Reduced Order Models on a Variational Multi-Scale Model of Navier–Stokes: foucs on wall law functions for boundary layer treatment preliminary study

Davide Torlo, Giovanni Stabile, Samuele Rubino, Gianluigi Rozza

MathLab, Mathematics Area, SISSA International School for Advanced Studies, Trieste, Italy
davidetorlo.it

CFC - Cannes - 26th April 2023

Motivation

Fluid Simulations

- Which **scale** can we approximate?
- Computational **costs** vs **accuracy**
- Large Eddy Simulations (LES)
- Variational Multi-Scale (**VMS**)
- Weak Boundary conditions and **Wall-Law** for boundary layers to improve accuracy
- Turbolence

Model order reduction

- **Parametric** context or time prediction
- Further reduce costs
- **Reduce** the computational cost for a new parameter/time
- Good approximation
- Challenges:
  - Representability (turbolence, moving discontinuities)
  - Stability

Motivation

Fluid Simulations

- Which **scale** can we approximate?
- Computational **costs** vs **accuracy**
- Large Eddy Simulations (LES)
- Variational Multi-Scale (**VMS**)
- Weak Boundary conditions and **Wall-Law** for boundary layers to improve accuracy
- Turbolence

Model order reduction

- **Parametric** context or time prediction
- Further reduce costs
- **Reduce** the computational cost for a new parameter/time
- Good approximation
- Challenges:
  - Representability (turbolence, moving discontinuities)
  - Stability

In this talk

- Wall Law model
- **POD-Galerkin** vs **POD-NN**
- Flow past cylinder

Motivation

Fluid Simulations

- Which **scale** can we approximate?
- Computational **costs** vs **accuracy**
- Large Eddy Simulations (LES)
- Variational Multi-Scale (**VMS**)
- Weak Boundary conditions and **Wall-Law** for boundary layers to improve accuracy
- Turbolence

Model order reduction

- **Parametric** context or time prediction
- Further reduce costs
- **Reduce** the computational cost for a new parameter/time
- Good approximation
- Challenges:
  - Representability (turbolence, moving discontinuities)
  - Stability

In this talk

- Wall Law model
- **POD-Galerkin** vs **POD-NN**
- Flow past cylinder

Not in this talk

- Not (really) turbolent
- No special techniques for advection dominated structures

Motivation

Fluid Simulations

- Which **scale** can we approximate?
- Computational **costs** vs **accuracy**
- Large Eddy Simulations (LES)
- Variational Multi-Scale (**VMS**)
- Weak Boundary conditions and **Wall-Law** for boundary layers to improve accuracy
- Turbolence

Model order reduction

- **Parametric** context or time prediction
- Further reduce costs
- **Reduce** the computational cost for a new parameter/time
- Good approximation
- Challenges:
  - Representability (turbolence, moving discontinuities)
  - Stability

In this talk

- Wall Law model
- **POD-Galerkin** vs **POD-NN**
- Flow past cylinder

Not in this talk

- Not (really) turbolent
- No special techniques for advection dominated structures

In the future

- VMS-Smagorinskij model
- Hyper-reduction
- NN for wall laws

Table of contents

1. Indroduction to Variational Multi-Scale (VMS) model
2. Boundary Conditions
3. Reduced Order Model: Galerkin Projection
4. Conclusions

Table of contents

1. Indroduction to Variational Multi-Scale (VMS) model
2. Boundary Conditions
3. Reduced Order Model: Galerkin Projection
4. Conclusions

Navier–Stokes VMS[1]

Navier–Stokes equations (strong)

$$\begin{cases} \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p - 2\,\mathrm{div}(\nu\nabla^s\underline{u}) = 0 \\ \nabla\cdot\underline{u} = 0 \\ \text{B.C. and I.C.} \end{cases}$$

[1] Y. Bazilevs, T.J.R. Hughes / Computers & Fluids 36 (2007) 12–26

Navier–Stokes VMS[1]

Navier–Stokes equations (strong)

$$\begin{cases} \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p - 2\,\mathrm{div}(\nu\nabla^s \underline{u}) = 0 \\ \nabla\cdot\underline{u} = 0 \\ \text{B.C. and I.C.} \end{cases}$$

Weak formulation

$$\begin{cases} \left(\underline{v}, \frac{\partial \underline{u}}{\partial t}\right)_\Omega - (\nabla\underline{v}, \underline{u}\otimes\underline{u})_\Omega + (q, \nabla\cdot\underline{u})_\Omega - \\ (\nabla\cdot\underline{v}, p)_\Omega + (\nabla^s\underline{v}, 2\nu\nabla^s\underline{u})_\Omega = 0 \\ \text{Dirichlet B.C. for } \underline{u} \text{ and } p \text{ and I.C.} \end{cases}$$

[1] Y. Bazilevs, T.J.R. Hughes / Computers & Fluids 36 (2007) 12–26

Navier–Stokes VMS[1]

Navier–Stokes equations (strong)

$$\begin{cases} \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} + \nabla p - 2\,\mathrm{div}(\nu\nabla^s \underline{u}) = 0 \\ \nabla\cdot\underline{u} = 0 \\ \text{B.C. and I.C.} \end{cases}$$

Weak formulation

$$\begin{cases} \left(\underline{v}, \frac{\partial \underline{u}}{\partial t}\right)_\Omega - (\nabla \underline{v}, \underline{u}\otimes\underline{u})_\Omega + (q, \nabla\cdot\underline{u})_\Omega - \\ (\nabla\cdot\underline{v}, p)_\Omega + (\nabla^s \underline{v}, 2\nu\nabla^s\underline{u})_\Omega = 0 \\ \text{Dirichlet B.C. for } \underline{u} \text{ and } p \text{ and I.C.} \end{cases}$$

Variational Multi-Scale Resiudal based

- $\underline{u} = \underline{u}_h + \underline{u}'$ (all variables)
- Residual based

$$\underline{u}_h' = -\tau_M \underline{r}_M(\underline{u}_h, p_h)$$
$$p_h' = -\tau_C r_C(\underline{u}_h)$$
$$\underline{r}_M(\underline{u}_h, p_h) = \frac{\partial \underline{u}_h}{\partial t} + \mathrm{div}(\underline{u}_h\otimes\underline{u}_h) + \nabla p_h - \mathrm{div}(2\nu\nabla^s\underline{u}_h)$$
$$\tau_M = \left(\frac{4}{\Delta t^2} + \underline{u}_h\cdot G\underline{u}_h + C_{inv}\nu^2 G : G\right)^{-\frac{1}{2}}$$
$$r_C(\underline{u}_h) = \mathrm{div}(\underline{u}_h), \qquad \tau_C = (\tau_M \underline{g}\cdot\underline{g})^{-1}$$
$$G = \left(\frac{d\xi}{dx}\right)^T \frac{d\xi}{dx}, \qquad \underline{g}_i = \sum_j \left(\frac{d\xi}{dx}\right)_{ji}$$

[1] Y. Bazilevs, T.J.R. Hughes / Computers & Fluids 36 (2007) 12–26

Navier–Stokes VMS[1]

Weak VMS formulation

$$
\begin{aligned}
&a^{VMS}(\underline{u}_h, p_h, \underline{v}_h, q_h) := \\
&\sum_e (\underline{v}_h, \partial_t \underline{u}_h)_{\Omega_e} - (\nabla \underline{v}_h, \underline{u}_h \otimes \underline{u}_h)_\Omega + (q_h, \nabla \cdot \underline{u}_h)_\Omega \\
&-\sum_e (\nabla \cdot \underline{v}_h, p_h)_{\Omega_e} + (\nabla^s \underline{v}_h, 2\nu \nabla^s \underline{u}_h)_\Omega \\
&+2\sum_e \left(\underline{u}_h \cdot \nabla^s \underline{v}_h, \underline{u}'_h\right)_{\Omega_e} - \sum_e \left(\nabla \underline{v}_h, \underline{u}'_h \otimes \underline{u}'_h\right)_{\Omega_e} \\
&+\sum_e \left(\nabla \cdot \underline{v}_h, p'_h\right)_{\Omega_e} = 0
\end{aligned}
$$

Variational Multi-Scale Resiudal based

- $\underline{u} = \underline{u}_h + \underline{u}'$ (all variables)
- Residual based

$$
\begin{aligned}
\underline{u}'_h &= -\tau_M \underline{r}_M(\underline{u}_h, p_h) \\
p'_h &= -\tau_C r_C(\underline{u}_h) \\
\underline{r}_M(\underline{u}_h, p_h) &= \frac{\partial \underline{u}_h}{\partial t} + \mathrm{div}(\underline{u}_h \otimes \underline{u}_h) + \nabla p_h \\
&\quad - \mathrm{div}(2\nu \nabla^s \underline{u}_h) \\
\tau_M &= \left(\frac{4}{\Delta t^2} + \underline{u}_h \cdot G \underline{u}_h + C_{inv} \nu^2 G : G \right)^{-\frac{1}{2}} \\
r_C(\underline{u}_h) &= \mathrm{div}(\underline{u}_h), \qquad \tau_C = (\tau_M \underline{g} \cdot \underline{g})^{-1} \\
G &= \left(\frac{d\xi}{dx}\right)^T \frac{d\xi}{dx}, \qquad \underline{g}_i = \sum_j \left(\frac{d\xi}{dx}\right)_{ji}
\end{aligned}
$$

[1] Y. Bazilevs, T.J.R. Hughes / Computers & Fluids 36 (2007) 12–26

Navier–Stokes VMS[2]

Weak VMS formulation

$$
\begin{aligned}
&a^{VMS}(\underline{u}_h, p_h, \underline{v}_h, q_h) := \\
&\sum_e (\underline{v}_h, \partial_t \underline{u}_h)_{\Omega_e} - (\nabla \underline{v}_h, \underline{u}_h \otimes \underline{u}_h)_\Omega + (q_h, \nabla \cdot \underline{u}_h)_\Omega \\
&- \sum_e (\nabla \cdot \underline{v}_h, p_h)_{\Omega_e} + (\nabla^s \underline{v}_h, 2\nu \nabla^s \underline{u}_h)_\Omega \\
&+2 \sum_e \left(\underline{u}_h \cdot \nabla^s \underline{v}_h, \underline{u}'_h\right)_{\Omega_e} - \sum_e \left(\nabla \underline{v}_h, \underline{u}'_h \otimes \underline{u}'_h\right)_{\Omega_e} \\
&+ \sum_e \left(\nabla \cdot \underline{v}_h, p'_h\right)_{\Omega_e} = 0
\end{aligned}
$$

Advantages

- Coarse scale $\Longrightarrow$ Discretized
- Fine scale $\Longrightarrow$ Modeled
- Extra accuracy by modeling higher order terms without solving them
- Stabilization effect: we can use $\mathbb{P}^p$ for both velocity and pressure (no need of $\mathbb{P}^1\mathbb{P}^2$ formulations)
- Duality: LES modeling and stabilization

[2] Y. Bazilevs, T.J.R. Hughes / Computers & Fluids 36 (2007) 12–26

Table of contents

1 Indroduction to Variational Multi-Scale (VMS) model

2 Boundary Conditions

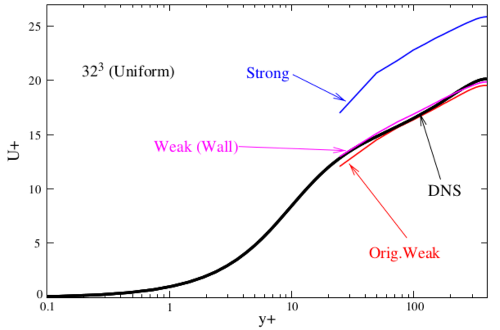
3 Reduced Order Model: Galerkin Projection

4 Conclusions

Boundary conditions

No slip Boundary conditions

- Can create boundary layers
- If strongly huge impact on the solution



(a) Mean stream-wise velocity

Weak Enforcement of no slip BC

$$a^{VMS}_{weakBC}(\underline{u}_h, p_h, \underline{v}_h, q_h) := a^{VMS}(\underline{u}_h, p_h, \underline{v}_h, q_h)$$
$$-\sum_b (\underline{v}_h, 2\nu\nabla^s \underline{u}_h \cdot \underline{n})_{\partial\Omega\cap\Gamma_b}$$
$$-\sum_b (2\nu\nabla^s \underline{v}_h \cdot \underline{n}, \underline{u}_h - \underline{0})_{\partial\Omega\cap\Gamma_b}$$
$$+\sum_b \left(\underline{v}_h \frac{C_b^I \nu}{h_b}, \underline{u}_h - \underline{0}\right)_{\partial\Omega\cap\Gamma_b} = 0$$

- Consistency term
- Adjoint consistency term
- Penalization of Dirichlet BC

Spalding Wall Law[3,4]

Weak penalty for no slip condition

$$\sum_b \left(\underline{v}_h \frac{C_b^I \nu}{h_b}, \underline{u}_h - \underline{0} \right)_{\partial\Omega\cap\Gamma_b}$$

- $C_b^I = 4$ user set coefficient
- $h_b = 2\left(\underline{n}^T G \underline{n}\right)^{-1/2}$ wall-normal element mesh size

Spalding Wall law

- More **physical intuition**
- Exploiting notion of fully developed turbulence
- Boundary layer typically not interesting for turbulent phenomena
- No-slip Dirichlet BC replaced by **traction Neumann boundary**

$$\sum_b \left(\underline{v}_h, u^{*2} \frac{\underline{u}_h}{\|\underline{u}_h\|} \right)_{\partial\Omega\cap\Gamma_b}$$

- u^{*2} magnitude of the **wall shear stress**
- Consistent with the "law of the wall"

[3] D.B. Spalding, A single formula for the law of the wall, J. Appl. Mech. 28 (1961) 444–458
[4] Y. Bazilevs et al. / Comput. Methods Appl. Mech. Engrg. 196 (2007) 4853–4862

Spalding Wall Law

Spalding Law

$$\sum_b \left(\underline{v}_h, u^{*2} \frac{\underline{u}_h}{\|\underline{u}_h\|} \right)_{\partial\Omega\cap\Gamma_b}$$

- Empirical relation between the mean fluid speed and the normal distance to the wall
- Spalding Law

$$y^+ \overset{!}{=} f(u^+) = u^+ + e^{-\chi B}\left(e^{\chi u^+} - 1 - \chi u^+ - \frac{(\chi u^+)^2}{2} - \frac{(\chi u^+)^3}{6} \right),$$

$$y^+ := \frac{y u^*}{\nu} \text{ distance from the wall in nondimensional wall units,}$$

$$u^+ := \frac{\|\underline{u}_h\|}{u^*} \text{ mean fluid speed in nondimensional wall units,}$$

$$\chi = 0.4, \; B = 5.5.$$

Spalding Wall Law

Spalding Law

$$\sum_b \left(\underline{v}_h, u^{*2} \frac{\underline{u}_h}{\|\underline{u}_h\|} \right)_{\partial\Omega\cap\Gamma_b} = \sum_b \left(\underline{v}_h \frac{u^{*2}}{\|\underline{u}_h\|}, \underline{u}_h - \underline{0} \right)_{\partial\Omega\cap\Gamma_b} = \sum_b (\underline{v}_h \tau_B, \underline{u}_h - \underline{0})_{\partial\Omega\cap\Gamma_b}, \quad \tau_B := \frac{u^{*2}}{\|\underline{u}_h\|}$$

- τ_B makes the connection with the weak formulation where $\tau_B = \frac{C_b^I \nu}{h_b}$
- One extra scalar nonlinear equation to be solved $y^+ \stackrel{!}{=} f(u^+)$ for each boundary cell (not too expensive)
- Rewrite the equation in terms of τ_B
- $r(\tau_B) = 0$ with $r'(\tau) > 0$ and $r''(\tau) < 0$ for $\tau > 0$
- Newton's method converges if τ^0 small enough (worst case bisection not too expensive)
- Initial guess $\tau^0 = \frac{C_b^I \nu}{h_b}$ (from weak formulation)

Full Order Model

FOM

- $\mathbb{P}^2 - \mathbb{P}^2$ **continuous Galerkin FEM** formulation
- Residual based Variational MultiScale discrete model
- Boundary consistency terms
- Weak penalty for no-slip BC
- **Spalding Wall Law** (very fast 1% cost)

Full Order Model

FOM

- $\mathbb{P}^2 - \mathbb{P}^2$ **continuous Galerkin FEM** formulation
- Residual based Variational MultiScale discrete model
- Boundary consistency terms
- Weak penalty for no-slip BC
- **Spalding Wall Law** (very fast 1% cost)

Test: Flow Past Cylinder

$\mathcal{R} = [0, 2.2] \times [-0.41, 0.41], \mathcal{C} = \mathcal{B}([0.2, 0], 0.05),$

$D = \mathcal{R} \backslash \mathcal{C}, \qquad T_{end} = 3, \qquad N_h = 3 \cdot 122145,$

$u_{in} = (\mu_1, 0), \quad \nu = \mu_2,$

Test 1: $Re \in [50, 2.5 \cdot 10^3],$

$\mu_1 \in [0.5, 5.0], \quad \mu_2 \in [2 \cdot 10^{-4}, 10^{-3}],$

No slip BC on top, bottom and circle,

Full Order Model

FOM

- $\mathbb{P}^2 - \mathbb{P}^2$ **continuous Galerkin FEM** formulation
- Residual based Variational MultiScale discrete model
- Boundary consistency terms
- Weak penalty for no-slip BC
- **Spalding Wall Law** (very fast 1% cost)

Test: Flow Past Cylinder

$\mathcal{R} = [0, 2.2] \times [-0.41, 0.41], \mathcal{C} = \mathcal{B}([0.2, 0], 0.05),$

$D = \mathcal{R} \backslash \mathcal{C}, \qquad T_{end} = 3, \qquad N_h = 3 \cdot 122145,$

$u_{in} = (\mu_1, 0), \quad \nu = \mu_2,$

Test 2: $Re \in [2.5 \cdot 10^3, 2.5 \cdot 10^5],$

$\mu_1 \in [0.5, 5.0], \quad \mu_2 \in [2 \cdot 10^{-6}, 2 \cdot 10^{-5}],$

No slip BC on top, bottom and circle,

Table of contents

1. Indroduction to Variational Multi-Scale (VMS) model
2. Boundary Conditions
3. Reduced Order Model: Galerkin Projection
4. Conclusions

Reduced Order Model[5]

Solution Manifold Compression

- Proper Orthogonal Decomposition (**POD**)
- Collection of **snapshots** $\{[u_h, p_h, \tau_{B,h}](t^{i_0}, \mu_1^{i_1}, \mu_2^{i_2})\}_{i\in\mathcal{T}} \in V_h$
- Generation of V_{RB} component by component
- No need of supremizer[5]

[5] Stabile, Giovanni, et al. "A reduced order variational multiscale approach for turbulent flows." Advances in Computational Mathematics 45 (2019): 2349-2368.

Reduced Order Model[5]

Solution Manifold Compression

- Proper Orthogonal Decomposition (**POD**)
- Collection of **snapshots** $\{[u_h, p_h, \tau_{B,h}](t^{i_0}, \mu_1^{i_1}, \mu_2^{i_2})\}_{i \in \mathcal{T}} \in V_h$
- Generation of V_{RB} component by component
- No need of supremizer[5]

Reconstruction

- **POD-Galerkin**
  - $F(u_h) = 0 \Longrightarrow V_{RB}^T F(V_{RB} u_{RB}) = 0$
  - Less equations
  - **Hyper-reduction** needed to decrease costs (not today)
  - **Physics** based
  - Less nonlinear iterations
  - For the moment no computational advantage
- **POD-NN**
  - NN learn map $u_{RB}(t, \mu_1, \mu_2)$ from training set
  - Very **fast** reconstruction
  - **No physics** involved
  - Needs **many snapshots** for different parameters (too expensive offline phase)

[5]Stabile, Giovanni, et al. "A reduced order variational multiscale approach for turbulent flows." Advances in Computational Mathematics 45 (2019): 2349-2368.

Reduced Order Model[5]

Solution Manifold Compression

- Proper Orthogonal Decomposition (**POD**)
- Collection of **snapshots** $\{[u_h, p_h, \tau_{B,h}](t^{i_0}, \mu_1^{i_1}, \mu_2^{i_2})\}_{i\in\mathcal{T}} \in V_h$
- Generation of V_{RB} component by component
- No need of supremizer[5]

Spalding coefficient reconstruction

- For $\tau_{B,h}$ POD-Galerkin and then Newton
- Problem: **Newton does not converge** for reduced $\tau_{B,RB}$ equation
- Multi-layer perceptron NN $u_{RB} \to \tau_{B,RB}$?

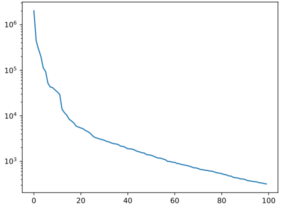
Reconstruction

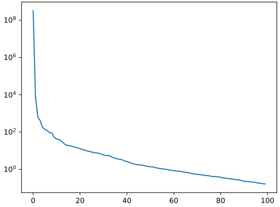
- **POD-Galerkin**
  - $F(u_h) = 0 \Longrightarrow V_{RB}^T F(V_{RB} u_{RB}) = 0$
  - Less equations
  - **Hyper-reduction** needed to decrease costs (not today)
  - **Physics** based
  - Less nonlinear iterations
  - For the moment no computational advantage
- **POD-NN**
  - NN learn map $u_{RB}(t, \mu_1, \mu_2)$ from training set
  - Very **fast** reconstruction
  - **No physics** involved
  - Needs **many snapshots** for different parameters (too expensive offline phase)

[5] Stabile, Giovanni, et al. "A reduced order variational multiscale approach for turbulent flows." Advances in Computational Mathematics 45 (2019): 2349-2368.

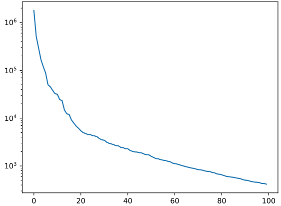
POD results: **Test1**, 20 parameters, 150 timesteps

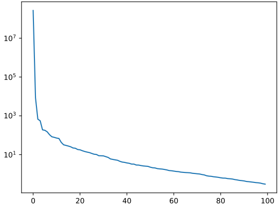
u p τ

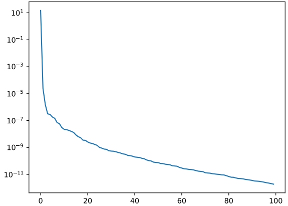




$\Leftarrow$ Weak BC, Spalding BC $\Downarrow$

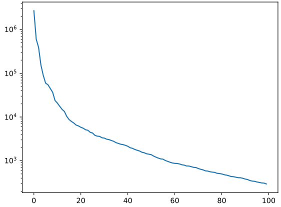


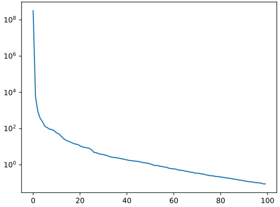




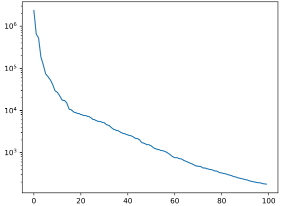
POD results: **Test2**, 20 parameters, 150 timesteps

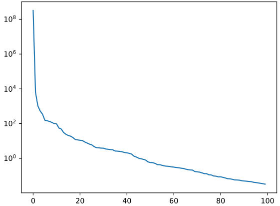
u | p | τ

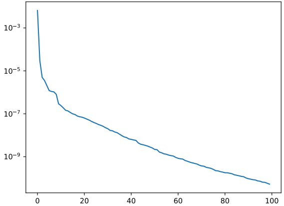




$\Leftarrow$ Weak BC, Spalding BC $\Downarrow$



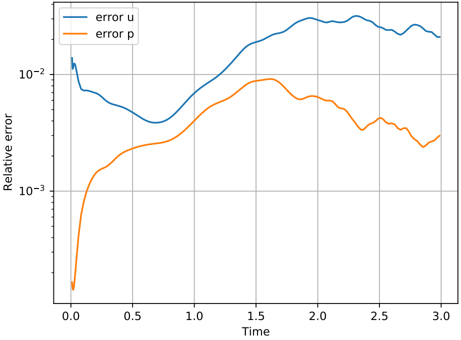




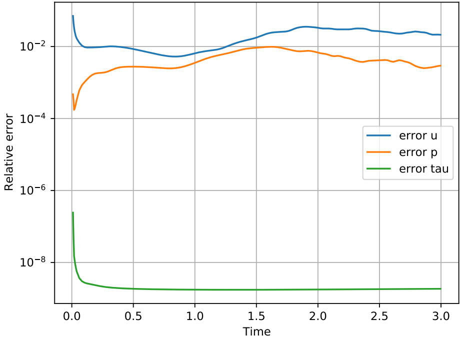
POD projection error: Test1, $(u_{in}, \nu) = (1, 0.001)$

$$N^u_{POD} = N^p_{POD} = 100, \; N^\tau_{POD} = 30$$

Weak BC



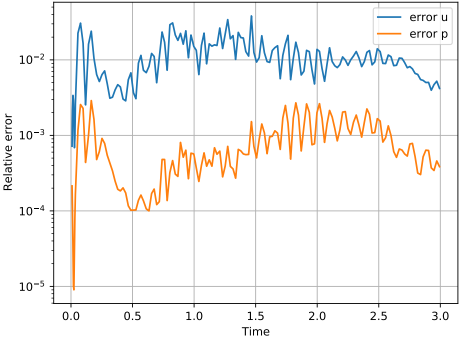
Spalding



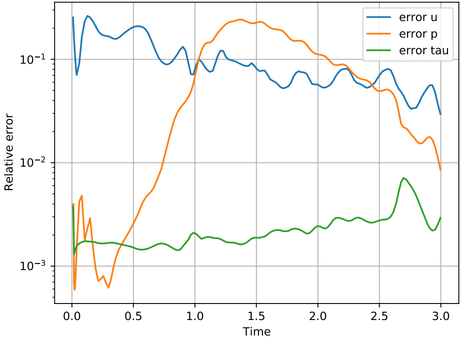
POD projection error: Test2, $(u_{in}, \nu) = (1, 2 \cdot 10^{-6})$

$$N^u_{POD} = N^p_{POD} = 100, \; N^\tau_{POD} = 30$$

Weak BC



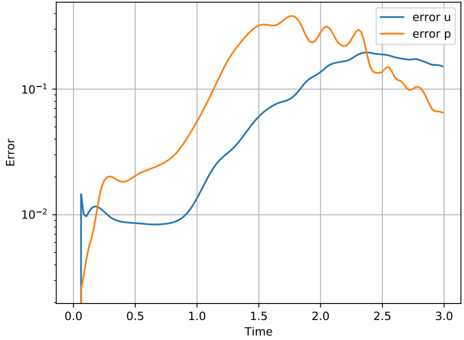
Spalding



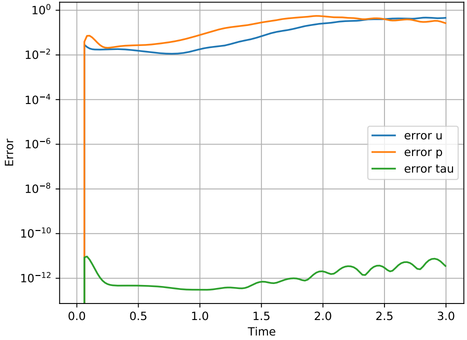
POD-Galerkin error: Test1, $(u_{in}, \nu) = (1, 0.001)$

$$N^u_{POD} = N^p_{POD} = 100,\ \tau \text{ exact}$$

Weak BC



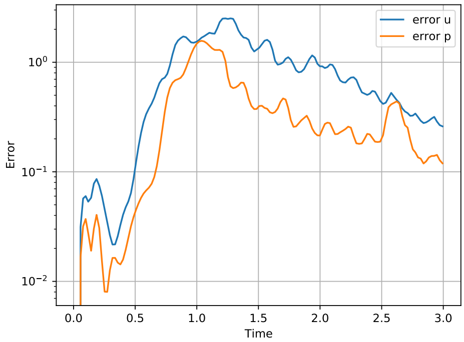
Spalding



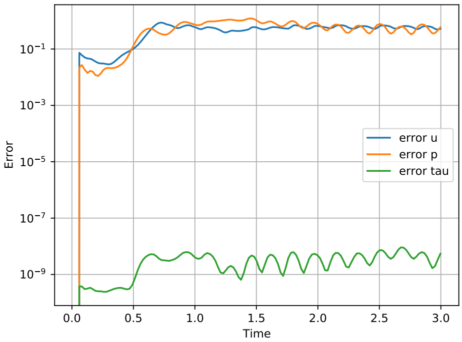
POD-Galerkin error: Test2, $(u_{in}, \nu) = (1, 2 \cdot 10^{-6})$

$$N^u_{POD} = N^p_{POD} = 100,\ \tau \text{ exact}$$

Weak BC



Spalding



Test1 Weak BC: POD-Galerkin (top) vs FOM (bottom)

Test1 Spalding BC: POD-Galerkin (top) vs FOM (bottom)

Test2 Weak BC: POD-Galerkin (top) vs FOM (bottom)

Test2 Spalding BC: POD-Galerkin (top) vs FOM (bottom)

Comments on POD-Galerkin

Vortex shedding start

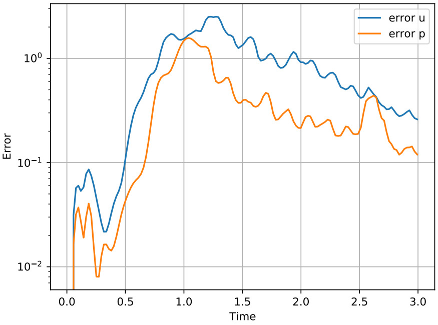
- The **vortex shedding** in FOM is dictated purely by the (triangular) mesh
- No reasons why also in ROM it should start at the same time and in the same direction
- Starting the ROM simulation after vortex shedding (at time $t = 1$)

Comments on POD-Galerkin

Vortex shedding start

- The **vortex shedding** in FOM is dictated purely by the (triangular) mesh
- No reasons why also in ROM it should start at the same time and in the same direction
- Starting the ROM simulation after vortex shedding (at time $t = 1$)

Weak BC Test2 ROM from $t = 0$

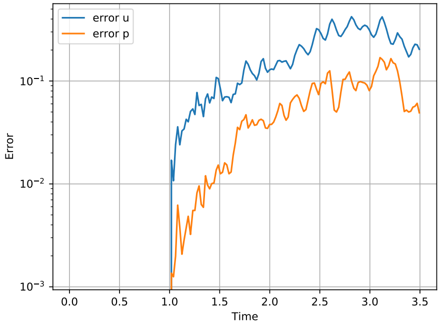


Comments on POD-Galerkin

Vortex shedding start

- The **vortex shedding** in FOM is dictated purely by the (triangular) mesh
- No reasons why also in ROM it should start at the same time and in the same direction
- Starting the ROM simulation after vortex shedding (at time $t = 1$)

Weak BC Test2 ROM from $t = 1$



Comments on POD-Galerkin

Weak BC Test2 ROM from $t = 1$

Comments on POD-Galerkin

Vortex shedding start

- The **vortex shedding** in FOM is dictated purely by the (triangular) mesh
- No reasons why also in ROM it should start at the same time and in the same direction
- Starting the ROM simulation after vortex shedding (at time $t = 1$)

Weak BC vs Spalding

- Spalding has larger error in representation
- Spalding has little worse behavior in POD-Galerkin
- In past simulations, τ_B computed as FOM (≈ 250 DOFs)

POD-NN

POD-NN

- Training set as POD: 20 params, 150 timesteps (3000 snapshots)
- Goal: learn map $(t, \mu_0, \mu_1) \to u_{RB}$
- NN setting
  - multi-layer perceptron
  - 4 hidden nodes
  - 100 neurons each
  - Various activation functions
- For u and p the loss struggle at decaying
- For τ already better results, but dangerous to be used alone (time consistency)

Prediction of τ

It might be safer to predict $\tau_B(u)$

- Learn $u_{RB} \to \tau_{B,RB}$
- NN as before
- Errors $\approx$ 6% on a test set
- Not really helpful in reducing the computational costs (solving for τ_B already cheap (1% of all costs)
- Still not physics based

Table of contents

1. Indroduction to Variational Multi-Scale (VMS) model
2. Boundary Conditions
3. Reduced Order Model: Galerkin Projection
4. Conclusions

Summary and perspectives[6]

Summary	Perspectives
• LES-VMS model for Navier-Stokes in FEM • Weak Boundary Conditions • Spalding Law • POD-Galerkin • POD-NN (just started)	• Hyper-reduction (EIM or overcollocation) • Extend to other models with Local Projection Stabilization (LPS) onto sub-filter scale[6] • 3D turbulent simulations • Improve the architecture for POD-NN to have a comparison with POD-Galerkin

[6] N. Ahmed, T. C. Rebollo, V. John and S. Rubino. Analysis of a Full Space–Time Discretization of the Navier–Stokes Equations by a Local Projection Stabilization Method. IMA Journal of Numerical Analysis, Vol. 37, pp. 1437–1467, 2017.

Literature

- Y. Bazilevs, T.J.R. Hughes. "Weak imposition of Dirichlet boundary conditions in fluid mechanics." Computers & Fluids 36 (2007) 12–26.
- Y. Bazilevs, C. Michler, V.M. Calo, T.J.R. Hughes. "Weak Dirichlet boundary conditions for wall-bounded turbulent flows." Comput. Methods Appl. Mech. Engrg. 196 (2007) 4853–4862.
- D.B. Spalding, A single formula for the law of the wall, J. Appl. Mech. 28 (1961) 444–458.
- Stabile, Giovanni, et al. "A reduced order variational multiscale approach for turbulent flows." Advances in Computational Mathematics 45 (2019): 2349-2368.
- N. Ahmed, T. C. Rebollo, V. John and S. Rubino. Analysis of a Full Space–Time Discretization of the Navier–Stokes Equations by a Local Projection Stabilization Method. IMA Journal of Numerical Analysis, Vol. 37, pp. 1437–1467, 2017.

Literature

- Y. Bazilevs, T.J.R. Hughes. "Weak imposition of Dirichlet boundary conditions in fluid mechanics." Computers & Fluids 36 (2007) 12–26.
- Y. Bazilevs, C. Michler, V.M. Calo, T.J.R. Hughes. "Weak Dirichlet boundary conditions for wall-bounded turbulent flows." Comput. Methods Appl. Mech. Engrg. 196 (2007) 4853–4862.
- D.B. Spalding, A single formula for the law of the wall, J. Appl. Mech. 28 (1961) 444–458.
- Stabile, Giovanni, et al. "A reduced order variational multiscale approach for turbulent flows." Advances in Computational Mathematics 45 (2019): 2349-2368.
- N. Ahmed, T. C. Rebollo, V. John and S. Rubino. Analysis of a Full Space–Time Discretization of the Navier–Stokes Equations by a Local Projection Stabilization Method. IMA Journal of Numerical Analysis, Vol. 37, pp. 1437–1467, 2017.

THANK YOU!