

- EXAM
- ORAL EXAM (THEORETICAL)
- CODING PROJECTS (PRACTICAL)

## ORDINARY DIFFERENTIAL EQUATIONS ODE

$$I = [t_0, t_{end}] \subset \mathbb{R}$$

$$y: I \rightarrow \mathbb{R}^S$$

$$F: I \times \mathbb{R}^S \rightarrow \mathbb{R}^S$$

$$\begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases} \approx \underbrace{F(y(t))}_{\text{AUTONOMOUS ODE}}$$

$$y^{(p)} = F(t, y, y', y'', \dots, y^{(p-1)})$$

HIGHER ORDER  
DERIVATIVE  
DIFE. EQ.

$$y \in \mathbb{R}^S$$

$$\begin{pmatrix} y \\ z_1 \\ \vdots \\ z_p \end{pmatrix} \in \mathbb{R}^{S(p+1)}$$

$$y' = z_1$$

$$y'' = z_1' = z_2$$

$$y''' = z_2' = z_3$$

$$y^{(p)} = z_p' = F(t, y, z_1, z_2, \dots, z_{p-1})$$

SYSTEM OF 1<sup>st</sup> DERIVATIVE DIFE. EQ.

$$\int_{t_0}^t \frac{dy}{dt} dt = y(t) - y(t_0) = y(t) - y_0$$

$$y(t) = y_0 + \int_{t_0}^t F(t, y(t)) dt$$

## DAHLQVIST'S PROBLEM

$$\begin{cases} y'(t) = -\lambda y(t) \\ y(t_0) = y_0 \end{cases}$$

$$\lambda \in \mathbb{R} \rightarrow \mathbb{C}$$

$$y(t) \neq 0 \quad y_0 > 0$$

$$\int_{t_0}^t \frac{y'(s)}{y(s)} ds = -\lambda \int_{t_0}^t ds$$

$$\int_{t_0}^t \frac{d}{ds} (\log(\gamma(s))) ds = -\lambda (t-t_0)$$

$$\log(\gamma(t)) - \log(\gamma(t_0)) = -\lambda (t-t_0)$$

$$\log\left(\frac{\gamma(t)}{\gamma(t_0)}\right) = -\lambda (t-t_0)$$

$$\frac{\gamma(t)}{\gamma(t_0)} = e^{-\lambda (t-t_0)}$$

$$\gamma(t) = \gamma_0 \underbrace{e^{-\lambda (t-t_0)}}_{\neq 0}$$

$$\gamma_0 > 0$$

$\lambda < 0 \rightarrow \text{exp. INCREASING}$

$\lambda > 0 \rightarrow \text{exp. to } 0$

PDS



$$C_1'(t) = \overbrace{C_2(t)}^{(2)} - \underbrace{5C_1(t)}_{(1)}$$

$$C_1, C_2 > 0$$

$$C_2'(t) = 5C_1(t) - C_2(t)$$

$$\sum_i C_i(t) = \Pi \quad \forall t$$

$$\frac{d}{dt} \left( \sum_i C_i(t) \right) = C_1' + C_2' = C_2 - 5C_1 + 5C_1 - C_2 = 0$$

$$C'(t) = A \cdot C(t) \quad A = \begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix}$$

$$C(0) = C_0$$

$$y' = \lambda y \rightarrow y = y_0 e^{\lambda t}$$

$$C(t) = \underbrace{e^{At}}_{\text{matrix}} C_0$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \quad \left[ \right]$$

$$A^2 = -6A$$

$$e^{At} = \sum_{k=1}^{\infty} \frac{(-6)^{k-1} A t^k}{k!} + 1$$

$$A^3 = -6A^2 = (-6)^2 A$$

$$A^k = (-6)^{k-1} A$$

$$e^{At} = \left( \sum_{k=0}^{\infty} \frac{(-\delta)^k \cdot t^k}{k!} \right) \frac{A}{-\delta} + 1$$

$$(e^{-\delta t} - 1) \frac{A}{-\delta} + 1$$

$$C(t) = C_0 + \frac{1 - e^{-\delta t}}{\delta} A C_0$$

$$x'(t) = \alpha x - \beta xy$$

$$y'(t) = \delta xy - \gamma y$$

$$\alpha, \beta, \gamma, \delta > 0$$

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \text{EQUILIBRIUM}$$

$$\eta(x, y) = \delta x - \gamma \log(x) + \beta y - \alpha \log(y)$$

$$\frac{d}{dt} \eta(x(t), y(t)) = \begin{pmatrix} \partial_x \eta \\ \partial_y \eta \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} =$$

$$\langle \nabla \eta, F(x, y) \rangle = 0 = \cancel{\delta \alpha x} - \cancel{\delta \beta xy} - \cancel{\gamma \alpha} + \cancel{\gamma \beta y} + \cancel{\beta \delta xy} - \cancel{\beta \gamma y} - \cancel{\alpha \delta x} + \cancel{\alpha \gamma} = 0$$

$$\nabla \eta = \begin{pmatrix} \delta - \gamma/x \\ \beta - \alpha/y \end{pmatrix}$$

$$\begin{cases} u'(t) = -\frac{v(t)}{n(t)} \\ v'(t) = \frac{u(t)}{n(t)} \end{cases} \quad n(t) = \sqrt{u^2(t) + v^2(t)}$$

$$\frac{n^2}{2}$$

$$\frac{d}{dt} \left( \frac{n^2}{2} \right) = \frac{d}{dt} \left( \frac{u^2 + v^2}{2} \right) = \frac{2u \cdot u' + 2v \cdot v'}{2} =$$

$$u \cdot u' + v \cdot v' = u \cdot \left( -\frac{v}{n} \right) + v \cdot \left( \frac{u}{n} \right) = 0$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos(\theta) u^0 - \sin(\theta) v^0 \\ \sin(\theta) v^0 + \cos(\theta) u^0 \end{pmatrix} \quad \theta = \frac{t}{\tau}$$

$$u' = -\frac{v}{\tau} - \alpha u$$

$$v' = \frac{u}{\tau} - \alpha v$$

$$\begin{aligned} \frac{d}{dt} \left( \frac{u^2}{2} \right) &= \frac{d}{dt} \left( \frac{u^2 + v^2}{2} \right) = u \cdot u' + v \cdot v' \\ &= -u \cdot \frac{v}{\tau} - \alpha u^2 + v \cdot \frac{u}{\tau} - \alpha v^2 = -\alpha (u^2 + v^2) \\ &= -2\alpha \frac{u^2 + v^2}{2} = -2\alpha \left( \frac{u^2}{2} \right) \end{aligned}$$

$$E = \frac{u^2}{2} \quad \frac{d}{dt} E = -2\alpha E$$

$$E = e^{-2\alpha t} E_0 \quad \alpha > 0$$

$$\frac{u^2}{2} = e^{-2\alpha t} \frac{u_0^2}{2} \Rightarrow u(t) = \underbrace{e^{-\alpha t}}_{\downarrow} u_0$$

$$\bullet \quad u' = -\sin(u)$$

$$\begin{cases} u' = v \\ v' = -\sin(u) \end{cases}$$

$$\gamma(t) = \frac{1}{2} v^2 - \cos(u)$$

$$\frac{d}{dt} \gamma(t) = \langle \nabla \gamma, \mathbf{F} \rangle = \begin{pmatrix} \sin(u) \\ v \end{pmatrix} \cdot \begin{pmatrix} v \\ -\sin(u) \end{pmatrix} =$$

$$v \cdot \sin(u) - v \cdot \sin(u) = 0$$

ROBERTSON

$$\begin{array}{ccc} & \xleftarrow{10^5 \cdot C_2 C_3} & \\ C_1 & \xrightarrow{\cos C_1} & C_2 \\ & \xleftarrow{3 \cdot 10^7 C_2^2} & \\ & C_3 & \end{array} \quad \frac{d}{dt} \left( \sum_i C_i \right) = 0$$

• PDE  $U(x, t) : \Omega \times [0, T] \rightarrow \mathbb{R}^d$

$$\partial_t U + \partial_{xx}(a(u)) + \partial_x b(u) + c(u) = 0$$

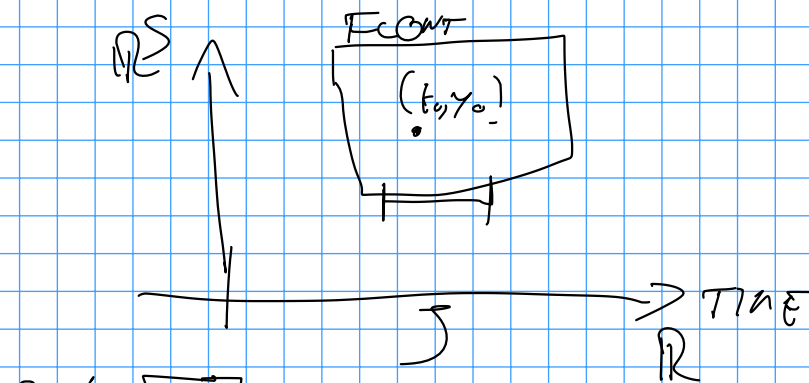
$$\underline{U} = \mathbb{R}^N \quad N \text{ \# DEF IN SPACE}$$

$$\partial_{xx} a(u) \approx D_2 A(U)$$

$$\partial_t \underline{U} = f(\underline{U}) \quad f(\underline{U}) = -(D_2 A(U) + \dots)$$

$$\partial_t u + \partial_x b(u) = 0 \quad \text{TVD}$$

$$U_i^n \quad TV(\underline{U}^n) := \sum_i |U_{i+1}^n - U_i^n|$$



$$\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}$$

$$\bullet y(\frac{1}{2}) = 0$$

$$\bullet y(t) = \frac{t^2}{4}$$

$$y' = \frac{t}{2}$$

$$\frac{t}{2}$$

$$F(t, y) : \mathbb{I} \times \mathbb{R}^S \rightarrow \mathbb{R}^S$$

$$\|F(t, y) - F(t, z)\| \leq L \|y - z\|$$

UNIFORMLY IN  $t \Rightarrow \exists!$  SOLUTION

$$y \in C^1(\mathbb{I})$$

$$\begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$$F \in C^{p-1} \quad p\text{-differentiable}$$

$y$  is  $p+1$  TIMES DIFFERENTIABLE

$$\frac{dy}{dt} = F$$

$$F^{(l+1)}(t, y) = \frac{\partial F^{(l)}}{\partial t}(t, y) + \frac{\partial F^{(l)}}{\partial y}(t, y)$$

$$\text{ODE} \quad I = [t_0, t_{\text{end}}] \quad y: I \rightarrow \mathbb{R}^S$$

$$F: I \times \mathbb{R}^S \rightarrow \mathbb{R}^S$$

$$\begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

$$t_0 = t^0 < t^1 < t^2 < \dots < t^N = t_{\text{end}}$$

$$[t^n, t^{n+1}] \quad y(t^{n+1}) = y(t^n) + \int_{t^n}^{t^{n+1}} F(t, y(t)) dt$$

$$\begin{matrix} \downarrow & \downarrow \\ y^n & y^{n+1} \end{matrix}$$

$$\approx y(t^n) + F(t^n, y(t^n)) \cdot \underbrace{(t^{n+1} - t^n)}_{\Delta t^n}$$

$$y^n \rightarrow y^{n+1}$$

$$y^{n+1} = y^n + \Delta t^n F(t^n, y^n)$$

$$y^0 = y_0$$

$$\begin{aligned} c_1^{n+1} + c_2^{n+1} &= c_1^n + \Delta t (-5c_1^n + c_2^n) \\ &\quad + c_2^n + \Delta t (-c_2^n + 5c_1^n) \end{aligned}$$

$$= c_1^n + c_2^n = c_1^0 + c_2^0 = \text{MASS} \quad \forall n$$

$$e_n = y(t^n) - y^n$$

$$y'(t^n)$$

CONSISTENCY ERROR

$$\varepsilon_n = y(t^{n+1}) - y(t^n) - \Delta t F(t^n, y(t^n)) =$$

$$= \int_{t^n}^{t^{n+1}} \underbrace{y'(t) - y'(t^n)}_{\text{constant}} dt$$

$$\omega(f, \Delta t) := \max_{t, t': |t-t'| < \Delta t} |f(t) - f(t')|$$

$$|\varepsilon_n| \leq \int_{t^n}^{t^{n+1}} \omega(y', \Delta t^n) dt = \Delta t^n \cdot \omega(y', \Delta t^n).$$

DOESN'T DEPEND ON  $t$

$$\begin{aligned} e_{n+1} &= y(t^{n+1}) - y^{n+1} = y(t^{n+1}) - y^n - \Delta t^n F(t^n, y^n) \\ &\quad + y(t^n) + \Delta t^n F(t^n, y(t^n)) \\ &\quad - y(t^n) - \Delta t^n F(t^n, y(t^n)) \\ &= \varepsilon_n + \underbrace{y(t^n) - y^n}_{e_n} + \Delta t^n [F(t^n, y(t^n)) - F(t^n, y^n)] \end{aligned}$$

$$|e_{n+1}| \leq |\varepsilon_n| + |e_n| + \Delta t^n L |y(t^n) - y^n| \leq \underbrace{|e_n|}_{e_n} + |\varepsilon_n| + \Delta t^n L |y(t^n) - y^n|$$

$$|F(t, y) - F(t, z)| \leq L |y - z| \quad (\text{HYPOTHESIS ON } F \text{ TO HAVE UNIQUENESS})$$

$$\leq |\varepsilon_n| + (1 + \Delta t^n L) |e_n|.$$

$$\leq \dots \leq e^{L|t^{n+1} - t^0|} |e_0| + \sum_{i=0}^n e^{L(t^{n+1} - t^{i+1})} |\varepsilon_i| \leq \dots$$

$$|\varepsilon_n| \leq \Delta t^n \omega(y', \Delta t^n)$$

$$\leq e^{L|t^n - t^0|} |e_0| + \underbrace{\omega(y', \Delta t^n)}_{\Delta t^n} \frac{e^{L(t^{n+1} - t^0)} - 1}{L}.$$

$$\omega(y', \Delta t) = \omega(F(\cdot, y(\cdot)), \Delta t)$$

$$\left[ \sum_{i=0}^n \Delta t e^{L \Delta t} = \underbrace{N \cdot \Delta t}_{T} e^{L \Delta t} = \frac{T}{\Delta t} \Delta t e^{L \Delta t} \dots \right]$$

$$\begin{aligned}
 w(y', \Delta t) &= \max_{t, t'; |t-t'| < \Delta t} |y'(t) - y'(t')| = (\text{AUTONOMOUS}) \\
 &= \max_{t, t'; |t-t'| < \Delta t} |F(y(t)) - F(y(t'))| \\
 &\leq L |y(t) - y(t')| \xrightarrow{\Delta t \rightarrow 0} 0 \\
 &\leq L C \cdot \Delta t
 \end{aligned}$$

$$\overbrace{|y(t) - y^n|}^{\max_{y'} \frac{1}{\Delta t}} \leq C_0 |y^0 - y(t^0)| + \Delta t \cdot C_1$$

$$\begin{aligned}
 e_1 = |y' - y(t')| &= |y^0 + \Delta t \overbrace{F(t^0, y^0)}^{y'(t_0)} \\
 &\quad - (y(t^0) + \Delta t y'(t_0) + \frac{\Delta t^2}{2} y''(t_0) + O(\Delta t^3))|
 \end{aligned}$$

$$= \left| \frac{\Delta t^2}{2} y''(t_0) \right| + O(\Delta t^3) = \underline{O(\Delta t^2)}$$

$$\begin{aligned}
 e_N &\approx \sum_{i=1}^N |y(t^i) - y^i| \leq N \cdot \frac{\Delta t^2}{2} \max |y''| \\
 &= \frac{\text{end}}{\Delta t} \cdot \frac{\Delta t^2}{2} \max |y''|
 \end{aligned}$$

DEF

P ORDER SCHEME

LARGEST P  $\Delta t \rightarrow 0$

$$|e_N| \leq C \cdot \Delta t^P$$

$$= \text{end} \cdot \frac{\Delta t}{2} \max |y''|$$

$$= C \cdot \Delta t = O(\Delta t)$$

FIRST ORDER

$$y'(t) = \Pi y(t) \quad \Pi \in \mathbb{R}^{S \times S} \quad \Delta t$$

$$\begin{aligned}
 y^n &= y^{n-1} + \Delta t \Pi y^{n-1} = (\mathbb{I} + \Delta t \Pi) y^{n-1} \\
 &= (\mathbb{I} + \Delta t \Pi)^n y^0
 \end{aligned}$$



$$|y^n| \stackrel{?}{\leq} |y^0| \leadsto \| (I + \Delta t \Pi)^n \|$$

$$\Pi \mapsto \hat{\Pi} = S^{-1} \Pi S$$

$$S^{-1} y^n = S^{-1} y^{n-1} + \Delta t S^{-1} \Pi_{\chi} y^{n-1}$$

$$S^{-1} y =: \hat{y}^n$$

$$\begin{aligned} \hat{y}^n &= \hat{y}^{n-1} + \Delta t \hat{\Pi} \hat{y}^{n-1} \\ &= (I + \Delta t \hat{\Pi}) \hat{y}^{n-1} \end{aligned}$$

$$\boxed{y'(t) = q y(t)}$$

$$q \in \mathbb{C}$$

$$\begin{aligned} q &\in \mathbb{R} \\ \boxed{1/q \leq 0} & \text{ Re}(q) < 0 \\ q &> 0 \end{aligned}$$

$$y_s^n \rightarrow \text{[Diagram of a box with points and a circle]}$$

$$\hat{\Pi} \in \mathbb{C}^{S \times S}$$

$$\begin{aligned} y^{n+1} &= y^n + q \Delta t y^n = (1 + q \Delta t) y^n \\ &= (1 + q \Delta t)^{n+1} y^0 \end{aligned}$$

$$|y^{n+1}| \leq |y^n| \leq \dots$$

$$|1 + q \Delta t| |y^n| \leq |y^n|$$

$$|1 + q \Delta t| \leq 1$$

$$\underline{\underline{z \in \mathbb{C}}}$$

$$|1 + z| \leq 1$$

$$R(z) = 1 + z$$

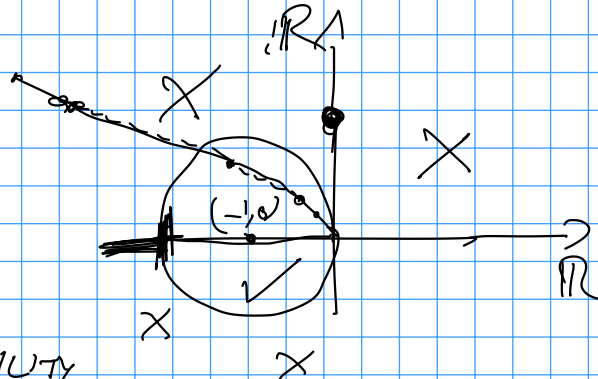
STABILITY  
FUNCTION

$$y^{n+1} = R(\Delta t q) \cdot y^n$$

$$z = \Delta t q$$

$$\{ |R(z)| \leq 1 \} \text{ STABILITY REGION}$$

$$q \in \mathbb{R} \quad |q| \Delta t \leq C \Rightarrow \Delta t \leq \frac{C}{|q|} \approx \frac{C}{|L|}$$



$$u' = v$$

$$v' = -u$$

$$u(0) = 0 \quad v(0) = 1$$

$$r^2 = u^2 + v^2 = \text{CONSTANT IN TIME}$$

$$\frac{d}{dt} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = 0$$

$$\underline{u} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{v}$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$y^{n+1} = y^n + \Delta t f(t^{n+1}, y^{n+1})$$

FIRST ORDER

LINEAR STABILITY

$$y' = q y \quad q \in \mathbb{C}$$

$$y^{n+1} = y^n + \Delta t q y^{n+1}$$

$$(1 - \Delta t q) y^{n+1} = y^n$$

$$z \in \mathbb{C}$$

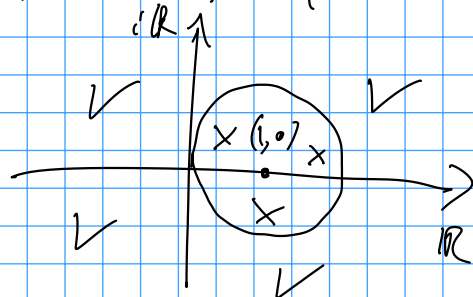
$$y^{n+1} = R(z) \cdot y^n = \frac{1}{1-z} y^n$$

$$R(z)$$

$$\sqrt{1}$$

$$1-z$$

$$\{|R(z)| \leq 1\} = \left\{ \frac{1}{|1-z|} \leq 1 \right\} = \{|1-z| \geq 1\}$$



$$y^{n+1} = y^n + f(y^{n+1}) \Delta t$$

$$y^{n+1} \leftarrow u^{(k)}$$



# IMPLICIT EULER + POSITIVITY

$\forall \Delta t$

PDS  $\frac{d}{dt} c_i = P_i(c) - D_i(c)$

$$\geq \sum_j \underbrace{p_{ij}(c)}_{\geq 0} - \underbrace{d_{ij}(c)}_{=0}$$

$p_{ij} = d_{ji} \geq 0$

1. CONSERVATION

$$\begin{aligned} \frac{d}{dt} \left( \sum_i c_i(t) \right) &= \sum_i \frac{d}{dt} c_i(t) = \sum_i \sum_j p_{ij}(c) - d_{ij}(c) \\ &= \sum_i \sum_j p_{ij} - \sum_i \sum_j d_{ji} = \sum_i \sum_j p_{ij} - p_{ji} = 0 \end{aligned}$$

• HOMEWORK CHECK THAT IMPLICIT EULER IS CONSERVATIVE.

2. POSITIVITY

$$\overbrace{c_i \rightarrow 0 \Rightarrow D_i(c) \rightarrow 0} \forall j \forall i$$
$$d_{ij}(c) \rightarrow 0$$

$$c_i(0) > 0 \Rightarrow c_i(t) \geq 0 \quad \forall i \quad \forall t$$

PROOF BY CONTRADICTION  $t: c_i(t) < 0$

$$\exists t^* \quad c_i(t^*) = 0 \quad c'_i(t^*) < 0$$

$$c'_i(t^*) = \underbrace{P_i(c(t^*))}_{\geq 0} - \underbrace{D_i(c(t^*))}_{=0} \geq 0 \quad \swarrow$$

## IMPLICIT EULER + CHEB PDS + POSITIVITY

$$y' = \Pi y \quad \Pi_{ii} < 0 \quad \Pi_{ij} \geq 0 \quad i \neq j$$

$$\sum_i \Pi_{ij} = 0$$

IMPLICIT EULER

$$(I - \Delta t \Pi) y^{n+1} = y^n \quad y^0 \geq 0 \Rightarrow y^{n+1} = A^{-1} y^n$$

$$A^{-1} \geq 0 \Leftrightarrow \forall i, j (A^{-1})_{ij} \geq 0$$

$$1. \quad 0 < A_{ii} = 1 - \Delta t \Pi_{ii} = 1 + \Delta t |\Pi_{ii}| = 1 + \Delta t \sum_{j \neq i} \Pi_{ji}$$

$$|A_{ii}| > \sum_{j \neq i} |A_{ji}|$$

$> 0$

$$\sum_j \Pi_{ji} = 0$$

$$0 > \Pi_{ii} = - \sum_{j \neq i} \overbrace{\Pi_{ji}}^{> 0} < 0$$

$> 0 \quad > 0 \quad > 0$

$$1 + \Delta t \sum_{j \neq i} \Pi_{ji} > \Delta t \sum_{j \neq i} \Pi_{ji} = \sum_{j \neq i} |A_{ji}|$$

A DIAG. DOMINANT BY COLUMN

•  $A^{-1} \geq 0$  ? JACOBI METHOD

$$A = D + (A - D) \quad D = \text{diag}(A)$$

$$A u = R \Leftrightarrow D u + (A - D) u = R$$

$$D u = R - (A - D) u \quad \checkmark \quad u = D^{-1} (D - A) u + D^{-1} R$$

$$D u^{(k)} = R - (A - D) u^{(k-1)}$$

$$u^{(k)} = D^{-1} (D - A) u^{(k-1)} + D^{-1} R$$

$$\|u - u^{(k)}\|_{\infty} = \|D^{-1} (D - A) (u^{(k-1)} - u)\|$$

$$\leq \|D^{-1} (D - A)\|_{\infty} \|u - u^{(k-1)}\|_{\infty}$$

$$\|B\|_{\infty} = \max_i \sum_j |B_{ji}|$$

$$B = D^{-1} (D - A) = (I - D^{-1} A)$$

$$(I - D^{-1}A)_{ji} = \begin{bmatrix} 0 & \dots \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & 0 \end{bmatrix}$$

$$\max_i \sum_{j \neq i} \frac{|A_{ji}|}{|A_{ii}|} = \max_i \frac{\sum_{j \neq i} |A_{ji}|}{|A_{ii}|} < 1$$

$$\|u - u^{(k)}\|_{\infty} < \|u - u^{(k-1)}\|_{\infty}$$

$$u^{(k)} \geq 0$$

$$Ay^{n+1} = y^n$$

$$u^{(k)} = \underbrace{D^{-1}}_{\geq 0} \underbrace{(D-A)}_{\substack{\geq 0 \\ \left\{ \begin{array}{ll} \Delta t \pi_{ij} > 0 & i \neq j \\ 0 & i = j \end{array} \right.}} \underbrace{u^{(k-1)}}_{\geq 0} + \underbrace{D^{-1}}_{\geq 0} \underbrace{y^n}_{\geq 0} \geq 0$$

$$u^{(k)} \rightarrow u = y^{n+1} \geq 0. \quad \square$$

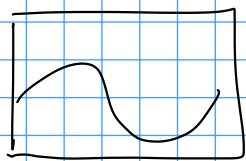
$$\begin{aligned} |e_n| &\leq e^{\int_0^n |e'|} |e_0| + \int_0^n \omega(y', dt) \cdot \sum_{i=0}^{n-1} \underbrace{e^{\int_0^n (t^n - t^{i+1})}} \\ &= \dots + \omega(\dots) \cdot \sum_{i=0}^{n-1} \int_{t^i}^{t^{i+1}} \underbrace{e^{\int_0^n (t^n - t^{i+1})}} dt \\ &\leq \dots + \omega(\dots) \sum_{i=0}^{n-1} \int_{t^i}^{t^{i+1}} e^{\int_0^n (t^n - t)} dt \\ &= \dots + \omega \dots \int_{t^0}^{t^n} e^{\int_0^n (t^n - t)} dt \\ &= \dots + \omega \left[ \frac{e^{\int_0^n (t^n - t)}}{-L} \right]_{t^0}^{t^n} \\ &= \dots + \omega \frac{e^{\int_0^n (t^n - t^0)} - 1}{L} \end{aligned}$$

$$\partial_t u + \partial_x F(u) = 0$$

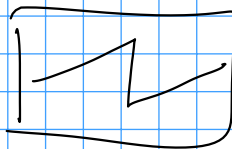
$$u: \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$$

$$F(u) = \begin{cases} u & \text{TRANSPORT} \\ \frac{u^2}{2} & \text{BURGER'S} \end{cases}$$

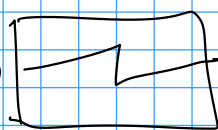
BURGER'S



$\Rightarrow$



$\Rightarrow$



TIME

$$TV(u^n) := \sum_j |u_j^n - u_{j-1}^n|$$

HARTEN FERN : INCREMENTAL FERN

$$\partial_t u_j = C_{j+1/2} (u_{j+1} - u_j) - D_{j-1/2} (u_j - u_{j-1})$$

$$C_{j+1/2}, D_{j-1/2} > 0 \quad \forall j$$

CHECK THAT IMPLICIT EULER DISCR OF INCREMENTAL FERN PDE IS TVD.

$$\text{i.e. } TV(u^{n+1}) \leq TV(u^n) \quad \forall \Delta t > 0.$$

$$u_j^{n+1} = u_j^n + \Delta t \left[ C_{j+1/2} (u_{j+1}^{n+1} - u_j^{n+1}) - D_{j-1/2} (u_j^{n+1} - u_{j-1}^{n+1}) \right]$$

$$u_{j+1}^{n+1} - u_j^{n+1} = u_{j+1}^n - u_j^n + \Delta t \left[ C_{j+3/2} (u_{j+2}^{n+1} - u_{j+1}^{n+1}) - C_{j+1/2} (u_{j+1}^{n+1} - u_j^{n+1}) - D_{j+1/2} (u_{j+1}^{n+1} - u_j^{n+1}) + D_{j-1/2} (u_j^{n+1} - u_{j-1}^{n+1}) \right]$$

$$\left[ 1 + \Delta t C_{j+1/2} + \Delta t D_{j+1/2} \right] (u_{j+1}^{n+1} - u_j^{n+1}) =$$

$$u_{j+1}^n - u_j^n + \Delta t \left[ C_{j+3/2} (u_{j+2}^{n+1} - u_{j+1}^{n+1}) + D_{j-1/2} (u_j^{n+1} - u_{j-1}^{n+1}) \right]$$

$> 0$

$$\sum_j (1 + \Delta t C_{j+\frac{1}{2}} + \Delta t D_{j+\frac{1}{2}}) |U_{j+1}^{n+1} - U_j^{n+1}|$$

$$\leq \underbrace{\sum_j |U_{j+1}^n - U_j^n|}_{TV(U^n)} + \Delta t \sum_j C_{j+\frac{1}{2}} |U_{j+2}^{n+1} - U_{j+1}^{n+1}|$$

$$+ \Delta t \sum_j D_{j-\frac{1}{2}} |U_j^{n+1} - U_{j-1}^{n+1}|$$

$$\sum_j C_{j+\frac{1}{2}} |U_{j+2}^{n+1} - U_{j+1}^{n+1}| = \sum_j C_{j+\frac{1}{2}} |U_{j+1}^{n+1} - U_j^{n+1}|$$

$$\sum_j D_{j-\frac{1}{2}} |U_j^{n+1} - U_{j-1}^{n+1}| = \sum_j D_{j+\frac{1}{2}} |U_{j+1}^{n+1} - U_j^{n+1}|$$

$$\sum_j |U_{j+1}^{n+1} - U_j^{n+1}| + \cancel{\Delta t \sum_j C_{j+\frac{1}{2}} |U_{j+1}^{n+1} - U_j^{n+1}|} + \cancel{\Delta t \sum_j D_{j+\frac{1}{2}} |U_{j+1}^{n+1} - U_j^{n+1}|}$$

$$\leq TV(U^n) + \cancel{\Delta t \sum_j C_{j+\frac{1}{2}} |U_{j+1}^{n+1} - U_j^{n+1}|} + \cancel{\Delta t \sum_j D_{j+\frac{1}{2}} |U_{j+1}^{n+1} - U_j^{n+1}|}$$

$$\Rightarrow TV(U^{n+1}) \leq TV(U^n) \quad \square$$

$$y' = \Pi y$$

$$\frac{y^{n+1} - y^n}{\Delta t} = \Pi y^{n+1} \Rightarrow (\mathbb{I} - \Delta t \Pi) y^{n+1} = y^n$$

$$e(\Delta t) = C \cdot \Delta t^p$$

$$\log(e(\Delta t)) = \log(C \cdot \Delta t^p) = \underbrace{\log(C)}_{\text{intercept}} + \underbrace{\log(\Delta t) \cdot p}_{\text{slope}}$$

$$|y^{n+1}| < |y^n|$$

$$\underline{y^n \mapsto y^{n+1}} \quad 2^{\text{nd}} \text{ order?} \quad \Theta \in \mathbb{R}$$

$$y^* = y^n + \Theta \Delta t F(t^n, y^n)$$

$$y^{n+1} = y^n + \Delta t \left[ \frac{2\Theta - 1}{2\Theta} F(t^n, y^n) + \frac{1}{2\Theta} F(t^{n+1}, y^*) \right]$$

# CONSISTENCY LOCAL ERROR

$y^n$  known exact  
 $\Downarrow$   
 Error  $y^{n+1}$ ?

$$y(t^{n+1}) = y^n + \Delta t y'(t^n) + \frac{\Delta t^2}{2} y''(t^n) + O(\Delta t^3)$$

$$y(t^*) = y(t + \theta \Delta t) = y^n + \theta \Delta t y'(t^n) + \frac{\theta^2 \Delta t^2}{2} y''(t^n) + O(\Delta t^3)$$

$$y^* = \overbrace{y^n + \theta \Delta t F(y^n)} + O(\Delta t^2) = y(t^*) + O(\Delta t^2)$$

$$y^{n+1} = y^n + \Delta t \left[ \frac{2\theta-1}{2\theta} F(y^n) + \frac{1}{2\theta} F(y^*) \right]$$

$$= y^n + \Delta t \left[ \frac{2\theta-1}{2\theta} F(y^n) + \frac{1}{2\theta} \left( F(y^n) + \frac{\partial F}{\partial y}(y^n) \theta \Delta t F(y^n) \right) \right]$$

$$= y^n + \Delta t F(y^n) + \frac{\cancel{\theta}}{\cancel{2\theta}} \Delta t^2 \frac{d}{dt} F(y(t^n)) =$$

$$= y^n + \Delta t F(y^n) + \frac{\Delta t^2}{2} \frac{d}{dt} y'(t^n)$$

$$= y^n + \Delta t y'(t^n) + \frac{\Delta t^2}{2} y''(t^n) = y(t^{n+1}) + O(\Delta t^3)$$

$$y(t^{n+1}) = y^n + y' \Delta t + \frac{\Delta t^2}{2} y'' + \frac{\Delta t^3}{6} y'''$$

$\Rightarrow$  GLOBAL ERROR  $O(\Delta t^2)$

## RUNGE-KUTTA

S STAGES

$$A \in \mathbb{R}^{S \times S}$$

$$c, b \in \mathbb{R}^S$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

$$A = (a_{kj})_{k,j=1}^S$$

$$\begin{cases} y^{(k)} = y^n + \Delta t \sum_{j=1}^S a_{kj} F(t^n + \Delta t c_j, y^{(j)}) & k=1, \dots, S \\ y^{n+1} = y^n + \Delta t \sum_{j=1}^S b_j F(t^n + \Delta t c_j, y^{(j)}) \end{cases}$$



$$\begin{array}{c} t^{n+1} \\ t^n + \Delta t c_2 \\ t^n + \Delta t c_3 \\ t^n \end{array}$$

$$\begin{array}{c} y^{n+1} \\ y^{(2)} \\ y^{(3)} \\ y^{(1)} \\ y^n \end{array}$$

EXPLICIT RK

$$y^{(k)} = y^n + \Delta t \sum_{j=1}^{k-1} a_{kj} F(y^{(j)})$$

Θ - SCHEME

$$\begin{cases} y^{(1)} = y^n + \Delta t \left[ \Theta \cdot F(y^{(1)}) + (1-\Theta) \cdot F(y^{(2)}) \right] \\ y^{(2)} = y^n + \Delta t \left[ \Theta \cdot F(y^{(1)}) + (1-\Theta) \cdot F(y^{(2)}) \right] \\ y^{n+1} = y^n + \Delta t \left[ \frac{2\Theta-1}{2\Theta} F(y^{(1)}) + \frac{1}{2\Theta} F(y^{(2)}) \right] \end{cases}$$

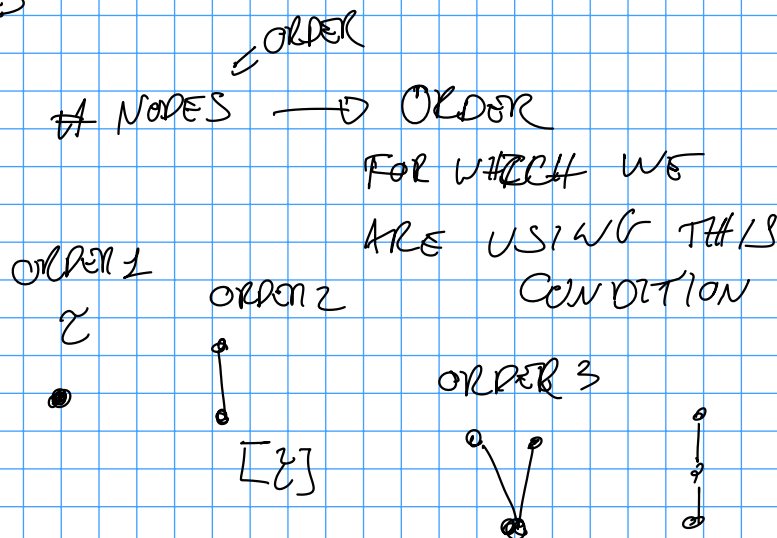
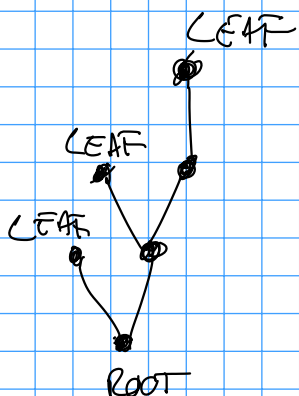
$$\begin{array}{c|cc} \Theta & \Theta & 1-\Theta \\ \hline 1-\Theta & \Theta & 1-\Theta \\ \hline \frac{2\Theta-1}{2\Theta} & \frac{1}{2\Theta} & \end{array}$$

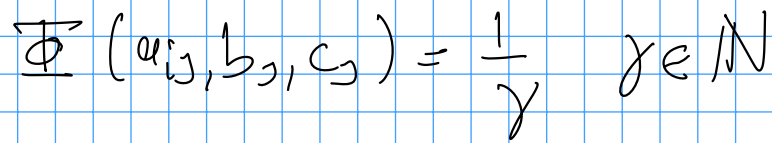
$$\begin{array}{c|c} \Theta & \Theta \\ \hline \frac{2\Theta-1}{2\Theta} & \frac{1}{2\Theta} \end{array}$$

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ c_2 & a_{21} & 0 & 0 & 0 \\ c_3 & a_{31} & a_{32} & 0 & 0 \\ c_4 & a_{41} & a_{42} & a_{43} & 0 \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array}$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

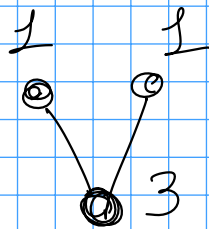
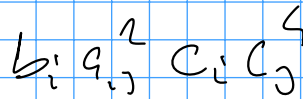
ROOTED TREES





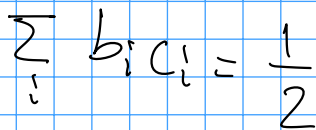
A handwritten diagram on blue grid paper showing a tree structure. A root node labeled  $b_i$  has three children:  $c_i$ ,  $c_j$ , and  $c_k$ . Node  $c_i$  has two children:  $a_i$  and  $a_j$ . Node  $c_j$  has one child:  $a_k$ . Node  $c_k$  has one child:  $a_l$ .

$\gamma = ?$



$$\mathbb{E}(A, b, c) = \frac{1}{\delta}$$

$$\sum_i b_i = \frac{1}{1} \quad \gamma = 1$$



$$\gamma = 2$$

$$\sum_i b_i c_i^2 = \frac{1}{3}$$

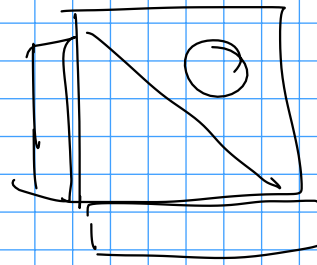
$$\begin{array}{c} 1 \text{ } C_3 \\ | \\ 2 \text{ } Q_1 \\ | \\ 3 \text{ } b_i \end{array}$$

$$\sum_{i,j} b_i a_{ij} C_j = \frac{1}{6}$$

S stages

EXPLICIT RK

$$\frac{s^2 + 3s - 2}{2}$$



5 STAGES  $\rightarrow$  ORDER  $\leq 4$

$$C_i = \sum_j a_{ij} \quad \leftarrow$$

(LINEAR)

STABILITY FOR RK METHODS

$$y' = qy \quad q \in \mathbb{C} \quad z = q \cdot \Delta t$$

$$y^{n+1} = R(z) \cdot y^n \quad S = \{z : |R(z)| \leq 1\}$$

$$Y \in \mathbb{R}^S$$

$$Y = \underline{1} \cdot y^n + \Delta t \underline{A} \cdot \underline{F(Y)}$$

$$y^{(k)} = y^n + \Delta t \sum_{j=1}^S a_{kj} F(y^{(j)})$$

$$F(y) = q \cdot y \quad F(Y) = q \cdot Y$$

$$\boxed{\text{RK}} \quad Y = \underline{1} y^n + z \cdot \underline{A} \cdot Y$$

$$\boxed{\phantom{A}} \cdot \boxed{\phantom{A}} = \boxed{\phantom{A}}$$

$$(\underline{I} - z \underline{A}) Y = y^n \underline{1}$$

$$y^{n+1} = y^n + \Delta t \sum_j b_j F(y^{(j)}) = y^n + \Delta t \cdot b^T \cdot F(Y) \\ = y^n + \Delta t \cdot q \cdot b^T Y$$

$$y^{n+1} = y^n + \Delta t q b^T Y = y^n + \Delta t q b^T (\underline{I} - z \underline{A})^{-1} \underline{1} y^n \\ = \left[ \underline{1} + z \cdot b^T (\underline{I} - z \underline{A})^{-1} \underline{1} \right] \cdot y^n$$

$$y^{n+1} = R(z) \cdot y^n \quad R(z) = \underline{1} + z b^T (\underline{I} - z \underline{A})^{-1} \underline{1}$$

$$\mathbb{I} = \mathbb{I} \quad z = 1 \quad (\sigma^{-1})$$

$$A = \mathbb{I}$$

$R(z) \rightarrow$  POLYNOMIAL FOR EXPLICIT RK  
 $\rightarrow$  FRACTION OF TWO POLY  
 IMPLICIT RK

$$\begin{array}{c|ccc} \textcircled{1} & & & \\ \hline 2/3 & 2/3 & & \\ \hline 2/3 & 1/3 & 1/3 & \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

ORDER 1

$$\sum b_i = 1 \Leftrightarrow \frac{1}{3} + 0 + \frac{2}{3} = 1 \checkmark$$

ORDER 2

$$\sum b_i c_i = \frac{1}{2}$$

$$0 \cdot \frac{1}{3} + \frac{2}{3} \cdot 0 + \frac{2}{3} \cdot \frac{2}{3} = \frac{1}{2} \checkmark$$

ORDER 3

$$\sum b_i c_i^2 = \frac{1}{3}$$

$$0 + 0 + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{1}{3} \checkmark$$

$$\begin{array}{c} 1 \text{ } c_j \\ 2 \text{ } a_{ij} \\ 3 \text{ } b_i \end{array} \quad \boxed{y=c}$$

$$\sum b_i a_{ij} c_j = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} =$$

$$= \frac{1}{9} + \frac{1}{18} = \frac{3}{18} = \frac{1}{6} \checkmark$$

ORDER 4

$$\sum b_i c_i^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} \neq \frac{1}{4} \quad \text{NOT ORDER 4}$$

STABILITY FUNCTION

$$R(z) = \mathbb{I} + \Delta t \underline{b}^T (\underline{\mathbb{I}} - z \underline{A})^{-1} \underline{1}$$

$$(\mathbb{I} - zA)^{-1}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 2/3 & 0 & 0 \\ 1/3 & 1/3 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2/9 & 0 & 0 \end{bmatrix} \quad A^3 = \underline{0}$$

$$A^k = \underline{0}$$

$$k \geq 3$$

$$(I - zA)^{-1} = \underbrace{I + zA + z^2 A^2}_{z=0} + \underbrace{z^3 A^3 + \dots}_{z=0}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ z\frac{2}{3} & 1 & 0 \\ z\frac{1}{3} + z^2\frac{2}{9} & z\frac{1}{3} & 1 \end{bmatrix}$$

$$R(z) = 1 + z \begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ z\frac{2}{3} & 1 & 0 \\ \frac{z}{3} + \frac{2}{9}z^2 & \frac{z}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 1 + z \left[ \frac{1}{4} + \frac{z}{4} + \frac{1}{6}z^2, \frac{z}{4}, 1, \frac{3}{4} \right] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

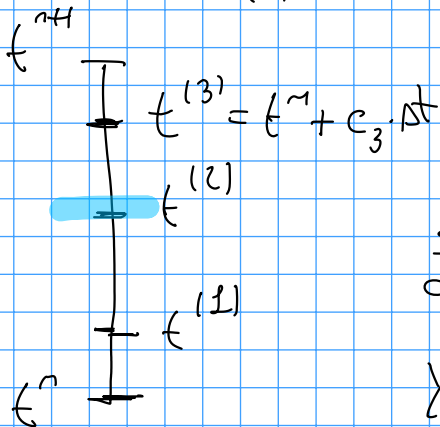
$$= 1 + z \left( \frac{1}{4} + \frac{z}{4} + \frac{1}{6}z^2 + \frac{z}{4} + \frac{3}{4} \right)$$

$$= 1 + z + \frac{z^2}{2} + \frac{1}{6}z^3$$

IMPLICIT RK

$$y^{(k)} = y^n + \sum_{j=1}^S a_{kj} F(y^{(j)}) \quad S \text{ EQUATIONS}$$

$$y(t) = \sum_{i=1}^S \varphi_i(t) \cdot \underbrace{y(t^n + c_i \Delta t)}_{y^{(i)}} + O(\Delta t^{p+1}) \quad S \times DIM \text{ equations}$$



$$\varphi_i(t^{(j)}) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\frac{dy}{dt} = F(y)$$

$$y(t) = y^n + \int_{t^n}^t F(y(s)) ds$$

$$y^{(i)}(t^{(i)}) = y^n + \int_{t^n}^{t^{(i)}} F(s, y(s)) ds$$

$$\approx y^n + \int_{t^n}^{t^{(i)}} \sum_{j=1}^S \phi_j(s) \cdot F(t^{(j)}, y^{(j)}) ds$$

INTERPOLATE F

$$= y^n + \sum_{j=1}^S \underbrace{\int_{t^n}^{t^{(i)}} \phi_j(s) ds}_{i = \Delta t a_{ij}} F(t^{(j)}, y^{(j)})$$

$$= y^n + \sum_{j=1}^S \Delta t a_{ij} F(t^{(j)}, y^{(j)})$$

$$y^{n+1} = y^n + \sum_{j=1}^S \underbrace{\int_{t^n}^{t^{n+1}} \phi_j(s) ds}_{i = \Delta t b_j} F(t^{(j)}, y^{(j)})$$

B, C, D CONDITIONS

$$B(p): \sum_{i=1}^S b_i c_i^{z-1} = \frac{1}{z} \quad \boxed{\forall z = 1, \dots, p}$$

$$\begin{array}{c} t^{n+1} \\ \vdots \\ t^{(i)} = t^n + \Delta t c_i \\ \vdots \\ t^n \end{array} \quad \begin{array}{c} \Delta t \\ \vdots \\ c_3 \rightarrow b_i \\ \vdots \\ c_2 \\ \vdots \\ c_1 \\ 0 \end{array}$$

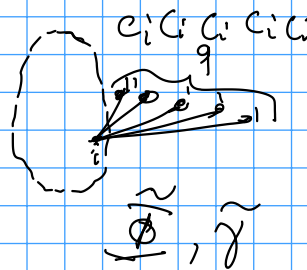
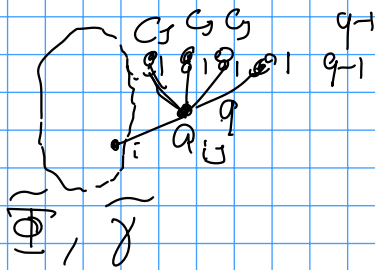
$$\int_0^1 t^{z-1} = \left[ \frac{t^z}{z} \right]_0^1 = \frac{1}{z}$$

QUADRATURE FORMULA  $b_i$  WEIGHTS  $c_i$  NODES

$$\int_0^1 t^{z-1} = \sum_i b_i c_i^{z-1} = \frac{1}{z}$$

$B(p)$  QUAD  $(b_i, c_i)$  IS EXACT POLY DEGREE  
UP TO  $p-1$

$$C(q): \sum_{j=1}^S a_j c_j^{q-1} = \frac{c_i^q}{q} \quad \forall i=1, \dots, S \quad \forall q=1, \dots, q$$



$$\Phi = \sum_{ij} \tilde{\Phi}_i \cdot a_{ij} \cdot c_j^{q-1}$$

$$\Phi = \sum_i \tilde{\Phi}_i \cdot c_i^q$$

$$\gamma = \tilde{\gamma} \cdot q$$

$$\gamma = \tilde{\gamma}$$

$$\gamma \Phi$$

$$=$$

$$\gamma \Phi$$

$$\Phi = \frac{1}{\gamma}$$

$$\sum_{ij} \tilde{\Phi}_i \cdot a_{ij} \cdot c_j^{q-1} \cdot q \cancel{\gamma} = \sum_i \tilde{\Phi}_i \cdot c_i^q \cdot \cancel{\gamma} \quad \forall \tilde{\Phi}_i$$

$$\Rightarrow \sum_j a_{ij} c_j^{q-1} \cdot \cancel{q} = \frac{c_i^q}{\cancel{q}}$$

$$\text{IF } B(p), C(\gamma), D(\gamma)$$

$$p \leq 2\gamma + 2 \quad p \leq \gamma + \gamma + 1 \quad \Rightarrow \text{THE METHOD IS OF ORDER } p.$$

$$B(zs) \quad C(s) \quad D(s) \quad \&$$

$$\begin{array}{ccc} B(zs-2) & \boxed{C(s)} & D(s-2) \\ 2s-1 & \boxed{C(s)} & D(s-1) \\ & C(s-1) & D(s) \end{array} \quad \text{LOBATTO IIIA}$$

$$y' = q \lambda$$

$$q \in \mathbb{C}$$

$$\Delta t \in \mathbb{R}^+$$

$$z = q \Delta t$$

$$y^{n+1} = R(z) \cdot y^n$$

$$R(z) = I + z \underline{b}^T (\underline{I} - z \underline{A})^{-1} \underline{1}$$

A-stability

$$\text{IF } \mathbb{C}^- \subset S$$

$$|y^{n+1}| \leq |y^n|$$

$$\boxed{\text{Re}(q) < 0}$$

$$\text{Re}(q) > 0?$$

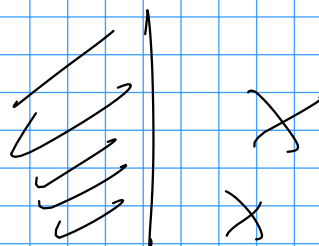


$$y' = -\underline{k} (y - \cos(t))$$

$$k = 2000$$

$$k \rightarrow \infty \quad y \rightarrow \cos(t)$$

→  
L-stability



$$R(z) = \frac{\text{poly (EXPlicit)}}{\text{FRACT. POLY (INPUT)}}$$

$$\lim_{z \rightarrow \infty} R(z) = 0 \Rightarrow \text{L-stable}$$

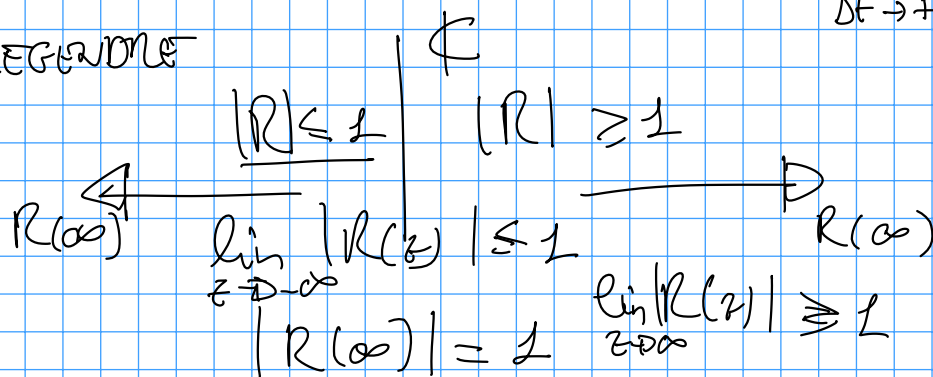
$$\operatorname{Re}(q) < 0$$

$$\Delta t \gg 1 \quad (z \rightarrow -\infty)$$

$$y' = q y$$

$$y^n \xrightarrow{\Delta t \rightarrow +\infty} 0$$

LEGEND



$$R(z) = \frac{a_0 + \dots + a_m z^m}{b_0 + \dots + b_n z^n}$$

$$z \rightarrow \infty$$

$$\frac{a_m z^m}{b_n z^n}$$

$$m > n$$

$$\lim |R| \rightarrow \infty$$

$$m = n$$

$$\lim |R| \rightarrow \frac{a_m}{b_m}$$

$$m < n$$

$$\lim \rightarrow 0$$

CRANK-NICOLSON

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$R(z)$$

$$(I - zA)^{-1}$$

$$\left[ \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -\frac{z}{2} & 1 - \frac{z}{2} & 0 & 1 \end{array} \right]$$

$$1 + z b^T \left[ \begin{array}{cc} 1 & 0 \\ \frac{z}{2-z} & \frac{z}{2-z} \end{array} \right] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left[ \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 - \frac{z}{2} & \frac{z}{2} & 1 \end{array} \right]$$

$$1 + z \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{2+z}{2-z} \end{bmatrix} = 1 + z \left( \frac{1}{2} + \frac{2+z}{2(2-z)} \right) \left[ \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{z}{2} \left( \frac{2}{2-z} \right) & \frac{2}{2-z} \end{array} \right]$$



$$= \frac{4 - 2z + 2z - z^2 + 2z + z^2}{2(2-z)} = \frac{z(2+z)}{2(2-z)} = \frac{2+z}{2-z}$$

$$\lim_{z \rightarrow 2} R(z) = \frac{2+2}{2-2} = -\frac{1}{0} \neq 0$$

$$\begin{array}{l} t^{n+1} \\ \uparrow \\ \tau_i = t^n + c_i \Delta t \\ \downarrow \\ t^n \end{array} \quad \begin{cases} u(\tau_i)' = f(\tau_i, u(\tau_i)) & i=1, \dots, S \\ u(t^n) = \gamma^n & (1) \end{cases}$$

$S+1$  CONDITIONS

$$u(t) \in \mathbb{P}_S \quad S = \# \text{ STAGES}$$

$$a_{ij} = \int_0^{c_i} \ell_j(t) dt \quad (2) \quad \ell_j(\tau_i) = \delta_{ij}$$

$$b_i = \int_0^1 \ell_i(t) dt \quad c_i = \tau_i$$

TH. CALCULATION needed (1) IS AN

IMPLICIT RK with  $A, b, c$  (2)  $[0, 1]$

$$u' \in \mathbb{P}_{S-1}$$

$$u'(t^n + \underbrace{\tau_j \Delta t}_{\in [t^n, t^{n+1}]}) = \sum_{j=1}^S \underbrace{\ell_j(z)}_{\in [0, 1]} \cdot \underbrace{u'(\tau_j)}_{[t^n, t^{n+1}]}$$

$$u(t^n + \tau \Delta t) = \gamma^n + \Delta t \int_0^\tau u'(t^n + s \Delta t) ds$$

$$= \gamma^n + \Delta t \int_0^\tau \sum_j \ell_j(s) \cdot u'(\tau_j) ds$$

$$= \gamma^n + \Delta t \sum_j \int_0^\tau \ell_j(s) ds \cdot u'(\tau_j)$$

$$= \gamma^n + \Delta t \sum_j \int_0^\tau \ell_j(s) ds \cdot f(\tau_j, u(\tau_j))$$

$$u^{(k)} := u(t^n + c_k \Delta t) = u(\tau_k)$$

$$u^{(k)} = y^n + \Delta t \sum_j a_{kj} \cdot f(\tau_j, u^{(k)})$$

$$y^{n+1} = u(t^{n+1}) = y^n + \Delta t \sum_j b_j f(\tau_j, u^{(j)}) \quad \boxtimes$$

$$B(p) \quad b_i = \int_0^1 \ell_j(s) ds \quad \tau_i = t^n + c_i \Delta t$$

$(c_i, b_i)$  QUADRATURE FORMULA

IS EXACT FOR POLY DEGREE  $P-1$

$$z = 1, \dots, P$$

$$\Rightarrow B(p)$$

$$\int_0^1 s^{z-1} ds = \sum_{j=1}^S c_j^{z-1} \cdot b_j$$

$$\left[ \frac{s^z}{z} \right]_0^1 = \frac{1}{z}$$

$$B(p): \sum_j c_j^{z-1} \cdot b_j = \frac{1}{z}$$

$$\forall z = 1, \dots, P \quad \checkmark$$

$C(s)$   
# OF STAGES

$$\sum_j a_{ij} c_j^{z-1} = \frac{c_i^z}{z} \quad \forall z = 1, \dots, S$$

$$s^{z-1} \in P_{S-1}$$

$$\underline{s^{z-1}} = \sum_j \ell_j(s) \cdot c_j^{z-1}$$

$$\sum_j a_{ij} c_j^{z-1} = \sum_j \underbrace{\int_0^{c_i} \ell_j(s) ds}_{a_{ij}} \cdot c_j^{z-1} = \int_0^{c_i} s^{z-1} ds$$

$$= \left[ \frac{s^z}{z} \right]_0^{c_i} = \frac{c_i^z}{z} \quad \boxtimes$$

$$\boxed{\text{THE}} \quad C(s) \text{ and } B(s+\gamma) \Rightarrow D(\gamma) \quad (3)$$

$\otimes$   $C(s) \quad B(2s) \Rightarrow D(s) \quad \text{LAGRANGE METHOD}$

$$B(p) = B(2s) \quad \wedge \quad C(\gamma) = C(s) \quad \wedge \quad D(\gamma) = D(s)$$

$$p = 2s$$

$$p \leq 2\gamma + 2$$

$$2s \leq 2s + 2 \quad \checkmark$$

$$2S = p = s + s + 1 = S + S + 1 = 2S + 1 \quad \checkmark$$

PROOF OF (3).

$$B(p): \sum_i b_i c_i^{z-1} = \frac{1}{z} \quad z=1, \dots, p$$

$$C(q): \sum_j a_{ij} c_j^{z-1} = \frac{c_i^z}{z} \quad z=1, \dots, q$$

$$D(q): \sum_i b_i c_i^{q-1} a_{ij} = \frac{b_j}{q} (1 - c_j^q)$$

$$d_j^{(z)} := \sum_i b_i c_i^{z-1} a_{ij} - \frac{b_j}{z} (1 - c_j^z) \quad \left[ q=1, \dots, s \right]$$

$$\text{GOAL } d_j^{(z)} = 0 \quad \forall j=1, \dots, s \quad \forall z=1, \dots, p$$

$$(4) \sum_{j=1}^s d_j^{(z)} \cdot c_j^{k-1} = 0 \quad \forall k=1, \dots, s \quad \forall z=1, \dots, p$$

$$\bigwedge_{j,k} \{ c_j^{k-1} \}_{j=1, \dots, s} \quad k=2, \dots, s$$

$c_j$  DISTINCT

$$\Rightarrow d_j^{(z)} = 0 \quad (5)$$

$$B(p): \sum_i b_i c_i^{z-1} = \frac{1}{z}$$

WE WILL PROVE (4) INSTEAD OF (5)

$$\sum_{j=1}^s d_j^{(z)} \cdot c_j^{k-1} = \sum_{j=1}^s \sum_{i=1}^s b_i c_i^{z-1} a_{ij} c_j^{k-1}$$

$$- \sum_{j=1}^s \frac{b_j}{z} (1 - c_j^z) \cdot c_j^{k-1} =$$

$$= \sum_{i=1}^s b_i c_i^{z-1} \sum_{j=1}^s a_{ij} c_j^{k-1} - \sum_{j=1}^s \frac{b_j}{z} c_j^{k-1} + \sum_{j=1}^s \frac{b_j}{z} c_j^{z+k-1}$$

$$= \sum_{i=1}^s b_i c_i^{z-1} \frac{c_i^k}{k} - \frac{1}{z} \sum_{j=1}^s b_j c_j^{k-1} + \frac{1}{z} \sum_{j=1}^s b_j c_j^{z+k-1}$$

$$= \frac{1}{k} \sum_{i=1}^s b_i c_i^{z+k-1} - \frac{1}{z} \cdot \frac{1}{k} + \frac{1}{z} \cdot \frac{1}{k+z}$$

$$= \frac{1}{k} \frac{1}{k+z} - \frac{1}{z} \frac{1}{k} + \frac{1}{z} \frac{1}{k+z} = \frac{z - (k+z) + k}{k z (k+z)} = 0 \quad \square$$

CRANK NICHOLSON  $C(S) \quad D(S-2) \quad \leftarrow$  COLLOCATION

II  $C(S-2) \quad D(S)$

III  $C \quad C(S-1) \quad D(S-1)$

$$y^{(k)} = y^r + \Delta t \sum q_k \downarrow F(y^{(k)})$$

$$y^{(n)} = y^r + \Delta t \sum b_j F(y^{(j)})$$

• STIFFLY ACCURATE IFF  $a_{sj} = b_j \quad \forall j$

$A \in \mathbb{R}^{S \times S} \Rightarrow$  NONLINEAR  $(S \times N) \times (S \times N)$   
 $N = \# \text{ DOF problem}$

$$O(N^2) \hookrightarrow O(N^3)$$

$$O((SN)^2) \hookrightarrow O((SN)^3)$$

$$O(S^2 \cdot N^2) \hookrightarrow O(S^2 \cdot N^3)$$

ORDER 6  
 $27 \cdot N^3$   
 DENSE  
 SSIM  
 $6 \cdot N^3$

S.  $O(N^2) \hookrightarrow S \cdot O(N^3)$  DIAGONALLY IMPLICIT

$\downarrow$   
 CRANK NICHOLSON

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

$$u^{(1)} = y^r$$

$$y^{(n)} \leftarrow u^{(2)} = y^r + \frac{\Delta t}{2} F(y^r) + \frac{\Delta t}{2} F(u^{(1)})$$

$$R(y^{(n)}) = y^{(n+1)} - y^r - \frac{\Delta t}{2} F(y^r) - \frac{\Delta t}{2} F(y^{(n)}) = 0$$

$$JR(y^{(n)}) = I - \frac{\Delta t}{2} JF(y^{(n)})$$

IMPLICIT RK  $\Rightarrow$  NEW LINEAR SYSTEMS!  
 PROBLEM!

$$\begin{cases} y^{(k)} = y^n + \Delta t \sum_j a_{kj} F(y^{(j)}) \\ y^{n+1} = y^n + \Delta t \sum_j b_j F(y^{(j)}) \end{cases} \quad \forall k \quad \text{DIRK}$$

$$\begin{cases} G^{(k)} = \Delta t F(y^n + \sum_{j=1}^k a_{kj} G^{(j)}) \\ y^{n+1} = y^n + \sum_j b_j G^{(j)} \end{cases}$$

DIRK

$$\begin{aligned} G^{(k)} &= \Delta t F(y^n + \sum_{j=1}^{k-1} \alpha_{kj} G^{(j)}) \\ &\quad + \Delta t \sum_{j=1}^k \gamma_{kj} J_y F(y^n) \cdot G^{(j)} \end{aligned}$$

$$\begin{aligned} [I - \Delta t \gamma_{kk} J_y F(y^n)] G^{(k)} &= \Delta t F(y^n + \sum_{j=1}^{k-1} \alpha_{kj} G^{(j)}) \\ &\quad + \Delta t \sum_{j=1}^{k-1} \gamma_{kj} J_y F(y^n) \cdot G^{(j)} \end{aligned}$$

$\gamma_{kk} = \gamma \quad \forall k$

LINEAR STABILITY

IMEX RK

IMPLICIT

EXPLICIT

$$\partial_t u = \underbrace{F(u)}_{\text{EXPLICIT}} + \underbrace{S(u)}_{\text{BAD GUY}}$$

EXPLICIT

BAD GUY

STIFF

0.01

0.01

$$\partial_t u + \partial_x F(u) = 0$$

$$\Delta t \leq \Delta x \cdot C$$

$$\partial_t u - \partial_{xx} u = 0$$

EXPLICIT

0.0001

$$\Delta t \leq \Delta x^2 \cdot C$$

$$\partial_{xxx} u$$

$$\Delta t \leq \Delta x^3$$

$$\partial_t u + \underbrace{\partial_x F(u)}_{\text{EX}} - \underbrace{\partial_{xx} u}_{\text{IM}} = 0$$

$$\partial_t u + F(u) + S(u) = 0$$

$$\frac{u^{n+1} - u^n}{\Delta t} + F(u^n) + S(u^{n+1}) = 0$$

IMEX RK

$\bar{A}, \bar{S}, \bar{C}$  EXPLICIT COEFF

$A, b, c$  IMPLICIT COEFF

---

LINEAR STABILITY

$$y' = \underbrace{\lambda_1}_{\text{EXPLICIT}} y + \underbrace{\lambda_2}_{\text{IMPLICIT}} y$$

$$z_1 = \Delta t \lambda_1 \in \mathbb{C}$$

$$z_2 = \Delta t \lambda_2 \in \mathbb{C}$$

$$y^{n+1} = R(z_1, z_2) \cdot y^n$$

$$R(z) : \mathbb{C} \rightarrow \mathbb{C}$$

$$|R(z)| : \mathbb{C} \rightarrow \mathbb{R} \quad \text{ZO} \quad \text{[Diagram of a circle in the complex plane with a point z on the boundary]}$$

$$|R(z_1, z_2)| : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R} \quad \text{ZO}$$

$$\operatorname{Re}(z_1) = 0 \quad z_1 \in i\mathbb{R}$$

$$|R(z_1, z_2)| : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

$$\operatorname{Im}(z_2) = 0 \quad z_2 \in \mathbb{R}$$

• RUSO 2000

$$\rightarrow z_2 \in \mathbb{C} : |R(z_1, z_2)| \leq 1 \quad \forall z_1 \in \{ |1+z_1| \leq 1 \}$$

$$\rightarrow z_1 \in \mathbb{C} : |R(z_1, z_2)| \leq 1 \quad \forall z_2 \in \mathbb{C}^-$$


---

$$y^n \longrightarrow y^{n+1}$$

$$y' = F(y)$$

$$y^{n-1}, y^{n-2}, y^{n-3}, \dots$$

ADAMS

$$\{F(t^{n-j}, y^{n-j})\}_{j=1}^k$$

GENERAL

⊕

$$\{y^{n-j}\}_{j=1}^k$$

⊕

$$y^{n+1} + \sum_{j=1}^k \alpha_j y^{n-j+1} = \Delta t \sum_{j=0}^k \beta_j F(t^{n-j+1}, y^{n-j+1})$$

STABILITY

$$\cancel{y^{n+1} = R(z) \cdot y^n} = y^{n+1} = R(z) \cdot y^n = \underline{R(z)^{n+1} \cdot y^0}$$

LINEAR STABILITY

$$\boxed{y' = q y}$$

$$q \in \mathbb{C} \wedge \operatorname{Re}(q) \leq 0$$

ANSWER:  $y^{n+1} = \zeta^{n+1} y^0 \uparrow$  STABLE  $\Leftrightarrow |\zeta| \leq 1$

$$y^{n+1} + \sum_{j=1}^k \alpha_j y^{n-j+1} = \underbrace{\Delta t q}_z \sum_{j=0}^k \beta_j y^{n-j+1}$$

$$\rightarrow \cancel{\zeta^{n+1}} + \sum_{j=1}^k \alpha_j \cancel{\zeta^{n-j+1}} \cdot \cancel{y^0} = z \cdot \sum_{j=0}^k \beta_j \cancel{\zeta^{n-j+1}}$$

DIVIDED BY  $\zeta^{n-k+1}$

$$\rightarrow \underbrace{\zeta^k} + \sum_{j=1}^k \alpha_j \zeta^{k-j} = z \cdot \sum_{j=0}^k \beta_j \zeta^{k-j}$$

$\zeta_i(z)$  SOLUTION  $\uparrow$

For  $i = 1, \dots, k$

$$|\zeta_i(z)| \leq 1 \quad \forall i$$

$$\begin{array}{c} z \in \mathbb{C} \\ \hline \end{array}$$

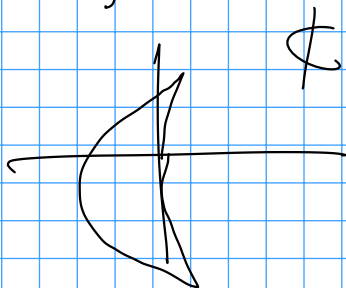
$$\zeta \rightarrow z$$

$$z = \frac{\zeta^k + \sum_{j=1}^k \alpha_j \zeta^{k-j}}{\sum_{j=0}^k \beta_j \zeta^{k-j}}$$

$$\zeta \in \mathbb{C}$$

$$\underline{|\zeta| \leq 1}$$

$$\boxed{\bar{z} \in \mathbb{C}}$$



## ACCURACY

### CONSISTENCY ERROR

$$y(t^{n+1}) + \sum_{j=1}^k \alpha_j y(t^{n-j+1}) - \Delta t \sum_{j=0}^k \beta_j \underbrace{F(t^{n-j+1}, y(t^{n-j+1}))}_{y'(t^{n-j+1})}$$

Taylor EXP in  $t^{n+1}$

$$= y(t^{n+1}) + \sum_{j=1}^k \alpha_j \sum_{\ell=0}^p \frac{(-j\Delta t)^\ell}{\ell!} y^{(\ell)}(t^{n+1})$$

$$- \Delta t \sum_{j=0}^k \beta_j \sum_{\ell=0}^{p-1} \frac{(-j\Delta t)^\ell}{\ell!} y^{(\ell+1)}(t^{n+1}) + \mathcal{O}(\Delta t^{p+1})$$

$$y(t^{n+1}) \left[ 1 + \sum_{j=1}^k \alpha_j \right] +$$

$$+ \Delta t y'(t^{n+1}) \left[ \sum_{j=1}^k -j \alpha_j + \sum_{j=0}^k \beta_j \right]$$

+ ...

FOR ORDER  $p$   $1 + \sum_{j=1}^k \alpha_j = 0$

$$\forall \ell = 1, \dots, p \quad \sum_{j=1}^k \alpha_j \frac{(-j)^\ell}{\ell!} - \sum_{j=0}^k \beta_j \frac{(-j)^{\ell-1}}{(\ell-1)!} = 0$$

ADAMS-BASHFORTH

VALUES  $F(t^{n-k+1}, y^{n-k+1})$

NO  $y^{n-k+1}$

$$y(t^{n+1}) = y(t^n) + \int_{t^n}^{t^{n+1}} F$$

$$= y(t^n) + \int_{t^n}^{t^{n+1}} P_n(t) dt$$

$$P_{n,k}(t^{n-j+1}) = F^{n-j+1} = F(t^{n-j+1}, y^{n-j+1}) \quad \forall j=1, \dots, k.$$

$$P_{n,k} \in \mathcal{P}_{k-1}$$



$$p_{n,k}(t) = \sum F^{n-j+1} \cdot L_{n,j,k}(t)$$

$$L_{n,j,k}(t) = \prod_{\substack{i=1 \\ i \neq j}}^k \frac{t - t^{n+1-i}}{t^{n+1-j} - t^{n+1-i}}$$

$$b_{n,j,k} = \frac{1}{t^{n+1} - t^n} \int_{t^n}^{t^{n+1}} L_{n,j,k}(t) dt \quad j=1, \dots, k$$

$$y^{n+1} = y^n + \Delta t \sum_{j=1}^k b_{n,j,k} F^{n-j+1}$$

$$F^{n+1} = F(t^{n+1}, y^{n+1})$$

STABILITY + ADMS + HIGH ORDER = NO!

$$y^{n+1} + \sum_{j=1}^k \alpha_j y^{n-j+1} = \Delta t \beta_0 F(t^{n+1}, y^{n+1})$$

BACKWARD DIFFERENCE FORMULA

BDF2  $y^{n+1} - \frac{3}{2} y^n + \frac{1}{2} y^{n-1} = \Delta t F(y^{n+1})$

1 STEP IMPLICIT EULER  $\mathcal{O}(\Delta t^2)$

N-1 STEPS BDF2  $\mathcal{O}(\Delta t^3)$

$$\mathcal{O}\left(\Delta t^{\frac{2}{3}} \frac{N}{\Delta t} + \Delta t^3\right) = \Delta t^2$$