

ADER and DeC: arbitrarily high order (explicit) methods for PDEs and ODEs

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Based on: Han Veiga, M., Öffner, P. & Torlo, D. *DeC and ADER: Similarities, Differences and a Unified Framework*. J Sci Comput 87, 2 (2021). <https://doi.org/10.1007/s10915-020-01397-5>

Outline

- 1 Motivation
- 2 DeC
- 3 ADER
- 4 Similarities
- 5 Simulations

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Motivation: high order accurate explicit method

We want to solve a hyperbolic PDE system for $u : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^D$

$$\partial_t u + \nabla_x \mathcal{F}(u) = 0. \quad (1)$$

Or ODE system for $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^S$

$$\partial_t \alpha + F(\alpha) = 0. \quad (2)$$

Applications:

- Fluids/transport
- Chemical/biological processes

How?

- Arbitrarily high order accurate
-

Motivation: high order accurate explicit method

We want to solve a hyperbolic PDE system for $m+1 \times 0 \dots m D$

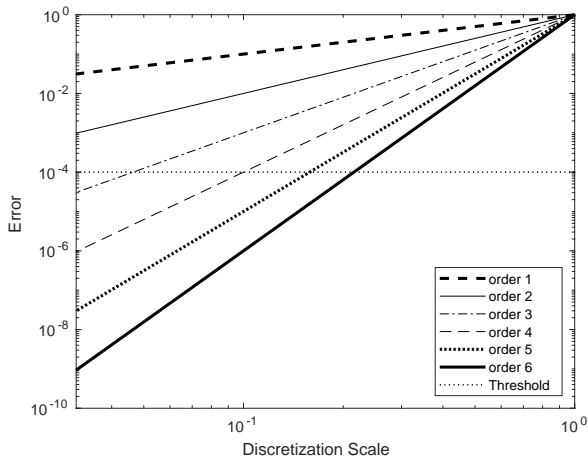
Or ODE system for

Applications:

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- Chemical/biology

How?

- Arbitrarily high order
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(1)

(2)

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Applications:

- Fluids/transport
- Chemical/biological processes

How?

- Arbitrarily high order accurate
- Explicit (if nonstiff problem)

Classical time integration: Runge–Kutta

$$\boldsymbol{\alpha}^{(1)} := \boldsymbol{\alpha}^n, \tag{3}$$

$$\boldsymbol{\alpha}^{(k)} := \boldsymbol{\alpha}^n + \sum_{s=1}^K A_{ks} F \left(t^n + b_s \Delta t, \boldsymbol{\alpha}^{(s)} \right), \quad \text{for } k = 2, \dots, K, \tag{4}$$

$$\boldsymbol{\alpha}^{n+1} := \sum_{k=1}^K \gamma_k \boldsymbol{\alpha}^{(k)}. \tag{5}$$

Classical time integration: Explicit Runge–Kutta

$$\alpha^{(k)} := \alpha^n + \sum_{s=1}^{k-1} A_{ks} F\left(t^n + b_s \Delta t, \alpha^{(s)}\right), \quad \text{for } k = 2, \dots, K.$$

- Easy to solve
- High orders involved:
 - Order conditions: system of many equations
 - Stages $K \geq d$ order of accuracy (e.g. RK44, RK65)

Classical time integration: Implicit Runge–Kutta

$$\alpha^{(k)} := \alpha^n + \sum_{s=1}^K A_{ks} F\left(t^n + b_s \Delta t, \alpha^{(s)}\right), \quad \text{for } k = 2, \dots, K.$$

- More complicated to solve for nonlinear systems
- High orders easily done:
 - Take a high order quadrature rule on $[t^n, t^{n+1}]$
 - Compute the coefficients accordingly, see Gauss–Legendre or Gauss–Lobatto polynomials
 - Order up to $d = 2K - 1$

Two iterative explicit arbitrarily high order accurate methods.

- ADER¹ for hyperbolic PDE, after a first analytic more complicated approach.
- Deferred Correction (DeC): introduced for explicit ODE², extended to implicit ODE³ and to hyperbolic PDE⁴.

¹M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. *Journal of Computational Physics*, 227(18):8209–8253, 2008.

²A. Dutt, L. Greengard, and V. Rokhlin. Spectral Deferred Correction Methods for Ordinary Differential Equations. *BIT Numerical Mathematics*, 40(2):241–266, 2000.

³M. L. Minion. Semi-implicit spectral deferred correction methods for ordinary differential equations. *Commun. Math. Sci.*, 1(3):471–500, 09 2003.

⁴R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. *Journal of Scientific Computing*, 73(2):461–494, Dec 2017.

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DeC high order time discretization: \mathcal{L}^2

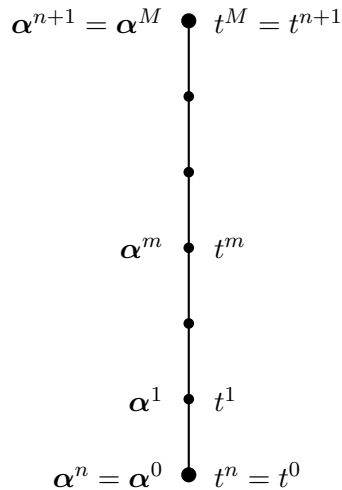
High order in time: we discretize our variable on $[t^n, t^{n+1}]$ in M substeps (α^m).

$$\partial_t \alpha + F(\alpha(t)) = 0.$$

Thanks to Picard–Lindelöf theorem, we can rewrite

$$\alpha^m = \alpha^0 - \int_{t^0}^{t^m} F(\alpha(t)) dt.$$

and if we want to reach order $r + 1$ we need $M = r$.

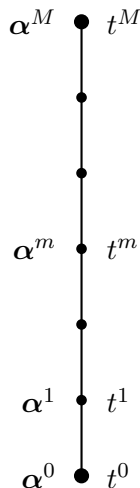


DeC high order time discretization: \mathcal{L}^2

More precisely, for each σ we want to solve $\mathcal{L}^2(\alpha^{n,0}, \dots, \alpha^{n,M}) = 0$, where

$$\mathcal{L}^2(\alpha^0, \dots, \alpha^M) = \begin{pmatrix} \alpha^M - \alpha^0 - \sum_{r=0}^M \int_{t^0}^{t^M} F(\alpha^r) \varphi_r(s) ds \\ \vdots \\ \alpha^1 - \alpha^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\alpha^r) \varphi_r(s) ds \end{pmatrix}$$

- $\mathcal{L}^2 = 0$ is a system of $M \times S$ coupled (non)linear equations
- \mathcal{L}^2 is an implicit method
- Not easy to solve directly $\mathcal{L}^2(\underline{\alpha}^*) = 0$
- High order ($\geq M + 1$), depending on points distribution

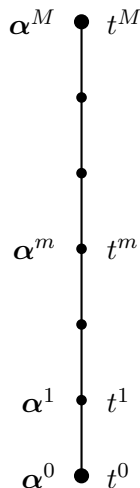


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$$\mathcal{L}^2(\alpha^0, \dots, \alpha^M) = \begin{pmatrix} \alpha^M - \alpha^0 - \Delta t \sum_{r=0}^M \theta_r^M F(\alpha^r) \\ \vdots \\ \alpha^1 - \alpha^0 - \Delta t \sum_{r=0}^M \theta_r^1 F(\alpha^r) \end{pmatrix}$$

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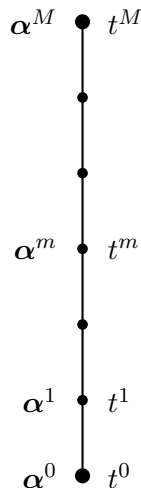


DeC low order time discretization: \mathcal{L}^1

Instead of solving the implicit system directly (difficult), we introduce a first order scheme $\mathcal{L}^1(\alpha^{n,0}, \dots, \alpha^{n,M})$:

$$\mathcal{L}^1(\alpha^0, \dots, \alpha^M) = \begin{pmatrix} \alpha^M - \alpha^0 - \Delta t \beta^M F(\alpha^0) \\ \vdots \\ \alpha^1 - \alpha^0 - \Delta t \beta^1 F(\alpha^0) \end{pmatrix}$$

- First order approximation
- Explicit Euler
- Easy to solve $\mathcal{L}^1(\underline{\alpha}) = 0$



Deferred Correction⁵

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

$$\underline{\alpha}^{0,(k)} := \underline{\alpha}(t^n), \quad k = 0, \dots, K,$$

$$\underline{\alpha}^{m,(0)} := \underline{\alpha}(t^n), \quad m = 1, \dots, M$$

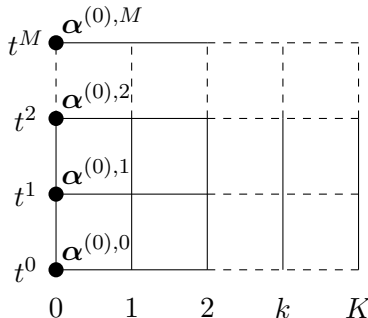
$$\mathcal{L}^1(\underline{\alpha}^{(k)}) = \mathcal{L}^1(\underline{\alpha}^{(k-1)}) - \mathcal{L}^2(\underline{\alpha}^{(k-1)}) \text{ with } k = 1, \dots, K.$$

- $\mathcal{L}^1(\underline{\alpha}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\alpha}) = 0$, high order $M + 1$.

Theorem (Convergence DeC)

- $\mathcal{L}^2(\underline{\alpha}^*) = 0$
- If \mathcal{L}^1 coercive with constant C_1
- If $\mathcal{L}^1 - \mathcal{L}^2$ Lipschitz with constant $C_2 \Delta t$

$$\text{Then } \|\underline{\alpha}^{(K)} - \underline{\alpha}^*\| \leq C(\Delta t)^K$$



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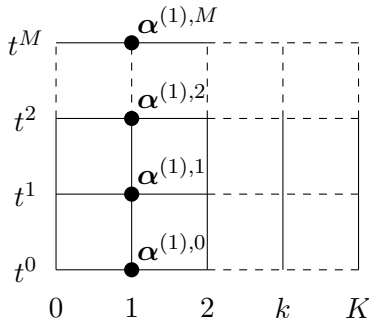
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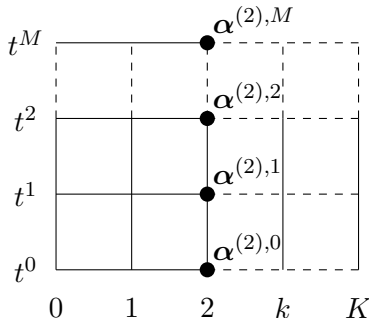
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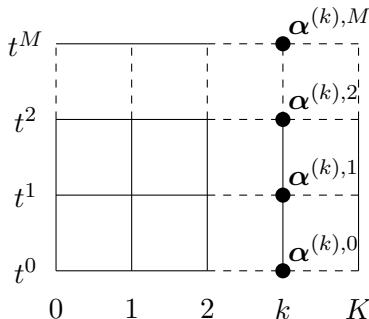
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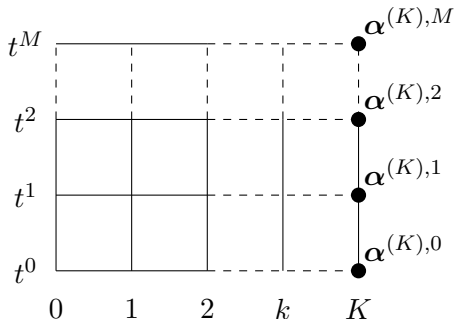
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Proof.

Let f^* be the solution of $\mathcal{L}^2(\underline{\alpha}^*) = 0$. We know that $\mathcal{L}^1(\underline{\alpha}^*) = \mathcal{L}^1(\underline{\alpha}^*) - \mathcal{L}^2(\underline{\alpha}^*)$, so that



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$$\begin{aligned}\mathcal{L}^1(\underline{\alpha}^{(k+1)}) - \mathcal{L}^1(\underline{\alpha}^*) &= \left(\mathcal{L}^1(\underline{\alpha}^{(k)}) - \mathcal{L}^2(\underline{\alpha}^{(k)}) \right) - (\mathcal{L}^1(\underline{\alpha}^*) - \mathcal{L}^2(\underline{\alpha}^*)) \\ \color{red}{C_1} \|\underline{\alpha}^{(k+1)} - \underline{\alpha}^*\| &\leq \|\mathcal{L}^1(\underline{\alpha}^{(k+1)}) - \mathcal{L}^1(\underline{\alpha}^*)\| = \\ &= \|\mathcal{L}^1(\underline{\alpha}^{(k)}) - \mathcal{L}^2(\underline{\alpha}^{(k)}) - (\mathcal{L}^1(\underline{\alpha}^*) - \mathcal{L}^2(\underline{\alpha}^*))\| \leq \\ &\leq \color{red}{C_2} \Delta \|\underline{\alpha}^{(k)} - \underline{\alpha}^*\|.\end{aligned}$$

$$\|\underline{\alpha}^{(k+1)} - \underline{\alpha}^*\| \leq \left(\frac{C_2}{C_1} \Delta \right) \|\underline{\alpha}^{(k)} - \underline{\alpha}^*\| \leq \left(\frac{C_2}{C_1} \Delta \right)^{k+1} \|\underline{\alpha}^{(0)} - \underline{\alpha}^*\|.$$

After K iteration we have an error at most of $\left(\frac{C_2}{C_1} \Delta \right)^K \|\underline{\alpha}^{(0)} - \underline{\alpha}^*\|$. □

DeC: Second order example

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Simplification of DeC for ODE

In practice

$$\mathcal{L}^1(\underline{\alpha}^{(k)}) = \mathcal{L}^1(\underline{\alpha}^{(k-1)}) - \mathcal{L}^2(\underline{\alpha}^{(k-1)}), \quad k = 1, \dots, K,$$

For $m = 1, \dots, M$

$$\begin{aligned} & \alpha^{(k),m}_- \alpha^0 - \beta^m \Delta t F(\alpha^0) - \alpha^{(k-1),m}_+ \alpha^0 + \beta^m \Delta t F(\alpha^0) \\ & + \alpha^{(k-1),m}_- \alpha^0 - \Delta t \sum_{r=0}^M \theta_r^m F(\alpha^{(k-1),r}) = 0 \end{aligned}$$

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DeC and residual distribution

Deferred Correction + Residual distribution

- Residual distribution (FV \Rightarrow FE) \Rightarrow High order in space
- Prediction/correction/iterations \Rightarrow High order in time
- Subtimesteps \Rightarrow High order in time

$$U_{\xi}^{m,(k+1)} = U_{\xi}^{m,(k)} - |C_p|^{-1} \sum_{E|\xi \in E} \left(\int_E \Phi_{\xi} \left(U^{m,(k)} - U^{n,0} \right) d\mathbf{x} + \Delta t \sum_{r=0}^M \theta_r^m \mathcal{R}_{\xi}^E(U^{r,(k)}) \right),$$

with

$$\sum_{\xi \in E} \mathcal{R}_{\xi}^E(u) = \int_E \nabla_{\mathbf{x}} F(u) d\mathbf{x}.$$

- The \mathcal{L}^2 operator contains also the complications of the spatial discretization (e.g. mass matrix)
- \mathcal{L}^1 operator further simplified up to a first order approximation (e.g. **mass lumping**)

Define \mathcal{L}^1 as

$$\mathcal{L}^1(\boldsymbol{\alpha}^0, \dots, \boldsymbol{\alpha}^M) = \begin{pmatrix} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \Delta t \beta^M F(\boldsymbol{\alpha}^0) \\ \vdots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \Delta t \beta^1 F(\boldsymbol{\alpha}^0) \end{pmatrix}$$

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$$\begin{aligned}\mathcal{L}^1(\boldsymbol{\alpha}^0, \dots, \boldsymbol{\alpha}^M) &= \begin{pmatrix} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \Delta t \beta^M (F(\boldsymbol{\alpha}^0) + \partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}^0)(\boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0)) \\ \vdots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \Delta t \beta^1 (F(\boldsymbol{\alpha}^0) + \partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}^0)(\boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0)) \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \Delta t \beta^M \partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}^0) \boldsymbol{\alpha}^M \\ \vdots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \Delta t \beta^1 \partial_{\boldsymbol{\alpha}} F(\boldsymbol{\alpha}^0) \boldsymbol{\alpha}^1 \end{pmatrix}\end{aligned}$$

$$\boldsymbol{\alpha}^{(k),m} - \boldsymbol{\alpha}^0 - \Delta t \sum_{r=0}^M \theta_r^m F(\boldsymbol{\alpha}^{(k-1),r}) = 0$$

DeC as RK

We can write DeC as RK defining $\underline{\theta}_0 = \{\theta_0^m\}_{m=1}^M$, $\underline{\theta}^M = \theta_r^M$ with $r \in 1, \dots, M$, denoting the vector $\underline{\theta}_r^{M,T} = (\theta_1^M, \dots, \theta_M^M)$. The Butcher tableau for an arbitrarily high order DeC approach is given by:

$$\begin{array}{c|cccccc}
 0 & 0 & & & & \\
 \underline{\beta} & \underline{\beta} & & & & \\
 \underline{\beta} & \underline{\theta}_0 & \underline{\tilde{\theta}} & & & \\
 \vdots & \underline{\theta}_0 & \underline{0} & \underline{\tilde{\theta}} & & \\
 \vdots & \underline{\theta}_0 & \underline{0} & \underline{0} & \underline{\tilde{\theta}} & \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
 \underline{\beta} & \underline{\theta}_0 & \underline{0} & \dots & \dots & \underline{0} & \underline{\tilde{\theta}} \\
 \hline
 & \underline{\theta}_0^M & \underline{0}^T & \dots & \dots & \underline{0}^T & \underline{\theta}_r^{M,T}
 \end{array} \quad . \quad (6)$$

- Choice of order
- Choice of point distributions t^0, \dots, t^M
- Computation of θ
- Loop for timesteps
- Loop for correction
- Loop for subimesteps

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- Cauchy–Kovalevskaya theorem
- Modern automatic version
- Space/time DG
- Prediction/Correction
- Fixed-point iteration process

Modern approach is DG in space time for hyperbolic problem

$$\partial_t u(x, t) + \nabla \cdot F(u(x, t)) = 0, \quad x \in \Omega \subset \mathbb{R}^d, \quad t > 0. \quad (7)$$

Prediction: iterative procedure

$$\int_{T^n \times V_i} \theta_{rs}(x, t) \partial_t \theta_{pq}(x, t) z^{pq} dx dt + \int_{T^n \times V_i} \theta_{rs}(x, t) \nabla_{\mathbf{x}} \cdot F(\theta_{pq}(x, t) z^{pq}) dx dt = 0.$$

Correction step: communication between cells

$$\int_{V_i} \Phi_r (u(t^{n+1}) - u(t^n)) dx + \int_{T^n \times \partial V_i} \Phi_r(x) \mathcal{G}(z^-, z^+) \cdot \mathbf{n} dS dt - \int_{T^n \times V_i} \nabla_{\mathbf{x}} \Phi_r \cdot F(z) dx dt = 0,$$

ADER: space-time discretization

Defining $\theta_{rs}(x, t) = \Phi_r(x)\phi_s(t)$ basis functions in space and time

$$\int_{T^n \times V_i} \theta_{rs}(x, t) \partial_t \theta_{pq}(x, t) u^{pq} dx dt + \int_{T^n \times V_i} \theta_{rs}(x, t) \nabla \cdot F(\theta_{pq}(x, t) u^{pq}) dx dt = 0. \quad (8)$$

This leads to

$$\underline{\underline{\underline{M}}}_{rspq} u^{pq} = \underline{\underline{r}}(\underline{\underline{\mathbf{u}}})_{rs}, \quad (9)$$

solved with fixed point iteration method.

+ Correction step where cells communication is allowed (derived from (8)).

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ADER: time integration method

Simplify! Take $\boldsymbol{\alpha}(t) = \sum_{m=0}^M \phi_m(t) \boldsymbol{\alpha}^m = \underline{\phi}(t)^T \underline{\boldsymbol{\alpha}}$

$$\int_{T^n} \psi(t) \partial_t \boldsymbol{\alpha}(t) dt - \int_{T^n} \psi(t) F(\boldsymbol{\alpha}(t)) dt = 0, \quad \forall \psi : T^n = [t^n, t^{n+1}] \rightarrow \mathbb{R}.$$

$$\mathcal{L}^2(\underline{\boldsymbol{\alpha}}) := \int_{T^n} \underline{\phi}(t) \partial_t \underline{\phi}(t)^T \underline{\boldsymbol{\alpha}} dt - \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\boldsymbol{\alpha}}) dt = 0$$

$$\underline{\phi}(t) = (\phi_0(t), \dots, \phi_M(t))^T$$

Quadrature...

$$\mathcal{L}^2(\underline{\boldsymbol{\alpha}}) := \underline{\underline{\mathbf{M}}} \underline{\boldsymbol{\alpha}} - \underline{r}(\underline{\boldsymbol{\alpha}}) = 0 \iff \underline{\underline{\mathbf{M}}} \underline{\boldsymbol{\alpha}} = \underline{r}(\underline{\boldsymbol{\alpha}}). \quad (10)$$

Nonlinear system of $M \times S$ equations

What goes into the mass matrix? Use of the integration by parts

$$\begin{aligned}\mathcal{L}^2(\underline{\alpha}) &:= \int_{T^n} \underline{\phi}(t) \partial_t \underline{\phi}(t)^T \underline{\alpha} dt + \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\alpha}) dt = \\ &\quad \underline{\phi}(t^{n+1}) \underline{\phi}(t^{n+1})^T \underline{\alpha} - \underline{\phi}(t^n) \underline{\alpha}^n - \int_{T^n} \partial_t \underline{\phi}(t) \underline{\phi}(t)^T \underline{\alpha} - \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\alpha}) dt\end{aligned}$$

$$\underline{\underline{\mathbf{M}}} = \underline{\phi}(t^{n+1}) \underline{\phi}(t^{n+1})^T - \int_{T^n} \partial_t \underline{\phi}(t) \underline{\phi}(t)^T$$

$$\underline{r}(\underline{\alpha}) = \underline{\phi}(t^n) \underline{\alpha}^n + \int_{T^n} \underline{\phi}(t) F(\underline{\phi}(t)^T \underline{\alpha}) dt$$

$$\underline{\underline{\mathbf{M}}} \underline{\alpha} = \underline{r}(\underline{\alpha})$$

ADER: Fixed point iteration

Iterative procedure to solve the problem for each time step

$$\underline{\alpha}^{(k)} = \underline{\underline{M}}^{-1} \underline{r}(\underline{\alpha}^{(k-1)}), \quad k = 1, \dots, \text{convergence} \quad (11)$$

with $\underline{\alpha}^{(0)} = \alpha(t^n)$.

Reconstruction step

$$\alpha(t^{n+1}) = \alpha(t^n) - \int_{T^n} F(\alpha^{(K)}(t)) dt.$$

- Convergence?
- How many steps K ?

ADER 2nd order

Example with 2 Gauss Legendre points and 2 iterations

Let us consider the timestep interval $[t^n, t^{n+1}]$, rescaled to $[0, 1]$.

Gauss-Legendre points quadrature and interpolation (in the interval $[0, 1]$)

$$\underline{t}_q = (t_q^0, t_q^1) = (t^0, t^1) = \left(\frac{\sqrt{3}-1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}} \right), \quad \underline{w} = (1/2, 1/2).$$

$$\underline{\phi}(t) = (\phi_0(t), \phi_1(t)) = \left(\frac{t - t^1}{t^0 - t^1}, \frac{t - t^0}{t^1 - t^0} \right).$$

Then, the mass matrix is given by

$$\underline{\underline{M}}_{m,l} = \phi_m(1)\phi_l(1) - \phi'_m(t^l)w_l, \quad m, l = 0, 1,$$

$$\underline{\underline{M}} = \begin{pmatrix} 1 & \frac{\sqrt{3}-1}{2} \\ -\frac{\sqrt{3}+1}{2} & 1 \end{pmatrix}.$$

ADER 2nd order

The right hand side is given

$$r(\underline{\alpha})_m = \alpha(0)\phi_m(0) + \Delta t F(\alpha(t^m))w_m, \quad m = 0, 1.$$

$$\underline{r}(\underline{\alpha}) = \alpha(0)\underline{\phi}(0) + \Delta t \begin{pmatrix} F(\alpha(t^1))w_1 \\ F(\alpha(t^2))w_2 \end{pmatrix}.$$

Then, the coefficients $\underline{\alpha}$ are given by

$$\underline{\alpha}^{(k+1)} = \underline{\underline{M}}^{-1} \underline{r}(\underline{\alpha}^{(k)}).$$

Finally, use $\underline{\alpha}^{(k+1)}$ to reconstruct the solution at the time step t^{n+1} :

$$\alpha^{n+1} = \underline{\phi}(1)^T \underline{\alpha}^{(k+1)}.$$

- Precompute $\underline{\underline{M}}$
- Precompute the rhs vector part using quadratures after a further approximation

$$\underline{r}(\underline{\alpha}) = \underline{\phi}(t^n)\underline{\alpha}^n + \int_{T^n} \underline{\phi}(t)F(\underline{\phi}(t)^T\underline{\alpha})dt \approx \underline{\phi}(t^n)\underline{\alpha}^n + \underbrace{\int_{T^n} \underline{\phi}(t)\underline{\phi}(t)^T dt}_{\text{Can be stored}} F(\underline{\alpha})$$

- Precompute the reconstruction coefficients $\underline{\phi}(1)^T$

Outline

- 1 Motivation
- 2 DeC
- 3 ADER
- 4 Similarities**
- 5 Simulations

ADER⁶ and DeC⁷: immediate similarities

- High order time-space discretization
- Start from a well known space discretization (FE/DG/FV)
- FE reconstruction in time
- System in time, with M equations
- Iterative method / K corrections
- Both high order explicit time integration methods (neglecting spatial discretization)

⁶M. Dumbser, D. S. Balsara, E. F. Toro, and C.-D. Munz. A unified framework for the construction of one-step finite volume and discontinuous galerkin schemes on unstructured meshes. *Journal of Computational Physics*, 227(18):8209–8253, 2008.

⁷R. Abgrall. High order schemes for hyperbolic problems using globally continuous approximation and avoiding mass matrices. *Journal of Scientific Computing*, 73(2):461–494, Dec 2017.

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$$\mathcal{L}^1(\underline{\alpha}^{(k)}) = \mathcal{L}^1(\underline{\alpha}^{(k-1)}) - \mathcal{L}^2(\underline{\alpha}^{(k-1)}), \quad k = 1, \dots, K,$$

$$\underline{\underline{M}}\underline{\alpha}^{(k)} - r(\alpha^{(k),0}) - \underline{\underline{M}}\underline{\alpha}^{(k-1)} + r(\alpha^{(k-1),0}) + \underline{\underline{M}}\underline{\alpha}^{(k-1)} - r(\underline{\alpha}^{(k-1)}) = 0$$

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$$\begin{aligned}\underline{\underline{M}}\underline{\alpha}^{(k)} - \cancel{r(\underline{\alpha}^{(k),0})} - \cancel{\underline{\underline{M}}\underline{\alpha}^{(k-1)}} + \cancel{r(\underline{\alpha}^{(k-1),0})} + \cancel{\underline{\underline{M}}\underline{\alpha}^{(k-1)}} - r(\underline{\alpha}^{(k-1)}) &= 0 \\ \underline{\underline{M}}\underline{\alpha}^{(k)} - r(\underline{\alpha}^{(k-1)}) &= 0.\end{aligned}$$

$$\begin{aligned}\mathcal{L}^2(\underline{\alpha}) &:= \underline{\underline{M}}\underline{\alpha} - r(\underline{\alpha}), \\ \mathcal{L}^1(\underline{\alpha}) &:= \underline{\underline{M}}\underline{\alpha} - r(\underline{\alpha}(t^n)).\end{aligned}$$

Apply the DeC Convergence theorem!

- \mathcal{L}^1 is coercive because $\underline{\underline{M}}$ is always invertible
- $\mathcal{L}^1 - \mathcal{L}^2$ is Lipschitz with constant $C\Delta t$ because they are consistent approx of the same problem
- Hence, after K iterations we obtain a K th order accurate approximation of $\underline{\alpha}^*$

$$\mathcal{L}^2(\boldsymbol{\alpha}^0, \dots, \boldsymbol{\alpha}^M) := \begin{cases} \boldsymbol{\alpha}^M - \boldsymbol{\alpha}^0 - \sum_{r=0}^M \int_{t^0}^{t^M} F(\boldsymbol{\alpha}^r) \varphi_r(s) ds \\ \dots \\ \boldsymbol{\alpha}^1 - \boldsymbol{\alpha}^0 - \sum_{r=0}^M \int_{t^0}^{t^1} F(\boldsymbol{\alpha}^r) \varphi_r(s) ds \end{cases} .$$

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$$\chi_{[t^0, t^m]}(t^m) \boldsymbol{\alpha}^m - \chi_{[t^0, t^m]}(t_0) \boldsymbol{\alpha}^0 - \int_{t^0}^{t^m} \chi_{[t^0, t^m]}(t) \sum_{r=0}^M F(\boldsymbol{\alpha}^r) \varphi_r(t) dt = 0$$

$$\int_{t^0}^{t^M} \chi_{[t^0, t^m]}(t) \partial_t (\boldsymbol{\alpha}(t)) dt - \int_{t^0}^{t^M} \chi_{[t^0, t^m]}(t) \sum_{r=0}^M F(\boldsymbol{\alpha}^r) \varphi_r(t) dt = 0,$$

$$\int_{T^n} \psi_m(t) \partial_t \boldsymbol{\alpha}(t) dt - \int_{T^n} \psi_m(t) F(\boldsymbol{\alpha}(t)) dt = 0.$$

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$$\chi_{[t^0, t^m]}(t^m) \boldsymbol{\alpha}^m - \chi_{[t^0, t^m]}(t_0) \boldsymbol{\alpha}^0 - \int_{t^0}^{t^m} \chi_{[t^0, t^m]}(t) \sum_{r=0}^M F(\boldsymbol{\alpha}^r) \varphi_r(t) dt = 0$$

$$\int_{t^0}^{t^M} \chi_{[t^0, t^m]}(t) \partial_t (\boldsymbol{\alpha}(t)) dt - \int_{t^0}^{t^M} \chi_{[t^0, t^m]}(t) \sum_{r=0}^M F(\boldsymbol{\alpha}^r) \varphi_r(t) dt = 0,$$

$$\int_{T^n} \psi_m(t) \partial_t \boldsymbol{\alpha}(t) dt - \int_{T^n} \psi_m(t) F(\boldsymbol{\alpha}(t)) dt = 0.$$

Runge Kutta vs DeC-ADER

Classical Runge Kutta (RK)

- One step method
- Internal stages

Explicit Runge Kutta

- + Simple to code
- Not easily generalizable to arbitrary order
- Stages $>$ order

Implicit Runge Kutta

- + Arbitrarily high order
- Require nonlinear solvers for nonlinear systems
- May not converge

DeC – ADER

- One step method
- Internal subimesteps
- Can be rewritten as explicit RK (for ODE)
- + Explicit
- + Simple to code
- + Iterations = order
- + Arbitrarily high order
- Large memory storage

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A-Stability

$$y'(t) = \lambda y(t) \quad y(0) = 1$$

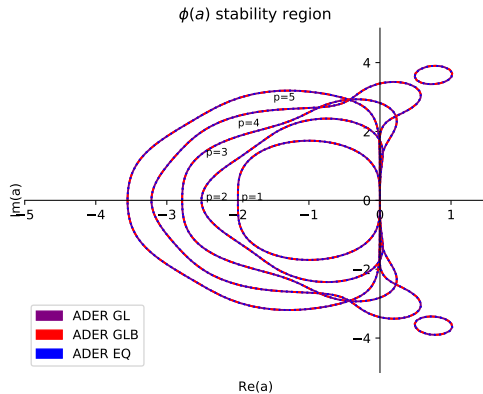
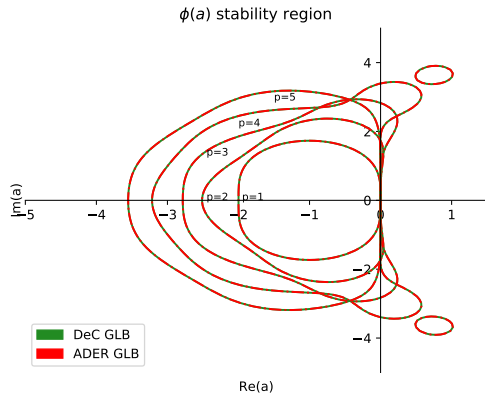
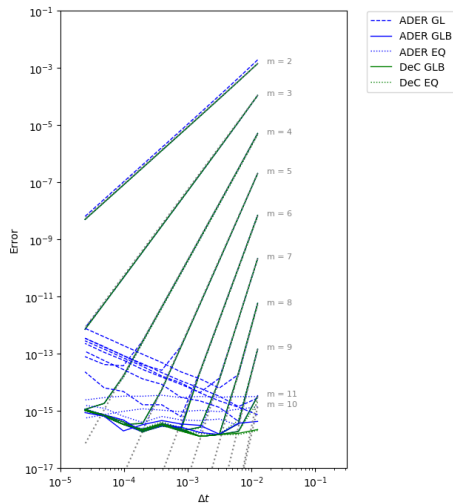


Figure: Stability region

Convergence

$$\begin{aligned}y'(t) &= -|y(t)|y(t), \\ y(0) &= 1, \\ t &\in [0, 0.1].\end{aligned}\tag{12}$$

Convergence curves for ADER and DeC, varying the approximation order and collocation of nodes for the subtimesteps for a scalar nonlinear ODE



Lotka–Volterra

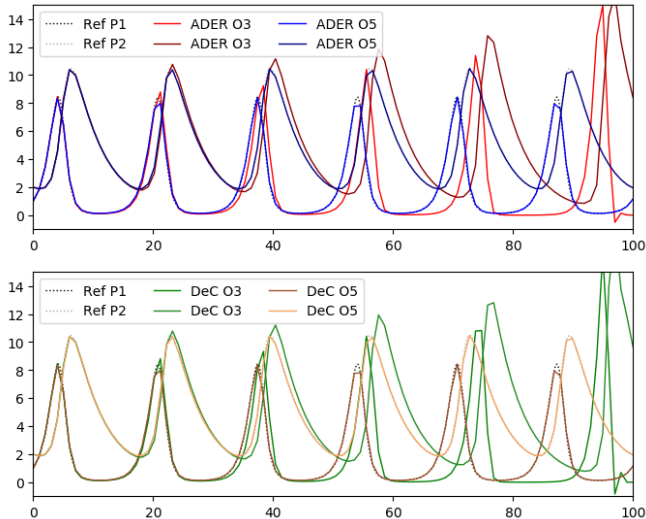


Figure: Numerical solution of the Lotka-Volterra system using ADER (top) and DeC (bottom) with Gauss-Lobatto nodes with timestep $\Delta T = 1$.

PDE: Burgers with spectral difference

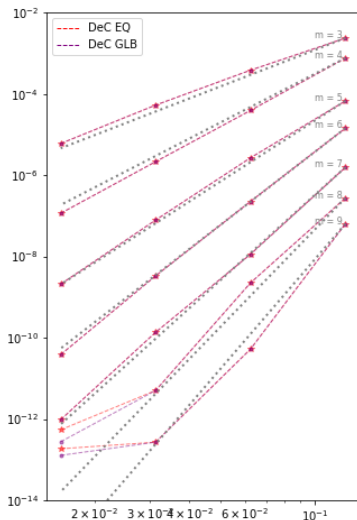
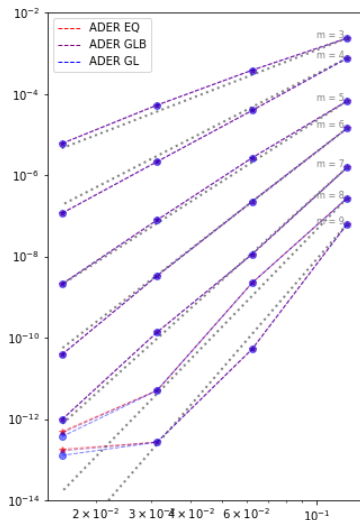


Figure: Convergence error for Burgers equations: Left ADER right DeC. Space discretization with spectral difference