

ADER and DeC:
arbitrarily high order (explicit)
methods for PDEs and ODEs



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Unified Framework*. J Sci Comput 87, 2 (2021).
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Outline

- 1 Motivation
- 2 DeC
- 3 ADER
- 4 Similarities
- 5 ADER stability and accuracy
- 6 Simulations
- 7 Efficient DeC (ADER)
- 8 An efficient Deferred Correction

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Motivation: high order accurate explicit method

Methods used to solve a hyperbolic PDE system for $u : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^D$

$$\partial_t u + \nabla_x \mathcal{F}(u) = 0. \quad (1)$$

Or ODE system for $\mathbf{u} : \mathbb{R}^+ \rightarrow \mathbb{R}^S$

$$\partial_t \mathbf{u} = F(\mathbf{u}). \quad (2)$$

Applications:

- Fluids/transport
- Chemical/biological processes

How?

- Arbitrarily high order accurate
-

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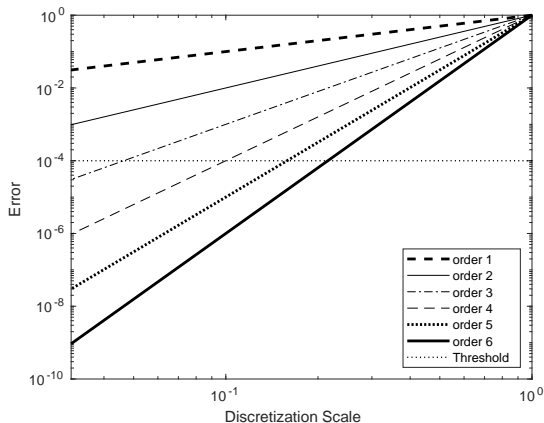
Or ODE system for u :

Applications:

- Fluids/transport
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How?

- Arbitrarily high order
-



(1)

(2)

Motivation: high order accurate explicit method

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Applications:

- Fluids/transport
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How?

- Arbitrarily high order accurate
- Explicit (if nonstiff problem)

Classical time integration: Runge–Kutta

$$\mathbf{u}^{(1)} := \mathbf{u}^n, \tag{3}$$

$$\mathbf{u}^{(k)} := \mathbf{u}^n + \sum_{s=1}^K a_{ks} F \left(t^n + c_s \Delta t, \mathbf{u}^{(s)} \right), \quad \text{for } k = 2, \dots, K, \tag{4}$$

$$\mathbf{u}^{n+1} := \mathbf{u}^n + \sum_{s=1}^K b_s F \left(t^n + c_s \Delta t, \mathbf{u}^{(s)} \right). \tag{5}$$

Classical time integration: Explicit Runge–Kutta

$$\mathbf{u}^{(k)} := \mathbf{u}^n + \sum_{s=1}^{k-1} a_{ks} F(t^n + c_s \Delta t, \mathbf{u}^{(s)}), \quad \text{for } k = 2, \dots, K.$$

- Easy to solve
- High orders involved:
 - Order conditions: system of many equations
 - Stages $K \geq d$ order of accuracy (e.g. RK44, RK65)

Classical time integration: Implicit Runge–Kutta

$$\mathbf{u}^{(k)} := \mathbf{u}^n + \sum_{s=1}^K a_{ks} F(t^n + c_s \Delta t, \mathbf{u}^{(s)}), \quad \text{for } k = 2, \dots, K.$$

- More complicated to solve for nonlinear systems
- High orders easily done:
 - Take a high order quadrature rule on $[t^n, t^{n+1}]$
 - Compute the coefficients accordingly, see Gauss–Legendre or Gauss–Lobatto polynomials
 - Order up to $d = 2K$

Two iterative explicit arbitrarily high order accurate methods.

- ADER¹ for hyperbolic PDE, after a first analytic more complicated approach.
- Deferred Correction (DeC): introduced for explicit ODE², extended to implicit ODE³ and to hyperbolic PDE⁴.

¹M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. *Journal of Computational Physics*, 227(18):8209–8253, 2008.

²A. Dutt, L. Greengard, and V. Rokhlin. Spectral Deferred Correction Methods for Ordinary Differential Equations. *BIT Numerical Mathematics*, 40(2):241–266, 2000.

³M. L. Minion. Semi-implicit spectral deferred correction methods for ordinary differential equations. *Commun. Math. Sci.*, 1(3):471–500, 09 2003.

⁴R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. *Journal of Scientific Computing*, 73(2):461–494, Dec 2017.

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DeC high order time discretization: \mathcal{L}^2

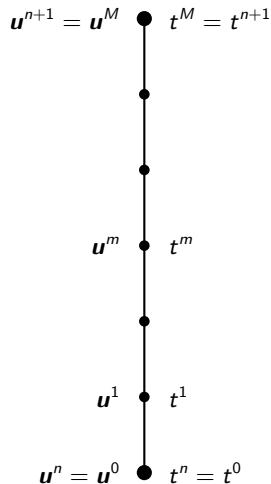
High order in time: we discretize our variable on $[t^n, t^{n+1}]$ in M substeps (\mathbf{u}^m).

$$\partial_t \mathbf{u} = F(\mathbf{u}(t)).$$

Thanks to Picard–Lindelöf theorem, we can rewrite

$$\mathbf{u}^m = \mathbf{u}^0 + \int_{t^0}^{t^m} F(\mathbf{u}(t)) dt.$$

and if we want to reach order $r + 1$ we need $M = r$.

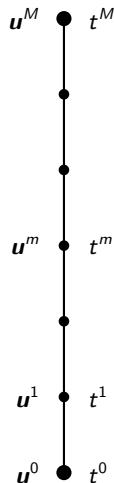


DeC high order time discretization: \mathcal{L}^2

More precisely, for each σ we want to solve $\mathcal{L}^2(\mathbf{u}^{n,0}, \dots, \mathbf{u}^{n,M}) = 0$, where

$$\mathcal{L}^2(\mathbf{u}^0, \dots, \mathbf{u}^M) = \begin{pmatrix} \mathbf{u}^M - \mathbf{u}^0 + \sum_{r=0}^M \int_{t^0}^{t^M} F(\mathbf{u}^r) \varphi_r(s) ds \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 + \sum_{r=0}^M \int_{t^0}^{t^1} F(\mathbf{u}^r) \varphi_r(s) ds \end{pmatrix}$$

- $\mathcal{L}^2 = 0$ is a system of $M \times S$ coupled (non)linear equations
- \mathcal{L}^2 is an implicit method (collocation method: Gauss, LobattoIIIA)
- Not easy to solve directly $\mathcal{L}^2(\underline{\mathbf{u}}^*) = 0$
- High order ($\geq M + 1$), depending on points distribution

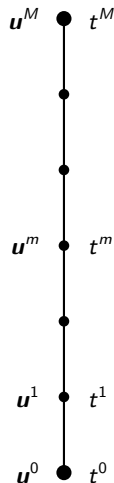


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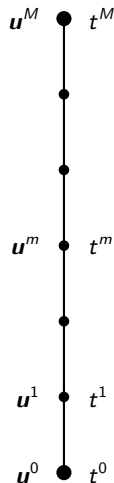


DeC low order time discretization: \mathcal{L}^1

Instead of solving the implicit system directly (difficult), we introduce a first order scheme $\mathcal{L}^1(\mathbf{u}^{n,0}, \dots, \mathbf{u}^{n,M})$:

$$\mathcal{L}^1(\mathbf{u}^0, \dots, \mathbf{u}^M) = \begin{pmatrix} \mathbf{u}^M - \mathbf{u}^0 + \Delta t \beta^M F(\mathbf{u}^0) \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 + \Delta t \beta^1 F(\mathbf{u}^0) \end{pmatrix}$$

- First order approximation
- Explicit Euler
- Easy to solve $\mathcal{L}^1(\underline{\mathbf{u}}) = 0$



Deferred Correction⁵

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

$$\underline{u}^{0,(k)} := \underline{u}(t^n), \quad k = 0, \dots, K,$$

$$\underline{u}^{m,(0)} := \underline{u}(t^n), \quad m = 1, \dots, M$$

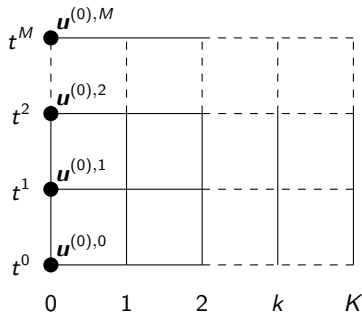
$$\mathcal{L}^1(\underline{u}^{(k)}) = \mathcal{L}^1(\underline{u}^{(k-1)}) - \mathcal{L}^2(\underline{u}^{(k-1)}) \text{ with } k = 1, \dots, K.$$

Theorem (Convergence DeC)

- $\mathcal{L}^2(\underline{u}^*) = 0$
- If \mathcal{L}^1 coercive with constant C_1
- If $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz with constant $C_2 \Delta t$

$$\text{Then } \|\underline{u}^{(K)} - \underline{u}^*\| \leq C(\Delta t)^K$$

- $\mathcal{L}^1(\underline{u}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{u}) = 0$, high order $M + 1$.



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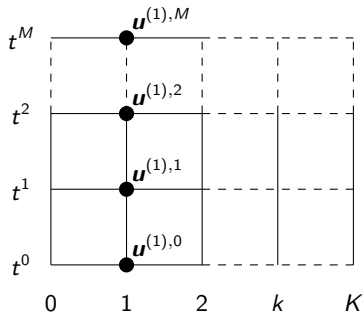
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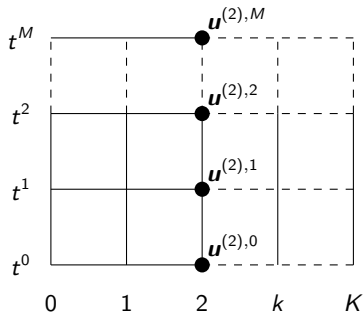
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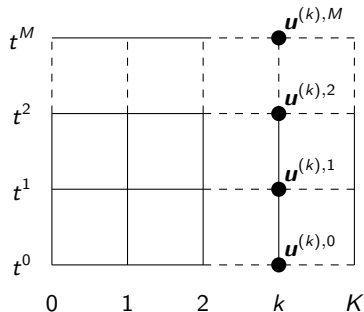
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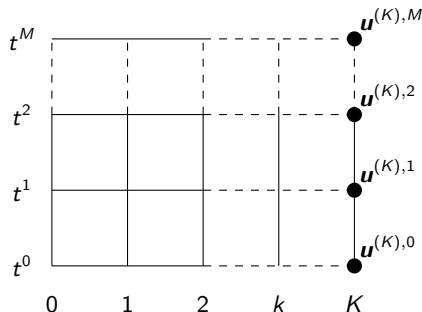
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Proof.

Let f^* be the solution of $\mathcal{L}^2(\underline{\mathbf{u}}^*) = 0$. We know that $\mathcal{L}^1(\underline{\mathbf{u}}^*) = \mathcal{L}^1(\underline{\mathbf{u}}^*) - \mathcal{L}^2(\underline{\mathbf{u}}^*)$, so that



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$$\begin{aligned}\mathcal{L}^1(\underline{u}^{(k+1)}) - \mathcal{L}^1(\underline{u}^*) &= (\mathcal{L}^1(\underline{u}^{(k)}) - \mathcal{L}^2(\underline{u}^{(k)})) - (\mathcal{L}^1(\underline{u}^*) - \mathcal{L}^2(\underline{u}^*)) \\ \color{red}{C_1} \|\underline{u}^{(k+1)} - \underline{u}^*\| &\leq \|\mathcal{L}^1(\underline{u}^{(k+1)}) - \mathcal{L}^1(\underline{u}^*)\| = \\ &= \|\mathcal{L}^1(\underline{u}^{(k)}) - \mathcal{L}^2(\underline{u}^{(k)}) - (\mathcal{L}^1(\underline{u}^*) - \mathcal{L}^2(\underline{u}^*))\| \leq \\ &\leq \color{red}{C_2} \Delta \|\underline{u}^{(k)} - \underline{u}^*\|. \\ \|\underline{u}^{(k+1)} - \underline{u}^*\| &\leq \left(\frac{C_2}{C_1} \Delta \right) \|\underline{u}^{(k)} - \underline{u}^*\| \leq \left(\frac{C_2}{C_1} \Delta \right)^{k+1} \|\underline{u}^{(0)} - \underline{u}^*\|.\end{aligned}$$

After K iteration we have an error at most of $\left(\frac{C_2}{C_1} \Delta \right)^K \|\underline{u}^{(0)} - \underline{u}^*\|$.



DeC: Second order example

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Simplification of DeC for ODE

In practice

$$\mathcal{L}^1(\underline{\mathbf{u}}^{(k)}) = \mathcal{L}^1(\underline{\mathbf{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\mathbf{u}}^{(k-1)}), \quad k = 1, \dots, K,$$

For $m = 1, \dots, M$

$$\begin{aligned} & \mathbf{u}^{(k),m} - \mathbf{u}^0 - \beta^m \Delta t F(\mathbf{u}^0) - \mathbf{u}^{(k-1),m} + \mathbf{u}^0 + \beta^m \Delta t F(\mathbf{u}^0) \\ & + \mathbf{u}^{(k-1),m} - \mathbf{u}^0 - \Delta t \sum_{r=0}^M \theta_r^m F(\mathbf{u}^{(k-1),r}) = 0 \end{aligned}$$

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Deferred Correction + Residual distribution

- Residual distribution (FV \Rightarrow FE) \Rightarrow High order in space
- Prediction/correction/iterations \Rightarrow High order in time
- Subtimesteps \Rightarrow High order in time

$$U_{\xi}^{m,(k+1)} = U_{\xi}^{m,(k)} - |C_p|^{-1} \sum_{E|\xi \in E} \left(\int_E \Phi_{\xi} (U^{m,(k)} - U^{n,0}) d\mathbf{x} + \Delta t \sum_{r=0}^M \theta_r^m \mathcal{R}_{\xi}^E(U^{r,(k)}) \right),$$

with

$$\sum_{\xi \in E} \mathcal{R}_{\xi}^E(u) = \int_E \nabla_{\mathbf{x}} F(u) d\mathbf{x}.$$

- The \mathcal{L}^2 operator contains also the complications of the spatial discretization (e.g. mass matrix)
- \mathcal{L}^1 operator further simplified up to a first order approximation (e.g. **mass lumping**)

\mathcal{L}^1 with mass lumping

Implicit simple DeC (Rosenbrock)

Define \mathcal{L}^1 as

$$\mathcal{L}^1(\mathbf{u}^0, \dots, \mathbf{u}^M) = \begin{pmatrix} \mathbf{u}^M - \mathbf{u}^0 - \Delta t \beta^M F(\mathbf{u}^0) \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 - \Delta t \beta^1 F(\mathbf{u}^0) \end{pmatrix}$$

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$$\begin{aligned}\mathcal{L}^1(\mathbf{u}^0, \dots, \mathbf{u}^M) &= \begin{pmatrix} \mathbf{u}^M - \mathbf{u}^0 - \Delta t \beta^M (F(\mathbf{u}^0) + \partial_{\mathbf{u}} F(\mathbf{u}^0)(\mathbf{u}^M - \mathbf{u}^0)) \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 - \Delta t \beta^1 (F(\mathbf{u}^0) + \partial_{\mathbf{u}} F(\mathbf{u}^0)(\mathbf{u}^1 - \mathbf{u}^0)) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{u}^M - \mathbf{u}^0 - \Delta t \beta^M \partial_{\mathbf{u}} F(\mathbf{u}^0) \mathbf{u}^M \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 - \Delta t \beta^1 \partial_{\mathbf{u}} F(\mathbf{u}^0) \mathbf{u}^1 \end{pmatrix}\end{aligned}$$

Implicit simple DeC (Rosenbrock)

$$\begin{aligned}\mathcal{L}^{1,m}(\mathbf{u}^0, \dots, \mathbf{u}^M) &= \mathbf{u}^m - \mathbf{u}^0 - \Delta t \beta^m \partial_{\mathbf{u}} F(\mathbf{u}^0) \mathbf{u}^m \\ \mathcal{L}^{2,m}(\mathbf{u}^0, \dots, \mathbf{u}^M) &= \mathbf{u}^m - \mathbf{u}^0 - \Delta t \sum_r \theta_r^m F(\mathbf{u}^r)\end{aligned}$$

Implicit simple DeC (Rosenbrock)

$$\mathbf{u}^{(k),m} - \mathbf{u}^0 - \Delta t \sum_{r=0}^M \theta_r^m F(\mathbf{u}^{(k-1),r}) = 0$$

DeC as RK

We can write DeC as RK defining $\underline{\theta}_0 = \{\theta_0^m\}_{m=1}^M$, $\underline{\theta}^M = \theta_r^M$ with $r \in 1, \dots, M$, denoting the vector $\underline{\theta}^{M,T} = (\theta_1^M, \dots, \theta_M^M)$. The Butcher tableau for an arbitrarily high order DeC approach is given by:

$$\begin{array}{c|cccccc}
 0 & 0 & & & & \\
 \underline{\beta} & \underline{\beta} & & & & \\
 \underline{\beta} & \underline{\theta}_0 & \underline{\tilde{\theta}} & & & \\
 \vdots & \underline{\theta}_0 & \underline{0} & \underline{\tilde{\theta}} & & \\
 \vdots & \underline{\theta}_0 & \underline{0} & \underline{0} & \underline{\tilde{\theta}} & \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
 \underline{\beta} & \underline{\theta}_0 & \underline{0} & \dots & \dots & \underline{0} & \underline{\tilde{\theta}} \\
 \hline
 & \underline{\theta}_0^M & \underline{0}^T & \dots & \dots & \underline{0}^T & \underline{\theta}^{M,T}
 \end{array} \quad (6)$$

Stability of (explicit) DeC

Idea: study the RK version!

$$u' = \lambda u \quad \Re(\lambda) < 0. \quad (7)$$

$$u_{n+1} = R(\lambda \Delta t) u_n, \quad R(z) = 1 + zb^T(I - zA)^{-1}\mathbf{1}, \quad z = \lambda \Delta t \quad (8)$$

Goal: find $z \in \mathbb{C}$ such that $|R(z)| < 1$.

Recall: stability function for explicit RK methods is a polynomial, indeed the inverse of $(I - zA)$ can be written in Taylor expansion as

$$(I - zA)^{-1} = \sum_{r=0}^{\infty} z^r A^r = I + zA + z^2 A^2 + \dots, \quad (9)$$

and, since A is strictly lower triangular, it is nilpotent. Hence, $R(z)$ is a polynomial in z with degree at most equal to S .

Theorem

If the RK method is of order P , then

$$R(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^P}{P!} + O(z^{P+1}). \quad (10)$$

The first $P + 1$ terms of the stability functions $R(\cdot)$ for explicit DeCs of order P are known.

Theorem

The stability function of any explicit DeC of order P (with P iterations) is

$$R(z) = \sum_{r=0}^P \frac{z^r}{r!} = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^P}{P!} \quad (11)$$

and does not depend on the distribution of the subtimenodes.

Proof (1/3)

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \star & 0 & 0 & \dots & 0 & 0 \\ \star & \star & 0 & \dots & 0 & 0 \\ \star & 0 & \star & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & 0 & 0 & \dots & \star & 0 \end{pmatrix},$$

Block structure of the matrix A

\star are some non-zero block matrices and the 0 are some zero block matrices.

The number of blocks in each line and row of these matrices is P , the order of the scheme.

Proof (2/3)

By induction, A^k has zeros in the upper triangular part, in the main block diagonal, and in all the $k - 1$ block diagonals below the main diagonal, i.e.,

$$(A^k)_{i,j} = 0 \quad , \text{ if } i < j + k,$$

where the indexes here refer to the blocks. Indeed, it is true that $A_{i,j} = 0$ if $i < j + 1$. Now, let us consider the entry $(A^{k+1})_{i,j}$ with $i < j + k + 1$, i.e., $i - k < j + 1$. It is defined as

$$(A^{k+1})_{i,j} = \sum_w (A^k)_{i,w} A_{w,j}. \quad (12)$$

Now, we can prove that all the terms of the sum are 0. Let $w < j + 1$, then $A_{w,j} = 0$ because of the structure of A ; while, if $w \geq j + 1 > i - k$, we have that $i < w + k$, so $(A^k)_{i,w} = 0$ by induction.

Proof (3/3)

In particular, this means that $A^P = \underline{0}$, because i is always smaller than $j + P$ as P is the number of the block matrices that we have. Hence,

$$(I - zA)^{-1} = \sum_{r=0}^{\infty} z^r A^r = \sum_{r=0}^{P-1} z^r A^r = I + zA + z^2 A^2 + \dots + z^{P-1} A^{P-1}. \quad (13)$$

Plugging this result into $R(z) = 1 + zb^T(I - zA)^{-1}\mathbf{1}$, the stability function $R(z)$ is a polynomial of degree P , the order of the scheme. All terms of order lower or equal to P must agree with the expansion of the exponential function, so it must be

$$R(z) = \sum_{r=0}^P \frac{z^r}{r!} = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^P}{P!}. \quad (14)$$

Note: no assumption on the distribution of the subtimenodes.

CODE

- Choice of iterations (P) and order
- Choice of point distributions t^0, \dots, t^M
- Computation of θ
- Loop for timesteps
- Loop for correction
- Loop for subimesteps

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ADER

- Cauchy–Kovalevskaya theorem
- Modern automatic version
- Space/time DG
- Prediction/Correction
- Fixed-point iteration process

Modern approach is DG in space time for hyperbolic problem

$$\partial_t u(x, t) + \nabla \cdot F(u(x, t)) = 0, x \in \Omega \subset \mathbb{R}^d, t > 0. \quad (15)$$

Prediction: iterative procedure

$$\int_{T^n \times V_i} \theta_{rs}(x, t) \partial_t \theta_{pq}(x, t) z^{pq} dx dt + \int_{T^n \times V_i} \theta_{rs}(x, t) \nabla_x \cdot F(\theta_{pq}(x, t) z^{pq}) dx dt = 0.$$

Correction step: communication between cells

$$\int_{V_i} \Phi_r(u(t^{n+1}) - u(t^n)) dx + \int_{T^n \times \partial V_i} \Phi_r(x) \mathcal{G}(z^-, z^+) \cdot \mathbf{n} dS dt - \int_{T^n \times V_i} \nabla_x \Phi_r \cdot F(z) dx dt = 0,$$

ADER: space-time discretization

Defining $\theta_{rs}(x, t) = \Phi_r(x)\phi_s(t)$ basis functions in space and time

$$\int_{T^n \times V_i} \theta_{rs}(x, t) \partial_t \theta_{pq}(x, t) u^{pq} dx dt + \int_{T^n \times V_i} \theta_{rs}(x, t) \nabla \cdot F(\theta_{pq}(x, t) u^{pq}) dx dt = 0. \quad (16)$$

ADER: space-time discretization

Defining $\theta_{rs}(x, t) = \Phi_r(x)\phi_s(t)$ basis functions in space and time

$$\int_{T^n \times V_i} \theta_{rs}(x, t) \partial_t \theta_{pq}(x, t) u^{pq} dx dt + \int_{T^n \times V_i} \theta_{rs}(x, t) \nabla \cdot F(\theta_{pq}(x, t) u^{pq}) dx dt = 0. \quad (16)$$

This leads to

$$\underline{\underline{M}}_{rspq} u^{pq} = \underline{\underline{r}}(\underline{\underline{u}})_{rs}, \quad (17)$$

solved with fixed point iteration method.

+ Correction step where cells communication is allowed (derived from (16)).

ADER: time integration method

Simplify! Take $\mathbf{u}(t) = \sum_{m=0}^M \phi_m(t) \mathbf{u}^m = \underline{\phi}(t)^T \underline{\mathbf{u}}$

$$\int_{T^n} \psi(t) \partial_t \mathbf{u}(t) dt - \int_{T^n} \psi(t) F(\mathbf{u}(t)) dt = 0, \quad \forall \psi : T^n = [t^n, t^{n+1}] \rightarrow \mathbb{R}.$$

$$\mathcal{L}^2(\underline{\mathbf{u}}) := \int_{T^n} \underline{\phi}(t) \partial_t \underline{\phi}(t)^T \underline{\mathbf{u}} dt - \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\mathbf{u}}) dt = 0$$

$$\underline{\phi}(t) = (\phi_0(t), \dots, \phi_M(t))^T$$

Quadrature...

$$\mathcal{L}^2(\underline{\mathbf{u}}) := \underline{\underline{\mathbf{M}}} \underline{\mathbf{u}} - \underline{r}(\underline{\mathbf{u}}) = 0 \iff \underline{\underline{\mathbf{M}}} \underline{\mathbf{u}} = \underline{r}(\underline{\mathbf{u}}). \quad (18)$$

Nonlinear system of $M \times S$ equations

ADER: Mass matrix

What goes into the mass matrix? Use of the integration by parts

$$\mathcal{L}^2(\underline{\mathbf{u}}) := \int_{T^n} \underline{\phi}(t) \partial_t \underline{\phi}(t)^T \underline{\mathbf{u}} dt + \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t))^T \underline{\mathbf{u}} dt =$$
$$\underline{\phi}(t^{n+1}) \underline{\phi}(t^{n+1})^T \underline{\mathbf{u}} - \underline{\phi}(t^n) \underline{\mathbf{u}}^n - \int_{T^n} \partial_t \underline{\phi}(t) \underline{\phi}(t)^T \underline{\mathbf{u}} dt - \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t))^T \underline{\mathbf{u}} dt$$

$$\underline{\underline{\mathbf{M}}} = \underline{\phi}(t^{n+1}) \underline{\phi}(t^{n+1})^T - \int_{T^n} \partial_t \underline{\phi}(t) \underline{\phi}(t)^T$$

$$\underline{r}(\underline{\mathbf{u}}) = \underline{\phi}(t^n) \underline{\mathbf{u}}^n + \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t))^T \underline{\mathbf{u}} dt$$

$$\underline{\underline{\mathbf{M}}} \underline{\mathbf{u}} = \underline{r}(\underline{\mathbf{u}})$$

ADER: Fixed point iteration

Iterative procedure to solve the problem for each time step

$$\underline{\underline{\mathbf{u}}}^{(k)} = \underline{\underline{\mathbf{M}}}^{-1} \underline{\underline{\mathbf{r}}}(\underline{\underline{\mathbf{u}}}^{(k-1)}), \quad k = 1, \dots, \text{convergence} \quad (19)$$

with $\underline{\underline{\mathbf{u}}}^{(0)} = \mathbf{u}(t^n)$.

Reconstruction step

$$\mathbf{u}(t^{n+1}) = \mathbf{u}(t^n) - \int_{T^n} F(\mathbf{u}^{(K)}(t)) dt.$$

- Convergence?
- How many steps K ?
- Accuracy \mathcal{L}^2 ?

ADER 2nd order

Example with 2 Gauss Legendre points, Lagrange polynomials and 2 iterations

Let us consider the timestep interval $[t^n, t^{n+1}]$, rescaled to $[0, 1]$.

Gauss-Legendre points quadrature and interpolation (in the interval $[0, 1]$)

$$\underline{t}_q = (t_q^0, t_q^1) = (t^0, t^1) = \left(\frac{\sqrt{3}-1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}} \right), \quad \underline{w} = (1/2, 1/2).$$

$$\underline{\phi}(t) = (\phi_0(t), \phi_1(t)) = \left(\frac{t - t^1}{t^0 - t^1}, \frac{t - t^0}{t^1 - t^0} \right).$$

Then, the mass matrix is given by

$$\underline{\underline{M}}_{m,l} = \phi_m(1)\phi_l(1) - \phi'_m(t^l)w_l, \quad m, l = 0, 1,$$

$$\underline{\underline{M}} = \begin{pmatrix} 1 & \frac{\sqrt{3}-1}{2} \\ -\frac{\sqrt{3}+1}{2} & 1 \end{pmatrix}.$$

ADER 2nd order

The right hand side is given

$$r(\underline{\mathbf{u}})_m = \alpha(0)\phi_m(0) + \Delta t F(\alpha(t^m))w_m, \quad m = 0, 1.$$

$$\underline{r}(\underline{\mathbf{u}}) = \alpha(0)\underline{\phi}(0) + \Delta t \begin{pmatrix} F(\alpha(t^1))w_1 \\ F(\alpha(t^2))w_2 \end{pmatrix}.$$

Then, the coefficients $\underline{\mathbf{u}}$ are given by

$$\underline{\mathbf{u}}^{(k+1)} = \underline{\underline{\mathbf{M}}}^{-1} \underline{r}(\underline{\mathbf{u}}^{(k)}).$$

Finally, use $\underline{\mathbf{u}}^{(k+1)}$ to reconstruct the solution at the time step t^{n+1} :

$$\mathbf{u}^{n+1} = \underline{\phi}(1)^T \underline{\mathbf{u}}^{(k+1)} = \mathbf{u}^n + \int_{T^n} \underline{\phi}(t)^T dt F(\underline{\mathbf{u}}^{(k)}).$$

- Choice: ϕ Lagrangian basis functions
- Different subimesteps: Gauss-Legendre, Gauss-Lobatto, equispaced
- Precompute $\underline{\underline{M}}$
- Precompute the rhs vector part using quadratures after a further approximation

$$\underline{r}(\underline{u}) = \underline{\phi}(t^n)\underline{u}^n + \int_{T^n} \underline{\phi}(t)F(\underline{\phi}(t)^T\underline{u})dt \approx \underline{\phi}(t^n)\underline{u}^n + \underbrace{\int_{T^n} \underline{\phi}(t)\underline{\phi}(t)^T dt}_{\text{Can be stored}} F(\underline{u})$$

- Precompute the reconstruction coefficients $\underline{\phi}(1)^T$

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ADER⁶ and DeC⁷: immediate similarities

- High order time-space discretization
- Start from a well known space discretization (FE/DG/FV)
- FE reconstruction in time
- System in time, with M equations
- Iterative method / K corrections

⁶M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. *Journal of Computational Physics*, 227(18):8209–8253, 2008.

⁷R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. *Journal of Scientific Computing*, 73(2):461–494, Dec 2017.

ADER⁶ and DeC⁷: immediate similarities

- High order time-space discretization
- Start from a well known space discretization (FE/DG/FV)
- FE reconstruction in time
- System in time, with M equations
- Iterative method / K corrections

- Both high order explicit time integration methods (neglecting spatial discretization)

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$$\mathcal{L}^2(\underline{\underline{u}}) := \underline{\underline{M}}\underline{\underline{u}} - r(\underline{\underline{u}}),$$

$$\mathcal{L}^1(\underline{\underline{u}}) := \underline{\underline{M}}\underline{\underline{u}} - r(\underline{\underline{u}}(t^n)).$$

$$\mathcal{L}^1(\underline{\underline{u}}^{(k)}) = \mathcal{L}^1(\underline{\underline{u}}^{(k-1)}) - \mathcal{L}^2(\underline{\underline{u}}^{(k-1)}), \quad k = 1, \dots, K,$$

$$\underline{\underline{M}}\underline{\underline{u}}^{(k)} - r(\underline{\underline{u}}^{(k),0}) - \underline{\underline{M}}\underline{\underline{u}}^{(k-1)} + r(\underline{\underline{u}}^{(k-1),0}) + \underline{\underline{M}}\underline{\underline{u}}^{(k-1)} - r(\underline{\underline{u}}^{(k-1)}) = 0$$

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$$\underline{\underline{M}}\underline{u}^{(k)} - \cancel{r(\underline{u}^{(k)}, \emptyset)} - \underline{\underline{M}}\underline{u}^{(k-1)} + \cancel{r(\underline{u}^{(k-1)}, \emptyset)} + \underline{\underline{M}}\underline{u}^{(k-1)} - r(\underline{u}^{(k-1)}) = 0$$

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$$\mathcal{L}^1(\underline{u}^{(k)}) = \mathcal{L}^1(\underline{u}^{(k-1)}) - \mathcal{L}^2(\underline{u}^{(k-1)}), \quad k = 1, \dots, K,$$

$$\begin{aligned} \underline{\underline{M}}\underline{u}^{(k)} - \cancel{r(\underline{u}^{(k),0})} - \cancel{\underline{\underline{M}}\underline{u}^{(k-1)}} + \cancel{r(\underline{u}^{(k-1),0})} + \cancel{\underline{\underline{M}}\underline{u}^{(k-1)}} - r(\underline{u}^{(k-1)}) &= 0 \\ \underline{\underline{M}}\underline{u}^{(k)} - r(\underline{u}^{(k-1)}) &= 0. \end{aligned}$$

$$\mathcal{L}^2(\underline{u}) := \underline{\underline{M}}\underline{u} - r(\underline{u}),$$

$$\mathcal{L}^1(\underline{u}) := \underline{\underline{M}}\underline{u} - r(\underline{u}(t^n)).$$

Apply the DeC Convergence theorem!

- \mathcal{L}^1 is coercive because $\underline{\underline{M}}$ is always invertible
- $\mathcal{L}^1 - \mathcal{L}^2$ is Lipschitz with constant $C\Delta t$ because they are consistent approx of the same problem
- Hence, after K iterations we obtain a K th order accurate approximation of \underline{u}^*

$$\mathcal{L}^2(\mathbf{u}^0, \dots, \mathbf{u}^M) := \begin{cases} \mathbf{u}^M - \mathbf{u}^0 - \sum_{r=0}^M \int_{t^0}^{t^M} F(\mathbf{u}^r) \varphi_r(s) ds \\ \dots \\ \mathbf{u}^1 - \mathbf{u}^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\mathbf{u}^r) \varphi_r(s) ds \end{cases} .$$

$$\mathcal{L}^2(\mathbf{u}^0, \dots, \mathbf{u}^M) := \begin{cases} \mathbf{u}^M - \mathbf{u}^0 - \sum_{r=0}^M \int_{t^0}^{t^M} F(\mathbf{u}^r) \varphi_r(s) ds \\ \dots \\ \mathbf{u}^1 - \mathbf{u}^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\mathbf{u}^r) \varphi_r(s) ds \end{cases} .$$

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$$\chi_{[t^0, t^m]}(t^m) \mathbf{u}^m - \chi_{[t^0, t^m]}(t_0) \mathbf{u}^0 - \int_{t^0}^{t^m} \chi_{[t^0, t^m]}(t) \sum_{r=0}^M F(\mathbf{u}^r) \varphi_r(t) dt = 0$$

$$\int_{t^0}^{t^M} \chi_{[t^0, t^m]}(t) \partial_t (\mathbf{u}(t)) dt - \int_{t^0}^{t^M} \chi_{[t^0, t^m]}(t) \sum_{r=0}^M F(\mathbf{u}^r) \varphi_r(t) dt = 0,$$

$$\int_{T^n} \psi_m(t) \partial_t \mathbf{u}(t) dt - \int_{T^n} \psi_m(t) F(\mathbf{u}(t)) dt = 0.$$

Classical Runge Kutta (RK)

- One step method
- Internal stages

Explicit Runge Kutta

- + Simple to code
- Not easily generalizable to arbitrary order
- Stages $>$ order

Implicit Runge Kutta

- + Arbitrarily high order
- Require nonlinear solvers for nonlinear systems
- May not converge

DeC – ADER

- One step method
- Internal subimesteps
- Can be rewritten as explicit RK (for ODE)
- + Explicit
- + Simple to code
- + Iterations = order
- + Arbitrarily high order
- Large memory storage

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Stability

Since ADER can be written as a DeC, the stability functions are given by the same formula as for DeC and the stability regions are the following.

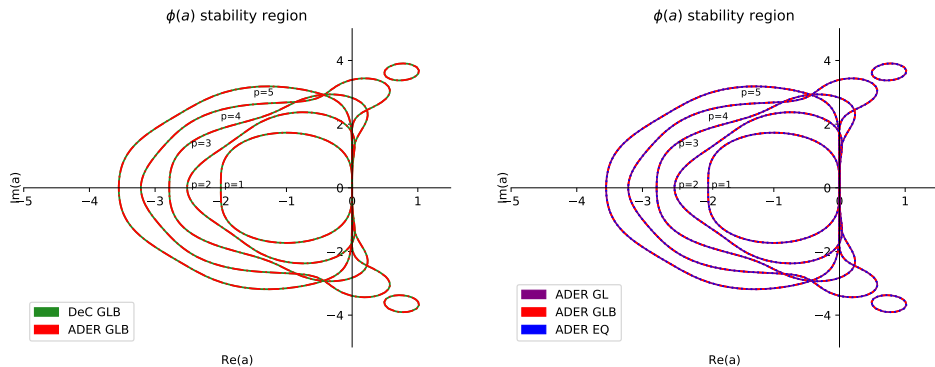


Figure: Stability region

Accuracy of ADER \mathcal{L}^2 operators

The two things that determine the accuracy of the ADER method are the iterations P and the accuracy of \mathcal{L}^2 .

Accuracy of ADER \mathcal{L}^2 for different distributions

- Equispaced: boring, minimum accuracy possible $M + 1$ nodes $p = M + 1$
- Gauss–Lobatto: this generates the LobattoIIIC methods, $M + 1$ nodes $p = 2M$
- Gauss–Legendre: this does not generate Gauss methods, $M + 1$ nodes $p = 2M + 1$

\mathcal{L}^2 ADER as RK

Here, we see \mathcal{L}^2 as an implicit RK

$$\mathcal{L}^{2,m}(\underline{\mathbf{u}}) = \underline{\underline{\mathbf{M}}}_j^m \underline{\mathbf{u}}^{(j)} - \underline{\phi}^m(t^n) \underline{\mathbf{u}}^n - \underbrace{\int_{T^n} \underline{\phi}^m(t) \underline{\phi}(t)_j dt}_{\Delta t \underline{\underline{\mathbf{R}}}_j^m} F(\underline{\mathbf{u}}^{(j)}) = 0$$

$$\tilde{\mathcal{L}}^{2,z}(\underline{\mathbf{u}}) = \underline{\mathbf{u}}^{(z)} - (\underline{\underline{\mathbf{M}}}^{-1})_m^z \underline{\phi}^m(t^n) \underline{\mathbf{u}}^n - \Delta t (\underline{\underline{\mathbf{M}}}^{-1})_m^z \underline{\underline{\mathbf{R}}}_j^m F(\underline{\mathbf{u}}^{(j)}) = 0$$

$$\underline{\mathbf{u}}^{(z)} = \underline{\mathbf{u}}^n + \Delta t a_{z,j} F(\underline{\mathbf{u}}^{(j)})$$

- $a_{mj} = (\underline{\underline{\mathbf{M}}}^{-1})_m^z \underline{\underline{\mathbf{R}}}_j^m$
- Prove that $(\underline{\underline{\mathbf{M}}}^{-1})_m^z \underline{\phi}^m(t^n) = 1$ for every z
- $c^m = \sum_r a_{mr} = t^m$
- $b_r = \frac{1}{\Delta t} \int_{T^m} \phi_r(t) dt = w_r$ quadrature weights

BCD conditions (Butcher 1964)

Define the conditions

$$B(p) : \sum_{i=1}^s b_i c_i^{z-1} = \frac{1}{z}, \quad z = 1, \dots, p; \quad (20)$$

$$C(\eta) : \sum_{j=1}^s a_{ij} c_j^{z-1} = \frac{c_i^z}{z}, \quad i = 1, \dots, s, z = 1, \dots, \eta; \quad (21)$$

$$D(\zeta) : \sum_{i=1}^s b_i c_i^{z-1} a_{ij} = \frac{b_j}{z} (1 - c_j^z), \quad j = 1, \dots, s, z = 1, \dots, \zeta. \quad (22)$$

Theorem (Butcher 1964)

If the coefficients b_i, c_i, a_{ij} of a RK scheme satisfy $B(p)$, $C(\eta)$ and $D(\zeta)$ with $p \leq \eta + \zeta + 1$ and $p \leq 2\eta + 2$, then the method is of order p .

$$C(s-1) D(s-1)$$

Lemma

\mathcal{L}^2 operator of ADER defined by Gauss–Lobatto or Gauss–Legendre points and quadrature (they coincide) with $s = M + 1$ stages satisfies $C(s-1)$ and $D(s-1)$.

Proof (1/4).

Interpolation with ϕ^j is exact for polynomials of degree $s-1$.

The quadrature is exact for polynomials of degree $2s-3$.

Recall that $\underline{\underline{A}} = \underline{\underline{M}}\underline{\underline{R}}$, Condition $C(s-1)$ reads

$$\underline{\underline{A}}\underline{\underline{c}}^{z-1} = \frac{1}{z}\underline{\underline{c}}^z \iff \underline{\underline{R}}\underline{\underline{c}}^{z-1} = \frac{1}{z}\underline{\underline{M}}\underline{\underline{c}}^z \iff \underline{\underline{X}} := \underline{\underline{R}}\underline{\underline{c}}^{z-1} - \frac{1}{z}\underline{\underline{M}}\underline{\underline{c}}^z = \underline{\underline{0}}, \quad z = 1, \dots, s-1.$$

Recall $b_m = t^m$, $c_m = w_m$, $\underline{\underline{R}}_{i,j} = \delta_{i,j}w_i$ and the definition of $\underline{\underline{M}}$

$$\underline{\underline{X}}_m := w_m(t^m)^{z-1} - \frac{1}{z} \left(\phi^m(1)\phi^j(1)(t^j)^z - \int_0^1 \frac{d}{d\xi} \phi^m(\xi)\phi^j(\xi)(t^j)^z d\xi \right).$$

$$C(s-1) D(s-1)$$

Proof (2/4).

Now, the interpolation of t^z with $z \leq s-1$ with basis functions ϕ^j is exact. Hence, we can substitute $\phi^j(\xi)(t^j)^z = \xi^z$ for all $z = 1, \dots, s-1$, obtaining

$$\mathcal{X}_m = w_m(t^m)^{z-1} - \frac{1}{z} \left(\phi^m(1)1^z - \int_0^1 \frac{d}{d\xi} \phi^m(\xi) \xi^z d\xi \right).$$

Using the exactness of the quadrature for polynomials of degree $2s-3$, both true for Gauss–Lobatto and Gauss–Legendre, we know that the previous integral is exactly computed as $\frac{d}{d\xi} \phi^m(\xi)$ is of degree at most $s-2$ and ξ^z is at most $s-1$. So, we can use integration by parts and obtain

$$\mathcal{X}_m = w_m(t^m)^{z-1} - \frac{1}{z} \left(\phi^m(0)0^z + \int_0^1 \phi^m(\xi) \frac{d}{d\xi} \xi^z d\xi \right) = w_m(t^m)^{z-1} - \int_0^1 \phi^m(\xi) \xi^{z-1} d\xi = 0$$

by the exactness of the quadrature rule and the definition of w_m . Note that the condition is sharp, since the interpolation is not anymore exact for $z = s$, hence $C(s)$ is not satisfied.

$$\underline{C}(s-1) \underline{D}(s-1)$$

Proof (3/4).

To prove $\underline{D}(s-1)$, we write explicitly the condition in matricial form, for all $z = 1, \dots, s-1$

$$\underline{bc}^{z-1} \underline{A} = \frac{1}{z} \underline{b(1-c^z)} \iff \underline{bc}^{z-1} \underline{\underline{M}}^{-1} \underline{\underline{R}} = \frac{1}{z} \underline{b(1-c^z)} \iff \underline{bc}^{z-1} = \frac{1}{z} \underline{b(1-c^z)} \underline{\underline{R}}^{-1} \underline{\underline{M}}.$$

Note that $b^m = w_m$ and $\underline{\underline{R}}_r^m = w_m \delta_r^m$, so $\underline{b(1-c^z)} \underline{\underline{R}}^{-1} = \underline{(1-c^z)}$. It is left to prove that

$$\mathcal{Y} := \underline{bc}^{z-1} - \frac{1}{z} \underline{(1-c^z)} \underline{\underline{M}} = \underline{0}.$$

$$\mathcal{Y}_m = w_m (t^m)^{z-1} - \frac{1}{z} \sum_{j=1}^s (1 - (t^j)^z) \left(\phi^j(1) \phi^m(1) - \int_0^1 \frac{d}{d\xi} \phi^j(\xi) \phi^m(\xi) d\xi \right).$$

Proof (4/4).

Let us observe that, since $z \leq s-1$, the polynomial is exactly represented by the Lagrangian interpolation $t^z = \sum_{j=1}^s \phi(t)(t^m)^j$. Hence, using the exactness of the quadrature for polynomials of degree at most $2s-3$, we have

$$\begin{aligned} \mathcal{Y}_m &= w_m(t^m)^{z-1} - \frac{1}{z} (1 - (1)^z) \phi^m(1) + \frac{1}{z} \int_0^1 \frac{d}{d\xi} (1 - (\xi)^z) \phi^m(\xi) d\xi \\ &= w_m(t^m)^{z-1} - \frac{1}{z} \int_0^1 z \xi^{z-1} \phi^m(\xi) d\xi = w_m(t^m)^{z-1} - w_m(t^m)^{z-1} = 0. \end{aligned}$$

Hence, ADER-Legendre and ADER-Lobatto satisfy $D(s-1)$. Note that the condition is sharp, since the interpolation is not anymore exact for $z = s$, hence $D(s)$ is not satisfied.

Remark (ADER-Legendre is no collocation method)

From the proof of previous Lemma, we can observe that ADER-Legendre methods do not satisfy $C(s)$, hence, the methods are not collocation methods and they do not coincide with Gauss–Legendre implicit RK methods.

Theorem

\mathcal{L}^2 of ADER with Gauss–Legendre is of order $2s - 1$.

Proof.

ADER-Legendre with $s = M + 1$ stages satisfies $B(2s)$ for the quadrature rule and, hence, it satisfies $B(2s - 1)$. For previous Lemma it also satisfies $C(s - 1)$ and $D(s - 1)$. Hence, Butcher's (1964) Theorem ($p \leq \eta + \zeta + 1$ and $p \leq 2\eta + 2$) guarantees that the method is of order $2s - 1$, since it is satisfied with $p = 2s - 1$ and $\eta = \zeta = s - 1$. □

Theorem

\mathcal{L}^2 of ADER with Gauss-Lobatto is of order $2s - 2$.

Proof.

The condition for $B(2s - 2)$ is satisfied as (c, b) is the Gauss-Lobatto quadrature with order $2s - 2$. Previous Lemma guarantees that ADER-Lobatto satisfies $B(2s - 2)$, $C(s - 1)$ and $D(s - 1)$, so Butcher's (1964) Theorem ($p \leq \eta + \zeta + 1$ and $p \leq 2\eta + 2$) is satisfied for order $p = 2s - 2$ and $\eta = \zeta = s - 1$. □

Theorem

\mathcal{L}^2 of ADER with Gauss-Lobatto is LobattoIIIC.

The Lobatto IIIC method is defined using the condition

$$a_{i1} = b_1 \quad \text{for } i = 1, \dots, s. \quad (23)$$

Lemma

\mathcal{L}^2 of ADER with Gauss-Lobatto satisfies (23).

Theorem (Chipman 1971)

Lobatto IIIC schemes (in particular RK a_{ij}) are uniquely determined by Gauss–Lobatto quadrature rule (c, b) , condition (23) and by $C(s - 1)$.

Lemma

\mathcal{L}^2 of ADER with Gauss-Lobatto satisfies (23).

Proof.

$$a_{i1} = \sum_j (\underline{\underline{\mathbf{M}}}^{-1})_{ij} \mathbb{R}_{j1} = b_1 = w_1 \iff$$

$$\sum_{i,j} \underline{\underline{\mathbf{M}}}_{ki} (\underline{\underline{\mathbf{M}}}^{-1})_{ij} \mathbb{R}_{j1} = \sum_i \underline{\underline{\mathbf{M}}}_{ki} w_1 \iff$$

$$\delta_{k1} w_1 = \mathbb{R}_{k1} = \sum_i \underline{\underline{\mathbf{M}}}_{ki} w_1$$

$$\sum_i \underline{\underline{\mathbf{M}}}_{ki} w_1 = \phi^m(1) w_1 - \int_0^1 \frac{d}{dt} \phi^m(\xi) w_1 dt = w_1 \phi^m(0) = w_1 \delta_{m,1}.$$



Outline

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- 2 DeC
- 3 ADER
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- 6 Simulations**
- 7 Efficient DeC (ADER)
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Applications

Usages

- Hyperbolic PDEs as explicit iterative methods (ADER: Toro, Dumbser, Klingenberg, Boscheri; DeC: Abgrall, Ricchiuto)
- IMEX solvers for hyperbolic with stiff sources (ADER: Dumbser, Boscheri; DeC: Abgrall, Torlo)
- IMEX solvers for hyperbolic with viscosity (treated implicitly) as compressible Navier Stokes (DeC: Minion, Dumbser, Zeifang)

IMEX

$$\partial_t u = F(u) + S(u)$$

$S(u)$ stiff to be treated implicitly

Advantages

- Arbitrary high order
- Unique framework to have matching between implicit and explicit terms
- Easy to code
- Iterative solver automatically included

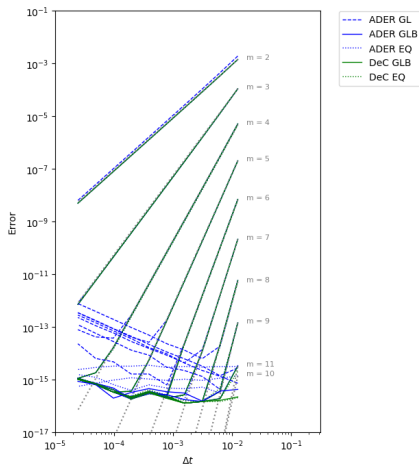
Disadvantages

- Explicit solver: many many stages
- Implicit: many stages
- Explicit: not amazing stability property (wrt SSP RK e.g.)

Convergence

$$\begin{aligned} y'(t) &= -|y(t)|y(t), \\ y(0) &= 1, \\ t &\in [0, 0.1]. \end{aligned} \tag{24}$$

Convergence curves for ADER and DeC, varying the approximation order and collocation of nodes for the sub-timesteps for a scalar nonlinear ODE



Lotka–Volterra

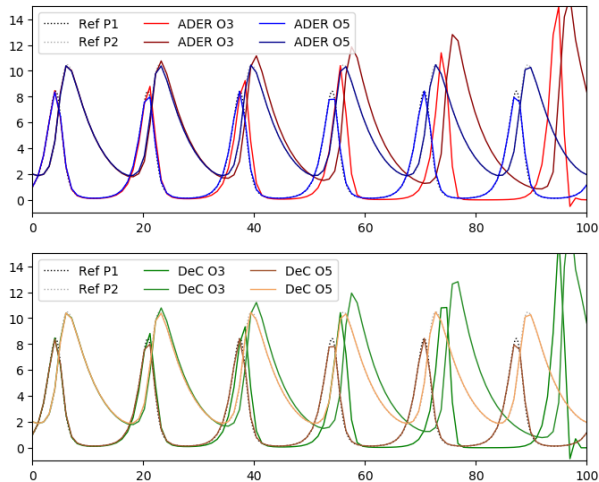


Figure: Numerical solution of the Lotka-Volterra system using ADER (top) and DeC (bottom) with Gauss-Lobatto nodes with timestep $\Delta T = 1$.

PDE: Burgers with spectral difference

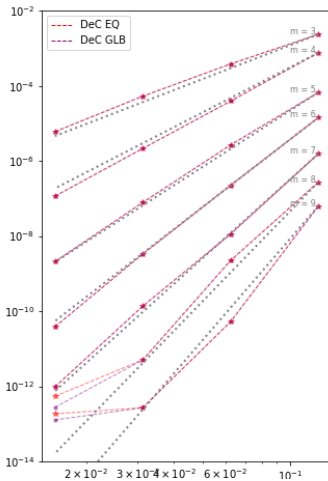
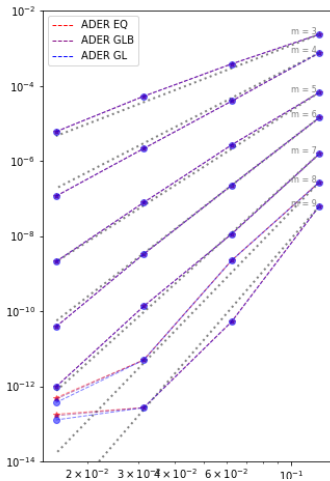


Figure: Convergence error for Burgers equations: Left ADER, right DeC. Space

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Reduce computational cost for explicit DeC

Literature

- Micalizzi, L., Torlo, D. *A new efficient explicit Deferred Correction framework: analysis and applications to hyperbolic PDEs and adaptivity.* arxiv.org/abs/2210.02976
- Micalizzi, L., Torlo, D., Boscheri, W. *Efficient iterative arbitrary high order methods: an adaptive bridge between low and high order.* arxiv.org/abs/2212.07783

Goal

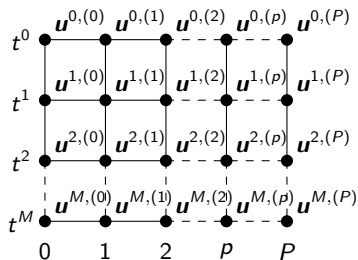
Reduce computational costs of explicit DeC.

$$\mathcal{L}^1(\underline{u}^{(p)}) = \mathcal{L}^1(\underline{u}^{(p-1)}) - \mathcal{L}^2(\underline{u}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

$$\underline{u}^{m,(p)} = \underline{u}^0 + \sum_{r=0}^M \theta_r^m F(t^r, \underline{u}^{r,(p-1)}), \quad \forall m = 1, \dots, M, \quad p = 1, \dots, P$$

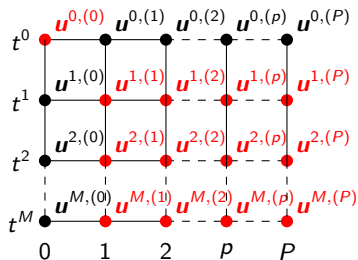
$$\mathcal{L}^1(\underline{u}^{(p)}) = \mathcal{L}^1(\underline{u}^{(p-1)}) - \mathcal{L}^2(\underline{u}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

$$\underline{u}^{m,(p)} = \underline{u}^0 + \sum_{r=0}^M \theta_r^m F(t^r, \underline{u}^{r,(p-1)}), \quad \forall m = 1, \dots, M, \quad p = 1, \dots, P$$



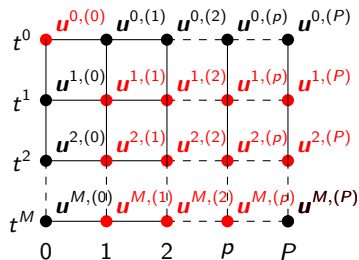
$$\mathcal{L}^1(\underline{u}^{(p)}) = \mathcal{L}^1(\underline{u}^{(p-1)}) - \mathcal{L}^2(\underline{u}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

$$\underline{u}^{m,(p)} = \underline{u}^0 + \sum_{r=0}^M \theta_r^m F(t^r, \underline{u}^{r,(p-1)}), \quad \forall m = 1, \dots, M, p = 1, \dots, P$$



$$\mathcal{L}^1(\underline{\mathbf{u}}^{(p)}) = \mathcal{L}^1(\underline{\mathbf{u}}^{(p-1)}) - \mathcal{L}^2(\underline{\mathbf{u}}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

$$\mathbf{u}^{m,(p)} = \mathbf{u}^0 + \sum_{r=0}^M \theta_r^m F(t^r, \mathbf{u}^{r,(p-1)}), \quad \forall m = 1, \dots, M, p = 1, \dots, P$$



\underline{c}	\mathbf{u}^0	$\mathbf{u}^{(1)}$	$\mathbf{u}^{(2)}$	$\mathbf{u}^{(3)}$	\dots	$\mathbf{u}^{(M-1)}$	$\mathbf{u}^{(M)}$	A
0	0							\mathbf{u}^0
$\underline{\beta}_{1:}$	$\underline{\beta}_{1:}$	$\underline{0}$						$\mathbf{u}^{(1)}$
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\Theta_{1:,1:}$	$\underline{0}$					$\mathbf{u}^{(2)}$
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{0}$	$\Theta_{1:,1:}$	$\underline{0}$				$\mathbf{u}^{(3)}$
	\vdots	\vdots		\ddots	\ddots			\vdots
	\vdots	\vdots			\ddots	\ddots		\vdots
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{0}$	\dots	\dots	$\underline{0}$	$\Theta_{1:,1:}$	$\underline{0}$	$\mathbf{u}^{(M)}$
\underline{b}	$\Theta_{M,0}$	$\underline{0}$	\dots	\dots	\dots	$\underline{0}$	$\Theta_{M,1:}$	$\mathbf{u}^{M,(M+1)}$

Large costs!

Large costs!

- DeC $S = M \cdot (P - 1) + 1$
 - DeC equi $S = (P - 1)^2 + 1$
 - DeC GLB $S = \left\lceil \frac{P}{2} \right\rceil (P - 1) + 1$

Equispaced

P	M	DeC
2	1	2
3	2	5
4	3	10
5	4	17
6	5	26
7	6	37
8	7	50
9	8	65
10	9	82

Gauss-Lobatto

P	M	DeC
2	1	2
3	2	5
4	2	7
5	3	13
6	3	16
7	4	25
8	4	29
9	5	41
10	5	46

Large costs!

- DeC $S = M \cdot (P - 1) + 1$
 - DeC equi $S = (P - 1)^2 + 1$
 - DeC GLB $S = \left\lceil \frac{P}{2} \right\rceil (P - 1) + 1$

Equispaced

P	M	DeC
2	1	2
3	2	5
4	3	10
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6	5	26
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Gauss-Lobatto

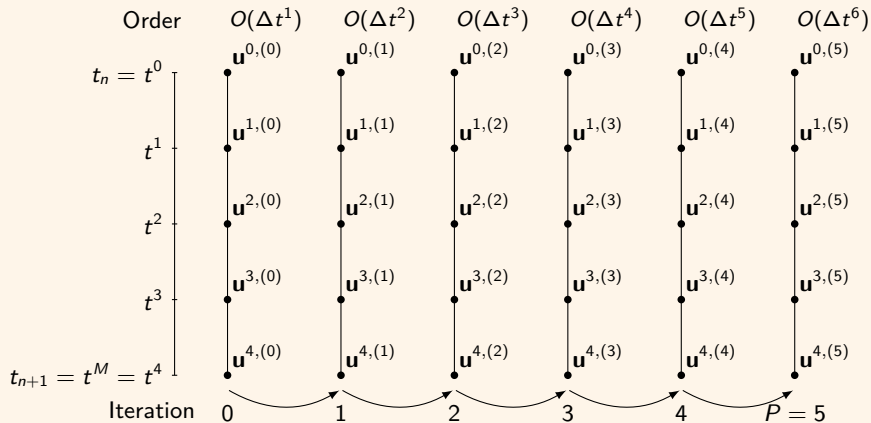
P	M	DeC
2	1	2
3	2	5
4	2	7
5	3	13
6	3	16
7	4	25
8	4	29
9	5	41
10	5	46

How can we save computational time?

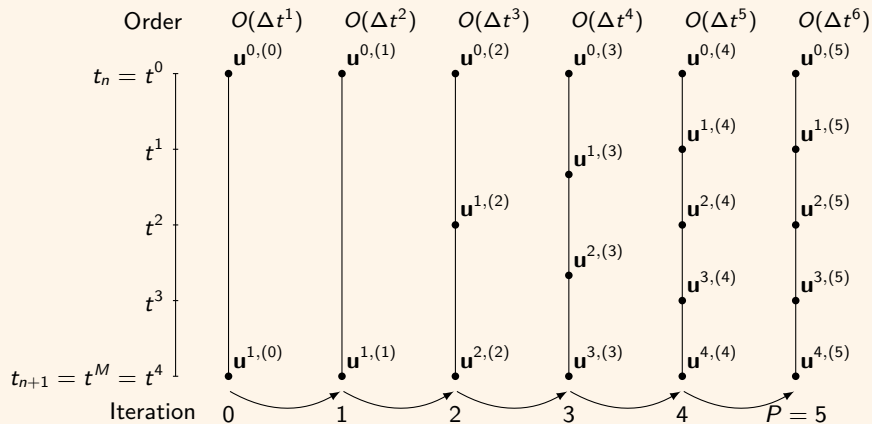
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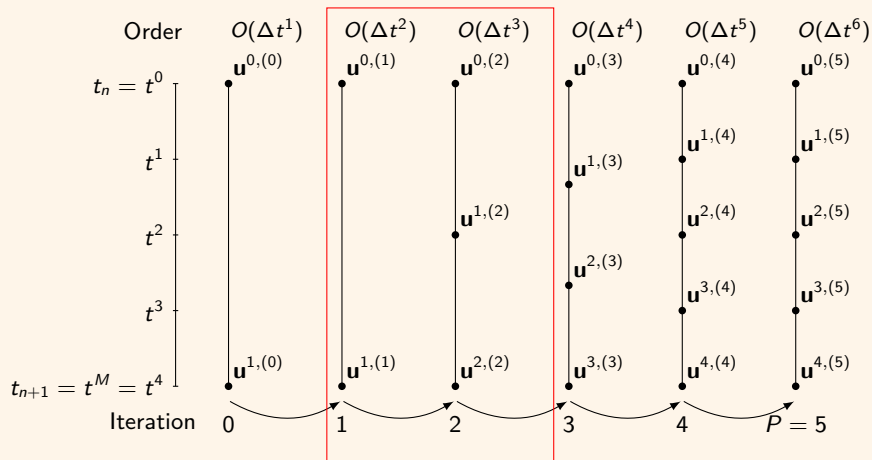
Idea for reduction of stages



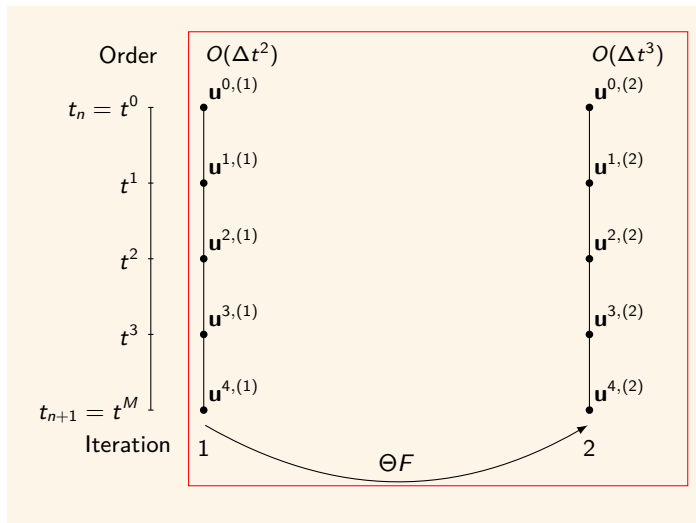
Idea for reduction of stages



Idea for reduction of stages



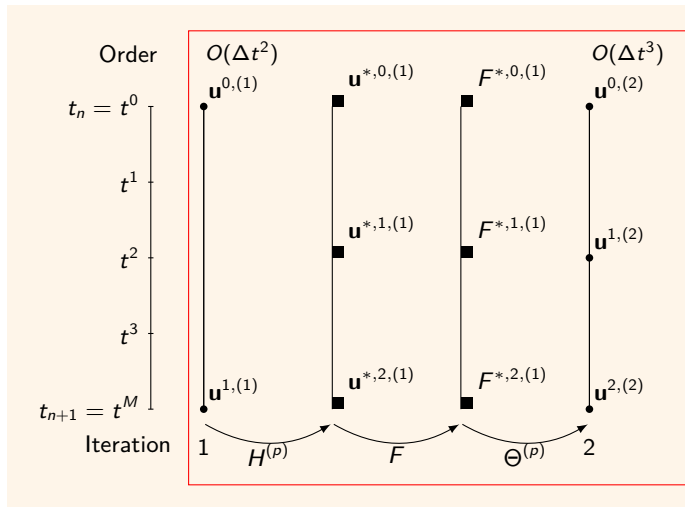
How to communicate between iterations?



DeC

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta F(\underline{u}^{(p-1)})$$

How to communicate between iterations?



DeC

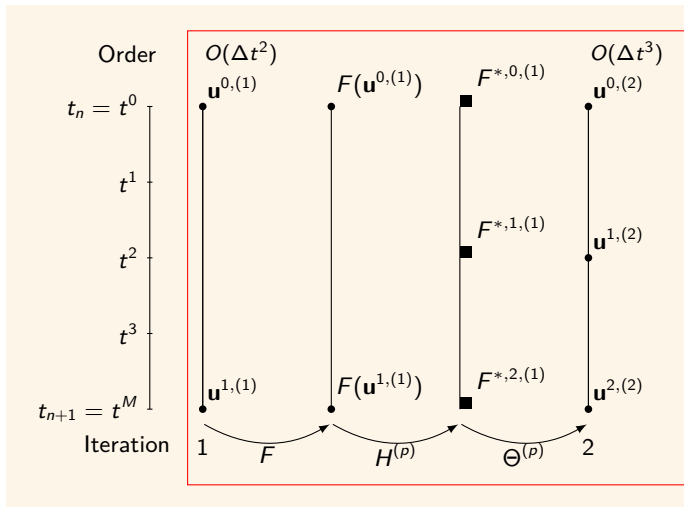
$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta F(\underline{u}^{(p-1)})$$

DeCu

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta^{(p)} F(H^{(p)} \underline{u}^{(p-1)})$$

$$H_{ij}^{(p)} = \phi_j^{(p-1)}(t^{i,(p)})$$

How to communicate between iterations?



DeC

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta F(\underline{u}^{(p-1)})$$

DeCu

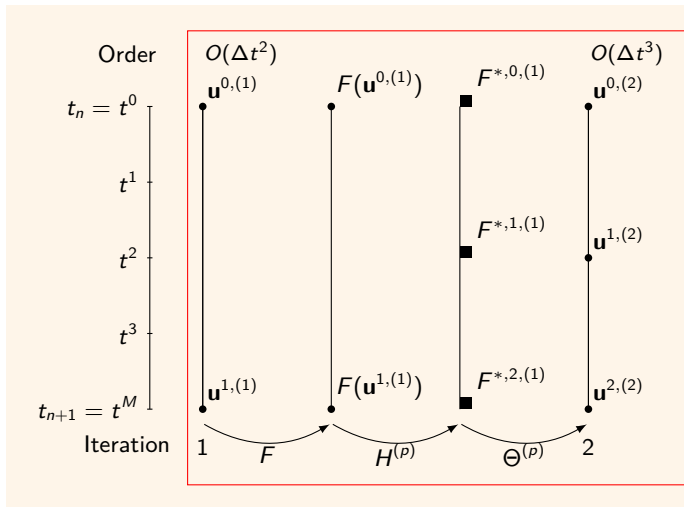
$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta^{(p)} F(H^{(p)} \underline{u}^{(p-1)})$$

DeCdu

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta^{(p)} H^{(p)} F(\underline{u}^{(p-1)})$$

$$H_{ij}^{(p)} = \phi_j^{(p-1)}(t^{i,(p)})$$

How to communicate between iterations?



DeC

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta F(\underline{u}^{(p-1)})$$

DeCu

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta^{(p)} F(H^{(p)} \underline{u}^{(p-1)})$$

$$\underline{u}^{*(p)} = \underline{u}^0 + \Delta t H^{(p)} \Theta^{*(p-1)} F(\underline{u}^{*(p-1)})$$

DeCdu

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta^{(p)} H^{(p)} F(\underline{u}^{(p-1)})$$

$$H_{ij}^{(p)} = \phi_j^{(p-1)}(t^{i,(p)})$$

$$\text{DeC} \quad S = M \cdot (P - 1) + 1$$

\underline{c}	\mathbf{u}^0	$\mathbf{u}^{(1)}$	$\mathbf{u}^{(2)}$	$\mathbf{u}^{(3)}$	\dots	$\mathbf{u}^{(M-1)}$	$\mathbf{u}^{(M)}$	A	dim
0	0							\mathbf{u}^0	1
$\underline{\beta}_{1:}$	$\underline{\beta}_{1:}$	$\underline{0}$						$\mathbf{u}^{(1)}$	M
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\Theta_{1:,1:}$	$\underline{0}$					$\mathbf{u}^{(2)}$	M
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{0}$	$\Theta_{1:,1:}$	$\underline{0}$				$\mathbf{u}^{(3)}$	M
	\vdots	\vdots		\ddots	\ddots			\vdots	M
	\vdots	\vdots			\ddots	\ddots		\vdots	M
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{0}$	\dots	\dots	$\underline{0}$	$\Theta_{1:,1:}$	$\underline{0}$	$\mathbf{u}^{(M)}$	M
\underline{b}	$\Theta_{M,0}$	$\underline{0}$	\dots	\dots	\dots	$\underline{0}$	$\Theta_{M,1:}$	$\mathbf{u}^{M,(M+1)}$	

Efficient DeC into RK framework

DeCu $S = M \cdot (P - 1) + 1 - \frac{(M-1)(M-2)}{2}$

\underline{c}	\mathbf{u}^0	$\mathbf{u}^{*(1)}$	$\mathbf{u}^{*(2)}$	$\mathbf{u}^{*(3)}$	\dots	$\mathbf{u}^{*(M-2)}$	$\mathbf{u}^{*(M-1)}$	$\mathbf{u}^{(M)}$	A	dim
0	0								\mathbf{u}^0	1
$\underline{\beta}_{1:}^{(2)}$	$\underline{\beta}_{1:}^{(2)}$	$\underline{0}$							$\mathbf{u}^{*(1)}$	2
$\underline{\beta}_{1:}^{(3)}$	$\underline{W}_{1:,0}^{(2)}$	$\underline{W}_{1:,1:}^{(2)}$	$\underline{0}$						$\mathbf{u}^{*(2)}$	3
$\underline{\beta}_{1:}^{(4)}$	$\underline{W}_{1:,0}^{(3)}$	$\underline{0}$	$\underline{W}_{1:,1:}^{(3)}$	$\underline{0}$					$\mathbf{u}^{*(3)}$	4
\vdots	\vdots	\vdots	\ddots	\ddots					\vdots	\vdots
\vdots	\vdots	\vdots	\ddots	\ddots					\vdots	\vdots
$\underline{\beta}_{1:}^{(M)}$	$\underline{W}_{1:,0}^{(M-1)}$	$\underline{0}$	\dots	\dots	$\underline{0}$	$\underline{W}_{1:,1:}^{(M-1)}$	$\underline{0}$	$\underline{0}$	$\mathbf{u}^{*(M-1)}$	M
$\underline{\beta}_{1:}^{(M)}$	$\underline{W}_{1:,0}^{(M)}$	$\underline{0}$	\dots	\dots	\dots	$\underline{0}$	$\underline{W}_{1:,1:}^{(M)}$	$\underline{0}$	$\mathbf{u}^{(M)}$	M
\underline{b}	$\underline{W}_{M,0}^{(M+1)}$	$\underline{0}$	\dots	\dots	\dots	\dots	$\underline{0}$	$\underline{W}_{M,1:}^{(M+1)}$	$\mathbf{u}^{M,(M+1)}$	

$$W^{(p)} := \begin{cases} H^{(p)} \Theta^{(p)} \in \mathbb{R}^{(p+2) \times (p+1)}, & \text{if } p = 2, \dots, M-1, \\ \Theta^{(M)} \in \mathbb{R}^{(M+1) \times (M+1)}, & \text{if } p \geq M. \end{cases}$$

Efficient DeC into RK framework

$$\text{DeCdu} \quad S = M \cdot (P - 1) + 1 - \frac{M(M-1)}{2}$$

\underline{c}	\mathbf{u}^0	$\mathbf{u}^{(1)}$	$\mathbf{u}^{(2)}$	$\mathbf{u}^{(3)}$	\dots	$\mathbf{u}^{(M-2)}$	$\mathbf{u}^{(M-1)}$	$\mathbf{u}^{(M)}$	A	dim
0	0								\mathbf{u}^0	1
$\beta_{1:}^{(1)}$	$\beta_{1:}^{(1)}$	$\underline{0}$							$\mathbf{u}^{(1)}$	1
$\beta_{1:}^{(2)}$	$Z_{1:,0}^{(2)}$	$Z_{1:,1:}^{(2)}$	$\underline{0}$						$\mathbf{u}^{(2)}$	2
$\beta_{1:}^{(3)}$	$Z_{1:,0}^{(3)}$	$\underline{0}$	$Z_{1:,1:}^{(3)}$	$\underline{0}$					$\mathbf{u}^{(3)}$	3
\vdots	\vdots	\vdots	\ddots	\ddots					\vdots	\vdots
\vdots	\vdots	\vdots	\ddots	\ddots	\ddots	\ddots			\vdots	\vdots
$\beta_{1:}^{(M-1)}$	$Z_{1:,0}^{(M-1)}$	$\underline{0}$	\dots	\dots	$\underline{0}$	$Z_{1:,1:}^{(M-1)}$	$\underline{0}$	$\underline{0}$	$\mathbf{u}^{(M-1)}$	$M - 1$
$\beta_{1:}^{(M)}$	$Z_{1:,0}^{(M)}$	$\underline{0}$	\dots	\dots	\dots	$\underline{0}$	$Z_{1:,1:}^{(M)}$	$\underline{0}$	$\mathbf{u}^{(M)}$	M
\underline{b}	$Z_{M,0}^{(M+1)}$	$\underline{0}$	\dots	\dots	\dots	\dots	$\underline{0}$	$Z_{M,1:}^{(M+1)}$	$\mathbf{u}^{M,(M+1)}$	

$$Z^{(p)} := \begin{cases} \Theta^{(p)} H^{(p-1)} \in \mathbb{R}^{(p+1) \times p}, & \text{if } p = 1, \dots, M, \\ \Theta^{(M)} \in \mathbb{R}^{(M+1) \times (M+1)}, & \text{if } p > M. \end{cases}$$

Equispaced

P	M	DeC	DeCu	DeCdu
2	1	2	2	2
3	2	5	5	4
4	3	10	9	7
5	4	17	14	11
6	5	26	20	16
7	6	37	27	22
8	7	50	35	29
9	8	65	44	37
10	9	82	54	46
11	10	101	65	56
12	11	122	77	67
13	12	145	90	79

Gauss-Lobatto

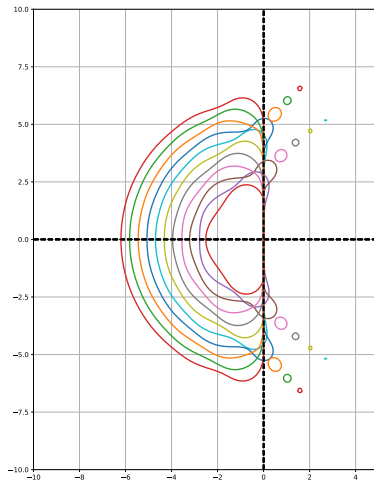
P	M	DeC	DeCu	DeCdu
2	1	2	2	2
3	2	5	5	4
4	2	7	7	6
5	3	13	12	10
6	3	16	15	13
7	4	25	22	19
8	4	29	26	23
9	5	41	35	31
10	5	46	40	36
11	6	61	51	46
12	6	67	57	52
13	7	85	70	64

DeC-DeCu-DeCdu

The **stability function** of DeC, DeCu, DeCdu of order P for any nodes distribution is

$$R(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^P}{P!}.$$

DeC, DeCu, DeCdu



Efficient DeC

- Code DeCu or DeCdu
- Check order of accuracy
- Write a code to obtain its RK matrix
- Check the stability function with nodepy
- Compare computational costs with original DeC