- EXAM
- ORAL EXAM (THEORETICAL)
- CODING PROJECTS (PRACTICAL)

ORDINARY DIFFERENTIAL EQUATIONS   ODE

$$I = [t_0, t_{end}] \subset \mathbb{R}$$

$$y : I \to \mathbb{R}^s \qquad\qquad F : I \times \mathbb{R}^s \to \mathbb{R}^s$$

$$\left\{\begin{array}{l} \dfrac{dy}{dt} = F(t, y(t)) \simeq \underbrace{F(y(t))}_{\text{AUTONOMOUS ODE}} \\ y(0) = y_0 \end{array}\right.$$

$$y^{(P)} = f(t, y, y', y'', \dots, y^{(P-1)}) \qquad \begin{array}{l}\text{HIGHER ORDER} \\ \text{DERIVATIVE} \\ \text{DIFF. EQ.}\end{array}$$

$$y \in \mathbb{R}^s$$

$$\begin{aligned} y' &= z_1 \\ y'' = z_1' &= z_2 \\ y''' = z_2' &= z_3 \\ &\;\;\vdots \\ y^{(P)} = z_p' &= f(t, y, z_1, z_2, \dots, z_{p-1}) \end{aligned} \qquad \left. \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \right] \qquad \begin{pmatrix} y \\ z_1 \\ \vdots \\ z_p \end{pmatrix} \in \mathbb{R}^{s(P+1)}$$

SYSTEM OF 1st DERIVATIVE DIFF. EQ.

$$\int_{t_0}^{t} \frac{dy}{dt} dt = y(t) - y(t_0) = y(t) - y_0$$

$$y(t) = y_0 + \int_{t_0}^{t} F(t, y(t)) \, dt$$

DAHLQUIST'S PROBLEM

$$\left\{\begin{array}{l} y'(t) = -\lambda \, y(t) \\ y(t_0) = y_0 \end{array}\right. \qquad \lambda \in \mathbb{R} \to \mathbb{C}$$

$$y(t) \neq 0 \qquad y_0 > 0$$

$$\int_{t_0}^{t} \frac{y'(s)}{y(s)} \, ds = -\lambda \int_{t_0}^{t} ds$$

$$\int_{t_0}^{t} \frac{d}{ds}\left(\log(y(s))\right) ds = -\lambda(t-t_0)$$

$$\log(y(t)) - \log(y(t_0)) = -\lambda(t-t_0)$$

$$\log\left(\frac{y(t)}{y(t_0)}\right) = -\lambda(t-t_0)$$

$$\frac{y(t)}{y(t_0)} = e^{\overbrace{-\lambda(t-t_0)}^{\geq 0}}$$

$$y(t) = y_0\, e^{\underbrace{-\lambda(t-t_0)}_{\neq 0}}$$

$$y_0 > 0$$

$$\lambda < 0 \quad\to\text{ exp. INCREASING}$$
$$\lambda > 0 \quad\to \text{exp. to } 0$$

PDS

$$\textcircled{c_1} \underset{\longleftarrow}{\overset{\longrightarrow}{}} \textcircled{c_2}$$

$$\begin{cases} c_1'(t) = \overbrace{c_2(t)} - \underline{5c_1(t)} \\ c_2'(t) = 5c_1(t) - c_2(t) \end{cases} \qquad c_1, c_2 > 0$$
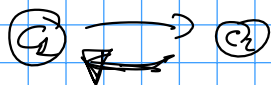
$$\sum_i c_i(t) = M \qquad \forall t$$

$$\frac{d}{dt}\left(\sum_i c_i(t)\right) = c_1' + c_2' = c_2 - 5c_1 + 5c_1 - c_2 = 0$$

$$c'(t) = A \cdot c(t) \qquad A = \begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix}$$

$$c(0) = c_0 \qquad y' = \lambda y \Rightarrow y = y_0 e^{\lambda t}$$

$$c(t) = \underline{e^{At}} c_0 \qquad e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \Big]$$

$$A^2 = -6A \qquad e^{At} = \sum_{k=1}^{\infty} \frac{(-6)^{k-1} A\, t^k}{k!} + 1$$

$$A^3 = -6\,A^2 = (-6)^2 A$$
$$A^k = (-6)^{k-1} A$$

$$e^{At} = \left(\sum_{k=0}^{\infty} \frac{(-\delta)^k \cdot t^k}{k!} - 1 \right) \underbrace{\frac{A}{-\delta}} + 1$$

$$\left(e^{-\delta t} - 1 \right) \frac{A}{-\delta} + 1$$

$$\underline{C(t) = C_0 + \frac{1 - e^{-\delta t}}{\delta} A C_0}$$

$$x'(t) = \alpha x - \beta x y$$
$$y'(t) = \delta x y - \gamma y \qquad\qquad \alpha, \beta, \gamma, \delta > 0$$

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \text{EQUILIBRIUM}$$

$$\eta(x,y) = \delta x - \gamma \log(x) + \beta y - \alpha \log(y)$$

$$\frac{d}{dt} \eta(x(t), y(t)) = \begin{pmatrix} \partial_x \eta \\ \partial_y \eta \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} =$$

$$\langle \nabla \eta, F(x,y) \rangle = 0 = \delta \alpha x - \delta \beta x y - \gamma \alpha$$
$$+ \gamma \beta y + \beta \delta x y$$
$$- \beta \gamma y - \alpha \delta x + \alpha \gamma$$

$$\nabla \eta = \begin{pmatrix} \delta - \gamma/x \\ \beta - \alpha/y \end{pmatrix} \qquad = 0$$

$$\begin{cases} u'(t) = - \dfrac{v(t)}{n(t)} \\ v'(t) = \dfrac{u(t)}{n(t)} \end{cases} \qquad n(t) = \sqrt{u^2(t) + v^2(t)}$$

$$\frac{n^2}{2}$$

$$\frac{d}{dt}\left(\frac{n^2}{2} \right) = \frac{d}{dt}\left(\frac{u^2 + v^2}{2} \right) = \frac{2 u \cdot u' + 2 v \cdot v'}{2} =$$

$$u \cdot u' + v \cdot v' = u \cdot \left(-\frac{v}{n} \right) + v \left(\frac{u}{n} \right) = 0$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{array}{l} \cos(\theta)\, u^0 - \sin(\theta)\, v^0 \\ \sin(\theta)\, v^0 + \cos(\theta)\, u^0 \end{array} \qquad \theta = \frac{t}{\eta}$$

$$u' = -\frac{v}{\eta} - \alpha u$$

$$v' = \frac{u}{\eta} - \alpha v$$

$$\frac{d}{dt}\left(\frac{\eta^2}{2}\right) = \frac{d}{dt}\left(\frac{u^2 + v^2}{2}\right) = u \cdot u' + v \cdot v'$$

$$= -u \cdot \frac{v}{\eta} - \alpha u^2 + v \cdot \frac{u}{\eta} - \alpha v^2 = -\alpha(u^2 + v^2)$$

$$= -2\alpha \frac{u^2 + v^2}{2} = -2\alpha \left(\frac{\eta^2}{2}\right)$$

$$E = \frac{\eta^2}{2} \qquad \frac{d}{dt} E = -2\alpha E$$

$$E = e^{-2\alpha t} E_0 \qquad\qquad \alpha > 0$$

$$\frac{\eta^2}{2} = e^{-2\alpha t} \frac{\eta_0^2}{2} \Rightarrow \eta(t) = \underbrace{e^{-\alpha t}}_{\searrow} \eta_0$$

- $u'' = -\sin(u)$

$$\begin{cases} u' = v \\ v' = -\sin(u) \end{cases} \qquad \eta(t) = \frac{1}{2} v^2 - \cos(u)$$

$$\frac{d}{dt} \eta(t) = \langle \nabla \eta, F \rangle = \begin{pmatrix} \sin(u) \\ v \end{pmatrix} \cdot \begin{pmatrix} v \\ -\sin(u) \end{pmatrix} =$$

$$v \cdot \sin(u) - v \cdot \sin(u) = 0$$

$$\underline{\text{ROBERTSON}}$$

$$\frac{d}{dt}\left(\sum_i c_i\right) = 0$$

$$C_1 \xrightarrow{\;0.04\, C_1\;} C_2$$

$$C_1 \xleftarrow{\;10^4 \cdot C_2 C_3\;}$$

$$C_2 \xrightarrow{\;3 \cdot 10^7 C_2^2\;} C_3$$

$$U(x,t) : \Omega \times [0,T] \to \mathbb{R}^d$$

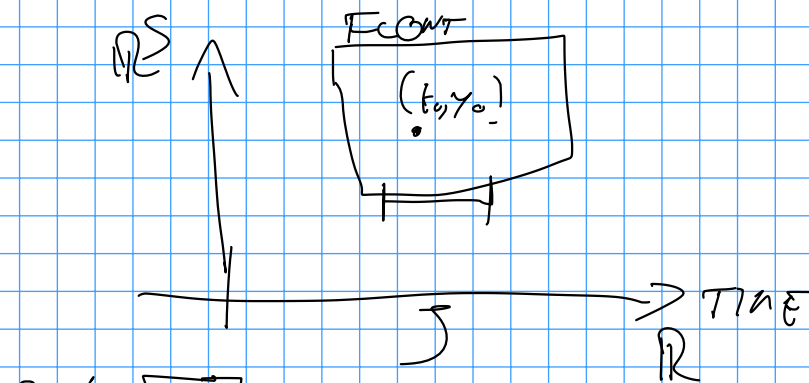$$\partial_t U + \partial_{xx}(a(U)) + \partial_x b(U) + c(U) = 0$$

$$\underline{U} = \mathbb{R}^N \qquad N \quad \# \text{ DOF IN SPACE}$$

$$\partial_{xx} a(U) \approx D_2 A(\underline{U})$$

$$\partial_t \underline{U} = f(\underline{U}) \qquad f(\underline{U}) = -(D_2 A(\underline{U}) + \_\_)$$

$$\partial_t U + \partial_x b(U) = 0 \qquad \underline{\underline{TVD}}$$

$$U_i^n \qquad TV(\underline{U}^n) := \sum_i |U_{i+1}^n - U_i^n|$$

$$\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}$$

• $y(t) = 0$

• $y(t) = \dfrac{t^2}{4}$ $\qquad \dfrac{t}{2}$

$$y' = 8\dfrac{t}{42}$$

$$F(t,y) : I \times \mathbb{R}^s \to \mathbb{R}^s$$

$$\|F(t,y) - F(t,z)\| \leq L \|y - z\|$$

UNIFORMLY IN t $\Rightarrow$ $\exists !$ SOLUTION

$$y \in C^1(I) \qquad \begin{cases} \dfrac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$F \in C^{p-1}$ p-differentiable

y is $p+1$ times differentiable

$$\frac{dy}{dt} = F$$

$$F^{(\ell+1)}(t,y) = \frac{\partial F^{(\ell)}}{\partial t}(t,y) + \frac{\partial F^{(\ell)}(t,y)}{\partial y}$$

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① ODE $I = [t_0, t_{end}]$ $y : I \to \mathbb{R}^S$

$$F : I \times \mathbb{R}^S \to \mathbb{R}^S$$

$$\begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$t_0 = t^0 < t^1 < t^2 < \dots \qquad < t^n < \dots \quad t^N = t_{end}$

$[t^n, t^{n+1}]$ $y(t^{n+1}) = y(t^n) + \int_{t^n}^{t^{n+1}} F(t, y(t)) dt$

$y^n \quad y^{n+1}$

$$\approx y(t^n) + F(t^n, y(t^n)) \cdot \underbrace{(t^{n+1} - t^n)}_{\Delta t^n}$$

$y^n \to y^{n+1}$ $y^{n+1} := y^n + \Delta t^n F(t^n, y^n)$

$$y^0 = y_0$$

$$c_1^{n+1} + c_2^{n+1} = c_1^n + \Delta t(-Sc_1^n + c_2^n)$$
$$+ c_2^n + \Delta t(-c_2^n + Sc_1^n)$$
$$= c_1^n + c_2^n = c_1^0 + c_2^0 = \text{MASS} \quad \forall n$$

$e_n = y(t^n) - y^n$ $y'(t^n)$

CONSISTENCY ERROR

$\varepsilon_n = y(t^{n+1}) - y(t^n) - \Delta t^n F(t^n, y(t^n)) =$

$$= \int_{t^n}^{t^{n+1}} y'(t) - y'(t') \, dt$$

$$\underbrace{\qquad\qquad}_{\text{COSTANT}}$$

$$\omega(f, \Delta t) := \max_{t, t'; |t-t'| < \Delta t} |f(t) - f(t')|$$

$$|\varepsilon_n| \le \int_{t^n}^{t^{n+1}} \underbrace{\omega(y', \Delta t^n)}_{\text{DOESN'T DEPEND ON } t} \, dt = \Delta t^n \cdot \omega(y', \Delta t^n).$$

$$e_{n+1} = y(t^{n+1}) - y^{n+1} = y(t^{n+1}) - y^n - \Delta t^n F(t^n, y^n)$$
$$+ y(t^n) + \Delta t\, F(t^n, y(t^n))$$
$$- y(t^n) - \Delta t\, F(t^n, y(t^n))$$

$$= \varepsilon_n + \underbrace{y(t^n) - y^n}_{e_n} + \Delta t \left[F(t^n, y(t^n)) - F(t^n, y^n) \right]$$

$$|e_{n+1}| \le |\varepsilon_n| + |e_n| + \Delta t\, L \, \underbrace{|y(t^n) - y^n|}_{e_n} \le \text{✶}$$

$$|F(t, y) - F(t, z)| \le L |y - z| \qquad \begin{array}{l}(\text{HYPOTHESIS ON} \\ F \text{ TO HAVE} \\ \text{UNIQUENESS})\end{array}$$

$$\text{✶} \le |\varepsilon_n| + (1 + \Delta t\, L) |e_n|. \qquad \Delta t$$

$$\le \cdots \le e^{L|t^{n+1} - t^0|} |e_0| + \sum_{i=0}^{n} e^{L(t^{n+1} - t^{i+1})} |\varepsilon_i| \le \text{✶✶}$$

$$|\varepsilon_n| \le \Delta t\, \omega(y', \Delta t)$$

$$\text{✶✶} \le e^{L|t^{n+1} - t^0|} |e_0| + \omega(y', \Delta t)' \frac{e^{L(t^{n+1} - t^0)} - 1}{L}.$$

$$\omega(y', \Delta t) = \omega(F(\cdot, y(\cdot)), \Delta t)$$

$$\left[\sum_{i=0}^{n} \Delta t\, e^{L\Delta t} = N \cdot \Delta t\, e^{L\Delta t} = \frac{T}{\Delta t} \Delta t\, e^{L\Delta t} \cdots \right]$$

$$w(y', \Delta t) = \max_{t, t'; |t-t'| < \Delta t} |y'(t) - y'(t')| = \quad \text{(AUTONO-} \\ \text{NOUS}$$

$$= \max_{t, t'; |t-t'| \le \Delta t} |F(y(t)) - F(y(t'))|$$

$$\le L |y(t) - y(t')| \xrightarrow[\Delta t \to 0]{} 0$$

$$\le L C \cdot \Delta t$$

$$\|= \\ \max y'$$

$$\boxed{|y(t^n) - y^n| \le C_0 |y^0 - y(t^0)| + \underset{\text{(highlighted)}}{\Delta t} \cdot C_1}$$

$$y'(t_0)$$

$$e_1 = |y^1 - y(t')| = |y^0 + \Delta t \, \overbrace{F(t^0, y^0)} \\ - (y(t^0) + \Delta t \, y'(t_0) + \frac{\Delta t^2}{2} y''(t_0) \\ + O(\Delta t^3)|$$

$$= \left| \frac{\Delta t^2}{2} y''(t_0) \right| + O(\Delta t^3) = O(\Delta t^2)$$

$$e_N \approx \sum_{i=1}^{N} |y(t^i) - y^i| \lesssim N \cdot \frac{\Delta t^2}{2} \max |y''|$$

$$= \frac{t_{end}}{\Delta t} \cdot \frac{\Delta t^2}{2} \max |y''|$$

$$= t_{end} \cdot \frac{\Delta t}{2} \max |y''|$$

DEF
P ORDER SCHEME
LARGEST P $\Delta t \to 0$
$|e_N| \le C \cdot \Delta t^P$

$$= C \cdot \Delta t = O(\Delta t)$$

FIRST ORDER

$$y'(t) = M y(t) \qquad M \in \mathbb{R}^{S \times S} \qquad \Delta t$$

$$y^n = y^{n-1} + \Delta t \, M \, y^{n-1} = (I + \Delta t \, M) y^{n-1}$$

$$= (I + \Delta t \, M)^n \, y^0$$

$$|y^n| \overset{?}{\underset{\leq}{}} |y^0| \quad \rightsquigarrow \quad \|(I + \delta t \Pi)^n\|$$

$$M \rightarrow \hat{A} = S^{-1} \Pi \cdot S \qquad \hat{A} = \begin{bmatrix} & & \\ & & \end{bmatrix} \in \mathbb{C}^{S \times S}$$

$$S^{-1} y^n = S^{-1} y^{n-1} + \delta t \, S^{-1} \underset{S \cdot S^{-1}}{\Pi} y^{n-1}$$

$$S^{-1} y^n =: \hat{y}^n$$

$$\hat{y}^n = \hat{y}^{n-1} + \delta t \, \hat{\Pi} \, \hat{y}^{n-1} \qquad y_S^n \rightarrow$$

$$= (I + \delta t \hat{\Pi}) \hat{y}^{n-1}$$

$$\boxed{y'(t) = q \, y(t)} \qquad q \in \mathbb{C} \qquad \begin{array}{c} q \in \mathbb{R} \\ \boxed{|q| \leq 0} \;\; Re(q) \leq 0 \\ q > 0 \end{array}$$

$$y^{n+1} = y^n + q \delta t \, y^n = (1 + q \delta t) y^n$$

$$= (1 + q \delta t)^{n+1} y^0$$

$$|y^{n+1}| \leq |y^n| \leq \cdots \qquad |y^{n+1}| = |y^n| \;\downarrow$$

$$|1 + q \delta t| \, |y^n| \leq |y^n|$$

$$\underline{|1 + q \delta t| \leq 1}$$

$$\underline{\underline{z \in \mathbb{C}}}$$
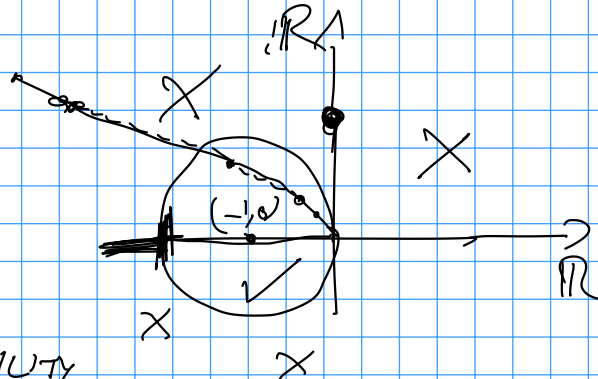
$$|1 + z| \leq 1$$

$$R(z) = 1 + z \qquad \text{STABILITY FUNCTION}$$

$$y^{n+1} = R(\delta t q) \cdot y^n \qquad \qquad z = \delta t q$$

$$\{ |R(z)| \leq 1 \} \quad \text{STABILITY REGION}$$

$$q \in \mathbb{R} \qquad |q| \delta t \leq C \quad \Rightarrow \quad \delta t \leq \frac{C}{|q|} \simeq \frac{C}{|\lambda|}$$

$$u' = v$$
$$v' = -u$$
$$u(0) = 0 \qquad v(0) = 1$$

$$r^2 = u^2 + v^2 = \text{CONSTANT IN TIME}$$

$$\frac{d}{dt}\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = 0$$

$$\underline{u} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{v}$$

$$\lambda^2 + 1 = 0$$
$$\lambda = \pm i$$

$$y^{n+1} = y^n + \Delta t\, f(t^{n+1}, y^{n+1}) \qquad \text{FIRST ORDER}$$

LINEAR STABILITY

$$y' = q y \qquad q \in \mathbb{C}$$

$$y^{n+1} = y^n + \Delta t q\, y^{n+1}$$

$$\left(1 - \underbrace{\Delta t q}_{z \in \mathbb{C}}\right) y^{n+1} = y^n$$

$$y^{n+1} = R(z) \cdot y^n = \overbrace{\frac{1}{1-z}}^{R(z)} y^n$$

$$\left\{ |R(z)| \leq 1 \right\} = \left\{ \frac{1}{|1-z|} \leq 1 \right\} = \left\{ |1-z| \geq 1 \right\}$$
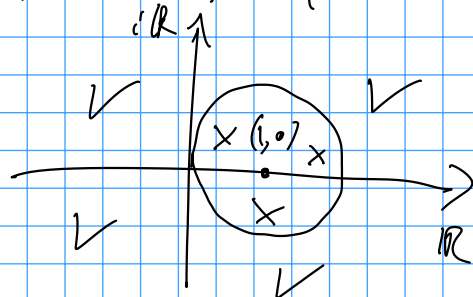
$$y^n = y^n + f(y^{n+1})\, \Delta t \qquad\qquad y^{n+1} \leftarrow u^{(k)}$$
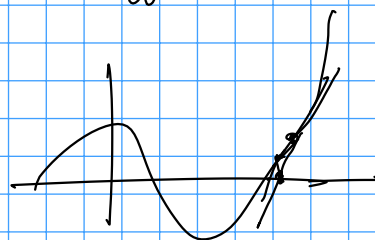
IMPLICIT EULER + POSITIVITY

$\forall \Delta t$

PDS
$$\frac{d}{dt} c_i = P_i(c) - D_i(c)$$

$$= \sum_j p_{ij}(c) - d_{ij}(c)$$

$$\underbrace{\qquad}_{\geq 0} \quad \underbrace{\qquad}_{\partial}$$

$$\boxed{p_{ij} = d_{ji}} \geq 0$$

1. CONSERVATION

$$\frac{d}{dt}\left(\sum_i c_i(t) \right) = \sum_i \frac{d}{dt} c_i(t) = \sum_i \sum_j p_{ij}(c) - d_{ij}(c)$$

$$= \sum_i \sum_j p_{ij} - \sum_i \sum_j d_{ji} = \sum_i \sum_j p_{ij} - p_{ij} = 0$$

- HOMEWORK   CHECK THAT   IMPLICIT EULER
  IS CONSERVATIVE.

2. POSITIVITY

$$\overbrace{c_i \to 0 \Rightarrow D_i(c) \to 0 \quad \forall j \; \forall i}$$

$$d_{ij}(c) \to 0$$

$$c_i(0) > 0 \Rightarrow c_i(t) \geq 0$$
$$\forall i \quad \forall t$$

PROOF BY CONTRADICTION $t_i \; c_i(t) < 0$

$$\exists \, t^* \quad c_i(t^*) = 0 \quad c_i'(t^*) < 0$$

$$c_i'(t^*) = \underbrace{P_i(c(t^*))}_{\geq 0} - \underbrace{D_i(c(t^*))}_{=0} \geq 0$$

IMPLICIT EULER + LINEAR PDS + POSITIVITY

$$y' = My \qquad M_{ii} < 0 \qquad M_{ij} \geq 0 \quad i \neq j$$

$$\sum_i M_{ij} = 0$$

INPLICIT EULER

$\underbrace{(I - \Delta t \Pi)}_{A} y^{n+1} = y^{n} \quad \overset{y^0 \geq 0}{\Longrightarrow} \quad y^{n+1} = \underbrace{A^{-1}}_{} y^{n}$

$A^{-1} \geq 0 \Longleftrightarrow \forall_{i,j} \, (A^{-1})_{ij} \geq 0$

1. $0 < A_{ii} = 1 - \underbrace{\Delta t \, \Pi_{ii}}_{<0} = 1 + \Delta t |\Pi_{ii}| = 1 + \Delta t \sum_{j \neq i} \Pi_{ji}$

$\boxed{|A_{ii}| > \sum_{j \neq i} |A_{ji}|} \quad \geq 0 \qquad \sum_{j} \Pi_{ji} = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad 0 > \Pi_{ii} = - \sum_{j \neq i} \overset{\geq 0}{\Pi_{ji}} < 0$

$\overset{\geq 0}{\Pi} \, \overset{\geq 0}{\Pi} \, \overset{\geq 0}{\Pi}$

$1 + \Delta t \sum_{j \neq i} \Pi_{ji} > \Delta t \sum_{j \neq i} \Pi_{ji} = \sum_{j \neq i} |A_{ji}|.$

A DIAG. DOMINANT BY COLUMN

• $A^{-1} \geq 0 \; ? \qquad$ JACOBI METHOD

$A = D + (A - D) \qquad D = diag(A)$

$\underline{A u = R} \Longleftrightarrow D u + (A - D) u = R$

$D u = R - \underbrace{(A - D) u} \quad \longleftarrow \checkmark \qquad \underline{u = D^{-1}(D - A) u + D^{-1} R}$

$D u^{(k)} = R - (A - D) u^{(k-1)}$

$u^{(k)} = D^{-1}(D - A) u^{(k-1)} + D^{-1} R$

$\| u - u^{(k)} \|_{\infty} = \| D^{-1}(D - A)(u^{(k-1)} - u) \|$

$\qquad\qquad \leq \| D^{-1}(D - A) \|_{\infty} \cdot \| u - u^{(k-1)} \|_{\infty}$

$\| B \|_{\infty} = \max_{i} \sum_{j} |B_{ji}| \qquad B = D^{-1}(D - A)$

$\qquad\qquad\qquad\qquad\qquad = (I - D^{-1} A)$

$$\left(I - D^{-1}A\right)_{ji} = \begin{bmatrix} 0 & \ddots \\ \ddots & \\ & \ddots & 0 \end{bmatrix}$$

$$\max_i \sum_{j \neq i} \frac{|A_{ji}|}{|A_{ii}|} = \max_i \frac{\sum_{j \neq i} |A_{ji}|}{|A_{ii}|} < 1$$

$$\|u - u^{(k)}\|_\infty < \|u - u^{(k-1)}\|_\infty$$

$$u^{(k)} \geq 0 \qquad\qquad Ay^{n+1} = y^n$$
$$\Downarrow$$

$$\underbrace{U^{(k)} = \underbrace{D^{-1}}_{\geq 0} \underbrace{(D-A)}_{\geq 0} \underbrace{u^{(k-1)}}_{\geq 0} + \underbrace{D^{-1}}_{\geq 0} \underbrace{y^n}_{\geq 0}}_{} \geq 0$$

$$\begin{cases} \Delta t \, \Pi_{ij} \geq 0 & i \neq j \\ 0 & i = j \end{cases}$$

$$u^{(k)} \longrightarrow u = y^{n+1} \geq 0. \qquad \boxtimes$$

$$|e_n| \leq e^{L|t^n - t^0|} |e_0| + \underline{\Delta t} \, \omega(y', \Delta t) \cdot \sum_{i=0}^{n-1} \underbrace{e^{L(t^n - t^{i+1})}}_{}$$

$$= \quad \cdots \quad + \omega(\cdots) \cdot \sum_{i=0}^{n-1} \int_{t^i}^{t^{i+1}} \underbrace{e^{L(t^n - t^{i+1})}}_{} dt$$

$$\leq \qquad\qquad + \omega(\cdots) \sum_{i=0}^{n-1} \overline{\int_{t^i}^{t^{i+1}} e^{L(t^n - t)} dt}$$

$$= \qquad\qquad + \omega \cdots \int_{t^0}^{t^n} e^{L(t^n - t)} dt$$

$$= \qquad\qquad + \omega \left[\frac{e^{L(t^n - t)}}{-L} \right]_{t^0}^{t^n}$$

$$= \qquad\qquad + \omega \, \frac{e^{L(t^n - t^0)} - 1}{L} \, .$$

$$\partial_t u + \partial_x F(u) = 0 \qquad u: \underset{\subset \mathbb{R}}{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$$

$$F(u) = \begin{cases} u & \text{TRASPORT} \\ \frac{u^2}{2} & \text{BURGERS} \\ \cdots \end{cases}$$

BURGERS
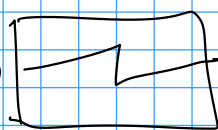

$$TV(u^n) := \sum_J |U_J - U_{J-1}|$$

HARTEN FORM : INCREMENTAL FORM
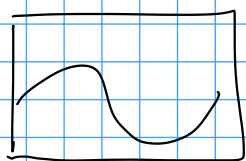
$$\partial_t U_J = C_{J+1/2}(U_{J+1} - U_J) - D_{J-1/2}(U_J - U_{J-1})$$

$$C_{J+1/2}, D_{J-1/2} > 0 \qquad \forall J$$

CHECK THAT IMPLICIT EULER DISCK
OF INCREMENTAL FORM PDE IS TVD.
I.E. $TV(U^{n+1}) \le TV(U^n) \quad \forall \Delta t > 0.$

$$U_J^{n+1} = U_J^n + \Delta t \Big[C_{J+1/2}(U_{J+1}^{n+1} - U_J^{n+1}) - D_{J-1/2}(U_J^{n+1} - U_{J-1}^{n+1}) \Big]$$

$$U_{J+1}^{n+1} - U_J^{n+1} = U_{J+1}^n - U_J^n + \Delta t \Big[C_{J+3/2}(U_{J+2}^{n+1} - U_{J+1}^{n+1}) - C_{J+1/2}(U_{J+1}^{n+1} - U_J^{n+1}) - D_{J+1/2}(U_{J+1}^{n+1} - U_J^{n+1}) + D_{J-1/2}(U_J^{n+1} - U_{J-1}^{n+1}) \Big]$$

$$\Big[1 + \Delta t\, C_{J+1/2} + \Delta t\, D_{J+1/2} \Big](U_{J+1}^{n+1} - U_J^{n+1}) =$$

$$\underset{\ge 0}{\Big\downarrow} \quad U_{J+1}^n - U_J^n + \Delta t \Big[C_{J+3/2}(U_{J+2}^{n+1} - U_{J+1}^{n+1}) + D_{J-1/2}(U_J^{n+1} - U_{J-1}^{n+1}) \Big]$$

$$\sum_j \left(1 + \Delta t\, C_{j+\frac{1}{2}} + \Delta t\, D_{j+\frac{1}{2}}\right) \left| U_{j+1}^{n+1} - U_j^{n+1} \right|$$

$$\leq \underbrace{\sum_j \left| U_{j+1}^n - U_j^n \right|}_{TV(U^n)} + \Delta t \sum_j C_{j+\frac{3}{2}} \left| U_{j+2}^{n+1} - U_{j+1}^{n+1} \right|$$

$$+ \Delta t \sum_j D_{j-\frac{1}{2}} \left| U_j^{n+1} - U_{j-1}^{n+1} \right|$$

$$\sum_j C_{j+\frac{3}{2}} \left| U_{j+2}^{n+1} - U_{j+1}^{n+1} \right| = \sum_j C_{j+\frac{1}{2}} \left| U_{j+1}^{n+1} - U_j^{n+1} \right|$$

$$\sum_j D_{j-\frac{1}{2}} \left| U_j^{n+1} - U_{j-1}^{n+1} \right| = \sum_j D_{j+\frac{1}{2}} \left| U_{j+1}^{n+1} - U_j^{n+1} \right|$$

$$\sum_j \left| U_{j+1}^{n+1} - U_j^{n+1} \right| + \Delta t \sum_j C_{j+\frac{1}{2}} \left| U_{j+1}^{n+1} - U_j^{n+1} \right| + \Delta t \sum_j D_{j+\frac{1}{2}} \left| U_{j+1}^{n} - U_j^{n} \right|$$

$$\leq TV(U^n) + \Delta t \sum_j C_{j+\frac{1}{2}} \left| U_{j+1}^{n+1} - U_j^{n+1} \right| + \Delta t \sum_j D_{j+\frac{1}{2}} \left| U_{j+1}^{n} - U_j^{n} \right|$$

$$\Rightarrow \quad TV(U^{n+1}) \leq TV(U^n) \qquad \boxtimes$$

$$y' = \Pi y$$

$$\frac{y^{n+1} - y^n}{\Delta t} = \Pi y^{n+1} \quad \Rightarrow \quad \underline{\left(\mathbb{I} - \Delta t\, \Pi\right) y^{n+1} = y^n}$$

$$e(\Delta t) = C \cdot \Delta t^p$$

$$\underbrace{\log(e(\Delta t))} = \log(C \cdot \Delta t^p) = \underline{\log(C)} + \underline{\log(\Delta t)} \cdot \boxed{p}$$

SLOPE

$$|y^{n+1}| < |y^n|$$

$$y^n \longmapsto y^{n+1} \qquad 2^{nd}\ \text{ORDER}\ ? \qquad \theta \in \mathbb{R}$$

$$y^* = y^n + \theta\, \Delta t\, F(t^n, y^n)$$

$$y^{n+1} = y^n + \Delta t \left[\frac{2\theta - 1}{2\theta} F(t^n, y^n) + \frac{1}{2\theta} F(t^n + \Delta t\, \theta, y^*) \right]$$

CONSISTENCY LOCAL ERROR y^n KNOWN

exact

ERROR y^{n+1} ?

$$y(t^{n+1}) = y^n + \Delta t\, y'(t^n)$$
$$+ \frac{\Delta t^2}{2} y''(t^n) + \mathcal{O}(\Delta t^3)$$

$$y(t^*) = y(t + \Theta \Delta t) = y^n + \Theta \Delta t\, y'(t^n) + \frac{\Theta^2 \Delta t^2}{2} y''(t^n)$$
$$+ \mathcal{O}(\Delta t^3)$$

$$y^* = y^n + \Theta \Delta t\, F(y^n) = y(t^*) + \mathcal{O}(\Delta t^2)$$

$$y^{n+1} = y^n + \Delta t \left[\frac{2\Theta - 1}{2\Theta} F(y^n) + \frac{1}{2\Theta} F(y^*) \right] \qquad y'(t^n)$$

$$= y^n + \Delta t \left[\frac{2\Theta - 1}{2\Theta} F(y^n) + \frac{1}{2\Theta} \left(F(y^n) + \frac{\partial F}{\partial y}(y^n) \Theta \Delta t\, F(y^n) \right) \right]$$

$$= y^n + \Delta t\, F(y^n) + \frac{\Theta}{2\Theta} \Delta t^2 \frac{d}{dt} F(y(t^n)) =$$

$$= y^n + \Delta t\, F(y^n) + \frac{\Delta t^2}{2} \frac{d}{dt} y'(t^n)$$

$$= y^n + \Delta t\, y'(t^n) + \frac{\Delta t^2}{2} y''(t^n) = y(t^{n+1}) + \mathcal{O}(\Delta t^3)$$

$$y(t^{n+1}) = y^n + y' \Delta t + \frac{\Delta t^2}{2} y'' + \frac{\Delta t^3}{6} y'''$$

$$\Rightarrow \text{GLOBAL ERROR } \mathcal{O}(\Delta t^2)$$

RUNGE-KUTTA

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

S STAGES $\quad A \in \mathbb{R}^{S \times S}$

$$c, b \in \mathbb{R}^S \qquad A = (a_{kj})_{k,j=1}^S$$

$$\begin{cases} y^{(k)} = y^n + \Delta t \sum_{j=1}^{S} a_{kj} F(t^n + \Delta t c_j, y^{(j)}) \quad k = 1, \dots, S \\ y^{n+1} = y^n + \Delta t \sum_{j=1}^{S} b_j F(t^n + \Delta t c_j, y^{(j)}) \end{cases}$$

t^{n+1} $\quad$ y^{n+1}

$t^n + \Delta t\, c_2$

$t^n + \Delta t\, c_4$ $\quad$ $t^n + \Delta t\, c_3$ $\quad$ $y^{(2)}$

$\qquad\qquad\qquad\qquad\qquad y^{(3)}$

$\qquad\qquad\qquad\qquad\qquad y^{(4)}$

t^n $\qquad\qquad\qquad\qquad\quad y^n$

EXPLICIT RK

$$y^{(k)} = y^n + \Delta t \sum_{j=1}^{k-1} a_{kj}\, F\!\left(y^{(j)}\right)$$

θ – SCHEME

$$\begin{cases} y^{(1)} = y^n + \Delta t\left[\, 0 \cdot F\!\left(y^{(1)}\right) + \theta \cdot F\!\left(y^{(2)}\right)\right] \\[4pt] y^{(2)} = y^n + \Delta t\left[\, \theta \cdot F\!\left(y^{(1)}\right) + 0 \cdot F\!\left(y^{(2)}\right)\right] \\[4pt] y^{n+1} = y^n + \Delta t\left[\, \dfrac{2\theta - 1}{2\theta}\, F\!\left(y^{(1)}\right) + \dfrac{1}{2\theta}\, F\!\left(y^{(2)}\right)\right] \end{cases}$$
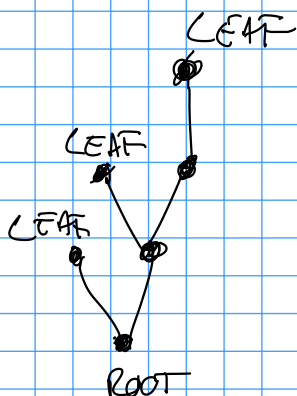
$$\begin{array}{c|cc} 0 & \theta & \theta \\ \theta & \theta & \theta \\ \hline & \dfrac{2\theta-1}{2\theta} & \dfrac{1}{2\theta} \end{array} \qquad\qquad \begin{array}{c|cc} 0 & & \\ \theta & \theta & \\ \hline & \dfrac{2\theta-1}{2\theta} & \dfrac{1}{2\theta} \end{array}$$

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ c_2 & a_{21} & 0 & 0 & 0 \\ c_3 & a_{31} & a_{32} & 0 & 0 \\ c_4 & a_{41} & a_{42} & a_{43} & 0 \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array} \qquad\qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$
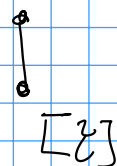
ROOTED TREES

LEAF

LEAF

LEAF

LEAF

ROOT

#NODES $\longrightarrow$ ORDER $\quad$ = ORDER

FOR WHICH WE ARE USING THIS CONDITION

ORDER 1

ORDER 2

$[2]$
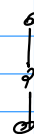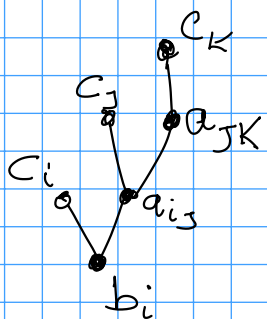
ORDER 3

$$\Phi(a_{ij}, b_j, c_j) = \frac{1}{\gamma} \qquad \gamma \in \mathbb{N}$$

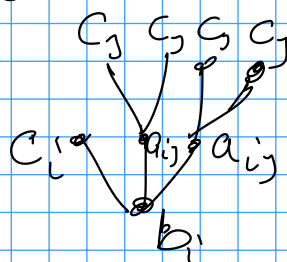$$\Phi \ ? \ \text{POLYNOMIAL}$$

$$\gamma \in \mathbb{N} \ ?$$

$$\forall$$

$$\sum_{i,J,K=1}^{s} b_i a_{ij} c_i a_{JK} c_J c_K = \Phi(a, b, c)$$
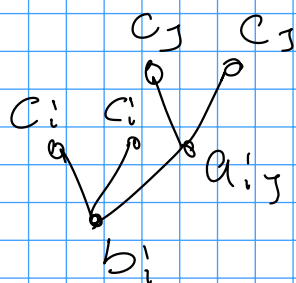
$$\Phi(A, b, c) = \sum_{i,J} b_i a_{ij} c_i^2 c_J^2$$

$$\boxed{\gamma = ?}$$

$$b_i a_{ij}^2 c_i c_J^4$$

$$2 = 1+1$$

$$5 = 4+1$$

$$\gamma = 80$$

$$\Phi(A, b, c) = \frac{1}{\gamma}$$

$$3$$

$$\tilde{i} \quad 1 \quad b_i \qquad \sum_i b_i = \frac{1}{1} \qquad \gamma = 1$$

$$1 \ c_i \qquad 2 \ b_i \qquad \sum_i b_i c_i = \frac{1}{2} \qquad \gamma = 2$$

$$c_i \quad c_i \qquad 3 \ b_i \qquad \sum_i b_i c_i^2 = \frac{1}{3}$$
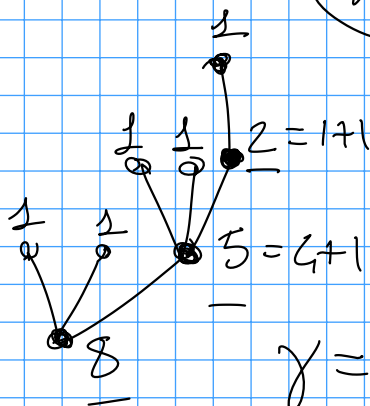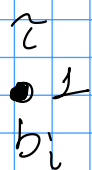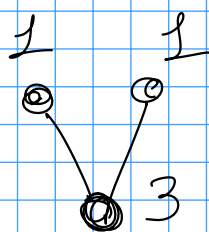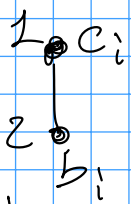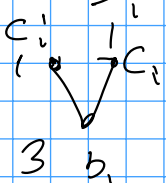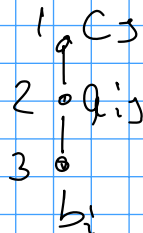
$$1 \ c_J \qquad 2 \ a_{ij} \qquad 3 \ b_i \qquad \sum_{iJ} b_i a_{ij} c_J = \frac{1}{6}$$

S Stages   EXPLICIT RK

$$\frac{S^2 + 3S - 2}{2}$$

S STAGES $\Rightarrow$ ORDER ≤ 4

$$\boxed{C_i = \sum_j a_{ij}} \quad \not\leftarrow$$
(LINEAR)
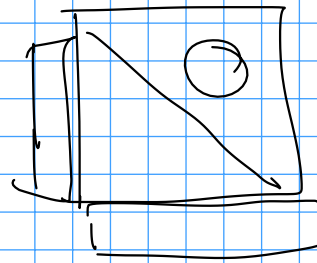
STABILITY FOR RK METHODS

$$y' = q y \qquad q \in \mathbb{C} \qquad z = q \cdot \Delta t$$

$$y^{m+1} = R(z) \cdot y^m \qquad S = \{ z : |R(z)| \leq 1 \}$$

$$\underline{Y} \in \mathbb{R}^S \qquad \underline{Y} = \underline{1} \cdot y^n + \Delta t \, \underline{\underline{A}} \cdot F(\underline{Y})$$

$$y^{(K)} = y^n + \Delta t \sum_{J=1}^S a_{KJ} \, F(y^{(J)})$$

$$F(y) = q \cdot y \qquad F(\underline{Y}) = q \cdot \underline{Y}$$

$$\boxed{Rk} \quad \underline{Y} = \underline{1} y^n + z \cdot \underline{\underline{A}} \cdot \underline{Y} \qquad \qquad \boxed{} \begin{bmatrix} \\ \\ \end{bmatrix} = \Box$$

$$(\underline{\underline{I}} - z \underline{\underline{A}}) \underline{Y} = y^n \underline{1}$$

$$y^{n+1} = y^n + \Delta t \sum_J b_J \, F(y^{(J)}) = y^m + \Delta t \cdot \underline{b}^T \cdot F(\underline{Y})$$

$$= y^n + \Delta t \cdot q \, \underline{b}^T \underline{Y}$$

$$y^{n+1} = y^n + \Delta t \, q \, \underline{b}^T \, \underline{Y} = y^n + \underbrace{\Delta t \, q}_{z} \, \underline{b}^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \cdot \underline{1} \, y^n$$

$$= \left[1 + z \cdot \underline{b}^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \cdot \underline{1} \right] \cdot y^n$$

$$y^{n+1} = R(z) \cdot y^n \qquad R(z) = 1 + z \, \underline{b}^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \underline{1}$$

$$\mathcal{I} = 1 \qquad z = \lambda \left(\sigma^{-1}\right)$$
$$A = 1$$

$R(z) \longrightarrow$ POLYNOMIAL FOR EXPLICIT RK
$\searrow$ FRACTION OF TWO POLY
$\qquad$ IMPLICIT RK

$$
\begin{array}{c|ccc}
0 & & & \\
2/3 & 2/3 & & \\
2/3 & 1/3 & 1/3 & \\
\hline
& \frac{1}{4} & 0 & \frac{3}{4}
\end{array}
$$

ORDER 1

$\sum b_i = 1 \Longleftrightarrow \frac{1}{4} + 0 + \frac{3}{4} = 1 \checkmark$

ORDER 2

$\sum_i b_i c_i = \frac{1}{2}$

$0 \cdot \frac{1}{4} + \frac{2}{3} \cdot 0 + \frac{\cancel{2}^1}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}_2} = \frac{1}{2} \checkmark$

ORDER 3

$\sum_i b_i c_i^2 = \frac{1}{3}$

$0 + 0 + \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{2}}{3} \cdot \frac{\cancel{2}}{\cancel{3}} = \frac{1}{3} \checkmark$

$\boxed{\gamma = c}$

$\sum_{i,j} b_i a_{ij} c_j = \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{2}}{3} \cdot \frac{1}{4} + \frac{\cancel{2}}{3} \cdot \frac{1}{3} \cdot \frac{1}{\cancel{4}_2} =$

$= \frac{1}{9} + \frac{1}{18} = \frac{3}{18} = \frac{1}{6} \checkmark$

ORDER 4

$\sum b_i c_i^3 = \frac{\cancel{3}}{4} \cdot \frac{\cancel{2}^1}{\cancel{3}} \cdot \frac{2}{3} = \frac{2}{9} \neq \frac{1}{4}$ NOT ORDER 4

STABILITY FUNCTION

$$R(z) = 1 + \Delta t \, \underline{b}^T \left(\underline{\underline{I}} - z\underline{\underline{A}}\right)^{-1} \underline{1}$$

$\left(\mathcal{I} - zA\right)^{-1}$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 2/3 & 0 & 0 \\ 1/3 & 1/3 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2/9 & 0 & 0 \end{bmatrix} \quad A^3 = \underline{\underline{0}}$$

$$A^k = \underline{\underline{0}}$$
$$k \geq 3$$

$$(I - zA)^{-1} = I + zA + z^2A^2 + z^3A^3 + \dots$$

$$\underbrace{\qquad}_{=0} \quad \underbrace{\qquad}_{=0}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ z\frac{2}{3} & 1 & 0 \\ z\frac{1}{3} + z^2\frac{2}{9} & z\frac{1}{3} & 1 \end{bmatrix}$$

$$R(z) = 1 + z \begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ z\frac{2}{3} & 1 & 0 \\ \frac{z}{3} + \frac{2}{9}z^2 & \frac{z}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 1 + z \begin{bmatrix} \frac{1}{4} + \frac{z}{4} + \frac{1}{6}z^2, & \frac{z}{4} + \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 1 + z \left(\frac{1}{4} + \frac{z}{4} + \frac{1}{6}z^2 + \frac{z}{4} + \frac{3}{4} \right)$$

$$= 1 + z + \frac{z^2}{2} + \frac{1}{6}z^3$$

IMPLICIT RK

$$y^{(k)} = y^n + \sum_{j=1}^{s} a_{kj} \, \underline{F(y^{(j)})} \qquad S \text{ EQUATIONS}$$

$$S \times DIM \text{ equations}$$

$$y(t) = \sum_{i=1}^{s} \varphi_i(t) \cdot \underbrace{y(t^n + c_i \Delta t)}_{y^{(i)}} + O(\Delta t^{p+1})$$

$$t^{(3)} = t^n + c_3 \Delta t$$

$$t^{(2)}$$

$$t^{(1)}$$
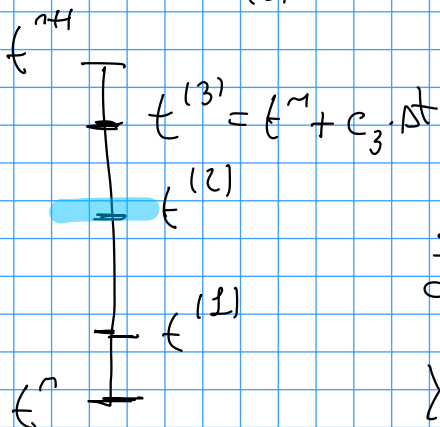
$$\varphi_i(t^{(j)}) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\frac{d}{dt} y = F(y)$$

$$y(t) = y^n + \int_{t^n}^{t} F(y(s)) \, ds$$

$$y^{(i)}\left(=y(t^{(i)})\right) = y^n + \int_{t^n}^{t^{(i)}} F(s, y(s))\,ds$$

$$\underset{\text{INTERPOLATE } F}{\approx} y^n + \int_{t^n}^{t^{(i)}} \sum_{J=1}^{S} \varphi_J(s) \cdot F(t^{(J)}, y^{(J)})\,ds$$

$$= y^n + \sum_{J=1}^{S} \underbrace{\int_{t^n}^{t^{(i)}} \varphi_J(s)\,ds}_{:= \Delta t\, a_{i,J}}\; F(t^{(J)}, y^{(J)})$$

$$= y^n + \sum_{J=1}^{S} \Delta t\, a_{i,J}\; F(t^{(J)}, y^{(J)})$$

$$y^{n+1} = y^n + \sum_{J=1}^{S} \underbrace{\int_{t^n}^{t^{n+1}} \varphi_J(s)\,ds}_{:= \Delta t\, b_J}\; F(t^{(J)}, y^{(J)})$$

$\overline{B, C, D}$ CONDITIONS $\not\!\!\Rightarrow$

$$B(p): \quad \sum_{i=1}^{S} b_i\, c_i^{z-1} = \frac{1}{z} \qquad \boxed{\forall\, z = 1, \ldots, P.}$$

t^{n+1} ... $t^{(i)} = t^n + \Delta t\, c_i$ ... t^n

$c_3 \to b_i$
c_2
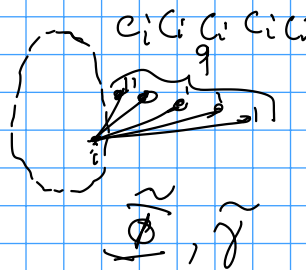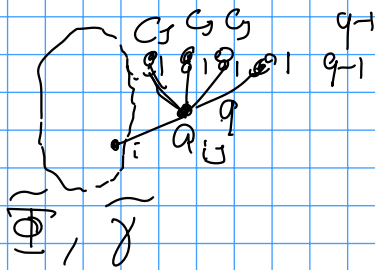c_1
0

$$\int_0^1 t^{z-1} = \left[\frac{t^z}{z}\right]_0^1 = \frac{1}{z}$$

QUADRATURE FORMULA b_i WEIGHTS c_i NODES

$$\int_0^1 t^{z-1} = \sum_i b_i\, c_i^{z-1} = \frac{1}{z}$$

$B(p)$ QUAD (b_i, c_i) IS EXACT <u>POLY</u> degree <u>UP TO P-1</u>

$$C(\eta): \quad \sum_{J=1}^{S} a_{ij}\, c_j^{q-1} = \frac{c_i^q}{q} \qquad \forall\, i = 1, \ldots, S \quad \forall\, q = 1, \ldots, \eta$$

$$\widetilde{\Phi}, \widetilde{\gamma} \qquad\qquad \widetilde{\widetilde{\Phi}}, \widetilde{\gamma}$$

$$\Phi = \sum_{ij} \widetilde{\Phi}_i \cdot a_{ij} \cdot c_j^{q-1} \qquad\qquad \Phi = \sum_i \widetilde{\widetilde{\Phi}}_i \cdot c_i^q$$

$$\gamma = \widetilde{\gamma} \cdot q \qquad\qquad \gamma = \widetilde{\gamma}$$

$$\gamma \cdot \Phi \;=\; \gamma \cdot \Phi \qquad\qquad \Phi = \frac{1}{\gamma}$$

$$\sum_{ij} \widetilde{\Phi}_i \, a_{ij} \, c_j^{q-1} \cdot q \, \cancel{\gamma} \;=\; \sum_i \widetilde{\widetilde{\Phi}}_i \, c_i^q \cdot \cancel{\gamma} \qquad \forall \widetilde{\Phi}_i$$

$$\Rightarrow \quad \sum_j a_{ij} \, c_j^{q-1} \cdot \cancel{q} \;=\; \frac{c_i^q}{q}$$

IF $\quad B(p), \quad C(\eta), \quad D(\zeta)$

$$p \le 2\eta + 2 \qquad p \le \zeta + \eta + 1 \qquad \Rightarrow \text{ THE METHOD IS OF ORDER } p.$$

$$B(2S) \quad C(S) \quad D(S) \qquad\qquad$$
$$B(2S-2) \quad \boxed{C(S)} \quad D(S-2) \qquad \text{LOBATTO IIIA}$$
$$2S-1 \quad \boxed{C(S)} \quad D(S-1)$$
$$C(S-1) \quad D(S)$$

$$y' = q\lambda \qquad q \in \mathbb{C} \qquad \Delta t \in \mathbb{R}^+$$

$$z = q\,\Delta t$$

$$y^{n+1} = R(z) \cdot y^n \qquad\qquad R(z) = 1 + z \,\underline{b}^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \underline{1}$$

$$\boxed{Re(q) < 0} \qquad Re(q) \ge 0\,?$$

$$\underline{\text{A-stability}}$$

IF $\quad \mathbb{C}^- \subset S \qquad |y^n| \le |y^n| \qquad \checkmark \quad \text{(X)} \quad \checkmark$

$$y' = -K(y - \cos(t)) \qquad K = 2000$$

$$K \to \infty \qquad y \neq \cos(t)$$

L-stability

$$R(z) < \begin{cases} \text{poly} & \text{(EXPLICIT)} \\ \text{FRACT. POLY} & \text{(IMPLICIT)} \end{cases}$$

$$\lim_{z \to \infty} R(z) = 0 \implies L\text{-stable} \qquad Re(q) < 0$$

$$y' = q\,y$$

$$\Delta t \gg 1 \quad (z \to -\infty) \qquad y^{nH} \to 0 \quad \Delta t \to +\infty$$

LEGENDRE

$$\overbrace{\qquad\qquad}^{} \quad |R| \leq 1 \quad \Big| \quad |R| \geq 1 \quad \longrightarrow$$

$$R(\infty) \qquad \lim_{z \to -\infty} |R(z)| \leq 1 \qquad R(\infty)$$

$$|R(\infty)| = 1 \qquad \lim_{z \to \infty} |R(z)| \geq 1$$
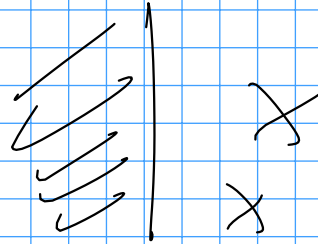
$$R(z) = \frac{a_0 + \cdots + a_m z^m}{b_0 + \cdots + b_n z^n}$$

$$z \to \infty \qquad \frac{a_m z^m}{b_n z^n}$$

$$\begin{array}{ll} m > n & \lim |R| \to \infty \\ m = n & \lim |R| \to \frac{a_m}{b_m} \\ m < n & \lim \to 0 \end{array}$$

CRANK-NICOLSON

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad R(z)$$

$$(I - zA)^{-1}$$

$$1 + z b^T \begin{bmatrix} 1 & 0 \\ \frac{z}{2-z} & \frac{2}{2-z} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \left| \begin{bmatrix} 1 & 0 \\ -\frac{z}{2} & 1-\frac{z}{2} \end{bmatrix} \right| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1-\frac{z}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{z}{2} & 1 \end{bmatrix}$$

$$1 + z \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{2+z}{2-z} \end{bmatrix} = 1 + z\left(\frac{1}{2} + \frac{2+z}{2(2-z)} \right) \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{z}{2}\left(\frac{2}{2-z}\right) & \frac{2}{2-z} \end{bmatrix}$$

$$= \frac{4 - 2z + 2z - z^2 + 2z + z^2}{2(2-z)} = \frac{z(2+z)}{z(2-z)} = \frac{2+z}{2-z}$$

$$\lim_{z \to \infty} R(z) = \frac{2+z}{2-z} = -1 \neq 0$$

$$t^{n+1} \quad \boxed{} \quad \begin{cases} u(\tau_i)' = f(\tau_i, u(\tau_i)) & i = 1, \dots, s \\ u(t^n) = \gamma^n & (1) \end{cases} \quad \begin{array}{c} s+1 \\ \text{CONDITIONS} \end{array}$$

$$\tau_i = t_n + c_i \, \Delta t$$

$$t^n \quad \boxed{}$$

$$\gamma^{n+1} = u(t^{n+1})$$

$$u(t) \in \mathbb{P}_s \qquad S = \# \text{STAGES}$$

$$a_{ij} = \int_0^{c_i} \ell_j(t) \, dt \qquad \ell_j(\tau_i) = \delta_{ij}$$

$$b_i = \int_0^1 \ell_i(t) \, dt \qquad (2) \qquad c_i = \tau_i$$

TH. COLLOCATION METHOD (1) IS AN INPLICIT RK WITH A, b, c (2) $\quad [0,1]$

$$u' \in \mathbb{P}_{S-1} \qquad u'(t^n + z \, \Delta t) = \sum_{j=1}^{s} \ell_j(z) \cdot u'(\tau_j)$$

$$\underbrace{}_{[0,1]}$$

$$G[t^n, t^{n+1}] \qquad [t^n, t^{n+1}]$$

$$u(t^n + z \, \Delta t) = \gamma^n + \Delta t \int_0^z u'(t^n + s \, \Delta t) \, ds$$

$$= \gamma^n + \Delta t \int_0^z \sum_j \ell_j(s) \cdot u'(\tau_j) \, ds$$

$$= \gamma^n + \Delta t \sum_j \int_0^z \ell_j(s) \, ds \cdot u'(\tau_j)$$

$$= \gamma^n + \Delta t \sum_j \int_0^z \ell_j(s) \, ds \; f(\tau_j, u(\tau_j))$$

$$u^{(K)} := u(t^n + c_K \, \Delta t) = u(\tau_K)$$

$$u^{(k)} = y^n + \Delta t \sum_J a_{kJ} \cdot f(\tau_J, u^{(k)})$$

$$y^{n+1} = u(t^{n+1}) = y^n + \Delta t \sum_J b_J f(\tau_J, u^{(J)}) \cdot \boxtimes$$

$$B(p) \qquad b_i = \int_0^1 \ell_J(s)\, ds \qquad \tau_i = t^n + c_i \Delta t$$

(c_i, b_i) QUADRATURE FORMULA

IS EXACT FOR POLY degree $P-1$

$$z = 1, \ldots, P \qquad \Rightarrow B(p)$$

$$\int_0^1 s^{z-1}\, ds = \sum_{J=1}^{S} c_J^{z-1} \cdot b_J$$

$$\underbrace{\left[\frac{s^z}{z}\right]_0^1 = \frac{1}{z}} \qquad B(p): \sum_J c_J^{z-1} \cdot b_J = \frac{1}{z}$$

$$\forall z = 1, \ldots, P \quad \boxtimes$$

$$C(S): \qquad \boxed{\sum_J a_{ij} c_J^{z-1}} = \frac{c_i^z}{z} \qquad \forall z = 1, \ldots, S$$

$\#$ OF STAGES

$$s^{z-1} \in \mathbb{P}_{S-1} \qquad \underline{s^{z-1} = \sum_J \ell_J(s) \cdot c_J^{z-1}}$$

$$\sum_J a_{ij} c_J^{z-1} = \sum_J \underbrace{\int_0^{c_i} \ell_J(s) \cdot c_J^{z-1}\, ds}_{a_{ij}} = \int_0^{c_i} s^{z-1}\, ds$$

$$= \left[\frac{s^z}{z}\right]_0^{c_i} = \frac{c_i^z}{z} \qquad \boxtimes$$

$\boxed{\text{THE}}$ $C(S)$ and $B(S+\gamma) \Rightarrow D(\gamma)$ $\qquad (3)$

$\boxed{\text{EXAMP}}$ $C(S)$ $B(2S) \Rightarrow D(S)$ LAGRANGE METHODS

$$B(p) = B(2S) \quad \wedge \quad C(\gamma) = C(S) \quad \wedge D(\gamma) = D(S)$$

$P = 2S \qquad\qquad P \le 2\gamma + 2 \qquad 2S \le 2S + 2 \checkmark$

$$2S = p \leq q + q + 1 = S + S + 1 = 2S + 1 \quad \checkmark$$

PROOF OF (3).

$$B(p): \quad \sum_i b_i c_i^{z-1} = \frac{1}{z} \qquad z = 1, \ldots, p$$

$$C(\eta): \quad \sum_j a_{ij} c_j^{z-1} = \frac{c_i^z}{z} \qquad z = 1, \ldots, \eta$$

$$D(\xi): \quad \sum_i b_i c_i^{q-1} a_{ij} = \frac{b_j}{q}(1 - c_j^q) \qquad q = 1, \ldots, \xi$$

$$d_j^{(z)} := \sum_i b_i c_i^{z-1} a_{ij} - \frac{b_j}{z}(1 - c_j^z)$$

GOAL $\quad d_j^{(z)} = 0 \qquad \forall j = 1, \ldots, S \qquad \forall z = 1, \ldots, \nu$

(4) $\quad \sum_{j=1}^{S} d_j^{(z)} \cdot c_j^{k-1} = 0 \qquad \forall k = 1, \ldots, S \qquad \forall z = 1, \ldots, \nu$

$$M_{jk} = \{ c_j^{k-1} \}_{\substack{j=1, \ldots, S \\ k=2, \ldots, S}} \qquad \underline{c_j \text{ DISTINCT}}$$

$$\underline{d}^{(z)} \cdot \underline{\underline{M}} = \underline{0}$$

$$\Rightarrow \quad \underline{d}^{(z)} = \underline{0} \quad (5) \qquad B(p): \sum_i b_i c_i^{z-1} = \frac{1}{z}$$

WE WILL PROVE (4) INSTEAD OF (5)

$$\sum_{j=1}^{S} d_j^{(z)} \cdot c_j^{k-1} = \sum_{j=1}^{S} \sum_{i=1}^{S} b_i c_i^{z-1} a_{ij} c_j^{k-1}$$

$$- \sum_{j=1}^{S} \frac{b_j}{z}(1 - c_j^z) \cdot c_j^{k-1} =$$

$$= \sum_{i=1}^{S} b_i c_i^{z-1} \sum_{j=1}^{S} a_{ij} c_j^{k-1} - \sum_{j=1}^{S} \frac{b_j}{z} c_j^{k-1} + \sum_{j=1}^{S} \frac{b_j}{z} c_j^{z+k-1}$$

$$= \sum_{i=1}^{S} b_i c_i^{z-1} \frac{c_i^k}{k} - \frac{1}{z} \sum_{j=1}^{S} b_j c_j^{k-1} + \frac{1}{z} \sum_j b_j c_j^{z+k-1}$$

$$= \frac{1}{k} \underbrace{\sum_{i=1}^{S} b_i c_i^{z+k-1}}_{\leq S + \nu - 1} - \frac{1}{z} \cdot \frac{1}{k} + \frac{1}{z} \cdot \frac{1}{k+z}$$

$$= \frac{1}{k} \cdot \frac{1}{k+z} - \frac{1}{z} \frac{1}{k} + \frac{1}{z} \cdot \frac{1}{k+z} = \frac{z - (k+z) + k}{k z (k+z)} = 0 \qquad \boxtimes$$

LOBATTO III A $\quad$ C (S) $\quad$ D(S-2) $\qquad \Leftarrow$ COLLOCATION

$\quad$ III B $\quad$ C (S-2) $\quad$ D(S)

$\quad$ III C $\quad$ C(S-1) $\quad$ D(S-1)

$$y^{(k)} = y^n + \Delta t \sum q_{kj} F(y^{(j)})$$

$$y^{n+1} = y^n + \Delta t \sum_j b_j F(y^{(j)})$$

• STIFFLY ACCURATE IFF $\quad a_{Sj} = b_j \quad \forall j$

$A \in \mathbb{R}^{S \times S} \Rightarrow$ NONLINEAR $\quad (S \times N) \times (S \times N)$

$\qquad N = \#$ DOF PROBLEM

$\mathcal{O}(N^2) \hookrightarrow \mathcal{O}(N^3)$

$O((SN)^2) \hookrightarrow O((S \cdot N)^3)$ $\qquad$ ORDER 6 $\qquad$ DIRK

$\mathcal{O}(S^2 \cdot N^2) \hookrightarrow \mathcal{O}(S^3 N^3)$ $\qquad 27 \cdot N^3$ $\qquad$ 6 STAGE $\quad$ $6 N^3$

$\underline{\underline{S \cdot \mathcal{O}(N^2)}} \hookrightarrow S \cdot \mathcal{O}(N^3)$ $\underline{\text{DIAGONALLY} \quad \text{IMPLICIT}}$

$\Downarrow$

CRANK NICOLSON

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

$U^{(1)} = y^n$

$y^{n+1} \Rightarrow U^{(2)} = y^n + \frac{\Delta t}{2} F(y^n) + \frac{\Delta t}{2} F(U^{(2)})$

$$R(y^{n+1}) = y^{n+1} - y^n - \frac{\Delta t}{2} F(y^n) - \frac{\Delta t}{2} F(y^{n+1}) = 0$$

$$J R(y^{n+1}) = I - \frac{\Delta t}{2} J F(y^{n+1})$$

IMPLICIT RK $\Rightarrow$ NEW LINEAR SYSTEMS !

$\qquad$ PROBLEM!

$$\begin{cases} y^{(k)} = y^n + \Delta t \sum_J a_{kJ} F(y^{(J)}) & \forall k \\ y^{n+1} = y^n + \Delta t \sum_J b_J F(y^{(J)}) & \boxed{DIRK} \end{cases}$$

$$\begin{cases} G^{(k)} = \Delta t\, F\left(y^n + \sum_{J=1}^{k} a_{kJ} G^{(J)}\right) \\ y^{n+1} = y^n + \sum_J b_J G^{(J)} \end{cases}$$

DIRK

$$G^{(k)} = \Delta t\, F\left(y^n + \sum_{J=1}^{k-1} \alpha_{kJ} G^{(J)}\right)$$

$$+ \Delta t \sum_{J=1}^{k} \gamma_{kJ} J_y F(y^n) \cdot G^{(J)}$$

$$\left[I - \Delta t\, \gamma_{kk} J_y F(y^n)\right] G^{(k)} = \Delta t\, F\left(y^n + \sum_{J=1}^{k-1} \alpha_{kJ} G^{(J)}\right)$$

$$+ \Delta t \sum_{J=1}^{k-1} \gamma_{kJ} J_y F(y^n) \cdot G^{(J)}$$

$$\gamma_{kk} = \gamma \quad \forall k$$

LINEAR STABILITY

IMEX RK   IMPLICIT   EXPLICIT

$$\partial_t u = \underbrace{F(u)}_{} + \underbrace{S(u)}_{}$$

EXPLICIT BAD GUY
STIFF
0.01   0.01

$$\Delta t \leq \Delta x \cdot C$$

$$\partial_t u + \partial_x F(u) = 0$$

EXPLICIT
0.0001

$$\partial_t u - \partial_{xx} u = 0$$

$$\Delta t \leq \Delta x^2 \cdot C$$

$$\partial_{xx} u$$

$$\Delta t \leq \Delta x^3$$

$$\partial_t u + \underbrace{\partial_x F(u)}_{EX} \underbrace{- \partial_{xx} u}_{IM} = 0$$

$$\partial_t q + F(u) + S(u) = 0$$

$$\frac{u^{n+1} - u^n}{\Delta t} + F(u^n) + S(u^{n+1}) = 0$$

IMEX RK

$\widetilde{A}, \widetilde{b}, \widetilde{c}$ EXPLICIT   COEFF

A, b, c IMPLICIT   COEFF

LINEAR STABILITY

$$Y' = \underbrace{\lambda_1 Y}_{\text{EXPLICIT}} + \underbrace{\lambda_2 Y}_{\text{IMPLICIT}}$$

$z_1 = \Delta t \, \lambda_1 \in \mathbb{C}$

$z_2 = \Delta t \, \lambda_2 \in \mathbb{C}$

$$y^{n+1} = R(z_1, z_2) \cdot y^n$$

$R(z) : \mathbb{C} \to \mathbb{C}$

$|R(z)| : \mathbb{C} \to \mathbb{R}$ 2D

$|R(z_1, z_2)| : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ 4D

$Re(z_1) = 0$ $z_1 \in i\mathbb{R}$ $|R(z_1, z_2)| : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$Im(z_2) = 0$ $z_2 \in \mathbb{R}$

● RUSSO 2000

$\to z_2 \in \mathbb{C}$; $|R(z_1, z_2)| \leq 1$ $\forall z_1 \in \{|1 + z_1| \leq 1\}$

$\to z_1 \in \mathbb{C}$; $|R(z_1, z_2)| \leq 1$ $\forall z_2 \in \mathbb{C}^-$

$y^n \longrightarrow y^{n+1}$ $y' = F(y)$

$y^{n-1}, y^{n-2}, y^{n-3} \ldots$

ADAMS $\left\{ F\left(t^{n-j}, y^{n-j} \right) \right\}_{j=1}^{k}$

GENERAL $\oplus$ $\left\{ y^{n-j} \right\}_{j=1}^{k}$

$$y^{n+1} + \sum_{j=1}^{k} \alpha_j \, y^{n-j+1} = \Delta t \sum_{j=0}^{k} \beta_j \, F\left(t^{n-j+1}, y^{n-j+1} \right)$$

STABILITY $~~~~~\cancel{y^{n+1} = R(z) \cdot y} = y^{n+1} = R(z) \cdot \hat{y} = \underline{R(z)^{n+1} \cdot y^0}$

LINEAR STABILITY $~~~~\boxed{y' = q\,y}~~~~ q \in \mathbb{C} \wedge \{\mathbb{R}e(q) \leq 0\}$

ANSATZ: $~~y^{n+1} = \zeta^{n+1} y^0 ~~~\uparrow~~~~~$ STABLE $(\Rightarrow)$ $|\zeta| \leq 1$

$$y^{n+1} + \sum_{j=1}^{k} \alpha_j \, y^{n-j+1} = \underset{z}{\underline{\Delta t \, q}} \sum_{j=0}^{k} \beta_j \cdot y^{n-j+1}$$

$\rightarrow \hspace{-0.3cm} D$ $~\underbrace{\zeta^{n+1} \cancel{y^0} + \sum_{j=1}^{k} \alpha_j \, \zeta^{n-j+1} \cdot \cancel{y_0} = z \cdot \sum_{j=0}^{k} \beta_j \, \zeta^{n-j+1} \cancel{y_0}}$

DIVIDED BY ζ^{n-k+1}

$\rightarrow \hspace{-0.3cm} D$ $~\underline{\zeta^k} + \sum_{j=1}^{k} \alpha_j \, \zeta^{k-j} = \underline{z} \cdot \sum_{j=0}^{k} \beta_j \, \zeta^{k-j}$

$\zeta_i (z)$ SOLUTION $\Uparrow$ $~~~~$ FOR $~~ i = 1, \dots, k$

$|\zeta_i (z)| \leq 1 \quad \forall i$ $~~~~~~~~~~~~~~~~~~~~~~ z \in \mathbb{C}$
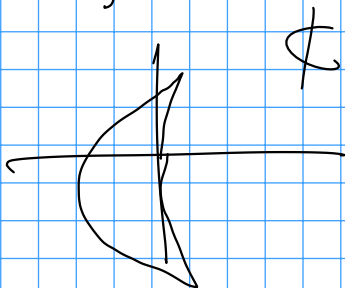
$\zeta \to z$

$$z = \frac{\zeta^k + \sum_{j=1}^{k} \alpha_j \, \zeta^{k-j}}{\sum_{j=0}^{k} \beta_j \, \zeta^{k-j}}$$

$\mathbb{C}$ $~~~~~~~~~~~ \zeta \in \mathbb{C} ~~~~ |\zeta| \leq 1$

$~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~ \boxed{z \in \mathbb{C}}$

ACCURACY

CONSISTENCY ERROR

$$y(t^{n+1}) + \sum_{j=1}^{k} \alpha_j \, y(t^{n-j+1}) - \Delta t \sum_{j=0}^{k} \beta_j \, F(t^{n-j+1}, y(t^{n-j+1}))$$

$$y^{l}(t^{n-j+1})$$

TAYLOR EXP IN t^{n+1}

$$= y(t^{n+1}) + \sum_{j=1}^{k} \alpha_j \sum_{\ell=0}^{P} \frac{(-j\Delta t)^{\ell}}{\ell!} \, y^{(\ell)}(t^{n+1})$$

$$- \Delta t \sum_{j=0}^{k} \beta_j \sum_{\ell=0}^{P-1} \frac{(-j\Delta t)^{\ell}}{\ell!} \, y^{(\ell+1)}(t^{n+1}) + \mathcal{O}(\Delta t^{P+1})$$

$$y(t^{n+1}) \left[1 + \sum_{j=1}^{k} \alpha_j \right] +$$

$$+ \Delta t \, y^{1}(t^{n+1}) \left[\sum_{j=1}^{k} -j \alpha_\ell + \sum_{j=0}^{k} \beta_j \right]$$

$$+ \ldots$$

FOR ORDER P $1 + \sum_{j=1}^{k} \alpha_j = 0$

$$\forall \ell = 1, \ldots, P \qquad \sum_{j=1}^{k} \alpha_j \frac{(-j)^{\ell}}{\ell!} - \sum_{j=0}^{k} \beta_j \frac{(-j)^{\ell-1}}{(\ell-1)!} = 0 \quad \boxtimes$$

ADAMS- BASHFERTth VALUES $F(t^{n-k+1}, y^{n-k+1})$

NO y^{n-k+1}

$$y(t^{n+1}) = y(t^n) + \int_{t^n}^{t^{n+1}} F$$

$$= y(t^n) + \int_{t^n}^{t^{n+1}} P_n(t) \, dt$$

$$P_{n,k}(t^{n-j+1}) = F^{n-j+1} = F(t^{n-j+1}, y^{n-j+1}) \quad \forall j = 1, \ldots, k.$$

$$P_{n,k} \in \mathbb{P}_{k-1}$$

$$P_{n,k}(t)_i = \sum F^{n-J+1} \cdot L_{n,J,K}(t)$$

$$L_{n,J,K}(t) = \prod_{\substack{i \neq J \\ i=1}}^{K} \frac{t - t^{n+1-i}}{t^{n+1-J} - t^{n+1-i}}$$

$$b_{n,J,K} = \frac{1}{t^{n+1} - t^n} \int_{t^n}^{t^{n+1}} L_{n,J,K}(t)\, dt \qquad J=1,\ldots,K$$

$$y^{n+1} = y^n + \Delta t^n \sum_{J=1}^{K} b_{n,J,K} F^{n-J+1}$$

$$F^{n+1} = F(t^{n+1}, y^{n+1})$$

STABILITY + ADAMS + HIGH ORDER = NO!

$$y^{n+1} + \sum_{J=1}^{K} \alpha_J y^{n-J+1} = \Delta t\, \beta_0 F(t^{n+1}, y^{n+1})$$

BACKWARD DIFFERENCE FORMULA

$$\underline{BDF2} \qquad y^{n+1} - \frac{3}{2} y^n + \underbrace{\frac{1}{2} y^{n-1}}_{} = \Delta t\, ?\, F(y^{n+1})$$

1 STEP IMPLICIT EULER $\mathcal{O}(\Delta t^2)$

N-1 STEPS   BDF2 $\mathcal{O}(\Delta t^3)$

$$\bigcirc \left(\Delta t^{\frac{2}{3}} \cdot \frac{N}{\Delta t} + \Delta t^2 \right) = \Delta t^2$$

RELAXATION RUNGE-KUTTA

- $\boxed{y' = F(y)}$ $\frac{1}{2} \langle y, y \rangle = $ KINETIC $= k$
                                                                  ENERGY

$$\frac{d}{dt} K(t) = \frac{d}{dt} \frac{1}{2} \langle y, y \rangle = \langle y, y' \rangle$$

$$= \underbrace{\langle y, F(y) \rangle \overset{(\leq)}{=} 0}_{\text{COMMON FOR ODES}}$$

$$\eta(y)$$

$$\frac{d}{dt} \eta(y) = \langle \partial_y \eta(y), y' \rangle$$

$$= \langle \underbrace{\partial_y \eta(y)}_{\substack{\text{ENTROPY} \\ \text{VARIABLES}}}, F(y) \rangle \overset{(\leq)}{=} 0$$

RELAX RK

$$\frac{c \mid A}{\mid b^T} \longrightarrow \begin{cases} y^{(k)} = y^n + \Delta t \sum_{J=1}^{S} a_{kJ} F(y^{(J)}) \\ y^{n+1} = y^n + \Delta t \sum_{J=1}^{S} b_J F(y^{(J)}) \end{cases}$$

$$y^{n+1} \approx y(t^{n+1})$$

$$\longrightarrow y_\gamma^{n+1} = y^n + \gamma \Delta t \sum_{J=1}^{S} b_J F(y^{(J)}) \qquad \gamma \in \mathbb{R}$$

$$\gamma \approx 1 \qquad\qquad\qquad\qquad\qquad\qquad \gamma \approx 1$$

ENERGY CASE $\quad \frac{d}{dt} \langle y, y \rangle \overset{(\leq)}{=} 0$

$$\langle y_\gamma^{n+1}, y_\gamma^{n+1} \rangle = \langle y^n, y^n \rangle + 2\gamma \Delta t \sum_{J=1}^{S} b_J \langle y^n, F(y^{(J)}) \rangle$$

$$+ \gamma^2 \Delta t^2 \sum_{J=1}^{S} \sum_{k=1}^{S} b_J b_k \langle F(y^{(J)}), F(y^{(k)}) \rangle$$

$$\frac{d}{dt} \langle y, y \rangle = 0 \Longleftarrow \quad \langle y, F(y) \rangle = 0$$

$$\leq 0 \Longleftarrow \quad \langle y, F(y) \rangle \leq 0$$

$$= \langle y^n, y^n \rangle + 2\gamma \Delta t \underbrace{\sum_{j=1}^{S} b_j \langle Y^{(j)}, F(Y^{(j)}) \rangle}_{= \, 0}$$

$$\overset{!}{=} \, 0 \quad \begin{cases} \cdots + 2\gamma \Delta t \sum_{j=1}^{S} b_j \langle y^n - y^{(j)}, \overset{(\xi)}{F(y^{(j)})} \rangle \\[4mm] + \gamma^2 \Delta t^2 \sum_{j=1}^{S} \sum_{k} b_j b_k \langle F^{(k)}, F^{(j)} \rangle \end{cases}$$

1 SOLUTION $\quad \gamma = 0 \quad$ (OBVIOUS)

2nd SOLUTION $\quad \gamma \neq 0$

$$\gamma = \frac{\Delta t \sum_{j=1}^{S} b_j \langle Y^{(j)} - y^n, F^{(j)} \rangle}{\Delta t^2 \sum_{j=1}^{S} \sum_{k=1}^{S} b_j b_k \langle F^{(k)}, F^{(j)} \rangle} \quad \Leftarrow$$

$$\longrightarrow \quad \Delta t^2 \sum_{j=1}^{S} \sum_{k=1}^{S} b_j b_k \langle F^{(k)}, F^{(j)} \rangle = \langle \Delta y, \Delta y \rangle$$

$$\gamma \Delta t \quad \leadsto \quad y_\gamma^{n+1} \approx y(t^n + \gamma \Delta t)$$

$$\Delta y_i = y^{n+1} - y^n = \Delta t \sum_{j=1}^{S} b_j F^{(j)}$$

GENERAL CASE $\qquad \underline{\eta(y)} \qquad \Big\| \quad \langle \partial_y \eta(y), F(y) \rangle$

$$\underbrace{t_+^n \gamma \Delta t}_{} \qquad \qquad \qquad \qquad \overset{= \, 0}{\leq \, 0}$$

$$\begin{cases} y^{(i)} = y^n + \Delta t \sum_{j=1}^{S} a_{ij} F(y^{(j)}) \\[4mm] y^{n+1} = y^n + \Delta t \sum_{j=1}^{S} b_j F(y^{(j)}) \end{cases}$$

$$\eta(y^{n+1}) - \eta(y^n) =$$

$$= \cancel{\eta(y^n)} - \cancel{\eta(y^n)} + \Delta t \sum_{j=1}^{S} b_j \langle F(y^{(j)}), \partial_y \eta(y^n) \rangle$$

$$+ \frac{\Delta t^2}{2} \sum_{i,j} b_i b_j \langle \underline{F(y^{(i)})}, \underline{\partial_y \eta(y^n)} \cdot \underline{F(y^{(j)})} \rangle$$

$$+ \mathcal{O}(\Delta t^3)$$

$$= \Delta t \sum_J b_J \underbrace{\langle F(y^{(J)}), \partial_y \eta(y^{(J)}) \rangle}_{\substack{(=0) \\ (\leq 0)}}$$

$$+ \Delta t \sum_J b_J \langle F^{(J)}, \underbrace{\partial_y \eta(y^n) - \partial_y \eta(y^{(J)})} \rangle$$

$$+ \frac{\Delta t^2}{2} \sum_{ij} b_J b_i \langle F^{(i)}, \eta_{yy}(y) F^{(J)} \rangle$$

$$+ \underbrace{O(\Delta t^3)}$$

$y \in \mathbb{R}^F$

$\eta : \mathbb{R}^I \to \mathbb{R}$

$\partial_y \eta : \mathbb{R}^I \to \mathbb{R}^I$

$\partial_{yy} \eta : \mathbb{R}^I \to \mathbb{R}^{I \times I}$

η CONVEX

$$\colorbox{cyan}{$\eta(y^{n+1}) - \eta(y^n) \stackrel{.}{=} \gamma \Delta t \sum_J b_J \langle F(y^{(J)}), \partial_y \eta(y^{(J)}) \rangle$}$$

∇ ↖ (arrow to below)

$$\underbrace{\phantom{\sum_J b_J \langle F(y^{(J)}), \partial_y \eta(y^{(J)}) \rangle}}_{\substack{(=0) \\ (\leq 0)}}$$

FIND A γ SUCH THAT

● DO WE HAVE A SOLUTION FOR γ ?

$\Omega'' > 0$

η CONVEX

$\Omega' < 0$

$\Omega' > 0$

$$\Omega(\gamma) = \eta\left(y^n + \gamma \Delta t \sum_J b_J F^{(J)}\right) - \eta(y^n)$$
$$- \gamma \Delta t \sum_J b_J \langle \partial_y \eta(y^{(J)}), F^{(J)} \rangle$$

$$\Omega'(0) < 0$$
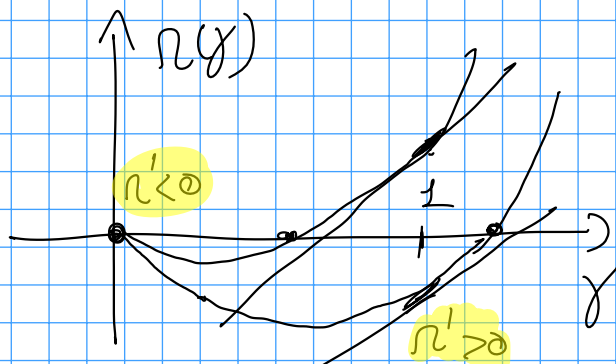
$$\Omega'(\gamma) = \left\langle \partial_y \eta\left(y^n + \gamma \Delta t \sum_J b_J F^J\right), \Delta t \sum_J b_J F^{(J)} \right\rangle$$
$$\longrightarrow \Delta t \sum_J b_J \langle \partial_y \eta(y^{(J)}), F^{(J)} \rangle$$

$$= \Delta t \sum_J b_J \left\langle F^{(J)}, \partial_y \eta\left(y^n + \gamma \Delta t \sum_J b_J F^J\right) - \partial_y \eta(y^{(J)}) \right\rangle$$

$$\partial_{yy} \eta \quad \text{POSITIVE DEFINITE}$$

$$\underline{\underline{\eta'(0)}} = -\Delta t \sum_i b_i \int_0^1 \eta_y\left(y^n + \nu \Delta t \sum_{k=1}^{s} a_{ik} F^{(k)}\right)$$

$$\left(F(y^{(i)}), \Delta t \sum a_{ij} F(y^{(j)})\right) d\nu$$

$$f(\nu) = \langle \eta_y\left(y^n + \nu \Delta t \sum_{k=1}^{s} a_{ik} F^{(k)}\right), F(y^{(i)}) \rangle$$

$$f'(\nu) = \eta_{yy}\left(y^n + \nu \Delta t \dots\right)\left(F(y^{(i)}), \Delta t \sum_{k=1}^{s} a_{ik} F^{(k)}\right)$$

$$f(\nu=1) = \langle \eta_y(y^{(i)}), F^{(i)} \rangle \qquad f(\nu=0) = \langle \eta_y(y^n), F^{(i)} \rangle$$

$$\eta'(0) = -\Delta t \sum_{i,j} b_i a_{ij} \int_0^1 \eta_{yy}(y^n + \nu \Delta t)\left(F(y^{(i)}), F(y^{(j)})\right)$$

$$= -\Delta t^2 \sum_{i,j} \underbrace{b_i a_{ij}}_{>0} \int_0^1 \underbrace{\eta_{yy}\left(F(y^n), F(y^n)\right)}_{>0}$$

$$+ O(\Delta t^3) \qquad < 0 \qquad \Delta t \text{ SMALL ENOUGH}$$

$$\int_0^1 \eta_{yy}\left(\overbrace{y^n + \nu \Delta t \sum b_i F^{(i)}}^{\approx y^n + O(\Delta t)}\right)\underbrace{\left(F(y^n), F(y^n)\right) d\nu}_{>0}$$

$$= -\int_0^1 d\nu\, \eta_{yy}(y^n)\left(F(y^n), F(y^n)\right) + O(\Delta t^3) < 0$$

IF $\quad \underbrace{\sum_{i,j} b_i a_{ij}}_{} > 0 \qquad \eta'' > 0 \qquad \Delta t \text{ SMALL}$

$C_i \bullet 1$
$\quad \downarrow \frac{1}{2}$
$\quad \bullet \frac{1}{2}$
b_i

$$\sum_{i,j} b_i a_{ij} = \sum_i b_i C_i = \underline{\underline{\frac{1}{2} > 0}} \qquad \boxed{C_i = \sum_j a_{ij}}$$

$\forall$ (AT LEAST) SECOND ORDER SCHEMES.

$$\eta'(0) < 0$$

$$\Sigma'(1) > 0$$

$$\eta'(1) = \Delta t \sum_i b_i \left\langle F(y^{(i)}), \eta'\left(y^n + \Delta t \sum_{k=1}^{S} b_k F^{(k)}\right) \right.$$
$$\left. - \eta'\left(y^n + \Delta t \sum_{j=1}^{S} a_{ij} F^{(j)}\right) \right\rangle$$

$$\nu = 0 \qquad \eta'\left(y^n + \Delta t \sum_j a_{ij} F^{(j)}\right)$$

$$\nu = 1 \qquad \eta'\left(y^n + \Delta t \sum_k b_k F^{(k)}\right)$$

$$f(\nu) = \left\langle \eta'\left(y^n + \nu \Delta t \sum_k b_k F^{(k)} + (1-\nu) \sum_j a_{ij} F^{(j)}\right), F^{(i)} \right\rangle$$

$$f'(\nu) = \eta''(\ast)\left(F^{(i)}, \Delta t \sum_j b_j F^{(j)} - \Delta t \sum_j a_{ij} F^{(j)}\right)$$

$$\underline{F^{(i)}}^T \qquad \underline{\eta''(\ast)} \qquad \underline{V}$$

$$\eta'(1) = \Delta t \sum_i b_i \int_0^1 \underbrace{F^{(i)}{}^T}_{\uparrow} \underbrace{\eta''(\ast)}_{\uparrow} \left(\Delta t \sum_j (b_j - a_{ij}) F^{(j)}\right)$$

$$= \Delta t^2 \sum_{i,j} b_i (b_j - a_{ij}) \underbrace{F(y^n) \cdot \eta''(y^n) \cdot F(y^n)}_{\geq 0} + O(\Delta t^3)$$

$$\eta'' : \mathbb{R}^I \to \mathbb{R}^{I \times I}$$

$$\text{IF} \sum_{i,j} b_i (b_j - a_{ij}) > 0$$

$$\underbrace{\sum_i b_i}_{1} \underbrace{\sum_j b_j}_{1} - \sum_{i,j} \overset{=\frac{1}{2}}{a_{ij} b_i} = 1 - \frac{1}{2} = \frac{1}{2} > 0 \checkmark$$

$$\gamma = O(\Delta t^{p-1})$$

$$O(\Delta t^{p+1}) \qquad \qquad \text{SCOPE} \quad O(\Delta t^2)$$

$$x = \frac{O(\Delta t^{p+1})}{\Theta(\Delta t^2)} \qquad \qquad \frac{O(\Delta t^{p+1})}{x} = O(\Delta t^2)$$

$O(\varepsilon) = x(q)$ $\qquad \lim_{\varepsilon \to 0} \frac{x}{\varepsilon} \in \mathbb{R}$

$\Theta(\varepsilon) = x(q)$ $\qquad 0 < \lim_{\varepsilon \to 0} \frac{x}{\varepsilon} < \infty$