- EXAM
- ORAL EXAM (THEORETICAL)
- CODING PROJECTS (PRACTICAL)

ORDINARY DIFFERENTIAL EQUATIONS ODE

$I = [t_0, t_{end}] \subset \mathbb{R}$

$y : I \to \mathbb{R}^S$ $\qquad F : I \times \mathbb{R}^S \to \mathbb{R}^S$

$$\begin{cases} \frac{dy}{dt} = F(t, y(t)) \simeq \underbrace{F(y(t))} \\ y(0) = y_0 \end{cases}$$

AUTONOMOUS ODE

$$y^{(p)} = \overbrace{f(t, y, y', y'', \dots, y^{(p-1)})}$$

HIGHER ORDER DERIVATIVE DIFF. EQ.

$y' = z_1$

$y'' = z_1' = z_2$

$y''' = z_2' = z_3$

$y^{(p)} = z_p' = f(t, y, z_1, z_2, \dots, z_{p-1})$

$y \in \mathbb{R}^S$

$$\begin{pmatrix} y \\ z_1 \\ \vdots \\ z_p \end{pmatrix} \in \mathbb{R}^{S(P+1)}$$

SYSTEM OF 1^{st} DERIVATIVE DIFF. EQ.

$$\int_{t_0}^{t} \frac{dy}{dt} dt = y(t) - y(t_0) = y(t) - y_0$$

$$y(t) = y_0 + \int_{t_0}^{t} F(t, y(t))\, dt$$

DAHLQUIST'S PROBLEM

$$\begin{cases} y'(t) = -\lambda y(t) \\ y(t_0) = y_0 \end{cases}$$

$\lambda \in \mathbb{R} \to \mathbb{C}$

$y(t) \neq 0 \qquad y_0 > 0$

$$\int_{t_0}^{t} \underbrace{\frac{y'(s)}{y(s)}} ds = -\lambda \int_{t_0}^{t} ds$$

$$\int_{t_0}^{t} \frac{d}{ds}\left(\log(y(s))\right) ds = -\lambda\,(t-t_0)$$

$$\log(y(t)) - \log(y(t_0)) = -\lambda\,(t-t_0)$$

$$\log\left(\frac{y(t)}{y(t_0)}\right) = -\lambda\,(t-t_0)$$

$$\frac{y(t)}{y(t_0)} = e^{-\lambda \overbrace{(t-t_0)}^{>0}}$$

$$y(t) = y_0 \underbrace{e^{\overline{-\lambda}(t-t_0)}}_{\neq 0} \qquad y_0 > 0$$

$\lambda < 0 \to$ exp. INCREASING

$\lambda > 0 \to$ exp. to 0

PDS

$c_1 \rightleftarrows c_2$

$$\begin{cases} c_1'(t) = \overline{c_2(t)} - \underline{5c_1(t)} \\ c_2'(t) = 5c_1(t) - c_2(t) \end{cases} \qquad c_1, c_2 > 0$$

$$\sum_i c_i(t) = m \qquad \forall t$$

$$\frac{d}{dt}\left(\sum_i c_i(t)\right) = c_1' + c_2' = c_2 - 5c_1 + 5c_1 - c_2 = 0$$

$$c'(t) = A \cdot c(t) \qquad A = \begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix}$$

$$c(0) = c_0 \qquad y' = \lambda y \Rightarrow y = y_0 e^{\lambda t}$$

$$c(t) = \underline{e^{At}} c_0 \qquad e^{At} = \left[\sum_{k=0}^{\infty} \frac{A^k t^k}{k!}\right]$$

$$A^2 = -6A \qquad e^{At} = \sum_{k=1}^{\infty} \frac{(-6)^{k-1} A t^k}{k!} + 1$$

$$A^3 = -6A^2 = (-6)^2 A$$

$$A^k = (-6)^{k-1} A$$

$$e^{At} = \Big(\underbrace{\sum_{k=0}^{\infty} \frac{(-6)^k \cdot t^k}{k!}} - 1 \Big) \frac{A}{-6} + 1$$

$$\left(e^{-6t} - 1\right) \frac{A}{-6} + 1$$

$$c(t) = C_0 + \frac{1 - e^{-6t}}{6} A C_0$$

$$x'(t) = \alpha x - \beta x y \qquad \alpha, \beta, \gamma, \delta > 0$$

$$y'(t) = \delta x y - \gamma y$$

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \text{EQUILIBRIUN}$$

$$\eta(x,y) = \delta x - \gamma \log(x) + \beta y - \alpha \log(y)$$

$$\frac{d}{dt} \eta(x(t), y(t)) = \begin{pmatrix} \partial_x \eta \\ \partial_y \eta \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} =$$

$$\langle \nabla \eta, F(x,y) \rangle = 0 = \cancel{\delta \alpha x} - \delta \cancel{\beta x y} - \cancel{\gamma \alpha}$$

$$+ \cancel{\gamma \beta y} + \cancel{\beta \delta} x y$$

$$- \beta \cancel{\gamma} y - \alpha \cancel{\delta} x + \alpha \cancel{\gamma}$$

$$= 0$$

$$\nabla \eta = \begin{pmatrix} \delta - \gamma / x \\ \beta - \alpha / y \end{pmatrix}$$

$$\begin{cases} u'(t) = -\dfrac{v(t)}{r(t)} \\ v'(t) = \dfrac{u(t)}{r(t)} \end{cases} \qquad r(t) = \sqrt{u^2(t) + v^2(t)}$$

$$\frac{r^2}{2}$$

$$\frac{d}{dt}\left(\frac{r^2}{2}\right) = \frac{d}{dt}\left(\frac{u^2 + v^2}{2}\right) = \frac{2 u \cdot u' + 2 v \cdot v'}{2} =$$

$$u \cdot u' + v \cdot v' = u \cdot \left(-\frac{v}{r}\right) + v \left(\frac{u}{r}\right) = 0$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{matrix} \cos(\theta)\, u^0 - \sin(\theta)\, v^0 \\ \sin(\theta)\, v^0 + \cos(\theta)\, u^0 \end{matrix} \qquad \theta = \frac{t}{r}$$

$$u' = -\frac{v}{r} - \alpha u$$

$$v' = \frac{u}{r} - \alpha v$$

$$\frac{d}{dt}\left(\frac{r^2}{2}\right) = \frac{d}{dt}\left(\frac{u^2+v^2}{2}\right) = u \cdot u' + v \cdot v'$$

$$= -u \cdot \cancel{\frac{v}{r}} - \alpha u^2 + v \cancel{\frac{u}{r}} - \alpha v^2 = -\alpha (u^2+v^2)$$

$$= -2\alpha \frac{u^2+v^2}{2} = -2\alpha \left(\frac{r^2}{2}\right)$$

$$E = \frac{r^2}{2} \qquad \frac{d}{dt} E = -2\alpha E$$

$$E = e^{-2\alpha t} E_0 \qquad \alpha > 0$$

$$\frac{r^2}{\cancel{2}} = e^{-2\alpha t} \frac{r_0^2}{\cancel{2}} \Rightarrow r(t) = \underbrace{e^{-\alpha t}}_{\to 0} r_0$$

- $u'' = -\sin(u)$

$$\begin{cases} u' = v \\ v' = -\sin(u) \end{cases} \qquad \eta(t) = \frac{1}{2} v^2 - \cos(u)$$

$$\frac{d}{dt} \eta(t) = \langle \nabla \eta, F \rangle = \begin{pmatrix} \sin(u) \\ v \end{pmatrix} \cdot \begin{pmatrix} v \\ -\sin(u) \end{pmatrix} =$$

$$v \cdot \sin(u) - v \cdot \sin(u) = 0$$

ROBERTSON

$$\frac{d}{dt}\left(\sum_i c_i\right) = 0$$

$$\begin{tikzcd} C_1 \arrow[r, bend right, "0.04\, c_1"'] & C_2 \arrow[l, bend right, "10^4 \cdot c_2 c_3"'] \arrow[d, "3 \cdot 10^7 c_2^2"] \\ & C_3 \end{tikzcd}$$

• PDE $\qquad u(x,t) : \Omega \times [0,T] \to \mathbb{R}^d$

$$\partial_t u + \partial_{xx}(a(u)) + \partial_x b(u) + c(u) = 0$$

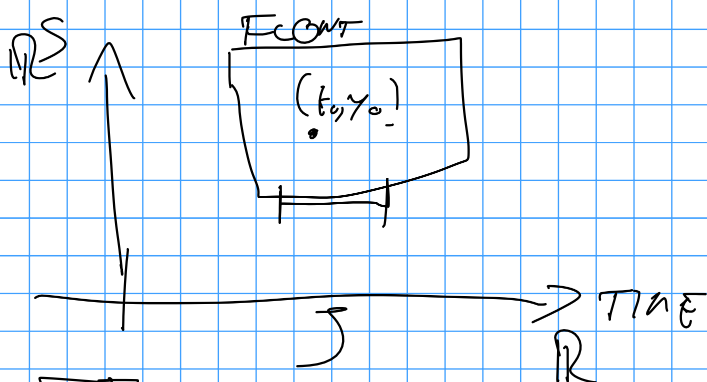
$$\underline{U} = \mathbb{R}^N \qquad N \ \# \text{ DOF IN SPACE}$$

$$\partial_{xx} a(u) \approx D_2 A(\underline{U})$$

$$\partial_t \underline{u} = f(\underline{u}) \qquad f(\underline{u}) = -(D_2 A(\underline{U}) + \dots)$$

$$\partial_t u + \partial_x b(u) = 0 \qquad \text{TVD}$$

$$U_i^n \qquad TV(\underline{U}^n) := \sum_i |U_{i+1}^n - U_i^n|$$



$$\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}$$

- $y(t) = 0$
- $y(t) = \frac{t^2}{4}$ $\qquad \frac{t}{2}$

$$y' = \frac{\cancel{2}t}{\cancel{4}2}$$

$$F(t,y) : I \times \mathbb{R}^S \to \mathbb{R}^S$$

$$\|F(t,y) - F(t,z)\| \leq L \|y - z\|$$

UNIFORMLY IN t $\Rightarrow$ $\exists!$ SOLUTION

$$y \in C^1(I) \qquad \begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$F \in C^{p-1}$ p-differentiable

y IS $p+1$ TIMES DIFFERENTIABLE

$$\frac{dy}{dt} = F$$

$$F^{(\ell+1)}(t,y) = \frac{\partial F^{(\ell)}}{\partial t}(t,y) + \frac{\partial F^{(\ell)}(t,y)}{\partial y}$$

ODE $\quad I = [t_0, t_{end}] \quad y : I \to \mathbb{R}^S$

$F : I \times \mathbb{R}^S \to \mathbb{R}^S$

$$\begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$t_0 = t^0 < t^1 < t^2 < \dots < t^n < \dots t^N = t_{end}$

$[t^n, t^{n+1}] \quad y(t^{n+1}) = y(t^n) + \int_{t^n}^{t^{n+1}} F(t, y(t))\,dt$

$\downarrow \quad \downarrow$

$y^n \quad y^{n+1}$

$$\approx y(t^n) + F(t^n, y(t^n)) \cdot \underbrace{(t^{n+1} - t^n)}_{\Delta t^n}$$

$y^n \to y^{n+1} \quad y^{n+1} := y^n + \Delta t^n F(t^n, y^n)$

$y^0 = y_0$

$$c_1^{n+1} + c_2^{n+1} = c_1^n + \Delta t(\cancel{-5c_1^n} + \cancel{c_2^n}) + c_2^n + \Delta t(\cancel{-c_2^n} + \cancel{5c_1^n})$$

$$= c_1^n + c_2^n = c_1^0 + c_2^0 = \text{MASS} \quad \forall n$$

$e_n = y(t^n) - y^n$

CONSISTENCY ERROR

$$\varepsilon_n = y(t^{n+1}) - \underline{y(t^n) - \Delta t \overbrace{F(t^n, y(t^n))}^{y'(t^n)}} =$$

$$= \int_{t^n}^{t^{n+1}} \underbrace{y'(t) - \underbrace{y'(t^n)}}_{\text{COSTANT}} \, dt$$

$$\omega(f, \Delta t) := \max_{t,t';\, |t-t'|<\Delta t} |f(t) - f(t')|$$

$$|\varepsilon_n| \le \int_{t^n}^{t^{n+1}} \underbrace{\omega(y', \Delta t^n)}_{\text{DOESN'T DEPEND ON } t} dt = \Delta t^n \cdot \omega(y', \Delta t^n).$$

$$e_{n+1} = y(t^{n+1}) - y^{n+1} = y(t^{n+1}) - y^n - \Delta t^n F(t^n, y^n)$$
$$+ y(t^n) + \Delta t\, F(t^n, y(t^n))$$
$$- y(t^n) - \Delta t\, F(t^n, y(t^n))$$

$$= \varepsilon_n + \overbrace{y(t^n) - y^n}^{e_n} + \Delta t\,[F(t, y(t^n)) - F(t, y^n)]$$

$$|e_{n+1}| \le |\varepsilon_n| + |e_n| + \Delta t\, L\, \underbrace{|y(t^n) - y^n|}_{e_n} \le (*)$$

$$|F(t, y) - F(t, z)| \le L|y - z|$$ (HYPOTHESIS ON F TO HAVE UNIQUENESS)

$$(*) \le |\varepsilon_n| + (1 + \Delta t L)\,|e_n|.$$

$$\le \dots \le e^{L|t^{n+1} - t^0|}|e_0| + \sum_{i=0}^{n} e^{L\overbrace{(t^{n+1} - t^{i+1})}^{\Delta t}} |\varepsilon_i| \le (*)$$

$$|\varepsilon_n| \le \Delta t\, \omega(y', \Delta t)$$

$$(*) \le e^{L|t^{n+1} - t^0|}|e_0| + \overline{\omega(y', \Delta t)}\, \frac{e^{L(t^{n+1} - t^0)} - 1}{L}.$$

$$\omega(y', \Delta t) = \omega(F(\cdot, y(\cdot)), \Delta t)$$

$$\left[\sum_{i=0}^{n} \Delta t\, e^{L\Delta t} = \underbrace{N \cdot \Delta t}\, e^{L\Delta t} = \frac{T}{\cancel{\Delta t}} \cancel{\Delta t}\, e^{L\Delta t} \dots\right]$$

$$w(y', \Delta t) = \max_{t,t' : |t-t'|<\Delta t} |y'(t) - y'(t')| = \quad \text{(AUTONOMOUS)}$$

$$= \max_{t,t' : |t-t'|\le\Delta t} |F(y(t)) - F(y(t'))|$$

$$\le L\, |y(t) - y(t')| \xrightarrow[\Delta t \to 0]{} 0$$

$$\le L\, \underset{\max_I y'}{\underset{\parallel}{C}} \cdot \Delta t$$

$$\boxed{|y(t^n) - y^n| \le C_0 |y^0 - y(t^0)| + \Delta t \cdot C_1}$$

$$e_1 = |\underbrace{y^1}| - y(t^1) | = | \cancel{y^0} + \cancel{\Delta t \overbrace{F(t^0, y^0)}^{y'(t_0)}}$$
$$- \left(\cancel{y(t^0)} + \cancel{\Delta t\, y'(t_0)} + \frac{\Delta t^2}{2} y''(t_0) \right.$$
$$\left. + O(\Delta t^3) \right|$$

$$= \left| \frac{\Delta t^2}{2} y''(t_0) \right| + O(\Delta t^3) = \underline{O(\Delta t^2)}$$

$$e_N \approx \sum_{i=1}^{N} |y(t^i) - y^i| \lesssim N \cdot \frac{\Delta t^2}{2} \max |y''|$$

$$= \frac{t_{end}}{\cancel{\Delta t}} \cdot \frac{\Delta t^{\cancel{2}}}{2} \max |y''|$$

$$= t_{end} \cdot \frac{\Delta t}{2} \max |y''|$$

$$= C \cdot \Delta t = O(\Delta t)$$

FIRST ORDER

DEF
P ORDER SCHEME
LARGEST P $\Delta t \to 0$
$|e_N| \le C \cdot \Delta t^P$

$$y'(t) = M y(t) \qquad M \in \mathbb{R}^{s \times s} \qquad \Delta t$$

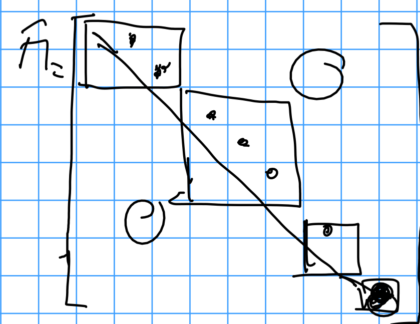
$$y^n = y^{n-1} + \Delta t M y^{n-1} = (I + \Delta t M) y^{n-1}$$

$$= (I + \Delta t M)^n y^0$$

$|y^n| \overset{?}{\leq} |y^0| \leadsto \|(I+\Delta t M)^n\|$

$\in \mathbb{C}^{S \times S}$

$M \to \hat{M} = S^{-1} M S$

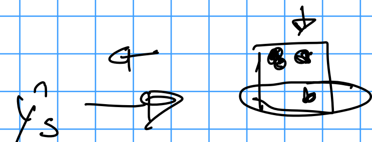


$S^{-1} y^n = S^{-1} y^{n-1} + \Delta t\, S^{-1} M \underset{S S^{-1}}{} y^{n-1}$

$S^{-1} y^n =: \hat{y}^n$

$\hat{y}^n = \hat{y}^{n-1} + \Delta t\, \hat{M}\, \hat{y}^{n-1}$

$= (I + \Delta t \hat{M})\, \hat{y}^{n-1}$



$\boxed{y'(t) = q\, y(t)}$ $\quad q \in \mathbb{C}$

$q \in \mathbb{R}$

$\boxed{q \leq 0}$ $\quad \mathrm{Re}(q) \leq 0$

$q > 0$

$y^{n+1} = y^n + q \Delta t\, y^n = (1 + q\Delta t)\, y^n$

$= (1 + q\Delta t)^{n+1}\, y^0$

$|y^{n+1}| \leq |y^n| \leq \ldots$

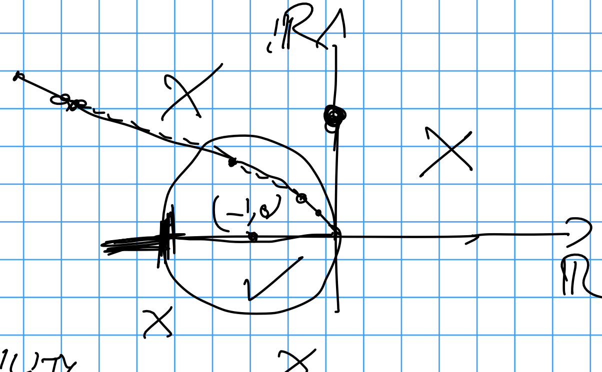
$|y^{n+1}| = |y^n|$

$|1 + q\Delta t|\, |y^n| \leq |y^n|$

$|1 + \underbrace{q\Delta t}| \leq 1$

$\underline{z \in \mathbb{C}}$

$|1 + z| \leq 1$



$R(z) = 1 + z$ STABILITY FUNCTION

$y^{n+1} = R(\Delta t q) \cdot y^n$ $\quad z = \Delta t q$

$\{ |R(z)| \leq 1 \}$ STABILITY REGION

$q \in \mathbb{R}$ $\quad |q| \Delta t \leq C \Rightarrow \Delta t \leq \frac{C}{|q|} \simeq \frac{C}{|\lambda|}$

$u' = v$

$v' = -u$

$u(0) = 0 \qquad v(0) = 1$

$r^2 = u^2 + v^2 =$ CONSTANT IN TIME

$$\frac{d}{dt} \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = 0$$

$$\underline{u} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{v}$$

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

$y^{n+1} = y^n + \Delta t\, f(t^{n+1}, y^{n+1})$ FIRST ORDER

LINEAR STABILITY

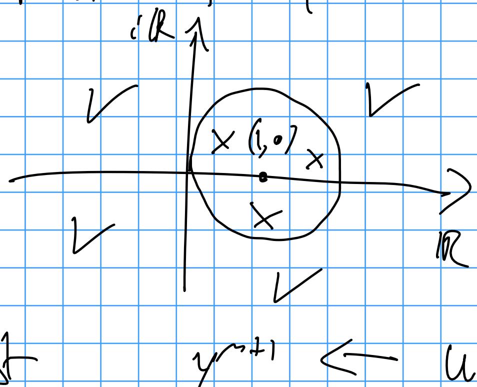
$y' = q y \qquad q \in \mathbb{C}$

$y^{n+1} = y^n + \Delta t\, q\, y^{n+1}$

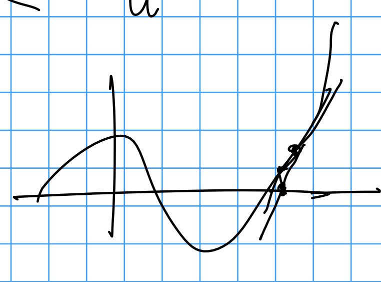
$$(1 - \underbrace{\Delta t q}_{z \in \mathbb{C}})\, y^{n+1} = y^n$$

$$y^{n+1} = R(z) \cdot y^n = \overbrace{\frac{1}{1-z}}^{R(z)} y^n$$

$$\{ |R(z)| \le 1 \} = \left\{ \frac{1}{|1-z|} \le 1 \right\} = \{ |1-z| \ge 1 \}$$



$y^{n+1} = y^n + f(y^{n+1})\, \Delta t \qquad y^{n+1} \leftarrow u^{(k)}$



IMPLICIT EULER + POSITIVITY

$\forall \Delta t$

PDS $\quad \frac{d}{dt} c_i = P_i(c) - D_i(c)$

$$= \sum_j \underset{\geq 0}{p_{ij}(c)} - \underset{\geq 0}{d_{ij}(c)}$$

$$\boxed{p_{ij} = d_{ji}} \geq 0$$

1. CONSERVATION

$$\frac{d}{dt}\left(\sum_i c_i(t)\right) = \sum_i \frac{d}{dt} c_i(t) = \sum_i \sum_j p_{ij}(c) - d_{ij}(c)$$

$$= \sum_i \sum_j p_{ij} - \sum_i \sum_j d_{ji} = \sum_i \sum_j p_{ij} - p_{ij} = 0$$

- HOMEWORK CHECK THAT IMPLICIT EULER IS CONSERVATIVE.

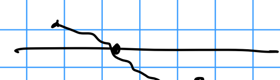
2. POSITIVITY $\quad \overline{c_i \to 0 \Rightarrow D_i(c) \to 0}\ \forall j\ \forall i$

$$d_{ij}(c) \to 0$$

$$c_i(0) > 0 \Rightarrow c_i(t) \geq 0$$
$$\forall_i \quad \forall t$$

PROOF BY CONTRADICTION $\quad t : c_i(t) < 0$

$$\exists\, t^* \quad c_i(t^*) = 0 \quad c_i'(t^*) < 0$$



$$c_i'(t^*) = \underbrace{P_i(c(t^*))}_{\geq 0} - \underbrace{D_i(c(t^*))}_{=0} \geq 0 \quad \lightning$$

IMPLICIT EULER + LINEAR PDS + POSITIVITY

$$y' = M y \qquad M_{ii} < 0 \qquad M_{ij} \geq 0 \quad i \neq j$$

$$\sum_i M_{ij} = 0$$

IMPLICIT EULER

$y^0 \geq 0$

$\underbrace{(I - \Delta t\, M)}_{A} y^{n+1} = y^n \Rightarrow y^{n+1} = \underbrace{A^{-1}} y^n$

$A^{-1} \geq 0 \iff \forall_{ij}\ (A^{-1})_{ij} \geq 0$

1. $0 < A_{ii} = 1 - \underbrace{\Delta t \underbrace{M_{ii}}_{<0}}_{>0} = 1 + \Delta t |M_{ii}| = 1 + \Delta t \sum_{j \neq i} M_{ji}$

$\boxed{|A_{ii}| > \sum_{j \neq i} |A_{ji}|}$

$\sum_j M_{ji} = 0$

$0 > M_{ii} = - \sum_{j \neq i} \overbrace{M_{ji}}^{>0} < 0$

$\underbrace{\overbrace{1}^{>0} + \overbrace{\Delta t}^{>0} \sum_{j \neq i} \overbrace{M_{ji}}^{>0}} > \Delta t \sum_{j \neq i} M_{ji} = \sum_{j \neq i} |A_{ji}|.$

A DIAG. DOMINANT BY COLUMN

- $A^{-1} \geq 0$?

JACOBI METHOD

$A = D + (A - D)$ $\quad D = \mathrm{diag}(A)$

$\underline{Au = R} \iff Du + \underbrace{(A-D)u} = R$

$Du = R - (A-D)u \leftarrow \checkmark$ $\quad u = D^{-1}(D-A)u + D^{-1}\cancel{R}$

$Du^{(k)} = R - (A-D)\, u^{(k-1)}$

$u^{(k)} = D^{-1}(D-A)\, u^{(k-1)} + D^{-1}\cancel{R}$

$\|u - u^{(k)}\|_\infty = \|D^{-1}(D-A)(u^{(k-1)} - u)\|$

$\leq \|D^{-1}(D-A)\|_\infty \cdot \|u - u^{(k-1)}\|_\infty$

$\|B\|_\infty = \max_i \sum_j |B_{ji}|$ $\quad B = D^{-1}(D-A)$

$= (I - D^{-1}A)$

$$\left(I - D^{-1}A\right)_{ji} = \begin{bmatrix} 0 & & \cdots \\ & 0 & \\ \cdots & & \ddots & \\ & & & 0 \end{bmatrix}$$

$$\max_i \sum_{j \neq i} \frac{|A_{ji}|}{|A_{ii}|} = \max_i \frac{\sum_{j \neq i} |A_{ji}|}{|A_{ii}|} < 1$$

$$\|u - u^{(k)}\|_\infty < \|u - u^{(k-1)}\|_\infty$$

$$u^{(k)} \geq 0 \qquad\qquad A y^{n+1} = y^n$$

$$\Downarrow$$

$$u^{(k)} = \underbrace{D^{-1}}_{\geq 0} \underbrace{(D-A)}_{\geq 0} \underbrace{u^{(k-1)}}_{\geq 0} + \underbrace{D^{-1}}_{\geq 0} \underbrace{y^n}_{\geq 0} \geq 0$$

$$\begin{cases} \Delta t \, \Pi_{ij} \geq 0 & i \neq j \\ 0 & i = j \end{cases}$$

$$u^{(k)} \to u = y^{n+1} \geq 0. \qquad \boxtimes$$

$$|e_n| \leq e^{L|t^n - t^0|} |e_0| + \underline{\Delta t}\, \omega(y', \Delta t) \cdot \sum_{i=0}^{n-1} \underbrace{e^{L(t^n - t^{i+1})}}$$

$$= \cdots + \omega(\ldots) \cdot \sum_{i=0}^{n-1} \int_{t^i}^{t^{i+1}} \underbrace{e^{L(t^n - \underbrace{t^{i+1}})}}\, dt$$

$$\leq \qquad + \omega(\ldots) \sum_{i=0}^{n-1} \int_{t^i}^{\overline{t^{i+1}}} e^{L(t^n - t)}\, dt$$

$$= \cdots + \omega \ldots \int_{t^0}^{t^n} e^{L(t^n - t)}\, dt$$

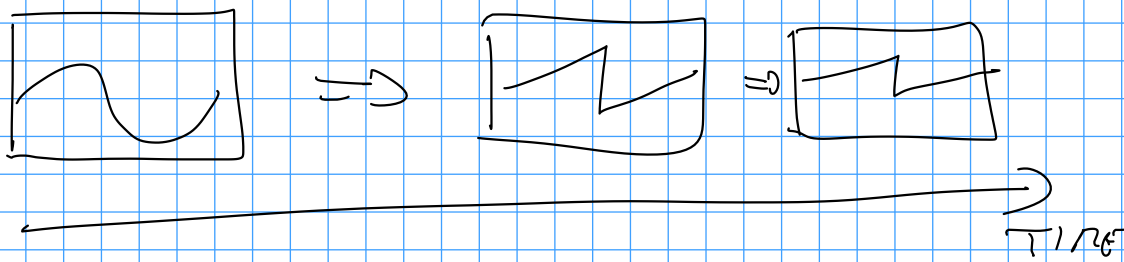
$$= \cdots + \omega \left[\frac{e^{L(t^n - t)}}{-L} \right]_{t^0}^{t^n}$$

$$= \cdots + \omega \, \frac{e^{L(t^n - t^0)} - 1}{L}.$$

$$\partial_t u + \partial_x F(u) = 0 \qquad u : \underset{\subset \mathbb{R}}{\Omega} \times \mathbb{R}^+ \to \mathbb{R}$$

$$F(u) = \begin{cases} u & \text{TRASPORT} \\ \frac{u^2}{2} & \text{BURGERS} \\ \dots \end{cases}$$

BURGERS



$$TV(U^n) := \sum_j |U_j^n - U_{j-1}^n|$$

HARTEN FORM : INCREMENTAL FORM

$$\partial_t U_j = C_{j+1/2}(U_{j+1} - U_j) - D_{j-1/2}(U_j - U_{j-1})$$

$$C_{j+1/2},\ D_{j-1/2} > 0 \qquad \forall j$$

CHECK THAT IMPLICIT EULER DISCR OF INCREMENTAL FORM PDE IS TVD.

I.E. $TV(U^{n+1}) \leq TV(U^n) \quad \forall \Delta t > 0.$

$$U_j^{n+1} = U_j^n + \Delta t \Big[C_{j+1/2}(U_{j+1}^{n+1} - U_j^{n+1}) - D_{j-1/2}(U_j^{n+1} - U_{j-1}^{n+1}) \Big]$$

$$U_{j+1}^{n+1} - U_j^{n+1} = U_{j+1}^n - U_j^n + \Delta t \Big[C_{j+3/2}(U_{j+2}^{n+1} - U_{j+1}^{n+1}) - C_{j+1/2}(U_{j+1}^{n+1} - U_j^{n+1}) - D_{j+1/2}(U_{j+1}^{n+1} - U_j^{n+1}) + D_{j-1/2}(U_j^{n+1} - U_{j-1}^{n+1}) \Big]$$

$$\underbrace{\Big[1 + \Delta t\, C_{j+1/2} + \Delta t\, D_{j+1/2}\Big]}_{>0} (U_{j+1}^{n+1} - U_j^{n+1}) = U_{j+1}^n - U_j^n + \Delta t \Big[C_{j+3/2}(U_{j+2}^{n+1} - U_{j+1}^{n+1}) + D_{j-1/2}(U_j^{n+1} - U_{j-1}^{n+1}) \Big]$$

$$\sum_j \left(1+\Delta t\, C_{j+1/2} + \Delta t\, D_{j+1/2}\right) \left| U_{j+1}^{n+1} - U_j^{n+1} \right|$$

$$\le \underbrace{\sum_j \left|U_{j+1}^{n} - U_j^{n}\right|}_{TV(U^n)} + \Delta t \sum_j C_{j+3/2} \left|U_{j+2}^{n+1} - U_{j+1}^{n+1}\right| + \Delta t \sum_j D_{j-1/2} \left|U_{j}^{n+1} - U_{j-1}^{n+1}\right|$$

$$\sum_j C_{j+3/2} \left|U_{j+2}^{n+1} - U_{j+1}^{n+1}\right| = \sum_j C_{j+1/2} \left|U_{j+1}^{n+1} - U_{j}^{n+1}\right|$$

$$\sum_j D_{j-1/2} \left|U_{j}^{n+1} - U_{j-1}^{n+1}\right| = \sum_j D_{j+1/2} \left|U_{j+1}^{n+1} - U_{j}^{n+1}\right|$$

$$\sum_j \left|U_{j+1}^{n+1} - U_j^{n+1}\right| + \cancel{\Delta t \sum_j C_{j+1/2} \left|U_{j+1}^{n+1} - U_{j}^{n+1}\right|} + \cancel{\Delta t \sum_j D_{j+1/2} \left|U_{j+1}^{n+1} - U_{j}^{n+1}\right|}$$

$$\le TV(U^n) + \cancel{\Delta t \sum_j C_{j+1/2} \left|U_{j+1}^{n+1} - U_{j}^{n+1}\right|} + \cancel{\Delta t \sum_j D_{j+1/2} \left|U_{j+1}^{n+1} - U_{j}^{n+1}\right|}$$

$$\Rightarrow TV(U^{n+1}) \le TV(U^n) \qquad \boxtimes$$

$$y' = \Lambda y$$

$$\frac{y^{n+1} - y^n}{\Delta t} = \Lambda y^{n+1} \Rightarrow \underbrace{(I - \Delta t \Lambda)}\, y^{n+1} = y^n$$

$$e(\Delta t) = C \cdot \Delta t^p$$

$$\underbrace{\log(e(\Delta t))} = \log(C \cdot \Delta t^p) = \log(C) + \underbrace{\log(\Delta t)} \cdot p$$

SLOPE

$$|y^{n+1}| < |y^n|$$

$y^n \longmapsto y^{n+1}$ $\quad$ 2^{nd} ORDER ? $\quad$ $\theta \in \mathbb{R}$

$$y^* = y^n + \theta \Delta t\, F(t^n, y^n)$$

$$y^{n+1} = y^n + \Delta t \left[\frac{2\theta - 1}{2\theta} F(t^n, y^n) + \frac{1}{2\theta} F(t^n + \Delta t\theta, y^*) \right]$$

CONSISTENCY LOCAL ERROR

y^n KNOWN EXACT $\Downarrow$ ERROR y^{n+1} ?

$$y(t^{n+1}) = y^n + \Delta t\, y'(t^n) + \frac{\Delta t^2}{2} y''(t^n) + O(\Delta t^3)$$

$$y(t^*) = y(t + \theta \Delta t) = y^n + \theta \Delta t\, y'(t^n) + \frac{\theta^2 \Delta t^2}{2} y''(t^n) + O(\Delta t^3)$$

$$\Downarrow$$

$$y^* = y^n + \theta \Delta t\, F(y^n) = y(t^*) + O(\Delta t^2)$$

$$y^{n+1} = y^n + \Delta t \left[\frac{2\theta - 1}{2\theta} F(y^n) + \frac{1}{2\theta} F(y^*) \right]$$

$$= y^n + \Delta t \left[\frac{2\theta - 1}{2\theta} F(y^n) + \frac{1}{2\theta} \left(F(y^n) + \frac{\partial F}{\partial y}(y^n)\, \theta \Delta t\, F(y^n) \right) \right]$$

$y''(t)$

$$= y^n + \Delta t\, F(y^n) + \frac{\cancel{\theta}}{2\cancel{\theta}} \Delta t^2 \frac{d}{dt} F(y(t^n)) =$$

$$= y^n + \Delta t\, F(y^n) + \frac{\Delta t^2}{2} \frac{d}{dt} y'(t^n)$$

$$= y^n + \Delta t\, y'(t^n) + \frac{\Delta t^2}{2} y''(t^n) = y(t^{n+1}) + O(\Delta t^3)$$

$$y(t^{n+1}) = y^n + y' \Delta t + \frac{\Delta t^2}{2} y'' + \underline{\frac{\Delta t^3}{6} y'''}$$

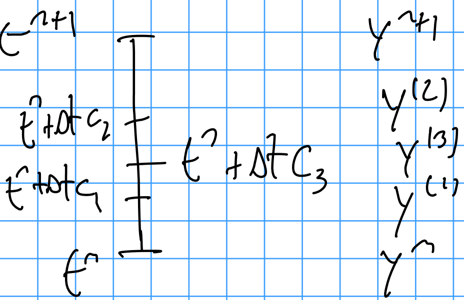
$\Rightarrow$ GLOBAL ERROR $O(\Delta t^2)$

RUNGE-KUTTA

S STAGES $\quad A \in \mathbb{R}^{S \times S}$ $\quad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$

$c, b \in \mathbb{R}^S \quad A = (a_{KJ})_{K,J=1}^{S}$

$$\begin{cases} y^{(K)} = y^n + \Delta t \sum_{J=1}^{S} a_{KJ} F(t^n + \Delta t c_J, y^{(J)}) & K = 1, \dots, S \\ y^{n+1} = y^n + \Delta t \sum_{J=1}^{S} b_J F(t^n + \Delta t c_J, y^{(J)}) \end{cases}$$



EXPLICIT RK

$$y^{(k)} = y^n + \Delta t \sum_{j=1}^{k-1} a_{kj} \, F(y^{(j)})$$

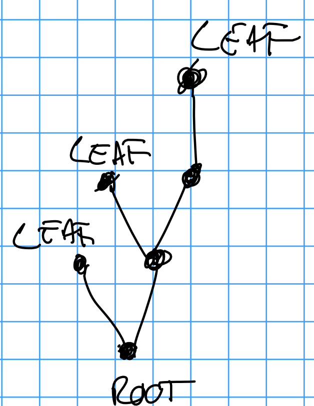
θ - SCHEME

$$\begin{cases} y^{(1)} = y^n + \Delta t \left[0 \cdot F(y^{(1)}) + 0 \cdot F(y^{(2)}) \right] \\ y^{(2)} = y^n + \Delta t \left[\theta \, F(y^{(1)}) + 0 \cdot F(y^{(2)}) \right] \\ y^{n+1} = y^n + \Delta t \left[\frac{2\theta - 1}{2\theta} F(y^{(1)}) + \frac{1}{2\theta} F(y^{(2)}) \right] \end{cases}$$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \theta & \theta & 0 \\ \hline & \frac{2\theta-1}{2\theta} & \frac{1}{2\theta} \end{array} \qquad \begin{array}{c|cc} 0 & & \\ \theta & \theta & \\ \hline & \frac{2\theta-1}{2\theta} & \frac{1}{2\theta} \end{array}$$

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ c_2 & a_{21} & 0 & 0 & 0 \\ c_3 & a_{31} & a_{32} & 0 & 0 \\ c_4 & a_{41} & a_{42} & a_{43} & 0 \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

ROOTED TREES



\# NODES $\longrightarrow$ ORDER $\leftarrow$ ORDER

FOR WHICH WE ARE USING THIS CONDITION

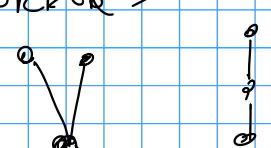
ORDER 1

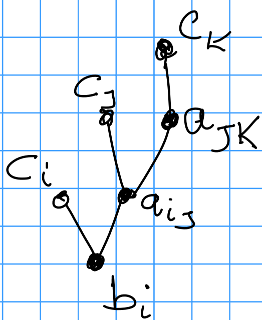
τ

ORDER 2

$[\tau]$

ORDER 3



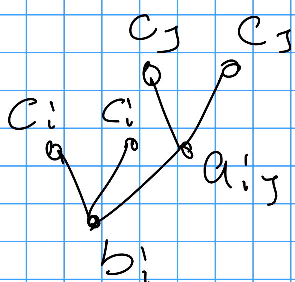


$\Phi(a_{ij}, b_j, c_j) = \frac{1}{\gamma}$ $\gamma \in \mathbb{N}$

Φ ? POLYNOMIAL

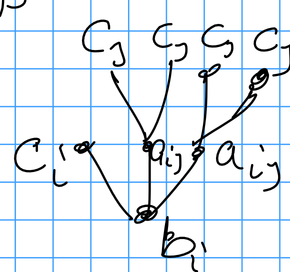
$\gamma \in \mathbb{N}$?

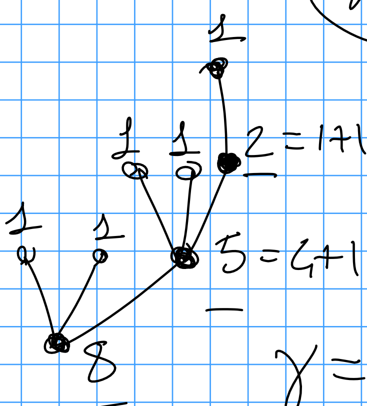
$$\sum_{i,j,k=1}^{s} b_i a_{ij} c_i a_{jk} c_j c_k = \Phi(a, b, c)$$



$$\Phi(A, b, c) = \sum_{i,j} b_i a_{ij} c_i^2 c_j^2$$

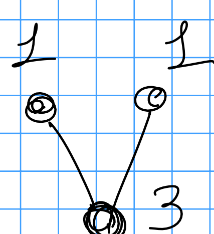
$\gamma = ?$



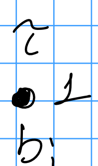


$b_i a_{ij}^2 c_i c_j^4$

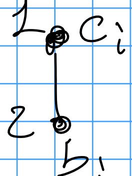
$\gamma = 80$



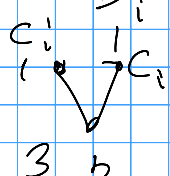
$\Phi(A, b, c) = \frac{1}{\gamma}$



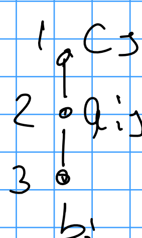
$\sum_i b_i = \frac{1}{1}$ $\gamma = 1$



$\sum_i b_i c_i = \frac{1}{2}$ $\gamma = 2$



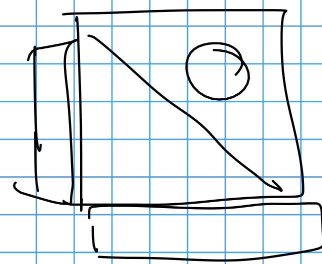
$\sum_i b_i c_i^2 = \frac{1}{3}$



$\sum_{i,j} b_i a_{ij} c_j = \frac{1}{6}$

S stages EXPLICIT RK

$$\frac{S^2+3S-2}{2}$$



5 STAGES $\Longrightarrow$ ORDER ≤ 4

$$\boxed{c_i = \sum_j a_{ij}} \quad \forall$$

(LINEAR)

STABILITY FOR RK METHODS

$y' = qy \qquad q \in \mathbb{C} \qquad z = q \cdot \Delta t$

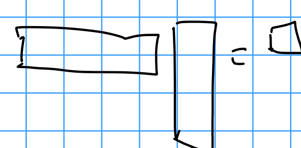
$y^{n+1} = R(z)\cdot y^n \qquad S = \{ z : |R(z)| \le 1 \}$

$\underline{Y} \in \mathbb{R}^S \qquad \underline{Y} = \underline{1}\cdot y^n + \Delta t\, \underline{\underline{A}} \cdot \underline{F(\underline{Y})}$

$$y^{(k)} = y^n + \Delta t \sum_{j=1}^{S} a_{kj} F(y^{(j)})$$

$F(y) = q\cdot y \qquad F(\underline{Y}) = q \cdot \underline{Y}$

$\boxed{\text{RK}} \qquad \underline{Y} = \underline{1}\, y^n + z \cdot \underline{\underline{A}} \cdot \underline{Y}$



$$(\underline{\underline{I}} - z\underline{\underline{A}})\,\underline{Y} = y^n\, \underline{1}$$

$$y^{n+1} = y^n + \Delta t \sum_j b_j F(y^{(j)}) = y^n + \Delta t \cdot b^T \cdot F(\underline{Y})$$

$$= y^n + \Delta t \cdot q\, b^T \underline{Y}$$

$$y^{n+1} = y^n + \Delta t\, q\, b^T \underline{Y} = y^n + \underbrace{\Delta t\, q}_{z}\, b^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \cdot \underline{1}\, y^n$$

$$= \left[1 + z \cdot b^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \cdot \underline{1} \right] \cdot y^n$$

$y^{n+1} = R(z) \cdot y^n \qquad R(z) = 1 + z\, b^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1}\, \underline{1}$

$I = 1 \qquad z = 1 \quad (\sigma^{-1})$

$A = 1$

$R(z) \rightarrow$ POLYNOMIAL FOR EXPLICIT RK

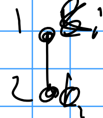
$\searrow$ FRACTION OF TWO POLY IMPLICIT RK

0			
$2/3$	$2/3$		
$2/3$	$1/3$	$1/3$	
	$\frac{1}{4}$	0	$\frac{3}{4}$

ORDER 1

$\sum b_i = 1 \iff \frac{1}{4} + 0 + \frac{3}{4} = 1$ ✓

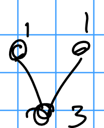
ORDER 2



$\sum_i b_i c_i = \frac{1}{2}$

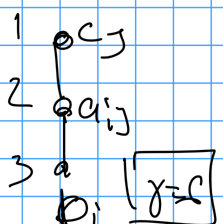
$0 \cdot \frac{1}{4} + \frac{2}{3} \cdot 0 + \frac{\cancel{2}^1}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}2} = \frac{1}{2}$ ✓

ORDER 3



$\sum_i b_i c_i^2 = \frac{1}{3}$

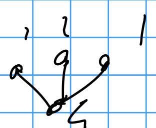
$0 + 0 + \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{2}}{3} \cdot \frac{\cancel{2}}{\cancel{3}} = \frac{1}{3}$ ✓



$\sum_{i,j} b_i a_{ij} c_j = \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{2}}{3} \frac{1}{\cancel{4}} + \frac{\cancel{2}^1}{3} \cdot \frac{1}{3} \cdot \frac{1}{\cancel{4}2} =$

$= \frac{1}{9} + \frac{1}{18} = \frac{3}{18} = \frac{1}{6}$ ✓

ORDER 4



$\sum b_i c_i^3 = \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{2}}{\cancel{3}} \frac{2}{3} \frac{2}{3} = \frac{2}{9} \neq \frac{1}{4}$ NOT ORDER 4

STABILITY FUNCTION

$R(z) = 1 + \Delta t\, \underline{b}^T (\underline{\underline{I}} - z\underline{\underline{A}})^{-1} \underline{1}$

$(I - zA)^{-1}$

$A = \begin{bmatrix} 0 & 0 & 0 \\ 2/3 & 0 & 0 \\ 1/3 & 1/3 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2/9 & 0 & 0 \end{bmatrix} \qquad A^3 = \underline{\underline{0}}$

$A^k = \underline{\underline{0}}$

$k \geq 3$

$$(I - zA)^{-1} = \underbrace{I + zA + z^2A^2}_{} + \underbrace{z^3A^3}_{=0} + \underbrace{\dots}_{=0}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ z\frac{2}{3} & 1 & 0 \\ z\frac{1}{3} + z^2\frac{2}{9} & z\frac{1}{3} & 1 \end{bmatrix}$$

$$R(z) = 1 + z\begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ z\frac{2}{3} & 1 & 0 \\ \frac{z}{3} + \frac{2}{9}z^2 & \frac{z}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 1 + z\begin{bmatrix} \frac{1}{4} + \frac{z}{4} + \frac{1}{6}z^2, & \frac{z}{4}, & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 1 + z\left(\frac{1}{4} + \frac{z}{4} + \frac{1}{6}z^2 + \frac{z}{4} + \frac{3}{4}\right)$$

$$= 1 + z + \frac{z^2}{2} + \frac{1}{6}z^3$$

IMPLICIT RK

$$y^{(k)} = y^n + \sum_{j=1}^{s} a_{kj} \, F(y^{(j)})$$ S EQUATIONS

S x DIM equations

$$y(t) = \sum_{i=1}^{s} \varphi_i(t) \, \underbrace{y(t^n + c_i \Delta t)}_{y^{(i)}} + O(\Delta t^{p+1})$$

t^{n+1}

$t^{(3)} = t^n + c_3 \Delta t$

$t^{(2)}$

$t^{(1)}$

t^n

$$\varphi_i(t^{(j)}) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\frac{d}{dt} y = F(y)$$

$$y(t) = y^n + \int_{t^n}^{t} F(y(s)) \, ds$$

$$y^{(i)} \left(\simeq y(t^{(i)}) \right) = y^n + \int_{t^n}^{t^{(i)}} F(s, y(s))\, ds$$

$$\underset{\text{INTERPOLATING } F}{\approx}\; y^n + \int_{t^n}^{t^{(i)}} \sum_{j=1}^{S} \varphi_j(s) \cdot F(t^{(j)}, y^{(j)})\, ds$$

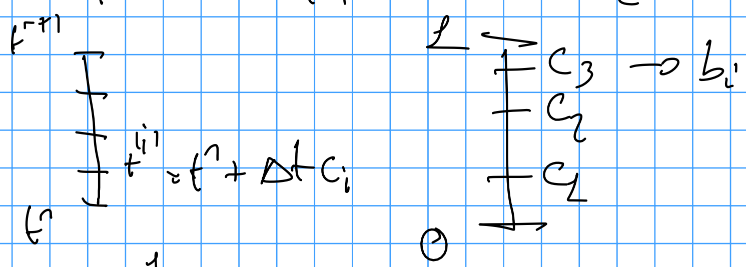
$$= y^n + \sum_{j=1}^{S} \underbrace{\int_{t^n}^{t^{(i)}} \varphi_j(s)\, ds}_{:= \Delta t\, a_{ij}} F(t^{(j)}, y^{(j)})$$

$$= y^n + \sum_{j=1}^{S} \Delta t\, a_{ij}\, F(t^{(j)}, y^{(j)})$$

$$y^{n+1} = y^n + \sum_{j=1}^{S} \underbrace{\int_{t^n}^{t^{n+1}} \varphi_j(s)\, ds}_{:= \Delta t\, b_j} F(t^{(j)}, y^{(j)})$$

B, C, D CONDITIONS

$$B(p): \quad \sum_{i=1}^{S} b_i c_i^{z-1} = \frac{1}{z} \quad \boxed{\forall z = 1, \dots, p}$$



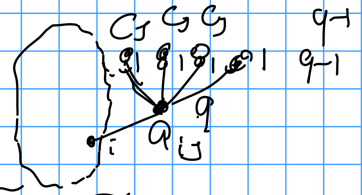
$$\int_0^1 t^{z-1} = \left[\frac{t^z}{z} \right]_0^1 = \frac{1}{z}$$

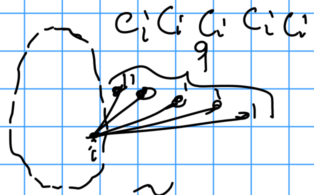
QUADRATURE FORMULA b_i WEIGHTS c_i NODES

$$\int_0^1 t^{z-1} = \sum_i b_i c_i^{z-1} = \frac{1}{z}$$

$B(p)$ QUAD (b_i, c_i) IS EXACT POLY degree UP TO $p-1$

$$C(\eta): \quad \sum_{j=1}^{S} a_{ij} c_j^{q-1} = \frac{c_i^q}{q} \qquad \forall i = 1, \dots, S \quad \forall q = 1, \dots, \eta$$





$\tilde{\Phi}, \tilde{\gamma}$ $\qquad$ $\tilde{\Phi}, \tilde{\gamma}$

$\Phi = \sum_{ij} \tilde{\Phi}_i \cdot a_{ij} \cdot c_j^{q-1}$ $\qquad$ $\Phi = \sum_i \tilde{\Phi}_i \, c_i^q$

$\gamma = \tilde{\gamma} \cdot q$ $\qquad$ $\gamma = \tilde{\gamma}$

$\gamma \Phi = \gamma \Phi$ $\qquad$ $\Phi = \frac{1}{\gamma}$

$$\sum_{ij} \tilde{\Phi}_i \, a_{ij} \, c_j^{q-1} \cdot q \, \cancel{\tilde{\gamma}} = \sum_i \tilde{\Phi}_i \, c_i^q \cdot \cancel{\tilde{\gamma}} \quad \forall \tilde{\Phi}_i$$

$$\Rightarrow \sum_j a_{ij} \, c_j^{q-1} \cdot \cancel{q} = \frac{c_i^q}{q}$$

IF $B(p)$, $C(\eta)$, $D(\xi)$

$p \leq 2\eta + 2$ $\quad$ $p \leq \xi + \eta + 1$ $\Rightarrow$ THE METHOD IS OF ORDER p.

$B(2S)$ $\quad$ $C(S)$ $\quad$ $D(S)$ $\leftarrow$

$B(2S-2)$ $\quad$ $\boxed{C(S)}$ $\quad$ $D(S-2)$ $\quad$ LOBATTO IIIA

$2S-1$ $\quad$ $C(S)$ $\quad$ $D(S-1)$

$C(S-1)$ $\quad$ $D(S)$