

A time-adaptive solver for pressure dominated flows in CFD and FSI: domain decomposition and model reduction

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Equations

- Fluid

$$\begin{cases} \rho_f [\partial_t u_f + (u_f \cdot \nabla) u_f] - \operatorname{div} \sigma_f(u_f, p_f) = b_f \\ -\operatorname{div} u_f = 0 \end{cases}$$

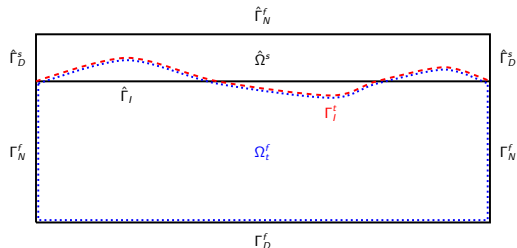
- Structure

$$\rho_s \partial_t^2 \hat{d}_s - \widehat{\operatorname{div}} \hat{P}(\hat{d}_s) = \hat{b}_s$$

- Interaction

- Continuity of the velocities u_f and $\frac{d}{dt} d_s$
- Balance of stresses
- Continuity of displacements (?)

- Boundary/initial conditions



Reduced Order Models for FSI with domain decomposition

Techniques

- Arbitrary Lagrangian-Eulerian (ALE) map for fluid changing domain

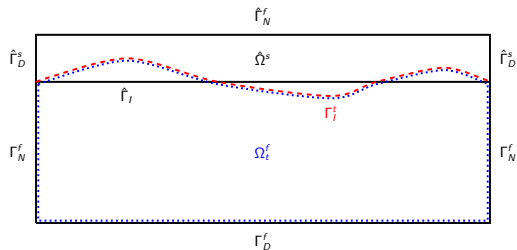
$$\mathcal{A}_t : \hat{\Omega}^f \rightarrow \Omega_t^f$$

$$\hat{x} \mapsto x = \hat{x} + \hat{d}_f(\hat{x}, t)$$

- Extension problem $\sigma^{ext}(\hat{d}_f(t)) = 0$
- Interface condition (ALE form) on $\hat{\Gamma}_I$
 - $\hat{u}_f = \frac{d}{dt} \hat{d}_s$
 - $\hat{J} \hat{\sigma}_f(\hat{u}_f, \hat{p}_f) \hat{F}^{-T} \hat{n}_f = -\hat{P}(\hat{d}_s) \hat{n}_s$
 - $\hat{d}_f = \hat{d}_s$
- Deformation gradient

$$\hat{F} := \hat{\nabla} \mathcal{A}_t$$

$$\hat{J} := \det \hat{F}$$



Cauchy stress tensor

$$\hat{\sigma}_f(\hat{u}_f, \hat{p}_f) = \hat{\sigma}_f^{du}(\hat{d}_f, \hat{u}_f) + \hat{\sigma}_f^p(\hat{p}_f)$$

$$\hat{\sigma}_f^{du}(\hat{d}_f, \hat{u}_f) := \rho_f \nu_f (\hat{\nabla} \hat{u}_f \hat{F}^{-1} + \hat{F}^{-T} \hat{\nabla}^T \hat{u}_f)$$

$$\hat{\sigma}_f^p(\hat{p}_f) := -\hat{p}_f I$$

Monolithic vs Partitioned

Monolithic

- Global functional spaces
- Interface continuity conditions incorporated into the spaces
- Fully coupled problem
- Computationally expensive (sometimes prohibitive)
- In general stable

Domain Decomposition / Partitioned / Segregated

- Local spaces
- Iterative procedure
- Use the state-of-the-art codes for each subcomponent
- Computationally effective
- Subject to possible stability issues

Model order reduction

- Both expensive
- Parametric context too expensive
- Reduce costs with MOR
- Intrusive (POD-Galerkin + hyper-reduction) vs non-intrusive (POD-NN)

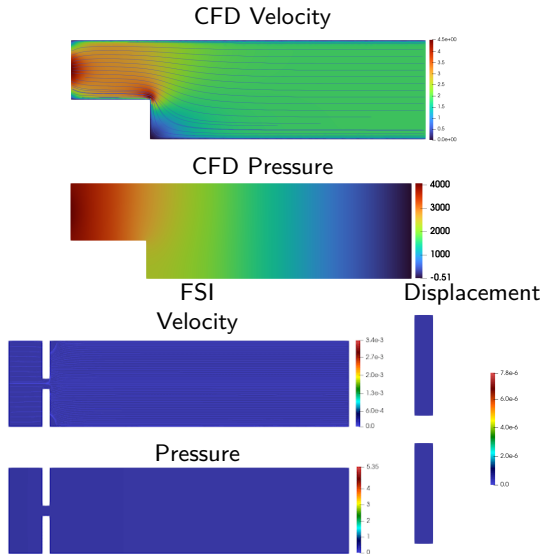
Model order reduction offline costs and tests

Computational problems

- Offline we need some FOM simulations
- Both monolithic and DD are anyway expensive
- Simulations can last some days to have reasonable accuracy

Test cases

- CFD test (no structure)
 - Backward facing step
 - Left inlet BC, right outflow, top/bottom walls
 - Pressure great changes
- FSI test
 - Double leaflets in channel (haemodynamics)
 - Left inlet BC through pressure, outflow right, wall top/bottom
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 - $\Delta t = 10^{-4}$ and $T_{end} = 0.01$



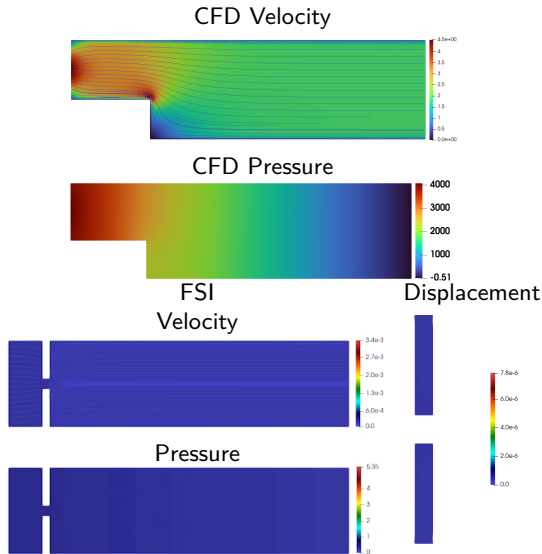
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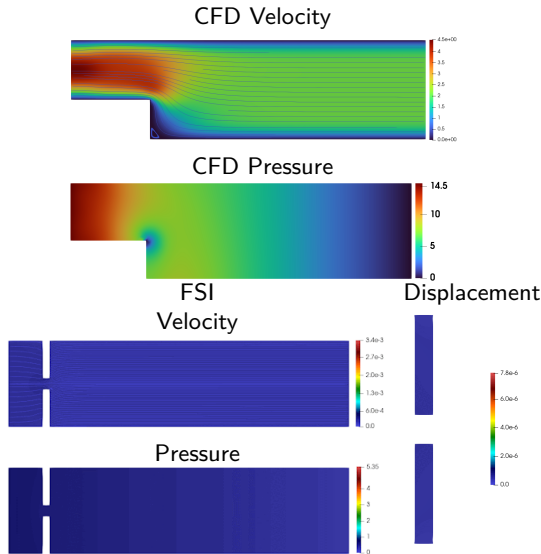
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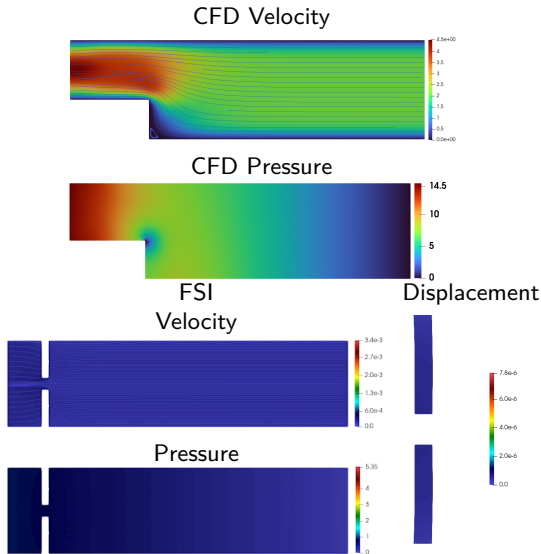
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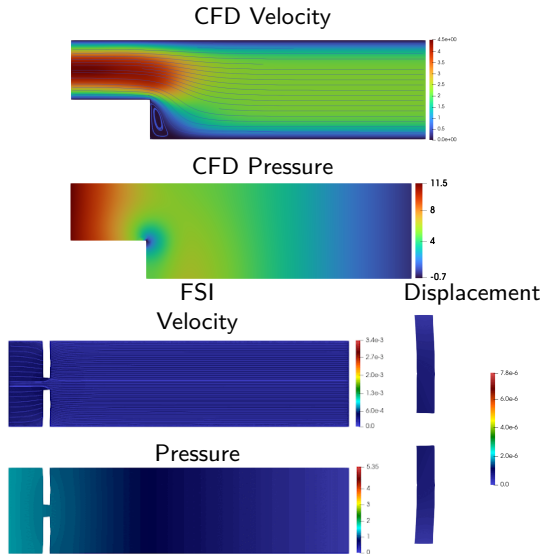
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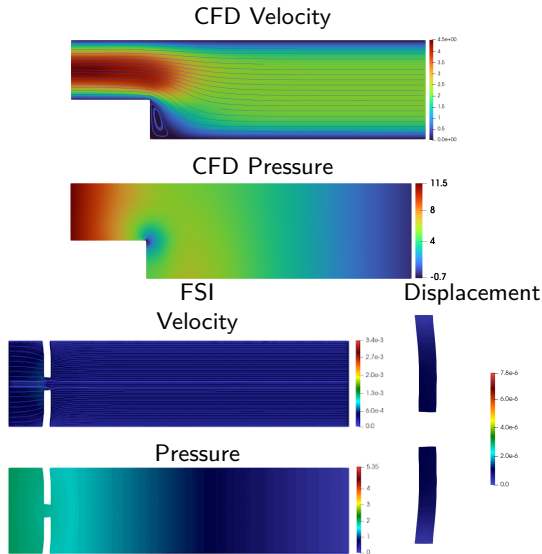
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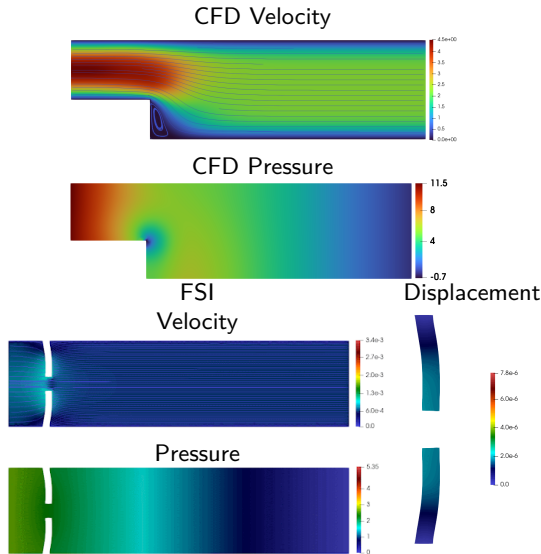
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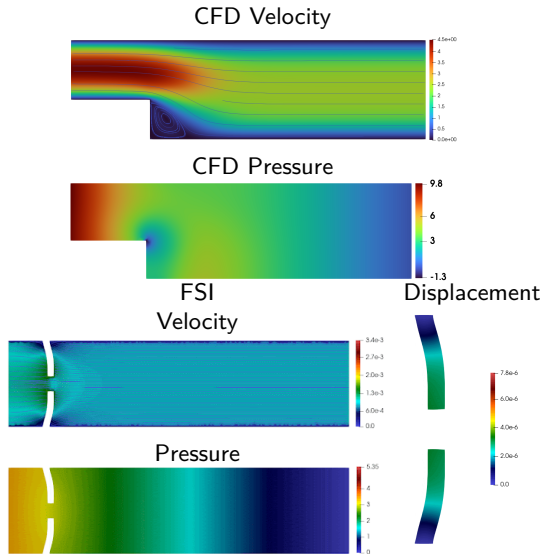
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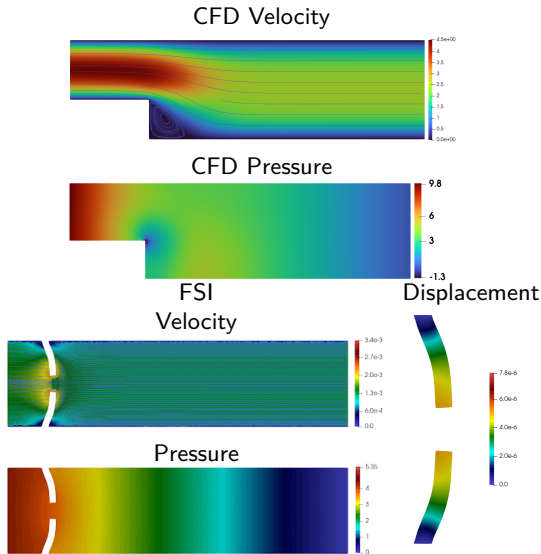
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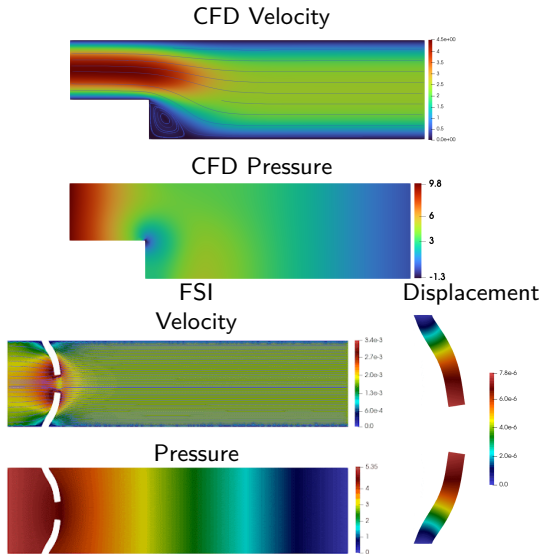
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Time adaptivity!

Classical time-adaptivity

- Two time integrators: A, B
- Either order of A different from order of B:
 $e_A = O(\Delta t^{p_A})$, $e_B = O(\Delta t^{p_B})$ with $p_A < p_B$
- Constant in front of the error is different in A and B (and proportionality is known):
 $e_A = C_A C \Delta t^p$, $e_B = C_B C \Delta t^p$ with $C_A \neq C_B$
- $e_A - e_B = u_A - u_B \approx e_A$
- Choose Δt according to error and tolerances
 - low error \implies increase Δt
 - large error \implies decrease Δt
- We will use implicit BDF2 and implicit BDF3
 - BDF2 is stable and cheap
 - BDF3 less stable, but good for estimator
 - Difference is in the way they discretize ∂_t

Saddle-point problems estimators

- ☺ $\partial_t u$ is in the equation
- ☹ $\partial_t p$ is not in the equation
 - Many estimators consider only u

Time adaptivity!

Our saddle-point estimator

- $\max_{\xi \in \{u, p, d\}} \|\xi^{n+1, \text{BDF2}} - \xi^{n+1, \text{BDF3}}\|_{L^2(\Omega)}$
- Cheaper way to compute ξ^{BDF3} still being 3rd order
- find $\tilde{\mathbf{U}}_h^{n+1, \text{BDF3}} = \mathbf{U}_h^{n+1, \text{BDF2}} + \delta \mathbf{U}_h^{n+1, \text{BDF3}} \in W$, where $\delta \mathbf{U}_h^{n+1, \text{BDF3}} \in W_0$ is the solution of the following linearised equation

$$J \left[R \left(\Xi_{n+1}^{\text{BDF3}} \left(\mathbf{U}_h^{\text{BDF2}} \right), \mathbf{U}_h^{n+1, \text{BDF2}} \right) \right] \left(\delta \mathbf{U}_h^{n+1, \text{BDF3}} \right) = -R \left(\Xi_{n+1}^{\text{BDF3}} \left(\mathbf{U}_h^{\text{BDF2}} \right), \mathbf{U}_h^{n+1, \text{BDF2}} \right)$$

- Nonlinear solver to get $\mathbf{U}_h^{n+1, \text{BDF2}}$
- 😊 Linear solver to get $\tilde{\mathbf{U}}_h^{n+1, \text{BDF3}}$

Backward-facing step flow CFD

- Incompressible Navier-Stokes
- $\mathbb{P}^2 - \mathbb{P}^1$ Taylor Hood FEM spaces
- Inflow $u_{in} = (\varphi(t)\frac{20}{9}(y-2)(5-y), 0)$ on Γ_{in}
- $\varphi(t) = 0.5(1 - \cos(\pi t))$ for $t \leq 1$, then $\varphi(t) = 1$
- Outflow homogeneous Neumann for all variables
- No slip wall on Γ_{wall} , i.e., Dirichlet homogeneous on u
- $\nu = 0.05$ with Reynolds number 300

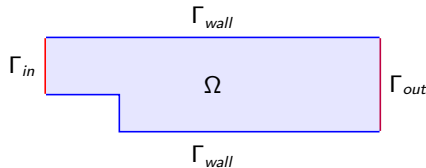
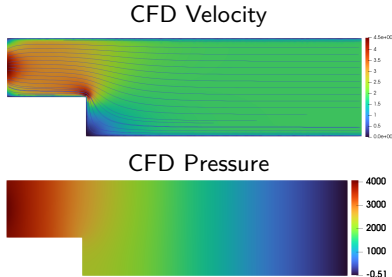


Figure: Backward-facing step domain



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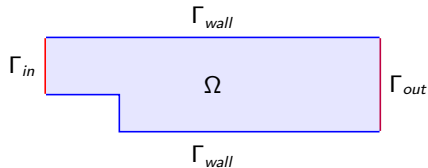
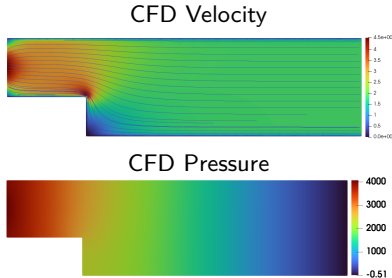


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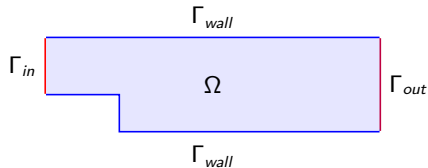
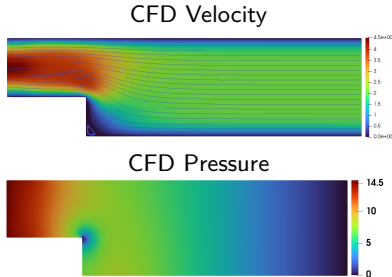


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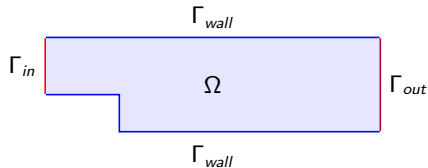
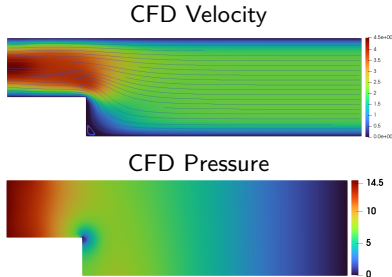


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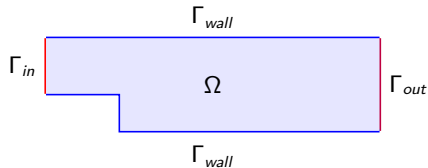
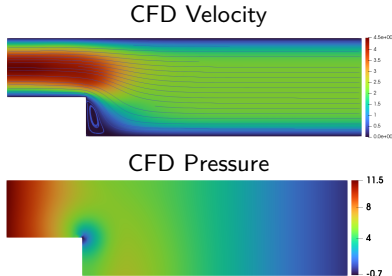


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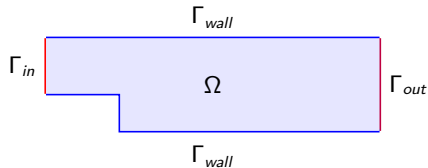
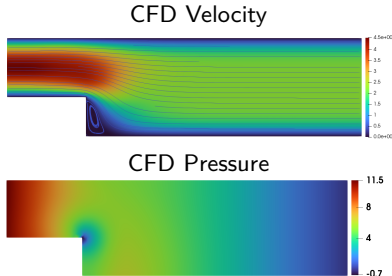


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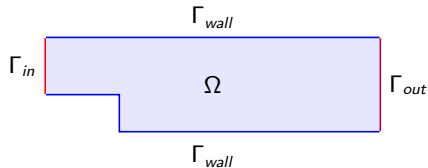
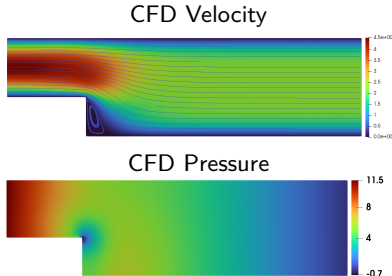


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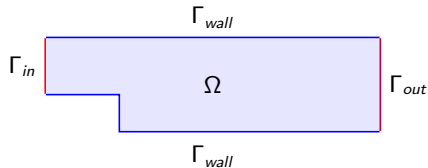
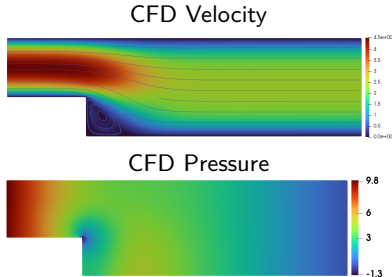
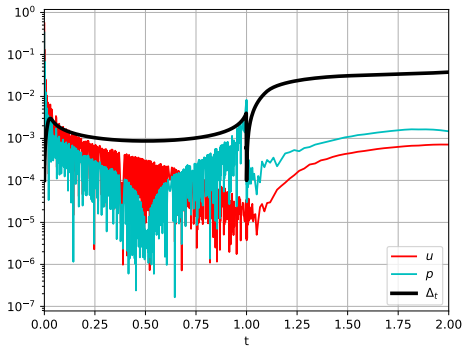


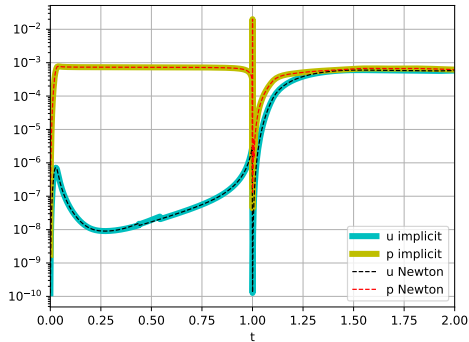
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Numerical simulations: CFD



(a) CFD-300: errors and timestep



(b) CFD-300: error estimators

Figure: Adaptive time-steps distribution and relative errors w.r.t. constant-timestep solution (left) and the comparison of implicit and linear implicit (Newton) time estimators (right)

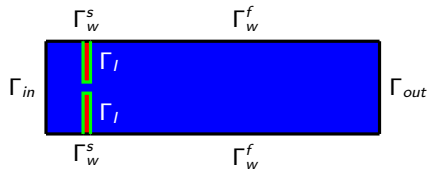
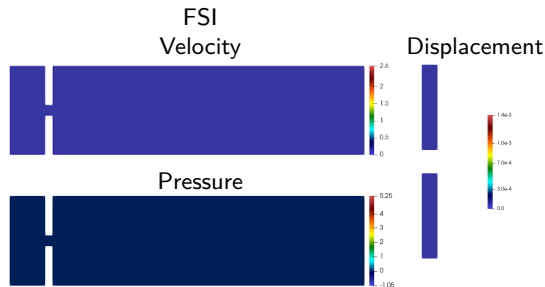


Figure: Reference FSI domain

Heamodynamic problem

- $\nu_f = 0.035, \rho_f = 1, \rho_s = 1.1$
- $\mathbb{P}_2 - \mathbb{P}_1 - \mathbb{P}_2$ with 67,390 DoFs
- $p_{in}(t) = \begin{cases} 5 \left(1 - \cos(\frac{\pi t}{0.2})\right) & \text{for } t \leq 0.1, \\ 5 & \text{for } t > 0.1. \end{cases}$
through a Neumann condition in Γ_{in}
 $u_N^f(\mathbf{x}, t) = -p_{in}(t)\mathbf{n}_f(\mathbf{x})$

☺ $T_{end} = 2$



Numerical simulation: FSI monolithic

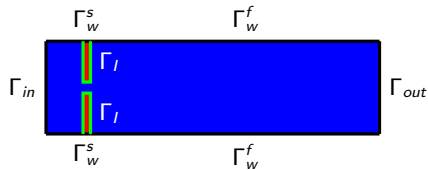
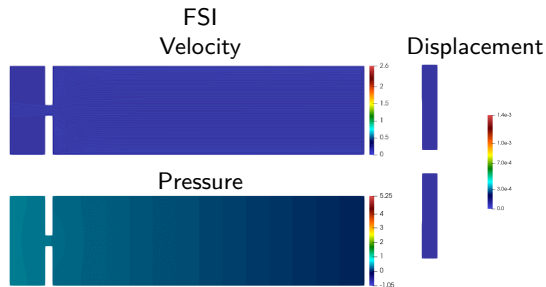


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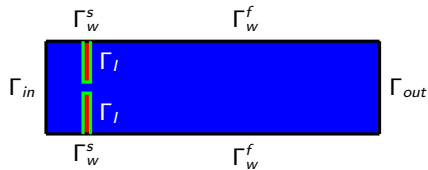
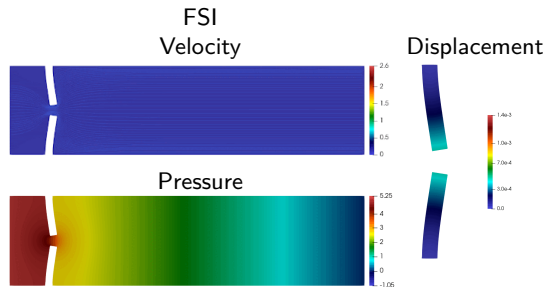


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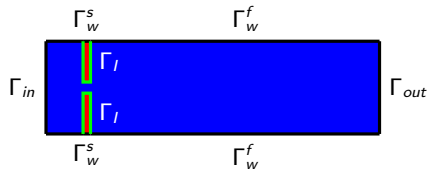
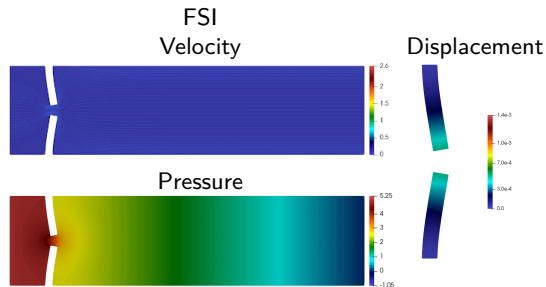


Figure: Reference FSI domain

Heamodynamic problem

- $\nu_f = 0.035, \rho_f = 1, \rho_s = 1.1$
- $\mathbb{P}_2 - \mathbb{P}_1 - \mathbb{P}_2$ with 67,390 DoFs
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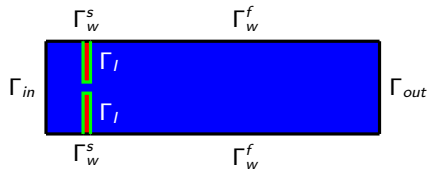
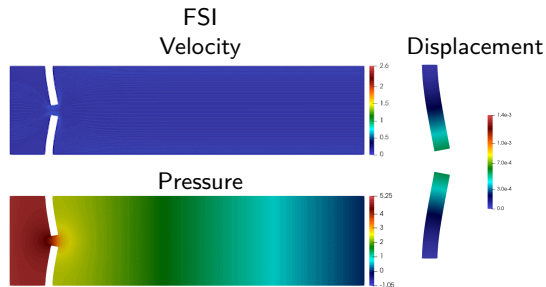


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Numerical simulation: FSI monolithic

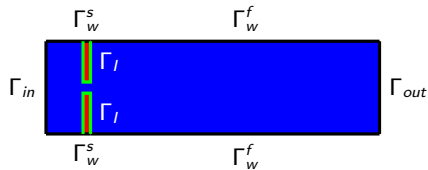
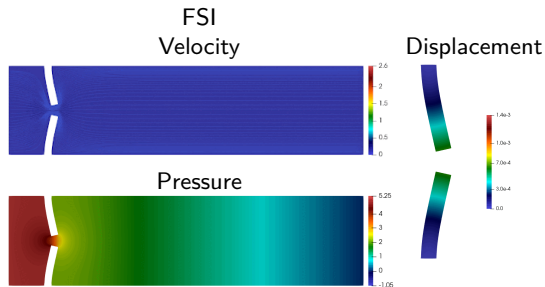


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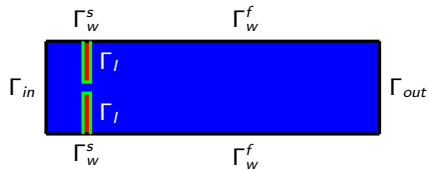
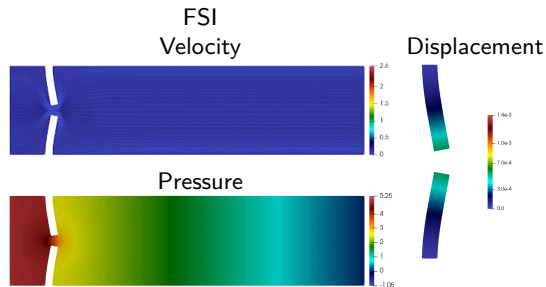


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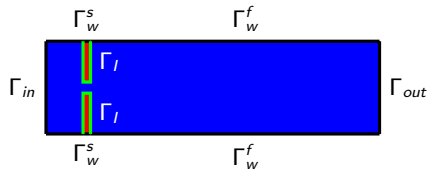
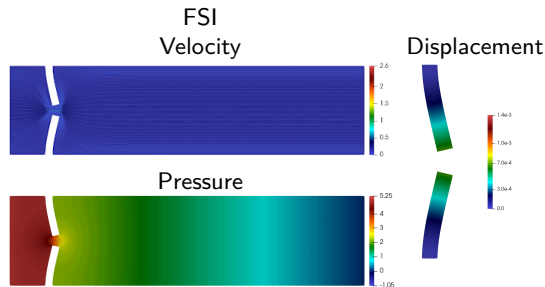


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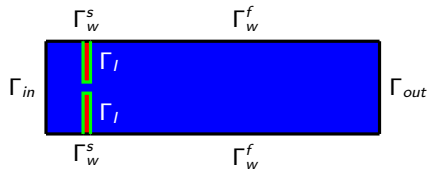
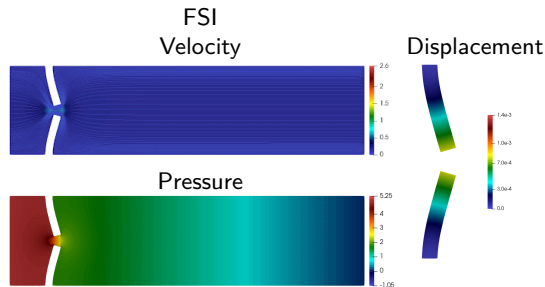


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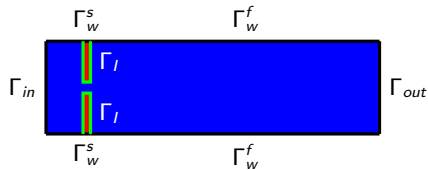
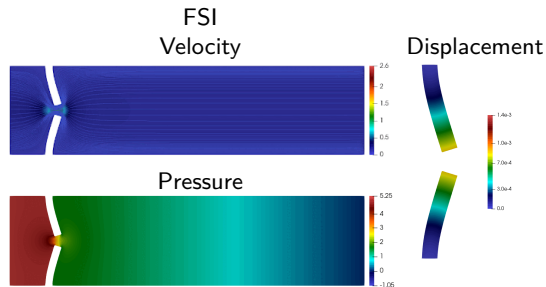


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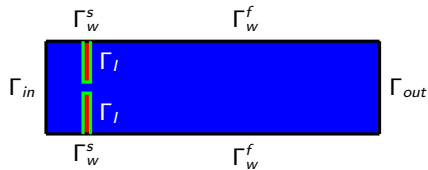
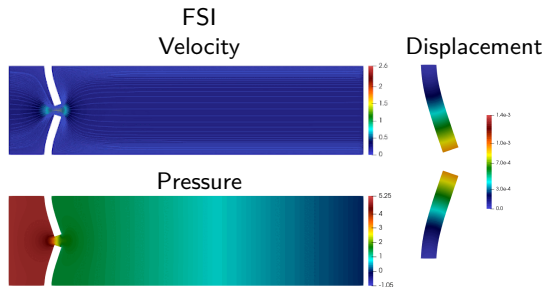


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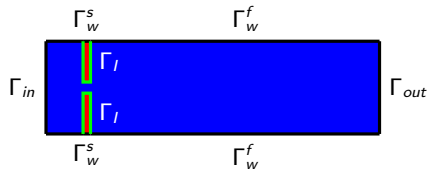
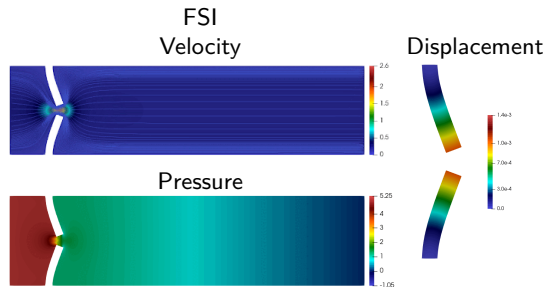


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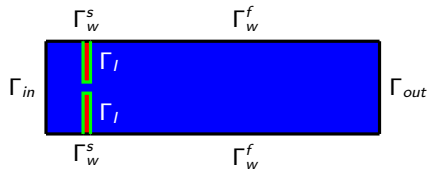
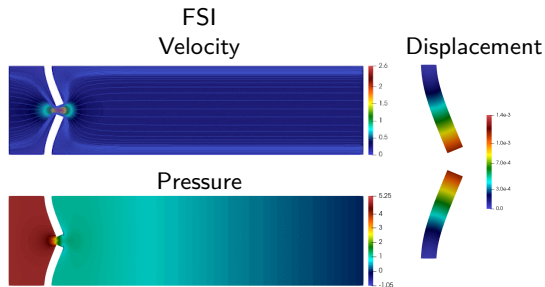


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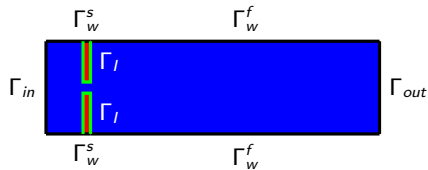
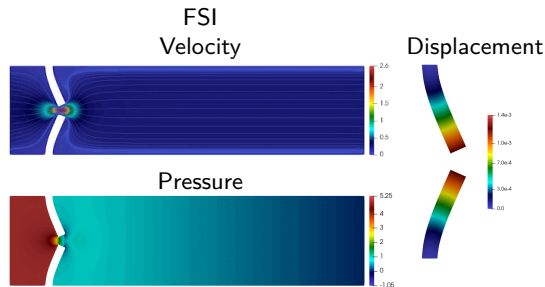


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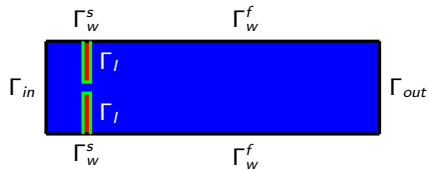
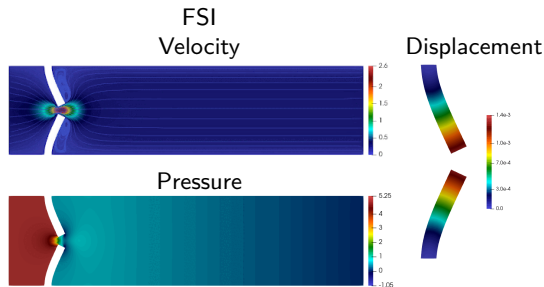


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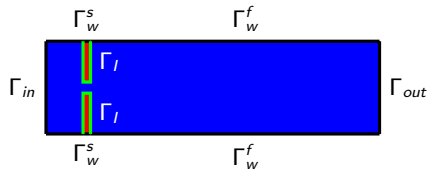
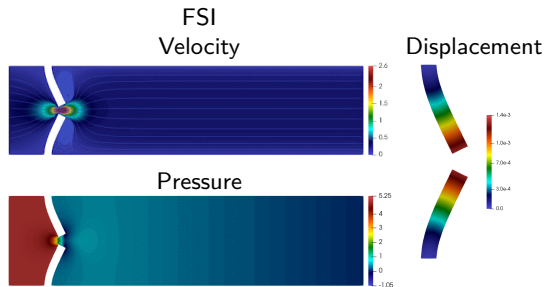


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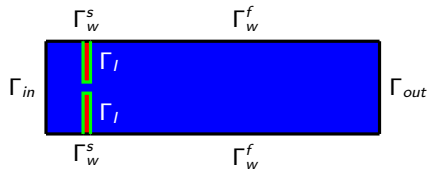
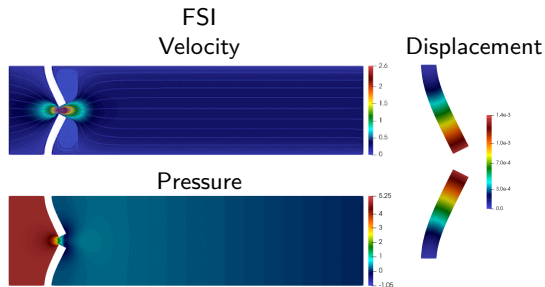


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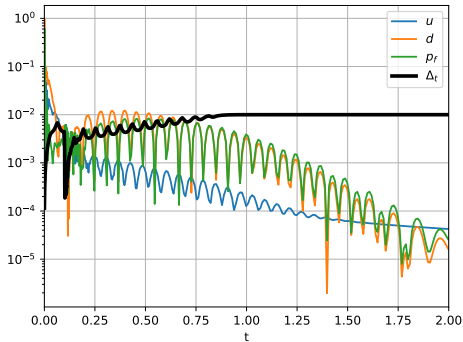
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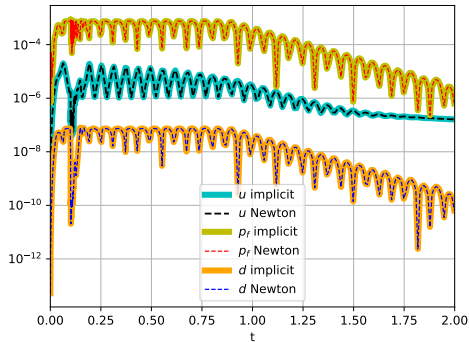
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Numerical simulations: FSI monolithic approach



(a) FSI-H2: errors and timestep



(b) FSI-H2: error estimators

Figure: Adaptive time-steps distribution and relative errors w.r.t. constant-timestep solution (left) and the comparison of implicit and LI time estimators (right)

Computational costs

Table: Computational cost comparison between constant BDF2 and time adaptive algorithms: total simulation time (left) and time of one evaluation of the error estimator (mean \pm std)

	Computational time		Estimator cost		Timesteps number		
Test	Constant	LI adaptive	Implicit	LI	Const.	Impl.	LI
CFD	12 hours	2 hours	2.1 ± 0.2 sec	1.68 ± 0.1 sec	20,000	976	976
FSI	132 hours	5 hours	37 ± 4 sec	25 ± 2 sec	20,000	347	360

Variational spaces

- $u_f \in V_f \subset H^1(\Omega^f)^2$
- $p_f \in Q_f = L^2(\Omega^f)$
- $d_f \in E_f \subset H_D^1(\Omega^f)^2$
- $d_s \in E_s \subset H_D^1(\Omega^s)^2$
- $u_s \in V_s \subset H_D^1(\Omega^s)^2$
- $g_I \in V_I \subset H^1(\Gamma_I)^2$

Interface variable

$$g_I := -P(d_s)\mathbf{n}_s =$$

$$J\sigma_f^{du}(d_f, u_f)F^{-T}\mathbf{n}_f + J\sigma_f^p(p_f)F^{-T}\mathbf{n}_f$$

Fluid

$$\begin{aligned} m_f(\partial_t u_f, v_f; d_f) + a_f(u_f, v_f; d_f) + c_f^{ALE}(\partial_t d_f, v_f, u_f; d_f) \\ + b_f^A(p_f, v_f; d_f) + c_f(u_f, u_f, v_f; d_f) \\ = f_f(v_f; d_f) + (u_N^f, v_f)_{\Gamma_N^f} + (g_I, v_f)_{\Gamma_I} \quad \forall v_f \in V_f, \\ b_f^B(u_f, q_f; d_f) = 0 \quad \forall q_f \in Q_f, \\ a_f^e(d_f, e_f) = 0, \quad \forall e_f \in E_f \\ d_f = d_s \text{ on } \Gamma_I. \end{aligned}$$

Structure

$$\begin{aligned} m_s(\partial_t u_s, v_s) + a_s(d_s, v_s) = f_s(v_s) + (d_N^s, v_s)_{\Gamma_N^s} - (g_I, v_s)_{\Gamma_I} \quad \forall v_s \in V_s, \\ (\partial_t d_s, e_s)_{\Omega_s} = (u_s, e_s)_{\Omega_s} \quad \forall e_s \in E_s. \end{aligned}$$

Objective Functional

$$J_\gamma(u_f, u_s; g_I) := \frac{1}{2} \int_{\Gamma_I} |u_f - u_s|^2 d\Gamma + \frac{\gamma}{2} \int_{\Gamma_I} |g_I|^2 d\Gamma$$

FSI + DD: Optimization strategy for Nonlinear Least Squares problems

Line search methods

- ☹ Not too good for our applications
- ☹ Gauss-Newton is “too local” - diverges after the first 12 time steps
- ☹ Finds infeasible directions for the nonlinear solvers
- ☹ Gradient-based methods stagnates too much and are not always able to overcome “flat” areas

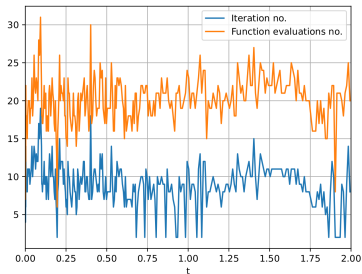
Trust region methods

- Compute (usually) quadratic approximation of the objective functional around the iterative point
- Update trust region radius depending on how well the model approximates the objective
- Solve model optimization problem within the trust region
- Subspace Interior Trust Region Method (STIR)^a

^aM. A. Branch et al. A Subspace, Interior, and Conjugate Gradient Method for Large-Scale Bound-Constrained Minimization Problems, 1999

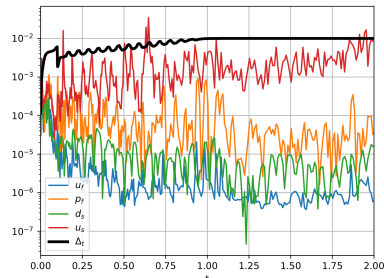
Simulations: Haemodynamics FSI DD STIR time-adaptive

Time adaptive \Rightarrow 302 timesteps
Iterations and **functional evaluations**
(only 1 gradient evaluation per iteration)

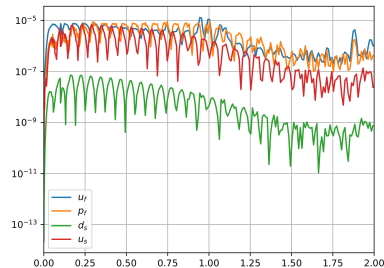


Relative errors

Reference solution:
monolithic with
 $\Delta t = 10^{-4}$



Estimators with linear implicit BDF2-BDF3 approach



ROM

- Intrusive
 - ☹ Galerkin Projection (some effort for Jacobian)
 - ☹ Hyper-reduction
- Nonintrusive: POD-NN
 - ☹ No further models
 - ☹ Exponential behaviors are difficult for NN
- Other questions
 - Time-adaptivity for ROM is necessary or a waste of time?

Perspectives

- More complex domains (multi-domain decomposition)
- Dimensionality reduction of the Jacobian
- Matrix-free methods
- Optimal control problems or inverse problems
- Heterogeneous coupling (FOM-ROM)

Bibliography

- Prusak, Nonino, Torlo, Ballarin and Rozza. An optimisation-based domain-decomposition reduced order model for the incompressible Navier-Stokes equations, CAMWA, 2023
- Prusak, Torlo, Nonino and Rozza. An optimisation-based domain-decomposition reduced order model for parameter-dependent non-stationary fluid dynamics problems, CAMWA, 2024
- I. Prusak. Application of optimisation-based domain-decomposition reduced order models to parameter-dependent fluid dynamics and multiphysics problems, PhD Thesis, 2023
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