- EXAM
- ORAL EXAM (THEORETICAL)
- CODING PROJECTS (PRACTICAL)

ORDINARY DIFFERENTIAL EQUATIONS ODE

$I = [t_0, t_{end}] \subset \mathbb{R}$

$y : I \to \mathbb{R}^s \qquad F : I \times \mathbb{R}^s \to \mathbb{R}^s$

$$\begin{cases} \dfrac{dy}{dt} = F(t, y(t)) \simeq \underbrace{F(y(t))}_{\text{AUTONOMOUS ODE}} \\ y(0) = y_0 \end{cases}$$

$$y^{(p)} = \overbrace{f(t, y, y', y'', \dots, y^{(p-1)})}$$

HIGHER ORDER DERIVATIVE DIFF. EQ.

$$\left.\begin{aligned} y' &= z_1 \\ y'' = z_1' &= z_2 \\ y''' = z_2' &= z_3 \\ y^{(p)} = z_p' &= f(t, y, z_1, z_2, \dots, z_{p-1}) \end{aligned}\right]$$

$y \in \mathbb{R}^s$

$$\begin{pmatrix} y \\ z_1 \\ \vdots \\ z_p \end{pmatrix} \in \mathbb{R}^{s(p+1)}$$

SYSTEM OF 1^{st} DERIVATIVE DIFF. EQ.

$$\int_{t_0}^{t} \frac{dy}{dt} dt = y(t) - y(t_0) = y(t) - y_0$$

$$y(t) = y_0 + \int_{t_0}^{t} F(t, y(t))\, dt$$

DAHLQUIST'S PROBLEM

$$\begin{cases} y'(t) = -\lambda y(t) \\ y(t_0) = y_0 \end{cases} \qquad \lambda \in \mathbb{R} \to \mathbb{C}$$

$y(t) \neq 0 \qquad y_0 \geq 0$

$$\int_{t_0}^{t} \underbrace{\frac{y'(s)}{y(s)}}\, ds = -\lambda \int_{t_0}^{t} ds$$

$$\int_{t_0}^{t} \frac{d}{ds}\Big(\log(y(s))\Big)\, ds = -\lambda\,(t-t_0)$$

$$\log(y(t)) - \log(y(t_0)) = -\lambda\,(t-t_0)$$

$$\log\left(\frac{y(t)}{y(t_0)}\right) = -\lambda\,(t-t_0)$$

$$\frac{y(t)}{y(t_0)} = e^{-\lambda\overbrace{(t-t_0)}^{>0}}$$

$$y(t) = y_0\, \underbrace{e^{-\lambda(t-t_0)}}_{\neq 0}$$

$y_0 > 0$

$\lambda < 0 \to$ exp. INCREASING

$\lambda > 0 \to$ exp. to 0

PDS

$$c_1 \rightleftarrows c_2$$

$$\begin{cases} c_1'(t) = c_2(t) - 5c_1(t) \\ c_2'(t) = 5c_1(t) - c_2(t) \end{cases} \qquad c_1, c_2 > 0$$

$$\sum_i c_i(t) = M \qquad \forall t$$

$$\frac{d}{dt}\left(\sum_i c_i(t)\right) = c_1' + c_2' = c_2 - 5c_1 + 5c_1 - c_2 = 0$$

$$c'(t) = A \cdot c(t) \qquad A = \begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix}$$

$$c(0) = c_0 \qquad y' = \lambda y \Rightarrow y = y_0 e^{\lambda t}$$

$$c(t) = \underline{e^{At}}\, c_0 \qquad e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

$$A^2 = -6A \qquad e^{At} = \sum_{k=1}^{\infty} \frac{(-6)^{k-1} A\, t^k}{k!} + I$$

$$A^3 = -6A^2 = (-6)^2 A$$

$$A^k = (-6)^{k-1} A$$

$$e^{At} = \Bigg(\underbrace{\sum_{k=0}^{\infty} \frac{(-6)^k \cdot t^k}{k!}} - 1 \Bigg) \frac{A}{-6} + 1$$

$$\left(e^{-6t} - 1\right) \frac{A}{-6} + 1$$

$$c(t) = C_0 + \frac{1 - e^{-6t}}{6} A C_0$$

$$x'(t) = \alpha x - \beta x y \qquad \alpha, \beta, \gamma, \delta > 0$$
$$y'(t) = \delta x y - \gamma y$$

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \text{EQUILIBRIUM}$$

$$\eta(x, y) = \delta x - \gamma \log(x) + \beta y - \alpha \log(y)$$

$$\frac{d}{dt} \eta(x(t), y(t)) = \begin{pmatrix} \partial_x \eta \\ \partial_y \eta \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} =$$

$$\langle \nabla \eta, F(x,y) \rangle = 0 = \cancel{\delta\alpha x} - \cancel{\delta\beta xy} - \cancel{\gamma\alpha}$$
$$+ \cancel{\gamma\beta y} + \cancel{\beta\delta xy}$$
$$- \cancel{\beta\gamma y} - \cancel{\alpha\delta x} + \cancel{\alpha\gamma}$$
$$= 0$$

$$\nabla \eta = \begin{pmatrix} \delta - \gamma/x \\ \beta - \alpha/y \end{pmatrix}$$

$$\begin{cases} u'(t) = -\dfrac{v(t)}{r(t)} \\ v'(t) = \dfrac{u(t)}{r(t)} \end{cases} \qquad r(t) = \sqrt{u^2(t) + v^2(t)}$$

$$\frac{r^2}{2}$$

$$\frac{d}{dt}\left(\frac{r^2}{2}\right) = \frac{d}{dt}\left(\frac{u^2 + v^2}{2}\right) = \frac{2 u \cdot u' + 2 v \cdot v'}{2} =$$

$$u \cdot u' + v \cdot v' = u \cdot \left(-\frac{v}{r}\right) + v \left(\frac{u}{r}\right) = 0$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{matrix} \cos(\theta)\, u^0 - \sin(\theta)\, v^0 \\ \sin(\theta)\, v^0 + \cos(\theta)\, u^0 \end{matrix} \qquad \theta = \frac{t}{r}$$

$$u' = -\frac{v}{r} - \alpha u$$

$$v' = \frac{u}{r} - \alpha v$$

$$\frac{d}{dt}\left(\frac{r^2}{2}\right) = \frac{d}{dt}\left(\frac{u^2+v^2}{2}\right) = u \cdot u' + v \cdot v'$$

$$= \cancel{-u \cdot \frac{v}{r}} - \alpha u^2 + \cancel{v\,\frac{u}{r}} - \alpha v^2 = -\alpha\left(u^2+v^2\right)$$

$$= -2\alpha\,\frac{u^2+v^2}{2} = -2\alpha\left(\frac{r^2}{2}\right)$$

$$E = \frac{r^2}{2} \qquad \frac{d}{dt} E = -2\alpha E$$

$$E = e^{-2\alpha t} E_0 \qquad \alpha > 0$$

$$\frac{r^2}{\cancel{2}} = e^{-2\alpha t}\,\frac{r_0^2}{\cancel{2}} \Rightarrow r(t) = \underbrace{e^{-\alpha t}}_{\to 0} r_0$$

- $u'' = -\sin(u)$

$$\begin{cases} u' = v \\ v' = -\sin(u) \end{cases} \qquad \eta(t) = \frac{1}{2} v^2 - \cos(u)$$

$$\frac{d}{dt}\eta(t) = \langle \nabla \eta, F \rangle = \begin{pmatrix} \sin(u) \\ v \end{pmatrix} \cdot \begin{pmatrix} v \\ -\sin(u) \end{pmatrix} =$$

$$v \cdot \sin(u) - v \cdot \sin(u) = 0$$

ROBERTSON

$$\begin{tikzcd} C_1 \arrow[r, bend right, "0.04\, c_1"'] & C_2 \arrow[l, bend right, "10^4 \cdot c_2 c_3"'] \arrow[d, "3 \cdot 10^7 c_2^2"] \\ & C_3 \end{tikzcd} \qquad \frac{d}{dt}\left(\sum_i c_i\right) = 0$$

• PDE $\quad U(x,t) : \Omega \times [0,T] \to \mathbb{R}^d$

$$\partial_t U + \partial_{xx}(a(U)) + \partial_x b(U) + c(U) = 0$$

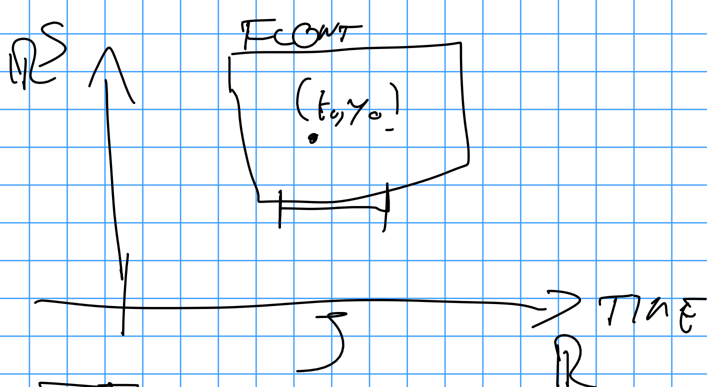
$\underline{U} = \mathbb{R}^N \qquad N$ # DOF IN SPACE

$$\partial_{xx} a(U) \approx D_2 A(\underline{U})$$

$$\partial_t \underline{U} = f(\underline{U}) \qquad f(\underline{U}) = -(D_2 A(\underline{U}) + \dots)$$

$\partial_t U + \partial_x b(U) = 0 \qquad$ TVD

$U_i^n \qquad TV(\underline{U}^n) := \sum_i |U_{i+1}^n - U_i^n|$



$$\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}$$

- $y(t) = 0$
- $y(t) = \frac{t^2}{4} \qquad \frac{t}{2}$

$$y' = \frac{2t}{\cancel{4}2}$$

$F(t,y) : I \times \mathbb{R}^S \to \mathbb{R}^S$

$$\|F(t,y) - F(t,z)\| \leq L \|y - z\|$$

UNIFORMLY IN $t \Rightarrow \exists!$ SOLUTION

$y \in C^1(I) \qquad \begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$

$F \in C^{p-1}$ p-differentiable

y IS $p+1$ TIMES DIFFERENTIABLE

$$\frac{dy}{dt} = F$$

$$F^{(\ell+1)}(t,y) = \frac{\partial F^{(\ell)}}{\partial t}(t,y) + \frac{\partial F^{(\ell)}(t,y)}{\partial y}$$

ODE $I = [t_0, t_{end}]$ $y : I \to \mathbb{R}^S$

$F : I \times \mathbb{R}^S \to \mathbb{R}^S$

$$\begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$t_0 = t^0 < t^1 < t^2 < \dots < t^n < \dots t^N = t_{end}$

$[t^n, t^{n+1}]$ $y(t^{n+1}) = y(t^n) + \int_{t^n}^{t^{n+1}} F(t, y(t))\,dt$

$\downarrow \quad \downarrow$

$y^n \quad y^{n+1}$ $\approx y(t^n) + F(t^n, y(t^n)) \cdot \underbrace{(t^{n+1} - t^n)}_{\Delta t^n}$

$y^n \to y^{n+1}$ $y^{n+1} := y^n + \Delta t^n F(t^n, y^n)$

$y^0 = y_0$

$$c_1^{n+1} + c_2^{n+1} = c_1^n + \Delta t(\cancel{-5c_1^n} + \cancel{c_2^n}) + c_2^n + \Delta t(\cancel{-c_2^n} + \cancel{5c_1^n})$$

$$= c_1^n + c_2^n = c_1^0 + c_2^0 = \text{MASS} \quad \forall n$$

$e_n = y(t^n) - y^n$

CONSISTENCY ERROR

$$\varepsilon_n = y(t^{n+1}) - y(t^n) - \Delta t \overbrace{F(t^n, y(t^n))}^{y'(t^n)} =$$

$$= \int_{t^n}^{t^{n+1}} \underbrace{y'(t) - \underbrace{y'(t^n)}}_{\text{COSTANT}} \, dt$$

$$\omega(f, \Delta t) := \max_{t,t' : |t-t'|<\Delta t} |f(t) - f(t')|$$

$$|\varepsilon_n| \le \int_{t^n}^{t^{n+1}} \underbrace{\omega(y', \Delta t^n)}_{\text{DOESN'T DEPEND ON } t} \, dt = \Delta t^n \cdot \omega(y', \Delta t^n).$$

$$e_{n+1} = y(t^{n+1}) - y^{n+1} = y(t^{n+1}) - y^n - \Delta t^n F(t^n, y^n)$$
$$+ y(t^n) + \Delta t\, F(t^n, y(t^n))$$
$$- y(t^n) - \Delta t\, F(t^n, y(t^n))$$

$$= \varepsilon_n + \overbrace{y(t^n) - y^n}^{e_n} + \Delta t \left[F(t, y(t^n)) - F(t, y^n) \right]$$

$$|e_{n+1}| \le |\varepsilon_n| + |e_n| + \Delta t\, L \, \underbrace{|y(t^n) - y^n|}_{e_n} \le \ast$$

$$|F(t, y) - F(t, z)| \le L |y - z|$$ (HYPOTHESIS ON F TO HAVE UNIQUENESS)

$$\ast \le |\varepsilon_n| + (1 + \Delta t L) |e_n|.$$

$$\le \dots \le e^{L|t^{n+1} - t^0|} |e_0| + \sum_{i=0}^{n} e^{L(\overbrace{t^{n+1} - t^{i+1}}^{\Delta t})} |\varepsilon_i| \le \circledast$$

$$|\varepsilon_n| \le \Delta t\, \omega(y', \Delta t)$$

$$\circledast \le e^{L|t^{n+1} - t^0|} |e_0| + \overline{\omega(y', \Delta t)}\, \frac{e^{L(t^{n+1} - t^0)} - 1}{L}.$$

$$\omega(y', \Delta t) = \omega(F(\cdot, y(\cdot)), \Delta t)$$

$$\left[\sum_{i=0}^{n} \Delta t\, e^{L \Delta t} = \underbrace{N \cdot \Delta t}\, e^{L \Delta t} = \frac{T}{\cancel{\Delta t}} \cancel{\Delta t}\, e^{L \Delta t} \dots \right]$$

$$w(y', \Delta t) = \max_{t,t' \,;\, |t-t'|<\Delta t} |y'(t) - y'(t')| = \text{(AUTONOMOUS)}$$

$$= \max_{t,t' \,;\, |t-t'|\le\Delta t} |F(y(t)) - F(y(t'))|$$

$$\le L\,|y(t) - y(t')| \xrightarrow[\Delta t \to 0]{} 0$$

$$\le L\,\underset{\substack{\max\limits_{I} y'}}{\underset{\parallel}{C}} \cdot \Delta t$$

$$\boxed{|y(t^n) - y^n| \le C_0\,|y^0 - y(t^0)| + \Delta t \cdot C_1}$$

$$e_1 = |\underbrace{y^1}_{} - y(t^1)| = |\cancel{y^0} + \cancel{\Delta t\,\overbrace{F(t^0, y^0)}^{y'(t_0)}}$$
$$- (\cancel{y(t^0)} + \cancel{\Delta t\, y'(t_0)} + \frac{\Delta t^2}{2} y''(t_0)$$
$$+ O(\Delta t^3)|$$

$$= \left|\frac{\Delta t^2}{2} y''(t_0)\right| + O(\Delta t^3) = O(\underline{\Delta t^2})$$

$$e_N \approx \sum_{i=1}^{N} |y(t^i) - y^i| \lesssim N \cdot \frac{\Delta t^2}{2} \max |y''|$$

$$= \frac{t_{end}}{\cancel{\Delta t}} \cdot \frac{\Delta t^{\cancel{2}}}{2} \max |y''|$$

$$= t_{end} \cdot \frac{\Delta t}{2} \max |y''|$$

$$= C \cdot \Delta t = O(\Delta t)$$

FIRST ORDER

DEF

p ORDER SCHEME

LARGEST p $\quad \Delta t \to 0$

$|e_N| \le C \cdot \Delta t^p$

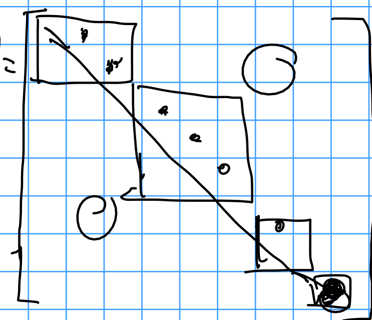
$$y'(t) = M y(t) \qquad M \in \mathbb{R}^{s \times s} \qquad \Delta t$$

$$y^n = y^{n-1} + \Delta t\, M y^{n-1} = (I + \Delta t\, M) y^{n-1}$$
$$= (I + \Delta t\, M)^n y^0$$

$$|y^n| \overset{?}{\leq} |y^0| \rightsquigarrow \|(I+\Delta t M)^n\|$$

$\in \mathbb{C}^{s \times s}$

$$M \to \hat{M} = S^{-1} M S$$

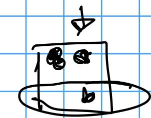
$\hat{M} =$ 

$$S^{-1} y^n = S^{-1} y^{n-1} + \Delta t\, S^{-1} M \underset{S S^{-1}}{\cdot} y^{n-1}$$

$$S^{-1} y^n =: \hat{y}^n$$

$$\hat{y}^n = \hat{y}^{n-1} + \Delta t\, \hat{M}\, \hat{y}^{n-1}$$

$$= (I + \Delta t \hat{M})\, \hat{y}^{n-1}$$

$\hat{y}^n_s \to$ 

$$\boxed{y'(t) = q\, y(t)} \qquad q \in \mathbb{C}$$

$q \in \mathbb{R}$, $\boxed{q \leq 0}$ $\mathrm{Re}(q) \leq 0$

$q > 0$

$$y^{n+1} = y^n + q \Delta t\, y^n = (1 + q\Delta t)\, y^n$$

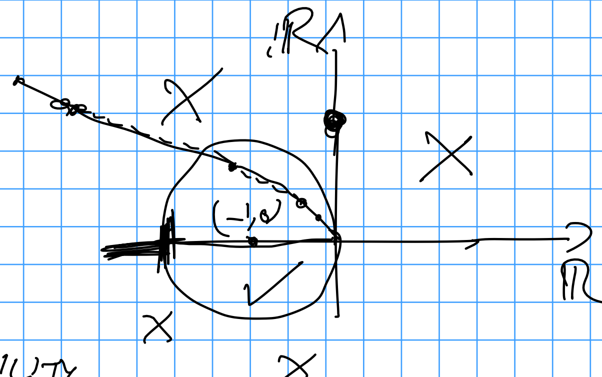
$$= (1 + q\Delta t)^{n+1} y^0$$

$$|y^{n+1}| \leq |y^n| \leq \dots \qquad |y^{n+1}| = |y^n|$$

$$|1 + q\Delta t|\, |y^n| \leq |y^n|$$

$$|1 + \underbrace{q \Delta t}_{z \in \mathbb{C}}| \leq 1$$

$$|1 + z| \leq 1$$



$R(z) = 1 + z$ STABILITY FUNCTION

$y^{n+1} = R(\Delta t q) \cdot y^n$ $\qquad z = \Delta t q$

$\{ |R(z)| \leq 1 \}$ STABILITY REGION

$$q \in \mathbb{R} \qquad |q| \Delta t \leq C \Rightarrow \Delta t \leq \frac{C}{|q|} \simeq \frac{C}{|\lambda|}$$

$u' = v$

$v' = -u$

$u(0) = 0 \quad v(0) = 1$

$r^2 = u^2 + v^2 = $ CONSTANT IN TIME

$$\frac{d}{dt} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = 0$$

$$\underline{u} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{v}$$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y^{n+1} = y^n + \Delta t\, f(t^{n+1}, y^{n+1})$ FIRST ORDER

LINEAR STABILITY

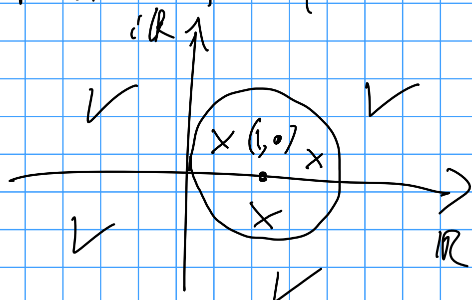
$y' = q y \qquad q \in \mathbb{C}$

$y^{n+1} = y^n + \Delta t\, q\, y^{n+1}$

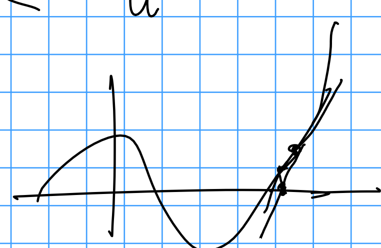
$(1 - \underbrace{\Delta t q}_{z \in \mathbb{C}}) y^{n+1} = y^n$

$$y^{n+1} = R(z) \cdot y^n = \overbrace{\frac{1}{1-z}}^{R(z)} y^n$$

$$\{ |R(z)| \le 1 \} = \left\{ \frac{1}{|1-z|} \le 1 \right\} = \{ |1-z| \ge 1 \}$$



$y^{n+1} = y^n + f(y^{n+1}) \Delta t \qquad y^{n+1} \leftarrow u^{(k)}$



IMPLICIT EULER + POSITIVITY

$\forall \Delta t$

PDS
$$\frac{d}{dt} c_i = P_i(c) - D_i(c)$$
$$= \sum_j \underbrace{p_{ij}(c)}_{\geq 0} - \underbrace{d_{ij}(c)}_{\geq 0}$$

$$\boxed{p_{ij} = d_{ji}} \geq 0$$

1. CONSERVATION

$$\frac{d}{dt}\left(\sum_i c_i(t) \right) = \sum_i \frac{d}{dt} c_i(t) = \sum_i \sum_j p_{ij}(c) - d_{ij}(c)$$

$$= \sum_i \sum_j p_{ij} - \sum_i \sum_j d_{ji} = \sum_i \sum_j p_{ij} - p_{ij} = 0$$

- HOMEWORK CHECK THAT IMPLICIT EULER IS CONSERVATIVE.

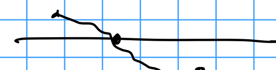
2. POSITIVITY $\quad \boxed{c_i \to 0 \Rightarrow D_i(c) \to 0} \quad \forall j \ \forall i$

$$d_{ij}(c) \to 0$$

$$c_i(0) > 0 \Rightarrow c_i(t) \geq 0 \quad \forall i \quad \forall t$$

PROOF BY CONTRADICTION $\quad t : c_i(t) < 0$

$$\exists\, t^* \quad c_i(t^*) = 0 \quad c_i'(t^*) < 0$$



$$c_i'(t^*) = \underbrace{P_i(c(t^*))}_{\geq 0} - \underbrace{D_i(c(t^*))}_{= 0} \geq 0 \quad ↯$$

IMPLICIT EULER + LINEAR PDS + POSITIVITY

$$y' = My \qquad M_{ii} < 0 \qquad M_{ij} \geq 0 \quad i \neq j$$

$$\sum_i M_{ij} = 0$$

IMPLICIT EULER

$$y^0 \geq 0$$

$$\underbrace{(I - \Delta t\, M)}_{A} y^{n+1} = y^n \implies y^{n+1} = \underbrace{A^{-1}}\, y^n$$

$$A^{-1} \geq 0 \iff \forall_{ij}\ (A^{-1})_{ij} \geq 0$$

1. $0 < A_{ii} = 1 - \Delta t \underbrace{\underbrace{M_{ii}}_{<0}}_{>0} = 1 + \Delta t\, |M_{ii}| = 1 + \Delta t \sum_{j \neq i} M_{ji}$

$$\boxed{|A_{ii}| > \sum_{j \neq i} |A_{ji}|}$$

$$\sum_j M_{ji} = 0$$

$$0 > M_{ii} = -\sum_{j \neq i} \overbrace{M_{ji}}^{\geq 0} < 0$$

$$\underbrace{\overbrace{1}^{>0} + \overbrace{\Delta t}^{>0} \sum_{j \neq i} \overbrace{M_{ji}}^{\geq 0}} > \Delta t \sum_{j \neq i} M_{ji} = \sum_{j \neq i} |A_{ji}|.$$

A DIAG. DOMINANT BY COLUMN

- $A^{-1} \geq 0$?

JACOBI METHOD

$$A = D + (A - D) \qquad D = \mathrm{diag}(A)$$

$$\underline{Au = R} \iff Du + \underbrace{(A-D)u} = R$$

$$Du = R - (A-D)u \quad \leftarrow \checkmark \qquad u = D^{-1}(D-A)u + D^{-1}R$$

$$Du^{(k)} = R - (A-D)\, u^{(k-1)}$$

$$u^{(k)} = D^{-1}(D-A)\, u^{(k-1)} + D^{-1}R$$

$$\|u - u^{(k)}\|_\infty = \|D^{-1}(D-A)(u^{(k-1)} - u)\|$$

$$\leq \|D^{-1}(D-A)\|_\infty \|u - u^{(k-1)}\|_\infty$$

$$\|B\|_\infty = \max_i \sum_j |B_{ji}| \qquad B = D^{-1}(D-A)$$

$$= (I - D^{-1}A)$$

$$(I - D^{-1}A)_{ji} = \begin{bmatrix} 0 & & \ddots \\ & 0 & \\ \ddots & & 0 \end{bmatrix}$$

$$\max_i \sum_{j \neq i} \frac{|A_{ji}|}{|A_{ii}|} = \max_i \frac{\sum_{j \neq i} |A_{ji}|}{|A_{ii}|} < 1$$

$$\|u - u^{(k)}\|_\infty < \|u - u^{(k-1)}\|_\infty$$

$$u^{(k)} \geq 0$$

$$Ay^{n+1} = y^n$$

$$\Downarrow$$

$$u^{(k)} = \underbrace{D^{-1}}_{>0} \underbrace{(D - A)}_{>0} \underbrace{u^{(k-1)}}_{>0} + \underbrace{D^{-1}}_{>0} \underbrace{y^n}_{>0} \geq 0$$

$$\begin{cases} \Delta t \, \Pi_{ij} > 0 & i \neq j \\ 0 & i = j \end{cases}$$

$$u^{(k)} \to u = y^{n+1} \geq 0. \qquad \boxtimes$$