ADER and DeC: arbitrarily high order (explicit) methods for PDEs and ODEs

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Based on: Han Veiga, M., Öffner, P. & Torlo, D. *DeC and ADER: Similarities, Differences and a Unified Framework.* J Sci Comput 87, 2 (2021). https://doi.org/10.1007/s10915-020-01397-5

Outline

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Motivation: high order accurate explicit method

We want to solve a hyperbolic PDE system for $u : \mathbb{R}^+ \times \Omega \to \mathbb{R}^D$

$$\partial_t u + \nabla_{\mathbf{x}} \mathcal{F}(u) = 0. \tag{1}$$

Or ODE system for $\boldsymbol{\alpha} : \mathbb{R}^+ \to \mathbb{R}^S$

$$\boxed{\partial_t \boldsymbol{\alpha} + F(\boldsymbol{\alpha}) = 0.} \tag{2}$$

Applications:

- Fluids/transport
- Chemical/biological processes

How?

- Arbitrarily high order accurate
-

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(1)

Or ODE system for

(2)

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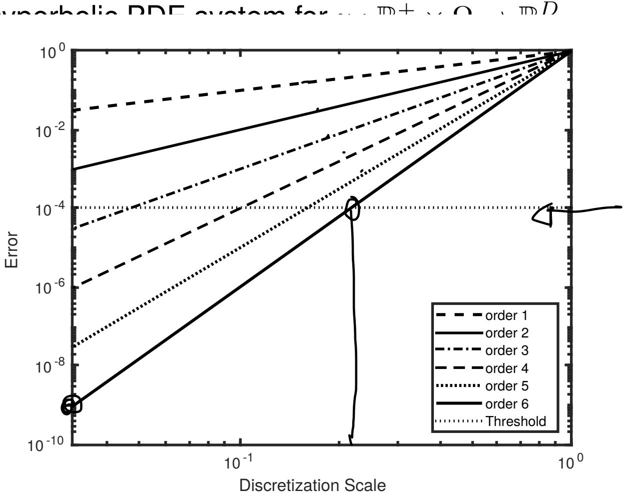
How?

- Arbitrarily high
-



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Applications:

- Fluids/transport
- Chemical/biological processes

How?

- Arbitrarily high order accurate
- Explicit (if nonstiff problem)

Classical time integration: Runge–Kutta

$\alpha_t = F(\alpha)$ $\qquad$ $\cancel{\alpha_t + F(\alpha) = 0}$

$$\boldsymbol{\alpha}^{(1)} := \boldsymbol{\alpha}^n, \tag{3}$$

$$\boldsymbol{\alpha}^{(k)} := \boldsymbol{\alpha}^n + \sum_{s=1}^{K} A_{ks} F\left(t^n + \overset{c_s}{\cancel{\beta_s}}\Delta t, \boldsymbol{\alpha}^{(s)}\right), \quad \text{for } k = 2, \dots, K, \tag{4}$$

$$\boldsymbol{\alpha}^{n+1} := \left(\sum_{k=1}^{K} \underset{b_k}{\cancel{\gamma_k}} \boldsymbol{\alpha}^{(k)}. \right) = \alpha^n + \sum_{s=1}^{K} b_k F(t^n + c_s \Delta t, \alpha^{(s)}) \tag{5}$$

Classical time integration: Explicit Runge–Kutta

$$\boldsymbol{\alpha}^{(k)} := \boldsymbol{\alpha}^n + \sum_{s=1}^{k-1} A_{ks} F\left(t^n + b_s \Delta t, \boldsymbol{\alpha}^{(s)}\right), \quad \text{for } k = 2, \dots, K.$$

- Easy to solve
- High orders involved:
  - Order conditions: system of many equations
  - Stages $K \geq d$ order of accuracy (e.g. RK44, RK65)

Classical time integration: Implicit Runge–Kutta

$$\boldsymbol{\alpha}^{(k)} := \boldsymbol{\alpha}^n + \sum_{s=1}^{K} A_{ks} F\left(t^n + b_s \Delta t, \boldsymbol{\alpha}^{(s)}\right), \quad \text{for } k = 2, \dots, K.$$

Dim · K

- More complicated to solve for nonlinear systems
- High orders easily done:
  - Take a high order quadrature rule on $[t^n, t^{n+1}]$
  - Compute the coefficients accordingly, see Gauss–Legendre or Gauss–Lobatto polynomials
  - Order up to $d = 2K - 1$

ADER and DeC

Two iterative explicit arbitrarily high order accurate methods.

- ADER[1] for hyperbolic PDE, after a first analytic more complicated approach.
- Deferred Correction (DeC): introduced for explicit ODE[2], extended to implicit ODE[3] and to hyperbolic PDE[4].

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[1] M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. Journal of Computational Physics, 227(18):8209–8253, 2008.

[2] A. Dutt, L. Greengard, and V. Rokhlin. Spectral Deferred Correction Methods for Ordinary Differential Equations. BIT Numerical Mathematics, 40(2):241–266, 2000.

[3] M. L. Minion. Semi-implicit spectral deferred correction methods for ordinary differential equations. Commun. Math. Sci., 1(3):471–500, 09 2003.

[4] R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. Journal of Scientific Computing, 73(2):461–494, Dec 2017.

Outline

DeC high order time discretization: $\mathcal{L}^2$

High order in time: we discretize our variable on $[t^n, t^{n+1}]$ in M substeps ($\boldsymbol{\alpha}^m$).

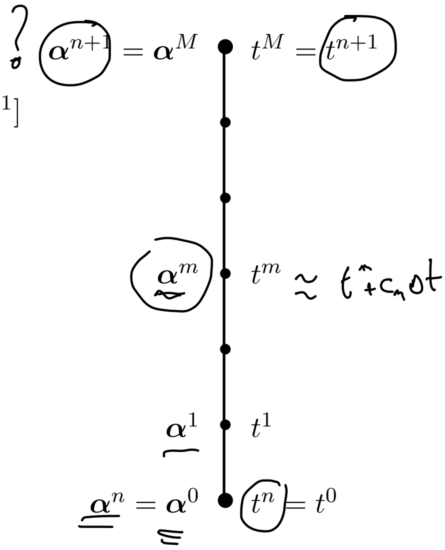
$$\partial_t \boldsymbol{\alpha} + F(\boldsymbol{\alpha}(t)) = 0.$$

Thanks to Picard–Lindelöf theorem, we can rewrite

$$\boldsymbol{\alpha}^m = \boldsymbol{\alpha}^0 - \int_{t^0}^{t^m} F(\boldsymbol{\alpha}(t))dt.$$

and if we want to reach order $r+1$ we need $M = r$.

EQUISPACED POINTS. $M \to$ ORDER M | G.L. $M \to 2M-1$

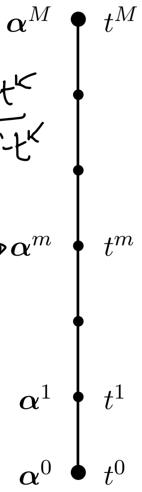


DeC high order time discretization: $\mathcal{L}^2$

More precisely, for each σ we want to solve $\mathcal{L}^2(\boldsymbol{\alpha}^{n,0},\dots,\boldsymbol{\alpha}^{n,M}) = 0$, where

$$\mathcal{L}^2(\boldsymbol{\alpha}^0,\dots,\boldsymbol{\alpha}^M) = \begin{pmatrix} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \sum_{r=0}^M \int_{t^0}^{t^M} \overbrace{F(\boldsymbol{\alpha}^r)\underline{\underline{\varphi_r(s)}}}\mathrm{d}s \\ \vdots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\boldsymbol{\alpha}^r)\varphi_r(s)\mathrm{d}s \end{pmatrix}$$

$$\varphi_r = \prod_{k\neq r} \frac{t-t^k}{t^r-t^k}$$



- $\mathcal{L}^2 = 0$ is a system of $M \times S$ coupled (non)linear equations
- $\mathcal{L}^2$ is an implicit method
- Not easy to solve directly $\mathcal{L}^2(\underline{\boldsymbol{\alpha}}^*) = 0$
- High order ($\geq M+1$), depending on points distribution

DeC high order time discretization: $\mathcal{L}^2$

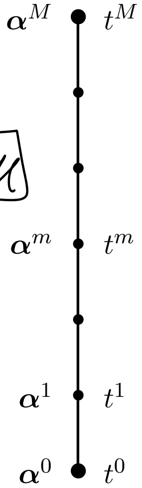
More precisely, for each σ we want to solve $\mathcal{L}^2(\boldsymbol{\alpha}^{n,0},\dots,\boldsymbol{\alpha}^{n,M}) = 0$, where

$\to \int_{t^0}^{t^m} \varphi_r(s)\,ds$

$$\mathcal{L}^2(\boldsymbol{\alpha}^0,\dots,\boldsymbol{\alpha}^M) = \begin{pmatrix} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \Delta t \sum_{r=0}^M \theta_r^M F(\boldsymbol{\alpha}^r) \\ \vdots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \Delta t \sum_{r=0}^M \theta_r^1 F(\boldsymbol{\alpha}^r) \end{pmatrix}$$

$\Theta =$

$\to \int_{t^0}^{t^1} \varphi_r(s)\,ds$



- $\boxed{\mathcal{L}^2 = 0}$ is a system of $M \times S$ coupled (non)linear equations
  $\to$ DIM $\alpha^l \in \mathbb{R}^S$
- $\mathcal{L}^2$ is an implicit method
- Not easy to solve directly $\mathcal{L}^2(\underline{\boldsymbol{\alpha}}^*) = 0$ $\to$ NON LINEAR SOLVER ?
- High order ($\geq M+1$), depending on points distribution

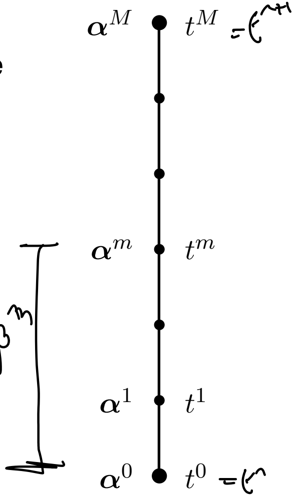
DeC low order time discretization: $\mathcal{L}^1$

Instead of solving the implicit system directly (difficult), we introduce a first order scheme $\mathcal{L}^1(\boldsymbol{\alpha}^{n,0},\dots,\boldsymbol{\alpha}^{n,M})$:

$$\mathcal{L}^1(\cancel{\boldsymbol{\alpha}^0},\dots,\boldsymbol{\alpha}^M) = \begin{pmatrix} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \Delta t \beta^M F(\boldsymbol{\alpha}^0) \\ \vdots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \Delta t \beta^1 F(\boldsymbol{\alpha}^0) \end{pmatrix}$$

$$\beta^m := \frac{t^m - t^0}{t^M - t^0}$$

- First order approximation
- Explicit Euler
- Easy to solve $\mathcal{L}^1(\underline{\boldsymbol{\alpha}}) = 0$



Deferred Correction[5]

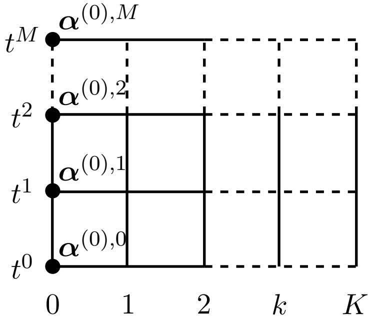
How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

- $\mathcal{L}^1(\underline{\boldsymbol{\alpha}}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\boldsymbol{\alpha}}) = 0$, high order $M+1$.

$$\boldsymbol{\alpha}^{0,(k)} := \boldsymbol{\alpha}(t^n), \quad k = 0, \dots, K,$$

$$\boldsymbol{\alpha}^{m,(0)} := \boldsymbol{\alpha}(t^n), \quad m = 1, \dots, M$$

$\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{\alpha}}^{(k-1)})$ with $k = 1, \dots, K$.



Theorem (Convergence DeC)

- $\mathcal{L}^2(\underline{\boldsymbol{\alpha}}^*) = 0$
- *If $\mathcal{L}^1$ coercive with constant C_1*
- *If $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz with constant $C_2 \Delta t$*

Then $\|\underline{\boldsymbol{\alpha}}^{(K)} - \underline{\boldsymbol{\alpha}}^*\| \leq C(\Delta t)^K$

[5] A. Dutt, L. Greengard, and V. Rokhlin. BIT Numerical Mathematics, 40(2):241–266, 2000.

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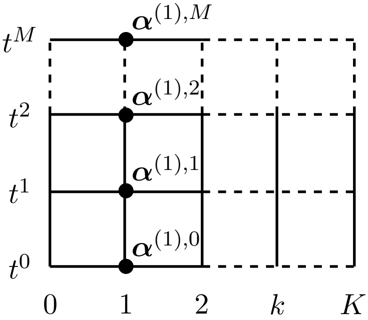

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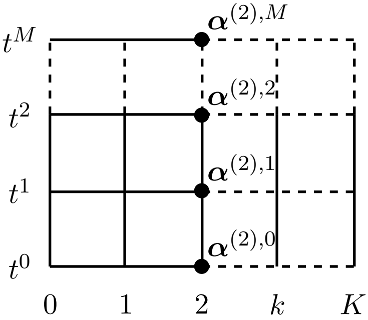

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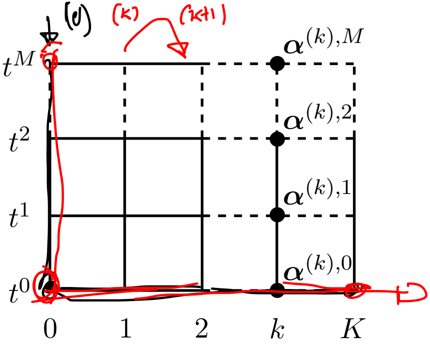

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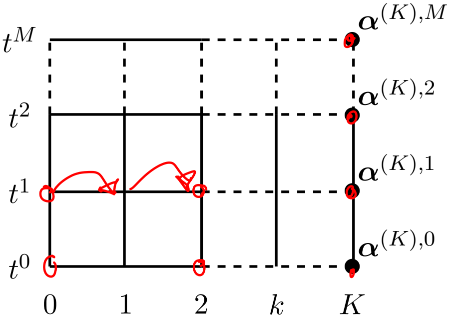

Theorem (Convergence DeC)

- $\boxed{\mathcal{L}^2(\underline{\boldsymbol{\alpha}}^*) = 0}$ $\quad (\alpha^{*,0}, \alpha^{*,1}, \dots \alpha^{*,M})^T$
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Proof.

Let f^* be the solution of $\mathcal{L}^2(\underline{\alpha}^*) = 0$. We know that $\mathcal{L}^1(\underline{\alpha}^*) = \mathcal{L}^1(\underline{\alpha}^*) - \mathcal{L}^2(\underline{\alpha}^*)$, so that

$$\|\alpha^{(k)} - \alpha^*\| \underset{\text{COERC.}}{\leq} C_1 \|\mathcal{L}^1(\alpha^{(k)}) - \mathcal{L}^1(\alpha^*)\| \overset{\text{DEF. ITER}}{\underset{\alpha^* \text{ SOL } \mathcal{L}^2=0}{=}}$$

$$= C_1 \|\mathcal{L}^1(\alpha^{(k-1)}) - \mathcal{L}^2(\alpha^{(k-1)}) - \mathcal{L}^1(\alpha^*) + \mathcal{L}^2(\alpha^*)\| \leq$$

$$\underset{\text{LIP. CONT}}{\leq} C_1 \Delta t C_2 \|\alpha^{(k-1)} - \alpha^*\|$$

$$\|\alpha^{(k)} - \alpha^*\| \leq (C_1 C_2)^K \cdot \Delta t^K \|\alpha^{(0)} - \alpha^*\|,$$

$\square$

DeC – Proof

Proof.

Let f^* be the solution of $\mathcal{L}^2(\underline{\boldsymbol{\alpha}}^*) = 0$. We know that $\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^*) = \mathcal{L}^1(\underline{\boldsymbol{\alpha}}^*) - \mathcal{L}^2(\underline{\boldsymbol{\alpha}}^*)$, so that

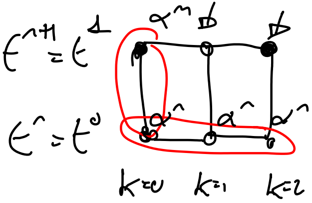
$$\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k+1)}) - \mathcal{L}^1(\underline{\boldsymbol{\alpha}}^*) = \left(\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k)}) - \mathcal{L}^2(\underline{\boldsymbol{\alpha}}^{(k)})\right) - \left(\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^*) - \mathcal{L}^2(\underline{\boldsymbol{\alpha}}^*)\right)$$

$$\begin{aligned}
{\color{red}C_1}||\underline{\boldsymbol{\alpha}}^{(k+1)} - \underline{\boldsymbol{\alpha}}^*|| \leq & ||\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k+1)}) - \mathcal{L}^1(\underline{\boldsymbol{\alpha}}^*)|| = \\
= & ||\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k)}) - \mathcal{L}^2(\underline{\boldsymbol{\alpha}}^{(k)}) - (\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^*) - \mathcal{L}^2(\underline{\boldsymbol{\alpha}}^*))|| \leq \\
\leq & {\color{red}C_2 \Delta}||\underline{\boldsymbol{\alpha}}^{(k)} - \underline{\boldsymbol{\alpha}}^*||.
\end{aligned}$$

$$||\underline{\boldsymbol{\alpha}}^{(k+1)} - \underline{\boldsymbol{\alpha}}^*|| \leq \left(\frac{C_2}{C_1}\Delta\right) ||\underline{\boldsymbol{\alpha}}^{(k)} - \underline{\boldsymbol{\alpha}}^*|| \leq \left(\frac{C_2}{C_1}\Delta\right)^{k+1} ||\underline{\boldsymbol{\alpha}}^{(0)} - \underline{\boldsymbol{\alpha}}^*||.$$

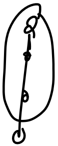
After K iteration we have an error at most of $\left(\frac{C_2}{C_1}\Delta\right)^K ||\underline{\boldsymbol{\alpha}}^{(0)} - \underline{\boldsymbol{\alpha}}^*||$. □

$$\alpha^{(0),m} = \alpha^{(k),0} = \alpha(t^n) = \alpha^n \quad \forall m \; \forall k$$



$$\mathcal{L}^2 = \alpha^1 - \alpha^0 - \Delta t \sum_{r=0}^{1} \theta_r^1 F(\alpha^r)$$

$$= \alpha^1 - \alpha^0 - \Delta t \, \frac{F(\alpha^0)+F(\alpha^1)}{2}$$



$$\mathcal{L}^1 = \alpha^1 - \alpha^0 - \Delta t \, F(\alpha^0)$$

$$\alpha^{(1),1} \Rightarrow \mathcal{L}^1(\alpha^{(1)}) = \mathcal{L}^1(\alpha^{(0)}) - \mathcal{L}^2(\alpha^{(0)})$$

$$\alpha^{(1),1} - \cancel{\alpha^0} - \cancel{\Delta t F(\alpha^0)} = \underbrace{\alpha^{(0),1}}_{\alpha^0} - \cancel{\alpha^0} - \cancel{\Delta t F(\alpha^0)} - \left(\cancel{\alpha^1} - \cancel{\alpha^0} - \Delta t \, \frac{F(\alpha^0)+F(\alpha^1)}{2}\right)$$

DeC: Second order example

$$\alpha^{(1),1} = \alpha^0 + \Delta t F(\alpha^0) \leftarrow$$

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$\alpha^{(2),1} = ?$$

$$\mathcal{L}^1(\alpha^{(2)}) = \mathcal{L}^1(\alpha^{(1)}) - \mathcal{L}^2(\alpha^{(1)})$$

$$\alpha^{(2),1} - \cancel{\alpha^0} - \cancel{\Delta t F(\alpha^0)} = \cancel{\alpha^{(1),1}} - \cancel{\alpha^0} - \cancel{\Delta t F(\alpha^0)} - \left[\cancel{\alpha^{(1),1}} - \alpha^0 - \Delta t \frac{F(\alpha^0) + F(\alpha^{(1)})}{2} \right]$$

$$\alpha^{(2),1} = \alpha^0 + \Delta t \frac{1}{2} \left(F(\alpha^0) + F(\alpha^{(1),1}) \right) \Big] \leftarrow$$

DeC: Second order example

$$(t^m - t^0)/(t^M - t^0)$$

$$\frac{\text{node}[m] - \text{node}[0]}{\underbrace{\text{node}[M+1]}_{=1} - \underbrace{\text{node}[0]}_{=0}} = \underline{\underline{\text{node}[m]}}$$

DeC: Second order example

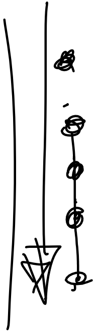
IF

COMP

$$K \cdot (M) = \text{STAGES RK}$$

$$\| $$

$$p \cdot (p-1) = p^2 - p$$



COST K

In practice

$$\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{\alpha}}^{(k-1)}), \qquad k = 1, \dots, K,$$

For $m = 1, \dots, M$

$$\boldsymbol{\alpha}^{(k),m} - \cancel{\boldsymbol{\alpha}^0} - \cancel{\beta^m \Delta t F(\boldsymbol{\alpha}^0)} - \cancel{\boldsymbol{\alpha}^{(k-1),m}} + \cancel{\boldsymbol{\alpha}^0} + \cancel{\beta^m \Delta t F(\boldsymbol{\alpha}^0)}$$

$$+ \cancel{\boldsymbol{\alpha}^{(k-1),m}} - \boldsymbol{\alpha}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{\alpha}^{(k-1),r}) = 0$$

Simplification of DeC for ODE

In practice

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$$\boldsymbol{\alpha}^{(k),m} - \cancel{\boldsymbol{\alpha}^0 - \beta^m \Delta t F(\boldsymbol{\alpha}^0)} - \boldsymbol{\alpha}^{(k-1),m} + \cancel{\boldsymbol{\alpha}^0 + \beta^m \Delta t F(\boldsymbol{\alpha}^0)}$$

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$$+ \cancel{\boldsymbol{\alpha}^{(k-1),m}} - \boldsymbol{\alpha}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{\alpha}^{(k-1),r}) = 0$$

$$\boxed{\boldsymbol{\alpha}^{(k),m} - \boldsymbol{\alpha}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{\alpha}^{(k-1),r}) = 0.}$$

$\forall k = 1, \dots, K$
$\forall m = 1, \dots, M$

DeC and residual distribution

Deferred Correction + Residual distribution

- Residual distribution (FV $\Rightarrow$ FE) $\Rightarrow$ High order in space
- Prediction/correction/iterations $\Rightarrow$ High order in time
- Subtimesteps $\Rightarrow$ High order in time

$$U_\xi^{m,(k+1)} = U_\xi^{m,(k)} - |C_p|^{-1} \sum_{\mathrm{E}|\xi\in\mathrm{E}} \left(\int_{\mathrm{E}} \Phi_\xi \left(U^{m,(k)} - U^{n,0} \right) \mathrm{d}\mathbf{x} + \Delta t \sum_{r=0}^{M} \theta_r^m \mathcal{R}_\xi^{\mathrm{E}}(U^{r,(k)}) \right),$$

with

$$\sum_{\xi\in\mathrm{E}} \mathcal{R}_\xi^{\mathrm{E}}(u) = \int_{\mathrm{E}} \nabla_{\mathbf{x}} F(u) \mathrm{d}\mathbf{x}.$$

- The $\mathcal{L}^2$ operator contains also the complications of the spatial discretization (e.g. mass matrix)
- $\mathcal{L}^1$ operator further simplified up to a first order approximation (e.g. **mass lumping**)

$\mathcal{L}^1$ with mass lumping

$$L^1 = \underset{\sim}{M} U_t + F_x(U^0)$$

$$= \underline{\underline{D\, U_t}}$$

$$\underline{\underline{D = M + O(\Delta x)}}$$

Implicit simple DeC

Define $\mathcal{L}^1$ as

$F(\alpha^n)$

$$\mathcal{L}^1(\boldsymbol{\alpha}^0,\dots,\boldsymbol{\alpha}^M) = \begin{pmatrix} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \Delta t \beta^M F(\boldsymbol{\alpha}^0) \\ \vdots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \Delta t \beta^1 F(\boldsymbol{\alpha}^0) \end{pmatrix}$$

$F(\alpha^m) \approx$

Define $\mathcal{L}^1$ as

$$
\mathcal{L}^1(\boldsymbol{\alpha}^0,\dots,\boldsymbol{\alpha}^M) = \begin{pmatrix} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \Delta t \beta^M \left(\cancel{F(\boldsymbol{\alpha}^0)} + \partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}^0)(\boldsymbol{\alpha}^M - \cancel{\boldsymbol{\alpha}^0})\right) \\ \vdots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \Delta t \beta^1 \left(\cancel{F(\boldsymbol{\alpha}^0)} + \partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}^0)(\boldsymbol{\alpha}^1 - \cancel{\boldsymbol{\alpha}^0})\right) \end{pmatrix}
$$

$$
= \begin{pmatrix} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \Delta t \beta^M \partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}^0)\boldsymbol{\alpha}^M \\ \vdots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \Delta t \beta^1 \partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}^0)\boldsymbol{\alpha}^1 \end{pmatrix}
$$

$\left(I - \Delta t \beta^m \partial_\alpha F(\alpha^0)\right)\alpha^{(k)} =$

$$\boldsymbol{\alpha}^{(k),m} - \boldsymbol{\alpha}^0 - \Delta t \sum_{r=0}^{M} \theta_r^m F(\boldsymbol{\alpha}^{(k-1),r}) = 0$$

α^0 | 0 | 0

$\alpha^{(1),1}$ | $\sum \theta_r^1$

$\alpha^{(1),2}$

$\alpha^{(1),M}$ | $\sum \theta_r^M$

$\alpha^{(2),1}$ | $\theta_0^1 \quad \theta_1^1 \quad \dots \quad \theta_M^1$

$\alpha^{(2),M}$ | $\theta_0^M \quad \theta_1^M \quad \theta_M^M$

$\alpha^{(k),m}$ | $\theta_0^m \quad 0\ 0\ 0\ 0\ 0\ 0 \quad \theta_1^m \dots \theta_M^m$

$\theta_0^M \quad 0\ 0\ 0\ 0 \quad \theta_1^M - \theta_M^M$

$\approx K \cdot M$

DeC as RK

We can write DeC as RK defining $\underline{\theta}_0 = \{\theta_0^m\}_{m=1}^M$, $\underline{\theta}^M = \theta_r^M$ with $r \in 1, \dots, M$, denoting the vector $\underline{\theta}_r^{M,T} = (\theta_1^M, \dots, \theta_M^M)$. The Butcher tableau for an arbitrarily high order DeC approach is given by:

$\alpha^{(3)} \rightarrow$

$$
\begin{array}{c|cccccccc}
0 & 0 & (1) & & & & & & \\
\beta & \beta & & & & & & & \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{\tilde{\theta}}} & (2) & & & & & \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} & & & & & \\
\vdots & \underline{\theta}_0 & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} & & & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & & & \\
\underline{\beta} & \underline{\theta}_0 & \underline{\underline{0}} & \dots & \dots & \underline{\underline{0}} & \underline{\underline{\tilde{\theta}}} & & \\
\hline
 & \theta_0^M & \underline{0}^T & \dots & & \dots & \underline{0}^T & \underline{\theta}_r^{M,T} &
\end{array} \quad . \tag{6}
$$

- Choice of order
- Choice of point distributions $t^0, \dots, t^M$
- Computation of θ
- Loop for timesteps
- Loop for correction
- Loop for subtimesteps

M $\quad$ $K=2$ $\quad$ t^0, t^1 $\quad$ $(0), (1), (2)$

ORDER p: $M = p-1$, $K = p$

$$\beta = \Delta t \cdot \frac{t^m - t^0}{t^M - t^0}$$

$$\theta^m_n = \int_0^{\beta^m} \varphi_n(s)\, ds \simeq \sum_i \omega_i \beta^m \varphi_n(s_i)$$

$t^n, t^{n+1}, \dots$

(K)

m

$$\mathcal{L}^{1,m,(k)} = \mathcal{L}^{1,m,(k-1)} - \mathcal{L}^{2,m,(k-1)}$$

Outline

- Cauchy–Kovalevskaya theorem
- Modern automatic version
- Space/time DG
- Prediction/Correction
- Fixed-point iteration process

Modern approach is DG in space time for hyperbolic problem

$$\partial_t u(x,t) + \nabla \cdot F(u(x,t)) = 0,\ x \in \Omega \subset \mathbb{R}^d,\ t > 0. \tag{7}$$

DG

$\|\mathcal{L}^2(\alpha^{(k)})\| \le tol$

Prediction: iterative procedure

$$\int_{T^n \times V_i} \theta_{rs}(x,t)\partial_t\theta_{pq}(x,t)z^{pq}\mathrm{d}x\mathrm{d}t + \int_{T^n \times V_i} \theta_{rs}(x,t)\nabla_{\mathbf{x}} \cdot F(\theta_{pq}(x,t)z^{pq})\mathrm{d}x\mathrm{d}t = 0.$$

Correction step: communication between cells

V_i V_{i+1}

$$\int_{V_i} \Phi_r \left(u(t^{n+1}) - u(t^n)\right) \mathrm{d}x + \int_{T^n \times \partial V_i} \Phi_r(x)\mathcal{G}(z^-, z^+) \cdot \mathbf{n} \,\mathrm{d}S \,\mathrm{d}t - \int_{T^n \times V_i} \nabla_{\mathbf{x}}\Phi_r \cdot F(z) \,\mathrm{d}x \,\mathrm{d}t = 0,$$

ADER: space-time discretization

Defining $\theta_{rs}(x,t) = \Phi_r(x)\phi_s(t)$ basis functions in space and time

$$\int_{T^n\times V_i} \theta_{rs}(x,t)\partial_t\theta_{pq}(x,t)u^{pq}\mathrm{d}x\mathrm{d}t + \int_{T^n\times V_i} \theta_{rs}(x,t)\nabla\cdot F(\theta_{pq}(x,t)u^{pq})\mathrm{d}x\mathrm{d}t = 0. \quad (8)$$

This leads to

$$\underline{\underline{\mathrm{M}}}_{rspq} u^{pq} = \underline{r}(\underline{\mathbf{u}})_{rs}, \quad (9)$$

solved with fixed point iteration method.
+ Correction step where cells communication is allowed (derived from (8)).

ADER: space-time discretization

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$$\int_{T^n \times V_i} \theta_{rs}(x,t)\partial_t \theta_{pq}(x,t) u^{pq} \mathrm{d}x\mathrm{d}t + \int_{T^n \times V_i} \theta_{rs}(x,t) \nabla \cdot F(\theta_{pq}(x,t)u^{pq}) \mathrm{d}x\mathrm{d}t = 0. \quad (8)$$

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+ Correction step where cells communication is allowed (derived from (8)).

$\alpha(t) = \sum_{0}^{n} \phi_m(t)\, \alpha^m$ $\alpha_t = F(\alpha)$ ϕ LAGRANGE INTERP FUNCTIONS

Simplify! Take $\boldsymbol{\alpha}(t) = \sum_{m=0}^{M} \phi_m(t)\boldsymbol{\alpha}^m = \underline{\phi}(t)^T\underline{\boldsymbol{\alpha}}$

$\psi = \phi$

$$\int_{T^n} \psi(t)\partial_t \boldsymbol{\alpha}(t)dt - \int_{T^n} \psi(t)F(\boldsymbol{\alpha}(t))dt = 0, \quad \forall \psi : T^n = [t^n, t^{n+1}] \to \mathbb{R}.$$

$\approx \underline{\phi}(t)^T \boxed{F(\alpha)}$

$$\mathcal{L}^2(\underline{\boldsymbol{\alpha}}) := \int_{T^n} \underline{\phi}(t)\partial_t \underline{\phi}(t)^T\underline{\boldsymbol{\alpha}}dt - \int_{T^n} \underline{\phi}(t)F(\underline{\phi}(t)^T\underline{\boldsymbol{\alpha}})dt = 0$$

$\alpha^* : \mathcal{L}^2(\alpha^*) = 0$

$$\underline{\phi}(t) = (\phi_0(t), \dots, \phi_M(t))^T$$

Quadrature...

$$\mathcal{L}^2(\underline{\boldsymbol{\alpha}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}} - \underline{r}(\underline{\boldsymbol{\alpha}}) = 0 \Longleftrightarrow \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}} = \underline{r}(\underline{\boldsymbol{\alpha}}). \tag{10}$$

Nonlinear system of $M \times S$ equations

ADER: Mass matrix

What goes into the mass matrix? Use of the integration by parts

$$\mathcal{L}^2(\underline{\boldsymbol{\alpha}}) := \int_{T^n} \underline{\phi}(t)\partial_t\underline{\phi}(t)^T\underline{\boldsymbol{\alpha}}dt + \int_{T^n} \underline{\phi}(t)F(\underline{\phi}(t)^T\underline{\boldsymbol{\alpha}})dt =$$

$\phi(t^n)^T\alpha = \alpha(t^n)$

$$\underline{\phi}(t^{n+1})\underline{\phi}(t^{n+1})^T\underline{\boldsymbol{\alpha}} - \underline{\phi}(t^n)\boldsymbol{\alpha}^n - \int_{T^n} \partial_t\underline{\phi}(t)\underline{\phi}(t)^T\underline{\boldsymbol{\alpha}} - \int_{T^n} \underline{\phi}(t)F(\underline{\phi}(t)^T\underline{\boldsymbol{\alpha}})dt$$

$$\underline{\underline{\mathrm{M}}} = \underline{\phi}(t^{n+1})\underline{\phi}(t^{n+1})^T - \int_{T^n} \partial_t\underline{\phi}(t)\underline{\phi}(t)^T$$

$$\underline{r}(\underline{\boldsymbol{\alpha}}) = \underline{\phi}(t^n)\boldsymbol{\alpha}^n + \int_{T^n} \underline{\phi}(t)F(\underline{\phi}(t)^T\underline{\boldsymbol{\alpha}})dt \approx \phi(t^n)\alpha^n + \int_{T^n}\phi(t)\phi^T F(\alpha)$$

$$\boxed{\underline{\underline{\mathrm{M}}}\boldsymbol{\alpha} = \underline{r}(\underline{\boldsymbol{\alpha}})}$$

ADER: Fixed point iteration

Iterative procedure to solve the problem for each time step

$$\underline{\boldsymbol{\alpha}}^{(k)} = \underline{\underline{\mathrm{M}}}^{-1} \underline{r}(\underline{\boldsymbol{\alpha}}^{(k-1)}), \quad k = 1, \dots, \text{convergence} \tag{11}$$

$\alpha^{(K)}$

with $\underline{\boldsymbol{\alpha}}^{(0)} = \boldsymbol{\alpha}(t^n)$.
Reconstruction step

$$\boldsymbol{\alpha}(t^{n+1}) = \boldsymbol{\alpha}(t^n) - \int_{T^n} F(\boldsymbol{\alpha}^{(K)}(t))dt. \approx \alpha^{(K)}(t^{n+1})$$

- Convergence?
- How many steps K?

ADER 2nd order

Example with 2 Gauss Legendre points and 2 iterations
Let us consider the timestep interval $[t^n, t^{n+1}]$, rescaled to $[0,1]$.
Gauss-Legendre points quadrature and interpolation (in the interval $[0,1]$)

t^{n+1} t^1 t^0

$$\underline{t}_q = (t_q^0, t_q^1) = (t^0, t^1) = \left(\frac{\sqrt{3}-1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}} \right), \quad \underline{w} = (1/2, 1/2)\,.$$

t^n

$$\underline{\phi}(t) = (\phi_0(t), \phi_1(t)) = \left(\frac{t-t^1}{t^0-t^1}, \frac{t-t^0}{t^1-t^0} \right).$$

Then, the mass matrix is given by

$\approx \int_{t^n} \phi_m' \phi_l = \sum_i \phi_m'(t^i)\, \phi_l(t^i)\, w_i$

$$\underline{\underline{\mathrm{M}}}_{m,l} = \phi_m(1)\phi_l(1) - \phi_m'(t^l) w_l, \quad m,l = 0,1,$$

$= \phi_m'(t^l)\, w_l$

$$\underline{\underline{\mathrm{M}}} = \begin{pmatrix} 1 & \frac{\sqrt{3}-1}{2} \\ -\frac{\sqrt{3}+1}{2} & 1 \end{pmatrix}.$$

ADER 2nd order

The right hand side is given

$\int F(\alpha(t)) \cdot \phi_m \simeq \sum_i F(\alpha(t^i)) \cdot \phi_m(t^i) w_i = F(\alpha(t^m)) w_m$

$$r(\underline{\boldsymbol{\alpha}})_m = \alpha(0)\phi_m(0) + \Delta t F(\alpha(t^m)) w_m, \quad m = 0, 1.$$

$$\underline{r}(\underline{\boldsymbol{\alpha}}) = \alpha(0)\underline{\phi}(0) + \Delta t \begin{pmatrix} F(\alpha(t^1))w_1 \\ F(\alpha(t^2))w_2. \end{pmatrix}.$$

Then, the coefficients $\underline{\boldsymbol{\alpha}}$ are given by

$$\boxed{\underline{\boldsymbol{\alpha}}^{(k+1)} = \underline{\underline{\mathrm{M}}}^{-1}\underline{r}(\underline{\boldsymbol{\alpha}}^{(k)}).}$$

Finally, use $\underline{\boldsymbol{\alpha}}^{(k+1)}$ to reconstruct the solution at the time step t^{n+1}:

$$\boxed{\alpha^{n+1} = \underline{\phi}(1)^T \underline{\boldsymbol{\alpha}}^{(k+1)}.}$$

CODE

- Precompute $\underline{\underline{\mathrm{M}}}$
- Precompute the rhs vector part using quadratures after a further approximation

$$\underline{r}(\underline{\boldsymbol{\alpha}}) = \underline{\phi}(t^n)\boldsymbol{\alpha}^n + \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\boldsymbol{\alpha}}) dt \approx \underline{\phi}(t^n)\boldsymbol{\alpha}^n + \underbrace{\int_{T^n} \underline{\phi}(t)\underline{\phi}(t)^T dt}_{\text{Can be stored}} F(\underline{\boldsymbol{\alpha}})$$

- Precompute the reconstruction coefficients $\underline{\phi}(1)^T$

Outline

ADER[6] and DeC[7]: immediate similarities

- High order time-(space) discretization
- Start from a well known space discretization (FE/DG/FV)
- FE reconstruction in time
- System in time, with M equations
- Iterative method / K corrections

- Both high order explicit time integration methods (neglecting spatial discretization)

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[6] M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. Journal of Computational Physics, 227(18):8209–8253, 2008.

[7] R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. Journal of Scientific Computing, 73(2):461–494, Dec 2017.

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ADER as DeC

ADER as DeC

$$\underline{\underline{\Pi}} \quad \underline{\alpha}^{(k)} = \Omega(\underline{\alpha}^{(k-1)})$$

$$\mathcal{L}^2(\underline{\boldsymbol{\alpha}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}} - r(\underline{\boldsymbol{\alpha}}),$$

$$\mathcal{L}^1(\underline{\boldsymbol{\alpha}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}} - \underline{r(\boldsymbol{\alpha}(t^n))}.$$

$$\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{\alpha}}^{(k-1)}), \qquad k = 1, \dots, K,$$

$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k)} - r(\underbrace{\cancel{\boldsymbol{\alpha}^{(k),0}}}_{\alpha(t^n)}) - \cancel{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k-1)}} + r(\underbrace{\cancel{\boldsymbol{\alpha}^{(k-1),0}}}_{\alpha(t^n)}) + \cancel{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k-1)}} - r(\underline{\boldsymbol{\alpha}}^{(k-1)}) = 0$$

$$\mathcal{L}^2(\underline{\boldsymbol{\alpha}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}} - r(\underline{\boldsymbol{\alpha}}),$$
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$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k)} - \cancel{r(\boldsymbol{\alpha}^{(k),0})} - \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k-1)} + \cancel{r(\boldsymbol{\alpha}^{(k-1),0})} + \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k-1)} - r(\underline{\boldsymbol{\alpha}}^{(k-1)}) = 0$$

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$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k)} - \cancel{r(\boldsymbol{\alpha}^{(k),0})} - \cancel{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k-1)}} + \cancel{r(\boldsymbol{\alpha}^{(k-1),0})} + \cancel{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k-1)}} - r(\underline{\boldsymbol{\alpha}}^{(k-1)}) = 0$$

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$$\mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k)}) = \mathcal{L}^1(\underline{\boldsymbol{\alpha}}^{(k-1)}) - \mathcal{L}^2(\underline{\boldsymbol{\alpha}}^{(k-1)}), \qquad k = 1, \dots, K,$$

$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k)} - \cancel{r(\boldsymbol{\alpha}^{(k),0})} - \cancel{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k-1)}} + \cancel{r(\boldsymbol{\alpha}^{(k-1),0})} + \cancel{\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k-1)}} - r(\underline{\boldsymbol{\alpha}}^{(k-1)}) = 0$$
$$\underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}}^{(k)} - r(\underline{\boldsymbol{\alpha}}^{(k-1)}) = 0.$$

$\mathcal{O}(\Delta t^P) + ODE$

$\mathcal{O}(\Delta t) + ODE$

$$\mathcal{L}^2(\underline{\boldsymbol{\alpha}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}} - r(\underline{\boldsymbol{\alpha}}),$$
$$\mathcal{L}^1(\underline{\boldsymbol{\alpha}}) := \underline{\underline{\mathrm{M}}}\underline{\boldsymbol{\alpha}} - r(\boldsymbol{\alpha}(t^n)).$$

$\|\mathcal{L}^1(u) - \mathcal{L}^1(v)\| = \|M(u-v)\|$
$\geq C\|u-v\|$

Apply the DeC Convergence theorem!

- $\mathcal{L}^1$ is coercive because $\underline{\underline{\mathrm{M}}}$ is always invertible
- $\mathcal{L}^1 - \mathcal{L}^2$ is Lipschitz with constant $C\Delta t$ because they are consistent approx of the same problem
- Hence, after K iterations we obtain a Kth order accurate approximation of $\underline{\boldsymbol{\alpha}}^*$

$K = P$ ORDER OF ACCURACY

DeC as ADER

Θ_r^r

$$\mathcal{L}^2(\boldsymbol{\alpha}^0,\dots,\boldsymbol{\alpha}^M) := \begin{cases} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \sum_{r=0}^M \int_{t^0}^{t^M} F(\boldsymbol{\alpha}^r)\varphi_r(s)\mathrm{d}s \\ \dots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\boldsymbol{\alpha}^r)\varphi_r(s)\mathrm{d}s \end{cases}.$$

$\alpha_t = F$ $\qquad$ $\varphi_r \Rightarrow$ $\quad \alpha(t) = \sum_r \varphi_r(t)\cdot\alpha^r$

$F(t) \approx \sum_r \varphi_r(t) F(\alpha^r)$

DeC as ADER

1) $\alpha^m - \alpha^0 \overset{?}{=} \int_{t^0}^{t^m} \chi_{[t^0, t^m]}(t) \cdot \partial_t \alpha(t)\, dt$

$= \int_{t^0}^{t^m} \partial_t \alpha(t)\, dt = \alpha(t^m) - \alpha(t^0)$

$= \alpha^m - \alpha^0$

DeC as ADER

DeC as ADER

$$\mathcal{L}^2(\boldsymbol{\alpha}^0,\dots,\boldsymbol{\alpha}^M) := \begin{cases} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \sum_{r=0}^M \int_{t^0}^{t^M} F(\boldsymbol{\alpha}^r)\varphi_r(s)\mathrm{d}s \\ \dots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\boldsymbol{\alpha}^r)\varphi_r(s)\mathrm{d}s \end{cases}.$$

$$\chi_{[t^0,t^m]}(t^m)\boldsymbol{\alpha}^m - \chi_{[t^0,t^m]}(t_0)\boldsymbol{\alpha}^0 - \int_{t^0}^{t^m} \chi_{[t^0,t^m]}(t)\sum_{r=0}^M F(\boldsymbol{\alpha}^r)\varphi_r(t)\mathrm{d}t = 0$$

$$\int_{t^0}^{t^M} \chi_{[t^0,t^m]}(t)\partial_t\left(\boldsymbol{\alpha}(t)\right)\mathrm{d}t - \int_{t^0}^{t^M} \chi_{[t^0,t^m]}(t)\sum_{r=0}^M F(\boldsymbol{\alpha}^r)\varphi_r(t)\mathrm{d}t = 0,$$

$$\int_{T^n} \psi_m(t)\partial_t\boldsymbol{\alpha}(t)\mathrm{d}t - \int_{T^n} \psi_m(t)F(\boldsymbol{\alpha}(t))\mathrm{d}t = 0.$$

$$\mathcal{L}^2(\boldsymbol{\alpha}^0,\dots,\boldsymbol{\alpha}^M) := \begin{cases} \underline{\boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0} - \sum_{r=0}^M \int_{t^0}^{t^M} F(\boldsymbol{\alpha}^r)\varphi_r(s)\mathrm{d}s \\ \dots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\boldsymbol{\alpha}^r)\varphi_r(s)\mathrm{d}s \end{cases}.$$

$$\chi_{[t^0,t^m]}(t) = \begin{cases} 1 & t\in[t^0,t^m] \\ 0 & \text{else} \end{cases}$$

$$\alpha^m - \alpha^0 - \sum_{r=0}^{n} \int_{t^0}^{t^m} \varphi_r(s) F(\alpha^r) ds$$

$$\chi_{[t^0,t^m]}(t^m)\boldsymbol{\alpha}^m - \chi_{[t^0,t^m]}(t_0)\boldsymbol{\alpha}^0 - \int_{t^0}^{t^m} \chi_{[t^0,t^m]}(t)\sum_{r=0}^M F(\boldsymbol{\alpha}^r)\varphi_r(t)\mathrm{d}t = 0$$

$$\int_{t^0}^{t^M} \chi_{[t^0,t^m]}(t)\partial_t\left(\boldsymbol{\alpha}(t)\right)\mathrm{d}t - \int_{t^0}^{t^M} \chi_{[t^0,t^m]}(t)\sum_{r=0}^M F(\boldsymbol{\alpha}^r)\varphi_r(t)\mathrm{d}t = 0,$$

$$\int_{T^n} \psi_m(t)\partial_t\boldsymbol{\alpha}(t)\mathrm{d}t - \int_{T^n} \psi_m(t)F(\boldsymbol{\alpha}(t))\mathrm{d}t = 0.$$

$$\alpha = \sum \varphi_r \alpha^r$$

Runge Kutta vs DeC–ADER

Classical Runge Kutta (RK)

- One step method
- Internal stages

Explicit Runge Kutta

+ Simple to code
- Not easily generalizable to arbitrary order
- Stages $>$ order

Implicit Runge Kutta

+ Arbitrarily high order
- Require nonlinear solvers for nonlinear systems
- May not converge

DeC – ADER

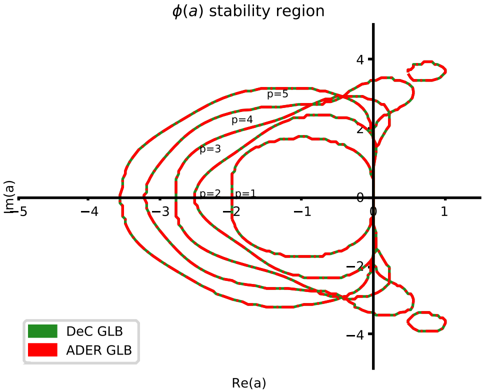
- One step method
- Internal subtimesteps
- Can be rewritten as explicit RK (for ODE)

+ Explicit
+ Simple to code
+ Iterations $=$ order
+ Arbitrarily high order
- Large memory storage

Outline

A–Stability

$$y'(t) = \lambda y(t) \qquad y(0) = 1$$



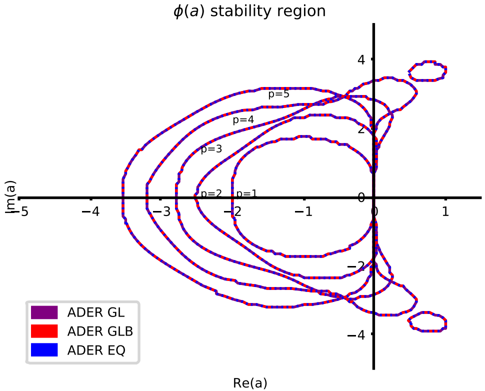
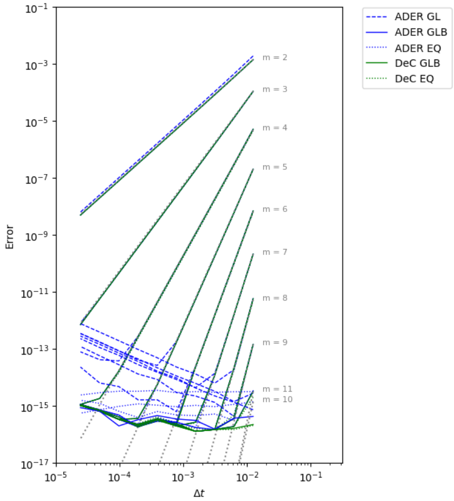


Figure: Stability region

Convergence

$$
\begin{aligned}
y'(t) &= -|y(t)|y(t), \\
y(0) &= 1, \\
t &\in [0, 0.1].
\end{aligned} \tag{12}
$$

Convergence curves for ADER and DeC, varying the approximation order and collocation of nodes for the subtimesteps for a scalar nonlinear ODE



Lotka–Volterra

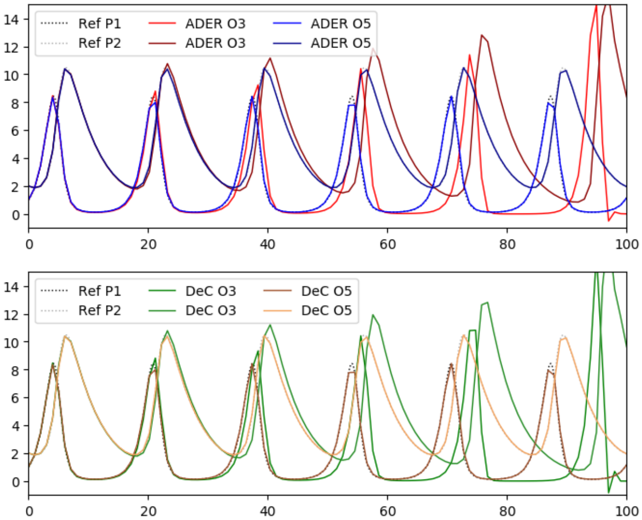
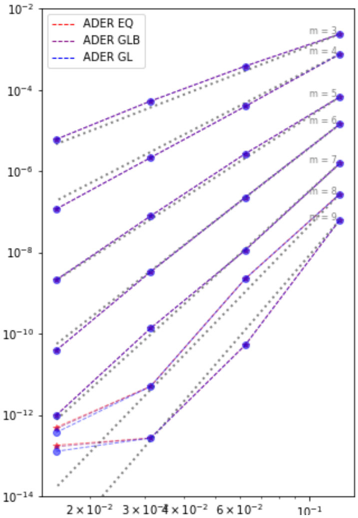


Figure: Numerical solution of the Lotka-Volterra system using ADER (top) and DeC (bottom) with Gauss-Lobatto nodes with timestep $\Delta T = 1$.

PDE: Burgers with spectral difference



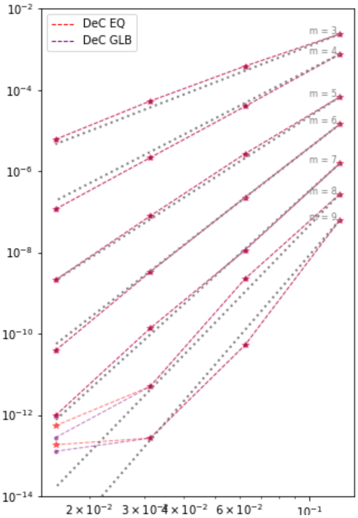


Figure: Convergence error for Burgers equations: Left ADER right DeC. Space discretization with spectral difference