$y : I \to \mathbb{R}^s \qquad s \geq 1$

$$\left.\begin{cases} \boxed{y' = F(t, y(t))} \\ y(0) = y_0 \end{cases}\right] \Rightarrow$$

INTEGRACE

$$\int_0^t y' \, dt = y(t) - y(0)$$

$$y^{(p)} = f(t, y(t), y'(t), \dots y^{(p-1)}(t))$$

$$\begin{cases} y'(t) = z_1(t) \\ z_1'(t) = y''(t) = z_2(t) \\ \vdots \\ z_{p-1}'(t) = z_p(t) \\ \underbrace{z_p'(t)}_{\tilde{y}'} = f(t, \underbrace{y(t), z_1, z_2, \dots z_{p-1}(t)}_{\tilde{y}(t)}) \end{cases}$$

$$\int_0^t y'(t) dt = \underbrace{y(t) - y(0) = \int_0^t F(t, y(t)) dt}$$

$$\begin{cases} y'(t) = -\lambda y(t) & \lambda \in \mathbb{R} \\ y(t_0) = y_0 \end{cases}$$

$$\int_0^t \frac{y'(s)}{y(s)} = -\int_0^t \lambda$$

$$\int_0^t \frac{d}{ds} \log(y(s)) ds = -\lambda(t - 0)$$

$$\log(y(t)) - \log(y(0)) = -\lambda t$$

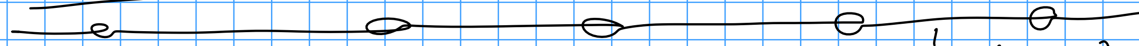
$$\log\left(\frac{y(t)}{y(0)}\right) = -\lambda t$$

$$y(t) = y(0) \cdot e^{-\lambda t}$$

$\lambda > 0 \qquad t \to +\infty \quad y \to 0$

$\lambda < 0 \qquad t \to +\infty \quad y \to \pm\infty$

$$\boxed{y' = -\lambda y \qquad \lambda > 0} \qquad \text{STABILE}$$



$$y' = -\lambda y \Rightarrow y = e^{-\lambda t}$$

$$\begin{cases} c_1'(t) = c_2(t) - 5c_1(t) \\ c_2'(t) = -c_2(t) + 5c_1(t) \\ c_1(0) = 0.9 \qquad c_2(0) = 0.1 \end{cases} \qquad \begin{cases} c' = Ac \\ c(0) = c^0 \end{cases}$$

- $c_1' + c_2' = 0 \qquad c_1(t) + c_2(t) = M \quad \forall t$

$$c(t) = \underbrace{e^{At}} c^0 \qquad e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

$$A^2 = -6A \qquad A^k = (-6)^{k-1} A$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{(-6)^{k-1} A\, t^k}{k!}$$

$$= 1 + \frac{A}{-6} \underbrace{\sum_{k=1}^{\infty} \frac{(-6)^k t^k}{k!}}_{e^{-6t} - 1} = 1 - \frac{1}{6} A \left(e^{-6t} - 1\right)$$

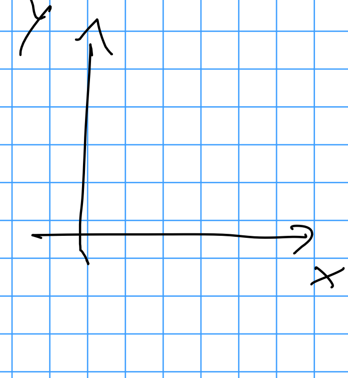
$$= 1 + \frac{1 - e^{-6t}}{6} A \qquad c(t) = c^0 + \frac{1 - e^{-6t}}{6} A c^0$$

- POSITIVITÀ
- CONSERVAZIONE MASSA

LOTKA-VOLTERRA

$$\begin{cases} x'(t) = \alpha x - \beta x y \\ y'(t) = \delta x y - \gamma y \end{cases} \qquad \alpha, \beta, \gamma, \delta > 0$$

$$g(x,y) = \delta x - \gamma \log(x) + \beta y - \alpha \log(y)$$



$$\frac{d}{dt}\eta(x(t), y(t)) = 0$$

$$\langle \nabla\eta, \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} \rangle = \langle \nabla\eta, F \rangle$$

$$\begin{cases} u'(t) = -\frac{v(t)}{r(t)} \\ v'(t) = \frac{u(t)}{r(t)} \\ r(t) = \sqrt{u^2(t)+v^2(t)} \end{cases}$$

$$u' = -\frac{v}{r} - \alpha u$$

$$v' = \frac{u}{r} - \alpha v$$

$$\frac{d}{dt}\left(\frac{r^2}{2}\right) = \frac{d}{dt}\left(\frac{u^2+v^2}{2}\right) = u \cdot \frac{d}{dt}u + v \cdot \frac{d}{dt}v$$

$$= u \cdot \left(-\frac{v}{r}\right) + v \cdot \frac{u}{r} = 0$$

$$= u \cdot \left(-\frac{v}{r} - \alpha u\right) + v \cdot \left(\frac{u}{r} - \alpha v\right) = -\alpha u^2 - \alpha v^2$$

$$= -\alpha r^2$$

$$\begin{cases} \frac{d}{dt}\left(\frac{r^2}{2}\right) = -\alpha r^2 \Rightarrow \\ r(0) = r_0 \end{cases}$$

$$\cancel{r} \cdot \frac{d}{dt} r = -\alpha r^{\cancel{2}}$$

$$r(t) = e^{-\alpha t} \cdot r_0$$

$$\theta(t) = \frac{e^{\alpha t} - 1}{\alpha r(t)}$$

$$u''(t) = -\sin(u) \qquad v = u'$$

$$\begin{cases} u' = v \\ v' (= u'') = -\sin(u) \end{cases}$$

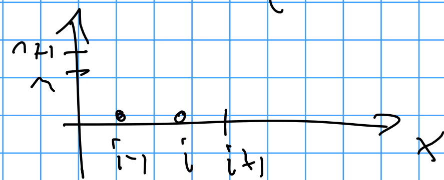
$$\boxed{\eta(t) = \frac{1}{2}v^2 - \cos(u)}$$

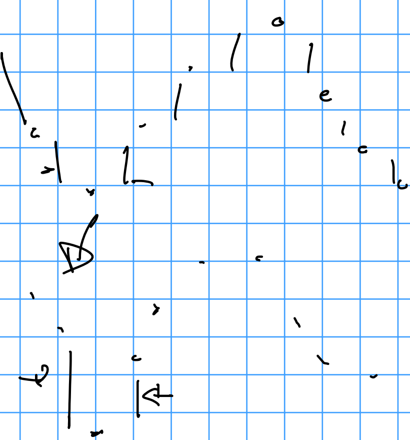
$$\nabla\eta = \begin{pmatrix} \sin(u) \\ v \end{pmatrix}$$

$$\eta'(t) = \langle \nabla_{(u,v)}\eta, \begin{pmatrix} u' \\ v' \end{pmatrix} \rangle = \sin(u) \cdot v + v \cdot (-\sin(u)) = 0$$

PDS

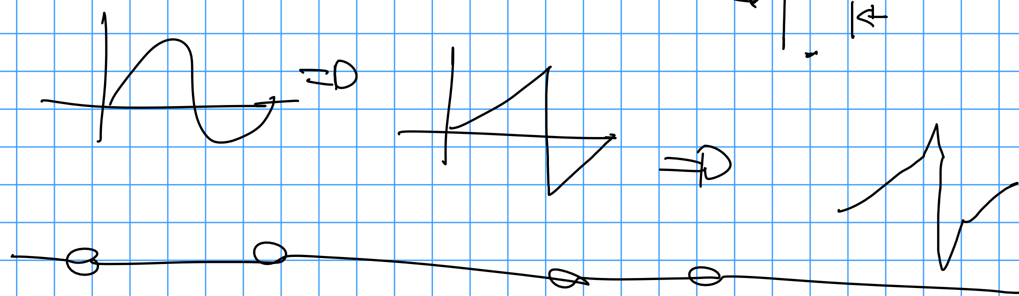
$\partial_t u + \partial_x f(u) = 0 \qquad u \in \mathbb{R}$





$\partial_t \sum_i |U_i - U_{i-1}| < 0$

BURGERS



$F : I \times \mathbb{R}^S \to \mathbb{R}^S$

$(t_0, y_0) \qquad y'(t) = F(t, y(t))$

$t > 0$

$y' = \sqrt{|y|} \qquad y(0) = 0$

$y(t) = 0$ ✓

$y(t) = \frac{t^2}{4} \qquad y' = \frac{t}{2} \qquad \sqrt{\frac{t^2}{4}} = \frac{t}{2}$

UNICITÀ F

$\|F(t,y) - F(t,z)\| \le \underline{\underline{L}}\, \|y - z\|$

$\Downarrow$

STABILITÀ

$y \in C^1$

$\frac{dy}{dt} = F(t, y(t))$

$\Longrightarrow$ se $F \in C^p$

$y \in C^{p+1} \qquad y^{(p+1)} = F^{(p)}(t, y(t))$

$$\begin{cases} \dfrac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases} \qquad y \in \mathbb{R}^S$$

$I = [t^0, t^N]$ $\quad T$

$t^0 \quad t^1 \quad t^2 \quad \cdots \quad t^N$

$$\frac{dy}{dt} \approx \frac{y^{n+1} - y^n}{\Delta t}$$

APPROX $\quad y^0 \quad y^1 \quad y^2 \quad \cdots \quad y^N$

EXACT $\approx \quad y(0) \quad y(t^1), \cdots \quad y(t^N)$

$$y^{n+1} = y^n + \Delta t \underbrace{F(t, y(t))} \approx y^n + \Delta t\, F(t^n, y^n)$$

$$y(t) = y(0) + \underbrace{\int_0^t F(s, y(s))\, ds}$$

$$y^1 \approx y(t^1) = y(0) + (t^1 - 0) \cdot F(t^0, y(0))$$

$$y^{n+1} = y^n + (t^{n+1} - t^n) \cdot F(t^n, y^n)$$

$$e_n = y(t^n) - y^n \qquad \frac{dy}{dt} = F(t, y(t))$$

$$\varepsilon_n = \underbrace{y(t^{n+1}) - y(t^n)} - \Delta t\, \underbrace{F(t^n, y(t^n))}$$

$$= \int_{t^n}^{t^{n+1}} \big(y'(t) - y'(t^n)\big)\, dt$$

$$|\varepsilon_n| \leq w(y', \Delta t) \cdot \Delta t$$

$$w(f, \Delta t) := \max_{\substack{t, t' \\ |t - t'| < \Delta t}} |f(t) - f(t')|$$

$$e_{n+1} \overset{?}{\sim} e_n \qquad = y^n + \Delta t\, F(t^n, y^n)$$

$$e_{n+1} = \overbrace{y(t^{n+1}) - y(t^n) - \Delta t\, F(t^n, y(t^n))}^{\varepsilon_n} - \overbrace{y^{n+1}}$$

$$+ y(t^n) + \Delta t\, F(t^n, y(t^n))$$

$$= \varepsilon_n + y(t^n) + \Delta t\, F(t^n, y(t^n)) - y^n - \Delta t\, F(t^n, y^n)$$

$$= \varepsilon_n + e_n + \Delta t \big(F(t^n, y(t^n)) - F(t^n, y^n)\big)$$

$$|e_{n+1}| \leq |\varepsilon_n| + |e_n| + \Delta t\, L \cdot \underbrace{|y(t^n) - y^n|}_{e_n}$$

$$\leq |\varepsilon_n| + (1+\Delta t L) \cdot |e_n|$$

$$|e_n| \leq e^{L(t^n - t^0)} \underbrace{|e_0|}_{} + \sum_{i=0}^{n-1} e^{L(t^{n+1} - t^n)} |\varepsilon_i|$$

$$y(0) - y^0 = 0$$

$$\leq e^{L(t^n - t^0)} |e_0| + \underbrace{\sum_{i=0}^{n-1} e^{L(t^{n+1} - t^n)}}_{} \cdot \Delta t\, \omega(y', \Delta t)$$

$$\frac{e^{L(t^n - t^0)} - 1}{L} \cdot \Delta t\, \omega(y', \Delta t)$$

$$\boxed{e \to 0 \qquad \Delta t \to 0} \qquad e \sim O(\Delta t)$$

$$e \approx c \cdot \Delta t$$

TAYLOR EXPANSION

$$e_1 = |y^1 - y(t^1)| = |\cancel{y^0} + \Delta t\, \cancel{F(t^0, y^0)}$$

$$- \left(\cancel{y(t^0)} + \cancel{\Delta t\, y'(t^0)} + \frac{\Delta t^2}{2} y''(t^0) \right.$$

$$+ O(\Delta t^3) | \qquad \Delta t = \frac{T}{N}$$

$$= \boxed{\frac{\Delta t^2}{2} \cdot |y''(t^0)|} + O(\Delta t^3) \qquad I = [0, T]$$

$$e_N \approx \sum_{i=1}^{N} |y(t^i) - y^i| \lessapprox N \cdot \frac{\Delta t^2}{2} \max_{t \in [t^0, t^N]} |y''(t)|$$

$$= \cancel{N} \cdot \frac{T}{\cancel{N}} \cdot \frac{\Delta t}{2} \cdot \max |y''(t)| = \underline{\underline{C \cdot \Delta t}}$$

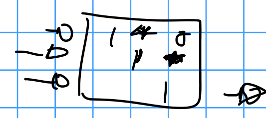
the maximum FIRST ORDER

ORDER OF ACCURACY $\forall p \quad \exists C$

$$|e_N| \leq C \cdot \Delta t^p \qquad \forall \Delta t \in \mathbb{R}^+$$

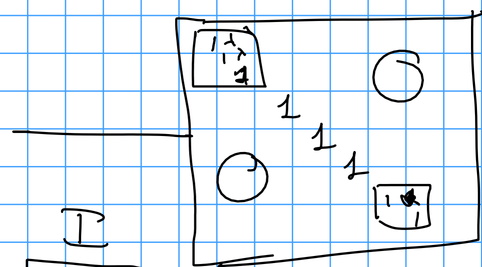
$y'(t) = M\, y(t) \qquad M \in \mathbb{R}^{s\times s}$

$$y^{n+1} = y^n + \Delta t\, M y^n = (I + \Delta t M)\, y^n$$
$$= (I + \Delta t M)^2\, y^{n-1}$$
$$= \underline{(I + \Delta t M)^{n+1}}\, y^0$$



$\hat{y}^n = S^{-1} y^n \qquad S \in \mathbb{R}^s$ INVERTIBLE

$\hat{M} = S^{-1} M S$



$= \hat{M}$

$$\hat{y}^{n+1} = S^{-1} y^{n+1} =$$
$$= \underbrace{S^{-1} y^n}_{\hat{y}^n} + \Delta t\, \underbrace{S^{-1} M S}_{\hat{M}} \cdot \underbrace{S^{-1} y^n}_{\hat{y}^n} = \underline{\underline{(I + \Delta t \hat{M})\, \hat{y}^n}}$$

$y'(t) = q\, y(t) \qquad q \in \mathbb{C}$ DAHLQUIST EQUATION

$$y^{n+1} = y^n + \Delta t q\, y^n = (1 + \Delta t q)\, y^n$$
$$= (1 + z)\, y^n \qquad z = \Delta t q \in \mathbb{C}$$

$$\boxed{Re(q) \le 0 \qquad \partial_t |y| \le 0}$$

$Re(q) > 0 \qquad |y(t)| \to +\infty$ DON'T CARE

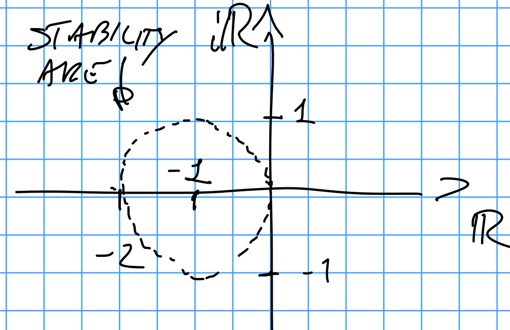
DISCRETE ANALOGOUS

$|y^{n+1}| \le |y^n| \qquad Re(q) \le 0$

$|(1+z)\, y^n| \le |y^n|$

$|\underbrace{1+z}| \le 1 \iff$

$z = \underline{\underline{\Delta t}} \cdot \underline{q}$



$y^{n+1} = R(z) \cdot y^n$ $R(z)$ STABILITY FUNCTION

$q \in \mathbb{R}$ $z = q\Delta t$ $q \le 0$

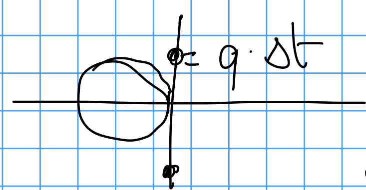
$|1 + \underbrace{q\Delta t}| \le 1$ $-1 - q\Delta t \le 1$

$\to |q|\Delta t < 2$ $\underbrace{-q}_{=|q|}\Delta t \le 2$

$\Delta t \le \frac{2}{|q|} = \frac{2}{L_f}$

$u'' = -u \Rightarrow u' = v$, $v' = -u$ $y'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y(t)$

$\lambda_i \in i\mathbb{R}$

$\to$ 

$q_i = \lambda_i = \pm i$

$\Delta t \to 0$ $q \in i\mathbb{R}$

$R(\Delta t q) > 1$

IMPLICIT EULER

$y^{n+1} = y^n + \Delta t\, f(t^{n+1}, y^{n+1})$

- NONLINEAR SOLV.
- FIRST ORDER

$y' = qy$ $q \in \mathbb{C}$

$y^{n+1} = y^n + \Delta t q y^{n+1}$

$(1 - \Delta t q) y^{n+1} = y^n$

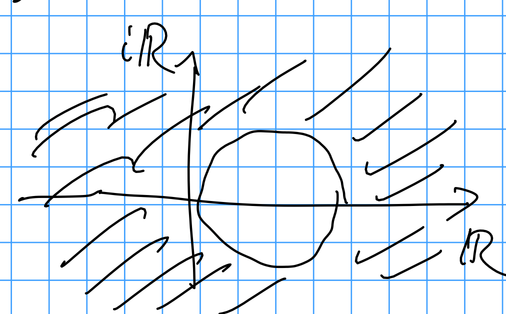
$y^{n+1} = \frac{1}{1 - \Delta t q} \cdot y^n = R(z) \cdot y^n$

STAB. FUNC.

$R(z) = \frac{1}{1-z}$ $z = q\Delta t$

$|R(z)| \le 1 \iff \left|\frac{1}{1-z}\right| \le 1 \iff |1-z| \ge 1$

$z \in \mathbb{C}$



PDS $\quad c_i'(t) = \sum_j p_{ij}(c) - d_{ij}(c)$

$$p_{ij}, d_{ij} \geq 0$$

$$p_{ij} = -d_{ij}$$

SYSTEM IS POSITIVE

$C(0) \geq 0 \quad$ IF $\quad d_{ij}(c) \xrightarrow[c_i \to 0]{} 0 \quad \Rightarrow \quad c_i(t) \geq 0$

$$y' = \Pi y \qquad \Pi_{ii} < 0 \qquad \Pi_{ij} \geq 0 \quad i \neq j$$

$$\boxed{\sum_i \Pi_{ij} = 0 \quad \text{PDS}}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}' = \begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

THEO.

$$A = I - \Delta t \Pi \qquad A^{-1} \geq 0$$

$$A_{ij} < 0 \quad i \neq j$$

1. $0 < A_{ii} = 1 + \Delta t |\Pi_{ii}| > \Delta t |\Pi_{ii}| = \Delta t \sum_{j \neq i} |\Pi_{ji}|$

$$|\Pi_{ii}| = \sum_{j \neq i} |\Pi_{ji}| \qquad \Rightarrow \text{NON SINGULAR}$$

2. JACOBI METHOD $\qquad B = D^{-1}(D - A) \qquad B_{ii} = 0$

$$D = \operatorname{diag}(A) \qquad B_{ji} = \frac{-A_{ji}}{A_{ii}}$$

$$\rho(B) \leq \|B\|_\infty = \max_i \sum_{j \neq i} B_{ji} = \max_i \sum_{j \neq i} \frac{|A_{ji}|}{|A_{ii}|} < 1$$

$$\boxed{\underline{\underline{A y^{n+1} = y^n}}}$$

$$w^{(k+1)} = D^{-1}\left(y^n + (D - A) w^{(k)}\right)$$

$$y^{n+1} = D^{-1}\left(y^n + (D - A) y^{n+1}\right)$$

$$= D^{-1}\left(\cancel{y^n} + D y^{n+1} - \cancel{y^n}\right)$$

$$= y^{n+1}$$

$$\underline{\underline{e}}^{(k+1)} = w^{(k+1)} - y^{n+1} = D^{-1}\left(y^n + (D - A) w^{(k)}\right) -$$

$$e^{(k+1)} < e^{(k)}$$

$$D^{-1}\left(y^n + (D-A)\,y^{n+1}\right)$$

$$D\,(D-A)\left(w^{(k)} - y^{n+1}\right)$$

$$\rho(B) < 1 \qquad = \underline{\underline{B\,e^{(k)}}}$$

$$(D-A)_{ij} = -A_{ij} > 0 \qquad i \neq j$$

$$w^{(k+1)} \geq 0$$

$$w^{(k+1)} = \underbrace{D^{-1}}_{\geq 0}\underbrace{\Big(\overbrace{y^n}^{\geq 0} + \overbrace{(D-A)}^{\geq 0}\,\overbrace{w^{(k)}}^{\text{BY IND } \geq 0}\Big)}_{\geq 0} \qquad \geq 0$$

$$y^{n+1} \geq 0 \qquad \text{IF } y^n \geq 0$$

$O(\Delta t)$ NOT TOO MUCH ACCURATE

$O(\Delta t^p)$ HIGH ORDER $p = 2$

ARBITRARILY HIGH ORDER $\forall p \in \mathbb{N}$

$$U^n \longrightarrow U^{n+1} \qquad U^{n-3}\; U^{n-2}\; U^{n-1}\; U^n \longrightarrow U^{n+1}$$

$\{y^{(s)}\}$

ONE STEP METHODS

$$\frac{dy}{dt} = F(t, y(t))$$

$$\frac{d^2y}{dt^2} = F^{(1)}(t, y(t))$$

θ METHOD $\qquad y^n \to y^{n+1} \qquad y^n \to y^* \to y^{n+1}$

$$y^* \approx y(t^n + \theta\,\Delta t) \qquad t^* = t^n + \theta\,\Delta t$$

$$y^* = y^n + \theta\,\Delta t\,F(t^n, y^n)$$

$$y^{n+1} = y^n + \Delta t\left(\frac{2\theta - 1}{2\theta}\,F(t^n, y^n) + \frac{1}{2\theta}\,F(t^*, y^*)\right)$$

TAYLOR EXPANSION EXACT VS APROX

$$y(t^{n+1}) = y^n + \Delta t\, y'(t^n) + \frac{\Delta t^2}{2} y''(t^n) + O(\Delta t^3)$$

$$y(t^*) = y^n + \theta \Delta t\, y'(t^n) + \underbrace{\frac{\theta^2 \Delta t^2}{2} y''(t^n)} + O(\Delta t^3)$$

$$y^* = y^n + \theta \Delta t F(y^n) = y(t^*) + O(\Delta t^2)$$

$$y^{n+1} = y^n + \Delta t \left(\frac{2\theta - 1}{2\theta} F(y^n) + \frac{1}{2\theta} F(y^*) \right) =$$

$$= y^n + \Delta t \left(\frac{2\theta-1}{2\theta} F(y^n) + \frac{1}{2\theta} F(y^n + \theta \Delta t\, y'(t^n)) \right) + O(\Delta t^3)$$

$$= y^n + \Delta t \left(\frac{2\theta-1}{2\theta} F(y^n) + \frac{1}{2\theta} \left[F(y^n) + \theta \Delta t\, y'(t^n) \cdot \frac{dF}{dy}(y^n) \right] \right)$$

$$= y^n + \Delta t \left(\underbrace{\frac{2\theta-1}{2\theta} y'(t^n) + \frac{1}{2\theta} y'(t^n)} + \frac{\Delta t}{2} \underbrace{y'(t^n) \frac{dF}{dt}(y^n)}_{\frac{d}{dt} y'(t^n)} \right)$$

$$y^n + \Delta t\, y'(t^n) + \frac{\Delta t^2}{2} y''(t^n) + \underbrace{O(\Delta t^3)}$$

$$= y(t^{n+1}) + \underbrace{O(\Delta t^3)}$$

$\Rightarrow$ SECOND ORDER

$y'(t^n) = F(y)$

$u_t + f(u)_x = 0$ $\quad F = f(u)_x$

$\Delta t^3 \quad T = \frac{N}{\Delta t} \cdot \Delta t^3 = \Delta t^2$

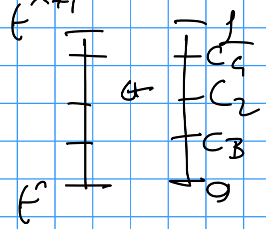
RK METHODS $\quad$ ONE STEP $\quad y^n \xrightarrow[y^{(1)}, \dots, y^{(k)}, \dots, y^{(s)}]{} y^{n+1}$

$y^n \to y^{n+1}$

$$\begin{cases} y^{(k)} = y^n + \Delta t \sum_{j=1}^{s} a_{kj} F(t^n + c_j \Delta t, y^{(j)}) \quad \forall k = 1, \dots, s \\ y^{n+1} = y^n + \Delta t \sum_{j=1}^{s} b_j F(t^n + c_j \Delta t, y^{(j)}) \end{cases}$$

$$y^{(k)} = y^n + \Delta t \sum_{j=1}^{k-1} a_{kj} F(t^n + c_j \Delta t, y^{(j)})$$

EXPLICIT RK



$$y^{n+1} = y^n + \int_{t^n}^{t^{n+1}} F$$

$$\simeq y^n + \Delta t \sum b_j F(y(t^n + \Delta t c_j))$$

$$\begin{array}{c|c} c & A \\ \hline & b \end{array}$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \theta & \theta & 0 \\ \hline & \frac{2\theta-1}{2\theta} & \frac{1}{2\theta} \end{array}$$

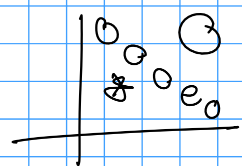
$y^n = y^n + 0 \quad (t^n + 0)$

$y^{\theta} = y^n + \theta \Delta t F(y^n) \quad t^n + \theta \Delta t$

$a_{21} = \theta$

$a_{22} = 0$

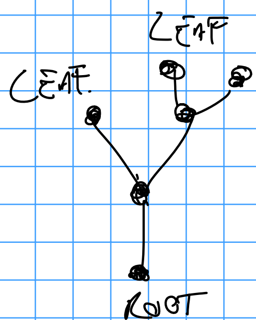
$$y^{n+1} = y^n + \Delta t \frac{2\theta - 1}{2\theta} F^n + \Delta t \frac{1}{2\theta} F^{\theta}$$



$\Rightarrow$ EXPLICIT

$$\boxed{a_{ij} = 0 \quad j \geq i}$$

ORDER = 6



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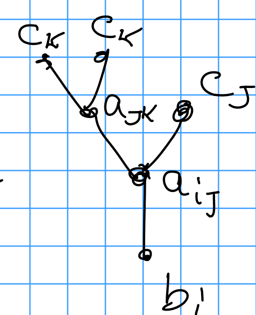
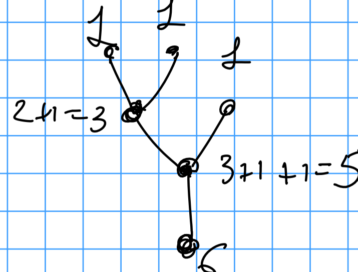


$$\Phi(t) = \sum_{i,j,k} b_i a_{ij} c_j a_{jk} c_k^2$$

ORDER = # NODES



$$\gamma(t) = 1 \cdot 3 \cdot 5 \cdot 6 = 90$$

$$\Phi(t) = \frac{1}{\gamma(t)}$$

$$\sum_{i,j,k} b_i a_{ij} c_j a_{jk} c_k^2 = \frac{1}{90}$$

$\rightarrow$ 1 OF THE CONDITIONS FOR ORDER 6

$$\boxed{c_i = \sum_j a_{ij}} \leftarrow$$

RK $\quad e \sim O(\Delta t^4)$

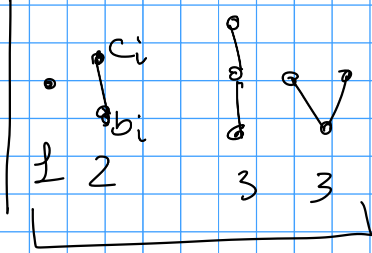
$\Delta t \qquad \Delta t_i = \dfrac{T}{N_i}$

$N_0 = 2 \qquad N_i = 2 \cdot N_{i-1}$

$$e \approx C \Delta t^p$$

$$\underbrace{\log(e)}\approx \log(C\Delta t^p) = \underbrace{p}\,\underbrace{\log(\Delta t)} + \log(C)$$

STABILITY FOR RK



$\underline{y' = qy} \qquad y \in \mathbb{R} \qquad q \in \mathbb{C}$

$F(y) = qy \qquad \mathrm{Re}(q) \leq 0$

$$y^{n+1} = R(\Delta t \cdot q)\, y^n = R(z)\, y^n \qquad z \in \mathbb{C}$$

$$S = \{ z \in \mathbb{C} : |R(z)| \leq 1 \}$$

$$y^{(k)} = y^n + \Delta t \; a_{kj} F(y^{(j)}) \qquad \underline{Y} = \begin{pmatrix} y^{(1)} \\ \vdots \\ y^{(s)} \end{pmatrix}$$

$$y^{n+1} = y^n + \Delta t\, b_j F(y^{(j)})$$

$$\underline{b} = (b_1, \ldots, b_s)^T$$

$$\underline{Y} = \mathbb{1} y^n + \Delta t\, q A \underline{Y}$$

$$(I - \Delta t\, q A)\underline{Y} = y^n \mathbb{1}$$

$$\underline{Y} = (I - \Delta t\, q A)^{-1} \mathbb{1}\, y^n$$

$$y^{n+1} = y^n + \Delta t\, \underline{b}^T F(\underline{Y}) =$$

$$y^n + \Delta t\, q\, b^T (I - \Delta t\, q A)^{-1} \mathbb{1}\, y^n$$

$$= (1 + \overset{z}{\cancel{\Delta t q}}\, b^T (I - \overset{z}{\cancel{\Delta t q}} A)^{-1} \mathbb{1}) \cdot y^n$$

$$= R(z) \cdot y^n$$

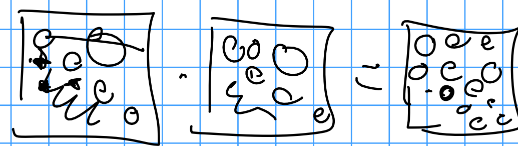
EXPL $R(z)$ POLYNOMIAL degree $\leq S$

IMPL $R(z)$ FRACTION OF POLYS degree $\leq S$

$$R(z) = 1 + z\, b^T \underbrace{(I - zA)^{-1}}\, \mathbb{1}$$

$$(I - zA)^{-1} = 1 + zA + (zA)^2 + (zA)^3 + \cdots$$

IF A EXP RK



$$A = \begin{bmatrix} 0 & 0 & 0 \\ 2/3 & 0 & 0 \\ 1/3 & 1/3 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2/9 & 0 & 0 \end{bmatrix} \qquad A^3 = \underline{0}$$

$$= I + zA + z^2 A^2 =$$

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6}$$

$$< 1$$

$$|y^{n+1}| = |R(z)|\, |y^n| < |y^n|$$

IMPLICIT RK $\quad U^n \to U^{n+1} \quad u' = F(t, u)$

$$\underbrace{y^{(k)} = U^n + \Delta t \sum_{j=1}^{S} a_{kj} F(t^n + c_j \Delta t, y^{(j)})} \quad \forall k = 1, .., S$$

(EXPL $a_{kj} = 0 \quad j \geq k$)

$$y^{(j)} \approx u(t^n + c_j \Delta t) \qquad c_j$$

$$y^{(j)} \approx U^n + \int_{t^n}^{t^n + c_j \Delta t} F(t, y(t))\, dt \qquad F^{(k)}$$

$$U^n + \sum_{k=1}^{S} \int_{t^n}^{t^n + c_j \Delta t} \varphi_k(t) \cdot F(t^n + c_k \Delta t, y^{(k)})\, dt$$

$$U^n + \sum_{k=1}^{S} \Delta t\, a_{jk}\, F(t^n + c_k \Delta t, y^{(k)})$$

$$U^{n+1} \simeq U^n + \sum_k \underbrace{\int_{t^n}^{t^{n+1}} \varphi_k(t)}_{b_k} F^{(k)} = U^n + \sum_k b_k F^{(k)}$$

$c_1 = 0$

$c_k \to$ GAUSS-LEGENDRE

$c_s = 1$

SIMPLIES THE TREE ORDER CONDITIONS

BUTCHER 1964

$$B(p): \quad \sum_{i=1}^{s} b_i c_i^{z-1} = \frac{1}{z} \qquad z = 1, \dots, p$$

(c_i, b_i) QUADRATURE FORMULA

NODES WEIGHTS

$B(p) \Leftrightarrow$ QUAD ORDER p

Let $\underline{x^{z-1}}$ for $z = 1, \dots p$

$$\int_0^1 x^{z-1}\,dx = \left[\frac{x^z}{z}\right]_0^1 = \frac{1}{z} \quad //$$

$$\sum_{i=1}^{s} c_i^{z-1} b_i \underset{B(p)}{=} \frac{1}{z}$$

$$C(\eta): \quad \sum_{j=1}^{s} a_{ij} c_j^{z-1} = \frac{c_i^z}{z} \qquad i = 1, \dots, s \quad z = 1, \dots, \eta$$

$$D(\xi): \quad \sum_{i=1}^{s} b_i c_i^{z-1} a_{ij} = \frac{b_j}{z} \cdot (1 - c_j^z) \qquad j = 1, \dots, s \quad z = 1, \dots, \xi$$

TEO:

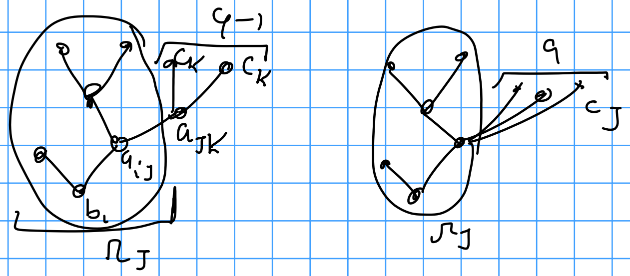
$B(p) \quad C(\eta) \quad D(\xi)$

$p \le \eta + \xi + 1$

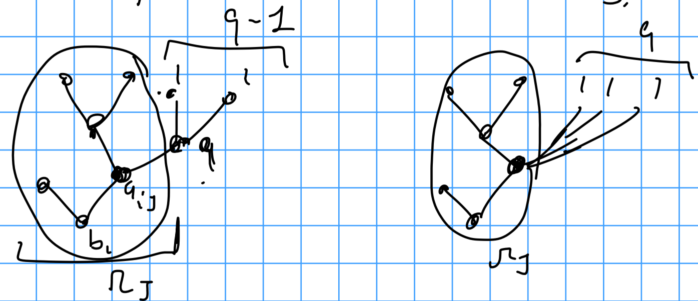
$p \le 2\eta + 2$

$\Rightarrow$ THE RK IS OF ORDER p

$C(q)$



$\Phi = \sum_{J,K} r_K a_{JK} c_K^{q-1}$ $\qquad \Phi = \sum_{J} r_J \cdot c_J^q$



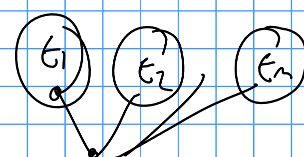
$\gamma = \gamma_r \cdot q \qquad \gamma = \gamma_r$

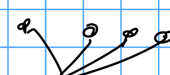
$$\Phi = \sum_{J,K} r_J \underbrace{a_{JK} c_K^{q-1}} = \frac{1}{\gamma_r \cdot q} \qquad \sum r_J c_J^q = \frac{1}{\gamma_r}$$

$$\underset{C(q)}{=} \sum r_J \frac{c_J^q}{\cancel{q}} = \frac{1}{\gamma_r \cdot \cancel{q}}$$

$D(q)$

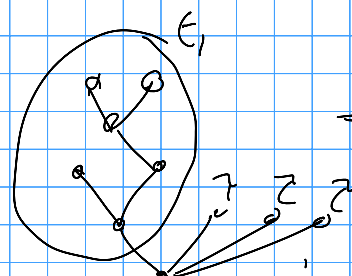
$t = [t_1, \ldots, t_n]$



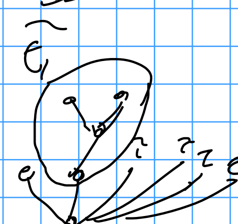


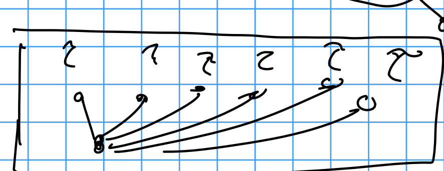
$\tau = \bullet$

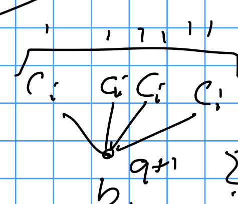
$C(n)$



$\Rightarrow$







$B(q+1)$

$\Rightarrow$

$\sum_i b_i c_i^q = \frac{1}{q+1}$

$y' = \lambda y \qquad Re(\lambda) \le 0 \qquad \lambda \in \mathbb{C}$

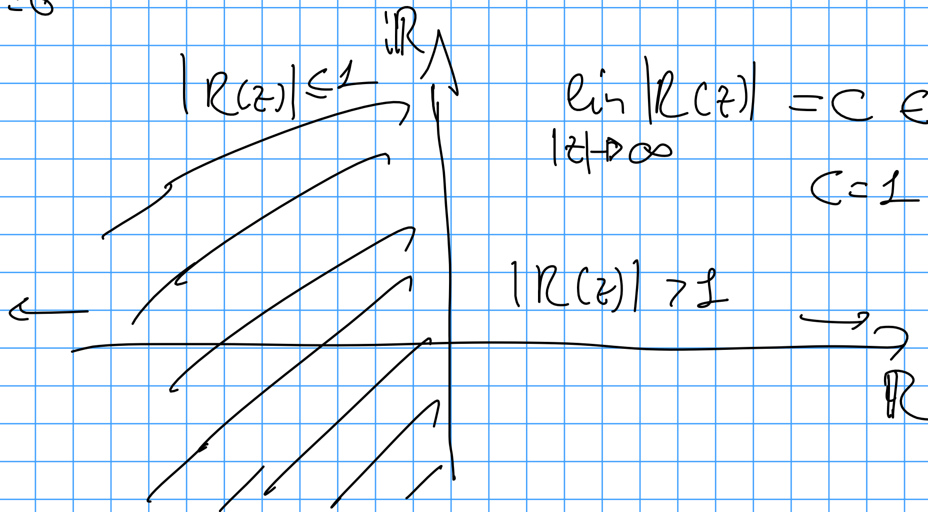
$y^{n+1} = R(h\lambda) \cdot y^n = R(z) \cdot y^n \qquad z \in \mathbb{C}$

$S = \{ |R(z)| \le 1 \}$

• A-stability RK IF $S \supseteq \mathbb{C}^- = \{ z : Re(z) \le 0 \}$

$u' = -k\,(u - \cos(t))$ $\quad$ $\underline{k = 2000}$

$u_0 = 0$

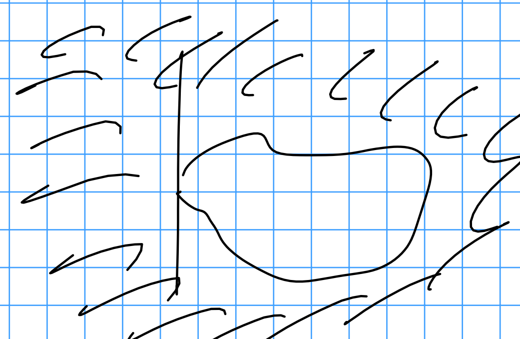


$\lim\limits_{|z| \to \infty} |R(z)| = C \in \mathbb{R}$

$C = 1$

$|R(z)| = 1 \qquad |y^{n+1}| = |y^n|$

$R(z) \xrightarrow[z \to +\infty]{} 0 \quad \Rightarrow$



$z \to +\infty \qquad y^{n+1} \to 0$

$\forall y^n$

• L-stability RK if $R(z) \xrightarrow[|z| \to +\infty]{} 0$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

$$R(z) = 1 + z\, b^T \underbrace{(I - zA)^{-1}}\, \underline{1} \qquad \frac{2-z}{2}$$

$$(I - zA)^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{z}{2} & 1 - \frac{z}{2} \end{pmatrix}^{-1} = \frac{2}{2-z} \begin{pmatrix} 1 - \frac{z}{2} & 0 \\ \frac{z}{2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{z}{2-z} & \frac{2}{2-z} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2+z}{2-z} \end{pmatrix}$$

$$R(z) = 1 + z b^T \begin{pmatrix} 1 \\ \frac{2+z}{2-z} \end{pmatrix} = 1 + z \left(\frac{1}{2} + \frac{2+z}{2(2-z)} \right)$$

$$= 1 + z \left(\frac{2 \cancel{-z} + 2 \cancel{+z}}{2(2-z)} \right) = 1 + \frac{2z}{2-z} = \frac{2-z+2z}{2-z}$$

$$= \frac{2+z}{2-z}$$

L-stable $\quad \lim_{z \to +\infty} R(z) = \lim_{z \to +\infty} \frac{2+z}{2-z} = -1 \neq 0$

COLLOCATION METHODS $\qquad u' = F(t,u)$

$\tau_i = t^n + c_i \Delta t \qquad \ell_i(\tau_j) = \delta_{ij} \qquad \tau_i$ UNIQUE

$$\begin{cases} \underline{u(\tau_i)'} = \underline{F(\tau_i, u(\tau_i))} & i = 1, \dots, s \\ u(t^n) = y^n \end{cases}$$

$y^{n+1} = u(t^{n+1}) \qquad u \in \mathbb{P}_s$

$u' \in \mathbb{P}_{s-1} \qquad u'(t^n + z\Delta t) = \sum_{j=1}^{s} \ell_j(z) \cdot u'(\tau_j)$

$$u(t^n + \Delta t\, z) = y^n + \Delta t \int_0^z u'(t^n + s\Delta t)\, ds$$

$$= y^n + \Delta t \sum_{j=1}^{s} \int_0^z \ell_j(s)\, ds \cdot u'(\tau_j)$$

$$= y^n + \Delta t \sum_{j=1}^{s} \underbrace{\int_0^z \ell_j(s)\, ds} \cdot F(\tau_j, u(\tau_j))$$

$u^{(j)}$

$z \leftarrow c_k \qquad u^{(k)} = y^n + \Delta t \sum_{j=1}^{s} \underbrace{\int_0^{c_k} \ell_j(s) ds}_{a_{kj} =} \cdot F(\tau_j, u^{(j)})$

$z=1 \qquad y^{n+1} = y^n + \Delta t \sum_{j=1}^{s} \underbrace{\int_0^1 \ell_j(s)\,ds}_{b_j} \; F(\tau_j, U^{(j)})$

$B(p): \quad \sum_{i=1}^{s} b_i c_i^{q-1} = \frac{1}{q} \qquad q=1,\dots,p$

- IF IRK = COLLOC. METH. $\underline{\text{WITH quad ord } p}$

$\Rightarrow B(p)$

QUAD FORMULA IS EXACT

$$\int_0^1 s^{z-1}\,ds = \sum c_j^{z-1} \cdot b_j \qquad \forall z \le p$$

$\parallel$

$\frac{1}{z} \longrightarrow B(p)$

$C(S)$ S STAGES OR RK = COLLOC.

$C(S): \rightarrow \underbrace{\sum_j a_{ij} c_j^{z-1}}_{} = \frac{c_i^z}{z} \qquad z=1,\dots,S \quad \forall i=1,\dots,S$

$a_{ij} = \int_0^{c_i} \ell_j(s)\,ds \qquad c_j$ NODES

$s^{z-1} \in \mathbb{P}_{S-1} \qquad S$ points

$s^{z-1} = \sum_{j=1}^{S} \ell_j(s) \cdot c_j^{z-1}$

$\Downarrow$

EXACT BC INTERP EXACT

$$\boxed{\sum_j a_{ij} c_j^{z-1}} = \sum_j \int_0^{c_i} \ell_j(s)\, c_j^{z-1}\,ds =$$

$$= \int_0^{c_i} s^{z-1}\,ds = \left[\frac{s^z}{z}\right]_0^{c_i} \boxed{= \frac{c_i^z}{z}}$$

LEMMA S-stage RK $c_1, \dots c_S \qquad b_1, \dots b_S$

$\Rightarrow \overline{C(S)} \wedge \overline{B(S+\nu)} \Rightarrow \overline{D(\nu)}$

$\left(D(S) \wedge B(S+\nu) \Rightarrow C(\nu) \right)$

$$B(s+\nu):\quad \sum_i b_i c_i^{z-1} = \frac{1}{z} \qquad z=1,\dots,s+\nu$$

$$C(s):\quad \sum_j a_{ij} c_j^{z-1} = \frac{c_i^z}{z} \qquad z=1,\dots,s \quad i=1,\dots,s$$

$$D(\nu):\quad \sum_{i=1}^{s} b_i c_i^{z-1} a_{ij} = \frac{b_j}{z}(1-c_j^z) \qquad z=1,\dots,\nu \quad j=1,\dots,s$$

$$\boxed{d_j^{(z)} := \sum_{i=1}^{s} b_i c_i^{z-1} a_{ij} - \frac{b_j}{z}(1-c_j^z)}$$

$$\Rightarrow \sum_{j=1}^{s} \underbrace{d_j^{(z)}} \cdot c_j^{k-1} = 0 \qquad \forall k=1,\dots,s \qquad z=1,\dots,\nu$$

$$= \sum_{i,j} b_i c_i^{z-1} a_{ij} c_j^{k-1} - \sum_j \frac{b_j}{z}(1-c_j^z)\cdot c_j^{k-1}$$

$$= \sum_i b_i c_i^{z-1} \cdot \frac{c_i^k}{k} - \sum_j \frac{b_j}{z}\cdot c_j^{k-1} + \sum_j \frac{b_j}{z}\cdot c_j^{z+k-1} \qquad \begin{matrix} k\le s \\ z\le \nu \end{matrix}$$

$$= \frac{1}{k}\sum_i b_i c_i^{z+k-1} - \frac{1}{zk} + \sum_j \frac{b_j}{z} c_j^{z+k-1} \qquad z+k-1 \le s+\nu-1$$

$$= \frac{1}{k}\cdot\frac{1}{z+k} - \frac{1}{zk} + \frac{1}{z}\cdot\frac{1}{z+k} = \frac{\cancel{z} - \cancel{(z+k)} + \cancel{k}}{zk(z+k)} = 0$$

$$[c_j^{k-1}] = \tilde{C}_{jk} \qquad \text{VANDERMOND MATRIX}$$

INVERTIBLE

$$\tilde{C}\cdot d^{(z)} = 0 \qquad \forall z=1,\dots,\nu$$

$$\Rightarrow d^{(z)} = 0 \qquad \forall z=1,\dots,\nu \qquad \boxtimes$$

GAUSS-LEGENDRE $(c_i, b_i)_{i=1}^{s}$ ORDER $2s$

COLLOCATION $C(s)$ ✓

$B(2s)$ ✓

$\nu = s$ $\quad C(s)$ ✓ $\quad B(s+s)$ ✓

$\Rightarrow D(s)$ ✓

$p = 2s \qquad \eta = s \qquad \xi = s$

$p \leq 2\eta + 2 \qquad p \leq \xi + \eta + 1$

$2s \leq 2s + 2$ ✓ $\qquad 2s \leq s + s + 1$ ✓

$\Rightarrow$ ORDER $p = 2s$

$s \rightarrow \underbrace{(c_i, b_i)}_{\text{GAUSS-LEGENDRE}} \Rightarrow B(2s)$ ✓ $\Rightarrow$ GOAL ORDER $2s$

$\Rightarrow$ QUADRATURE ORDER $2s$

COLLOCATION METHOD

$\Rightarrow C(s)$ ✓ $\qquad \int_0 (\cdot \quad \cdot \quad \cdot)_1$

$\Rightarrow$ LEMMA (WANNER, HAIRER) $\qquad D(s)$ ✓

$\Rightarrow$ BUTCHER (1964) $\Rightarrow$ ORDER $2s$

$p = 2s \qquad 2s \leq 2s + 2$ ✓

$\eta = s \qquad 2s \leq s + s + 1$ ✓ $\qquad \boxtimes$

$\xi = s$

$\Rightarrow$ NOT $\angle$-STABLE

RADAU IA $\qquad \frac{d^{s-1}}{dx^{s-1}} \left(x^s (x-1)^{s-1} \right)$

$\Rightarrow$

ZEROS OF POLY = NODES c_i

RADAU IA NO COLLOCATION

$D(s)$ CONDITIONS $\qquad 2s-1$ ACCURACY

$c_i = \sum_{j=1}^{s} a_{ij}$ GIVEN $\qquad b_i$ = QUADRATURE WEIGHTS

$a_{ij} = \underline{\underline{s \times s}}$ UNKNOWNS $\qquad \underline{D(s)} \Rightarrow \underline{\underline{s \times s}}$ equations

$\Rightarrow$ <u>UNIQUE SOLUTION</u>.

$B(2S-1) + D(S) \Rightarrow C(S-1)$

$\Rightarrow$ ORDER $2S-1 = p$ $\quad \xi = S \quad \nu = S-1$

$p = 2S-1 \leq 2\nu + 2 = 2S-2+2 = 2S$ ✓

$p = 2S-1 \leq \nu + \xi + 1 = S + S - 1 + 1 = 2S$ ✓

$\Rightarrow$ ORDER $2S-1$ ✓

RADAU $\underline{\text{IIA}}$ $\quad \underline{\underline{C(S)}}$ $\quad b_i, c_i \Rightarrow$ QUADRATURE

a_{ij} $S \times S$ COEFF $\quad$ $2S-1$ ACCURACY

$C(S)$ $S \times S$ EQUATIONS

UNIQUE DEFINITION

$$A_{ij} \underbrace{[c_j^{q-1}]} = \frac{c_i^q}{q}$$

$\Downarrow$

$$\sum_j A_{ij} \underbrace{C_{jq}} = R_{iq} \Leftrightarrow A_{ij} = \sum_q R_{iq} [C^{-1}]_{qj}$$

C VANDERMONDE MATRIX IF c_i DISTINCT

$\Rightarrow C$ IS INVERTIBLE ✓

DEFINITION RADAU IIA

$C(S) \leftarrow$

COLLOCATION METHOD ON $(b_i, c_i) \Rightarrow C(S)$

$B(2S-1) + C(S) \Rightarrow D(S-1)$

$\Rightarrow$ ORDER $2S-1$

LOBATTO $\Rightarrow (c_i, b_i)$ GAUSS-LOBATTO NODES/QUADRATURES

$$\frac{d^{S-2}}{dx^{S-2}} \left(x^{S-1}(x-1)^{S-1} \right) \quad \Rightarrow \text{ORDER } 2S-2$$

$0, 1 \in \{c_i\}$

$B(2S-2)$

a_{ij}

$\Rightarrow$ $\underline{\text{IIIA}}$ $C(S)$ (COLLOCATION METHOD)

$\Rightarrow D(S-2) \Rightarrow$

$P = 2S-2 \leq 2\eta + 2 = 2S+2$ ✓

$P = 2S-2 \leq \eta + \xi + 1 = S + S-2+1 = 2S-1$ ✓

$\Rightarrow$ ORDER $2S-2$ ✓

III B $D(S) \Rightarrow$ UNIQUE METHOD

$+ \; C(S-2) \qquad \Rightarrow$ ORDER $2S-2$ ✓

III C $\quad C(S-1) \Rightarrow S \times (S-1)$ equations

$a_{SJ} = b_J \quad S$ CONDITIONS

$\Rightarrow$ UNIQUE

$C(S-1) + B(2S-2) \Rightarrow D(S-1)$

$P = 2S-2 \leq 2\eta + 2 = 2S-2+2 = 2S$ ✓

$P = 2S-2 \leq \eta + \xi + 1 = S-1+S-1+1 = 2S-1$ ✓

$\Rightarrow$ ORDER $2S-2$ ✓

$$y^{(1)} \approx u(t^n)$$

$$\cancel{\times}\; u(t^n)$$

$$y^{(S)} = u^n + \Delta t \sum_{J=1}^{S} a_{SJ} \, F(y^{(J)})$$

$$y^{n+1} = u^n + \Delta t \sum_J b_J \, F(y^{(J)}) \qquad \boxed{S \times \dim}$$

DIAGONALLY IMPLICIT RUNGE KUTTA

$N_{ODE} \qquad S \times N_{ODE} = D \quad$ DIMENSION

(NON) LINEAR SYSTEM

COST $\mathcal{O}(D^2) \sim \mathcal{O}(D^3)$

$\mathcal{O}(S^2 N_{ODE}^2) \sim \mathcal{O}(S^3 N_{ODE}^3)$

LOBATTO

$S = 4 \qquad \Rightarrow \qquad 16 \cdot \underline{\mathcal{O}(N_{ODE}^2)} \sim 64 \, \mathcal{O}(N_{ODE}^3)$

DIRK $\quad \Rightarrow S = 10 \qquad 10 \cdot \mathcal{O}(N_{ODE}^2) \sim 10 \cdot \mathcal{O}(N_{ODE}^3)$

$$u^{(k)} = U^n + \Delta t \sum_{j=1}^{k} a_{kj} F(U^{(j)})$$

$U^{(1)} \Rightarrow$ (NON)LINEAR SYS DIMER N_{ODE}

$U^{(2)} \to N_{ODE} \quad \forall k$

C.N.

$$U^{(1)} = U^n$$

$$U^{(2)} = U^n + \Delta t \left(\frac{1}{2} F(U^{(1)}) + \frac{1}{2} F(U^{(2)}) \right)$$

$$R(U^{(2)}) = U^{(2)} - U^n - \Delta t \left(\frac{F(U^{(1)}) + F(U^{(2)})}{2} \right)$$

NEWTON METHOD $\quad JR\,?$

$$JR(U^{(2)}) = Id - \frac{\Delta t}{2} JF(U^{(2)})$$

$$\boxed{U^{(2),N_N} = U^{(2),N_N-1} - JR^{-1} R(U^{(2),N_N-1})}$$

ROSENBROCK METHODS

$$G^{(k)} = \Delta t F(y^{(k)})$$

$$\underline{G^{(k)}} = \Delta t \, F\left(y^n + \Delta t \sum_{j=1}^{k} a_{kj} G^{(j)}\right)$$

$$\approx \left[\Delta t F\left(y^n + \Delta t \sum_{j=1}^{k-1} \alpha_{kj} G^{(j)}\right) + \Delta t \sum_{j=1}^{k} \gamma_{kj} \underline{JF(y^n)} \, G^{(j)} \right]$$

$$(I - \Delta t \gamma_{kk} J) G^{(k)} = [\quad]$$

DIRK

SDIRK $\qquad \gamma_{kk} = \gamma \quad \forall k$

$$(I - \Delta t \gamma J(y^n))$$

STABILITY REGION

$$y' = \lambda y \qquad Re(\lambda) \le 0 \qquad z = \lambda \Delta t$$

$$G^{(k)} = z y^n + z \sum_j \underbrace{(\alpha_{kj} + \gamma_{kj})}_{a_{kj}} G^{(j)}$$

$$(I - zA) \underline{G} = z \cdot \mathbb{1} y^n$$

$$y^{n+1} = y^n + b^T G = y^n + z b^T (I - zA)^{-1} \mathbb{1} y^n$$

$$\underbrace{\left(1 + z b^T (I - zA)^{-1} \mathbb{1}\right)}_{R(z)} y^n$$

IMPLICIT - EXPLICIT RUNGE KUTTA

ADVECTION TERMS $\partial_x F(u)$ — EXPLICIT $\boxed{\Delta t \le C \cdot \Delta x}$ STABILITY

DIFFUSION TERMS $\boxed{d\, \partial_{xx} u}$ — EXPLICIT $\Delta t \le C \cdot \Delta x^2$ STABLE

TOO LONG

DISP TERMS $\boxed{d\, \partial_{xxx} u}$ — $\Delta t \le C \cdot \Delta x^3$

$$\boxed{\partial_t u + F(u) + S(u) = 0}$$

$S(u) \to \infty$

EXPLICIT - STABLE

$\boxed{\Delta t \sim \varepsilon}$ $\quad \Delta t \to 0$

TOO LONG

$$\frac{u - f}{\varepsilon} \xrightarrow[\varepsilon \to 0]{} \infty$$

$$\frac{U^{n+1} - U^n}{\Delta t} + \underbrace{F(U^n)} + \underbrace{S(U^{n+1})} = 0$$

$$y^{(k)} = y^n + \Delta t \sum_{j=1}^{k-1} \tilde{a}_{kj} F(y^{(j)}) + \Delta t \sum_{j=1}^{s} a_{kj} S(y^{(j)})$$

$$y^{n+1} = y^n + \Delta t \sum_j \tilde{b}_j F(y^j) + \Delta t \sum b_j S(y^{(j)})$$

$\begin{array}{c|c} \tilde{c} & \tilde{A} \\ \hline & \tilde{b} \end{array}$ || EXPLICIT $\quad \begin{array}{c|c} c & A \\ \hline & b \end{array}$ || IMPLICIT

IMEX RK $\quad c = \tilde{c}$

$$\partial_t y = \underbrace{\lambda_1 y}_{\text{NON-STIFF}} + \underbrace{\lambda_2 y}_{\text{STIFF}}$$

① SIMPLIFICATION
$\lambda_1 \in i\mathbb{R}$
$\lambda_2 \in \mathbb{R}^-$

$$R(z_1, z_2) = 1 + (z_1 \hat{b}^T + z_2 b^T)(I - z_1 \hat{A} - z_2 A)^{-1} \mathbb{1}$$

$$R : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

$$|R| : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^+$$

② HUNSDORFER
$z_1 \in \{ |1 + z_1| \le 1 \}$

③ z_2

$\Rightarrow z_2 \quad |R(z_1, z_2)| < 1$

$\Rightarrow z_1 \Rightarrow$ STABILITY

RK ONE-STEP

$$U^n \xrightarrow{\text{STAGES}} U^{n+1}$$

MULTI-STEP

$$\dots U^{n-2},\ U^{n-1},\ U^n,\ U^{n+1}$$

$$F(U^{n-2}) \dots F(U^{n-1})$$

$\sim \quad U^0 \to U^1$

GENERAL LINEAR M.S. METHOD

$$y^{n+1} + \sum_{j=1}^{K} \alpha_j y^{n-j+1} = \Delta t \sum_{j=0}^{K} \beta_j F(t^{n-j+1}, y^{n-j+1})$$

CHEAP
COMPUT RHS

$$\underbrace{y^{n-K+1} \quad \dots \quad y^{n-1} \quad y^n}_{K} \quad y^{n+1}$$

$$F(y^{n-K+1}) \quad \dots \quad F(y^n) \quad F(y^{n+1})$$

$$j = K \quad \dots \quad 2 \quad 1 \quad 0$$

STABILITY $\quad y' = q y \quad \mathrm{Re}(q) \le 0$

$$\underbrace{y^{n+1} = \xi y^n = \xi^n y^1}_{R(q)} \qquad |\xi| \le 1$$

$$y^{n+1} + \sum_{J=1}^{K} \alpha_J \, y^{n-J+1} = \Delta t q \sum_{J=0}^{K} \beta_J \, y^{n-J+1}$$

$$\underbrace{\xi^K + \alpha_1 \xi^{K-1} + \alpha_2 \xi^{K-2} + \dots \alpha_K \xi^0} = \underbrace{\Delta t q}_{z \in \mathbb{C}} \underbrace{\left(\beta_0 \xi^K + \dots + \beta_K \xi^0\right)}$$

$|\xi|$?

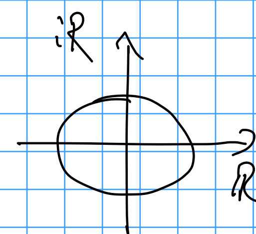
POLY degree K

FIND ZEROS ξ_i

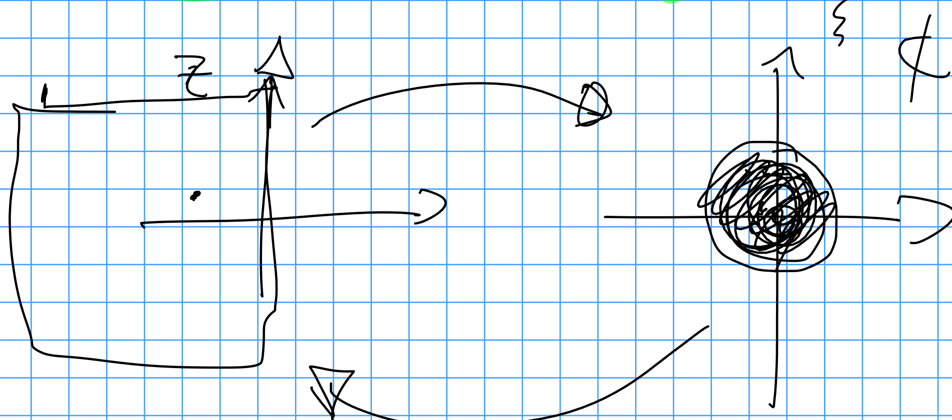
EACH ZERO $|\xi_i| \le 1$

MULTIPLE ZERO $|\xi_i| < 1$

$z \mapsto \xi$ $\quad |\xi| \le 1$



$$z = \frac{\xi^K + \alpha_1 \xi^{K-1} + \dots + \alpha_K \xi^0}{\beta_0 \xi^K + \dots + \beta_K \xi^0}$$



ACCURACY $\quad y' = F(y)$

$$y^{n+1} + \sum_{J=1}^{K} \alpha_J \, y(t^{n-J+1}) - \Delta t \sum_{J=0}^{K} \beta_J \underbrace{y'(t^{n-J+1})}$$

$$= y(t^{n+1}) + \sum_{J=1}^{K} \alpha_J \sum_{\ell=0}^{P} \frac{(-J\Delta t)^\ell}{\ell!} \cdot y^{(\ell)}(t^{n+1})$$

$$- \Delta t \sum_{J=1}^{K} \beta_J \sum_{\ell=0}^{P-1} \frac{(-J\Delta t)^\ell}{\ell!} \, y^{(\ell+1)}(t^{n+1})$$

$$\underbrace{+ O(\Delta t^{P+1})}$$

$\ell = 0$

$$= y(t^{n+1}) \left(1 + \sum_{J=1}^{K} \alpha_J \overbrace{\frac{(-J\Delta t)^0}{0!}}^{=1} \right.$$

$$+\, y'(t^{n+1}) \left(\underbrace{\sum_{J=1}^{k} \alpha_J \frac{-J\Delta t}{1}}_{\ell=1} - \underbrace{\Delta t \sum_{J=0}^{k} \beta_J}_{\ell=0} \right)$$

$$+\, \Delta t^{\ell}\, y^{(\ell)}(t^{n+1}) \left[\underbrace{\sum_{J=1}^{k} \alpha_J \frac{(-J)^{\ell}}{\ell!}}_{\ell=\ell} - \sum_{J=0}^{k} \beta_J \frac{(-J)^{\ell-1}}{(\ell-1)!} \right]$$

ORDER 0 $\Rightarrow$ $\quad 1 + \sum \alpha_J = 0$

ORDER 1 $\Rightarrow$ $\quad -\sum_{J=1}^{k} \alpha_J J - \sum_{J=0}^{k} \beta_J = 0$

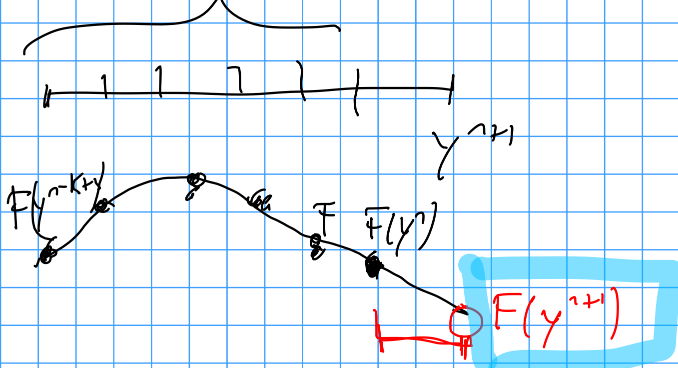
ORDER ℓ $\quad \sum_{J=1}^{k} \alpha_J \frac{(-J)^{\ell}}{\ell!} - \sum_{J=0}^{k} \beta_J \frac{(-J)^{\ell-1}}{(\ell-1)!} = 0$

$\alpha_1, \ldots, \alpha_k \qquad \beta_0, \ldots, \beta_k \qquad 2k+1$ COEFF

ORDER p $\Rightarrow$ p equations

ADAM BASHFORTH EXPLICIT

$$y^{n+1} = y^n + \int_{t^n}^{t^{n+1}} F(t, y(t)) \approx y^n + \int_{t^n}^{t^{n+1}} P_n(t)$$



$P_{n,k}$ = POLY DEGREE $k-1$ INTERPOLATES

$$P_{n,k}(t^{n-J+1}) = F(y^{n-J+1}) \qquad J = 1, \ldots, k$$

$$P_{n,k}(t) = \sum_{J=1}^{k} L_{n,J,k}(t) \cdot F(y^{n-J+1}) \qquad b_{n,J,k} =$$

$$\int_{t^n}^{t^{n+1}} P = \sum_{J=1}^{k} F(y^{n-J+1}) \overbrace{\int_{t^n}^{t^{n+1}} L_{n,J,k}(t)\, dt}$$

$$y^{n+1} = y^n + \Delta t \sum_{j=1}^{k} b_{j,k} \, F^{n-j+1}$$

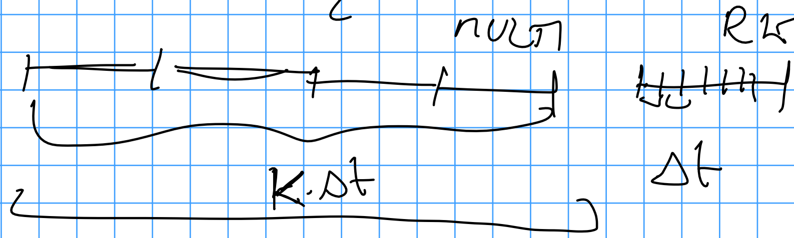
$k=0$ $\quad F^n \quad \Rightarrow y^{n+1} = y^n + \Delta t F^n$ EXP. EUL

$$\underbrace{\alpha_j = 0 \qquad \beta_j = b_{j,k}}_{\Rightarrow \mathcal{O}(\Delta t^{k+1})}$$

ORDER K

ORDER 2

$$y^{n+1} = y^n + \Delta t \, \frac{3}{2} F^n - \Delta t \, \frac{1}{2} F^{n-1}$$



BOUND K·Δt $\Rightarrow$ ORDER ↗ K↗ STAB REG ↘

ADAMS-MOULTON METHOD (IMPLICIT)

$P_{n,k}$ POLY degree K $\qquad P_{n,k}(t^{n-j+1}) = F^{n-j+1}$

$j = 0, \dots k$

$$y^{n+1} = y^n + \Delta t \sum_{j=0}^{k} b_{n,j,k} \, F^{n-j+1}$$

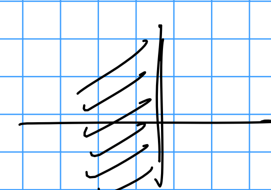
$$\underbrace{y^{n+1}} = y^n + \underbrace{\Delta t \, b_{0,k} F^{n+1}}_{\text{IMPL}} + \underbrace{\Delta t \sum_{j=1}^{k} F^{n-j+1} \, b_{j,k}}_{\text{EXPL}}$$

IF $\Delta t \, b_{0,k} \ll 1 \Rightarrow \exists!$ SOLUTION

COMPUT EFF = COMP EFF IMPLICIT EULER

$k=0 \Rightarrow$ IMPLICIT EULER

$k=1 \Rightarrow y^{n+1} = y^n + \Delta t \, \dfrac{F^{n+1} + F^n}{2}$



AB

$$y^{n+1} = y^n + \sum_{j=1}^{k} b^{AB}_{j,k} F^{n-j+1}$$

$$\underbrace{y^{n+1}} = y^n + \underbrace{\Delta t\, b_{0,k} F^{n+1}}_{\cancel{\text{IMPL}}\ \text{EXPL}} \overbrace{+ \Delta t \sum_{j=1}^{k} F^{n-j+1} b_{j,k}}^{}$$

EXPL

$$y^{n+1} + \sum_{j=1}^{k} \alpha_j y^{n-j+1} = \Delta t \sum_{j=0}^{k} \beta_j F^{n-j+1}$$

$\alpha_1 = -1 \qquad \alpha_j = 0 \qquad j = 2, \dots, k$

BDF $\quad \beta_0 = 1 \qquad \beta_j = 0 \qquad j = 1, \dots, k$

α ?

$$\begin{cases} \sum \alpha_j = -1 \\ \sum j\, \alpha_j = -\beta_0 \\ \sum \alpha_j \dfrac{(-j)^\ell}{\ell!} = 0 \qquad \ell = 2, \dots, p \end{cases}$$

ORDER ≤ 6 NOT A-STABLE FROM ORDER 3

$$\boxed{y^{n+1} = +\frac{4}{3} y^n - \frac{1}{3} y^{n-1} + \frac{2}{3} \Delta t F^{n+1}}$$ ORDER 2

STRUCTURE PRESERVING METHODS

- POSITIVITY
- ENERGY/ENTROPY
- (PDE) WEAK SOL

} PHYSICAL $\Rightarrow$ ALSO IN NUMERICAL

RELAXATION (RUNGE-KUTTA)

$y' = F(y) \qquad \langle y, F(y)\rangle = 0 \; (\le)$

$$\frac{d}{dt}\frac{1}{2}\langle y, y\rangle = \langle y, y'\rangle = \langle y, F\rangle = 0 \; (\le)$$

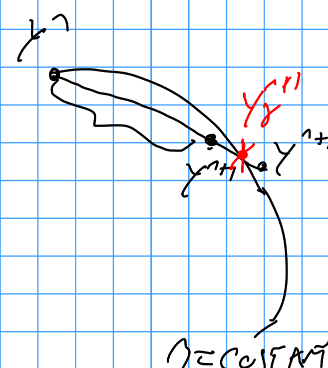
$$\frac{d}{dt}\eta(y) = \langle \underbrace{\eta'(y)}_{\partial_y \eta(y)}, \underbrace{y'}_{\frac{d}{dt}y} \rangle = \langle \eta', F\rangle = 0 \; (\le)$$

RK $\quad \begin{array}{c|c} c & A \\ \hline & b \end{array} \quad$ GOAL: MODIFY RK

i $\eta(y^{n+1}) = \eta(y^n)$? $(\le)$

$$y^{(k)} = y^n + \Delta t \sum_{j=1}^{s} a_{kj} F^{(j)} \qquad k = 1, \dots, s$$

$$y^{n+1} = y^n + \Delta t \sum_{j=1}^{s} b_j F^{(j)}$$



$$\underbrace{\boxed{y^{n+1}_\gamma} = y^n + \gamma^n \Delta t \sum_{j=1}^{s} b_j F^{(j)}}$$

VERIFY OUR CONSTRAINT

FIND γ

ENERGY CASE $\quad \langle y^{n+1}_\gamma, y^{n+1}_\gamma \rangle \overset{!}{=} \langle y^n, y^n\rangle \; (\le)$

$$\langle y^{n+1}_\gamma, y^{n+1}_\gamma \rangle = \langle y^n, y^n\rangle + 2\gamma^n \Delta t \sum_{j=1}^{s} b_j \langle y^n, F^{(j)}\rangle + \gamma^2 \Delta t^2 \sum_{j=1}^{s}\sum_{k=1}^{s} b_j b_k \langle F^{(j)}, F^{(k)}\rangle = *$$

$\langle y, F\rangle = 0 \; (\le) \qquad \langle y^n, F(y^{(j)})\rangle \overset{?}{=} 0$

$$\langle y^{(j)}, F(y^{(j)})\rangle = 0 \; (\le)$$

$=0 \quad (\le 0)$

$$* = \langle y^n, y^n\rangle + 2\gamma \Delta t \sum b_j \langle y^{(j)}, F^{(j)}\rangle$$

$$+ 2\gamma \Delta t \sum b_j \langle y^n - y^{(j)}, F^{(j)}\rangle$$

$$+\gamma^2 \Delta t^2 \sum_{j,k=1}^{s} b_j b_k \langle F^{(j)}, F^{(k)} \rangle \Big]$$ SIGN ?

SUPPOSE $b_j \geq 0$

SET γ i $\quad 2\cancel{\gamma}\Delta t \sum b_j \langle y^n - y^{(j)}, F^{(j)} \rangle + \gamma^{\cancel{2}} \Delta t^2 \cdot$

$$\cdot \sum b_j b_k \langle F^{(j)}, F^{(k)} \rangle = 0$$

$\gamma_1 = 0$ NOT INTERESTING

$$\gamma = -\frac{2\Delta t \sum_j b_j \langle y^n - y^{(j)}, F^{(j)} \rangle}{\Delta t^2 \sum_{j,k} b_j b_k \langle F^{(j)}, F^{(k)} \rangle}$$

$\Rightarrow$ RELAX RK

$y^{(k)} =$ CLASSICAL RK

COMPUTE γ

$$y_\gamma^{n+1} = y^n + \gamma \Delta t \sum b_j F^{(j)}$$

$$\boxed{\langle y_\gamma^{n+1}, y_\gamma^{n+1} \rangle \underset{(\leq)}{=} \langle y^n, y^n \rangle}$$

$$y_\gamma^{n+1} \approx \cancel{y(t^n + \Delta t)}$$

$$\approx \underline{y(t^n + \gamma \Delta t)}$$

$\gamma \approx 1 \qquad \gamma = 1 + O(\Delta t^p)$

$\Rightarrow$ ORDER ACCURACY DOESN'T CHANGE

$y' = F(y) \qquad\qquad \eta = 0 \qquad \underline{\langle D\eta, F \rangle = 0}$

$$\underline{\overline{\langle y, F(y) \rangle}} \underset{(\leq)}{=} 0$$

$y' = F(y)$ $\quad \boxed{\partial_t \eta(y(t)) \ \le 0,\ = 0}$ $\quad \eta: \mathbb{R}^D \to \mathbb{R}$

$\eta_y: \mathbb{R}^D \to \mathbb{R}^D$

$\langle \eta_y(y(t)), y'(t) \rangle = \underbrace{\langle \eta_y(y(t)), F(y(t)) \rangle}$ $\ (\le) 0,\ = 0$

$\eta_{yy}: \mathbb{R}^D \to \mathbb{R}^{D \times D}$

$y^{(i)} = y^n + \Delta t \sum_j a_{ij} F(y^{(j)})$

$y^{n+1} = y^n + \Delta t \sum_j b_j F(y^{(j)})$

$\eta(y^{n+1}) - \eta(y^n) = \eta(y^n + \Delta t\, \overline{b_j F(y^{(j)})}) - \eta(y^n)$

$= \cancel{\eta(y^n)} + \Delta t \sum_j b_j \langle \eta_y(y^n), F(y^{(j)}) \rangle$

$+ \frac{\Delta t^2}{2} \underbrace{\eta_{yy}(y^n)}_{A} \Big(\underbrace{\sum_j b_j F(y^{(j)})}_{x}, \underbrace{\sum_k b_k F(y^{(k)})}_{y} \Big)$

$x^T A y$

$+ O(\Delta t^3)$ $\quad - \cancel{\eta(y^n)}$

$= \Delta t \sum_j b_j \overline{\langle \eta_y(y^{(j)}), F(y^{(j)}) \rangle}$ $\ = 0,\ (\le) 0$

$+ \Delta t \sum_j b_j \langle \eta_y(y^n) - \eta_y(y^{(j)}), F(y^{(j)}) \rangle$

$+ \frac{\Delta t^2}{2} \eta_{yy}(-, -)$

RRK

$$\begin{cases} y^{(i)} = y^n + \Delta t \sum_j a_{ij} F(y^{(j)}) \\ y^{n+1}_\gamma = y^n + \gamma^n \Delta t \sum_j b_j F(y^{(j)}) \\ R(\gamma^n) = \eta(y^{n+1}_\gamma) - \eta(y^n) - \gamma_n \Delta t \sum_j b_j \langle \eta_y(y^{(j)}), F^{(j)} \rangle \overset{!}{=} 0 \end{cases}$$

$R: \mathbb{R} \to \mathbb{R}$

η ~~NONLINEAR~~ / NON QUADRATIC

$\Rightarrow$ "MORE DIFFICULT" FIND γ^n

1) $\exists !$ SOLUTION $\Omega(\gamma)=0$? $\gamma \neq 0$

2) $\gamma = 1 + \mathcal{O}(\Delta t^{p-1}) \Rightarrow$ SAME ORDER OF ACCURACY

1) $\sum_{i,j} b_i a_{ij} > 0$

$$\boxed{\begin{array}{l}\text{ORDER } 2 \\ \sum b_i a_{ij} = \frac{1}{2} > 0\end{array}}$$

$\boxed{\eta_{yy} > 0}$ Δt SMALL ENOUGH $\Rightarrow$ $\boxed{\Omega'(0) < 0}$

$$\Omega(\gamma) = \eta\big(y^n + \gamma \Delta t \sum_j b_j F^j\big) - \eta(y^n) - \gamma \Delta t \sum_j b_j \langle \eta_y(y^j), F^j \rangle$$

$$\Omega'(\gamma) = \langle \eta_y\big(y^n + \gamma\Delta t \sum_j b_j F^j\big), \Delta t \sum_j b_j F^j \rangle - \Delta t \sum_j b_j \langle \eta_y(y^j), F^j \rangle$$

$$\Omega'(0) = \langle \eta_y(y^n), \Delta t \sum_j b_j F^j \rangle - \Delta t \sum_j b_j \langle \eta_y(y^j), F^j \rangle$$

$$= \sum_j \Delta t\, b_j \langle \eta_y(y^n) - \eta_y(y^{(j)}), \underbrace{F(y^{(j)})}_{} \rangle$$

$$= -\sum_j \Delta t\, b_j \int_0^1 \eta_{yy}\big(y^n + s \underbrace{\Delta t \sum_k a_{jk} F^{(k)}}_{}\big) \Big\{ \Delta t \sum_k a_{jk} F(y^{(k)}), F(y^{(j)}) \Big\}\, ds$$

$$= -\sum_j \sum_k \Delta t^2\, b_j a_{jk} \int_0^1 \eta_{yy}(y^n) \{ F(y^{(k)}), F(y^{(j)}) \}\, ds + \mathcal{O}(\Delta t^3)$$

$$= -\sum_j \sum_k \Delta t^2\, b_j a_{jk}\, \eta_{yy}(y^n) \{ F(y^{(k)}), F(y^{(j)}) \} + \mathcal{O}(\Delta t^3)$$

$$= -\underbrace{\sum_j \sum_k \underbrace{\Delta t^2}_{\geq 0} b_j a_{jk}}_{> 0} \underbrace{\eta_{yy}(y^n) \{ F(y^n), F(y^n) \}}_{> 0} + \underbrace{\mathcal{O}(\Delta t^3)}_{?}$$

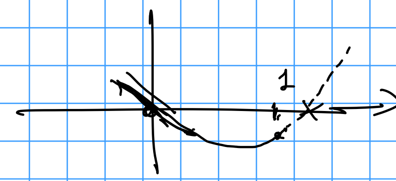
< 0 (UNLESS TRIVIAL CONSTANT CASES)

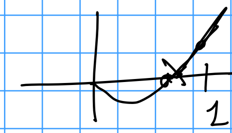
IF $\Delta t \ll 1 \Rightarrow \Delta t^2$ TERM DETERMINES SIGN

$\Omega'(0) < 0$.

$\boxtimes$

$\underline{R'(1) > 0}$ $\qquad$ $R(0) = 0$





Hyp: $\sum_{i,j} b_i (\underbrace{b_j}_{1} - a_{ij}) > 0$ $\qquad$ $\boxed{\text{ORDER } 2}$

$$\sum b_i = 1 \qquad \sum_i b_i \sum_j b_j = 1$$

$$\sum_{i,j} b_i a_{ij} = \frac{1}{2} \qquad \sum b_i (b_j - a_{ij}) = 1 - \frac{1}{2} = \frac{1}{2} > 0$$

$$R(\gamma) = \eta\Big(y^n + \gamma \Delta t \sum_j b_j F^j\Big) - \eta(y^n) - \gamma \Delta t \sum_j b_j \langle \eta_y(y^j), F^j \rangle$$

$$R'(\gamma) = \Big\langle \eta_y\Big(y^n + \gamma \Delta t \sum_j b_j F^j\Big), \Delta t \sum_j b_j F^j \Big\rangle - \Delta t \sum_j b_j \langle \eta_y(y^j), F^j \rangle$$

$$R'(1) = \Big\langle \eta_y\Big(y^n + \Delta t \sum_k b_k F^k\Big), \Delta t \sum_j b_j F^j \Big\rangle - \Delta t \sum_j b_j \langle \eta_y(y^j), F^j \rangle$$

$$= \Delta t \sum_j b_j \Big\langle \eta_y\Big(y^n + \Delta t \sum_k b_k F^k\Big) - \eta_y(y^j), F^j \Big\rangle$$

$$= \Delta t \sum_j b_j \int_0^1 \eta_{yy}\big(y^{n+1} + s(y^j - y^{n+1})\big) \Big\{ \cancel{y^n} + \Delta t \sum_k \underbrace{b_k} F^k - \cancel{y^n} - \Delta t \sum_k \underbrace{a_{jk}} F^k, F^j \Big\} ds$$

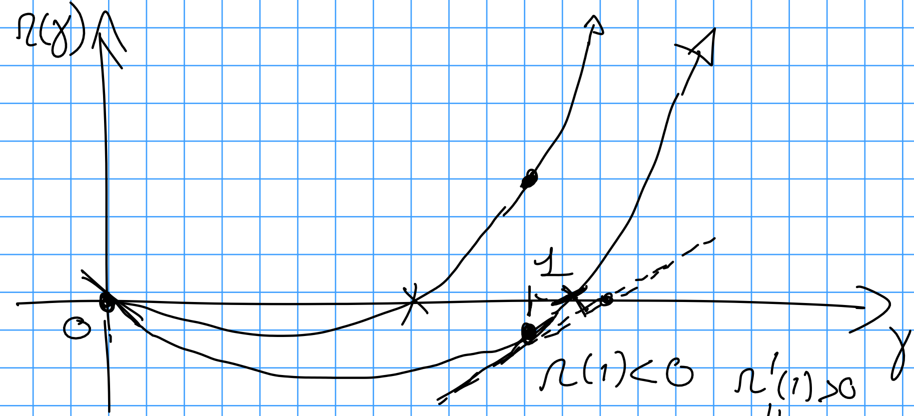
$$= \Delta t^2 \sum_j b_j \sum_k (b_k - a_{jk}) \quad \eta_{yy}(y^n) \{F^k, F^j\}$$

$$+ O(\Delta t^3)$$

$$= \underbrace{\Delta t^2}_{>0} \underbrace{\sum_{j,k} b_j (b_k - a_{jk})}_{>0} \underbrace{\eta_{yy}(y^n) \{F^n, F^n\}}_{>0} + O(\Delta t^3)$$

> 0

$R'(1) > 0$



$\Omega(0)=0$ $\quad \Omega''>0 \Rightarrow \partial_\gamma \Omega' > 0$

$$\Omega'(\gamma) = \langle \eta_y(y^n+\gamma\Delta t\sum_J b_J F^J), \Delta t \sum_J b_J F^J\rangle - \Delta t \sum_J b_J \langle \eta_y(y^J), F^J\rangle$$

$$\Omega''(\gamma) = \eta_{yy}(y^n+\gamma\Delta t\sum_J b_J F^J)\{\underbrace{\Delta t\sum_J b_J F^J}, \underbrace{\Delta t\sum_J b_J F^J}\} > 0$$

- ACCURACY $\qquad$ RK ORDER p

$y^1 - y(t^1) = O(\Delta t^{p+1})$ $\qquad y^{N_t} - y(t^{N_t}) = O(\Delta t^p)$

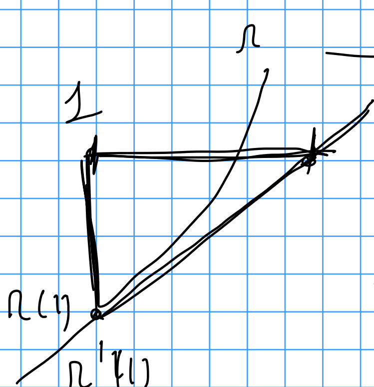
$y_\gamma^{n+1} = y(t_\gamma^{n+1}) + O(\Delta t^{p+1})$ IF $y^n = y(t^n)$

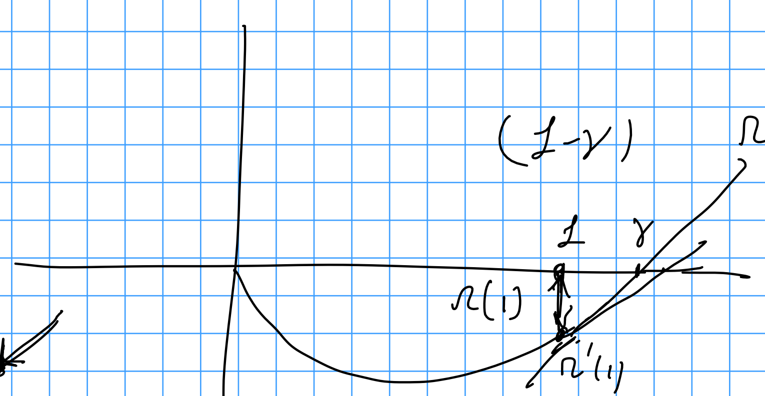
y^{n+1}

IF $\boxed{\gamma_n = 1 + O(\Delta t^{p-1})}$ $\Rightarrow y_\gamma^{n+1} \approx y(t^n+\gamma\Delta t) + O(\Delta t^{p+1})$

$y^{n+1} = y(t^n+\Delta t) + O(\Delta t^{p+1})$

$\Omega(1)$





$\Omega(1) = O(\Delta t^{p+1})$ $\qquad (1-\gamma) \le O(\Delta t^{p-1})$

$\Omega'(1) = O(\Delta t^2)$

$$r(1) = \eta(y^{n+1}) - \eta(y^n) - \Delta t \sum_j b_j \langle \eta_y(y^{(j)}), F^j \rangle$$

$$= \int_{t^n}^{t^{n+1}} \frac{d}{dt}\big(\eta(y(t))\big)\, dt - \Delta t \sum_j b_j \langle \eta_y(y^j), F^j \rangle$$

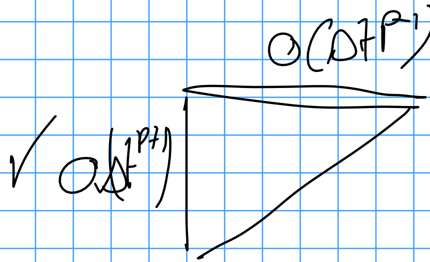
$$= \int_{t^n}^{t^{n+1}} \langle \eta_y(y(t)), F(y(t)) \rangle\, dt - \Delta t \sum_j b_j \dots$$

$$= \cancel{\Delta t \sum_j \langle \eta_y(y^j), F(y^j) \rangle b_j} + O(\Delta t^{p+1})$$

$$\cancel{- \Delta t \sum_j \langle \eta_y(y^j), F(y^j) \rangle b_j} \qquad 0$$

BECAUSE ORDER p

$= \Delta t + \int_{t^n}^{t^{n+1}}$



$$\frac{O(\Delta t^{p+1})}{O(\Delta t^{p-1})} = \underbrace{O(\Delta t^2)}$$

$$r'(1) = \Theta(\Delta t^2) \iff \lim_{\Delta t \to 0} \frac{r'(1)}{\Delta t^2} = C$$

$$0 < C < \infty$$

NON LINEAR OSCILL

$$\eta = \frac{y^2}{2} = \frac{u^2 + v^2}{2}$$

$$\eta' = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \eta'' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > 0$$

PENDULUM

$$\eta = \frac{1}{2} v^2 - \cos(u) \qquad \eta' = \begin{pmatrix} \sin(u) \\ v \end{pmatrix}$$

$$\eta'' = \begin{pmatrix} \cos(u) & 0 \\ 0 & 1 \end{pmatrix} > 0 \quad \text{IF} \quad u \simeq 0$$

$$\eta(x,y) = \delta x - \gamma \log(x) + \beta y - \alpha \log(y)$$

$$\eta' = \begin{pmatrix} \delta - \gamma \frac{1}{x} \\ \beta - \alpha \frac{1}{y} \end{pmatrix} \qquad \eta'' = \begin{pmatrix} \frac{\gamma}{x^2} & 0 \\ 0 & \frac{\alpha}{y^2} \end{pmatrix} > 0$$