ADER and DeC: arbitrarily high order (explicit) methods for PDEs and ODEs

Davide Torlo

SISSA Mathlab

Based on: Han Veiga, M., Öffner, P. & Torlo, D. *DeC and ADER: Similarities, Differences and a Unified Framework.* J Sci Comput 87, 2 (2021). https://doi.org/10.1007/s10915-020-01397-5

Outline

Outline

Motivation: high order accurate explicit method

Methods used to solve a hyperbolic PDE system for $u : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^D$

$$\partial_t u + \nabla_{\mathbf{x}} \mathcal{F}(u) = 0. \tag{1}$$

Or ODE system for $\boldsymbol{u} : \mathbb{R}^+ \rightarrow \mathbb{R}^S$

$$\partial_t \boldsymbol{u} = F(\boldsymbol{u}). \tag{2}$$

Applications:

- Fluids/transport
- Chemical/biological processes

How?

- Arbitrarily high order accurate
-

Motivation: high order accurate explicit method

Methods used to so

Or ODE system for

(1)

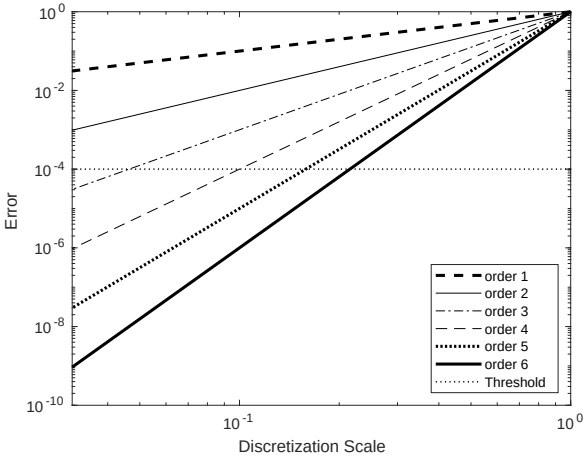
(2)

Applications:

- Fluids/transport
- Chemical/biolog

How?

- Arbitrarily high
-



Motivation: high order accurate explicit method

Methods used to solve a hyperbolic PDE system for $u : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^D$

$$\partial_t u + \nabla_{\mathbf{x}} \mathcal{F}(u) = 0. \tag{1}$$

Or ODE system for $\boldsymbol{u} : \mathbb{R}^+ \rightarrow \mathbb{R}^S$

$$\partial_t \boldsymbol{u} = F(\boldsymbol{u}). \tag{2}$$

Applications:

- Fluids/transport
- Chemical/biological processes

How?

- Arbitrarily high order accurate
- Explicit (if nonstiff problem)

Classical time integration: Runge–Kutta

$$\boldsymbol{u}^{(1)} := \boldsymbol{u}^n, \tag{3}$$

$$\boldsymbol{u}^{(k)} := \boldsymbol{u}^n + \sum_{s=1}^{K} a_{ks} F\left(t^n + c_s \Delta t, \boldsymbol{u}^{(s)}\right), \quad \text{for } k = 2, \ldots, K, \tag{4}$$

$$\boldsymbol{u}^{n+1} := \boldsymbol{u}^n + \sum_{s=1}^{K} b_s F\left(t^n + c_s \Delta t, \boldsymbol{u}^{(s)}\right). \tag{5}$$

Classical time integration: Explicit Runge–Kutta

$$\boldsymbol{u}^{(k)} := \boldsymbol{u}^n + \sum_{s=1}^{k-1} a_{ks} F\left(t^n + c_s \Delta t, \boldsymbol{u}^{(s)}\right), \quad \text{for } k = 2, \ldots, K.$$

- Easy to solve
- High orders involved:
  - Order conditions: system of many equations
  - Stages $K \geq d$ order of accuracy (e.g. RK44, RK65)

Classical time integration: Implicit Runge–Kutta

$$\boldsymbol{u}^{(k)} := \boldsymbol{u}^n + \sum_{s=1}^{K} a_{ks} F\left(t^n + c_s \Delta t, \boldsymbol{u}^{(s)}\right), \quad \text{for } k = 2, \ldots, K.$$

- More complicated to solve for nonlinear systems
- High orders easily done:
  - Take a high order quadrature rule on $[t^n, t^{n+1}]$
  - Compute the coefficients accordingly, see Gauss–Legendre or Gauss–Lobatto polynomials
  - Order up to $d = 2K$

ADER and DeC

Two iterative explicit arbitrarily high order accurate methods.

- ADER[1] for hyperbolic PDE, after a first analytic more complicated approach.
- Deferred Correction (DeC): introduced for explicit ODE[2], extended to implicit ODE[3] and to hyperbolic PDE[4].

[1] M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. Journal of Computational Physics, 227(18):8209–8253, 2008.

[2] A. Dutt, L. Greengard, and V. Rokhlin. Spectral Deferred Correction Methods for Ordinary Differential Equations. BIT Numerical Mathematics, 40(2):241–266, 2000.

[3] M. L. Minion. Semi-implicit spectral deferred correction methods for ordinary differential equations. Commun. Math. Sci., 1(3):471–500, 09 2003.

[4] R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. Journal of Scientific Computing, 73(2):461–494, Dec 2017.

Outline

DeC high order time discretization: $\mathcal{L}^2$

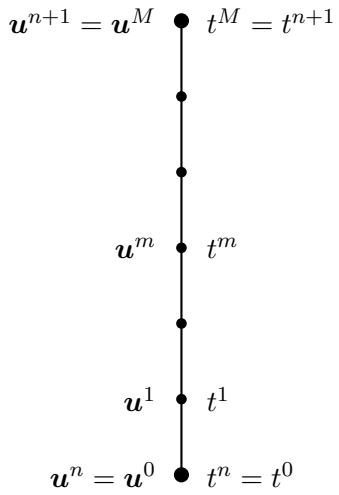
High order in time: we discretize our variable on $[t^n, t^{n+1}]$ in M substeps ($\boldsymbol{u}^m$).

$$\partial_t \boldsymbol{u} = F(\boldsymbol{u}(t)).$$

Thanks to Picard–Lindelöf theorem, we can rewrite

$$\boldsymbol{u}^m = \boldsymbol{u}^0 + \int_{t^0}^{t^m} F(\boldsymbol{u}(t))dt.$$

and if we want to reach order $r+1$ we need $M = r$.

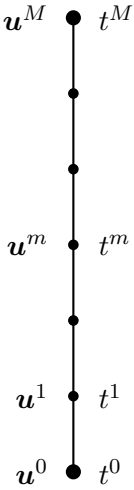


DeC high order time discretization: $\mathcal{L}^2$

More precisely, for each σ we want to solve $\mathcal{L}^2(\boldsymbol{u}^{n,0},\dots,\boldsymbol{u}^{n,M})=0$, where

$$\mathcal{L}^2(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M)=\begin{pmatrix}\boldsymbol{u}^M-\boldsymbol{u}^0+\sum_{r=0}^M\int_{t^0}^{t^M}F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s\\ \vdots\\ \boldsymbol{u}^1-\boldsymbol{u}^0+\sum_{r=0}^M\int_{t^0}^{t^1}F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s\end{pmatrix}$$

- $\mathcal{L}^2=0$ is a system of $M\times S$ coupled (non)linear equations
- $\mathcal{L}^2$ is an implicit method (collocation method: Gauss, LobattoIIIA)
- Not easy to solve directly $\mathcal{L}^2(\underline{\boldsymbol{u}}^*)=0$
- High order ($\geq M+1$), depending on points distribution

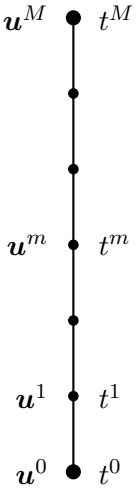


DeC high order time discretization: $\mathcal{L}^2$

More precisely, for each σ we want to solve $\mathcal{L}^2(\boldsymbol{u}^{n,0},\dots,\boldsymbol{u}^{n,M}) = 0$, where

$$\mathcal{L}^2(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) = \begin{pmatrix} \boldsymbol{u}^M - \boldsymbol{u}^0 + \Delta t \sum_{r=0}^M \theta_r^M F(\boldsymbol{u}^r) \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 + \Delta t \sum_{r=0}^M \theta_r^1 F(\boldsymbol{u}^r) \end{pmatrix}$$

- $\mathcal{L}^2 = 0$ is a system of $M \times S$ coupled (non)linear equations
- $\mathcal{L}^2$ is an implicit method (collocation method: Gauss, LobattoIIIA)
- Not easy to solve directly $\mathcal{L}^2(\underline{\boldsymbol{u}}^*) = 0$
- High order ($\geq M+1$), depending on points distribution

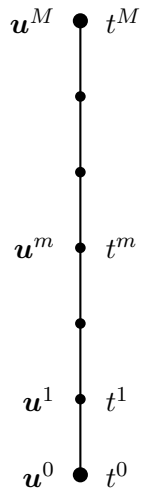


DeC low order time discretization: $\mathcal{L}^1$

Instead of solving the implicit system directly (difficult), we introduce a first order scheme $\mathcal{L}^1(\boldsymbol{u}^{n,0}, \ldots, \boldsymbol{u}^{n,M})$:

$$\mathcal{L}^1(\boldsymbol{u}^0, \ldots, \boldsymbol{u}^M) = \begin{pmatrix} \boldsymbol{u}^M - \boldsymbol{u}^0 + \Delta t \beta^M F(\boldsymbol{u}^0) \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 + \Delta t \beta^1 F(\boldsymbol{u}^0) \end{pmatrix}$$

- First order approximation
- Explicit Euler
- Easy to solve $\mathcal{L}^1(\underline{\boldsymbol{u}}) = 0$



Deferred Correction[5]

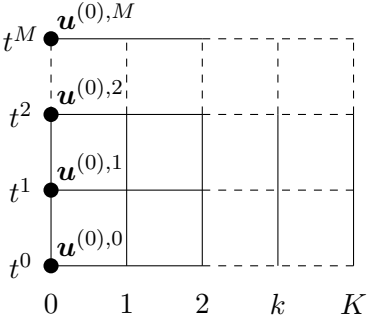
How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

$\boldsymbol{u}^{0,(k)} := \boldsymbol{u}(t^n), \quad k = 0, \ldots, K,$

$\boldsymbol{u}^{m,(0)} := \boldsymbol{u}(t^n), \quad m = 1, \ldots, M$

$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)})$ with $k = 1, \ldots, K.$

- $\mathcal{L}^1(\underline{\boldsymbol{u}}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\boldsymbol{u}}) = 0$, high order $M + 1$.



Theorem (Convergence DeC)

- $\mathcal{L}^2(\underline{\boldsymbol{u}}^*) = 0$
- *If* $\mathcal{L}^1$ *coercive with constant* C_1
- *If* $\mathcal{L}^1 - \mathcal{L}^2$ *Lipschitz with constant* $C_2 \Delta t$

Then $\|\underline{\boldsymbol{u}}^{(K)} - \underline{\boldsymbol{u}}^*\| \leq C(\Delta t)^K$

[5] A. Dutt, L. Greengard, and V. Rokhlin. BIT Numerical Mathematics, 40(2):241–266, 2000.

Deferred Correction[5]

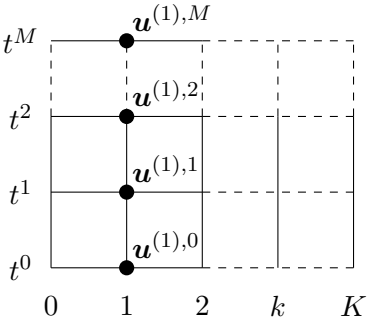
How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

$\boldsymbol{u}^{0,(k)} := \boldsymbol{u}(t^n), \quad k = 0, \dots, K,$

$\boldsymbol{u}^{m,(0)} := \boldsymbol{u}(t^n), \quad m = 1, \dots, M$

$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)})$ with $k = 1, \dots, K.$

- $\mathcal{L}^1(\underline{\boldsymbol{u}}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\boldsymbol{u}}) = 0$, high order $M + 1$.



Theorem (Convergence DeC)

- $\mathcal{L}^2(\underline{\boldsymbol{u}}^*) = 0$
- *If* $\mathcal{L}^1$ *coercive with constant* C_1
- *If* $\mathcal{L}^1 - \mathcal{L}^2$ *Lipschitz with constant* $C_2 \Delta t$

Then $\|\underline{\boldsymbol{u}}^{(K)} - \underline{\boldsymbol{u}}^*\| \leq C(\Delta t)^K$

[5] A. Dutt, L. Greengard, and V. Rokhlin. BIT Numerical Mathematics, 40(2):241–266, 2000.

Deferred Correction[5]

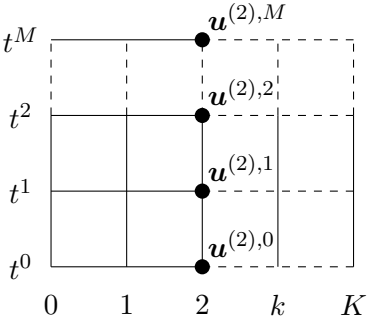
How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

$\boldsymbol{u}^{0,(k)} := \boldsymbol{u}(t^n), \quad k = 0, \dots, K,$

$\boldsymbol{u}^{m,(0)} := \boldsymbol{u}(t^n), \quad m = 1, \dots, M$

$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)})$ with $k = 1, \dots, K.$

- $\mathcal{L}^1(\underline{\boldsymbol{u}}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\boldsymbol{u}}) = 0$, high order $M + 1$.



Theorem (Convergence DeC)

- $\mathcal{L}^2(\underline{\boldsymbol{u}}^*) = 0$
- *If* $\mathcal{L}^1$ *coercive with constant* C_1
- *If* $\mathcal{L}^1 - \mathcal{L}^2$ *Lipschitz with constant* $C_2 \Delta t$

Then $\|\underline{\boldsymbol{u}}^{(K)} - \underline{\boldsymbol{u}}^*\| \leq C(\Delta t)^K$

[5] A. Dutt, L. Greengard, and V. Rokhlin. BIT Numerical Mathematics, 40(2):241–266, 2000.

Deferred Correction[5]

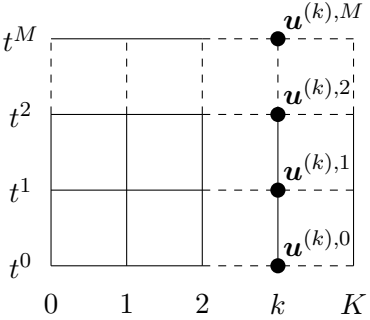
How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

$\boldsymbol{u}^{0,(k)} := \boldsymbol{u}(t^n), \quad k = 0, \dots, K,$

$\boldsymbol{u}^{m,(0)} := \boldsymbol{u}(t^n), \quad m = 1, \dots, M$

$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)})$ with $k = 1, \dots, K.$

- $\mathcal{L}^1(\underline{\boldsymbol{u}}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\boldsymbol{u}}) = 0$, high order $M + 1$.



Theorem (Convergence DeC)

- $\mathcal{L}^2(\underline{\boldsymbol{u}}^*) = 0$
- *If* $\mathcal{L}^1$ *coercive with constant* C_1
- *If* $\mathcal{L}^1 - \mathcal{L}^2$ *Lipschitz with constant* $C_2 \Delta t$

Then $\|\underline{\boldsymbol{u}}^{(K)} - \underline{\boldsymbol{u}}^*\| \leq C(\Delta t)^K$

[5] A. Dutt, L. Greengard, and V. Rokhlin. BIT Numerical Mathematics, 40(2):241–266, 2000.

Deferred Correction[5]

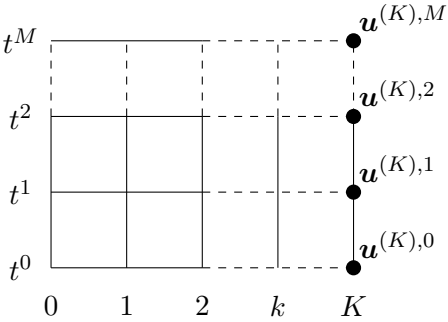
How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

$\boldsymbol{u}^{0,(k)} := \boldsymbol{u}(t^n), \quad k = 0, \dots, K,$

$\boldsymbol{u}^{m,(0)} := \boldsymbol{u}(t^n), \quad m = 1, \dots, M$

$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)})$ with $k = 1, \dots, K.$

- $\mathcal{L}^1(\underline{\boldsymbol{u}}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\boldsymbol{u}}) = 0$, high order $M + 1$.



Theorem (Convergence DeC)

- $\mathcal{L}^2(\underline{\boldsymbol{u}}^*) = 0$
- *If* $\mathcal{L}^1$ *coercive with constant* C_1
- *If* $\mathcal{L}^1 - \mathcal{L}^2$ *Lipschitz with constant* $C_2 \Delta t$

Then $\|\underline{\boldsymbol{u}}^{(K)} - \underline{\boldsymbol{u}}^*\| \leq C(\Delta t)^K$

[5] A. Dutt, L. Greengard, and V. Rokhlin. BIT Numerical Mathematics, 40(2):241–266, 2000.

DeC – Proof

Proof.

Let f^* be the solution of $\mathcal{L}^2(\underline{\boldsymbol{u}}^*) = 0$. We know that $\mathcal{L}^1(\underline{\boldsymbol{u}}^*) = \mathcal{L}^1(\underline{\boldsymbol{u}}^*) - \mathcal{L}^2(\underline{\boldsymbol{u}}^*)$, so that

□

DeC – Proof

Proof.

Let f^* be the solution of $\mathcal{L}^2(\underline{\boldsymbol{u}}^*) = 0$. We know that $\mathcal{L}^1(\underline{\boldsymbol{u}}^*) = \mathcal{L}^1(\underline{\boldsymbol{u}}^*) - \mathcal{L}^2(\underline{\boldsymbol{u}}^*)$, so that

$$\begin{aligned}
\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k+1)}) - \mathcal{L}^1(\underline{\boldsymbol{u}}^*) &= \left(\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k)})\right) - \left(\mathcal{L}^1(\underline{\boldsymbol{u}}^*) - \mathcal{L}^2(\underline{\boldsymbol{u}}^*)\right) \\
C_1||\underline{\boldsymbol{u}}^{(k+1)} - \underline{\boldsymbol{u}}^*|| &\leq ||\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k+1)}) - \mathcal{L}^1(\underline{\boldsymbol{u}}^*)|| = \\
&= ||\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k)}) - (\mathcal{L}^1(\underline{\boldsymbol{u}}^*) - \mathcal{L}^2(\underline{\boldsymbol{u}}^*))|| \leq \\
&\leq C_2\Delta||\underline{\boldsymbol{u}}^{(k)} - \underline{\boldsymbol{u}}^*||. \\
||\underline{\boldsymbol{u}}^{(k+1)} - \underline{\boldsymbol{u}}^*|| &\leq \left(\frac{C_2}{C_1}\Delta\right)||\underline{\boldsymbol{u}}^{(k)} - \underline{\boldsymbol{u}}^*|| \leq \left(\frac{C_2}{C_1}\Delta\right)^{k+1}||\underline{\boldsymbol{u}}^{(0)} - \underline{\boldsymbol{u}}^*||.
\end{aligned}$$

After K iteration we have an error at most of $\left(\frac{C_2}{C_1}\Delta\right)^K ||\underline{\boldsymbol{u}}^{(0)} - \underline{\boldsymbol{u}}^*||$. □

DeC: Second order example

DeC: Second order example

DeC: Second order example

DeC: Second order example

Simplification of DeC for ODE

In practice

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

For $m = 1, \dots, M$

$$\boldsymbol{u}^{(k),m} - \boldsymbol{u}^0 - \beta^m \Delta t F(\boldsymbol{u}^0) - \boldsymbol{u}^{(k-1),m} + \boldsymbol{u}^0 + \beta^m \Delta t F(\boldsymbol{u}^0)$$
$$+ \boldsymbol{u}^{(k-1),m} - \boldsymbol{u}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{u}^{(k-1),r}) = 0$$

Simplification of DeC for ODE

In practice

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

For $m = 1, \dots, M$

$$\boldsymbol{u}^{(k),m} - \cancel{\boldsymbol{u}^0 - \beta^m \Delta t F(\boldsymbol{u}^0)} - \boldsymbol{u}^{(k-1),m} + \cancel{\boldsymbol{u}^0 + \beta^m \Delta t F(\boldsymbol{u}^0)}$$
$$+ \boldsymbol{u}^{(k-1),m} - \boldsymbol{u}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{u}^{(k-1),r}) = 0$$

Simplification of DeC for ODE

In practice

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

For $m = 1, \dots, M$

$$\boldsymbol{u}^{(k),m} - \cancel{\boldsymbol{u}^0 - \beta^m \Delta t F(\boldsymbol{u}^0)} - \cancel{\boldsymbol{u}^{(k-1),m}} + \cancel{\boldsymbol{u}^0 + \beta^m \Delta t F(\boldsymbol{u}^0)}$$
$$+ \cancel{\boldsymbol{u}^{(k-1),m}} - \boldsymbol{u}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{u}^{(k-1),r}) = 0$$

Simplification of DeC for ODE

In practice

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

For $m = 1, \dots, M$

$$\begin{aligned}
&\boldsymbol{u}^{(k),m} - \cancel{\boldsymbol{u}^0 - \beta^m \Delta t F(\boldsymbol{u}^0)} - \cancel{\boldsymbol{u}^{(k-1),m}} + \cancel{\boldsymbol{u}^0 + \beta^m \Delta t F(\boldsymbol{u}^0)} \\
&+ \cancel{\boldsymbol{u}^{(k-1),m}} - \boldsymbol{u}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{u}^{(k-1),r}) = 0 \\
&\boldsymbol{u}^{(k),m} - \boldsymbol{u}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{u}^{(k-1),r}) = 0.
\end{aligned}$$

DeC and residual distribution

Deferred Correction + Residual distribution

- Residual distribution (FV $\Rightarrow$ FE) $\Rightarrow$ High order in space
- Prediction/correction/iterations $\Rightarrow$ High order in time
- Subtimesteps $\Rightarrow$ High order in time

$$U_{\xi}^{m,(k+1)} = U_{\xi}^{m,(k)} - |C_p|^{-1} \sum_{\mathrm{E}|\xi\in\mathrm{E}} \left(\int_{\mathrm{E}} \Phi_{\xi} \left(U^{m,(k)} - U^{n,0} \right) \mathrm{d}\mathbf{x} + \Delta t \sum_{r=0}^{M} \theta_r^m \mathcal{R}_{\xi}^{\mathrm{E}}(U^{r,(k)}) \right),$$

with

$$\sum_{\xi\in\mathrm{E}} \mathcal{R}_{\xi}^{\mathrm{E}}(u) = \int_{\mathrm{E}} \nabla_{\mathbf{x}} F(u) \mathrm{d}\mathbf{x}.$$

- The $\mathcal{L}^2$ operator contains also the complications of the spatial discretization (e.g. mass matrix)
- $\mathcal{L}^1$ operator further simplified up to a first order approximation (e.g. **mass lumping**)

$\mathcal{L}^1$ with mass lumping

Implicit simple DeC (Rosenbrock)

Define $\mathcal{L}^1$ as

$$\mathcal{L}^1(\boldsymbol{u}^0, \ldots, \boldsymbol{u}^M) = \begin{pmatrix} \boldsymbol{u}^M - \boldsymbol{u}^0 - \Delta t \beta^M F(\boldsymbol{u}^0) \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \Delta t \beta^1 F(\boldsymbol{u}^0) \end{pmatrix}$$

Implicit simple DeC (Rosenbrock)

Define $\mathcal{L}^1$ as

$$\mathcal{L}^1(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) = \begin{pmatrix} \boldsymbol{u}^M - \boldsymbol{u}^0 - \Delta t \beta^M \left(F(\boldsymbol{u}^0) + \partial_{\boldsymbol{u}} F(\boldsymbol{u}^0)(\boldsymbol{u}^M - \boldsymbol{u}^0)\right) \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \Delta t \beta^1 \left(F(\boldsymbol{u}^0) + \partial_{\boldsymbol{u}} F(\boldsymbol{u}^0)(\boldsymbol{u}^1 - \boldsymbol{u}^0)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{u}^M - \boldsymbol{u}^0 - \Delta t \beta^M \partial_{\boldsymbol{u}} F(\boldsymbol{u}^0)\boldsymbol{u}^M \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \Delta t \beta^1 \partial_{\boldsymbol{u}} F(\boldsymbol{u}^0)\boldsymbol{u}^1 \end{pmatrix}$$

Implicit simple DeC (Rosenbrock)

$$\mathcal{L}^{1,m}(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) = \boldsymbol{u}^m - \boldsymbol{u}^0 - \Delta t \beta^m \partial_{\boldsymbol{u}} F(\boldsymbol{u}^0)\boldsymbol{u}^m$$
$$\mathcal{L}^{2,m}(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) = \boldsymbol{u}^m - \boldsymbol{u}^0 - \Delta t \sum_r \theta_r^m F(\boldsymbol{u}^r)$$

Implicit simple DeC (Rosenbrock)

DeC as RK

$$\boldsymbol{u}^{(k),m} - \boldsymbol{u}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{u}^{(k-1),r}) = 0$$

DeC as RK

DeC as RK

We can write DeC as RK defining $\underline{\theta}_0 = \{\theta_0^m\}_{m=1}^M$, $\underline{\theta}^M = \theta_r^M$ with $r \in 1, \ldots, M$, denoting the vector $\underline{\theta}_r^{M,T} = (\theta_1^M, \ldots, \theta_M^M)$. The Butcher tableau for an arbitrarily high order DeC approach is given by:

$$
\begin{array}{c|cccccc}
0 & 0 & & & & & \\
\underline{\beta} & \underline{\beta} & & & & & \\
\underline{\beta} & \underline{\theta}_0 & \underline{\tilde{\theta}} & & & & \\
\vdots & \underline{\theta}_0 & \underline{0} & \underline{\tilde{\theta}} & & & \\
\vdots & \underline{\theta}_0 & \underline{0} & \underline{0} & \underline{\tilde{\theta}} & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\
\underline{\beta} & \underline{\theta}_0 & \underline{0} & \ldots & \ldots & \underline{0} & \underline{\tilde{\theta}} \\
\hline
 & \theta_0^M & \underline{0}^T & \ldots & & \ldots & \underline{0}^T \quad \underline{\theta}_r^{M,T}
\end{array}
\qquad (6)
$$

Stability of (explicit) DeC

Idea: study the RK version!

$$u' = \lambda u \qquad \Re(\lambda) < 0. \tag{7}$$

$$u_{n+1} = R(\lambda \Delta t) u_n, \qquad R(z) = 1 + z b^T (I - zA)^{-1} \mathbf{1}, \qquad z = \lambda \Delta t \tag{8}$$

Goal: find $z \in \mathbb{C}$ such that $|R(z)| < 1$.
Recall: stability function for explicit RK methods is a polynomial, indeed the inverse of $(I - zA)$ can be written in Taylor expansion as

$$(I - zA)^{-1} = \sum_{r=0}^{\infty} z^r A^s = I + zA + z^2 A^2 + \dots, \tag{9}$$

and, since A is strictly lower triangular, it is nilpotent. Hence, $R(z)$ is a polynomial in z with degree at most equal to S.

Stability of (explicit) DeC

Theorem

If the RK method is of order P, then

$$R(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^P}{P!} + O(z^{P+1}). \tag{10}$$

The first $P + 1$ terms of the stability functions $R(\cdot)$ for explicit DeCs of order P are known.

Theorem

The stability function of any explicit DeC of order P (with P iterations) is

$$R(z) = \sum_{r=0}^{P} \frac{z^r}{r!} = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^P}{P!} \tag{11}$$

and does not depend on the distribution of the subtimenodes.

Stability of (explicit) DeC

Proof (1/3)

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \star & 0 & 0 & \dots & 0 & 0 \\ \star & \star & 0 & \dots & 0 & 0 \\ \star & 0 & \star & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & 0 & 0 & \cdots & \star & 0 \end{pmatrix},$$

Block structure of the matrix A

$\star$ are some non-zero block matrices and the 0 are some zero block matrices.

The number of blocks in each line and row of these matrices is P, the order of the scheme.

Proof (2/3)

By induction, A^k has zeros in the upper triangular part, in the main block diagonal, and in all the $k-1$ block diagonals below the main diagonal, i.e.,

$$(A^k)_{i,j} = 0 \quad \text{, if } i < j + k,$$

where the indexes here refer to the blocks. Indeed, it is true that $A_{i,j} = 0$ if $i < j + 1$. Now, let us consider the entry $(A^{k+1})_{i,j}$ with $i < j + k + 1$, i.e., $i - k < j + 1$. It is defined as

$$(A^{k+1})_{i,j} = \sum_w (A^k)_{i,w} A_{w,j}. \tag{12}$$

Now, we can prove that all the terms of the sum are 0. Let $w < j + 1$, then $A_{w,j} = 0$ because of the structure of A; while, if $w \geq j + 1 > i - k$, we have that $i < w + k$, so $(A^k)_{i,w} = 0$ by induction.

Proof (3/3)

In particular, this means that $A^P = \underline{\underline{0}}$, because i is always smaller than $j + P$ as P is the number of the block matrices that we have. Hence,

$$(I - zA)^{-1} = \sum_{r=0}^{\infty} z^r A^s = \sum_{r=0}^{P-1} z^r A^s = I + zA + z^2 A^2 + \cdots + z^{P-1} A^{P-1}. \tag{13}$$

Plugging this result into $R(z) = 1 + zb^T (I - zA)^{-1} \mathbf{1}$, the stability function $R(z)$ is a polynomial of degree P, the order of the scheme. All terms of order lower or equal to P must agree with the expansion of the exponential function, so it must be

$$R(z) = \sum_{r=0}^{P} \frac{z^r}{r!} = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^P}{P!}. \tag{14}$$

Note: no assumption on the distribution of the subtimenodes.

CODE

- Choice of iterations (P) and order
- Choice of point distributions $t^0, \ldots, t^M$
- Computation of θ
- Loop for timesteps
- Loop for correction
- Loop for subtimesteps

Outline

ADER

- Cauchy–Kovalevskaya theorem
- Modern automatic version
- Space/time DG
- Prediction/Correction
- Fixed-point iteration process

Modern approach is DG in space time for hyperbolic problem

$$\partial_t u(x,t) + \nabla \cdot F(u(x,t)) = 0,\ x \in \Omega \subset \mathbb{R}^d,\ t > 0. \tag{15}$$

Prediction: iterative procedure

$$\int_{T^n \times V_i} \theta_{rs}(x,t)\partial_t \theta_{pq}(x,t) z^{pq} \mathrm{d}x\mathrm{d}t + \int_{T^n \times V_i} \theta_{rs}(x,t)\nabla_{\mathbf{x}} \cdot F(\theta_{pq}(x,t)z^{pq})\mathrm{d}x\mathrm{d}t = 0.$$

Correction step: communication between cells

$$\int_{V_i} \Phi_r \left(u(t^{n+1}) - u(t^n)\right) \mathrm{d}x + \int_{T^n \times \partial V_i} \Phi_r(x)\mathcal{G}(z^-, z^+) \cdot \mathbf{n}\, \mathrm{d}S\, \mathrm{d}t - \int_{T^n \times V_i} \nabla_{\mathbf{x}} \Phi_r \cdot F(z)\, \mathrm{d}x\, \mathrm{d}t = 0,$$

ADER: space-time discretization

Defining $\theta_{rs}(x,t) = \Phi_r(x)\phi_s(t)$ basis functions in space and time

$$\int_{T^n \times V_i} \theta_{rs}(x,t)\partial_t \theta_{pq}(x,t) u^{pq} \mathrm{d}x\mathrm{d}t + \int_{T^n \times V_i} \theta_{rs}(x,t) \nabla \cdot F(\theta_{pq}(x,t)u^{pq}) \mathrm{d}x\mathrm{d}t = 0. \tag{16}$$

This leads to

$$\underline{\underline{\underline{\mathrm{M}}}}_{rspq} u^{pq} = \underline{r}(\underline{\mathbf{u}})_{rs}, \tag{17}$$

solved with fixed point iteration method.
+ Correction step where cells communication is allowed (derived from (16)).

ADER: space-time discretization

Defining $\theta_{rs}(x,t) = \Phi_r(x)\phi_s(t)$ basis functions in space and time

$$\int_{T^n \times V_i} \theta_{rs}(x,t)\partial_t \theta_{pq}(x,t) u^{pq} \mathrm{d}x\mathrm{d}t + \int_{T^n \times V_i} \theta_{rs}(x,t) \nabla \cdot F(\theta_{pq}(x,t)u^{pq}) \mathrm{d}x\mathrm{d}t = 0. \tag{16}$$

This leads to

$$\underset{\equiv\equiv rspq}{\mathrm{M}} \; u^{pq} = \underline{\underline{r}}(\underline{\underline{\mathbf{u}}})_{rs}, \tag{17}$$

solved with fixed point iteration method.
+ Correction step where cells communication is allowed (derived from (16)).

ADER: time integration method

Simplify! Take $\boldsymbol{u}(t) = \sum_{m=0}^{M} \phi_m(t)\boldsymbol{u}^m = \underline{\phi}(t)^T \underline{\boldsymbol{u}}$

$$\int_{T^n} \psi(t)\partial_t \boldsymbol{u}(t)dt - \int_{T^n} \psi(t)F(\boldsymbol{u}(t))dt = 0, \quad \forall \psi : T^n = [t^n, t^{n+1}] \to \mathbb{R}.$$

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \int_{T^n} \underline{\phi}(t)\partial_t \underline{\phi}(t)^T \underline{\boldsymbol{u}}dt - \int_{T^n} \underline{\phi}(t)F(\underline{\phi}(t)^T \underline{\boldsymbol{u}})dt = 0$$

$$\underline{\phi}(t) = (\phi_0(t), \ldots, \phi_M(t))^T$$

Quadrature...

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - \underline{r}(\underline{\boldsymbol{u}}) = 0 \Longleftrightarrow \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} = \underline{r}(\underline{\boldsymbol{u}}). \tag{18}$$

Nonlinear system of $M \times S$ equations

ADER: Mass matrix

What goes into the mass matrix? Use of the integration by parts

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \int_{T^n} \underline{\phi}(t)\partial_t \underline{\phi}(t)^T \underline{\boldsymbol{u}} dt + \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\boldsymbol{u}}) dt =$$

$$\underline{\phi}(t^{n+1})\underline{\phi}(t^{n+1})^T \underline{\boldsymbol{u}} - \underline{\phi}(t^n)\boldsymbol{u}^n - \int_{T^n} \partial_t \underline{\phi}(t)\underline{\phi}(t)^T \underline{\boldsymbol{u}} - \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\boldsymbol{u}}) dt$$

$$\underline{\underline{\mathrm{M}}} = \underline{\phi}(t^{n+1})\underline{\phi}(t^{n+1})^T - \int_{T^n} \partial_t \underline{\phi}(t)\underline{\phi}(t)^T$$

$$\underline{r}(\underline{\boldsymbol{u}}) = \underline{\phi}(t^n)\boldsymbol{u}^n + \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\boldsymbol{u}}) dt$$

$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} = \underline{r}(\underline{\boldsymbol{u}})$$

ADER: Fixed point iteration

Iterative procedure to solve the problem for each time step

$$\underline{\boldsymbol{u}}^{(k)} = \underline{\underline{\mathrm{M}}}^{-1}\underline{r}(\underline{\boldsymbol{u}}^{(k-1)}), \quad k = 1, \ldots, \text{convergence} \tag{19}$$

with $\underline{\boldsymbol{u}}^{(0)} = \boldsymbol{u}(t^n)$.
Reconstruction step

$$\boldsymbol{u}(t^{n+1}) = \boldsymbol{u}(t^n) - \int_{T^n} F(\boldsymbol{u}^{(K)}(t))dt.$$

- Convergence?
- How many steps K?
- Accuracy $\mathcal{L}^2$?

ADER 2nd order

Example with 2 Gauss Legendre points, Lagrange polynomials and 2 iterations
Let us consider the timestep interval $[t^n, t^{n+1}]$, rescaled to $[0, 1]$.
Gauss-Legendre points quadrature and interpolation (in the interval $[0, 1]$)

$$\underline{t}_q = \left(t_q^0, t_q^1\right) = \left(t^0, t^1\right) = \left(\frac{\sqrt{3}-1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}}\right), \quad \underline{w} = (1/2, 1/2)\,.$$

$$\underline{\phi}(t) = (\phi_0(t), \phi_1(t)) = \left(\frac{t-t^1}{t^0-t^1}, \frac{t-t^0}{t^1-t^0}\right).$$

Then, the mass matrix is given by

$$\underline{\underline{\mathrm{M}}}_{m,l} = \phi_m(1)\phi_l(1) - \phi_m'(t^l)w_l, \quad m, l = 0, 1,$$

$$\underline{\underline{\mathrm{M}}} = \begin{pmatrix} 1 & \frac{\sqrt{3}-1}{2} \\ -\frac{\sqrt{3}+1}{2} & 1 \end{pmatrix}.$$

ADER 2nd order

The right hand side is given

$$r(\underline{\boldsymbol{u}})_m = \alpha(0)\phi_m(0) + \Delta t F(\alpha(t^m))w_m, \quad m = 0, 1.$$

$$\underline{r}(\underline{\boldsymbol{u}}) = \alpha(0)\underline{\phi}(0) + \Delta t \begin{pmatrix} F(\alpha(t^1))w_1 \\ F(\alpha(t^2))w_2. \end{pmatrix}.$$

Then, the coefficients $\underline{\boldsymbol{u}}$ are given by

$$\underline{\boldsymbol{u}}^{(k+1)} = \underline{\underline{\mathrm{M}}}^{-1}\underline{r}(\underline{\boldsymbol{u}}^{(k)}).$$

Finally, use $\underline{\boldsymbol{u}}^{(k+1)}$ to reconstruct the solution at the time step t^{n+1}:

$$\boldsymbol{u}^{n+1} = \underline{\phi}(1)^T\underline{\boldsymbol{u}}^{(k+1)} = \boldsymbol{u}^n + \int_{T^n} \underline{\phi}(t)^T dt\, F(\underline{\boldsymbol{u}}^{(k)}).$$

CODE

- Choice: ϕ Lagrangian basis functions
- Different subtimesteps: Gauss-Legendre, Gauss–Lobatto, equispaced
- Precompute $\underline{\underline{\mathrm{M}}}$
- Precompute the rhs vector part using quadratures after a further approximation

$$\underline{r}(\underline{\boldsymbol{u}}) = \underline{\phi}(t^n)\boldsymbol{u}^n + \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\boldsymbol{u}}) dt \approx \underline{\phi}(t^n)\boldsymbol{u}^n + \underbrace{\int_{T^n} \underline{\phi}(t)\underline{\phi}(t)^T dt}_{\text{Can be stored}} F(\underline{\boldsymbol{u}})$$

- Precompute the reconstruction coefficients $\underline{\phi}(1)^T$

Outline

ADER[6] and DeC[7]: immediate similarities

- High order time-space discretization
- Start from a well known space discretization (FE/DG/FV)
- FE reconstruction in time
- System in time, with M equations
- Iterative method / K corrections

- Both high order explicit time integration methods (neglecting spatial discretization)

[6] M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. Journal of Computational Physics, 227(18):8209–8253, 2008.

[7] R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. Journal of Scientific Computing, 73(2):461–494, Dec 2017.

ADER[6] and DeC[7]: immediate similarities

- High order time-space discretization
- Start from a well known space discretization (FE/DG/FV)
- FE reconstruction in time
- System in time, with M equations
- Iterative method / K corrections

- Both high order explicit time integration methods (neglecting spatial discretization)

[6] M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. Journal of Computational Physics, 227(18):8209–8253, 2008.

[7] R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. Journal of Scientific Computing, 73(2):461–494, Dec 2017.

ADER as DeC

ADER as DeC

ADER as DeC

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\underline{\boldsymbol{u}}),$$
$$\mathcal{L}^1(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\boldsymbol{u}(t^n)).$$

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k)} - r(\boldsymbol{u}^{(k),0}) - \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)} + r(\boldsymbol{u}^{(k-1),0}) + \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)} - r(\underline{\boldsymbol{u}}^{(k-1)}) = 0$$

ADER as DeC

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\underline{\boldsymbol{u}}),$$
$$\mathcal{L}^1(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\boldsymbol{u}(t^n)).$$

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k)} - \cancel{r(\boldsymbol{u}^{(k),0})} - \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)} + \cancel{r(\boldsymbol{u}^{(k-1),0})} + \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)} - r(\underline{\boldsymbol{u}}^{(k-1)}) = 0$$

ADER as DeC

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\underline{\boldsymbol{u}}),$$
$$\mathcal{L}^1(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\boldsymbol{u}(t^n)).$$

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k)} - \cancel{r(\boldsymbol{u}^{(k),0})} - \underline{\underline{\mathrm{M}}}\cancel{\underline{\boldsymbol{u}}^{(k-1)}} + \cancel{r(\boldsymbol{u}^{(k-1),0})} + \underline{\underline{\mathrm{M}}}\cancel{\underline{\boldsymbol{u}}^{(k-1)}} - r(\underline{\boldsymbol{u}}^{(k-1)}) = 0$$

ADER as DeC

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\underline{\boldsymbol{u}}),$$
$$\mathcal{L}^1(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\boldsymbol{u}(t^n)).$$

$$\mathcal{L}^1(\underline{\boldsymbol{u}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{u}}^{(k-1)}), \qquad k = 1, \dots, K,$$

$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k)} - \cancel{r(\boldsymbol{u}^{(k),0})} - \cancel{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)}} + \cancel{r(\boldsymbol{u}^{(k-1),0})} + \cancel{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k-1)}} - r(\underline{\boldsymbol{u}}^{(k-1)}) = 0$$
$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}}^{(k)} - r(\underline{\boldsymbol{u}}^{(k-1)}) = 0.$$

ADER as DeC

$$\mathcal{L}^2(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\underline{\boldsymbol{u}}),$$
$$\mathcal{L}^1(\underline{\boldsymbol{u}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{u}} - r(\boldsymbol{u}(t^n)).$$

Apply the DeC Convergence theorem!

- $\mathcal{L}^1$ is coercive because $\underline{\underline{\mathrm{M}}}$ is always invertible
- $\mathcal{L}^1 - \mathcal{L}^2$ is Lipschitz with constant $C\Delta t$ because they are consistent approx of the same problem
- Hence, after K iterations we obtain a Kth order accurate approximation of $\underline{\boldsymbol{u}}^*$

DeC as ADER

$$\mathcal{L}^2(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) := \begin{cases} \boldsymbol{u}^M - \boldsymbol{u}^0 - \sum_{r=0}^{M} \int_{t^0}^{t^M} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \\ \dots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \sum_{r=0}^{M} \int_{t^0}^{t^1} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \end{cases} .$$

DeC as ADER

DeC as ADER

DeC as ADER

$$\mathcal{L}^2(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) := \begin{cases} \boldsymbol{u}^M - \boldsymbol{u}^0 - \sum_{r=0}^{M} \int_{t^0}^{t^M} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \\ \dots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \sum_{r=0}^{M} \int_{t^0}^{t^1} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \end{cases} .$$

$$\chi_{[t^0,t^m]}(t^m)\boldsymbol{u}^m - \chi_{[t^0,t^m]}(t_0)\boldsymbol{u}^0 - \int_{t^0}^{t^m} \chi_{[t^0,t^m]}(t) \sum_{r=0}^{M} F(\boldsymbol{u}^r)\varphi_r(t)\mathrm{d}t = 0$$

$$\int_{t^0}^{t^M} \chi_{[t^0,t^m]}(t)\partial_t\left(\boldsymbol{u}(t)\right)\mathrm{d}t - \int_{t^0}^{t^M} \chi_{[t^0,t^m]}(t) \sum_{r=0}^{M} F(\boldsymbol{u}^r)\varphi_r(t)\mathrm{d}t = 0,$$

$$\int_{T^n} \psi_m(t)\partial_t \boldsymbol{u}(t)\mathrm{d}t - \int_{T^n} \psi_m(t) F(\boldsymbol{u}(t))\mathrm{d}t = 0.$$

DeC as ADER

$$\mathcal{L}^2(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) := \begin{cases} \boldsymbol{u}^M - \boldsymbol{u}^0 - \sum_{r=0}^{M} \int_{t^0}^{t^M} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \\ \dots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \sum_{r=0}^{M} \int_{t^0}^{t^1} F(\boldsymbol{u}^r)\varphi_r(s)\mathrm{d}s \end{cases}.$$

$$\chi_{[t^0,t^m]}(t^m)\boldsymbol{u}^m - \chi_{[t^0,t^m]}(t_0)\boldsymbol{u}^0 - \int_{t^0}^{t^m} \chi_{[t^0,t^m]}(t) \sum_{r=0}^{M} F(\boldsymbol{u}^r)\varphi_r(t)\mathrm{d}t = 0$$

$$\int_{t^0}^{t^M} \chi_{[t^0,t^m]}(t)\partial_t\left(\boldsymbol{u}(t)\right)\mathrm{d}t - \int_{t^0}^{t^M} \chi_{[t^0,t^m]}(t) \sum_{r=0}^{M} F(\boldsymbol{u}^r)\varphi_r(t)\mathrm{d}t = 0,$$

$$\int_{T^n} \psi_m(t)\partial_t\boldsymbol{u}(t)\mathrm{d}t - \int_{T^n} \psi_m(t)F(\boldsymbol{u}(t))\mathrm{d}t = 0.$$

Runge Kutta vs DeC–ADER

Classical Runge Kutta (RK)

- One step method
- Internal stages

Explicit Runge Kutta

+ Simple to code
- Not easily generalizable to arbitrary order
- Stages $>$ order

Implicit Runge Kutta

+ Arbitrarily high order
- Require nonlinear solvers for nonlinear systems
- May not converge

DeC – ADER

- One step method
- Internal subtimesteps
- Can be rewritten as explicit RK (for ODE)

+ Explicit
+ Simple to code
+ Iterations $=$ order
+ Arbitrarily high order
- Large memory storage

Outline

Stability

Since ADER can be written as a DeC, the stability functions are given by the same formula as for DeC and the stability regions are the following.

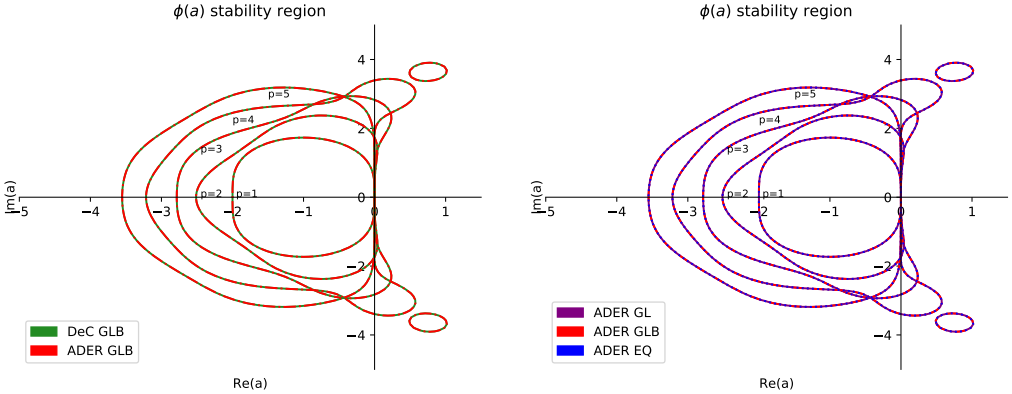


Figure: Stability region

Accuracy of ADER $\mathcal{L}^2$ operators

The two things that determine the accuracy of the ADER method are the iterations P and the accuracy of $\mathcal{L}^2$.

Accuracy of ADER $\mathcal{L}^2$ for different distributions

- Equispaced: boring, minimum accuracy possible $M+1$ nodes $p = M+1$
- Guass–Lobatto: this generates the LobattoIIIC methods, $M+1$ nodes $p = 2M$
- Gauss–Legendre: this does not generate Gauss methods, $M+1$ nodes $p = 2M+1$

$\mathcal{L}^2$ ADER as RK

Here, we see $\mathcal{L}^2$ as an implicit RK

$$\mathcal{L}^{2,m}(\underline{\boldsymbol{u}}) = \underline{\underline{\mathrm{M}}}^m_j \boldsymbol{u}^{(j)} - \underline{\phi}^m(t^n)\boldsymbol{u}^n - \underbrace{\int_{T^n} \underline{\phi}^m(t)\underline{\phi}(t)_j dt}_{\Delta t \underline{\underline{\mathrm{R}}}^m_j} F(\boldsymbol{u}^{(j)}) = 0$$

$$\tilde{\mathcal{L}}^{2,z}(\underline{\boldsymbol{u}}) = \boldsymbol{u}^{(z)} - (\underline{\underline{\mathrm{M}}}^{-1})^z_m \underline{\phi}^m(t^n)\boldsymbol{u}^n - \Delta t (\underline{\underline{\mathrm{M}}}^{-1})^z_m \underline{\underline{\mathrm{R}}}^m_j F(\boldsymbol{u}^{(j)}) = 0$$

$$\boldsymbol{u}^{(z)} = \boldsymbol{u}^n + \Delta t a_{z,j} F(\boldsymbol{u}^{(j)})$$

- $a_{mj} = (\underline{\underline{\mathrm{M}}}^{-1})^z_m \underline{\underline{\mathrm{R}}}^m_j$
- Prove that $(\underline{\underline{\mathrm{M}}}^{-1})^z_m \underline{\phi}^m(t^n) = 1$ for every z
- $c^m = \sum_r a_{mr} = t^m$
- $b_r = \frac{1}{\Delta t}\int_{T^m} \phi_r(t)dt = w_r$ quadrature weights

$\mathcal{L}^2$ ADER as RK

$\mathcal{L}^2$ ADER as RK

BCD conditions (Butcher 1964)

Define the conditions

$$B(p): \quad \sum_{i=1}^{s} b_i c_i^{z-1} = \frac{1}{z}, \qquad z = 1, \dots, p; \tag{20}$$

$$C(\eta): \quad \sum_{j=1}^{s} a_{ij} c_j^{z-1} = \frac{c_i^z}{z}, \qquad i = 1, \dots, s,\ z = 1, \dots, \eta; \tag{21}$$

$$D(\zeta): \quad \sum_{i=1}^{s} b_i c_i^{z-1} a_{ij} = \frac{b_j}{z}(1 - c_j^z), \qquad j = 1, \dots, s,\ z = 1, \dots, \zeta. \tag{22}$$

Theorem (Butcher 1964)

If the coefficients b_i, c_i, a_{ij} of a RK scheme satisfy $B(p)$, $C(\eta)$ and $D(\zeta)$ with $p \leq \eta + \zeta + 1$ and $p \leq 2\eta + 2$, then the method is of order p.

Lemma

$\mathcal{L}^2$ operator of ADER defined by Gauss–Lobatto or Gauss–Legendre points and quadrature (they coincide) with $s = M + 1$ stages satisfies $C(s-1)$ and $D(s-1)$.

Proof (1/4).

Interpolation with ϕ^j is exact for polynomials of degree $s-1$.
The quadrature is exact for polynomials of degree $2s-3$.
Recall that $\underline{\underline{A}} = \underline{\underline{\mathrm{MR}}}$, Condition $C(s-1)$ reads

$$\underline{\underline{A}}\,\underline{\underline{c^{z-1}}} = \frac{1}{z}\underline{c^z} \iff \underline{\underline{\mathrm{R}}}\,\underline{\underline{c^{z-1}}} = \frac{1}{z}\underline{\underline{\mathrm{M}}}\,\underline{c^z} \iff \underline{\mathcal{X}} := \underline{\underline{\mathrm{R}}}\,\underline{\underline{c^{z-1}}} - \frac{1}{z}\underline{\underline{\mathrm{M}}}\,\underline{c^z} = \underline{0}, \qquad z = 1, \dots, s-1.$$

Recall $b_m = t^m$, $c_m = w_m$, $\underline{\underline{\mathrm{R}}}_{i,j} = \delta_{i,j} w_i$ and the definition of $\underline{\underline{\mathrm{M}}}$

$$\mathcal{X}_m := w_m (t^m)^{z-1} - \frac{1}{z}\left(\phi^m(1)\phi^j(1)(t^j)^z - \int_0^1 \frac{d}{d\xi}\phi^m(\xi)\phi^j(\xi)(t^j)^z d\xi\right).$$

Proof (2/4).

Now, the interpolation of t^z with $z \leq s-1$ with basis functions ϕ^j is exact. Hence, we can substitute $\phi^j(\xi)(t^j)^z = \xi^z$ for all $z = 1, \dots, s-1$, obtaining

$$\mathcal{X}_m = w_m (t^m)^{z-1} - \frac{1}{z}\left(\phi^m(1)1^z - \int_0^1 \frac{d}{d\xi}\phi^m(\xi)\xi^z d\xi\right).$$

Using the exactness of the quadrature for polynomials of degree $2s-3$, both true for Gauss–Lobatto and Gauss–Legendre, we know that the previous integral is exactly computed as $\frac{d}{d\xi}\phi^m(\xi)$ is of degree at most $s-2$ and ξ^z is at most $s-1$. So, we can use integration by parts and obtain

$$\mathcal{X}_m = w_m (t^m)^{z-1} - \frac{1}{z}\left(\phi^m(0)0^z + \int_0^1 \phi^m(\xi)\frac{d}{d\xi}\xi^z d\xi\right) = w_m (t^m)^{z-1} - \int_0^1 \phi^m(\xi)\xi^{z-1} d\xi = 0$$

by the exactness of the quadrature rule and the definition of w_m. Note that the condition is sharp, since the interpolation is not anymore exact for $z = s$, hence $C(s)$ is not satisfied.

Proof (3/4).

To prove $D(s-1)$, we write explicitly the condition in matricial form, for all $z = 1, \dots, s-1$

$$\underline{bc^{z-1}}\,\underline{\underline{A}} = \frac{1}{z}\underline{b(1-c^z)} \iff \underline{bc^{z-1}}\,\underline{\underline{\mathrm{M}}}^{-1}\,\underline{\underline{\mathrm{R}}} = \frac{1}{z}\underline{b(1-c^z)} \iff \underline{bc^{z-1}} = \frac{1}{z}\underline{b(1-c^z)}\,\underline{\underline{\mathrm{R}}}^{-1}\,\underline{\underline{\mathrm{M}}}.$$

Note that $b^m = w_m$ and $\underline{\underline{\mathrm{R}}}^m_r = w_m \delta^m_r$, so $\underline{b(1-c^z)}\,\underline{\underline{\mathrm{R}}}^{-1} = \underline{(1-c^z)}$. It is left to prove that

$$\mathcal{Y} := \underline{bc^{z-1}} - \frac{1}{z}\underline{(1-c^z)\underline{\underline{\mathrm{M}}}} = \underline{0}.$$

$$\mathcal{Y}_m = w_m (t^m)^{z-1} - \frac{1}{z}\sum_{j=1}^{s}\left(1-(t^j)^z\right)\left(\phi^j(1)\phi^m(1) - \int_0^1 \frac{d}{d\xi}\phi^j(\xi)\phi^m(\xi)d\xi\right).$$

Proof (4/4).

Let us observe that, since $z \leq s-1$, the polynomial is exactly represented by the Lagrangian interpolation $t^z = \sum_{j=1}^{s} \phi(t)(t^m)^z$. Hence, using the exactness of the quadrature for polynomials of degree at most $2s-3$, we have

$$\begin{aligned}\mathcal{Y}_m =& w_m(t^m)^{z-1} - \frac{1}{z}\left(1-(1)^z\right)\phi^m(1) + \frac{1}{z}\int_0^1 \frac{d}{d\xi}\left(1-(\xi)^z\right)\phi^m(\xi)d\xi \\ =& w_m(t^m)^{z-1} - \frac{1}{z}\int_0^1 z\,\xi^{z-1}\phi^m(\xi)d\xi = w_m(t^m)^{z-1} - w_m(t^m)^{z-1} = 0.\end{aligned}$$

Hence, ADER-Legendre and ADER-Lobatto satisfy $D(s-1)$. Note that the condition is sharp, since the interpolation is not anymore exact for $z=s$, hence $D(s)$ is not satisfied.

ADER Gauss–Legendre $\mathcal{L}^2$

Remark (ADER-Legendre is no collocation method)

From the proof of previous Lemma, we can observe that ADER-Legendre methods do not satisfy $C(s)$, hence, the methods are not collocation methods and they do not coincide with Gauss–Legendre implicit RK methods.

Theorem

$\mathcal{L}^2$ of ADER with Gauss–Legendre is of order $2s-1$.

Proof.

ADER-Legendre with $s = M + 1$ stages satisfies $B(2s)$ for the quadrature rule and, hence, it satisfies $B(2s-1)$. For previous Lemma it also satisfies $C(s-1)$ and $D(s-1)$. Hence, Butcher's (1964) Theorem ($p \leq \eta + \zeta + 1$ and $p \leq 2\eta + 2$) guarantees that the method is of order $2s-1$, since it is satisfied with $p = 2s-1$ and $\eta = \zeta = s-1$. □

ADER Gauss–Lobatto $\mathcal{L}^2$

Theorem

$\mathcal{L}^2$ of ADER with Gauss-Lobatto is of order $2s-2$.

Proof.

The condition for $B(2s-2)$ is satisfied as (c,b) is the Gauss–Lobatto quadrature with order $2s-2$. Previous Lemma guarantees that ADER-Lobatto satisfies $B(2s-2)$, $C(s-1)$ and $D(s-1)$, so Butcher's (1964) Theorem ($p \leq \eta + \zeta + 1$ and $p \leq 2\eta + 2$) is satisfied for order $p = 2s-2$ and $\eta = \zeta = s-1$. □

ADER Gauss–Lobatto $\mathcal{L}^2$

Theorem

$\mathcal{L}^2$ of ADER with Gauss-Lobatto is LobattoIIIC.

The Lobatto IIIC method is defined using the condition

$$a_{i1} = b_1 \qquad \text{for } i = 1, \dots, s. \tag{23}$$

Lemma

$\mathcal{L}^2$ of ADER with Gauss-Lobatto satisfies (23)*.*

Theorem (Chipman 1971)

Lobatto IIIC schemes (in particular RK a_{ij}) are uniquely determined by Gauss–Lobatto quadrature rule (c, b), condition (23) *and by $C(s-1)$.*

ADER Gauss–Lobatto $\mathcal{L}^2$

Lemma

$\mathcal{L}^2$ of ADER with Gauss-Lobatto satisfies (23)*.*

Proof.

$$a_{i1} = \sum_j (\underline{\underline{\mathrm{M}}}^{-1})_{ij} \mathbb{R}_{j1} = b_1 = w_1 \iff$$

$$\sum_{i,j} \underline{\underline{\mathrm{M}}}_{ki} (\underline{\underline{\mathrm{M}}}^{-1})_{ij} \mathbb{R}_{j1} = \sum_i \underline{\underline{\mathrm{M}}}_{ki} w_1 \iff$$

$$\delta_{k1} w_1 = \mathbb{R}_{k1} = \sum_i \underline{\underline{\mathrm{M}}}_{ki} w_1$$

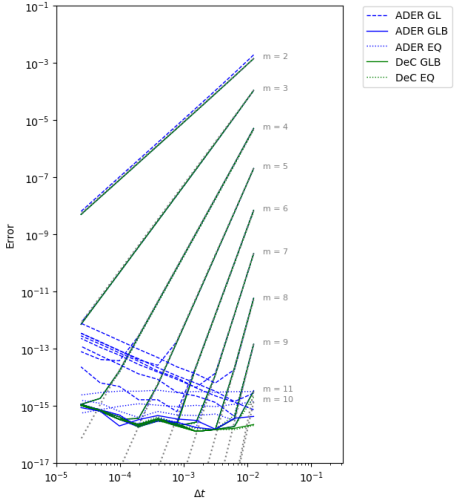
$$\sum_i \underline{\underline{\mathrm{M}}}_{ki} w_1 = \phi^m(1) w_1 - \int_0^1 \frac{d}{dt} \phi^m(\xi) w_1 dt = w_1 \phi^m(0) = w_1 \delta_{m,1}.$$

Outline

Convergence

$$
\begin{aligned}
y'(t) &= -|y(t)|y(t), \\
y(0) &= 1, \\
t &\in [0, 0.1].
\end{aligned}
\tag{24}
$$

Convergence curves for ADER and DeC, varying the approximation order and collocation of nodes for the subtimesteps for a scalar nonlinear ODE



Lotka–Volterra

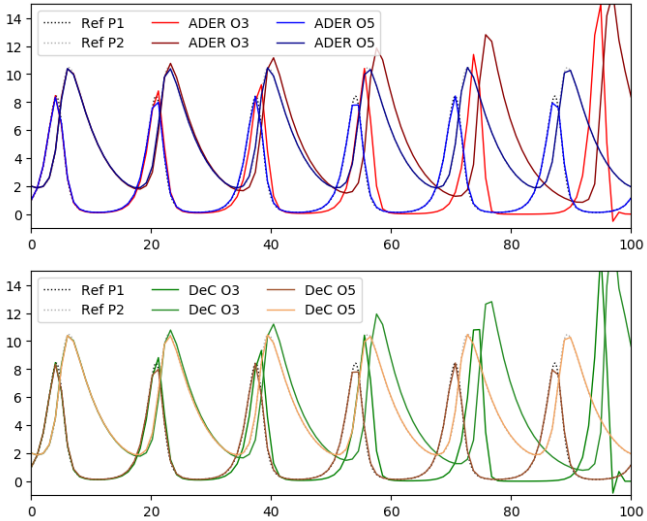
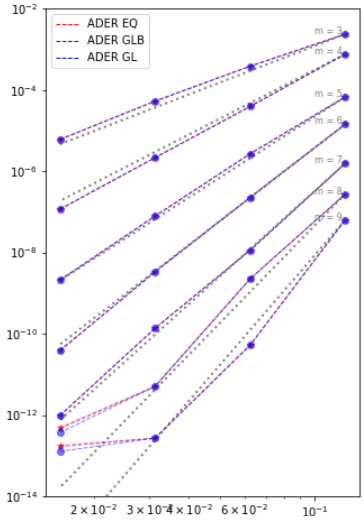


Figure: Numerical solution of the Lotka-Volterra system using ADER (top) and DeC (bottom) with Gauss-Lobatto nodes with timestep $\Delta T = 1$.

PDE: Burgers with spectral difference



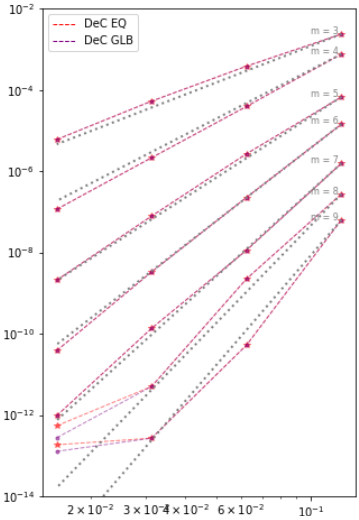


Figure: Convergence error for Burgers equations: Left ADER right DeC. Space discretization with spectral difference