(u^{ex}) + O(\Delta t) - \mathcal{L}^2(u^{ex}) + O(\Delta t^p)\| \leq \underline{O(\Delta t)}$

$\shortparallel$

$\|\cancel{v^n} - \cancel{u^0} - \Delta t \underline{\beta^n} F(v^0) - \cancel{v^n} + \cancel{v^0} + \Delta t \sum \theta_n^m F(v^m)\| = \Delta t \| \underbrace{\sum \theta_n^m} (F(v^m) - F(v^0))\|$

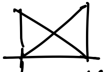
$\sum \theta_n^m = \beta^n$

$\leq c \cdot \Delta t \cdot \|\underline{F(v) - F(v^0)}\| \leq C_2 \Delta t \|v - v^0\|$

# DeC: Second order example

$$\begin{array}{lllll} t^1 & U^1 & U^{1,(0)} & U^{1,(1)} & U^{1,(2)} \\ t^0 & U^0 & U^{0,(0)} & U^{0,(1)} & U^{0,(2)} = U(t^n) \\ & k= & 0 & 1 & 2 \end{array}$$

$u^{1,(0)} = u(t^n)$ $\quad \theta^1_r := \int_{t^0}^{t^1} \varphi_r \, dt \, \frac{1}{\Delta t}$



$t^0$ $\quad$ $t^1$

$\theta^1_r = \left[\frac{1}{2} \ \ \frac{1}{2}\right] \quad r=0,1$

$k=1 \quad \mathcal{L}^1(u^{(1)}) = \mathcal{L}^1(u^{(0)}) - \mathcal{L}^2(u^{(0)})$

$$u^{(1),1} - \cancel{U^0} + \cancel{\Delta t \beta^1 F(U^0)} = \cancel{u^{(0),1}} - \cancel{U^0} + \cancel{\Delta t \beta^1 F(U^0)} - \left[\cancel{u^{(0),1}} - U^0 + \frac{\Delta t}{2}\left(F(U^{(0),0}) + F(U^{(0),1})\right)\right]$$

$$= U^0 - \Delta t F(U^0)$$

$$U^{(2),1} - \cancel{U^0} + \cancel{\Delta t \beta^1 F(U^0)} = \cancel{U^{(1),1}} - \cancel{U^0} + \cancel{\Delta t \beta^1 F(U^0)} - \left[\cancel{U^{(1),1}} - U^0 + \frac{\Delta t}{2}\left[F(U^{(1),0}) + F(U^{(1),1})\right]\right]$$

$$= U^0 - \frac{\Delta t}{2}\left[F(U^0) + F(U^{(1),1})\right]$$

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

# DeC: Second order example

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## Simplification of DeC for ODE

In practice

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

For $m = 1, \dots, M$

$$\boldsymbol{u}^{(k),m} - \cancel{\boldsymbol{u}^0} - \cancel{\beta^m \Delta t F(\boldsymbol{u}^0)} - \cancel{\boldsymbol{u}^{(k-1),m}} + \cancel{\boldsymbol{u}^0} + \cancel{\beta^m \Delta t F(\boldsymbol{u}^0)}$$

$$+ \cancel{\boldsymbol{u}^{(k-1),m}} - \boldsymbol{u}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{u}^{(k-1),r}) = 0$$

## Simplification of DeC for ODE

In practice

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

For $m = 1, \dots, M$

$$\boldsymbol{u}^{(k),m} \cancel{- \boldsymbol{u}^0 - \beta^m \Delta t F(\boldsymbol{u}^0)} - \boldsymbol{u}^{(k-1),m} \cancel{+ \boldsymbol{u}^0 + \beta^m \Delta t F(\boldsymbol{u}^0)}$$

$$+ \boldsymbol{u}^{(k-1),m} - \boldsymbol{u}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{u}^{(k-1),r}) = 0$$

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For $m = 1, \dots, M$

$$\boldsymbol{u}^{(k),m} - \cancel{\boldsymbol{u}^0 - \beta^m \Delta t F(\boldsymbol{u}^0)} - \cancel{\boldsymbol{u}^{(k-1),m}} + \cancel{\boldsymbol{u}^0 + \beta^m \Delta t F(\boldsymbol{u}^0)}$$

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$$\boldsymbol{u}^{(k),m} - \boldsymbol{u}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{u}^{(k-1),r}) = 0.$$

$$\underline{u}^{(k)} - u^0 \underline{1} - \Delta t\, \underline{\underline{\Theta}} \cdot F(\underline{u}^{(k-1)}) = 0$$

## DeC and residual distribution

Deferred Correction + Residual distribution

$\mathcal{L}^1$ LUMPED MASS MATRIX

$D_{ii} = \sum_j M_{ij}$

- Residual distribution (FV $\Rightarrow$ FE) $\Rightarrow$ High order in space
- Prediction/correction/iterations $\Rightarrow$ High order in time
- Subtimesteps $\Rightarrow$ High order in time

$$U_\xi^{m,(k+1)} = U_\xi^{m,(k)} - |C_p|^{-1} \sum_{\mathrm{E}|\xi\in\mathrm{E}} \left( \int_{\mathrm{E}} \Phi_\xi \left(U^{m,(k)} - U^{n,0}\right) \mathrm{d}\mathbf{x} + \Delta t \sum_{r=0}^{M} \theta_r^m \mathcal{R}_\xi^{\mathrm{E}}(U^{r,(k)}) \right),$$

with

$$\sum_{\xi\in\mathrm{E}} \mathcal{R}_\xi^{\mathrm{E}}(u) = \int_{\mathrm{E}} \nabla_{\mathbf{x}} F(u) \mathrm{d}\mathbf{x}.$$

- The $\mathcal{L}^2$ operator contains also the complications of the spatial discretization (e.g. mass matrix)
- $\mathcal{L}^1$ operator further simplified up to a first order approximation (e.g. **mass lumping**)

## $\mathcal{L}^1$ with mass lumping

$$\mathcal{L}^1 = D(u^m - u^0) + \Delta t\, \beta^m \cdot F(u^0)$$

$$\mathcal{L}^2 = M(u^m - u^0) + \Delta t \sum \theta_r^m F(u^r)$$

$$\mathcal{L}^1(u^{(k)}) - \mathcal{L}^1(u^{(k-1)}) + \mathcal{L}^2(u^{(k-1)}) =$$

$$D(u^{m,(k)} - u^0) - \underline{D(u^{m,(k-1)}} - u^0) + \underline{M(u^{m,(k-1)}} - u^0) + \Delta t \sum \theta_r^m F(u^r) = 0$$

# Implicit simple DeC (Rosenbrock)

Define $\mathcal{L}^1$ as

$$\mathcal{L}^1(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) = \begin{pmatrix} \boldsymbol{u}^M - \boldsymbol{u}^0 - \Delta t \beta^M \underbrace{F(\boldsymbol{u}^0)}_{F(U^M)} \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \Delta t \beta^1 \underbrace{F(\boldsymbol{u}^0)}_{F(U^1)} \end{pmatrix}$$

IMPLICIT DEC

$\mathcal{L}^1$ IMPLICIT EULER

$$\mathcal{L}^{1,m}(U^{(k)}) - \mathcal{L}^{1,m}(U^{(k-1)}) + \mathcal{L}^{2,m}(U^{(k-1)}) = 0$$

$$= U^{(k),m} - U^0 + \Delta t \beta^m F(U^{(k),m}) - \cancel{U^{(k-1),m}} \cancel{+U^0} - \Delta t \beta^m F(U^{(k-1),m}) + \cancel{U^{(k-1),m}} \cancel{-U^0} + \Delta t \sum_r \theta_r^m F(U^{(k-1),r})$$

UNCOUPLING THE SUBTIMESTEPS WRT $\mathcal{L}^2 = 0$ FULLY IMPLICIT

BIRK

# Implicit simple DeC (Rosenbrock)

Define $\mathcal{L}^1$ as

JACOBIAN

$$
\begin{aligned}
\mathcal{L}^1(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) &= \begin{pmatrix} \boldsymbol{u}^M - \boldsymbol{u}^0 - \Delta t \beta^M \left(F(\boldsymbol{u}^0) + \partial_{\boldsymbol{u}} F(\boldsymbol{u}^0)(\boxed{\boldsymbol{u}^M} - \boldsymbol{u}^0)\right) \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \Delta t \beta^1 \left(F(\boldsymbol{u}^0) + \partial_{\boldsymbol{u}} F(\boldsymbol{u}^0)(\boldsymbol{u}^1 - \boldsymbol{u}^0)\right) \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{u}^M - \boldsymbol{u}^0 - \Delta t \beta^M \partial_{\boldsymbol{u}} F(\boldsymbol{u}^0)\boldsymbol{u}^M \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \Delta t \beta^1 \partial_{\boldsymbol{u}} F(\boldsymbol{u}^0)\boldsymbol{u}^1 \end{pmatrix}
\end{aligned}
$$

# Implicit simple DeC (Rosenbrock)

$$\mathcal{L}^{1,m}(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) = \boldsymbol{u}^m - \boldsymbol{u}^0 - \Delta t \beta^m \partial_{\boldsymbol{u}} F(\boldsymbol{u}^0)\boldsymbol{u}^m$$

$$\mathcal{L}^{2,m}(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) = \boldsymbol{u}^m - \boldsymbol{u}^0 - \Delta t \sum_r \theta_r^m F(\boldsymbol{u}^r)$$

$$\mathcal{L}^{1,m}(\underline{U}^{(K)}) - \mathcal{L}^{1,m}(\underline{U}^{(K-1)}) + \mathcal{L}^{2,m}(\underline{U}^{(K-1)}) =$$

$$= U^{m,(K)} - \cancel{U^0} + \Delta t\, \beta^m \partial_u F(U^0) U^{m,(K)} - \cancel{U^{m,(K-1)}} + \cancel{U^0} - \Delta t\, \beta^m \partial_u F(U^0) U^{m,(K-1)}$$

$$+ \cancel{U^{m,(K-1)}} - U^0 + \Delta t \sum_r \Theta_r^m F(U^{r,(K-1)})$$

$$= \underbrace{\left[ I + \Delta t\, \beta^m \underline{\partial_u F(U^0)} \right] U^{m,(K)}}_{\text{LINEAR IMPLICIT TERM}} - \Delta t\, \beta^m \partial_u F(U^0) U^{m,(K-1)} - U^0 + \Delta t \sum \Theta_r^m F(U^{r,(K-1)}) = 0$$

# Implicit simple DeC (Rosenbrock)

# DeC as RK

$$\boldsymbol{u}^{(k),m} - \boldsymbol{u}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{u}^{(k-1),r}) = 0$$

## DeC as RK

$$u^{m,(k)} = u^0 + \Delta t \sum_{r=0}^{M} \theta_r^m F(u^{r,(k-1)})$$

$$u^{0,(k)} = u(t^n)$$

$$+ \Delta t\, \theta_0^m F(u^0) + \Delta t \sum_{r=1}^{M} \theta_r^m F(u^{r,(k-1)})$$

$$\underline{u} = \begin{pmatrix} u^M \\ \vdots \\ u^1 \end{pmatrix}$$

$$\underline{u}^{(k)} = u^0 \cdot \underline{1} + \Delta t\, \underline{\theta}_0 \cdot F(u^0) + \Delta t\, \underline{\underline{\tilde{\theta}}} F(\underline{u}^{(k-1)})$$

$$(\tilde{\theta})_{m,r} = \theta_r^m \qquad m = 1, \dots, M \quad r = 1, \dots, M$$

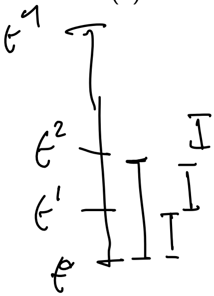
# DeC as RK

We can write DeC as RK defining $\underline{\theta}_0 = \{\theta_0^m\}_{m=1}^M$, $\underline{\theta}^M = \theta_r^M$ with $r \in 1, \dots, M$, denoting the vector $\underline{\theta}_r^{M,T} = (\theta_1^M, \dots, \theta_M^M)$. The Butcher tableau for an arbitrarily high order DeC approach is given by:

$$
\begin{array}{c|c|cccccccc}
 & & U^0 & U^{(1)} & & & & & & \\
U^0 & 0 & 0 & & & & & & & \\
U^{(1)} & \underline{\beta} & \underline{\beta} & & & & & & & \\
U^{(2)} & \underline{\beta} & \underline{\theta}_0 & \underline{\underline{\tilde{\theta}}} & & & & & & \\
U^{(3)} & \vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} & & & & & \\
 & \vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} & & & & \\
 & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & & \\
U^{(K-1)} & \underline{\beta} & \underline{\theta}_0 & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} & \\
\hline
U^{(K)} & & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\theta}^{M,T}
\end{array}
\qquad (6)
$$

$(K-1)\cdot M + 1$

$$U_{n+1} = U^{(K),M} = U^0 + \Delta t\, \theta_0^M F(U^0) + \Delta t \sum_r \underbrace{\theta_r^M}\, F(U^{r,(K-1)})$$



IMPLICIT DEC $\mathcal{L}$ IMPLICIT EULER $F(U$

$$\mathcal{L}^{1,m}(\underline{U}^{(k)}) - \mathcal{L}^{1,m}(\underline{U}^{(k-1)}) + \mathcal{L}^{2,m}(U^{(k-1)}) = 0$$

$$= \underline{U}^{(k),m} - U^0 + \underline{\Delta t\, \beta^m F(\underline{U}^{(k),m})} - \cancel{U^{(k-1),m}} \cancel{+U^0} - \underline{\Delta t\, \beta^m F(U^{(k-1),m})} + \cancel{U^{(k-1),m}} \cancel{-U^0} + \underline{\Delta t \sum_r \Theta^m_r F(U^{(k-1),r})}$$

UNCOUPLING THE SUBTIMESTEPS WRT $\mathcal{L}^2 = 0$ FULLY IMPLICIT

BIRK

$$B: B_{ii} = \beta^i$$

$$\begin{array}{c|ccc}
0 & 0 & & \\
\beta & \underline{\theta} & \underline{B} & \\
\beta & \underline{\theta_0} & \underline{\theta}\text{-}\underline{B} & B \\
\vdots & \theta_c & & B \\
\vdots & & & \theta\text{-}B \quad B \\
\beta & \theta_e & & \theta\text{-}B \quad B \\
\vdots & & & \\
\hline
\end{array}$$

DIRK

## Stability of (explicit) DeC

Idea: study the RK version!

$$u' = \lambda u \qquad \Re(\lambda) < 0. \tag{7}$$

$$u_{n+1} = R(\lambda \Delta t) u_n, \quad R(z) = 1 + z b^T (I - zA)^{-1} \mathbf{1}, \qquad z = \lambda \Delta t \tag{8}$$

Goal: find $z \in \mathbb{C}$ such that $|R(z)| < 1$.
Recall: stability function for explicit RK methods is a polynomial, indeed the inverse of $(I - zA)$ can be written in Taylor expansion as

$$(I - zA)^{-1} = \sum_{r=0}^{\infty} z^r A^s = I + zA + z^2 A^2 + \dots, \tag{9}$$

and, since $A$ is strictly lower triangular, it is nilpotent. Hence, $R(z)$ is a polynomial in $z$ with degree at most equal to $S$.

## Stability of (explicit) DeC

**Theorem**

*If the RK method is of order $P$, then*

$$R(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^P}{P!} + O(z^{P+1}). \tag{10}$$

The first $P+1$ terms of the stability functions $R(\cdot)$ for explicit DeCs of order $P$ are known.

**Theorem**

*The stability function of any explicit DeC of order $P$ (with $P$ iterations) is*

$$R(z) = \sum_{r=0}^{P} \frac{z^r}{r!} = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^P}{P!} \tag{11}$$

*and does not depend on the distribution of the subtimenodes.*

## Proof (1/3)

$$A = \left.\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \star & 0 & 0 & \dots & 0 & 0 \\ \star & \star & 0 & \dots & 0 & 0 \\ \star & 0 & \star & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & 0 & 0 & \cdots & \star & 0 \end{pmatrix}\right\} P$$

Block structure of the matrix $A$

$\star$ are some non-zero block matrices and the 0 are some zero block matrices.

The number of blocks in each line and row of these matrices is $P$, the order of the scheme.

$$\begin{pmatrix} 0 & & & & \\ \star & 0 & & & \\ & \star & 0 & & \\ & & \star & 0 & \\ & & & \star & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & & & & \\ 0 & 0 & & & \\ \star & 0 & 0 & & \\ & \star & 0 & 0 & \\ & & \star & 0 & 0 \end{pmatrix}$$

$$A^P = \underline{0}$$

## Stability of (explicit) DeC

### Proof (2/3)

By induction, $A^k$ has zeros in the upper triangular part, in the main block diagonal, and in all the $k-1$ block diagonals below the main diagonal, i.e.,

$$(A^k)_{i,j} = 0 \quad \text{, if } i < j+k,$$

where the indexes here refer to the blocks. Indeed, it is true that $A_{i,j}=0$ if $i<j+1$. Now, let us consider the entry $(A^{k+1})_{i,j}$ with $i<j+k+1$, i.e., $i-k<j+1$. It is defined as

$$(A^{k+1})_{i,j} = \sum_w (A^k)_{i,w} A_{w,j}. \tag{12}$$

Now, we can prove that all the terms of the sum are 0. Let $w<j+1$, then $A_{w,j}=0$ because of the structure of $A$; while, if $w \geq j+1 > i-k$, we have that $i<w+k$, so $(A^k)_{i,w}=0$ by induction.

## Stability of (explicit) DeC

$R(z) = 1 + z b^T (I - zA)^{-1} 1$

### Proof (3/3)

In particular, this means that $A^P = \underline{\underline{0}}$, because $i$ is always smaller than $j + P$ as $P$ is the number of the block matrices that we have. Hence,

$$\underline{(I - zA)^{-1}} = \sum_{r=0}^{\infty} z^r A^s = \sum_{r=0}^{P-1} z^r A^s = I + zA + z^2 A^2 + \dots + z^{P-1} A^{P-1}. \tag{13}$$

Plugging this result into $R(z) = 1 + zb^T (I - zA)^{-1}\mathbf{1}$, the stability function $R(z)$ is a polynomial of degree $P$, the order of the scheme. All terms of order lower or equal to $P$ must agree with the expansion of the exponential function, so it must be

$$R(z) = \sum_{r=0}^{P} \frac{z^r}{r!} = \underline{1 + z + \frac{z^2}{2!} + \dots + \frac{z^P}{P!}}. \tag{14}$$

Note: no assumption on the distribution of the subtimenodes.

# CODE

- Choice of iterations ($P$) and order
- Choice of point distributions $t^0, \dots, t^M$
- Computation of $\theta$
- Loop for timesteps
- Loop for correction
- Loop for subtimesteps

# Outline

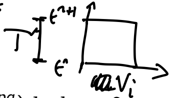
## ADER ARBITRARY DERIVATIVE

- Cauchy–Kovalevskaya theorem
- Modern automatic version
- Space/time DG
- Prediction/Correction
- Fixed-point iteration process

Modern approach is DG in space time for hyperbolic problem

$$\partial_t u(x,t) + \nabla_{\mathbf{x}} \cdot F(u(x,t)) = 0,\ x \in \Omega \subset \mathbb{R}^d,\ t>0. \quad (15)$$

$$\theta_{rs}(x,t) = \varphi_r(x)\cdot\psi_s(t)$$




Prediction: iterative procedure

$$\int_{T^n\times V_i} \theta_{rs}(x,t)\partial_t \sum_{pq}\theta_{pq}(x,t) z^{pq}\mathrm{d}x\mathrm{d}t + \int_{T^n\times V_i} \theta_{rs}(x,t)\nabla_{\mathbf{x}}\cdot F(\theta_{pq}(x,t)z^{pq})\mathrm{d}x\mathrm{d}t = 0.$$

Correction step: communication between cells

$$\int_{V_i} \Phi_r \left(u(t^{n+1}) - u(t^n)\right)\mathrm{d}x + \int_{T^n\times\partial V_i} \Phi_r(x)\mathcal{G}(z^-,z^+)\cdot\mathbf{n}\,\mathrm{d}S\,\mathrm{d}t - \int_{T^n\times V_i} \nabla_{\mathbf{x}}\Phi_r\cdot F(z)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

## ADER: space-time discretization

Defining $\theta_{rs}(x,t) = \Phi_r(x)\phi_s(t)$ basis functions in space and time

$$\int_{T^n\times V_i} \theta_{rs}(x,t)\partial_t\theta_{pq}(x,t)u^{pq}\mathrm{d}x\mathrm{d}t + \int_{T^n\times V_i} \theta_{rs}(x,t)\nabla\cdot F(\theta_{pq}(x,t)u^{pq})\mathrm{d}x\mathrm{d}t = 0. \tag{16}$$

# ADER: space-time discretization

Defining $\theta_{rs}(x,t) = \Phi_r(x)\phi_s(t)$ basis functions in space and time

$$\overbrace{\int_{T^n \times V_i}} \overbrace{\theta_{rs}(x,t)\partial_t \theta_{pq}(x,t)} u^{pq} \overbrace{\mathrm{d}x\mathrm{d}t} + \int_{T^n \times V_i} \theta_{rs}(x,t) \nabla \cdot F(\theta_{pq}(x,t)u^{pq})\mathrm{d}x\mathrm{d}t = 0. \tag{16}$$

$M_{rspq}$

This leads to

$$\sum_{pq} \underline{\underline{\underline{\mathrm{M}}}}_{rspq} u^{pq} = \underline{\underline{r}}(\underline{\underline{\mathbf{u}}})_{rs}, \qquad \forall r,s \tag{17}$$

solved with fixed point iteration method.

$+$ Correction step where cells communication is allowed (derived from (16)).

## ADER: $\boxed{\text{time}}$ integration method

LAGRANGIAN BASIS F. $\phi_m$

$\boxed{\text{Simplify!}}$ Take $\boldsymbol{u}(t) = \sum_{m=0}^{M} \phi_m(t)\boldsymbol{u}^m = \underline{\phi}(t)^T \underline{\boldsymbol{u}}$

$\partial_t u = F(u)$

$$\int_{T^n} \psi(t)\partial_t \boldsymbol{u}(t)dt - \int_{T^n} \psi(t)F(\boldsymbol{u}(t))dt = 0, \quad \forall \psi : T^n = [t^n, t^{n+1}] \to \mathbb{R}.$$

$$\to \mathcal{L}^2(\underline{\boldsymbol{u}}) := \int_{T^n} \underline{\phi}(t)\partial_t \underline{\phi}(t)^T \underline{\boldsymbol{u}}dt - \int_{T^n} \underline{\phi}(t)F(\underline{\phi}(t)^T \underline{\boldsymbol{u}})dt = 0$$

$$\underline{\phi}(t) = (\phi_0(t), \dots, \phi_M(t))^T$$

Quadrature...

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \boxed{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - \underline{r}(\underline{\boldsymbol{u}}) = 0} \iff \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} = \underline{r}(\underline{\boldsymbol{u}}). \tag{18}$$

Nonlinear system of $M \times S$ equations

# ADER: Mass matrix

What goes into the mass matrix? Use of the integration by parts

$U(t) = \sum \phi_n(t)\, U^n$

$\underbrace{\sum \phi_n(t^n)\, U^n}_{\boxed{U^n}}$

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \int_{T^n} \underline{\phi}(t)\partial_t\underline{\phi}(t)^T\underline{\boldsymbol{u}}dt + \int_{T^n}\underline{\phi}(t)F(\underline{\phi}(t)^T\underline{\boldsymbol{u}})dt =$$

$\underline{\phi} = \begin{pmatrix} \phi_0 \\ \vdots \\ \phi_M \end{pmatrix}$

$$\underline{\phi}(t^{n+1})\underline{\phi}(t^{n+1})^T\underline{\boldsymbol{u}} - \underline{\phi}(t^n)\boxed{\underline{\boldsymbol{u}}^n} - \int_{T^n}\partial_t\underline{\phi}(t)\underline{\phi}(t)^T\underline{\boldsymbol{u}} + \int_{T^n}\underline{\phi}(t)F(\underline{\phi}(t)^T\underline{\boldsymbol{u}})dt$$

$$\underline{\underline{\mathrm{M}}} = \underline{\phi}(t^{n+1})\underline{\phi}(t^{n+1})^T - \int_{T^n}\partial_t\underline{\phi}(t)\underline{\phi}(t)^T$$

$$\underline{r}(\underline{\boldsymbol{u}}) = \underline{\phi}(t^n)\underline{\boldsymbol{u}}^n + \int_{T^n}\underline{\phi}(t)F(\underline{\phi}(t)^T\underline{\boldsymbol{u}})dt$$

$$\boxed{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} = \underline{r}(\underline{\boldsymbol{u}})}$$

# ADER: Fixed point iteration

Iterative procedure to solve the problem for each time step

$\underline{\underline{M}}\,\underline{U} = \underline{r}(\underline{u})$

$$\underline{\boldsymbol{u}}^{(k)} = \underline{\underline{\mathrm{M}}}^{-1}\underline{r}(\underline{\boldsymbol{u}}^{(k-1)}), \quad k = 1, \dots, \text{convergence} \tag{19}$$

with $\underline{\boldsymbol{u}}^{(0)} = \boldsymbol{u}(t^n)$.

$\underline{U} = \underline{\underline{M}}^{-1}\underline{r}(\underline{u})$

Reconstruction step

$$\boldsymbol{u}(t^{n+1}) = \boldsymbol{u}(t^n) - \int_{T^n} F(\boldsymbol{u}^{(K)}(t))dt.$$

$$\boxed{\underline{U}^{(k)} = \underline{\underline{M}}^{-1}\underline{r}(\underline{u}^{(k-1)})}$$

$\Updownarrow$

$u(t^{n+1}) = \sum_r \phi_r(t^{n+1}) \cdot U^r$

- Convergence?
- How many steps $K$?
- Accuracy $\mathcal{L}^2$?

ADER as a DeC $\Rightarrow$

ADER TRENTO

# ADER 2nd order

Example with 2 Gauss Legendre points, Lagrange polynomials and 2 iterations
Let us consider the timestep interval $[t^n, t^{n+1}]$, rescaled to $[0,1]$.
Gauss-Legendre points quadrature and interpolation (in the interval $[0,1]$)

$$\underline{t}_q = (t_q^0, t_q^1) = (t^0, t^1) = \left( \frac{\sqrt{3}-1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}} \right), \quad \underline{w} = (1/2, 1/2).$$

$\phi_\ell(t^\ell) = 1$

$$\underline{\phi}(t) = (\phi_0(t), \phi_1(t)) = \left( \frac{t - t^1}{t^0 - t^1}, \frac{t - t^0}{t^1 - t^0} \right).$$

$\phi_\ell(t^k) = \delta_{\ell k}$

$\int_0^1 \phi'_m(t)\, \phi_\ell(t)\, dt = w_\ell \cdot \phi'_m(t^\ell)$

Then, the mass matrix is given by

$$\underline{\underline{\mathrm{M}}}_{m,l} = \phi_m(1)\phi_l(1) - \boxed{\phi'_m(t^l) w_l}, \quad m, l = 0, 1,$$

$$\underline{\underline{\mathrm{M}}} = \begin{pmatrix} 1 & \frac{\sqrt{3}-1}{2} \\ -\frac{\sqrt{3}+1}{2} & 1 \end{pmatrix}.$$

## ADER 2nd order

The right hand side is given

$$r(\underline{\boldsymbol{u}})_m = \overset{u}{\cancel{\alpha}}(0)\phi_m(0) + \Delta t F(\alpha(t^m))w_m, \quad m = 0, 1.$$

$$\underline{r}(\underline{\boldsymbol{u}}) = \overset{u}{\cancel{\alpha}}(0)\underline{\phi}(0) + \Delta t \begin{pmatrix} F(\alpha(t^1))w_1 \\ F(\alpha(t^2))w_2. \end{pmatrix}.$$

Then, the coefficients $\underline{\boldsymbol{u}}$ are given by

$$\boxed{\underline{\boldsymbol{u}}^{(k+1)} = \underline{\underline{\mathrm{M}}}^{-1}\underline{r}(\underline{\boldsymbol{u}}^{(k)}).}$$

Finally, use $\underline{\boldsymbol{u}}^{(k+1)}$ to reconstruct the solution at the time step $t^{n+1}$:

$$\underbrace{\boldsymbol{u}^{n+1} = \underline{\phi}(1)^T\underline{\boldsymbol{u}}^{(k+1)}}_{} = \underbrace{\boldsymbol{u}^n + \int_{T^n} \underline{\phi}(t)^T dt\, F(\underline{\boldsymbol{u}}^{(k)}).}_{}$$

# CODE

- Choice: $\phi$ Lagrangian basis functions
- Different subtimesteps: Gauss-Legendre, Gauss–Lobatto, equispaced
- Precompute $\underline{\underline{\mathrm{M}}}$
- Precompute the rhs vector part using quadratures after a further approximation

$$\underline{r}(\underline{\boldsymbol{u}}) = \underline{\phi}(t^n)\boldsymbol{u}^n + \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\boldsymbol{u}}) dt \approx \underline{\phi}(t^n)\boldsymbol{u}^n + \underbrace{\overbrace{\int_{T^n} \underline{\phi}(t)\underline{\phi}(t)^T dt}^{R}}_{\text{Can be stored}} F(\underline{\boldsymbol{u}})$$

- Precompute the reconstruction coefficients $\underline{\phi}(1)^T$

$\tilde{\phi}_i(s) = \phi_i\left(\frac{t-t^n}{\Delta t}\right)$

$dt = \Delta t\, ds$

$$\rightarrow M_{ij} = \phi_i(t^{n+1})\phi_j(t^{n+1}) - \int_{t^n}^{t^{n+1}} \phi_i'(t)\phi_j(t)\,dt = \tilde{\phi}_i(1)\tilde{\phi}_j(1) \dots - \frac{\cancel{\Delta t}}{\cancel{\Delta t}}\int_0^1 \tilde{\phi}_i'(s)\tilde{\phi}_j(s)\,ds$$

$$\Delta t\, R_{ij} := \int_{t^n}^{t^{n+1}} \phi_i(t)\phi_j(t)\,dt = \Delta t \cdot \int_0^1 \tilde{\phi}_i(s)\tilde{\phi}_j(s)\,ds$$

$$\underline{\underline{M}}\,\underline{u}^{(k)} = \underline{\Phi}(t^n)u^n + R\,F(u^{(k-1)}) \qquad \underline{u}^{(k)} = \overbrace{\underline{\underline{M}}^{-1}\underline{\phi}(t^n)}^{1} \cdot u^n + \overbrace{\underline{\underline{M}}^{-1}\underline{\underline{R}}}^{\text{evalMat}} F(u^{(k-1)})$$

## Outline

$$u^{(k)} = \underline{1} u^n + \underline{E}\, F(u^{(k-1)})$$

# ADER[6] and DeC[7]: immediate similarities

- High order time-(space) discretization
- Start from a well known space discretization (FE/DG/FV)
- FE reconstruction in time
- System in time, with $M$ equations (n+1 )
- Iterative method / $K$ corrections

[6] M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. Journal of Computational Physics, 227(18):8209–8253, 2008.

[7] R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. Journal of Scientific Computing, 73(2):461–494, Dec 2017.

# ADER[6] and DeC[7]: immediate similarities

- High order time-space discretization
- Start from a well known space discretization (FE/DG/FV)
- FE reconstruction in time
- System in time, with $M$ equations
- Iterative method / $K$ corrections
- Both high order explicit time integration methods (neglecting spatial discretization)

[6] M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. Journal of Computational Physics, 227(18):8209–8253, 2008.

[7] R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. Journal of Scientific Computing, 73(2):461–494, Dec 2017.

# ADER as DeC

$$? \rightarrow \underline{\underline{M}}\, \underline{u}^{(k)} = \underline{\Phi}(0) \cdot U_n + \underbrace{\Delta t}_{\vee} \underline{\underline{R}} \cdot F(\underline{u}^{(k-1)})$$

$$\mathcal{L}^2(\underline{u}) = \underline{\underline{M}}\, \underline{u} - \underline{\Phi}(0) \cdot U_n - \Delta t \underline{\underline{R}} F(\underline{u})$$

$$\mathcal{L}^2(\underline{u}) = \underline{u} - u_n \underline{1} - \Delta t \underline{\underline{M}}^{-1} \underline{\underline{R}} F(\underline{u})$$

$$\mathcal{L}^1(\underline{u}) = \underline{\underline{M}}\, \underline{u} - \phi(0) \cdot U_n - \Delta t \underline{\underline{R}} F(U_n) \cdot \underline{1}$$

EXPLICIT EULER

$$\mathcal{L}^1(\underline{u}) = \underline{u} - \underline{1} \cdot u_n - \Delta t \underline{\underline{M}}^{-1} \underline{\underline{R}} F(u_n) \cdot \underline{1}$$

DeC $\mathcal{L}^1, \mathcal{L}^2$

$$\mathcal{L}^1(\underline{u}^{(k)}) = \mathcal{L}^1(\underline{u}^{(k-1)}) - \mathcal{L}^2(\underline{u}^{(k-1)})$$

$$\underline{\underline{M}}\, \underline{u}^{(k)} - \cancel{\underline{\Phi}(0) U_n} - \cancel{\Delta t \underline{\underline{R}} F(U_n) \cdot \underline{1}} = \cancel{\underline{\underline{M}}\, \underline{u}^{(k-1)}} - \cancel{\underline{\Phi}(0) U_n} - \cancel{\Delta t \underline{\underline{R}} F(U_n) \cdot \underline{1}}$$

$$- \cancel{\underline{\underline{M}}\, \underline{u}^{(k-1)}} + \underline{\Phi}(0) U_n + \Delta t \underline{\underline{R}} F(\underline{u}^{(k-1)})$$

$$\underline{\underline{M}}\, \underline{u}^{(k)} = \underline{\Phi}(0) U_n + \Delta t \underline{\underline{R}} F(\underline{u}^{(k-1)}) \quad \checkmark$$

# ADER as DeC

# ADER as DeC

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\underline{\boldsymbol{u}}),$$
$$\mathcal{L}^1(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\boldsymbol{u}(t^n)).$$

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k)} - r(\boldsymbol{u}^{(k),0}) - \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)} + r(\boldsymbol{u}^{(k-1),0}) + \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)} - r(\underline{\boldsymbol{u}}^{(k-1)}) = 0$$

# ADER as DeC

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$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k)} - \cancel{r(\boldsymbol{u}^{(k),0})} - \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)} + \cancel{r(\boldsymbol{u}^{(k-1),0})} + \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)} - r(\underline{\boldsymbol{u}}^{(k-1)}) = 0$$

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$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\underline{\boldsymbol{u}}),$$
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$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k)} - \cancel{r(\boldsymbol{u}^{(k),0})} - \cancel{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)}} + \cancel{r(\boldsymbol{u}^{(k-1),0})} + \cancel{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)}} - r(\underline{\boldsymbol{u}}^{(k-1)}) = 0$$
$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k)} - r(\underline{\boldsymbol{u}}^{(k-1)}) = 0.$$

# ADER as DeC

$M_{ij} = \varphi_i(1)\varphi_j(1) - \int_0^1 \partial_t \varphi_i(t)\varphi_j(t)dt$

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\underline{\boldsymbol{u}}),$$
$$\mathcal{L}^1(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\boldsymbol{u}(t^n)).$$

Apply the DeC Convergence theorem!

- $\mathcal{L}^1$ is coercive because $\underline{\underline{\mathrm{M}}}$ is always invertible
- $\mathcal{L}^1 - \mathcal{L}^2$ is Lipschitz with constant $C\Delta t$ because they are consistent approx of the same problem
- Hence, after $K$ iterations we obtain a $K$th order accurate approximation of $\underline{\boldsymbol{u}}^*$

# DeC as ADER

$$\mathcal{L}^2(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) := \begin{cases} \boldsymbol{u}^M - \boldsymbol{u}^0 - \sum_{r=0}^M \int_{t^0}^{t^M} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \\ \dots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \end{cases}.$$

# DeC as ADER

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## DeC as ADER

$$\mathcal{L}^2(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) := \begin{cases} \boldsymbol{u}^M - \boldsymbol{u}^0 - \sum_{r=0}^M \int_{t^0}^{t^M} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \\ \dots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \end{cases}.$$

## DeC as "ADER"

$$\mathcal{L}^2(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) := \begin{cases} \boldsymbol{u}^M - \boldsymbol{u}^0 - \sum_{r=0}^M \int_{t^0}^{t^M} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \\ \dots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \end{cases}.$$

$$\chi_{[t^0,t^m]}(t^m)\boldsymbol{u}^m - \chi_{[t^0,t^m]}(t_0)\boldsymbol{u}^0 - \int_{t^0}^{t^m} \chi_{[t^0,t^m]}(t) \sum_{r=0}^M F(\boldsymbol{u}^r)\varphi_r(t)\mathrm{d}t = 0$$

$$\int_{t^0}^{t^M} \chi_{[t^0,t^m]}(t)\partial_t\left(\boldsymbol{u}(t)\right)\mathrm{d}t - \int_{t^0}^{t^M} \chi_{[t^0,t^m]}(t) \sum_{r=0}^M F(\boldsymbol{u}^r)\varphi_r(t)\mathrm{d}t = 0,$$

$$\int_{T^n} \psi_m(t)\partial_t \boldsymbol{u}(t)\mathrm{d}t - \int_{T^n} \psi_m(t) F(\boldsymbol{u}(t))\mathrm{d}t = 0.$$

## Runge Kutta vs DeC–ADER

### Classical Runge Kutta (RK)

- One step method
- Internal stages

Explicit Runge Kutta

\+ Simple to code

\- Not easily generalizable to arbitrary order

\- Stages $>$ order

Implicit Runge Kutta

\+ Arbitrarily high order

\- Require nonlinear solvers for nonlinear systems

\- May not converge

### DeC – ADER

- One step method
- Internal subtimesteps
- Can be rewritten as explicit RK (for ODE)

\+ Explicit

\+ Simple to code

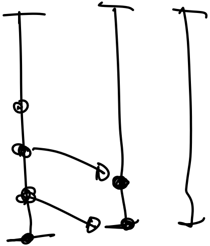
\+ Iterations $=$ order

\+ Arbitrarily high order

\- Large memory storage

# Outline





# Stability

$$u^{(k)} = u_n - \Delta t \underbrace{\underline{M}^{-1}\underline{R}}_{\underline{E}} F(\underline{u}^{(k-1)})$$

Since ADER can be written as a DeC, the stability functions are given by the same formula as for DeC and the stability regions are the following.

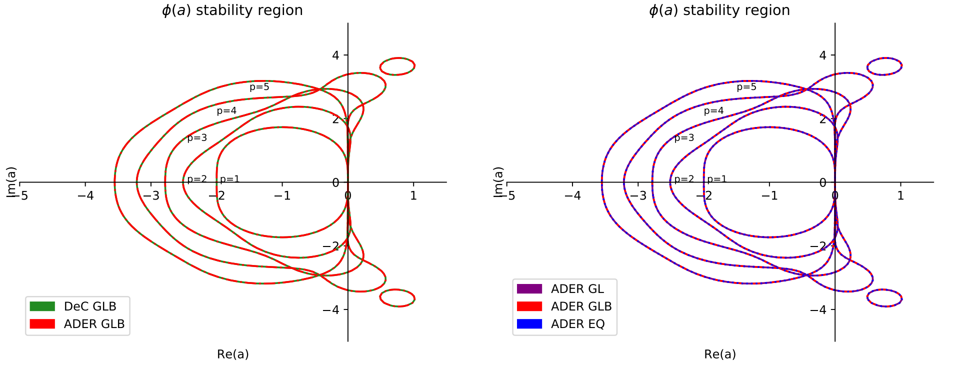



Figure: Stability region

## Accuracy of ADER $\mathcal{L}^2$ operators

The two things that determine the accuracy of the ADER method are the iterations $P$ and the accuracy of $\mathcal{L}^2$.

### Accuracy of ADER $\mathcal{L}^2$ for different distributions

- Equispaced: boring, minimum accuracy possible $M+1$ nodes $p = M+1$
- Guass–Lobatto: this generates the LobattoIIIC methods, $M+1$ nodes $p = 2M$
- Gauss–Legendre: this does not generate Gauss methods, $M+1$ nodes $p = 2M+1$

DEC $\mathcal{L}^2_{(y)=0}$ COLLOCATION METHOD
EQUISPACED $M+1$
G. LOBATTO $2M$
G. LEGENDRE $2M+2$

$M+1$ POINTS

B, C, D

$C(S) = C(M+1)$ B( ORDER OF QUAD FORMULA)

$B(\nu+S) + C(S) \Rightarrow D(\nu)$

# $\mathcal{L}^2$ ADER as RK

Here, we see $\mathcal{L}^2$ as an implicit RK

$$\mathcal{L}^{2,m}(\underline{\boldsymbol{u}}) = \underline{\underline{\mathrm{M}}}^m_j \boldsymbol{u}^{(j)} - \underline{\phi}^m(t^n)\boldsymbol{u}^n - \underbrace{\int_{T^n} \underline{\phi}^m(t)\underline{\phi}(t)_j dt}_{\Delta t \underline{\underline{\mathrm{R}}}^m_j} F(\boldsymbol{u}^{(j)}) = 0$$

$$\tilde{\mathcal{L}}^{2,z}(\underline{\boldsymbol{u}}) = \boldsymbol{u}^{(z)} - (\underline{\underline{\mathrm{M}}}^{-1})^z_m \underline{\phi}^m(t^n)\boldsymbol{u}^n - \Delta t (\underline{\underline{\mathrm{M}}}^{-1})^z_m \underline{\underline{\mathrm{R}}}^m_j F(\boldsymbol{u}^{(j)}) = 0$$

$$\boldsymbol{u}^{(z)} = \boldsymbol{u}^n + \Delta t a_{z,j} F(\boldsymbol{u}^{(j)})$$

- $a_{mj} = (\underline{\underline{\mathrm{M}}}^{-1})^z_m \underline{\underline{\mathrm{R}}}^m_j$
- Prove that $(\underline{\underline{\mathrm{M}}}^{-1})^z_m \underline{\phi}^m(t^n) = 1$ for every $z$
- $c^m = \sum_r a_{mr} = t^m$
- $b_r = \frac{1}{\Delta t}\int_{T^m} \phi_r(t)dt = w_r$ quadrature weights

# $\mathcal{L}^2$ ADER as RK

# $\mathcal{L}^2$ ADER as RK

## BCD conditions (Butcher 1964)

$c_j = \sum_i a_{ji}$ $\underline{U} = u_n \cdot \underline{1} + \Delta t \underline{\underline{E}} \cdot F(\underline{u})$

$U^{(k)} = U_n + \Delta t \sum_{j=1}^{S} a_{kj} F(u^{(j)})$

$U^{n+1} = U_n + \Delta t \sum_{j=1}^{S} b_j F(U^{(j)})$

Define the conditions

$$B(p): \quad \sum_{i=1}^{s} b_i c_i^{z-1} = \frac{1}{z}, \qquad z = 1, \dots, p; \tag{20}$$

$$C(\eta): \quad \sum_{j=1}^{s} a_{ij} c_j^{z-1} = \frac{c_i^z}{z}, \qquad i = 1, \dots, s,\ z = 1, \dots, \eta; \tag{21}$$

$$D(\zeta): \quad \sum_{i=1}^{s} b_i c_i^{z-1} a_{ij} = \frac{b_j}{z}(1 - c_j^z), \qquad j = 1, \dots, s,\ z = 1, \dots, \zeta. \tag{22}$$

$\underline{\underline{A}} = \underline{\underline{E}} = \boxed{\underline{\underline{\Phi}}^{-1} \underline{\underline{R}}}$ $\quad b_i = w_i = \int_0^1 \varphi_i(t)\,dt$

### Theorem (Butcher 1964)

*If the coefficients $b_i, c_i, a_{ij}$ of a RK scheme satisfy $B(p)$, $C(\eta)$ and $D(\zeta)$ with $p \leq \eta + \zeta + 1$ and $p \leq 2\eta + 2$, then the method is of order $p$.*

$C(s-1)$ $D(s-1)$ $S = \#\text{STAGES} = M+1$ $\quad - \sum_j a_{ij} c_j^{z-1} = \frac{1}{z} c_i^z \quad \forall\, i=1,\dots,s$ $\quad 0,\dots,M$

$\underline{\underline{A}}\, \underline{c}^{z-1}$

## Lemma

*$\mathcal{L}^2$ operator of ADER defined by Gauss–Lobatto or Gauss–Legendre points and quadrature (they coincide) with $s = M+1$ stages satisfies $C(s-1)$ and $D(s-1)$.*

## Proof (1/4).

Interpolation with $\phi^j$ is exact for polynomials of degree $s-1$. $\quad$ s POLYNOMIALS $\Rightarrow \mathbb{P}_{s-1}$

The quadrature is exact for polynomials of degree $2s-3$. $\quad$ (GLB) $\quad$ (GLG $2s-1$)

Recall that $\underline{\underline{A}} = \underline{\underline{M}}^{-1}\underline{\underline{R}}$, Condition $C(s-1)$ reads

$$\boxed{\underline{\underline{A}}\,\underline{c}^{z-1}} = \frac{1}{z}\underline{c}^z \iff \underline{\underline{R}}\,\underline{c}^{z-1} = \frac{1}{z}\underline{\underline{M}}\,\underline{c}^z \iff \underline{\mathcal{X}} := \underline{\underline{R}}\,\underline{c}^{z-1} - \frac{1}{z}\underline{\underline{M}}\,\underline{c}^z \overset{?}{=} \underline{0}, \qquad z = 1,\dots,s-1.$$

Recall $b_m = w_m$, $c_m = t^m$, $\underline{\underline{R}}_{i,j} = \delta_{i,j} w_i$ and the definition of $\underline{\underline{M}} = \underline{\phi}(1)\underline{\phi}^T(1) - \int \underline{\phi}'(t)\underline{\phi}^T(t)\,dt$

$$\mathcal{X}_m := w_m (t^m)^{z-1} - \underbrace{\frac{1}{z}}_{\sum_j}\left(\underbrace{\underbrace{\phi^m(1)\phi^j(1)}(t^j)^z}_{1^z} - \int_0^1 \frac{d}{d\xi}\phi^m(\xi)\underbrace{\phi^j(\xi)(t^j)^z}_{\to t^z} d\xi\right).$$

$$t^z = \sum \phi^j(t)\,(t^j)^z$$

## $C(s-1)$ $D(s-1)$

$\mathbb{P}_{s-1}$ $\quad$ $s \to$ $\quad$ $\phi_j \in \mathbb{P}_{s-1}$

### Proof (2/4).

Now, the interpolation of $t^z$ with $\boxed{z \leq s-1}$ with basis functions $\phi^j$ is exact. Hence, we can substitute $\phi^j(\xi)(t^j)^z = \xi^z$ for all $z = 1, \dots, s-1$, obtaining

$\mathbb{P}_{s-2}$ $\quad$ $\in \mathbb{P}_{s-1}$

$$-\int_0^1 \phi_m^1 \cdot \xi^z = -\phi(1)1^z + \phi(0)0^z + \int_0^1 \phi_m \cdot \frac{d}{d\xi}(\xi^z)\, d\xi$$

$$\mathcal{X}_m = w_m(t^m)^{z-1} - \frac{1}{z}\left(\phi^m(1)1^z - \int_0^1 \frac{d}{d\xi}\phi^m(\xi)\xi^z d\xi\right).$$

$\mathbb{P}_{2s-3}$

Using the exactness of the quadrature for polynomials of degree $2s-3$, both true for Gauss–Lobatto and Gauss–Legendre, we know that the previous integral is exactly computed as $\frac{d}{d\xi}\phi^m(\xi)$ is of degree at most $s-2$ and $\xi^z$ is at most $s-1$. So, we can use integration by parts and obtain

$0$ $\quad$ $z \cdot \xi^{z-1}$ $\quad$ $\xi^m$ $\quad$ $\phi^m(t^j) = \delta_{mj}$

$$\mathcal{X}_m = w_m(t^m)^{z-1} - \frac{1}{z}\left(\phi^m(0)0^z + \int_0^1 \phi^m(\xi)\frac{d}{d\xi}\xi^z d\xi\right) = w_m(t^m)^{z-1} - \int_0^1 \phi^m(\xi)\xi^{z-1} d\xi = 0$$

$\to w_m \cdot (t^m)^{z-1}$

by the exactness of the quadrature rule and the definition of $w_m$. Note that the condition is sharp, since the interpolation is not anymore exact for $z = s$, hence $C(s)$ is not satisfied.

$C(s-1)$ ✓

# $C(s-1)$ $D(s-1)$

$R = \int \underbrace{\phi_i}_{P_{s-1}} \underbrace{\phi_j}_{P_{s-1}}$ $P_{s-2}$

## Proof (3/4).

To prove $D(s-1)$, we write explicitly the condition in matricial form, for all $z = 1, \dots, s-1$

$$\underline{bc^{z-1}}\,\underline{\underline{A}} = \frac{1}{z}\underline{b(1-c^z)} \iff \underline{bc^{z-1}}\,\underline{\underline{\mathrm{M}}}^{-1}\,\underline{\underline{\mathrm{R}}} = \frac{1}{z}\underline{b(1-c^z)} \iff \underline{bc^{z-1}} = \frac{1}{z}\underline{b(1-c^z)}\,\underline{\underline{\mathrm{R}}}^{-1}\,\underline{\underline{\mathrm{M}}}.$$

Note that $b^m = w_m$ and $\underline{\underline{\mathrm{R}}}^m_r = w_m\delta^m_r$, so $\underline{b(1-c^z)}\,\underline{\underline{\mathrm{R}}}^{-1} = \underline{(1-c^z)}$. It is left to prove that

$$\mathcal{Y} := \underline{bc^{z-1}} - \frac{1}{z}\underline{(1-c^z)}\underline{\underline{\mathrm{M}}} = \underline{0}.$$

$$\mathcal{Y}_m = w_m(t^m)^{z-1} - \frac{1}{z}\underbrace{\sum_{j=1}^{s}\left(1-(t^j)^z\right)}_{P_{s-1}}\left(\phi^j(1)\phi^m(1) - \int_0^1 \frac{d}{d\xi}\phi^j(\xi)\phi^m(\xi)d\xi\right).$$

$z \leq s-1$

**Proof (4/4).**

Let us observe that, since $z \leq s-1$, the polynomial is exactly represented by the Lagrangian interpolation $t^z = \sum_{j=1}^{s} \phi(t)(t^m)^z$. Hence, using the exactness of the quadrature for polynomials of degree at most $2s-3$, we have

$$\begin{aligned}\mathcal{Y}_m =& w_m (t^m)^{z-1} - \frac{1}{z}\left(1-(1)^z\right)\phi^m(1) + \frac{1}{z}\int_0^1 \frac{d}{d\xi}\left(1-(\xi)^z\right)\phi^m(\xi)d\xi \\ =& w_m (t^m)^{z-1} - \frac{1}{z}\int_0^1 z\,\xi^{z-1}\phi^m(\xi)d\xi = w_m(t^m)^{z-1} - w_m(t^m)^{z-1} = 0.\end{aligned}$$

Hence, ADER-Legendre and ADER-Lobatto satisfy $D(s-1)$. Note that the condition is sharp, since the interpolation is not anymore exact for $z=s$, hence $D(s)$ is not satisfied.

## ADER Gauss–Legendre $\mathcal{L}^2$

**Remark (ADER-Legendre is no collocation method)**

*From the proof of previous Lemma, we can observe that ADER-Legendre methods do not satisfy $C(s)$, hence, the methods are not collocation methods and they do not coincide with Gauss–Legendre implicit RK methods.*

**Theorem**

*$\mathcal{L}^2$ of ADER with Gauss–Legendre is of order $2s-1$.*

**Proof.**

ADER-Legendre with $s = M+1$ stages satisfies $B(2s)$ for the quadrature rule and, hence, it satisfies $B(2s-1)$. For previous Lemma it also satisfies $C(s-1)$ and $D(s-1)$. Hence, Butcher's (1964) Theorem ($p \leq \eta+\zeta+1$ and $p \leq 2\eta+2$) guarantees that the method is of order $2s-1$, since it is satisfied with $p = 2s-1$ and $\eta = \zeta = s-1$. $\square$

$p = 2s-1$ $\quad$ $B(2s) \Rightarrow B(2s-1)$ ✓ $\quad$ $p \leq \eta + \zeta + 1$ $\quad$ $p = 2s-1$, $\eta = s-1$, $\zeta = s-1$

$2s-1 \leq s-1+s-1+1$

$2s-1 \leq 2s-1$ ✓

$2s-1 = p \leq 2\eta+2 = 2s \cancel{-2} \cancel{+2}$ ✓

$\boxtimes$

# ADER Gauss–Lobatto $\mathcal{L}^2$

**Theorem**

*$\mathcal{L}^2$ of ADER with Gauss-Lobatto is of order $2s-2$.*

**Proof.**

The condition for $B(2s-2)$ is satisfied as $(c,b)$ is the Gauss–Lobatto quadrature with order $2s-2$. Previous Lemma guarantees that ADER-Lobatto satisfies $B(2s-2)$, $C(s-1)$ and $D(s-1)$, so Butcher's (1964) Theorem ($p \leq \eta+\zeta+1$ and $p \leq 2\eta+2$) is satisfied for order $p=2s-2$ and $\eta=\zeta=s-1$. □

# ADER Gauss–Lobatto $\mathcal{L}^2$

**Theorem**

*$\mathcal{L}^2$ of ADER with Gauss-Lobatto is LobattoIIIC.*

The Lobatto IIIC method is defined using the condition

$$\boxed{a_{i1} = b_1 \qquad \text{for } i = 1, \dots, s.} \tag{23}$$

**Lemma**

*$\mathcal{L}^2$ of ADER with Gauss-Lobatto satisfies (23).*

**Theorem (Chipman 1971)**

*Lobatto IIIC schemes (in particular RK $a_{ij}$) are uniquely determined by Gauss–Lobatto quadrature rule $(c, b)$, condition (23) and by $C(s-1)$.*

# ADER Gauss–Lobatto $\mathcal{L}^2$

**Lemma**

*$\mathcal{L}^2$ of ADER with Gauss-Lobatto satisfies* (23).

**Proof.**

$$a_{i1} = \sum_j (\underline{\underline{\mathrm{M}}}^{-1})_{ij}\mathbb{R}_{j1} = b_1 = w_1 \iff$$

$$\sum_{i,j} \underline{\underline{\mathrm{M}}}_{ki}(\underline{\underline{\mathrm{M}}}^{-1})_{ij}\mathbb{R}_{j1} = \sum_i \underline{\underline{\mathrm{M}}}_{ki} w_1 \iff$$

$$\delta_{k1} w_1 = \mathbb{R}_{k1} = \sum_i \underline{\underline{\mathrm{M}}}_{ki} w_1$$

m=1

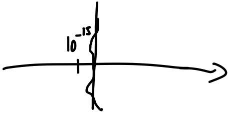
$$\sum_i \underline{\underline{\mathrm{M}}}_{ki} w_1 = \phi^m(1) w_1 - \int_0^1 \frac{d}{dt}\phi^m(\xi) w_1 dt = w_1 \phi^m(0) = w_1 \delta_{m,1}.$$

□

# Summary of results on $\mathcal{L}^2 = 0$

$n+1=S$

$\not{+}$ $2S-2$




| Method | DeC | | ADER | | |
|---|---|---|---|---|---|
| Nodes | Equispaced | Gauss–Lobatto | Equispaced | Gauss–Lobatto | Gauss–Legendre |
| Order | $M+1$ | $2M$ | $M+1$ | $2M$ ✓ | $2M+1$[8] |
| Known method | $\boxed{\text{Collocation}}$ | Lobatto IIIA | | Lobatto IIIC ✓ | |
| A–stability | ☹ | ☺ | $\boxed{???}$ | ☺ ✓ | ☺[9] |

L-STABLE

?

[8] M. Han Veiga, L. Micalizzi and D. T.. "On improving the efficiency of ADER methods." AMC, 466, page 128426, (2024)

[9] P. Öffner, L. Petri, D.T.. "Analysis for Implicit and Implicit-Explicit ADER and DeC Methods for Ordinary Differential Equations, Advection-Diffusion and Advection-Dispersion Equations" (2024)

# Outline

# Applications

## Usages

- Hyperbolic PDEs as explicit iterative methods (ADER: Toro, Dumbser, Klingenberg, Boscheri; DeC: Abgrall, Ricchiuto)
- IMEX solvers for hyperbolic with stiff sources (ADER: Dumbser, Boscheri; DeC: Abgrall, Torlo)
- IMEX solvers for hyperbolic with viscosity (treated implicitly) as compressible Navier Stokes (DeC: Minion, Dumbser, Zeifang)

## IMEX

$\partial_t u = F(u) + S(u)$
$S(u)$ stiff to be treated implicitly

## Advantages

- Arbitrary high order
- Unique framework to have matching between implicit and explicit terms
- Easy to code
- Iterative solver automatically included

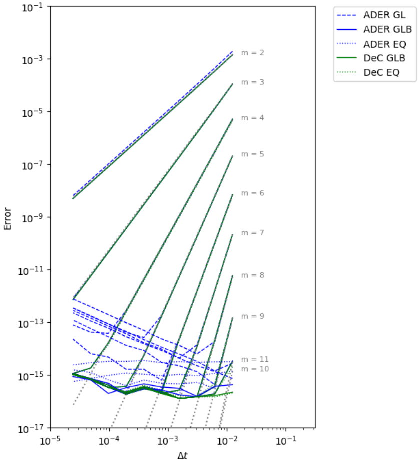
## Disadvantages

- Explicit solver: many many stages
- Implicit: many stages
- Explicit: not amazing stability property (wrt SSP RK e.g.)

# Convergence

$$
\begin{aligned}
y'(t) &= -|y(t)|y(t), \\
y(0) &= 1, \\
t &\in [0, 0.1].
\end{aligned}
\tag{24}
$$

Convergence curves for ADER and DeC, varying the approximation order and collocation of nodes for the subtimesteps for a scalar nonlinear ODE



# Lotka–Volterra

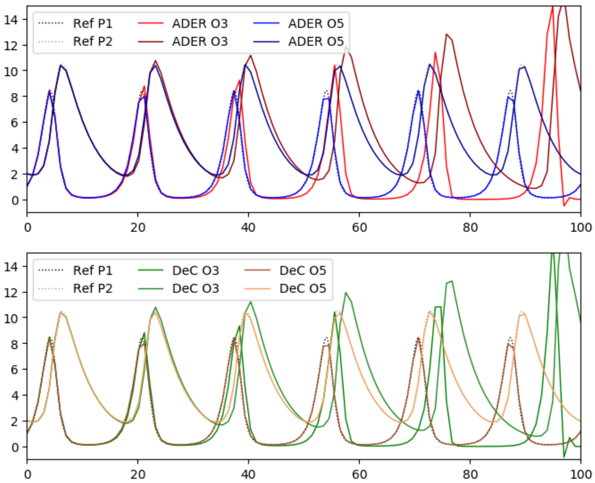
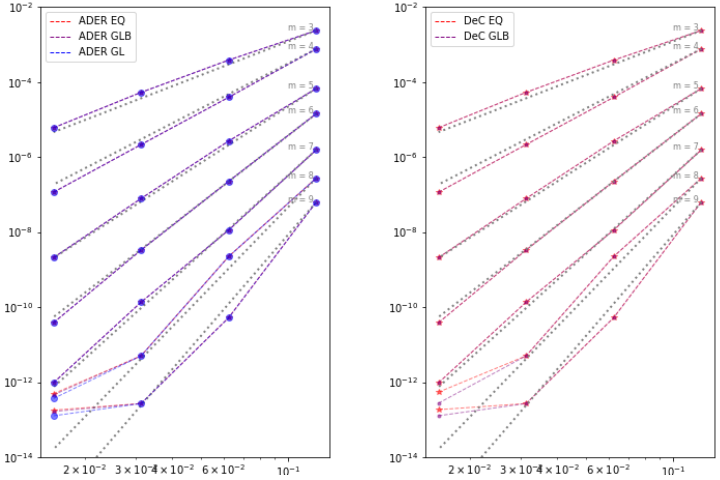



Figure: Numerical solution of the Lotka-Volterra system using ADER (top) and DeC (bottom) with Gauss-Lobatto nodes with timestep $\Delta T = 1$.

## PDE: Burgers with spectral difference




Convergence error for Burgers equations: Left ADER right DeC. Space discretization with spectral difference

# Outline

## Reduce computational cost for explicit DeC

### Literature

- *L. Micalizzi and D. Torlo. "A new efficient explicit Deferred Correction framework: analysis and applications to hyperbolic PDEs and adaptivity. " Commun. Appl. Math. Comput. (2023).* arxiv.org/abs/2210.02976
- *L. Micalizzi, D. Torlo and W. Boscheri. "Efficient iterative arbitrary high order methods: an adaptive bridge between low and high order." Commun. Appl. Math. Comput. (2023)* arxiv.org/abs/2212.07783
- *M. Han Veiga, L. Micalizzi and D. Torlo. "On improving the efficiency of ADER methods." Applied Mathematics and Computation, 466, page 128426, 2024.* arxiv.org/abs/2305.13065

### Goal

Reduce computational costs of explicit DeC/ADER.

# DeC as RK for ODEs

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

$$\boldsymbol{u}^{m,(p)} = \boldsymbol{u}^0 + \sum_{r=0}^{M} \theta_r^m F(t^r, \boldsymbol{u}^{r,(p-1)}), \qquad \forall m = 1, \dots, M, \; p = 1, \dots, P$$

## DeC as RK for ODEs

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

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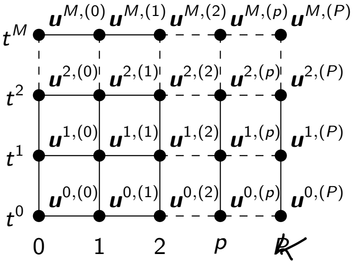

## DeC as RK for ODEs

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

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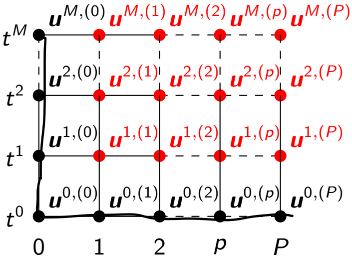
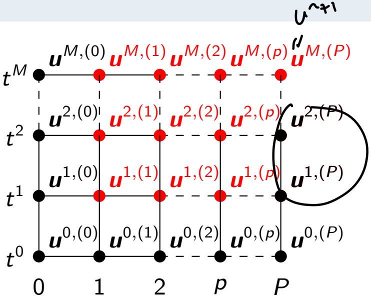


# DeC as RK for ODEs

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(p)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(p-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

$$\boldsymbol{u}^{m,(p)} = \boldsymbol{u}^0 + \sum_{r=0}^{M} \theta_r^m F(t^r, \boldsymbol{u}^{r,(p-1)}), \qquad \forall m = 1, \dots, M,\ p = 1, \dots, P$$




| $\underline{c}$ | $\mathbf{u}^0$ | $\mathbf{u}^{(1)}$ | $\mathbf{u}^{(2)}$ | $\mathbf{u}^{(3)}$ | $\cdots$ | $\mathbf{u}^{(M-1)}$ | $\mathbf{u}^{(M)}$ | A |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | $\Theta_0$ | | | | | | $\mathbf{u}^0$ |
| $\underline{\beta}_{1:}$ | $\underline{\beta}_{1:}$ | $\underline{\underline{0}}$ | $\tilde{\Theta}$ | | | | | $\mathbf{u}^{(1)}$ |
| $\underline{\beta}_{1:}$ | $\Theta_{1:,0}$ | $\Theta_{1:,1:}$ | $\underline{\underline{0}}$ | | | | | $\mathbf{u}^{(2)}$ |
| $\underline{\beta}_{1:}$ | $\Theta_{1:,0}$ | $\underline{\underline{0}}$ | $\Theta_{1:,1:}$ | $\underline{\underline{0}}$ | | | | $\mathbf{u}^{(3)}$ |
| | $\vdots$ | $\vdots$ | | $\ddots$ | $\ddots$ | | | $\vdots$ |
| | $\vdots$ | $\vdots$ | | | $\ddots$ | $\ddots$ | | $\vdots$ |
| $\underline{\beta}_{1:}$ | $\Theta_{1:,0}$ | $\underline{\underline{0}}$ | $\cdots$ | $\cdots$ | $\underline{\underline{0}}$ | $\Theta_{1:,1:}$ | $\underline{\underline{0}}$ | $\mathbf{u}^{(M)}$ |
| $\underline{b}$ | $\Theta_{M,0}$ | $\underline{0}$ | $\cdots$ | $\cdots$ | $\cdots$ | $\underline{0}$ | $\Theta_{M,1:}$ | $\mathbf{u}^{M,(M+1)}$ |

# Costs

**Large costs!**

# Costs

**Large costs!**

- DeC $S = M \cdot (P-1) + 1$
  - DeC equi $S = (P-1)^2 + 1$
  - DeC GLB $S = \left\lceil \frac{P}{2} \right\rceil (P-1) + 1$

Equispaced

| $P$ | $M$ | DeC |
|---|---|---|
| 2 | 1 | 2 |
| 3 | 2 | 5 |
| 4 | 3 | 10 |
| 5 | 4 | 17 |
| 6 | 5 | 26 |
| 7 | 6 | 37 |
| 8 | 7 | 50 |
| 9 | 8 | 65 |
| 10 | 9 | 82 |

Gauss–Lobatto

| $P$ | $M$ | DeC |
|---|---|---|
| 2 | 1 | 2 |
| 3 | 2 | 5 |
| 4 | 2 | 7 |
| $\boxed{5}$ | 3 | $\boxed{13}$ |
| 6 | 3 | 16 |
| 7 | 4 | 25 |
| 8 | 4 | 29 |
| 9 | 5 | 41 |
| 10 | 5 | 46 |

(SSPRK 10 4)

## Costs

**Large costs!**

- DeC $S = M \cdot (P-1)+1$
  - DeC equi $S=(P-1)^2+1$
  - DeC GLB $S=\left\lceil \frac{P}{2} \right\rceil (P-1)+1$

Equispaced

| $P$ | $M$ | DeC |
|---|---|---|
| 2 | 1 | 2 |
| 3 | 2 | 5 |
| 4 | 3 | 10 |
| 5 | 4 | 17 |
| 6 | 5 | 26 |
| 7 | 6 | 37 |
| 8 | 7 | 50 |
| 9 | 8 | 65 |
| 10 | 9 | 82 |

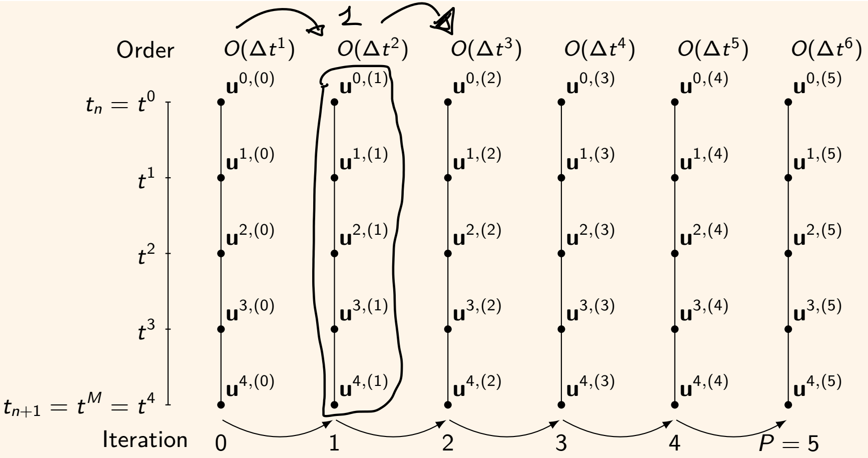
Gauss–Lobatto

| $P$ | $M$ | DeC |
|---|---|---|
| 2 | 1 | 2 |
| 3 | 2 | 5 |
| 4 | 2 | 7 |
| 5 | 3 | 13 |
| 6 | 3 | 16 |
| 7 | 4 | 25 |
| 8 | 4 | 29 |
| 9 | 5 | 41 |
| 10 | 5 | 46 |

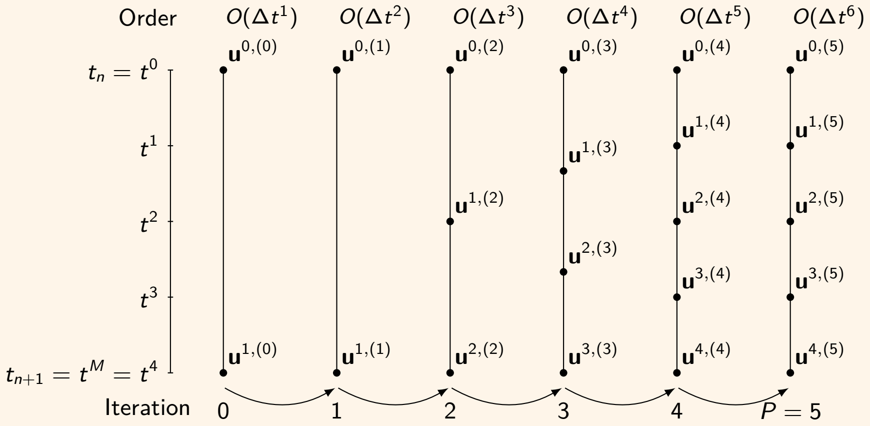
**How can we save computational time?**

# Outline

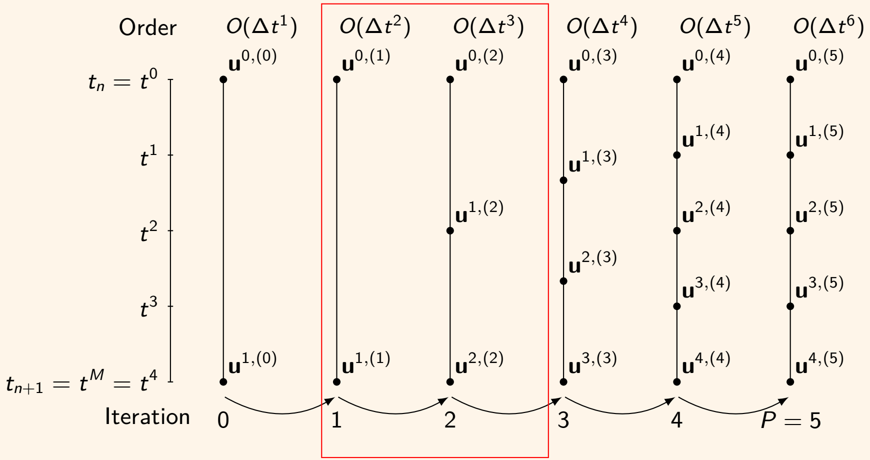
## Idea for reduction of stages



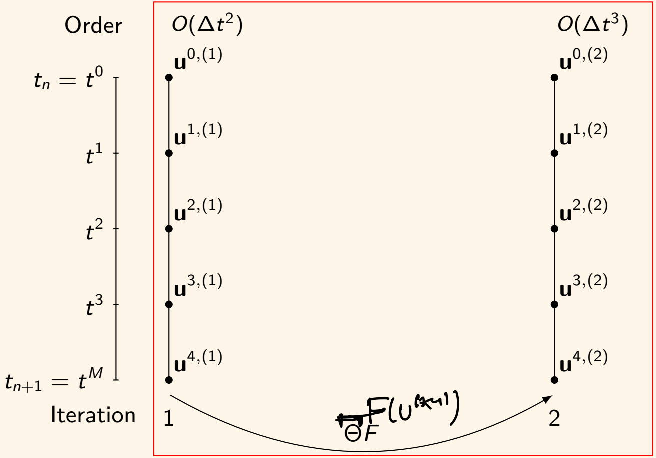
## Idea for reduction of stages



## Idea for reduction of stages



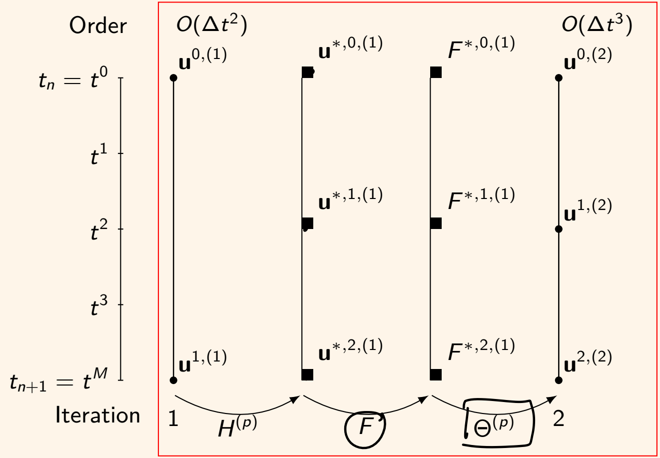
# How to communicate between iterations?




## DeC

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t \Theta F(\underline{\boldsymbol{u}}^{(p-1)})$$

# How to communicate between iterations?




**DeC**

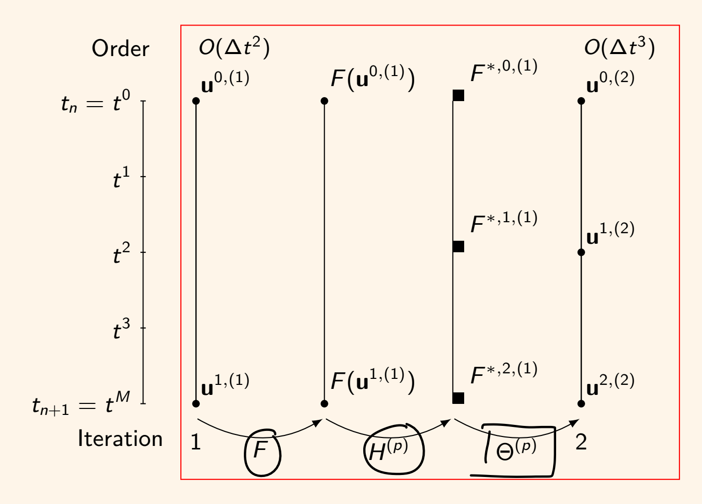
$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t \Theta F(\underline{\boldsymbol{u}}^{(p-1)})$$

**DeCu**

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t \Theta^{(p)} F(H^{(p)} \underline{\boldsymbol{u}}^{(p-1)})$$

$$H_{ij}^{(p)} = \phi_j^{(p-1)}(t^{i,(p)})$$

# How to communicate between iterations?




**DeC**

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^0 + \Delta t \Theta F(\underline{\boldsymbol{u}}^{(p-1)})$$

**DeCu**

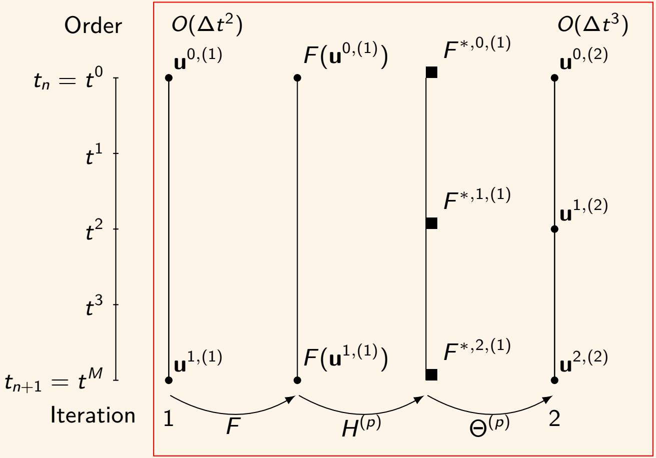
$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^0 + \Delta t \Theta^{(p)} F(H^{(p)} \underline{\boldsymbol{u}}^{(p-1)})$$

**DeCdu**

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^0 + \Delta t \Theta^{(p)} H^{(p)} F(\underline{\boldsymbol{u}}^{(p-1)})$$

$$H_{ij}^{(p)} = \phi_j^{(p-1)}(t^{i,(p)})$$

## How to communicate between iterations?




### DeC

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t \Theta F(\underline{\boldsymbol{u}}^{(p-1)})$$

### DeCu

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t \Theta^{(p)} F(H^{(p)} \underline{\boldsymbol{u}}^{(p-1)})$$

$$\underline{\boldsymbol{u}}^{*(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t H^{(p)} \Theta^{*(p-1)} F(\underline{\boldsymbol{u}}^{*(p-1)})$$

### DeCdu

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t \Theta^{(p)} H^{(p)} F(\underline{\boldsymbol{u}}^{(p-1)})$$

$$H^{(p)}_{ij} = \phi_j^{(p-1)}(t^{i,(p)})$$

## Efficient DeC into RK framework

**DeC** $\quad S = M \cdot (P-1) + 1$

| $\underline{c}$ | $\mathbf{u}^0$ | $\mathbf{u}^{(1)}$ | $\mathbf{u}^{(2)}$ | $\mathbf{u}^{(3)}$ | $\dots$ | $\mathbf{u}^{(M-1)}$ | $\mathbf{u}^{(M)}$ | A | dim |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | | | | $\mathbf{u}^0$ | 1 |
| $\underline{\beta}_{1:}$ | $\underline{\beta}_{1:}$ | $\underline{\underline{0}}$ | | | | | | $\mathbf{u}^{(1)}$ | M |
| $\underline{\beta}_{1:}$ | $\Theta_{1:,0}$ | $\Theta_{1:,1:}$ | $\underline{\underline{0}}$ | | | | | $\mathbf{u}^{(2)}$ | M |
| $\underline{\beta}_{1:}$ | $\Theta_{1:,0}$ | $\underline{\underline{0}}$ | $\Theta_{1:,1:}$ | $\underline{\underline{0}}$ | | | | $\mathbf{u}^{(3)}$ | M |
| | $\vdots$ | $\vdots$ | | $\ddots$ | $\ddots$ | | | $\vdots$ | M |
| | $\vdots$ | $\vdots$ | | | $\ddots$ | $\ddots$ | | $\vdots$ | M |
| $\underline{\beta}_{1:}$ | $\Theta_{1:,0}$ | $\underline{\underline{0}}$ | $\dots$ | $\dots$ | $\underline{\underline{0}}$ | $\Theta_{1:,1:}$ | $\underline{\underline{0}}$ | $\mathbf{u}^{(M)}$ | M |
| $\underline{b}$ | $\Theta_{M,0}$ | $\underline{0}$ | $\dots$ | $\dots$ | $\dots$ | $\underline{0}$ | $\Theta_{M,1:}$ | $\mathbf{u}^{M,(M+1)}$ | |

$$\textbf{DeCu} \qquad S = M\cdot(P-1)+1 - \frac{(M-1)(M-2)}{2}$$

| $\underline{c}$ | $\mathbf{u}^0$ | $\mathbf{u}^{*(1)}$ | $\mathbf{u}^{*(2)}$ | $\mathbf{u}^{*(3)}$ | $\dots$ | $\mathbf{u}^{*(M-2)}$ | $\mathbf{u}^{*(M-1)}$ | $\mathbf{u}^{(M)}$ | A | dim |
|---|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | | | | | | | | $\mathbf{u}^0$ | 1 |
| $\underline{\beta}^{(2)}_{1:}$ | $\underline{\beta}^{(2)}_{1:}$ | $\underline{\underline{0}}$ | | | | | | | $\mathbf{u}^{*(1)}$ | 2 |
| $\underline{\beta}^{(3)}_{1:}$ | $W^{(2)}_{1:,0}$ | $W^{(2)}_{1:,1:}$ | $\underline{\underline{0}}$ | | | | | | $\mathbf{u}^{*(2)}$ | 3 |
| $\underline{\beta}^{(4)}_{1:}$ | $W^{(3)}_{1:,0}$ | $\underline{\underline{0}}$ | $W^{(3)}_{1:,1:}$ | $\underline{\underline{0}}$ | | | | | $\mathbf{u}^{*(3)}$ | 4 |
| | $\vdots$ | $\vdots$ | | $\ddots$ | $\ddots$ | | | | $\vdots$ | $\vdots$ |
| | $\vdots$ | $\vdots$ | | | $\ddots$ | $\ddots$ | | | $\vdots$ | $\vdots$ |
| $\underline{\beta}^{(M)}_{1:}$ | $W^{(M-1)}_{1:,0}$ | $\underline{\underline{0}}$ | $\cdots$ | $\cdots$ | $\underline{\underline{0}}$ | $W^{(M-1)}_{1:,1:}$ | $\underline{\underline{0}}$ | $\underline{\underline{0}}$ | $\mathbf{u}^{*(M-1)}$ | $M$ |
| $\underline{\beta}^{(M)}_{1:}$ | $W^{(M)}_{1:,0}$ | $\underline{\underline{0}}$ | $\cdots$ | $\cdots$ | $\cdots$ | $\underline{\underline{0}}$ | $W^{(M)}_{1:,1:}$ | $\underline{\underline{0}}$ | $\mathbf{u}^{(M)}$ | $M$ |
| $\underline{b}$ | $W^{(M+1)}_{M,0}$ | $\underline{0}$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\underline{0}$ | $W^{(M+1)}_{M,1:}$ | $\mathbf{u}^{M,(M+1)}$ | |

$$W^{(p)} := \begin{cases} H^{(p)}\Theta^{(p)} \in \mathbb{R}^{(p+2)\times(p+1)}, & \text{if } p = 2,\dots,M-1,\\ \Theta^{(M)} \in \mathbb{R}^{(M+1)\times(M+1)}, & \text{if } p \geq M. \end{cases}$$

# Efficient DeC into RK framework

**DeCdu** $S = M \cdot (P-1) + 1 - \frac{M(M-1)}{2}$

| $\underline{c}$ | $\mathbf{u}^0$ | $\mathbf{u}^{(1)}$ | $\mathbf{u}^{(2)}$ | $\mathbf{u}^{(3)}$ | $\dots$ | $\mathbf{u}^{(M-2)}$ | $\mathbf{u}^{(M-1)}$ | $\mathbf{u}^{(M)}$ | A | dim |
|---|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | | | | | | | | $\mathbf{u}^0$ | 1 |
| $\underline{\beta}^{(1)}_{1:}$ | $\underline{\beta}^{(1)}_{1:}$ | $\underline{\underline{0}}$ | | | | | | | $\mathbf{u}^{(1)}$ | 1 |
| $\underline{\beta}^{(2)}_{1:}$ | $Z^{(2)}_{1:,0}$ | $Z^{(2)}_{1:,1:}$ | $\underline{\underline{0}}$ | | | | | | $\mathbf{u}^{(2)}$ | 2 |
| $\underline{\beta}^{(3)}_{1:}$ | $Z^{(3)}_{1:,0}$ | $\underline{\underline{0}}$ | $Z^{(3)}_{1:,1:}$ | $\underline{\underline{0}}$ | | | | | $\mathbf{u}^{(3)}$ | 3 |
| | $\vdots$ | $\vdots$ | | $\ddots$ | $\ddots$ | | | | $\vdots$ | $\vdots$ |
| | $\vdots$ | $\vdots$ | | | $\ddots$ | $\ddots$ | | | $\vdots$ | $\vdots$ |
| $\underline{\beta}^{(M-1)}_{1:}$ | $Z^{(M-1)}_{1:,0}$ | $\underline{\underline{0}}$ | $\cdots$ | $\cdots$ | $\underline{\underline{0}}$ | $Z^{(M-1)}_{1:,1:}$ | $\underline{\underline{0}}$ | $\underline{\underline{0}}$ | $\mathbf{u}^{(M-1)}$ | $M-1$ |
| $\underline{\beta}^{(M)}_{1:}$ | $Z^{(M)}_{1:,0}$ | $\underline{\underline{0}}$ | $\cdots$ | $\cdots$ | $\cdots$ | $\underline{\underline{0}}$ | $Z^{(M)}_{1:,1:}$ | $\underline{\underline{0}}$ | $\mathbf{u}^{(M)}$ | $M$ |
| $\underline{b}$ | $Z^{(M+1)}_{M,0}$ | $\underline{0}$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\underline{0}$ | $Z^{(M+1)}_{M,1:}$ | $\mathbf{u}^{M,(M+1)}$ | |

$$Z^{(p)} := \begin{cases} \Theta^{(p)} H^{(p-1)} \in \mathbb{R}^{(p+1)\times p}, & \text{if } p = 1, \dots, M, \\ \Theta^{(M)} \in \mathbb{R}^{(M+1)\times(M+1)}, & \text{if } p > M. \end{cases}$$

# Computational costs reduction: RK stages

## Equispaced

| P | M | DeC | DeCu | DeCdu |
|---|---|---|---|---|
| 2 | 1 | 2 | 2 | 2 |
| 3 | 2 | 5 | 5 | 4 |
| 4 | 3 | 10 | 9 | 7 |
| 5 | 4 | 17 | 14 | 11 |
| 6 | 5 | 26 | 20 | 16 |
| 7 | 6 | 37 | 27 | 22 |
| 8 | 7 | 50 | 35 | 29 |
| 9 | 8 | 65 | 44 | 37 |
| 10 | 9 | 82 | 54 | 46 |
| 11 | 10 | 101 | 65 | 56 |
| 12 | 11 | 122 | 77 | 67 |
| 13 | 12 | 145 | 90 | 79 |

## Gauss-Lobatto

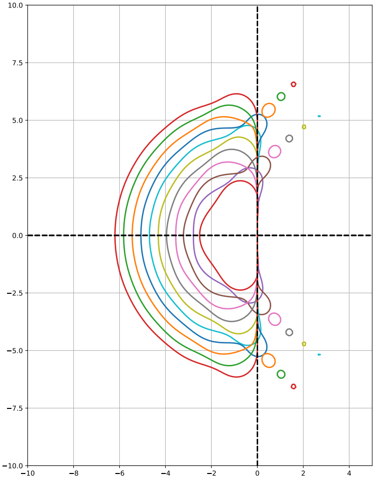
| P | M | DeC | DeCu | DeCdu |
|---|---|---|---|---|
| 2 | 1 | 2 | 2 | 2 |
| 3 | 2 | 5 | 5 | 4 |
| 4 | 2 | 7 | 7 | 6 |
| 5 | 3 | 13 | 12 | 10 |
| 6 | 3 | 16 | 15 | 13 |
| 7 | 4 | 25 | 22 | 19 |
| 8 | 4 | 29 | 26 | 23 |
| 9 | 5 | 41 | 35 | 31 |
| 10 | 5 | 46 | 40 | 36 |
| 11 | 6 | 61 | 51 | 46 |
| 12 | 6 | 67 | 57 | 52 |
| 13 | 7 | 85 | 70 | 64 |

## DeC, DeCu, DeCdu

### DeC-DeCu-DeCdu

The **stability function** of DeC, DeCu, DeCdu of order $P$ for any nodes distribution is

$$R(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^P}{P!}.$$



# Exercise

## Efficient DeC

- Code DeCu or DeCdu
- Check order of accuracy
- Write a code to obtain its RK matrix
- Check the stability function with nodepy
- Compare computational costs with original DeC

# Outline

# Summary: ADER/DeC $\boxed{\mathcal{L}^2 = 0}$

## DeC

- Integral form $U^m - U^0 - \int F(u)$
- Collocation methods
- Order ($M+1$ equispaced, $2M$ GLB)
- Stability (A-stability for GLB: Lobatto IIIA)

## ADER

- Weak form $\int \Phi u - \Phi F(u)$
- Not collocation methods
- Order ($M+1$ equispaced, $2M$ GLB, $2M+1$ GLG)
- Stability (A-stability for GLB GLG, I don't know for equi)

$\mathcal{L}^1$ EXPLICIT EULER

$\mathcal{L}^2(U^{(k)}) = \mathcal{L}^1(U^{(k-1)}) - \mathcal{L}^2(U^{(k-1)})$

$\boxed{R(z)}$

$d-2 \leq m \leq d$

PADÉ $(S-1, S)$ & $2S-1$

LOBATTO IIIC

# Summary: ADER/DeC iterative