

Reduced Order Models on a Variational Multi-Scale Model of Navier–Stokes: focus on wall law functions for boundary layer treatment preliminary study



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Fluid Simulations

- Which **scale** can we approximate?
- Computational **costs** vs **accuracy**
- Large Eddy Simulations (LES)
- Variational Multi-Scale (**VMS**)
- Weak Boundary conditions and **Wall-Law** for boundary layers to improve accuracy
- Turbulence

Model order reduction

- **Parametric** context or time prediction
- Further reduce costs
- **Reduce** the computational cost for a new parameter/time
- Good approximation
- Challenges:
 - Representability (turbulence, moving discontinuities)
 - Stability

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In this talk

- Wall Law model
- **POD-Galerkin** vs **POD-NN**
- Flow past cylinder

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In the future

- VMS-Smagorinsky model
- Hyper-reduction
- NN for wall laws

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① Introduction to Variational Multi-Scale (VMS) model

② Boundary Conditions

③ Reduced Order Model: Galerkin Projection

④ Conclusions

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Navier–Stokes equations (strong)

$$\begin{cases} \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} + \nabla p - 2 \operatorname{div}(\nu \nabla^s \underline{u}) = 0 \\ \nabla \cdot \underline{u} = 0 \\ \text{B.C. and I.C.} \end{cases}$$

¹Y. Bazilevs, T.J.R. Hughes / Computers & Fluids 36 (2007) 12–26

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Weak formulation

$$\begin{cases} \left(\underline{v}, \frac{\partial \underline{u}}{\partial t} \right)_\Omega - (\nabla \underline{v}, \underline{u} \otimes \underline{u})_\Omega + (q, \nabla \cdot \underline{u})_\Omega - \\ (\nabla \cdot \underline{v}, p)_\Omega + (\nabla^s \underline{v}, 2\nu \nabla^s \underline{u})_\Omega = 0 \\ \text{Dirichlet B.C. for } \underline{u} \text{ and } p \text{ and I.C.} \end{cases}$$

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Navier–Stokes VMS¹

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Variational Multi-Scale Residual based

- $\underline{u} = \underline{u}_h + \underline{u}'$ (all variables)
- Residual based

$$\underline{u}'_h = -\tau_M r_M(\underline{u}_h, p_h)$$

$$p'_h = -\tau_C r_C(\underline{u}_h)$$

$$\begin{aligned} r_M(\underline{u}_h, p_h) &= \frac{\partial \underline{u}_h}{\partial t} + \operatorname{div}(\underline{u}_h \otimes \underline{u}_h) + \nabla p_h \\ &\quad - \operatorname{div}(2\nu \nabla^s \underline{u}_h) \end{aligned}$$

$$\tau_M = \left(\frac{4}{\Delta t^2} + \underline{u}_h \cdot G \underline{u}_h + C_{inv} \nu^2 G : G \right)^{-\frac{1}{2}}$$

$$r_C(\underline{u}_h) = \operatorname{div}(\underline{u}_h), \quad \tau_C = (\tau_M \underline{g} \cdot \underline{g})^{-1}$$

$$G = \left(\frac{d\xi}{dx} \right)^T \frac{d\xi}{dx}, \quad \underline{g}_i = \sum_j \left(\frac{d\xi}{dx} \right)_{ji}$$

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Weak VMS formulation

$$\begin{aligned}
 a^{\text{VMS}}(\underline{u}_h, p_h, \underline{v}_h, q_h) := & \\
 & \sum_e (\underline{v}_h, \partial_t \underline{u}_h)_{\Omega_e} - (\nabla \underline{v}_h, \underline{u}_h \otimes \underline{u}_h)_\Omega + (q_h, \nabla \cdot \underline{u}_h)_\Omega \\
 & - \sum_e (\nabla \cdot \underline{v}_h, p_h)_{\Omega_e} + (\nabla^s \underline{v}_h, 2\nu \nabla^s \underline{u}_h)_\Omega \\
 & + 2 \sum_e (\underline{u}_h \cdot \nabla^s \underline{v}_h, \underline{u}'_h)_{\Omega_e} - \sum_e (\nabla \underline{v}_h, \underline{u}'_h \otimes \underline{u}'_h)_{\Omega_e} \\
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 \end{aligned}$$

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Advantages

- Coarse scale \implies Discretized
- Fine scale \implies Modeled
- Extra accuracy by modeling higher order terms without solving them
- Stabilization effect: we can use \mathbb{P}^p for both velocity and pressure (no need of $\mathbb{P}^1\mathbb{P}^2$ formulations)
- Duality: LES modeling and stabilization

²Y. Bazilevs, T.J.R. Hughes / Computers & Fluids 36 (2007) 12–26

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1 Introduction to Variational Multi-Scale (VMS) model

2 Boundary Conditions

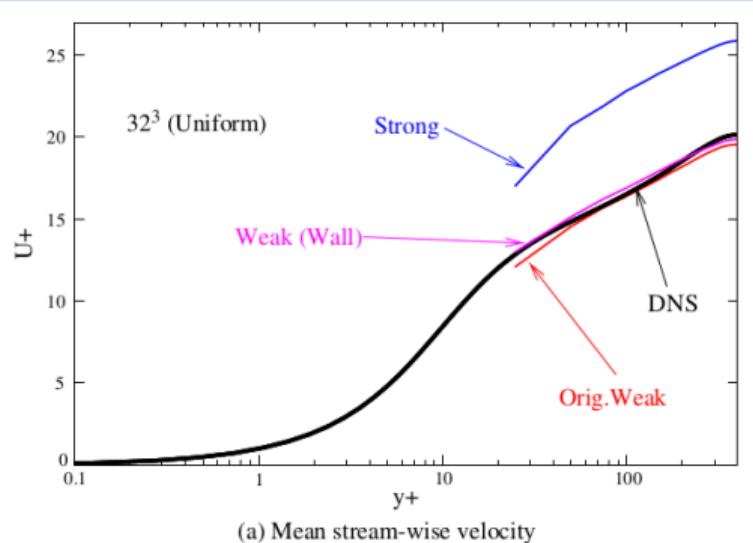
3 Reduced Order Model: Galerkin Projection

4 Conclusions

Boundary conditions

No slip Boundary conditions

- Can create boundary layers
- If strongly huge impact on the solution



Weak Enforcement of no slip BC

$$\begin{aligned} & a_{\text{weakBC}}^{\text{VMS}}(\underline{u}_h, p_h, \underline{v}_h, q_h) := a^{\text{VMS}}(\underline{u}_h, p_h, \underline{v}_h, q_h) \\ & - \sum_b (\underline{v}_h, 2\nu \nabla^s \underline{u}_h \cdot \underline{n})_{\partial\Omega \cap \Gamma_b} \\ & - \sum_b (2\nu \nabla^s \underline{v}_h \cdot \underline{n}, \underline{u}_h - \underline{0})_{\partial\Omega \cap \Gamma_b} \\ & + \sum_b \left(\underline{v}_h \frac{C_b' \nu}{h_b}, \underline{u}_h - \underline{0} \right)_{\partial\Omega \cap \Gamma_b} = 0 \end{aligned}$$

- Consistency term
- Adjoint consistency term
- Penalization of Dirichlet BC

Spalding Wall Law³⁴

Weak penalty for no slip condition

$$\sum_b \left(\underline{v}_h \frac{C'_b \nu}{h_b}, \underline{u}_h - \underline{0} \right)_{\partial\Omega \cap \Gamma_b}$$

- $C'_b = 4$ user set coefficient
- $h_b = 2 (\underline{n}^T G \underline{n})^{-1/2}$ wall-normal element mesh size

Spalding Wall law

- More **physical intuition**
- Exploiting notion of fully developed turbulence
- Boundary layer typically not interesting for turbulent phenomena
- No-slip Dirichlet BC replaced by **traction Neumann boundary**

$$\sum_b \left(\underline{v}_h, u^{*2} \frac{\underline{u}_h}{\|\underline{u}_h\|} \right)_{\partial\Omega \cap \Gamma_b}$$

- u^{*2} magnitude of the **wall shear stress**
- Consistent with the “law of the wall”

³D.B. Spalding, A single formula for the law of the wall, J. Appl. Mech. 28 (1961) 444–458

⁴Y. Bazilevs et al. / Comput. Methods Appl. Mech. Engrg. 196 (2007) 4853–4862

Spalding Wall Law

Spalding Law

$$\sum_b \left(\underline{v}_h, u^{*2} \frac{\underline{u}_h}{\|\underline{u}_h\|} \right)_{\partial\Omega \cap \Gamma_b}$$

- Empirical relation between the mean fluid speed and the normal distance to the wall
- Spalding Law

$$y^+ \stackrel{!}{=} f(u^+) = u^+ + e^{-\chi B} \left(e^{\chi u^+} - 1 - \chi u^+ - \frac{(\chi u^+)^2}{2} - \frac{(\chi u^+)^3}{6} \right),$$

y^+ := $\frac{yu^*}{\nu}$ distance from the wall in nondimensional wall units,

u^+ := $\frac{\|\underline{u}_h\|}{u^*}$ mean fluid speed in nondimensional wall units,

$$\chi = 0.4, B = 5.5.$$

Spalding Law

$$\sum_b \left(\underline{v}_h, u^{*2} \frac{\underline{u}_h}{\|\underline{u}_h\|} \right)_{\partial\Omega \cap \Gamma_b} = \sum_b \left(\underline{v}_h \frac{u^{*2}}{\|\underline{u}_h\|}, \underline{u}_h - \underline{0} \right)_{\partial\Omega \cap \Gamma_b} = \sum_b (\underline{v}_h \tau_B, \underline{u}_h - \underline{0})_{\partial\Omega \cap \Gamma_b}, \quad \tau_B := \frac{u^{*2}}{\|\underline{u}_h\|}$$

- τ_B makes the connection with the weak formulation where $\tau_B = \frac{C_b^l \nu}{h_b}$
- One extra scalar nonlinear equation to be solved $y^+ \stackrel{!}{=} f(u^+)$ for each boundary cell (not too expensive)
- Rewrite the equation in terms of τ_B
- $r(\tau_B) = 0$ with $r'(\tau) > 0$ and $r''(\tau) < 0$ for $\tau > 0$
- Newton's method converges if τ^0 small enough (worst case bisection not too expensive)
- Initial guess $\tau^0 = \frac{C_b^l \nu}{h_b}$ (from weak formulation)

FOM

- $\mathbb{P}^2 - \mathbb{P}^2$ **continuous Galerkin FEM** formulation
- Residual based Variational MultiScale discrete model
- Boundary consistency terms
- Weak penalty for no-slip BC
- **Spalding Wall Law** (very fast 1% cost)

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Test: Flow Past Cylinder

$\mathcal{R} = [0, 2.2] \times [-0.41, 0.41]$, $\mathcal{C} = \mathcal{B}([0.2, 0], 0.05)$,
 $D = \mathcal{R} \setminus \mathcal{C}$, $T_{end} = 3$, $N_h = 3 \cdot 122145$,
 $u_{in} = (\mu_1, 0)$, $\nu = \mu_2$,
Test 1: $Re \in [50, 2.5 \cdot 10^3]$,
 $\mu_1 \in [0.5, 5.0]$, $\mu_2 \in [2 \cdot 10^{-4}, 10^{-3}]$,
No slip BC on top, bottom and circle,

FOM

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Test 2: $Re \in [2.5 \cdot 10^3, 2.5 \cdot 10^5]$,
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Solution Manifold Compression

- Proper Orthogonal Decomposition (**POD**)
- Collection of **snapshots**
 $\{[u_h, p_h, \tau_{B,h}](t^{i_0}, \mu_1^{i_1}, \mu_2^{i_2})\}_{i \in \mathcal{T}} \in V_h$
- Generation of **V_{RB}** component by component
- No need of supremizer⁵

⁵Stabile, Giovanni, et al. "A reduced order variational multiscale approach for turbulent flows." *Advances in Computational Mathematics* 45 (2019): 2349-2368.

Reduced Order Model⁵

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Reconstruction

- **POD-Galerkin**
 - $F(u_h) = 0 \implies V_{RB}^T F(V_{RB} u_{RB}) = 0$
 - Less equations
 - **Hyper-reduction** needed to decrease costs (not today)
 - **Physics** based
 - Less nonlinear iterations
 - For the moment no computational advantage
- **POD-NN**
 - NN learn map $u_{RB}(t, \mu_1, \mu_2)$ from training set
 - Very **fast** reconstruction
 - **No physics** involved
 - Needs **many snapshots** for different parameters (too expensive offline phase)

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Spalding coefficient reconstruction

- For $\tau_{B,h}$ POD-Galerkin and then Newton
- Problem: **Newton does not converge** for reduced $\tau_{B,RB}$ equation
- Multi-layer perceptron NN $u_{RB} \rightarrow \tau_{B,RB}$?

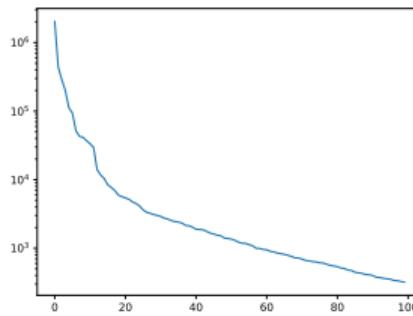
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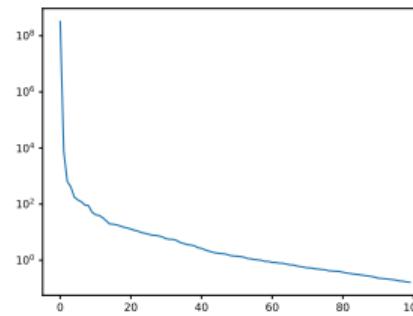
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POD results: **Test1**, 20 parameters, 150 timesteps

u

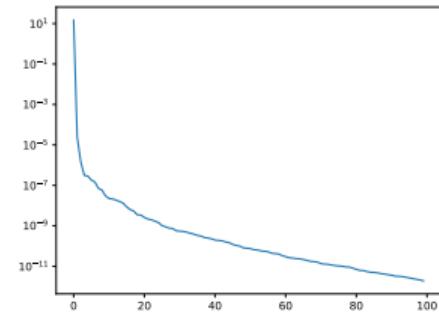
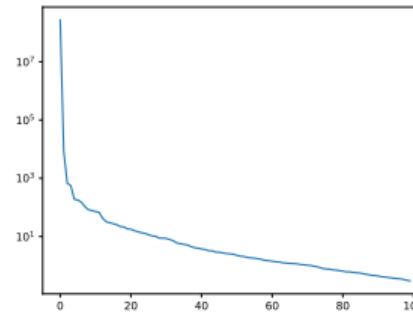
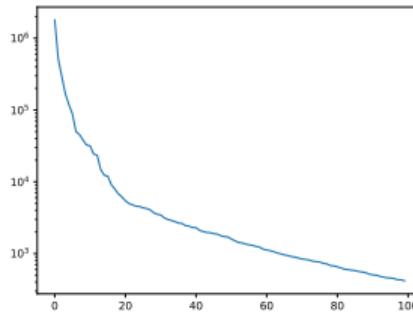


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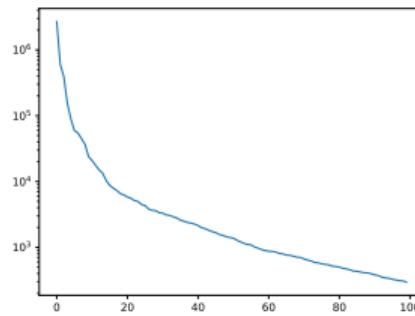
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⇐ Weak BC, Spalding BC ↓

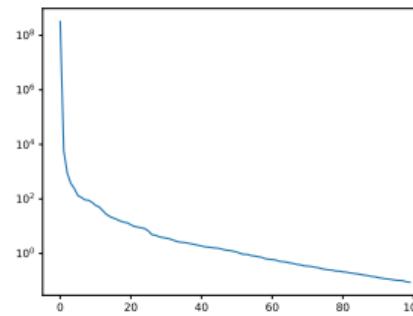


POD results: **Test2**, 20 parameters, 150 timesteps

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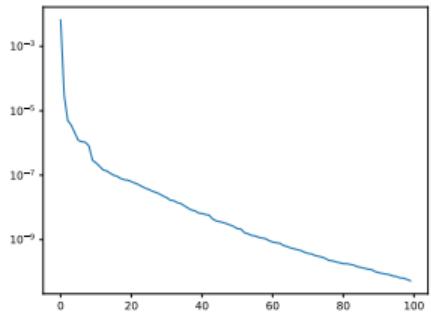
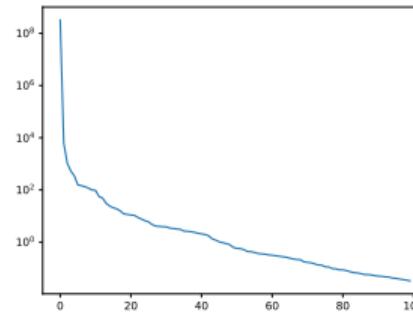
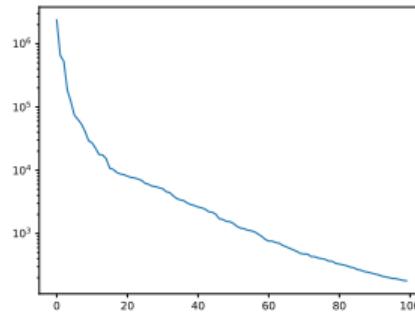


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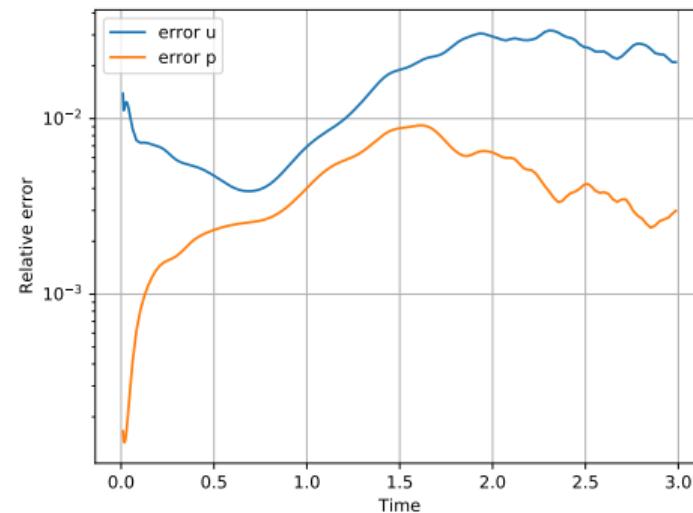
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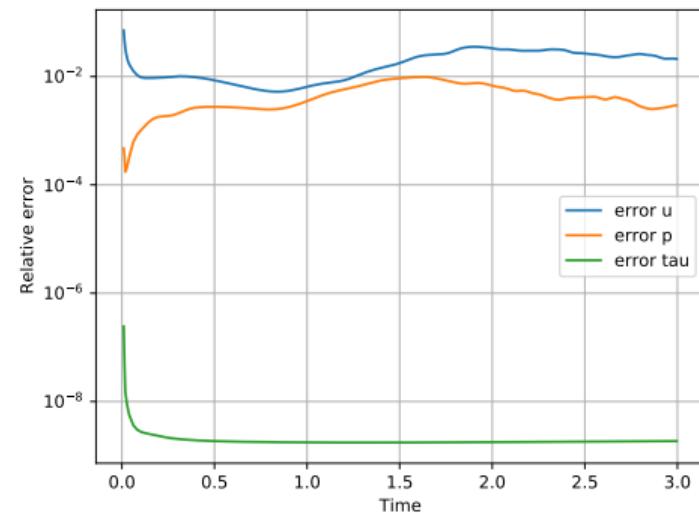
POD projection error: Test1, $(u_{in}, \nu) = (1, 0.001)$

$$N_{POD}^u = N_{POD}^p = 100, N_{POD}^\tau = 30$$

Weak BC



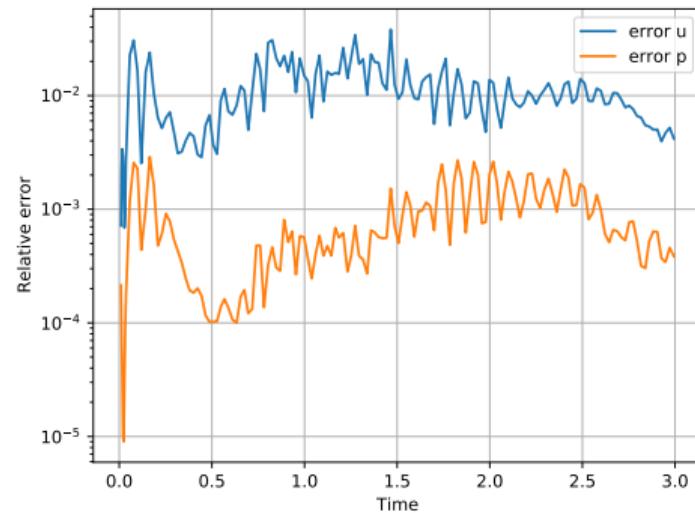
Spalding



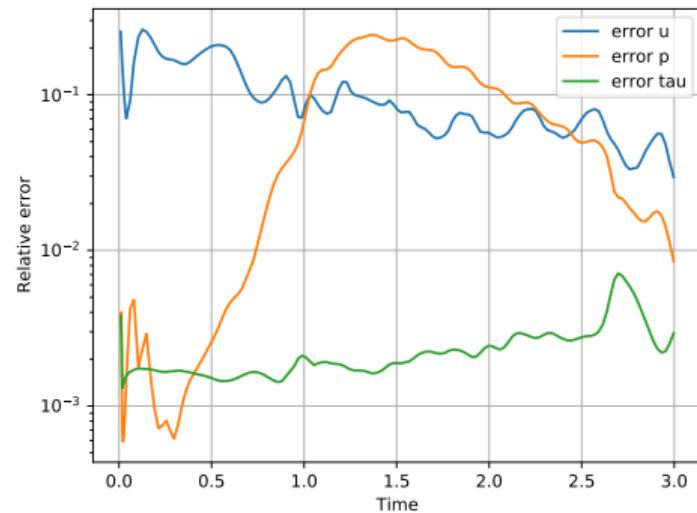
POD projection error: Test2, $(u_{in}, \nu) = (1, 2 \cdot 10^{-6})$

$$N_{POD}^u = N_{POD}^p = 100, N_{POD}^\tau = 30$$

Weak BC



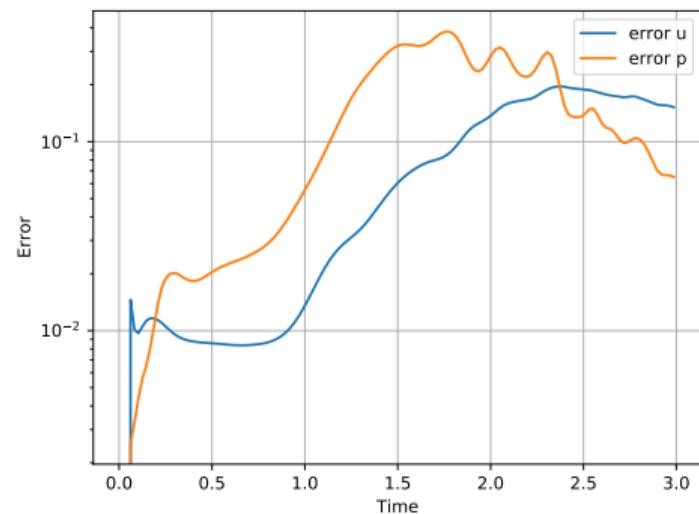
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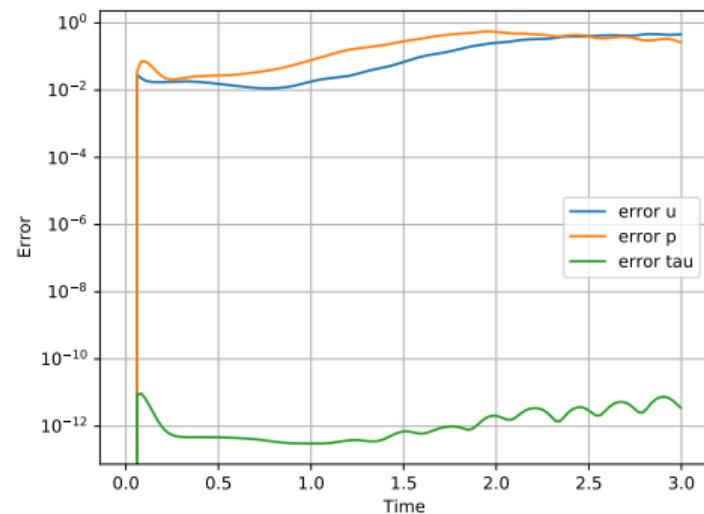
POD-Galerkin error: **Test1**, $(u_{in}, \nu) = (1, 0.001)$

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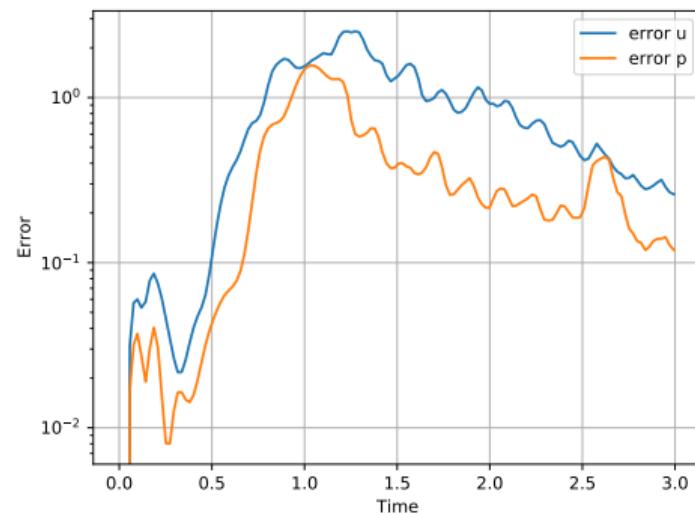


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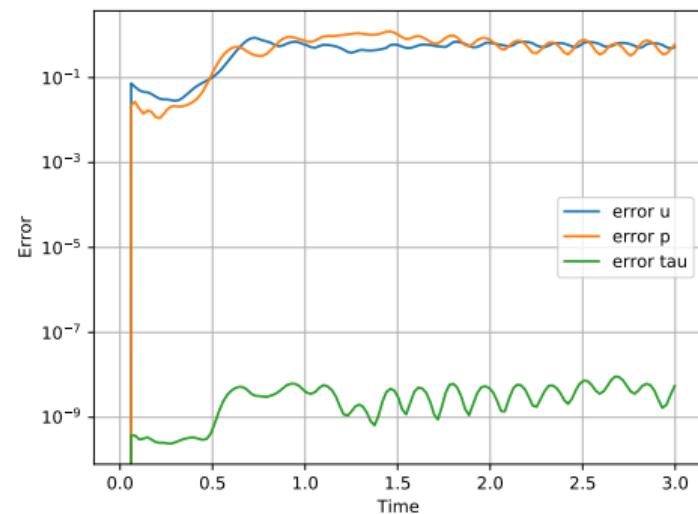


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Weak BC



Spalding



Test1 Weak BC: POD-Galerkin (top) vs FOM (bottom)

Test1 Spalding BC: POD-Galerkin (top) vs FOM (bottom)

Test2 Weak BC: POD-Galerkin (top) vs FOM (bottom)

Test2 Spalding BC: POD-Galerkin (top) vs FOM (bottom)

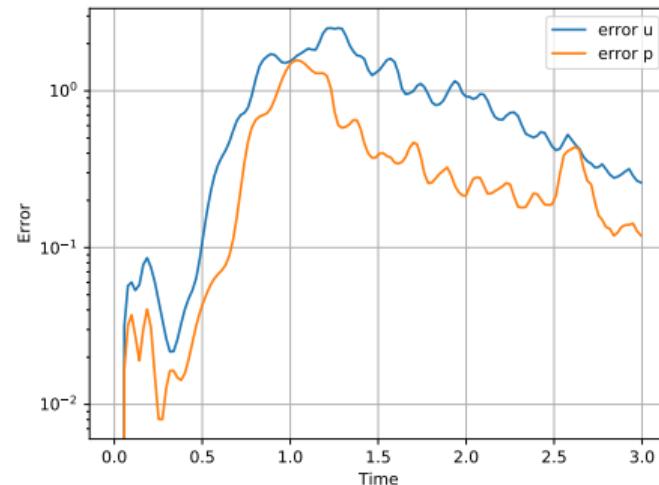
Vortex shedding start

- The **vortex shedding** in FOM is dictated purely by the (triangular) mesh
- No reasons why also in ROM it should start at the same time and in the same direction
- Starting the ROM simulation after vortex shedding (at time $t = 1$)

Weak BC Test2 ROM from $t = 0$

Vortex shedding start

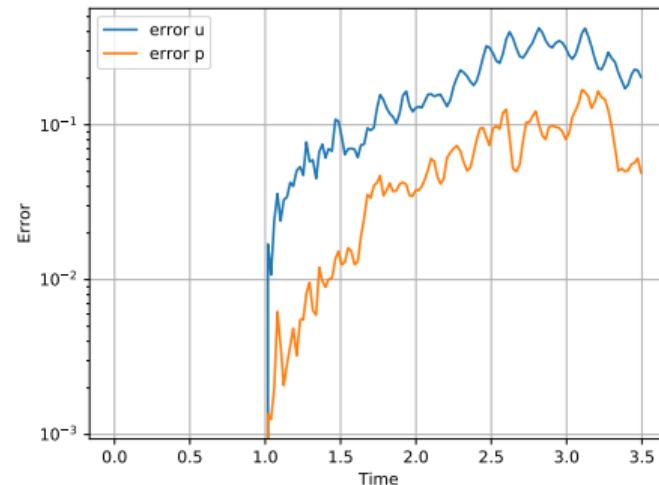
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Weak BC vs Spalding

- Spalding has larger error in representation
- Spalding has little worse behavior in POD-Galerkin
- In past simulations, τ_B computed as FOM (≈ 250 DOFs)

POD-NN

- Training set as POD: 20 params, 150 timesteps (3000 snapshots)
- Goal: learn map $(t, \mu_0, \mu_1) \rightarrow u_{RB}$
- NN setting
 - multi-layer perceptron
 - 4 hidden nodes
 - 100 neurons each
 - Various activation functions
- For u and p the loss struggle at decaying
- For τ already better results, but dangerous to be used alone (time consistency)

Prediction of τ

It might be safer to predict $\tau_B(u)$

- Learn $u_{RB} \rightarrow \tau_{B,RB}$
- NN as before
- Errors $\approx 6\%$ on a test set
- Not really helpful in reducing the computational costs (solving for τ_B already cheap (1% of all costs))
- Still not physics based

Table of contents

① Introduction to Variational Multi-Scale (VMS) model

② Boundary Conditions

③ Reduced Order Model: Galerkin Projection

④ Conclusions

Summary and perspectives⁶

Summary	Perspectives
<ul style="list-style-type: none">• LES-VMS model for Navier-Stokes in FEM• Weak Boundary Conditions• Spalding Law• POD-Galerkin• POD-NN (just started)	<ul style="list-style-type: none">• Hyper-reduction (EIM or overcollocation)• Extend to other models with Local Projection Stabilization (LPS) onto sub-filter scale⁶• 3D turbulent simulations• Improve the architecture for POD-NN to have a comparison with POD-Galerkin

⁶N. Ahmed, T. C. Rebollo, V. John and S. Rubino. Analysis of a Full Space–Time Discretization of the Navier–Stokes Equations by a Local Projection Stabilization Method. IMA Journal of Numerical Analysis, Vol. 37, pp. 1437–1467, 2017.

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THANK YOU!