- EXAM
- ORAL EXAM (THEORETICAL)
- CODING PROJECTS (PRACTICAL)

ORDINARY DIFFERENTIAL EQUATIONS ODE

$I = [t_0, t_{end}] \subset \mathbb{R}$

$y : I \to \mathbb{R}^S$ $\qquad F : I \times \mathbb{R}^S \to \mathbb{R}^S$

$$\begin{cases} \frac{dy}{dt} = F(t, y(t)) \simeq \underbrace{F(y(t))}_{\text{AUTONOMOUS ODE}} \\ y(0) = y_0 \end{cases}$$

$$y^{(p)} = \overline{f(t, y, y', y'', \dots, y^{(p-1)})}$$

HIGHER ORDER DERIVATIVE DIFF. EQ.

$$\left.\begin{aligned} y' &= z_1 \\ y'' = z_1' &= z_2 \\ y''' = z_2' &= z_3 \\ y^{(p)} = z_p' &= f(t, y, z_1, z_2, \dots, z_{p-1}) \end{aligned}\right]$$

$y \in \mathbb{R}^S$

$$\begin{pmatrix} y \\ z_1 \\ \vdots \\ z_p \end{pmatrix} \in \mathbb{R}^{S(p+1)}$$

SYSTEM OF 1st DERIVATIVE DIFF. EQ.

$$\int_{t_0}^{t} \frac{dy}{dt} dt = y(t) - y(t_0) = y(t) - y_0$$

$$y(t) = y_0 + \int_{t_0}^{t} F(t, y(t))\, dt$$

DAHLQUIST'S PROBLEM

$$\begin{cases} y'(t) = -\lambda y(t) \\ y(t_0) = y_0 \end{cases}$$

$\lambda \in \mathbb{R} \to \mathbb{C}$

$y(t) \neq 0 \qquad y_0 \geq 0$

$$\int_{t_0}^{t} \underbrace{\frac{y'(s)}{y(s)}}_{} ds = -\lambda \int_{t_0}^{t} ds$$

$$\int_{t_0}^{t} \frac{d}{ds}\left(\log(y(s))\right) ds = -\lambda (t-t_0)$$

$$\log(y(t)) - \log(y(t_0)) = -\lambda (t-t_0)$$

$$\log\left(\frac{y(t)}{y(t_0)}\right) = -\lambda (t-t_0)$$

$$\frac{y(t)}{y(t_0)} = e^{-\lambda(t-t_0)}$$

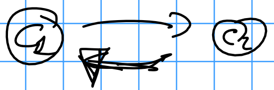
$$y(t) = y_0 \underbrace{e^{-\lambda \overbrace{(t-t_0)}^{>0}}}_{\neq 0}$$

$y_0 > 0$

$\lambda < 0$ → exp. INCREASING

$\lambda > 0$ → exp. to 0

PDS



$$\begin{cases} c_1'(t) = \overline{c_2(t)} - \underline{5c_1(t)} \\ c_2'(t) = 5c_1(t) - c_2(t) \end{cases}$$

$c_1, c_2 > 0$

$$\sum_i c_i(t) = M \qquad \forall t$$

$$\frac{d}{dt}\left(\sum_i c_i(t)\right) = c_1' + c_2' = c_2 - 5c_1 + 5c_1 - c_2 = 0$$

$$c'(t) = A \cdot c(t) \qquad A = \begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix}$$

$$c(0) = c_0 \qquad y' = \lambda y \rightarrow y = y_0 e^{\lambda t}$$

$$c(t) = \underline{e^{At}} c_0 \qquad e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

$$A^2 = -6A \qquad e^{At} = \sum_{k=1}^{\infty} \frac{(-6)^{k-1} A t^k}{k!} + 1$$

$$A^3 = -6A^2 = (-6)^2 A$$

$$A^k = (-6)^{k-1} A$$

$$e^{At} = \left(\underbrace{\sum_{k=0}^{\infty} \frac{(-6)^k \cdot t^k}{k!}} - 1 \right) \frac{A}{-6} + 1$$

$$\left(e^{-6t} - 1\right) \frac{A}{-6} + 1$$

$$c(t) = C_0 + \frac{1 - e^{-6t}}{6} A C_0$$

$$x'(t) = \alpha x - \beta x y$$
$$y'(t) = \delta x y - \gamma y$$

$\alpha, \beta, \gamma, \delta > 0$

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \text{EQUILIBRIUN}$$

$$\eta(x, y) = \delta x - \gamma \log(x) + \beta y - \alpha \log(y)$$

$$\frac{d}{dt} \eta(x(t), y(t)) = \begin{pmatrix} \partial_x \eta \\ \partial_y \eta \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} =$$

$$\langle \nabla \eta, F(x,y) \rangle = 0 = \cancel{\delta \alpha x} - \cancel{\delta \beta x y} - \cancel{\gamma \alpha}$$
$$+ \cancel{\gamma \beta y} + \cancel{\beta \delta x y}$$
$$- \cancel{\beta \gamma y} - \cancel{\alpha \delta x} + \cancel{\alpha \gamma}$$
$$= 0$$

$$\nabla \eta = \begin{pmatrix} \delta - \gamma / x \\ \beta - \alpha / y \end{pmatrix}$$

$$\begin{cases} u'(t) = -\dfrac{v(t)}{n(t)} \\ v'(t) = \dfrac{u(t)}{n(t)} \end{cases} \qquad n(t) = \sqrt{u^2(t) + v^2(t)}$$

$$\frac{n^2}{2}$$

$$\frac{d}{dt}\left(\frac{n^2}{2}\right) = \frac{d}{dt}\left(\frac{u^2 + v^2}{2}\right) = \frac{2u \cdot u' + 2v \cdot v'}{2} =$$

$$u \cdot u' + v \cdot v' = u \cdot \left(-\frac{v}{n}\right) + v\left(\frac{u}{n}\right) = 0$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{matrix} \cos(\theta)\, u^0 - \sin(\theta)\, v^0 \\ \sin(\theta)\, v^0 + \cos(\theta)\, u^0 \end{matrix} \qquad \theta = \frac{t}{r}$$

$$u' = -\frac{v}{r} - \alpha u$$

$$v' = \frac{u}{r} - \alpha v$$

$$\frac{d}{dt}\left(\frac{r^2}{2}\right) = \frac{d}{dt}\left(\frac{u^2+v^2}{2}\right) = u \cdot u' + v \cdot v'$$

$$= -u \cdot \cancel{\frac{v}{r}} - \alpha u^2 + v \cancel{\frac{u}{r}} - \alpha v^2 = -\alpha(u^2+v^2)$$

$$= -2\alpha \frac{u^2+v^2}{2} = -2\alpha\left(\frac{r^2}{2}\right)$$

$$E = \frac{r^2}{2} \qquad \frac{d}{dt} E = -2\alpha E$$

$$E = e^{-2\alpha t} E_0 \qquad \alpha > 0$$

$$\frac{r^2}{\cancel{2}} = e^{-2\alpha t} \frac{r_0^2}{\cancel{2}} \Rightarrow r(t) = \underbrace{e^{-\alpha t}}_{\searrow} r_0$$

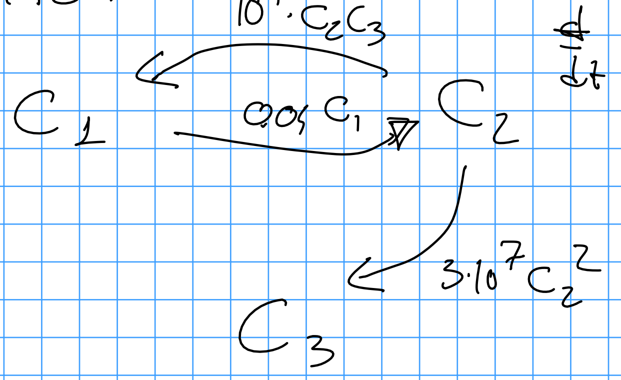
- $u'' = -\sin(u)$

$$\begin{cases} u' = v \\ v' = -\sin(u) \end{cases} \qquad \eta(t) = \frac{1}{2} v^2 - \cos(u)$$

$$\frac{d}{dt} \eta(t) = \langle \nabla \eta, F \rangle = \begin{pmatrix} \sin(u) \\ v \end{pmatrix} \cdot \begin{pmatrix} v \\ -\sin(u) \end{pmatrix} =$$

$$v \cdot \sin(u) - v \cdot \sin(u) = 0$$

ROBERTSON



$$\frac{d}{dt}\left(\sum_i c_i\right) = 0$$

• PDE $U(x,t) : \Omega \times [0,T] \to \mathbb{R}^d$

$$\partial_t U + \partial_{xx}(a(U)) + \partial_x b(U) + c(U) = 0$$

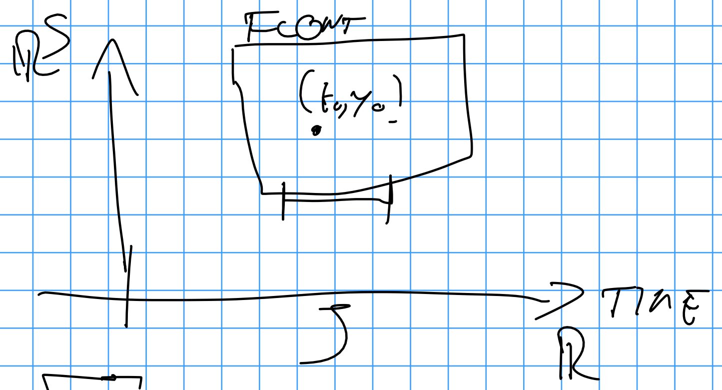
$\underline{U} = \mathbb{R}^N$ N # DOF IN SPACE

$$\partial_{xx} a(U) \approx D_2 A(\underline{U})$$

$\partial_t \underline{U} = f(\underline{U})$ $f(\underline{U}) = -(D_2 A(\underline{U}) + \dots)$

$\partial_t U + \partial_x b(U) = 0$ $\underline{\underline{\text{TVD}}}$

U_i^n $TV(\underline{U}^n) := \sum_i |U_{i+1}^n - U_i^n|$



$$\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}$$

- $y(t) = 0$
- $y(t) = \frac{t^2}{4}$ // $\frac{t}{2}$

$y' = \frac{\cancel{2}t}{\cancel{4}2}$

$F(t,y) : I \times \mathbb{R}^S \to \mathbb{R}^S$

$$\|F(t,y) - F(t,z)\| \leq L\,\|y-z\|$$

UNIFORMLY IN t $\Rightarrow \exists!$ SOLUTION

$y \in C^1(I)$ $\begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$

$F \in C^{p-1}$ p-differentiable

Y IS P+1 TIMES DIFFERENTIABLE

$$\frac{dy}{dt} = F$$

$$F^{(\ell+1)}(t,y) = \frac{\partial F^{(\ell)}}{\partial t}(t,y) + \frac{\partial F^{(\ell)}(t,y)}{\partial y}$$

ODE $\quad I = [t_0, t_{end}] \quad y : I \to \mathbb{R}^S$

$F : I \times \mathbb{R}^S \to \mathbb{R}^S$

$$\begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$t_0 = t^0 < t^1 < t^2 < \dots < t^n < \dots \; t^N = t_{end}$

$$\begin{tikzcd} {[t^n,} \arrow[d] & {t^{n+1}]} \arrow[d] \\ y^n \arrow[r, bend right] & y^{n+1} \end{tikzcd}$$

$$y(t^{n+1}) = y(t^n) + \int_{t^n}^{t^{n+1}} F(t, y(t))\,dt$$

$$\approx y(t^n) + F(t^n, y(t^n)) \cdot \underbrace{(t^{n+1} - t^n)}_{\Delta t^n}$$

$y^n \to y^{n+1}$

$$y^{n+1} := y^n + \Delta t^n \, F(t^n, y^n)$$

$$y^0 = y_0$$

$$c_1^{n+1} + c_2^{n+1} = c_1^n + \Delta t\left(\cancel{-5c_1^n} + \cancel{c_2^n}\right)$$

$$+ c_2^n + \Delta t\left(\cancel{-c_2^n} + \cancel{5c_1^n}\right)$$

$$= c_1^n + c_2^n = c_1^0 + c_2^0 = \text{MASS} \quad \forall n$$

$$e_n = y(t^n) - y^n$$

CONSISTENCY ERROR

$$\varepsilon_n = y(t^{n+1}) - \underline{y(t^n) - \Delta t \overbrace{F(t^n, y(t^n))}^{y'(t^n)}} =$$

$$= \int_{t^n}^{t^{n+1}} \underbrace{y'(t) - \underbrace{y'(t^n)}_{}}_{\text{COSTANT}} \, dt$$

$$\omega(f, \underline{\Delta t}) := \max_{t,t' :\, |t-t'|<\Delta t} |f(t) - f(t')|$$

$$|\varepsilon_n| \le \int_{t^n}^{t^{n+1}} \underbrace{\omega(y', \Delta t^n)}_{\text{DOESN'T DEPEND ON } t} \, dt = \Delta t^n \cdot \omega(y', \Delta t^n).$$

$$\underline{e_{n+1}} = \underline{y(t^{n+1})} - \underline{y^{n+1}} = \underline{y(t^{n+1})} - y^n - \Delta t^n F(t^n, y^n)$$

$$+ y(t^n) + \Delta t \, F(t^n, y(t^n))$$

$$- y(t^n) - \Delta t \, F(t^n, y(t^n))$$

$$= \varepsilon_n + \overbrace{y(t^n) - y^n}^{e_n} + \Delta t \left[F(t, y(t^n)) - F(t, y^n) \right]$$

$$|e_{n+1}| \le |\varepsilon_n| + |e_n| + \Delta t \, L \, |\underbrace{y(t^n) - y^n}_{e_n}| \le \ast$$

$$|F(t,y) - F(t,z)| \le L|y-z|$$ (HYPOTHESIS ON F TO HAVE UNIQUENESS)

$$\ast \le \underbrace{|\varepsilon_n|}_{} + (1 + \Delta t L) \underbrace{|e_n|}_{}.$$

$$\le \dots \le e^{L|t^{n+1}-t^0|} |e_0| + \sum_{i=0}^{n} e^{L(\overbrace{t^{n+1} - t^{i+1}}^{\Delta t})} |\varepsilon_i| \le \circledast$$

$$|\varepsilon_n| \le \Delta t \, \omega(y', \Delta t)$$

$$\circledast \le \underbrace{e^{L|t^{n+1}-t^0|} |e_0|}_{} + \overbrace{\omega(y', \Delta t)}^{} \, \frac{e^{L(t^{n+1}-t^0)} - 1}{L}.$$

$$\omega(y', \Delta t) = \omega(F(\cdot, y(\cdot)), \Delta t)$$

$$\left[\sum_{i=0}^{n} \Delta t \, e^{L \Delta t} = \underbrace{N \cdot \Delta t}_{} \, e^{L \Delta t} = \frac{T}{\cancel{\Delta t}} \cancel{\Delta t} \, e^{L \textcircled{\Delta t}} \dots \right]$$

$$w(y', \Delta t) = \max_{t,t' : |t-t'| < \Delta t} |y'(t) - y'(t')| = $$ (AUTONOMOUS)

$$= \max_{t,t' : |t-t'| \le \Delta t} |F(y(t)) - F(y(t'))|$$

$$\le L\, |y(t) - y(t')| \xrightarrow[\Delta t \to 0]{} 0$$

$$\le L \underset{\substack{\| \\ \max_I y'}}{C} \cdot \Delta t$$

$$\boxed{|y(t^n) - y^n| \le C_0 |y^0 - y(t^0)| + \Delta t \cdot C_1}$$

$$e_1 = |\underbrace{y^1}_{} - y(t^1)| = |\cancel{y^0} + \cancel{\Delta t \overset{y'(t_0)}{\overbrace{F(t^0, y^0)}}}$$

$$- \left(\cancel{y(t^0)} + \cancel{\Delta t\, y'(t_0)} + \frac{\Delta t^2}{2} y''(t_0)\right.$$

$$\left. + O(\Delta t^3)\right|$$

$$= \left|\frac{\Delta t^2}{2} y''(t_0)\right| + O(\Delta t^3) = \underline{O(\Delta t^2)}$$

$$e_N \approx \sum_{i=1}^{N} |y(t^i) - y^i| \lesssim N \cdot \frac{\Delta t^2}{2} \max |y''|$$

$$= \frac{t_{end}}{\cancel{\Delta t}} \cdot \frac{\Delta t^{\cancel{2}}}{2} \max |y''|$$

$$= t_{end} \cdot \frac{\Delta t}{2} \max |y''|$$

$$= C \cdot \Delta t = O(\Delta t)$$

FIRST ORDER

DEF
P ORDER SCHEME
LARGEST P $\Delta t \to 0$
$|e_N| \le C \cdot \Delta t^P$

$y'(t) = M y(t) \qquad M \in \mathbb{R}^{s \times s} \qquad \Delta t$

$$y^n = y^{n-1} + \Delta t M y^{n-1} = (I + \Delta t M) y^{n-1}$$

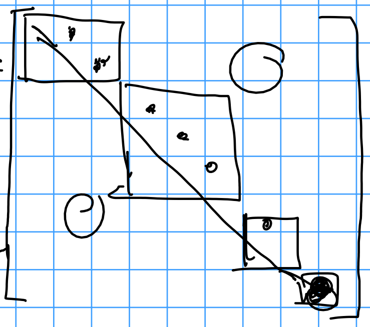
$$= (I + \Delta t M)^n y^0$$

$|y^n| \overset{?}{\leq} |y^0| \rightsquigarrow \|(I+\Delta t M)^n\|$

$\in \mathbb{C}^{S \times S}$

$M \to \hat{M} = S^{-1} M S$

$\hat{M} =$



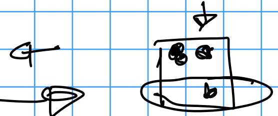
$S^{-1} y^n = S^{-1} y^{n-1} + \Delta t\, S^{-1} M \underset{S \cdot S^{-1}}{\wedge} y^{n-1}$

$S^{-1} y^n =: \underline{\hat{y}^n}$

$\hat{y}^n = \hat{y}^{n-1} + \Delta t\, \hat{M}\, \hat{y}^{n-1}$

$= (I + \Delta t \hat{M})\, \hat{y}^{n-1}$

$\hat{y}_s \to$



$\boxed{y'(t) = q\, y(t)}$ $\quad q \in \mathbb{C}$

$q \in \mathbb{R}$

$\boxed{q \leq 0}$ $\mathrm{Re}(q) \leq 0$

$q > 0$

$y^{n+1} = y^n + q \Delta t\, y^n = (1 + q\Delta t)\, y^n$

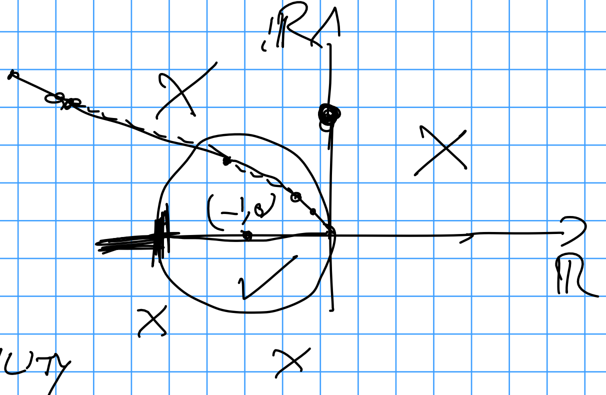
$= (1 + q \Delta t)^{n+1} y^0$

$|y^{n+1}| \leq |y^n| \leq \dots$ $\quad |y^{n+1}| = |y^n| \downarrow$

$|1 + q\Delta t|\, |y^n| \leq |y^n|$

$|1 + \underbrace{q \Delta t}_{\underline{\underline{z \in \mathbb{C}}}}| \leq 1$

$|1 + z| \leq 1$



$R(z) = 1 + z$ STABILITY FUNCTION

$y^{n+1} = R(\Delta t q) \cdot y^n$ $\quad z = \Delta t q$

$\{ |R(z)| \leq 1 \}$ STABILITY REGION

$q \in \mathbb{R}$ $\quad |q| \Delta t \leq C \Rightarrow \Delta t \leq \frac{C}{|q|} \simeq \frac{C}{|\lambda|}$

$u' = v$

$v' = -u$

$u(0) = 0 \quad v(0) = 1$

$r^2 = u^2 + v^2$ = CONSTANT IN TIME

$$\frac{d}{dt} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = 0$$

$$\underline{u} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{v}$$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y^{n+1} = y^n + \Delta t f(t^{n+1}, y^{n+1})$ FIRST ORDER

LINEAR STABILITY

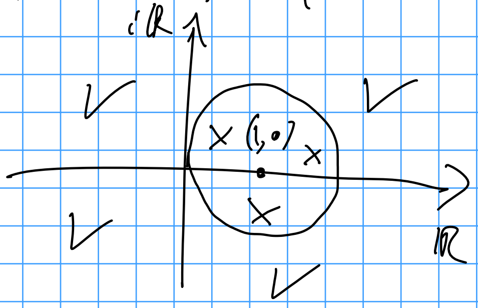
$y' = qy \qquad q \in \mathbb{C}$

$y^{n+1} = y^n + \Delta t q y^{n+1}$

$$(1 - \underbrace{\Delta t q}_{z \in \mathbb{C}}) y^{n+1} = y^n$$

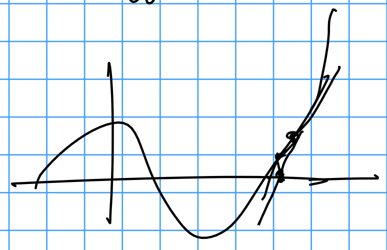
$$y^{n+1} = R(z) \cdot y^n = \overbrace{\frac{1}{1-z}}^{R(z)} y^n$$

$$\{|R(z)| \leq 1\} = \left\{\frac{1}{|1-z|} \leq 1\right\} = \{|1-z| \geq 1\}$$



$y^{n+1} = y^n + f(y^{n+1}) \Delta t$

$y^{n+1} \leftarrow u^{(k)}$



IMPLICIT EULER + POSITIVITY

$\forall \Delta t$

PDS $$\frac{d}{dt} c_i = P_i(c) - D_i(c)$$

$$= \sum_j \underset{\overset{\vee}{0}}{p_{ij}(c)} - \underset{\overset{\vee}{0}}{d_{ij}(c)}$$

$$\boxed{p_{ij} = d_{ji}} \geq 0$$

1. CONSERVATION

$$\frac{d}{dt}\left(\sum_i c_i(t)\right) = \sum_i \frac{d}{dt} c_i(t) = \sum_i \sum_j p_{ij}(c) - d_{ij}(c)$$

$$= \sum_i \sum_j p_{ij} - \sum_i \sum_j d_{ji} = \sum_i \sum_j p_{ij} - p_{ij} = 0$$

- HOMEWORK CHECK THAT IMPLICIT EULER IS CONSERVATIVE.

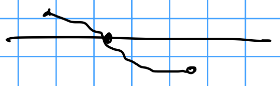
2. POSITIVITY $\quad \boxed{c_i \to 0 \Rightarrow D_i(c) \to 0 \quad \forall j \ \forall i}$

$$d_{ij}(c) \to 0$$

$$c_i(0) > 0 \Rightarrow c_i(t) \geq 0 \quad \forall i \quad \forall t$$

PROOF By CONTRADICTION $\quad t : c_i(t) < 0$

$$\exists\, t^* \quad c_i(t^*) = 0 \quad c_i'(t^*) < 0$$



$$c_i'(t^*) = \underbrace{P_i(c(t^*))}_{\geq 0} - \underbrace{D_i(c(t^*))}_{= 0} \geq 0 \quad ↯$$

IMPLICIT EULER + LINEAR PDS + POSITIVITY

$$y' = My \qquad M_{ii} < 0 \qquad M_{ij} \geq 0 \quad i \neq j$$

$$\sum_i M_{ij} = 0$$

IMPLICIT EULER

$y^0 \geq 0$

$(\underbrace{I - \Delta t M}_{A}) y^{n+1} = y^n \Rightarrow y^{n+1} = \underbrace{A^{-1}} y^n$

$A^{-1} \geq 0 \iff \forall_{ij} (A^{-1})_{ij} \geq 0$

1. $0 < A_{ii} = \underbrace{1 - \Delta t \underbrace{M_{ii}}_{<0}}_{>0} = 1 + \Delta t |M_{ii}| = 1 + \Delta t \sum_{j \neq i} M_{ji}$

$\boxed{|A_{ii}| > \sum_{j \neq i} |A_{ji}|}$

$\sum_j M_{ji} = 0$

$0 > M_{ii} = -\sum_{j \neq i} \overbrace{M_{ji}}^{\geq 0} < 0$

$\underbrace{\overbrace{1}^{>0} + \overbrace{\Delta t}^{>0} \overbrace{\sum_{j \neq i} M_{ji}}^{\geq 0}} > \Delta t \sum_{j \neq i} M_{ji} = \sum_{j \neq i} |A_{ji}|.$

A DIAG. DOMINANT BY COLUMN

- $A^{-1} \geq 0$?   JACOBI METHOD

$A = D + (A - D)$ $D = \text{diag}(A)$

$\underline{Au = R} \iff Du + \underbrace{(A-D)}u = R$

$Du = R - (A-D)u \Leftarrow ✓$ $u = D^{-1}(D-A)u + D^{-1}\cancel{R}$

$Du^{(k)} = R - (A-D)u^{(k-1)}$

$u^{(k)} = D^{-1}(D-A)u^{(k-1)} + D^{-1}\cancel{R}$

$\|u - u^{(k)}\|_\infty = \|D^{-1}(D-A)(u^{(k-1)} - u)\|$

$\leq \|D^{-1}(D-A)\|_\infty \|u - u^{(k-1)}\|_\infty$

$\|B\|_\infty = \max_i \sum_j |B_{ji}|$ $B = D^{-1}(D-A)$

$= (I - D^{-1}A)$

$$\left(I - D^{-1}A\right)_{ji} = \begin{bmatrix} 0 & & \ddots \\ & 0 & \\ \ddots & & 0 \end{bmatrix}$$

$$\max_i \sum_{j\neq i} \frac{|A_{ji}|}{|A_{ii}|} = \max_i \frac{\sum_{j\neq i} |A_{ji}|}{|A_{ii}|} < 1$$

$$\|u - u^{(k)}\|_\infty < \|u - u^{(k-1)}\|_\infty$$

$$u^{(k)} \geq 0 \qquad\qquad A y^{n+1} = y^n$$

$$\Downarrow$$

$$U^{(k)} = \underbrace{D^{-1}}_{\geq 0}\underbrace{(D-A)}_{\geq 0}\underbrace{U^{(k-1)}}_{\geq 0} + \underbrace{D^{-1}}_{\geq 0}\underbrace{y^n}_{\geq 0} \geq 0$$

$$\begin{cases} \Delta t \Pi_{ij} \geq 0 & i \neq j \\ 0 & i = j \end{cases}$$

$$u^{(k)} \to u = y^{n+1} \geq 0. \qquad \boxtimes$$

$$|e_n| \leq e^{L|t^n - t^0|}|e_0| + \underline{\Delta t}\, \omega(y', \Delta t) \cdot \sum_{i=0}^{n-1} \underbrace{e^{L(t^n - t^{i+1})}}$$

$$= \cdots + \omega(\ldots) \cdot \sum_{i=0}^{n-1} \int_{t^i}^{t^{i+1}} \underbrace{e^{L(t^n - \underbrace{t^{i+1}})}} dt$$

$$\leq \qquad + \omega(\ldots) \sum_{i=0}^{n-1} \int_{t^i}^{t^{i+1}} e^{L(t^n - t)} dt$$

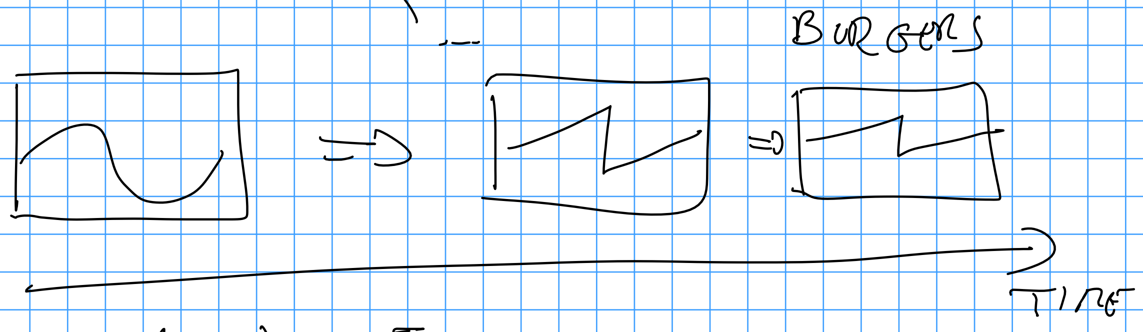
$$= \cdots + \omega \cdots \int_{t^0}^{t^n} e^{L(t^n - t)} dt$$

$$= \cdots + \omega \left[\frac{e^{L(t^n - t)}}{-L} \right]_{t^0}^{t^n}$$

$$= \cdots + \omega \frac{e^{L(t^n - t^0)} - 1}{L}.$$

$\partial_t u + \partial_x F(u) = 0 \qquad u : \underset{\subset \mathbb{R}}{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$

$F(u) = \begin{cases} u & \text{TRASPORT} \\ \frac{u^2}{2} & \text{BURGERS} \\ \dots \end{cases}$



$TV(U^n) := \sum_J \left| U_J^n - U_{J-1}^n \right|$

HARTEN FORM : INCREMENTAL FORM

$\partial_t U_J = C_{J+1/2}(U_{J+1} - U_J) - D_{J-1/2}(U_J - U_{J-1})$

$C_{J+1/2},\ D_{J-1/2} > 0 \qquad \forall J$

CHECK THAT IMPLICIT EULER DISCR OF INCREMENTAL FORM PDE IS TVD.

I.E. $TV(U^{n+1}) \leq TV(U^n) \quad \forall \Delta t > 0.$

$$U_J^{n+1} = U_J^n + \Delta t\Big[C_{J+1/2}(U_{J+1}^{n+1} - U_J^{n+1}) - D_{J-1/2}(U_J^{n+1} - U_{J-1}^{n+1})\Big]$$

$$U_{J+1}^{n+1} - U_J^{n+1} = U_{J+1}^n - U_J^n + \Delta t\Big[C_{J+3/2}(U_{J+2}^{n+1} - U_{J+1}^{n+1}) - C_{J+1/2}(U_{J+1}^{n+1} - U_J^{n+1}) - D_{J+1/2}(U_{J+1}^{n+1} - U_J^{n+1}) + D_{J-1/2}(U_J^{n+1} - U_{J-1}^{n+1})\Big]$$

$$\underbrace{\Big[1 + \Delta t\, C_{J+1/2} + \Delta t\, D_{J+1/2}\Big]}_{>0}(U_{J+1}^{n+1} - U_J^{n+1}) = U_{J+1}^n - U_J^n + \Delta t\Big[C_{J+3/2}(U_{J+2}^{n+1} - U_{J+1}^{n+1}) + D_{J-1/2}(U_J^{n+1} - U_{J-1}^{n+1})\Big]$$

$$\sum_J \left(1+\Delta t\, C_{J+1/2}+\Delta t\, D_{J+1/2}\right)\left|U^{n+1}_{J+1}-U^{n+1}_J\right|$$

$$\leq \underbrace{\sum_J \left|U^n_{J+1}-U^n_J\right|}_{TV(U^n)} + \Delta t\sum_J C_{J+3/2}\left|U^{n+1}_{J+2}-U^{n+1}_{J+1}\right| + \Delta t\sum_J D_{J-1/2}\left|U^{n+1}_J-U^{n+1}_{J-1}\right|$$

$$\sum_J C_{J+3/2}\left|U^{n+1}_{J+2}-U^{n+1}_{J+1}\right| = \sum_J C_{J+1/2}\left|U^{n+1}_{J+1}-U^{n+1}_J\right|$$

$$\sum_J D_{J-1/2}\left|U^{n+1}_J-U^{n+1}_{J-1}\right| = \sum_J D_{J+1/2}\left|U^{n+1}_{J+1}-U^{n+1}_J\right|$$

$$\sum_J \left|U^{n+1}_{J+1}-U^{n+1}_J\right| + \cancel{\Delta t\sum_J C_{J+1/2}\left|U^{n+1}_{J+1}-U^{n+1}_J\right|} + \cancel{\Delta t\sum_J D_{J+1/2}\left|U^{n}_{J+1}-U^{n}_J\right|}$$

$$\leq TV(U^n) + \cancel{\Delta t\sum_J C_{J+1/2}\left|U^{n+1}_{J+1}-U^{n+1}_J\right|} + \cancel{\Delta t\sum_J D_{J+1/2}\left|U^{n+1}_{J+1}-U^{n+1}_J\right|}$$

$$\Rightarrow TV(U^{n+1}) \leq TV(U^n) \qquad \boxtimes$$

$$y' = \lambda y$$

$$\frac{y^{n+1}-y^n}{\Delta t} = \lambda y^{n+1} \Rightarrow \underbrace{(I-\Delta t\lambda)}\, y^{n+1} = y^n$$

$$e(\Delta t) = C\cdot\Delta t^p$$

$$\underbrace{\log(e(\Delta t))} = \log(C\cdot\Delta t^p) = \log(C) + \underbrace{\log(\Delta t)}\cdot\boxed{p}$$

SLOPE

$$|y^{n+1}| < |y^n|$$

$y^n \longmapsto y^{n+1}$ $\qquad$ 2^{nd} ORDER ? $\qquad$ $\theta\in\mathbb{R}$

$$y^* = y^n + \theta\Delta t\, F(t^n, y^n)$$

$$y^{n+1} = y^n + \Delta t\left[\frac{2\theta-1}{2\theta}F(t^n,y^n) + \frac{1}{2\theta}F(t^n+\Delta t\theta, y^*)\right]$$

CONSISTENCY LOCAL ERROR

y^n KNOWN EXACT

$\Downarrow$

ERROR y^{n+1} ?

$$y(t^{n+1}) = y^n + \Delta t\, y'(t^n) + \frac{\Delta t^2}{2} y''(t^n) + O(\Delta t^3)$$

$$y(t^*) = y(t + \theta \Delta t) = y^n + \theta \Delta t\, y'(t^n) + \frac{\theta^2 \Delta t^2}{2} y''(t^n) + O(\Delta t^3)$$

$\Downarrow$

$$y^* = \overbrace{y^n + \theta \Delta t F(y^n)} = y(t^*) + O(\Delta t^2)$$

$$y^{n+1} = y^n + \Delta t \left[\frac{2\theta - 1}{2\theta} F(y^n) + \frac{1}{2\theta} F(y^*) \right]$$

$$= y^n + \Delta t \left[\frac{2\theta - 1}{2\theta} F(y^n) + \frac{1}{2\theta} \left(F(y^n) + \frac{\partial F}{\partial y}(y^n)\, \theta \Delta t \overbrace{F(y^n)}^{y''(t^n)} \right) \right]$$

$$= y^n + \Delta t F(y^n) + \frac{\cancel{\theta}}{2\cancel{\theta}} \Delta t^2 \frac{d}{dt} F(y(t^n)) =$$

$$= y^n + \Delta t F(y^n) + \frac{\Delta t^2}{2} \frac{d}{dt} y'(t^n)$$

$$= y^n + \Delta t\, y'(t^n) + \frac{\Delta t^2}{2} y''(t^n) = y(t^{n+1}) + O(\Delta t^3)$$

$$y(t^{n+1}) = y^n + y' \Delta t + \frac{\Delta t^2}{2} y'' + \underline{\frac{\Delta t^3}{6} y'''}$$

$\Rightarrow$ GLOBAL ERROR $O(\Delta t^2)$

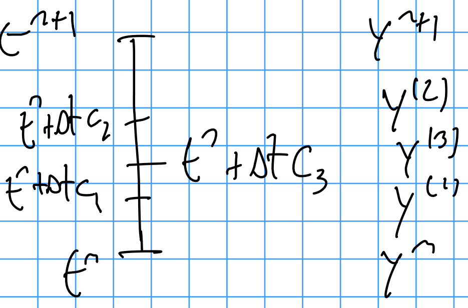
RUNGE-KUTTA

S STAGES $\quad A \in \mathbb{R}^{S \times S}$ $\quad c, b \in \mathbb{R}^S$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

$$A = (a_{KJ})_{K,J=1}^{S}$$

$$\begin{cases} y^{(K)} = y^n + \Delta t \sum_{J=1}^{S} a_{KJ} F(t^n + \Delta t c_J, y^{(J)}) & K = 1, \dots, S \\ y^{n+1} = y^n + \Delta t \sum_{J=1}^{S} b_J F(t^n + \Delta t c_J, y^{(J)}) \end{cases}$$



EXPLICIT RK

$$y^{(k)} = y^n + \Delta t \sum_{j=1}^{k-1} a_{kj} F(y^{(j)})$$

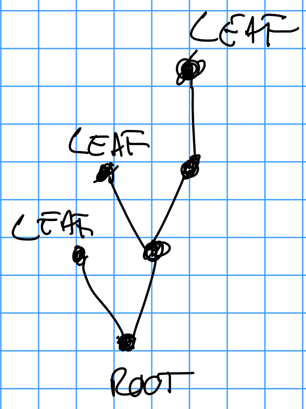
θ - SCHEME

$$\begin{cases} y^{(1)} = y^n + \Delta t \left[0 \cdot F(y^{(1)}) + 0 \cdot F(y^{(2)}) \right] \\ y^{(2)} = y^n + \Delta t \left[\theta \, F(y^{(1)}) + 0 \cdot F(y^{(2)}) \right] \\ y^{n+1} = y^n + \Delta t \left[\frac{2\theta - 1}{2\theta} F(y^{(1)}) + \frac{1}{2\theta} F(y^{(2)}) \right] \end{cases}$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \theta & \theta & 0 \\ \hline & \frac{2\theta-1}{2\theta} & \frac{1}{2\theta} \end{array} \qquad \begin{array}{c|cc} 0 & & \\ \theta & \theta & \\ \hline & \frac{2\theta-1}{2\theta} & \frac{1}{2\theta} \end{array}$$

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ c_2 & a_{21} & 0 & 0 & 0 \\ c_3 & a_{31} & a_{32} & 0 & 0 \\ c_4 & a_{41} & a_{42} & a_{43} & 0 \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

ROOTED TREES

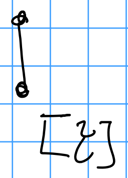


$\# \text{ NODES} \xrightarrow{\text{ORDER}} \text{ORDER}$

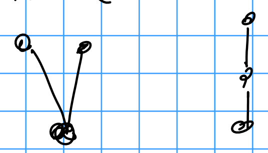
FOR WHICH WE ARE USING THIS CONDITION

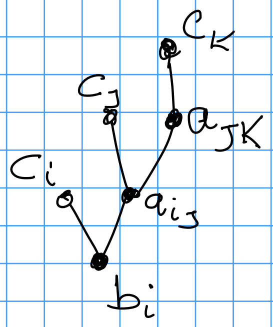
ORDER 1

ORDER 2



ORDER 3





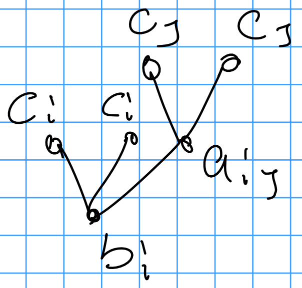
$\Phi(a_{iJ}, b_J, c_J) = \frac{1}{\gamma}$ $\gamma \in \mathbb{N}$

Φ ? POLYNOMIAL

$\gamma \in \mathbb{N}$?

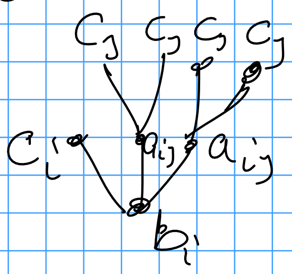
↓

$$\sum_{i,J,K=1}^{s} b_i a_{iJ} c_i a_{JK} c_J c_K = \Phi(a, b, c)$$

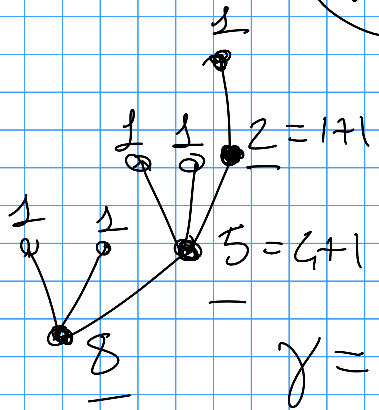


$\Phi(A, b, c) = \sum_{i,J} b_i a_{iJ} c_i^2 c_J^2$

$\boxed{\gamma = ?}$

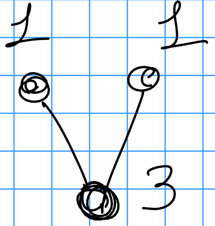


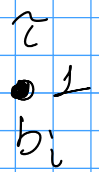
$b_i a_{iJ}^2 c_i c_J^4$



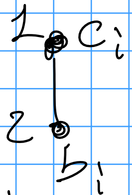
$\gamma = 80$

$\Phi(A, b, c) = \frac{1}{\gamma}$

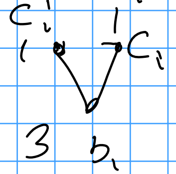




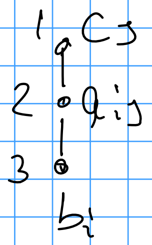
$\sum_i b_i = \frac{1}{1}$ $\gamma = 1$



$\sum_i b_i c_i = \frac{1}{2}$ $\gamma = 2$



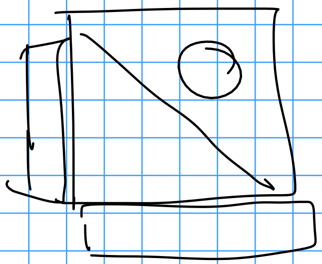
$\sum_i b_i c_i^2 = \frac{1}{3}$



$\sum_{i,J} b_i a_{iJ} c_J = \frac{1}{6}$

S stages   EXPLICIT   RK

$$\frac{S^2+3S-2}{2}$$



5 STAGES $\Longrightarrow$ ORDER ≤ 4

$$\boxed{C_i = \sum_j a_{ij}} \nleftarrow$$

(LINEAR)

STABILITY FOR RK METHODS

$y' = qY$ $q \in \mathbb{C}$ $z = q \cdot \Delta t$

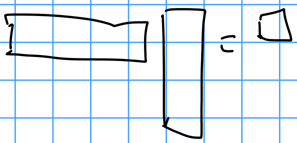
$y^{n+1} = R(z) \cdot y^n$ $S = \{z : |R(z)| \le 1\}$

$\underline{Y} \in \mathbb{R}^S$ $\underline{Y} = \underline{1} \cdot y^n + \Delta t\, \underline{\underline{A}} \cdot \underline{F(\underline{Y})}$

$$y^{(k)} = y^n + \Delta t \sum_{j=1}^{S} a_{kj} F(y^{(j)})$$

$F(y) = q \cdot y$ $F(\underline{Y}) = q \cdot \underline{Y}$

$\boxed{RK}$ $\underline{Y} = \underline{1} y^n + z \cdot \underline{\underline{A}} \cdot \underline{Y}$



$$(\underline{\underline{I}} - z\underline{\underline{A}})\underline{Y} = y^n \underline{1}$$

$$y^{n+1} = y^n + \Delta t \sum_j b_j F(y^{(j)}) = y^n + \Delta t \cdot b^T \cdot F(\underline{Y})$$

$$= y^n + \Delta t \cdot q\, \underline{b}^T \underline{Y}$$

$$y^{n+1} = y^n + \Delta t\, q\, b^T \underline{Y} = y^n + \underbrace{\Delta t\, q}_{z}\, b^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \cdot \underline{1}\, y^n$$

$$= \left[1 + z \cdot b^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \cdot \underline{1}\right] \cdot y^n$$

$y^{n+1} = R(z) \cdot y^n$ $R(z) = 1 + z\, b^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \underline{1}$

$I = 1$ $z = 1$ (σ^{-1})

$A = 1$

$R(z) \to$ POLYNOMIAL FOR EXPLICIT RK

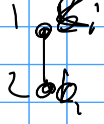
$\to$ FRACTION OF TWO POLY IMPLICIT RK

$$\begin{array}{c|ccc} 0 & & & \\ 2/3 & 2/3 & & \\ 2/3 & 1/3 & 1/3 & \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$$

ORDER 1

$\sum b_i = 1 \iff \frac{1}{4} + 0 + \frac{3}{4} = 1$ ✓

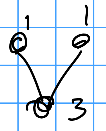
ORDER 2



$\sum_i b_i c_i = \frac{1}{2}$

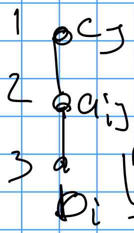
$0 \cdot \frac{1}{4} + \frac{2}{3} \cdot 0 + \frac{\cancel{2}^{1}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}_{2}} = \frac{1}{2}$ ✓

ORDER 3



$\sum_i b_i c_i^2 = \frac{1}{3}$

$0 + 0 + \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{2}}{3} \cdot \frac{\cancel{2}}{\cancel{3}} = \frac{1}{3}$ ✓

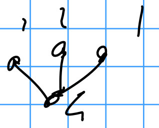


$\boxed{\gamma=6}$

$\sum_{i,j} b_i a_{ij} c_j = \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{2}}{3} \frac{1}{\cancel{4}} + \frac{\cancel{2}^{1}}{3} \cdot \frac{1}{3} \cdot \frac{1}{\cancel{4}_{2}} =$

$= \frac{1}{9} + \frac{1}{18} = \frac{3}{18} = \frac{1}{6}$ ✓

ORDER 4



$\frac{1}{4}$

$\sum b_i c_i^3 = \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{2}}{\cancel{3}} \frac{\cancel{2}}{3} \frac{2}{3} = \frac{2}{9} \neq \frac{1}{4}$ NOT ORDER 4

STABILITY FUNCTION

$$R(z) = 1 + \Delta t \, \underline{b}^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \underline{1}$$

$$(I - zA)^{-1}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 2/3 & 0 & 0 \\ 1/3 & 1/3 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2/9 & 0 & 0 \end{bmatrix} \quad A^3 = \underline{\underline{0}}$$

$$A^k = \underline{\underline{0}}$$

$$k \geq 3$$

$$(I - zA)^{-1} = \underbrace{I + zA + z^2A^2}_{} + \underbrace{z^3A^3}_{=0} + \underbrace{\cdots\cdots}_{=0}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ z\frac{2}{3} & 1 & 0 \\ z\frac{1}{3} + z^2\frac{2}{9} & z\frac{1}{3} & 1 \end{bmatrix}$$

$$R(z) = 1 + z \begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ z\frac{2}{3} & 1 & 0 \\ \frac{z}{3} + \frac{2}{9}z^2 & \frac{z}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 1 + z \begin{bmatrix} \frac{1}{4} + \frac{z}{4} + \frac{1}{6}z^2, & \frac{z}{4}, & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 1 + z \left(\frac{1}{4} + \frac{z}{4} + \frac{1}{6}z^2 + \frac{z}{4} + \frac{3}{4} \right)$$

$$= 1 + z + \frac{z^2}{2} + \frac{1}{6}z^3$$

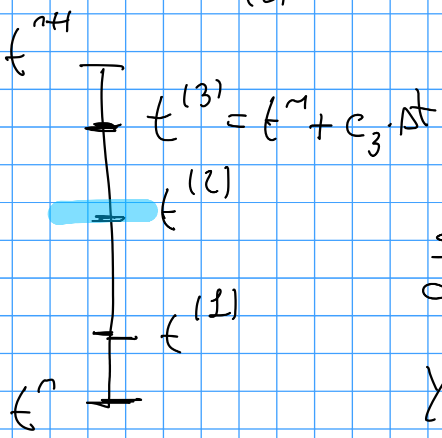
IMPLICIT RK

$$y^{(k)} = y^n + \sum_{j=1}^{S} a_{kj} \, F(\underline{y^{(j)}})$$

S EQUATIONS

S x DIM equations

$$Y(t) = \sum_{i=1}^{S} \varphi_i(t) \, \underbrace{y(t^n + c_i \Delta t)}_{y^{(i)}} + O(\Delta t^{p+1})$$



$$\varphi_i(t^{(j)}) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\frac{d}{dt} y = F(y)$$

$$y(t) = y^n + \int_{t^n}^{t} F(y(s)) \, ds$$

$$y^{(i)} \left(= y(t^{(i)})\right) = y^n + \int_{t^n}^{t^{(i)}} F(s, y(s))\, ds$$

$$\underset{\text{INTERPOLATING } F}{\approx} y^n + \int_{t^n}^{t^{(i)}} \sum_{j=1}^{S} \varphi_j(s) \cdot F(t^{(j)}, y^{(j)})\, ds$$

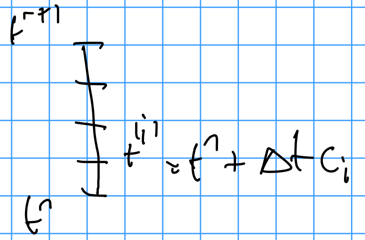
$$= y^n + \sum_{j=1}^{S} \underbrace{\int_{t^n}^{t^{(i)}} \varphi_j(s)\, ds}_{:= \Delta t\, a_{ij}} \; F(t^{(j)}, y^{(j)})$$

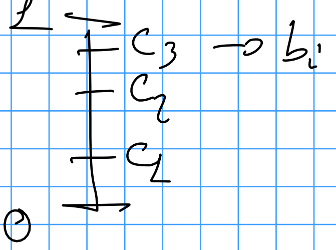
$$= y^n + \sum_{j=1}^{S} \Delta t\, a_{ij} \; F(t^{(j)}, y^{(j)})$$

$$y^{n+1} = y^n + \sum_{j=1}^{S} \underbrace{\int_{t^n}^{t^{n+1}} \varphi_j(s)\, ds}_{:= \Delta t\, b_j} \; F(t^{(j)}, y^{(j)})$$

B, C, D CONDITIONS

$$B(p): \quad \sum_{i=1}^{S} b_i c_i^{z-1} = \frac{1}{z} \quad \boxed{\forall\, z = 1, \dots, p.}$$





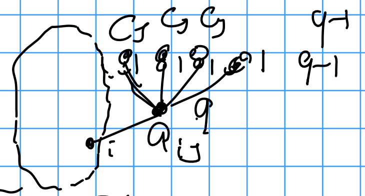
$$\int_0^1 t^{z-1} = \left[\frac{t^z}{z}\right]_0^1 = \frac{1}{z}$$

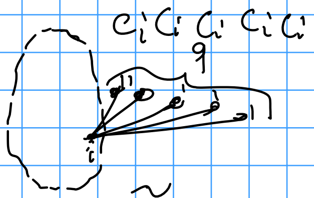
QUADRATURE FORMULA b_i WEIGHTS c_i NODES

$$\int_0^1 t^{z-1} = \sum_i b_i c_i^{z-1} = \frac{1}{z}$$

$B(p)$ QUAD (b_i, c_i) IS EXACT POLY degree UP TO $p-1$

$$C(\eta): \quad \sum_{j=1}^{S} a_{ij} c_j^{q-1} = \frac{c_i^q}{q} \qquad \forall i = 1, \dots, S \quad \forall q = 1, \dots, \eta$$





$\tilde{\Phi}, \tilde{\gamma}$ $\tilde{\Phi}, \tilde{\gamma}$

$\Phi = \sum_{ij} \tilde{\Phi}_i \cdot a_{ij} \cdot c_j^{q-1}$ $\Phi = \sum_i \tilde{\Phi}_i\, c_i^q$

$\gamma = \tilde{\gamma} \cdot q$ $\gamma = \tilde{\gamma}$

$$\Phi = \frac{1}{\gamma}$$

$$\gamma \Phi \quad = \quad \gamma \Phi$$

$$\sum_{ij} \tilde{\Phi}_i a_{ij} c_j^{q-1} \cdot q \cancel{\tilde{\gamma}} = \sum_i \tilde{\Phi}_i\, c_i^q \cdot \cancel{\tilde{\gamma}} \quad \forall \tilde{\Phi}_i$$

$$\Rightarrow \sum_j a_{ij} c_j^{q-1} \cdot \cancel{q} = \frac{c_i^q}{q}$$

IF $B(p)$, $C(\eta)$, $D(\xi)$

$p \leq 2\eta + 2$ $p \leq \xi + \eta + 1$ $\Rightarrow$ THE METHOD IS OF ORDER p.

$B(2S)$ $C(S)$ $D(S)$ ←

$B(2S-2)$ $\boxed{C(S)}$ $D(S-2)$ LOBATTO IIIA

$2S-1$ $C(S)$ $D(S-1)$

$C(S-1)$ $D(S)$

$y' = q\lambda$ $q \in \mathbb{C}$ $\Delta t \in \mathbb{R}^+$

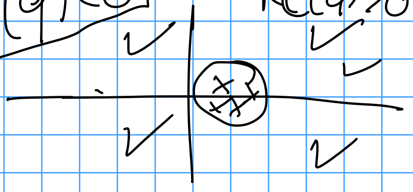
$z = q\Delta t$

$y^{n+1} = R(z) \cdot y^n$ $R(z) = 1 + z\, \underline{b}^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \underline{1}$

$\boxed{Re(q) < 0}$ $Re(q) > 0$?

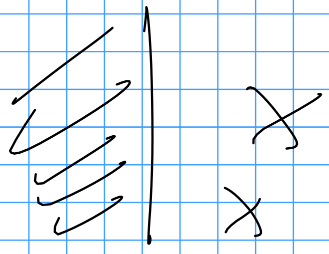
A-stability

IF $\mathbb{C}^- \subset S$ $|y^{n+1}| \leq |y^n|$



$y' = -\underline{k}(y - \cos(t))$ $k = 2000$

$k \to \infty$ $y = \cos(t)$



L-stability

$R(z) <$ POLY (EXPLICIT)
FRACT. POLY (IMPLICIT)

$\lim_{z \to \infty} R(z) = 0 \Rightarrow$ L-stable

$Re(q) < 0$

$y' = q y$

$y^{n+1} \to 0$
$\Delta t \to +\infty$

$\Delta t >> 1$ $(z \to -\infty)$

LEGENDRE

$\mathbb{C}$

$\underline{|R| \le 1}$ | $|R| \ge 1$

$R(\infty) \leftarrow$ $\rightarrow R(\infty)$

$\lim_{z \to -\infty} |R(z)| \le 1$

$|R(\infty)| = 1$

$\lim_{z \to \infty} |R(z)| \ge 1$

$$R(z) = \frac{a_0 + \_ \quad + a_m z^m}{b_0 + \quad \_\_\_ + b_n z^n}$$

$z \to \infty$ $\frac{a_m z^m}{b_n z^n}$

$m > n$ $\lim |R| \to \infty$

$m = n$ $\lim |R| \to \frac{a_m}{b_n}$

$m < n$ $\lim \to 0$

CRANK-NICOLSON

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

$R(z)$

$(I - zA)^{-1}$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -\frac{z}{2} & 1-\frac{z}{2} & 0 & 1 \end{array}\right]$$

$$\underbrace{1 + z b^T \begin{bmatrix} 1 & 0 \\ \frac{z}{2-z} & \frac{2}{2-z} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

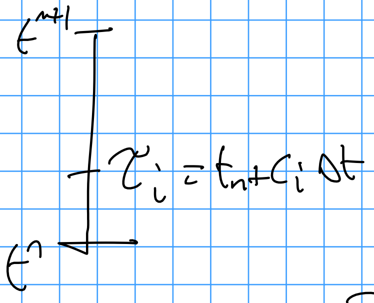
$$\left\| \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1-\frac{z}{2} & \frac{z}{2} & 1 \end{array}\right] \right.$$

$$1 + z \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{2+z}{2-z} \end{bmatrix} = 1 + z\left(\frac{1}{2} + \frac{2+z}{2(2-z)}\right)$$

$$\left\| \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{z}{2}\left(\frac{2}{2-z}\right) & \frac{2}{2-z} \end{array}\right] \right.$$

$$= \frac{4 \cancel{-2z} \cancel{+2z} \cancel{-z^2} + 2z \cancel{+z^2}}{2(2-z)} = \frac{\cancel{z}(2+z)}{\cancel{z}(2-z)} = \frac{2+z}{2-z}$$

$$\lim_{z\to\infty} R(z) = \frac{2+z}{2-z} = -1 \neq 0$$



$$\begin{cases} u(\tau_i)' = f(\tau_i, u(\tau_i)) & i = 1 \dots s \\ u(t^n) = y^n & (1) \end{cases}$$

$s+1$ CONDITIONS

$$y^{n+1} = u(t^{n+1})$$

$$\underline{u(t) \in \mathbb{P}_s}$$

s = # STAGES

$$a_{ij} = \int_0^{c_i} \ell_j(t)\,dt \qquad \ell_j(\tau_i) = \delta_{ij}$$

(2)

$$b_i = \int_0^1 \ell_i(t)\,dt \qquad c_i = \tau_i$$

TH. COLLOCATION METHOD (1) IS AN IMPLICIT RK WITH A, b, c (2)

$$u' \in \mathbb{P}_{s-1} \qquad u'(t^n + \underbrace{z}_{\underbrace{[0,1]}_{\in [t^n, t^{n+1}]}} \Delta t) = \sum_{j=1}^{s} \underset{\overset{\uparrow}{[0,1]}}{\ell_j(z)} \cdot \underset{\overset{\uparrow}{[t^n, t^{n+1}]}}{u'(\tau_j)}$$

$$u(t^n + z\Delta t) = y^n + \Delta t \int_0^z u'(t^n + s\Delta t)\,ds$$

$$= y^n + \Delta t \int_0^z \underbrace{\sum}_{j} \ell_j(s) \cdot \underbrace{u'(\tau_j)}\,ds$$

$$= y^n + \Delta t \sum_j \int_0^z \ell_j(s)\,ds \cdot u'(\tau_j)$$

$$= y^n + \Delta t \sum_j \int_0^z \ell_j(s)\,ds\; f(\tau_j, u(\tau_j))$$

$$u^{(k)} := u(t^n + c_k \Delta t) = u(\tau_k)$$

$$u^{(k)} = y^n + \Delta t \sum_J a_{kJ} \cdot f(\tau_J, u^{(k)})$$

$$y^{n+1} = u(t^{n+1}) = y^n + \Delta t \sum_J b_J \, f(\tau_J, u^{(J)}) \, . \quad \boxtimes$$

$B(p)$ $\qquad b_i = \int_0^1 \ell_J(s)\, ds \qquad \tau_i = t^n + c_i \Delta t$

(c_i, b_i) QUADRATURE FORMULA IS EXACT FOR POLY degree $P-1$

$z = 1, \dots, p$ $\quad \Rightarrow B(p)$

$$\underset{\|}{\int_0^1 s^{z-1}\, ds} = \sum_{J=1}^{S} c_J^{z-1} \cdot b_J$$

$$\left[\frac{s^z}{z}\right]_0^1 = \frac{1}{z} \qquad B(p) ; \ \sum_J c_J^{z-1} \cdot b_J = \frac{1}{z}$$

$\forall\, z = 1, \dots, P$ ✓ $\boxtimes$

$C(S)$;
$\#$ OF STAGES

$$\sum_J a_{iJ} c_J^{z-1} = \frac{c_i^z}{z} \qquad \forall\, z = 1, \dots, S$$

$$s^{z-1} \in \mathbb{P}_{S-1} \qquad s^{z-1} = \sum_J \ell_J(s) \cdot c_J^{z-1}$$

$$\sum_J a_{iJ} c_J^{z-1} = \sum_J \underbrace{\int_0^{c_i} \ell_J(s)}_{a_{iJ}} \cdot c_J^{z-1} \underbrace{ds} = \int_0^{c_i} s^{z-1}\, ds$$

$$= \left[\frac{s^z}{z}\right]_0^{c_i} = \frac{c_i^z}{z} \qquad \boxtimes$$

THM $C(S)$ and $B(S+\nu) \Rightarrow D(\nu)$ $\qquad (3)$

EXAMP ● $C(S)$ $\quad B(2S) \Rightarrow D(S)$ $\qquad$ LAGRANGE METHODS

$$B(p) = B(2S) \ \wedge \ C(\eta) = C(S) \ \wedge \ D(\zeta) = D(S)$$

$$P = 2S \qquad p \leq 2\eta + 2 \qquad 2S \leq 2S + 2 \ ✓$$

$$2s = p \le \xi + \eta + 1 = s + s + 1 = 2s + 1 \quad \checkmark$$

PROOF OF (3).

$B(p)$: $\sum_i b_i c_i^{z-1} = \frac{1}{z}$ $\quad z = 1, \dots, p$

$C(\eta)$: $\sum_j a_{ij} c_j^{z-1} = \frac{c_i^z}{z}$ $\quad z = 1, \dots, \eta$

$D(\xi)$: $\sum_i b_i c_i^{q-1} a_{ij} = \frac{b_j}{q}(1 - c_j^q)$ $\quad \boxed{q = 1, \dots, \xi}$

$$d_j^{(z)} := \sum_i b_i c_i^{z-1} a_{ij} - \frac{b_j}{z}(1 - c_j^z)$$

GOAL $d_j^{(z)} = 0 \quad \forall j = 1, \dots, s \quad \forall z = 1, \dots, \nu$

(4) $\sum_{j=1}^{s} d_j^{(z)} \cdot \underbrace{c_j^{k-1}} = 0 \quad \forall k = 1, \dots, s \quad \forall z = 1, \dots, \nu$

$M_{jk} = \{c_j^{k-1}\}_{\substack{j=1,\dots,s \\ k=1,\dots,s}}$ $\qquad$ $\underline{c_j \text{ DISTINCT}}$

$\underline{d}^{(z)} \cdot \underline{\underline{M}} = \underline{0}$

$\Rightarrow \underline{d}^{(z)} = \underline{0}$ (5) $\qquad$ $B(p)$: $\sum_i b_i c_i^{z-1} = \frac{1}{z}$

WE WILL PROVE (4) INSTEAD OF (5)

$$\sum_{j=1}^{s} d_j^{(z)} \cdot c_j^{k-1} = \sum_{j=1}^{s} \sum_{i=1}^{s} b_i c_i^{z-1} a_{ij} c_j^{k-1} - \sum_{j=1}^{s} \frac{b_j}{z}(1 - c_j^z) \cdot c_j^{k-1} =$$

$$= \sum_{i=1}^{s} b_i c_i^{z-1} \sum_{j=1}^{s} a_{ij} c_j^{k-1} - \sum_{j=1}^{s} \frac{b_j}{z} c_j^{k-1} + \sum_{j=1}^{s} \frac{b_j}{z} c_j^{z+k-1}$$

$$= \sum_{i=1}^{s} b_i c_i^{z-1} \frac{c_i^k}{k} - \frac{1}{z} \sum_{j=1}^{s} b_j c_j^{k-1} + \frac{1}{z} \sum_j b_j c_j^{z+k-1}$$

$$= \frac{1}{k} \sum_{i=1}^{s} b_i c_i^{\overbrace{z+k-1}^{\le s+\nu-1}} - \frac{1}{z} \cdot \frac{1}{k} + \frac{1}{z} \cdot \frac{1}{k+z}$$

$$= \frac{1}{k} \frac{1}{k+z} - \frac{1}{z} \frac{1}{k} + \frac{1}{z} \frac{1}{k+z} = \frac{z - (k+z) + k}{k z (k+z)} = 0$$

☒

LOBATTO IIIA $C(S)$ $D(S-2)$ ← COLLOCATION

IIIB $C(S-2)$ $D(S)$

IIIC $C(S-1)$ $D(S-1)$

$$y^{(k)} = y^n + \Delta t \sum a_{kj} F(y^{(j)})$$

$$y^{n+1} = y^n + \Delta t \sum_j b_j F(y^{(j)})$$

STIFFLY ACCURATE IFF $a_{sj} = b_j \quad \forall j$

$A \in \mathbb{R}^{S\times S}$ ⇒ NONLINEAR $(S\times N)\times(S\times N)$

N = # DOF PROBLEM

$O(N^2) \longleftrightarrow O(N^3)$

$O((S N)^2) \longleftrightarrow O((S\cdot N)^3)$ ORDER 6

$O(S^2\cdot N^2) \longleftrightarrow O(S^3\cdot N^3)$ $27\cdot N^3$ DIRK 6 STAGE $6\cdot N^3$

$\underset{=}{S}\cdot \underbrace{O(N^2)} \longleftrightarrow S\cdot O(N^3)$ DIAGONALLY IMPLICIT

↓

CRANK NICOLSON

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

$$U^{(1)} = y^n$$

$$y^{n+1} = \underline{U}^{(2)} = y^n + \frac{\Delta t}{2} F(y^n) + \frac{\Delta t}{2} F(\underline{u}^{(2)})$$

$$R(y^{n+1}) = y^{n+1} - y^n - \frac{\Delta t}{2} F(y^n) - \frac{\Delta t}{2} F(y^{n+1}) = 0$$

$$JR(y^{n+1}) = I - \frac{\Delta t}{2} JF(y^{n+1})$$

IMPLICIT RK ⇒ NON LINEAR SYSTEMS!

PROBLEM!

$$\begin{cases} y^{(k)} = y^n + \Delta t \sum_j a_{kj} \underbrace{F(y^{(j)})} & \forall k \\ y^{n+1} = y^n + \Delta t \sum_j b_j F(y^{(j)}) & \boxed{\text{DIRK}} \end{cases}$$

$$\begin{cases} G^{(k)} = \Delta t\, F\left(y^n + \sum_{j=1}^{k} a_{kj} G^{(j)}\right) \\ y^{n+1} = y^n + \sum_j b_j G^{(j)} \end{cases}$$

DIRK

$$\underline{\underline{G}}^{(k)} = \Delta t\, F\left(y^n + \sum_{j=1}^{k-1} \alpha_{kj} G^{(j)}\right) + \Delta t \sum_{j=1}^{k} \gamma_{kj} J_y F(y^n) \cdot \underline{\underline{G}}^{(j)}$$

$$\underline{\left[I - \Delta t \gamma_{kk} J_y F(y^n)\right]} G^{(k)} = \Delta t\, F\left(y^n + \sum_{j=1}^{k-1} \alpha_{kj} G^{(j)}\right) + \Delta t \sum_{j=1}^{k-1} \gamma_{kj} J_y F(y) \cdot G^{(j)}$$

$$\gamma_{kk} = \gamma \quad \forall k$$

LINEAR STABILITY

IMEX RK IMPLICIT EXPLICIT

$$\partial_t u = \underbrace{F(u)}_{\text{EXPLICIT}} + \underbrace{S(u)}_{\text{BAD GUY}}$$

STIFF

0.01 0.01

$$\partial_t u + \partial_x F(u) = 0 \qquad \Delta t \leq \Delta x \cdot C$$

EXPLICIT

0.0001

$$\partial_t u - \partial_{xx} u = 0 \qquad \Delta t \leq \Delta x^2 \cdot C$$

$$\partial_{xxx} u \qquad \Delta t \leq \Delta x^3$$

$$\partial_t u + \underbrace{\partial_x F(u)}_{\text{EX}} \underbrace{- d\partial_{xx} u}_{\text{IM}} = 0$$

$$\partial_t u + F(u) + S(u) = 0$$

$$\frac{u^{n+1} - u^n}{\Delta t} + F(u^n) + S(u^{n+1}) = 0$$

IMEX RK

$\tilde{A}, \tilde{b}, \tilde{c}$ EXPLICIT COEFF

A, b, c IMPLICIT COEFF

LINEAR STABILITY

$$y' = \underbrace{\lambda_1}_{\text{EXPLICIT}} y + \underbrace{\lambda_2}_{\text{IMPLICIT}} y$$

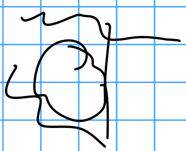
$z_1 = \Delta t \lambda_1 \in \mathbb{C}$

$z_2 = \Delta t \lambda_2 \in \mathbb{C}$

$y^{n+1} = R(z_1, z_2) \cdot y^n$

$R(z) : \mathbb{C} \to \mathbb{C}$

$|R(z)| : \mathbb{C} \to \mathbb{R}$ 2D



$|R(z_1, z_2)| : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ 4D

$Re(z_1) = 0$ $z_1 \in i\mathbb{R}$ $|R(z_1, z_2)| : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$Im(z_2) = 0$ $z_2 \in \mathbb{R}$

- RUSSO 2000

$\to z_2 \in \mathbb{C} : |R(z_1, z_2)| \leq 1 \quad \forall z_1 \in \{|1 + z_1| \leq 1\}$

$\to z_1 \in \mathbb{C} : |R(z_1, z_2)| \leq 1 \quad \forall z_2 \in \mathbb{C}^-$

$y^n \longrightarrow y^{n+1}$ $y' = F(y)$

$y^{n-1}, y^{n-2}, y^{n-3} \dots$

ADAMS $\{F(t^{n-j}, y^{n-j})\}_{j=1}^{k}$

GENERAL $\textcircled{F}$ $\{y^{n-j}\}_{j=1}^{k}$

$\Downarrow$

$$y^{n+1} + \sum_{j=1}^{k} \alpha_j \, y^{n-j+1} = \Delta t \sum_{j=0}^{k} \beta_j \, F(t^{n-j+1}, y^{n-j+1})$$

STABILITY ~~$y^{n+1} = R(z)\cdot y^n$~~ $= y^{n+1} = R(z)\cdot y^n = \underline{R(z)^{n+1} \cdot y^0}$

LINEAR STABILITY $\boxed{y' = q\, y}$ $\quad q \in \mathbb{C} \cap \{Re(q) \le 0\}$

ANSATZ: $y^{n+1} = \underbrace{\xi^{n+1}} y^0$ $\quad \uparrow$ $\quad$ STABLE $\Leftrightarrow$ $|\xi| \le 1$

$$y^{n+1} + \sum_{j=1}^{k} \alpha_j \, y^{n-j+1} = \underbrace{\Delta t\, q}_{z} \sum_{j=0}^{k} \beta_j \cdot y^{n-j+1}$$

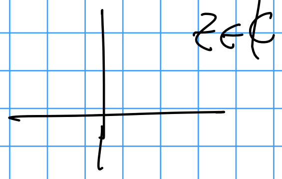
$$\Rightarrow \underbrace{\xi^{n+1} \cancel{y^0} + \sum_{j=1}^{k} \alpha_j \, \xi^{n-j+1} \cdot \cancel{y_0} = z \cdot \sum_{j=0}^{k} \beta_j \, \xi^{n-j+1} \cancel{y_0}}$$

DIVIDED BY ξ^{n-k+1}

$$\Rightarrow \underline{\xi^{k}} + \sum_{j=1}^{k} \alpha_j \, \xi^{k-j} = \underline{z} \cdot \sum_{j=0}^{k} \beta_j \, \xi^{k-j}$$

$\xi_i(z)$ SOLUTION $\uparrow$ FOR $i = 1, \dots, k$

$|\xi_i(z)| \le 1 \quad \forall i$

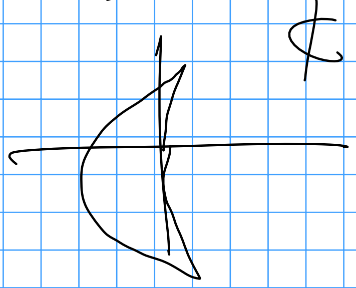


$\xi \to z$

$$z = \frac{\xi^{k} + \sum_{j=1}^{k} \alpha_j \, \xi^{k-j}}{\sum_{j=0}^{k} \beta_j \, \xi^{k-j}}$$

$\xi \in \mathbb{C}$ $\quad \underline{|\xi| \le 1}$

$\boxed{\textcircled{z} \in \mathbb{C}}$



ACCURACY

CONSISTENCY ERROR

$y'(t^{n-j+1})$

$$y(t^{n+1}) + \sum_{j=1}^{k} \alpha_j\, y(t^{n-j+1}) - \Delta t \sum_{j=0}^{k} \beta_j \underbrace{F(t^{n-j+1}, y(t^{n-j+1}))}$$

TAYLOR EXP IN t^{n+1}

$$= y(t^{n+1}) + \sum_{j=1}^{k} \alpha_j \sum_{\ell=0}^{p} \frac{(-j\Delta t)^{\ell}}{\ell!}\, y^{(\ell)}(t^{n+1})$$

$$- \Delta t \sum_{j=0}^{k} \beta_j \sum_{\ell=0}^{p-1} \frac{(-j\Delta t)^{\ell}}{\ell!}\, y^{(\ell+1)}(t^{n+1}) + O(\Delta t^{p+1})$$

$$y(t^{n+1})\left[1 + \sum_{j=1}^{k} \alpha_j\right] +$$

$$+ \Delta t\, y'(t^{n+1})\left[\sum_{j=1}^{k} -j\alpha_\ell + \sum_{j=0}^{k} \beta_j\right]$$

$+ \dots$

FOR ORDER P $\quad 1 + \sum_{j=1}^{k} \alpha_j = 0$

$\forall \ell = 1, \dots, p \quad \sum_{j=1}^{k} \alpha_j \frac{(-j)^{\ell}}{\ell!} - \sum_{j=0}^{k} \beta_j \frac{(-j)^{\ell-1}}{(\ell-1)!} = 0$ ☒

ADAMS-BASHFORTH

VALUES $F(t^{n-k+1}, y^{n-k+1})$

NO y^{n-k+1}

$$y(t^{n+1}) = y(t^{n}) + \int_{t^n}^{t^{n+1}} F$$

$$= y(t^{n}) + \int_{t^n}^{t^{n+1}} P_n(t)\, dt$$

$$P_{n,k}(t^{n-j+1}) = F^{n-j+1} = F(t^{n-j+1}, y^{n-j+1}) \quad \forall j = 1, \dots, k$$

$$P_{n,k} \in \mathbb{P}_{k-1}$$

$$P_{n,k}(t)_i = \sum F^{n-J+1} \cdot L_{n,J,k}(t)$$

$$L_{n,J,k}(t) = \prod_{\substack{i \neq J \\ i=1}}^{k} \frac{t - t^{n+1-i}}{t^{n+1-J} - t^{n+1-i}}$$

$$b_{n,J,k} = \frac{1}{t^{n+1}-t^{n}} \int_{t^n}^{t^{n+1}} L_{n,J,k}(t)\, dt \qquad J=1,\dots,k$$

$$y^{n+1} = y^n + \Delta t^n \sum_{J=1}^{k} b_{n,J,k} F^{n-J+1}$$

$$F^{n+1} = F(t^{n+1}, y^{n+1})$$

STABILITY + ADAMS + HIGH ORDER = NO!

$$y^{n+1} + \sum_{J=1}^{k} \alpha_J y^{n-J+1} = \Delta t\, \beta_0 F(t^{n+1}, y^{n+1})$$

BACKWARD DIFFERENCE FORMULA

BDF 2 $$y^{n+1} - \frac{3}{2} y^n + \underbrace{\frac{1}{2} y^{n-1}}_{} = \Delta t\, ?\, F(y^{n+1})$$

1 STEP IMPLICIT EULER $O(\Delta t^2)$

N-1 STEPS BDF2 $O(\Delta t^3)$

$$O\left(\Delta t^{\overset{2}{\cancel{3}}} \frac{N}{\cancel{\Delta t}} + \Delta t^3 \right) = \Delta t^2$$

RELAXATION RUNGE-KUTTA

- $\boxed{y' = F(y)}$ $\frac{1}{2}\langle y, y \rangle$ = KINETIC ENERGY = K

$$\frac{d}{dt} K(t) = \frac{d}{dt} \frac{1}{2} \langle y, y \rangle = \langle y, y' \rangle$$

$$= \underbrace{\langle y, F(y) \rangle \overset{(\le)}{=} 0}_{\text{COMMON FOR ODES}}$$

$\eta(y)$

$$\frac{d}{dt} \eta(y) = \langle \partial_y \eta(y), y' \rangle$$

$$= \langle \underbrace{\partial_y \eta(y)}_{\text{ENTROPY VARIABLES}}, F(y) \rangle \overset{(\le)}{=} 0$$

RELAX RK

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \to \begin{cases} y^{(k)} = y^n + \Delta t \sum_{j=1}^{s} a_{kj} F(y^{(j)}) \\ y^{n+1} = y^n + \Delta t \sum_{j=1}^{s} b_j F(y^{(j)}) \end{cases}$$

$$y^{n+1} \simeq y(t^{n+1})$$

$$\to y^{n+1}_{\gamma} = y^n + \gamma \Delta t \sum_{j=1}^{s} b_j F(y^{(j)}) \qquad \gamma \in \mathbb{R}$$

$$\gamma \approx 1 \qquad \gamma \approx 1$$

ENERGY CASE $\frac{d}{dt} \langle y, y \rangle \overset{(\le)}{=} 0$

$$\langle y^{n+1}_{\gamma}, y^{n+1}_{\gamma} \rangle = \langle y^n, y^n \rangle + 2\gamma \Delta t \sum_{j=1}^{s} b_j \langle y^n, F(y^{(j)}) \rangle$$

$$+ \gamma^2 \Delta t^2 \sum_{j=1}^{s} \sum_{k=1}^{s} b_j b_k \langle F(y^{(j)}), F(y^{(k)}) \rangle$$

$$\frac{d}{dt} \langle y, y \rangle = 0 \Leftarrow \langle y, F(y) \rangle = 0$$

$$\le 0 \Leftarrow \langle y, F(y) \rangle \le 0$$

$$= \langle y^n, y^n \rangle + 2\gamma \Delta t \sum_{j=1}^{S} b_j \underbrace{\langle y^{(j)}, F(y^{(j)}) \rangle}_{\substack{=0 \\ (\leq)}}$$

$$\overset{!}{=} 0 \begin{cases} + 2\gamma \Delta t \sum_{j=1}^{S} b_j \langle y^n - y^{(j)}, F(y^{(j)}) \rangle \\ + \gamma^2 \Delta t^2 \sum_{j=1}^{S} \sum_k b_j b_k \langle F^{(k)}, F^{(j)} \rangle \end{cases}$$

1 SOLUTION $\gamma = 0$ (OBVIOUS)

2nd SOLUTION $\gamma \neq 0$

$$\gamma = \frac{\Delta t \sum_{j=1}^{S} b_j \langle y^{(j)} - y^n, F^{(j)} \rangle}{\Delta t^2 \sum_{j=1}^{S} \sum_{k=1}^{S} b_j b_k \langle F^{(k)}, F^{(j)} \rangle}$$

$\rightarrow \Delta t^2 \sum_{j=1}^{S} \sum_{k=1}^{S} b_j b_k \langle F^{(k)}, F^{(j)} \rangle = \langle \Delta y, \Delta y \rangle$

$$\gamma \Delta t \rightsquigarrow y^{n+1}_{\gamma} \approx y(t^n + \underline{\gamma \Delta t})$$

$$\Delta y := y^{n+1} - y^n = \Delta t \sum_{j=1}^{S} b_j F^{(j)}$$

GENERAL CASE $\underline{\eta(y)}$ $\quad \Big\| \quad \langle \partial_y \eta(y), F(y) \rangle \begin{matrix} = 0 \\ \leq 0 \end{matrix}$

$\underline{t^n + \gamma \Delta t}$

$$\begin{cases} y^{(i)} = y^n + \Delta t \sum_{j=1}^{S} a_{ij} F(y^{(j)}) \\ y^{n+1} = y^n + \Delta t \sum_{j=1}^{S} b_j F(y^{(j)}) \end{cases}$$

$\eta(y^{n+1}) - \eta(y^n) =$

$$= \cancel{\eta(y^n)} - \cancel{\eta(y^n)} + \Delta t \sum_{j=1}^{S} b_j \overbrace{\langle F(y^{(j)}), \partial_y \eta(y^n) \rangle}$$

$$+ \frac{\Delta t^2}{2} \sum_{i,j} b_i b_j \langle \underline{F(y^{(i)})}, \underline{\partial_{yy} \eta(y^n)} \cdot \underline{F(y^{(j)})} \rangle$$

$$+ \mathcal{O}(\Delta t^3)$$

$$= \Delta t \sum_J b_J \underbrace{\langle F(y^{(J)}), \partial_y \eta(y^{(J)}) \rangle}_{\substack{(=0)\\(\leq 0)}}$$

$$+ \Delta t \sum_J b_J \langle F^{(J)}, \partial_y \eta(y^n) - \partial_y \eta(y^{(J)}) \rangle$$

$$+ \frac{\Delta t^2}{2} \sum_{i,J} b_J b_i \langle F^{(i)}, \eta_{yy}(y) F^{(J)} \rangle$$

$$+ \mathcal{O}(\Delta t^3)$$

$y \in \mathbb{R}^I$

$\eta : \mathbb{R}^I \to \mathbb{R}$

$\partial_y \eta : \mathbb{R}^I \to \mathbb{R}^I$

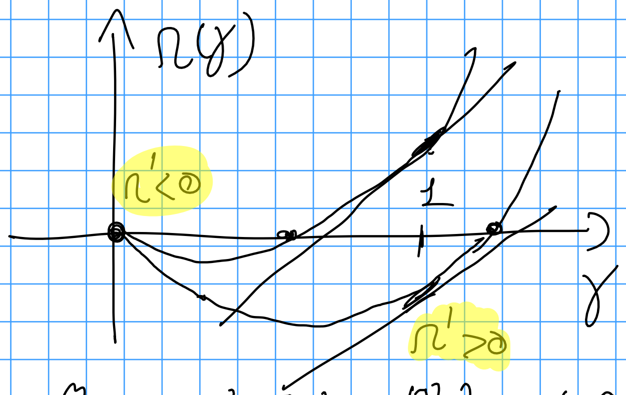
$\partial_{yy} \eta : \mathbb{R}^I \to \mathbb{R}^{I \times I}$

η CONVEX

$$\eta(y_\gamma^{n+1}) - \eta(y^n) \overset{!}{=} \gamma \Delta t \sum_J b_J \underbrace{\langle F(y^{(J)}), \partial_y \eta(y^{(J)}) \rangle}_{\substack{(=0)\\(\leq 0)}}$$

FIND A γ SUCH THAT

- DO WE HAVE A SOLUTION FOR γ?



$\Omega'' > 0$

η CONVEX

$$\Omega(\gamma) = \eta(y^n + \gamma \Delta t \sum_J b_J F^{(J)}) - \eta(y^n) - \gamma \Delta t \sum_J b_J \langle \partial_y \eta(y^{(J)}), F^{(J)} \rangle$$

$$\Omega'(0) < 0$$

$$\Omega'(\gamma) = \langle \partial_y \eta(y^n + \gamma \Delta t \sum_J b_J F^J), \Delta t \sum_J b_J F^{(J)} \rangle - \Delta t \sum_J b_J \langle \partial_y \eta(y^{(J)}), F^{(J)} \rangle$$

$$= \Delta t \sum_J b_J \langle F^{(J)}, \partial_y \eta(y^n + \overset{0}{\gamma} \Delta t \sum_J b_J F^J) - \partial_y \eta(y^{(J)}) \rangle$$

$\partial_{yy} \eta$ POSITIVE DEFINITE

$$\Omega'(0) = -\Delta t \sum_i b_i \int_0^1 \eta_{yy}\left(y^n + \nu \Delta t \sum_{k=1}^{s} a_{ik} F^{(k)}\right)$$

$$\left(F(y^{(i)}),\ \Delta t \sum a_{ij} F(y^{(j)})\right) d\nu$$

$$f(\nu) = \langle \eta_y (y^n + \nu \underbrace{\Delta t \sum_{k=1}^{s} a_{ik} F^{(k)}}), F(y^{(i)}) \rangle$$

$$f'(\nu) = \eta_{yy}(y^n + \nu \Delta t \ldots)\left(F(y^{(i)}), \underline{\Delta t \sum_{k=1}^{s} a_{ik} F^{(k)}}\right)$$

$$f(\nu=1) = \langle \eta_y(y^{(i)}), F^{(i)} \rangle \qquad f(\nu=0) = \langle \eta_y(y^n), F^{(i)} \rangle$$

$$\Omega'(0) = -\Delta t \sum_{i,j} b_i a_{ij} \int_0^1 \eta_{yy}(y^n + \nu \Delta t)\left(F(y^{(i)}), F(y^{(j)})\right)$$

$$= -\Delta t^2 \overbrace{\sum_{i,j} b_i a_{ij}}^{>0} \underbrace{\int_0^1 \eta_{yy} \left(F(y^n), F(y^n)\right)}_{>0}$$

$$+ O(\Delta t^3) \quad < 0$$

Δt SMALL ENOUGH

$$\int_0^1 \underbrace{\eta_{yy}(\overbrace{y^n + \nu \Delta t \sum_i b_i F^{(i)}}^{\simeq y^n + O(\Delta t)}) \left(F(y^n), F(y^n)\right)} d\nu$$

$$= -\cancel{\int_0^1 d\nu}\ \eta_{yy}(y^n) \overbrace{\left(F(y^n), F(y^n)\right)}^{>0} + O(\Delta t^3) < 0$$

IF $\underbrace{\sum_{i,j} b_i a_{ij} > 0}$ $\quad \eta'' > 0 \quad$ Δt SMALL

$$\boxed{c_i = \sum_j a_{ij}}$$

c_i ● 1

● 2

b_i

$$\sum_{i,j} b_i a_{ij} = \underline{\sum_i b_i c_i} = \frac{1}{2} > 0$$

∀ (AT LEAST) SECOND ORDER SCHEMES.

$$\Omega'(0) < 0$$

$\Omega'(1) > 0$

$$\Omega'(1) = \Delta t \sum_i b_i \left\langle F(y^{(i)}), \overbrace{\eta'\Big(y^n + \Delta t \sum_{k=1}^{s} b_k F^{(k)}\Big) - \underbrace{\eta'\Big(y^n + \Delta t \sum_{j=1}^{s} a_{ij} F^{(j)}\Big)}}\right\rangle$$

$\nu = 0 \quad \eta'\Big(y^n + \Delta t \sum_j a_{ij} F^{(j)}\Big)$

$\nu = 1 \quad \eta'\Big(y^n + \Delta t \sum_k b_k F^{(k)}\Big)$

$$f(\nu) = \Big\langle \eta'\Big(\underbrace{y^n + \nu \Delta t \sum_k b_k F^{(k)} + (1-\nu) \overset{\Delta t}{\downarrow}\sum_j a_{ij} F^{(j)}}_{*}\Big), F^{(i)} \Big\rangle$$

$$f'(\nu) = \eta''(*)\Big(F^{(i)}, \underbrace{\Delta t \sum_j b_j F^{(j)} - \Delta t \sum_j a_{ij} F^{(j)}}_{V}\Big)$$

$\underline{F^{(i)T}} \quad \underline{\eta''(*)} \quad \underline{V}$

$$\Omega'(1) = \Delta t \sum_i b_i \int_0^1 \underset{\uparrow}{F^{(i)T}}\, \underset{\uparrow}{\eta''(*)} \Big(\underline{\Delta t \sum_j (b_j - a_{ij})} \overset{*}{F^{(j)}}\Big)$$

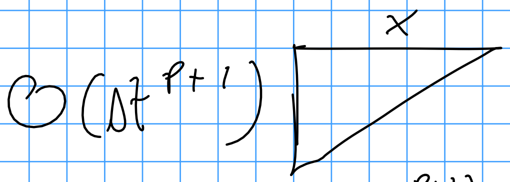
$$= \Delta t^2 \sum_{i,j} b_i (b_j - a_{ij}) \underbrace{F(y^n) \cdot \eta''(y^n) \cdot F(y^n)}_{\geq 0} + O(\Delta t^3)$$

$\eta'' : \mathbb{R}^I \to \mathbb{R}^{I \times I}$

IF $\sum_{i,j} b_i (b_j - a_{ij}) > 0$

$$\underbrace{\sum_i b_i}_{1} \underbrace{\sum_j b_j}_{1} - \overbrace{\sum_{i,j} a_{ij} b_i}^{=\frac{1}{2}} = 1 - \frac{1}{2} = \frac{1}{2} > 0 \checkmark$$

$\gamma = O(\Delta t^{p-1})$



SCOPE $O(\Delta t^2)$

$\underline{x = \dfrac{O(\Delta t^{p+1})}{\Theta(\Delta t^2)}}$

$\dfrac{O(\Delta t^{p+1})}{x} = O(\Delta t^2)$

$O(\varepsilon) = x(q)$ $\qquad \lim_{\varepsilon \to 0} \frac{x}{\varepsilon} \in \mathbb{R}$

$\Theta(\varepsilon) = x(q)$ $\qquad 0 < \lim_{\varepsilon \to 0} \frac{x}{\varepsilon} < \infty$

POSITIVITY PRESERVING METHODS FOR PRODUCTION–DESTRUCTION SYSTEMS

PDS $\quad c_i \quad \frac{d}{dt} c_i = P_i(c) - D_i(c)$

$i = 1, \ldots, I$ $\qquad P_i(c), D_i(c) > 0$

- CONSERVATIVE IF $P_i(c) = \sum_{j=1}^{I} p_{ij}(c)$

$$D_i(c) = \sum_{j=1}^{I} d_{ij}(c)$$

$$p_{ij}(c) = d_{ji}(c)$$

$$\sum_i \frac{d}{dt} c_i = \sum_i \sum_j p_{ij}(c) - \sum_i \sum_j d_{ij}(c)$$

$$\| \qquad = \underbrace{\sum_i \sum_j p_{ij}(c)} - \underbrace{\sum_i \sum_j p_{ji}(c)} = 0$$

$$\frac{d}{dt}\left(\sum_i c_i\right) = 0$$

- POSITIVITY: IF $c_i(0) > 0$
  AND IF $\left(c_i \to 0 \Rightarrow D_i(c) \to 0\right)$

$\Rightarrow \quad c_i(t) \geq 0 \quad \forall\, t > 0$

ODE $\frac{d}{dt} c_i = P_i(c) - \underbrace{D_i(c)}$. PROOF BY CONTRADICT.

- EXAMPLE: SIR

SUSCEPTIBLE, INFECTED, RECOVERED

$$\partial_t S = -\beta \frac{SI}{N} \qquad N = S + I + R$$

$$\partial_t I = \beta \frac{SI}{N} - \gamma I \qquad S \to 0$$

$$\partial_t R = \gamma I \qquad \beta \cdot \frac{SI}{N} \to 0$$

$$I \to 0 \quad \gamma I \to 0$$

	IMPL EU	EXPL EU
- ONE STEP METHOD	✓	✓
- UNCONDITIONALLY POSITIVE $\forall \Delta t$	✓	✗ $(\Delta t <.)$
- UNCONDITIONALLY CONSERVATIVE	✓	✓
- HIGH ORDER	✗	✗
- LINEARLY IMPLICIT (AT MOST)	✗	✓

EXPLICIT EULER

$$c_i^{n+1} = c_i^n + \sum_j p_{ij}(c^n) - \sum_j d_{ij}(c^n)$$

- CONSERVATION

$$\sum_i c_i^{n+1} = \sum_i c_i^n + \underbrace{\Delta t \sum_i \sum_j p_{ij}(c^n) - \Delta t \sum_i \sum_j \underbrace{d_{ij}(c^n)}_{p_{ji}(c^n)}}_{=0}$$

$$P(c^n) - D(c^n) \neq 0 \Rightarrow \exists\, i:$$

$$D_i(c^n) > P_i(c^n) \geq 0$$

$$\Delta t > \frac{c_i^n}{\underbrace{D_i(c^n) - P_i(c^n)}_{>0}} > 0$$

$$c_i^{n+1} = c_i^n + \Delta t\, P_i(c^n) - \Delta t\, D_i(c^n)$$

$$= c_i^n + \Delta t \underbrace{\left(P_i(c^n) - D_i(c^n)\right)}_{<0}$$

$$< c_i^n + \frac{c_i^n}{D_i(c^n) - P_i(c^n)} \left(P_i(c^n) - D_i(c^n)\right)$$

$$= c_i^n - c_i^n = 0 \qquad c_i^{n+1} < 0 .$$

• PATANKAR'S TRICK

$$c_i^{n+1} = c_i^n + \Delta t\, P_i(c^n) - \Delta t\, D_i(c^n) \cdot \frac{c_i^{n+1}}{c_i^n}$$

$$\frac{c_i^{n+1}}{c_i^n} = 1 + O(\Delta t) \Rightarrow$$ SAME ORDER OF ACCURACY

$$c_i^{n+1} \underbrace{\left[1 + \Delta t \frac{\overbrace{D_i(c^n)}^{>0}}{\underbrace{c_i^n}_{>0}}\right]}_{>0} = \underbrace{\underbrace{c_i^n}_{>0} + \Delta t \underbrace{P_i(c^n)}_{>0}}_{>0}$$

$$c_i^{n+1} = \frac{c_i^n + \Delta t\, P_i(c^n)}{1 + \Delta t \frac{D_i(c^n)}{c_i^n}} > 0 \quad \forall i = 1, \dots, I$$

$$\forall \Delta t > 0$$

• LOST CONSERVATION

MODIFIED PATANKAR METHODS 2003

$$c_i^{n+1} = c_i^n + \Delta t \sum_j p_{ij}(c^n) \cdot \frac{c_j^{n+1}}{c_j^n} - \Delta t \sum_j d_{ij}(c^n) \cdot \frac{c_i^{n+1}}{c_i^n}$$

GOAL FOR THE CONSERVATION

$$\Delta t \sum_i \sum_j p_{ij}(c^n) \frac{c_j^{n+1}}{c_j^n} - \Delta t \sum_i \sum_j d_{ij}(c^n) \frac{c_i^{n+1}}{c_i^n} =$$

$$\parallel$$

$$- \Delta t \sum_i \sum_j p_{ji}(c^n) \frac{c_i^{n+1}}{c_i^n}$$

$$- \Delta t \sum_i \sum_j p_{ij}(c^n) \frac{c_j^{n+1}}{c_j^n} \overset{!}{=} 0$$

$$\forall i$$

$$\underline{M} \rightarrow \Delta$$

$$c_i^{n+1} = c_i^n + \underbrace{\Delta t \sum_j p_{ij}(c^n)\cdot \frac{c_j^{n+1}}{c_j^n}}_{\Downarrow\ \underline{M}} - \underbrace{\Delta t \sum_j d_{ij}(c^n)\cdot \frac{c_i^{n+1}}{c_i^n}}_{\underline{M}}$$

LINEAR SYSTEM

$$\sum_i \underline{M}_{ki}\cdot c_i^{n+1} = c_k^n \qquad p_{ii} = d_{ii} = 0$$

$$\underline{M}\,\underline{c}^{n+1} = \underline{c}^n$$

$$M_{ii} = 1 + \Delta t \sum_j \frac{d_{ij}(c^n)}{c_i^n} > 0$$

$$M_{ij} = -\Delta t \frac{p_{ij}(c^n)}{c_j^n} \le 0 \qquad j \neq i$$

- $\underline{M}$:
  - POSITIVE DIAGONAL
  - NON POSITIVE OFF-DIAGONAL TERMS
  - STRICTLY DIAGONALLY DOMINANT BY COLUMNS

$$\Downarrow$$

∃ INVERSE AND $M^{-1} \geq 0$

$$|M_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{I} |M_{ji}|$$

DIAGONALLY DOMINANT BY COLUMNS

$$|M_{ii}| = 1 + \Delta t \sum_{j\neq i} \frac{d_{ij}(c^n)}{c_i^n} >$$

$$> \Delta t \sum_{j\neq i} \frac{d_{ij}(c^n)}{c_i^n} = \Delta t \sum_{j\neq i} \frac{p_{ji}(c^n)}{c_i^n}$$

$$= -\sum_{j\neq i} \Pi_{ji} = \sum_{j\neq i} |\Pi_{ji}| \quad \boxtimes$$

2003 RK2 2018/2019

THERE IS NO MPRK33 3STAGES 3ORDER

2020

MPDeC

PROBLEM

$\theta_r^m < 0$

↓

DeC

$$c_i^{m,(k)} = c_i^0 + \Delta t \sum_{r=0}^{M} \theta_r^m \cdot \sum_{j=1}^{I} \left[p_{ij}(c^{r,(k-1)}) \frac{c_j^{m,(k)}}{c_j^{m,(k-1)}} - d_{ij}(c^{r,(k-1)}) \cdot \frac{c_i^{m,(k)}}{c_i^{m,(k-1)}} \right]$$

$$c_i^{n,(k)} = c_i^{n,(k-1)} + O(\Delta t^{k-1})$$

$$\frac{c_i^{n(k)}}{c_i^{n(k-1)}} = 1 + O(\Delta t^{k-1}) \Rightarrow$$ KEEPS THE ORDER OF ACCURACY

$$c_i^{m,(k)} = c_i^0 + \Delta t \sum_{r=0}^{M} \theta_r^m \cdot \sum_{j=1}^{I} \left[p_{ij}(c^{r,(k-1)}) \frac{c^{m,(k)}_{\gamma(j,i,\theta_r^m)}}{c^{m,(k-1)}_{\gamma(j,i,\theta_r^m)}} - d_{ij}(c^{r,(k-1)}) \cdot \frac{c^{m,(k)}_{\gamma(i,j,\theta_r^m)}}{c^{m,(k-1)}_{\gamma(i,j,\theta_r^m)}} \right]$$

$$\gamma(a,b,\theta) = \begin{cases} a & \theta \geq 0 \\ b & \theta < 0 \end{cases}$$

$$y^{(K)} = y^{n} + \Delta t \sum_{J} a_{KJ} F^{(J)}$$

$$y^{(K)} = \sum_{J=0}^{K-1} \left[y^{(J)} \cdot \alpha_{KJ} + \Delta t\, \beta_{KJ} F^{(J)} \right]$$

$$= \sum_{J=0}^{K-1} \alpha_{KJ} \underbrace{\left[y^{(J)} + \Delta t \frac{\beta_{KJ}}{\alpha_{KJ}} \cdot F^{(J)} \right]}_{}$$

EXPLICIT EULER

WITH 7. STEP $\boxed{\Delta t \frac{\beta_{KJ}}{\alpha_{KJ}}}$

CONVEX COMBINATION EXP. EUL STEPS

$\Downarrow$

- $\underline{\alpha_{KJ}} > 0$ $\quad \sum_{J} \alpha_{KJ} = 1$

EXPL. RK

$$\begin{array}{c|c} c & A \\ \hline & b \end{array} \quad \frac{S\cdot(S-1)}{2}$$

- $\underline{\beta_{KJ}} > 0$

$\alpha, \beta \quad \frac{S\cdot(S-1)}{2} \cdot 2$

- $\Delta t \frac{\beta_{KJ}}{\alpha_{KJ}} < \Delta t_{eE}$ $\quad C := \min_{J \le K \le S} \frac{\alpha_{KJ}}{\beta_{KJ}}$

C STRONG STABILITY PRES. COEFFICIENT.

$$\Delta t < C \cdot \Delta t_{eE}$$