High order IMEX deferred correction residual distribution schemes for stiff relaxation problems

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1^{st} April 2019

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Acknowledgments: This research is funded by the ITN ModShockComp.

Outline

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Motivation: relaxed systems

What we want to solve is an hyperbolic relaxation system:

$$\begin{aligned} &\partial_t u + \nabla_x \cdot A(u) = \frac{S(u)}{\varepsilon} \text{ or} \\ &\partial_t u + H(u)\nabla_x u = \frac{S(u)}{\varepsilon} \end{aligned} \tag{1}$$

Applications:

- Kinetic models
- Multiphase flows
- Viscoelasticity problems

Goal

A scheme that is

- Asymptotic preserving:

$$
\begin{tikzcd}
\mathcal{F}^{\varepsilon}_{\delta} \arrow[r, "\varepsilon \to 0"] \arrow[d, "\delta \to 0"'] & \mathcal{F}^{0}_{\delta} \arrow[d, "\delta \to 0"] \\
\mathcal{F}^{\varepsilon} \arrow[r, "\varepsilon \to 0"] & \mathcal{F}^{0}
\end{tikzcd}
$$

- High order in space and time
- Computationally explicit (as much as possible, no mass matrix)

Kinetic Models

Kinetic relaxation models by D. Aregba-Driollet and R. Natalini[1].
Hyperbolic limit equation is

$$u_t + \sum_{d=1}^{D} \partial_{x_d} A_d(u) = 0, \quad u : \Omega \to \mathbb{R}^K.$$

$$\begin{tikzcd}
\Omega \subset \mathbb{R}^D \arrow[r, "u"] \arrow[dr, bend right, "f"'] & \mathbb{R}^K \arrow[loop right, "A_d"] \arrow[d, bend right, "M"'] \\
& \mathbb{R}^L \arrow[u, bend right, "P"'] \arrow[loop right, "\Lambda_d"]
\end{tikzcd}$$

Relaxation system

$$f_t^\varepsilon + \sum_{d=1}^{D} \Lambda_d \partial_{x_d} f^\varepsilon = \frac{1}{\varepsilon}\left(M(Pf^\varepsilon) - f^\varepsilon\right), \quad f^\varepsilon : \Omega \to \mathbb{R}^L$$

$$Pf^\varepsilon \to u, \quad P(M(u)) = u, \quad P\Lambda_d M(u) = A_d(u).$$

[1] D. Aregba-Driollet and R. Natalini. Discrete kinetic schemes for multidimensional systems of conservation laws. SIAM J. Numer. Anal., 37(6):1973–2004, 2000.

We have to find M, P, Λ that respect previous conditions.
$L = N \times K$ with $P = (I_K, \dots, I_K)$ N blocks of identity matrices in $\mathbb{R}^K$.
$f_n \in \mathbb{R}^K$ with $n = 1, \dots, N$

$$\Lambda_d = diag(\Lambda_1^{(d)}, \dots, \Lambda_N^{(d)}) \qquad \Lambda_n^{(d)} = \lambda_n^{(d)} I_K, \quad \text{for } \lambda_n^{(d)} \in \mathbb{R}.$$

With this formalism we can rewrite (6) as

$$\begin{cases} \partial_t f_n^\varepsilon + \sum_{d=1}^D \Lambda_n^{(d)} \partial_{x_d} f_n^\varepsilon = \frac{1}{\varepsilon} \left(M_n(u^\varepsilon) - f_n^\varepsilon \right), & \forall n = 1, \dots, N \\ u^\varepsilon = \sum_{n=1}^N f_n^\varepsilon \end{cases} . \tag{2}$$

Kinetic model – DRM

Let us present the *diagonal relaxation method (DRM)*. Here $N = D + 1$. Then we have to define maxwellians M_n and matrices $\Lambda_j^{(d)}$. Take $\lambda > 0$ and

$$\Lambda_j^{(d)} = \begin{cases} -\lambda I_K & j = d \\ \lambda I_K & j = D+1 \\ 0 & \text{else} \end{cases} .$$

The Maxwellians can be defined as follows:

$$\begin{cases} M_{D+1}(u) = \left(u + \frac{1}{\lambda} \sum_{d=1}^{D} A_d(u) \right) / (D+1) \\ M_j(u) = -\frac{1}{\lambda} A_j(u) + M_{D+1}(u) \end{cases}$$

In 1D scalar case it is the well known Jin–Xin relaxation system. Important: we have to choose λ according to Whitham's subcharacteristic condition.

1 mass fraction equation
2 Euler systems (+ 2 EOS)

$$
\begin{aligned}
&\partial_t \alpha_g && = - V_i \partial_x \alpha_g \quad + \mu \Delta P && \text{(3a)}\\
&\partial_t \alpha_g \rho_g + \partial_x \alpha_g \rho_g u_g && = 0 && \text{(3b)}\\
&\partial_t \alpha_g \rho_g u_g + \partial_x (\alpha_g \rho_g u_g^2 + \alpha_g P_g) && = P_i \partial_x \alpha_g \quad - \lambda \Delta u && \text{(3c)}\\
&\partial_t \alpha_g \rho_g E_g + \partial_x u_g (\alpha_g \rho_g E_g + \alpha_g P_g) && = P_i V_i \partial_x \alpha_g \quad + \mu P_i \Delta P - \lambda V_i \Delta u && \text{(3d)}\\
&\partial_t \alpha_l \rho_l + \partial_x \alpha_l \rho_l u_l && = 0 && \text{(3e)}\\
&\partial_t \alpha_l \rho_l u_l + \partial_x (\alpha_l \rho_l u_l^2 + \alpha_l P_l) && = P_i \partial_x \alpha_l \quad + \lambda \Delta u && \text{(3f)}\\
&\partial_t \alpha_l \rho_l E_l + \partial_x u_l (\alpha_l \rho_l E_l + \alpha_l P_l) && = P_i V_i \partial_x \alpha_l \quad - \mu P_i \Delta P + \lambda V_i \Delta u && \text{(3g)}
\end{aligned}
$$

EOS: $\rho E = \frac{P+\gamma P_\infty}{\gamma - 1} + \frac{1}{2}\rho u^2$
$\lambda, \mu \to \infty$ relaxation parameters
$\Delta f = f_g - f_l$

Outline

IMEX discretization

Stiff source term $\Rightarrow$ oscillations when $\varepsilon \ll \Delta t$
$\Delta t \approx \varepsilon$ not feasible
IMEX approach: IMplicit for source term, EXplicit for advection term

$$\frac{u^{n+1}-u^n}{\Delta t} + \nabla_x \cdot F(u)^n = \frac{S(u)^{n+1}}{\varepsilon} \tag{4}$$

IMEX discretization - Kinetic model

Stiff source term $\Rightarrow$ oscillations when $\varepsilon \ll \Delta t$
$\Delta t \approx \varepsilon$ not feasible
IMEX approach: IMplicit for source term, EXplicit for advection term

$$\frac{f^{n+1,\varepsilon} - f^{n,\varepsilon}}{\Delta t} + \sum_{d=1}^{D} \Lambda_d \partial_{x_d} f^{n,\varepsilon} = \frac{1}{\varepsilon}\left(M(Pf^{n+1,\varepsilon}) - f^{n+1,\varepsilon}\right) \qquad (5)$$
$$f^{0,\varepsilon}(x) = f_0^{\varepsilon}(x)$$

How to treat non-linear implicit functions?
Recall: $PM(u) = u$ and $Pf^{\varepsilon} = u^{\varepsilon}$, so

$$\frac{u^{n+1,\varepsilon} - u^{n,\varepsilon}}{\Delta t} + \sum_{d=1}^{D} P\Lambda_d \partial_{x_d} f^{n,\varepsilon} = 0. \qquad (6)$$

Find $u^{n+1,\varepsilon}$ and substitute it in (5).
IMEX formulation = $\mathcal{L}^1$ (first order accurate).

IMEX discretization - Multiphase flows

$$
\begin{aligned}
\frac{\alpha_g^{n+1}-\alpha_g^n}{\Delta t} & &&= -V_i\partial_x\alpha_g^n + \mu\Delta P^{n+1}\\
\frac{\alpha_g\rho_g^{n+1}-\alpha_g\rho_g^n}{\Delta t} + & \partial_x \alpha_g\rho_g u_g^n &&= 0\\
\frac{\alpha_g\rho_g u_g^{n+1}-\alpha_g\rho_g u_g^n}{\Delta t} + & \partial_x(\alpha_g\rho_g u_g^2+\alpha_g P_g)^n &&= P_i\partial_x\alpha_g^n \quad - \lambda\Delta u^{n+1}\\
\underbrace{\frac{\alpha_g\rho_g E_g^{n+1}-\alpha_g\rho_g E_g^n}{\Delta t}}_{\text{time derivative}} + & \underbrace{\partial_x u_g(\alpha_g\rho_g E_g+\alpha_g P_g)^n}_{\text{conservative flux}} &&= \underbrace{P_iV_i\partial_x\alpha_g^n}_{\text{non cons}} + \underbrace{\mu P_i\Delta P^{n+1} - \lambda V_i\Delta u^{n+1}}_{\text{stiff source}}
\end{aligned}
$$

- IMEX approach: IMplicit stiff source term, EXplicit fluxes
- Difficulties: non linear implicit system ($\alpha_g^{n+1}P_g^{n+1} + \mu P_i\Delta P^{n+1}$)
- Non linear solver
- Discretization of non conservative terms

Outline

Residual Distribution

- High order
- Easy to code
- FE based
- Compact stencil
- No need of Riemann solver
- No need of conservative variables
- Can recast some other FV, FE schemes

$$\partial_t U + \nabla_x \cdot A(U) = S(U)$$

$$V_h = \{U \in L^2(\Omega_h, \mathbb{R}^D) \cap \mathcal{C}^0(\Omega_h),\, U|_K \in \mathbb{P}^k,\, \forall K \in \Omega_h\}.$$

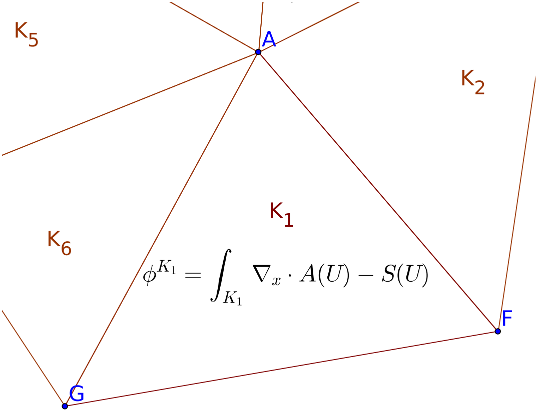
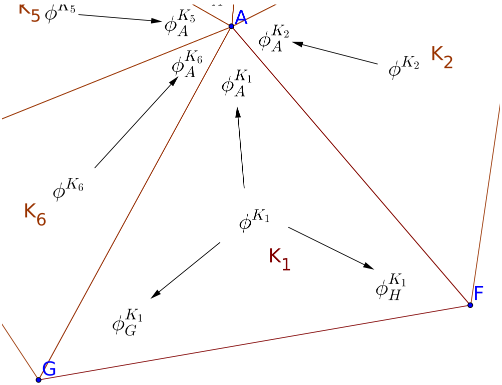


Figure: Defining total residual, nodal residuals and building the RD scheme



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Residual Distribution - Spatial Discretization

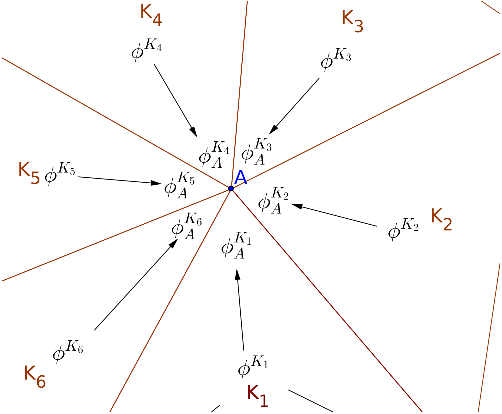


Figure: Defining total residual, nodal residuals and building the RD scheme

Residual Distribution - Spatial Discretization

1. Define $\forall K \in \Omega_h$ a fluctuation term (total residual) $\phi^K = \int_K \nabla \cdot A(U) - S(U) dx$
2. Define a nodal residual ϕ^K_σ $\forall \sigma \in K$:

$$\phi^K = \sum_{\sigma \in K} \phi^K_\sigma, \quad \forall K \in \Omega_h. \tag{7}$$

3. The resulting scheme is

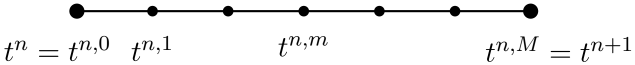
$$\sum_{K|\sigma \in K} \phi^K_\sigma = 0, \quad \forall \sigma \in D_h. \tag{8}$$

Remark: the definition of the nodal residuals leads to the scheme!
We use as Galerkin, Rusanov, PSI limiter, jump stabilization.

Outline

High order RD schemes

To get high order in space: $\varphi|_K \in \mathbb{P}^r \Rightarrow r+1$ order.
High order in time we should discretize our variable on $[t^n, t^{n+1}]$ in M substeps ($U_\sigma^{n,m}$).



Thanks to Picard–Lindelöf theorem, we can rewrite

$$U_\sigma^{n,m} = U_\sigma^{n,0} + \int_{t^n}^{t^{n,m}} \nabla \cdot A(U(x,s)) - S(U(x,s))ds$$

and if we want to reach order $r+1$ we need $M = r$.

High order RD schemes

More precisely, for each σ we want to solve $\mathcal{L}^2_\sigma(U^{n,0},\dots,U^{n,M})=0$, where

$$\mathcal{L}^2_\sigma(U^{n,0},\dots,U^{n,M}) =$$
$$= \begin{pmatrix} \sum_{K\ni\sigma}\left(\int\limits_K \varphi_\sigma(U^{n,1}(x)-U^{n,0}(x))dx + \int\limits_{t^{n,0}}^{t^{n,1}} \mathcal{I}_M(\phi^K_\sigma(U^{n,0}),\dots,\phi^K_\sigma(U^{n,M}),s)ds\right) \\ \vdots \\ \sum_{K\ni\sigma}\left(\int\limits_K \varphi_\sigma(U^{n,M}(x)-U^{n,0}(x))dx + \int\limits_{t^{n,0}}^{t^{n,M}} \mathcal{I}_M(\phi^K_\sigma(U^{n,0}),\dots,\phi^K_\sigma(U^{n,M}),s)ds\right) \end{pmatrix}$$

which is a fully implicit system of M equations with M unknowns (times #DoFs).

Low order RD

Instead of solving the implicit system directly (difficult), we introduce a first order scheme $\mathcal{L}^1_\sigma(U^{n,0},\dots,U^{n,M})$:

$$\mathcal{L}^1_\sigma(U^{n,0},\dots,U^{n,M}) =$$

$$= \begin{pmatrix} \sum_{K\ni\sigma} \left((U_\sigma^{n,1} - U_\sigma^{n,0}) \int\limits_K \varphi_\sigma dx + \int\limits_{t^{n,0}}^{t^{n,1}} \mathcal{I}_0(\phi_\sigma^K(U^{n,0}, U^{n,1}), s) ds \right) \\ \vdots \\ \sum_{K\ni\sigma} \left((U_\sigma^{n,M} - U_\sigma^{n,0}) \int\limits_K \varphi_\sigma dx + \int\limits_{t^{n,0}}^{t^{n,M}} \mathcal{I}_0(\phi_\sigma^K(U^{n,0}, U^{n,M}), s) ds \right) \end{pmatrix}$$

- IMEX discretization
- mass lumping on implicit terms (time derivative and source term)
- easy to be solved (explicit or small implicit systems)
- stable

Deferred Correction

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

- $\mathcal{L}^1(U^{n+1}, U^n) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(U^{n+1}, U^n) = 0$, order r (>1).

DeC method [2]

- Compute prediction $U^{(1)}: \ \mathcal{L}^1(U^{(1)}, U^n) = 0$.
- Compute corrections $U^{(j)}$ for $j = 2, \dots, K$:
  $\mathcal{L}^1(U^{(j)}, U^n) = \mathcal{L}^1(U^{(j-1)}, U^n) - \mathcal{L}^2(U^{(j-1)}, U^n)$.
- $U^{n+1} := U^{(K)}$.

Order of convergence $\min(r, K)$
Implicit $\mathcal{L}^1$ and explicit $\mathcal{L}^2$.

[2] A. Dutt, L. Greengard, and V. Rokhlin. Spectral Deferred Correction Methods for Ordinary Differential Equations. BIT Numerical Mathematics, 40(2):241–266, 2000.

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Deferred Correction

Theorem (Deferred Correction convergence)

Given the DeC procedure. If

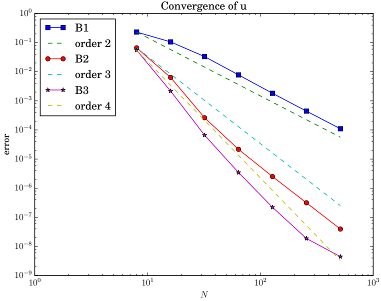
- $\mathcal{L}^1$ *is coercive with constant* α_1
- $\mathcal{L}^2 - \mathcal{L}^1$ *is Lipschitz continuous with constant* $\alpha_2 \Delta$
- $\exists ! U^*_\Delta$ *such that* $\mathcal{L}^2(U^*_\Delta) = 0$.

Then if $\eta = \frac{\alpha_2}{\alpha_1}\Delta < 1$*, the deferred correction is converging to* U^*_Δ *and after* K *iterations the error is smaller than* η^K *times the original error.*

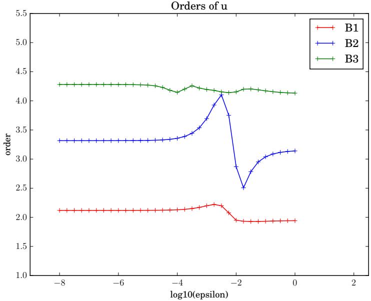
Outline

$$u_t + u_x = 0, \quad x \in [0,1], \quad t \in [0,T],\, T = 0.12, \quad u_0(x) = e^{-80(x-0.4)^2},$$

outflow BC, $\lambda = 1.5$, $\varepsilon = 10^{-10}$, $\theta_1 = 1$, $\theta_2 = 5$ (derivative stabilization).



(a) Scalar 1D convergence



(b) Order varying relaxation parameter

Figure: Scalar linear 1D test

Numerical tests: Euler equation

Next simulations will be over the Euler equation

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (E+p)v \end{pmatrix}_x = 0, \qquad x \in [0,1],\, t \in [0,T] \tag{9}$$

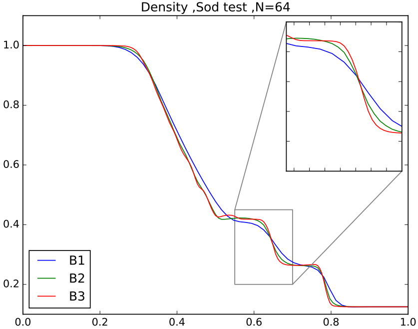
ρ is the density, v the speed, p the pressure and E the total energy.
The system is closed by the equation of state

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2. \tag{10}$$

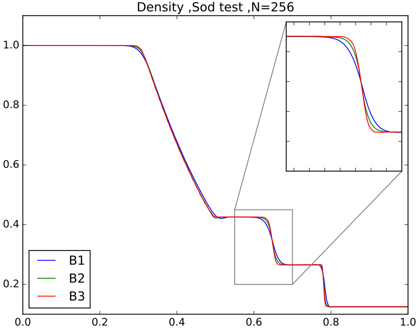
Numerical tests: Sod shock test

$\gamma = 1.4$, $T = 0.16$, outflow BC, $\varepsilon = 10^{-9}$, $\lambda = 2$, CFL = 0.2.
For $\mathbb{B}^1$ $\theta_1 = 1$, for $\mathbb{B}^2$ $\theta_1 = 1$, $\theta_2 = 0.5$, for $\mathbb{B}^3$ $\theta_1 = 2.5$, $\theta_2 = 4$.

$$\rho_0 = \mathbb{1}_{[0,0.5]}(x) + 0.1\mathbb{1}_{[0.5,1]}(x), \quad v_0 = 0, \quad p_0 = \mathbb{1}_{[0,0.5]}(x) + 0.125\mathbb{1}_{[0.5,1]}(x).$$



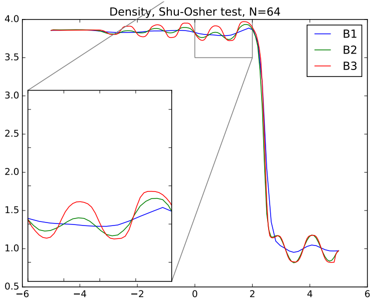
(a) $N = 64$



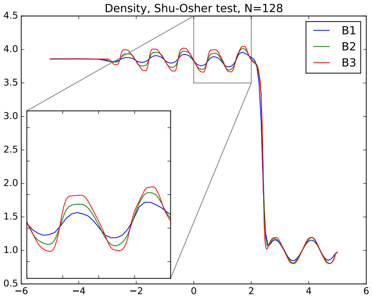
(b) $N = 256$

$\gamma = 1.4$, $T = 1.8$, outflow BC $\varepsilon = 10^{-9}$, $\lambda = 3$, CFL=0.1.
For $\mathbb{B}^1$ $\theta_1 = 0.5$, for $\mathbb{B}^2$ $\theta_1 = 0.8$, $\theta_2 = 1$, for $\mathbb{B}^3$ $\theta_1 = 3$, $\theta_2 = 1$.
The initial conditions are

$$\begin{pmatrix} \rho_0 \\ v_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 3.857143 \\ 2.629369 \\ 10.333333 \end{pmatrix} x \in [-5, -4], \quad \begin{pmatrix} \rho_0 \\ v_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 1 + 0.2\sin(5x) \\ 0 \\ 1 \end{pmatrix} \text{ else.}$$



(c) $N = 64$

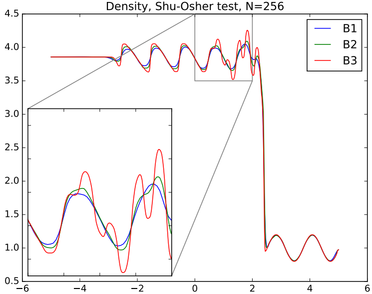


(d) $N = 128$

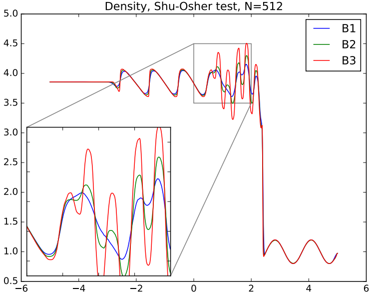
Numerical tests: Shu–Osher test

$\gamma = 1.4$, $T = 1.8$, outflow BC $\varepsilon = 10^{-9}$, $\lambda = 3$, CFL=0.1.
For $\mathbb{B}^1$ $\theta_1 = 0.5$, for $\mathbb{B}^2$ $\theta_1 = 0.8$, $\theta_2 = 1$, for $\mathbb{B}^3$ $\theta_1 = 3$, $\theta_2 = 1$.
The initial conditions are

$$\begin{pmatrix} \rho_0 \\ v_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 3.857143 \\ 2.629369 \\ 10.333333 \end{pmatrix} x \in [-5, -4], \quad \begin{pmatrix} \rho_0 \\ v_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 1 + 0.2\sin(5x) \\ 0 \\ 1 \end{pmatrix} \text{ else.}$$



(e) $N = 256$



(f) $N = 512$

Numerical tests 2D: Euler equation

Euler equation in 2D domain

$$\partial_t U(\mathbf{x},t) + \partial_x f(U(\mathbf{x},t)) + \partial_y g(U(\mathbf{x},t)) = 0, \; \mathbf{x} = (x,y) \in \Omega \subset \mathbb{R}^2,$$
$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad f(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix}, \quad g(U) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix} \tag{11}$$

ρ is the density, u is the speed in x direction, v is the speed in y direction, E the total energy and p the pressure.
The closing EOS is:

$$p = (\gamma - 1)\Big(E - \frac{1}{2}\rho(u^2 + v^2)\Big). \tag{12}$$

Numerical tests 2D: Steady vortex for convergence

Initial conditions and solution for all $t \in [0, \infty)$ are

$$\begin{pmatrix} \rho_0 \\ u_0 \\ v_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{\gamma-1}{\gamma}\frac{1}{2}\left(\frac{5}{2\pi}\right)^2 e^{\frac{1-r^2}{2}}\right)^{\frac{1}{\gamma-1}} \\ \frac{5}{2\pi}(-y)e^{\frac{1-r^2}{2}} \\ \frac{5}{2\pi}(x)e^{\frac{1-r^2}{2}} \\ \rho_0^\gamma \end{pmatrix}.$$

Here $r^2 = x^2 + y^2$, the boundary conditions are outflow and $T = 1$.
$\gamma = 1.4$, $\varepsilon = 10^{-9}$, $\lambda = 1.4$ and CFL = 0.1.
For $\mathbb{B}^1$ $\theta_1 = 0.1$, for $\mathbb{B}^2$ $\theta_1 = 0.01,\ \theta_2 = 0$, for $\mathbb{B}^3$ $\theta_1 = 0.001,\ \theta_2 = 0$.

Numerical tests 2D: Steady vortex for convergence

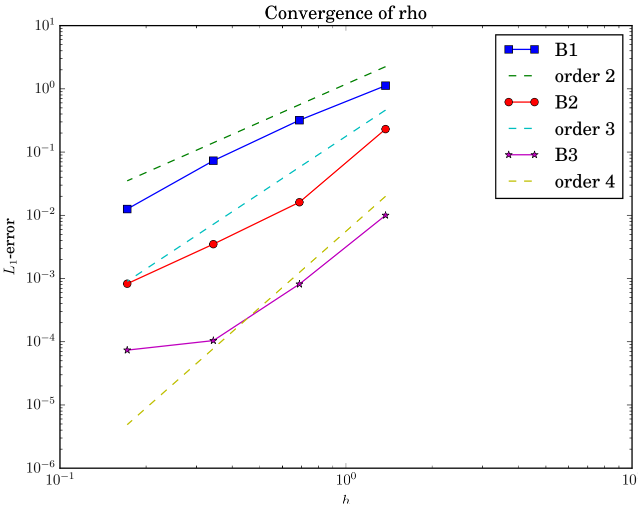


Figure: 2D convergence

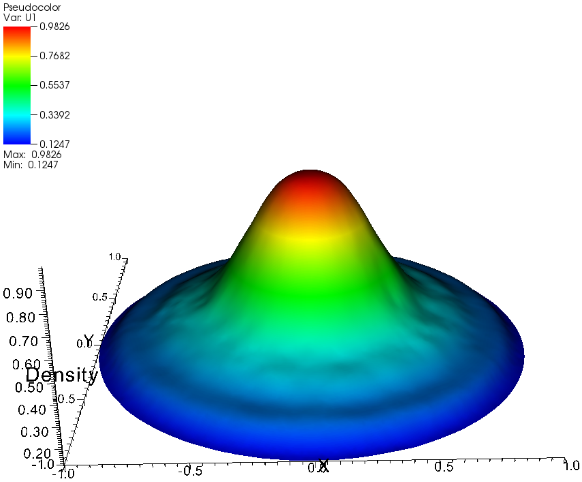
Numerical tests 2D: Sod shock test

Initial conditions are

$$\begin{pmatrix} \rho_0 \\ u_0 \\ v_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ if } r < \frac{1}{2}, \qquad \begin{pmatrix} \rho_0 \\ u_0 \\ v_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} 0.125 \\ 0 \\ 0 \\ 0.1 \end{pmatrix} \text{ if } r \geq \frac{1}{2}.$$

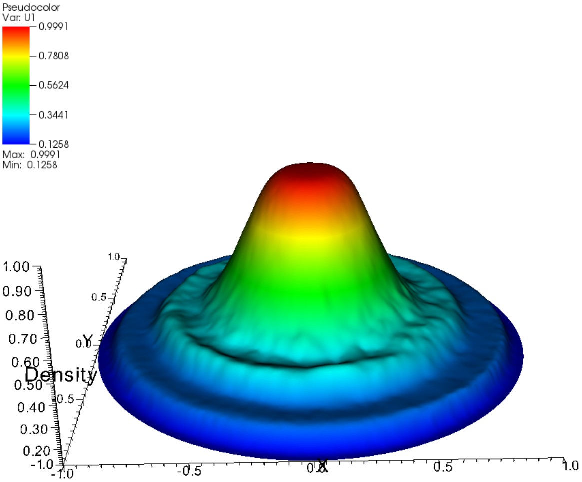
Here $r^2 = x^2 + y^2$, $\gamma = 1.4$, $\varepsilon = 10^{-9}$, $\lambda = 1.4$, CFL = 0.1, $T = 0.25$ and outflow boundary conditions.
For $\mathbb{B}^1$ $\theta_1 = 0.1$, for $\mathbb{B}^2$ $\theta_1 = 0.1,\ \theta_2 = 0.0001$, for $\mathbb{B}^3$ $\theta_1 = 0.01,\ \theta_2 = 0.0001$.

Numerical tests 2D: Sod shock test



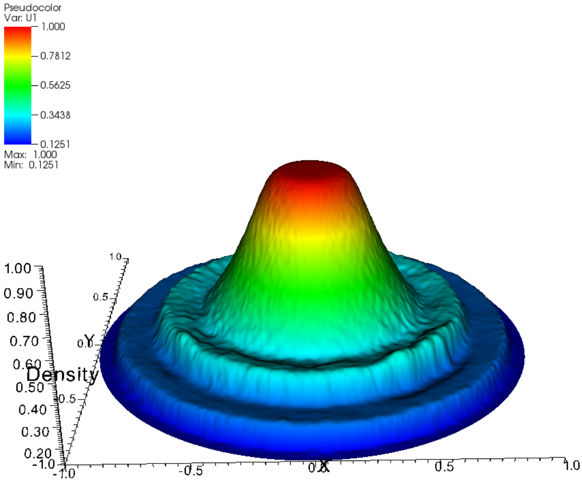
(a) $\mathbb{B}^1, N = 13548$

Numerical tests 2D: Sod shock test



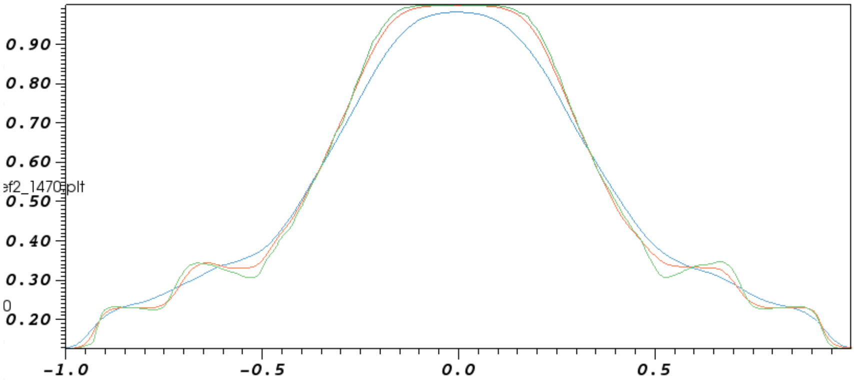
(b) $\mathbb{B}^2, N = 13548$

Numerical tests 2D: Sod shock test



(c) $\mathbb{B}^3, N = 13548$

Numerical tests 2D: Sod shock test



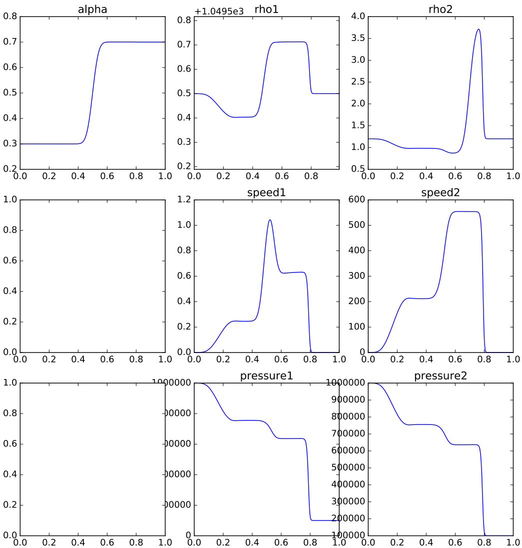
(d) Slices of $\mathbb{B}^1$ (blue), $\mathbb{B}^2$ (red) and $\mathbb{B}^3$ (green), $N = 13548$

Numerical tests 1D: Multiphase flow

Test1: pressure and mass fraction discontinuity
$\mu = 10^9$, $\lambda = 0$, $T = 350\mu\text{s}$, EOS: $\rho E = \frac{P+\gamma P_\infty}{\gamma-1} + \frac{1}{2}\rho u^2$
Rusanov scheme

		Phase 1 Liquid $\gamma = 4.4, P_\infty = 6 \cdot 10^8$	Phase 2 Air $\gamma = 1.4, P_\infty = 0$
IC	$x < 0.5$	$\alpha = 0.3$ $\rho = 1050\text{kg}/\text{m}^3$ $u = 0\text{m}/\text{s}$ $P = 10^6\text{Pa}$	$\alpha = 0.7$ $\rho = 1.2\text{kg}/\text{m}^3$ $u = 0\text{m}/\text{s}$ $P = 10^6\text{Pa}$
	$x > 0.5$	$\alpha = 0.7$ $\rho = 1050\text{kg}/\text{m}^3$ $u = 0\text{m}/\text{s}$ $P = 10^5\text{Pa}$	$\alpha = 0.3$ $\rho = 1.2\text{kg}/\text{m}^3$ $u = 0\text{m}/\text{s}$ $P = 10^5\text{Pa}$

Numerical tests 1D: Multiphase flow

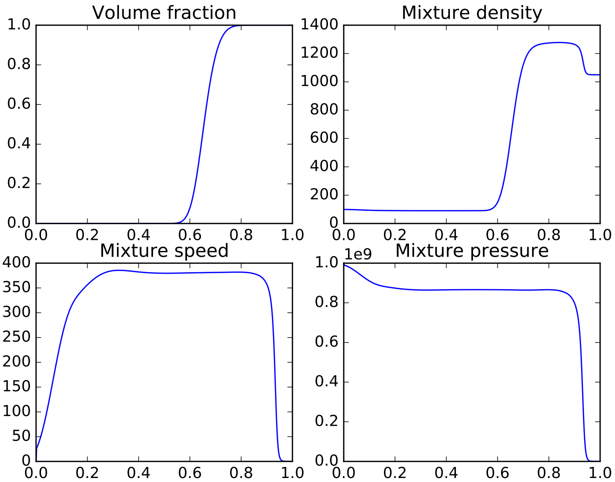


Numerical tests 1D: Multiphase flow

Test2: separated phases with high pressure discontinuity
$\mu = 10^9$, $\lambda = 10^7$, $T = 150\mu\text{s}$, EOS: $\rho E = \frac{P+\gamma P_\infty}{\gamma - 1} + \frac{1}{2}\rho u^2$
Rusanov scheme

		Phase 1 Liquid $\gamma = 4.4, P_\infty = 6 \cdot 10^8$	Phase 2 Air $\gamma = 1.4, P_\infty = 0$
IC	$x < 0.5$	$\alpha = 0.000001$ $\rho = 1050\text{kg}/\text{m}^3$ $u = 0\text{m}/\text{s}$ $P = 10^9\text{Pa}$	$\alpha = 0.999999$ $\rho = 100\text{kg}/\text{m}^3$ $u = 0\text{m}/\text{s}$ $P = 10^9\text{Pa}$
	$x > 0.5$	$\alpha = 0.999999$ $\rho = 1050\text{kg}/\text{m}^3$ $u = 0\text{m}/\text{s}$ $P = 10^5\text{Pa}$	$\alpha = 0.000001$ $\rho = 100\text{kg}/\text{m}^3$ $u = 0\text{m}/\text{s}$ $P = 10^5\text{Pa}$

Numerical tests 1D: Multiphase flow



Outline

Conclusion and perspective

Conclusions

- Asymptotic preserving
- IMEX
- Residual Distribution
- Deferred Correction

Perspective

- High order multiphase flows
- MOOD for multiphase flows
- Compare with high order RK IMEX schemes
- Reduced basis algorithms on the scheme

Thank you for the attention!

Whitham's subcharacteristic condition

$$f_t^\varepsilon + \sum_{d=1}^{D} \Lambda_d \partial_{x_d} f^\varepsilon = \frac{1}{\varepsilon}\left(M(Pf^\varepsilon) - f^\varepsilon\right), \qquad f^\varepsilon : \Omega \to \mathbb{R}^L$$

If we call $u^\varepsilon = Pf^\varepsilon$, $v_d^\varepsilon = P\Lambda_d f^\varepsilon$ we have from (6) that

$$\begin{cases} \partial_t u^\varepsilon + \sum_{j=1}^D \partial_{x_j} v_j^\varepsilon = 0 \\ \partial_t v_d^\varepsilon + \sum_{j=1}^D \partial_{x_j} (P\Lambda_j \Lambda_d f^\varepsilon) = \frac{1}{\varepsilon}(A_d(u^\varepsilon) - v_d^\varepsilon) \end{cases}.$$

If we do a Taylor expansion in ε we get

$$v_d^\varepsilon = A_d(u^\varepsilon) - \varepsilon \left(\partial_t v_d^\varepsilon + \sum_{j=1}^D \partial_{x_j} (P\Lambda_d\Lambda_j f^\varepsilon) \right) \tag{13}$$

$$= A_d(u^\varepsilon) - \varepsilon \left(\partial_t v_d^\varepsilon + \sum_{j=1}^D \partial_{x_j} (P\Lambda_d\Lambda_j M(u^\varepsilon)) \right) + \mathcal{O}(\varepsilon^2). \tag{14}$$

Whitham's condition

$$\partial_t u^\varepsilon + \sum_{d=1}^{D} \partial_{x_d} A_d(u^\varepsilon) = \varepsilon \sum_{d=1}^{D} \partial_{x_d} \left(\partial_t v_d^\varepsilon + \sum_{j=1}^{D} \partial_{x_j} (P\Lambda_d \Lambda_j M(u^\varepsilon)) \right) + \mathcal{O}(\varepsilon^2)$$

$$\partial_t u^\varepsilon + \sum_{d=1}^{D} \partial_{x_d} A_d(u^\varepsilon) = \varepsilon \sum_{d=1}^{D} \partial_{x_d} \left(\sum_{j=1}^{D} B_{dj}(u^\varepsilon) \partial_{x_j} u^\varepsilon \right) + \mathcal{O}(\varepsilon^2).$$

For this case, the Whitham's subcharacteristic condition[3] becomes

$$B_{jd} := P\Lambda_d \Lambda_j M'(u) - A_d'(u) A_j'(u), \qquad \sum_{j,d=1}^{D} (B_{dj}\xi_j, \xi_d) \geq 0.$$

[3] **natalini**.

Problems: convection parameter

How to set the convection parameter automatically?
To verify Whitham's subcharacteristic condition we have to

$$B_{jd} := P\Lambda_d\Lambda_j M'(u) - A'_d(u)A'_j(u), \qquad \sum_{j,d=1}^{D} (B_{dj}\xi_j, \xi_d) \geq 0.$$

In DRM for 2D systems, we have:

$$\Lambda_1 = \begin{pmatrix} -\lambda I_K & 0_K & 0_K \\ 0_K & 0_K & 0_K \\ 0_K & 0_K & \lambda I_K \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0_K & 0_K & 0_K \\ 0_K & -\lambda I_K & 0_K \\ 0_K & 0_K & \lambda I_K \end{pmatrix}$$

$$P\Lambda_1 = (-\lambda I_K, 0_K, \lambda I_K), \quad P\Lambda_2 = (0_K, -\lambda I_K, \lambda I_K)$$

$$P\Lambda_1\Lambda_1 = (\lambda^2 I_K, 0_K, \lambda^2 I_K), \quad P\Lambda_2\Lambda_2 = (0_K, \lambda^2 I_K, \lambda^2 I_K)$$

$$P\Lambda_1\Lambda_2 = P\Lambda_2\Lambda_1 = (0_K, 0_K, \lambda^2 I_K)$$

Problems: convection parameter

Moreover we now that

$$\mathbb{R}^{(K,K\cdot N)} \ni M'(u) =$$

$$= \begin{pmatrix} \frac{u}{3} + \frac{1}{3\lambda}(-2A_1 + A_2) \\ \frac{u}{3} + \frac{1}{3\lambda}(A_1 - 2A_2) \\ \frac{u}{3} + \frac{1}{3\lambda}(A_1 + A_2) \end{pmatrix}' = \frac{1}{3} \begin{pmatrix} I_K + \frac{1}{\lambda}(-2A_1' + A_2') \\ I_K + \frac{1}{\lambda}(A_1' - 2A_2') \\ I_K + \frac{1}{\lambda}(A_1' + A_2') \end{pmatrix}.$$

So, if we compute the B matrices we get

$$B_{11} = \frac{2}{3}\lambda^2 I_K + \lambda(\frac{2}{3}A_2' - \frac{1}{3}A_1') - A_1' A_1'^T$$

$$B_{12/21} = \frac{1}{3}\lambda^2 I_K + \lambda(\frac{1}{3}A_2' + \frac{1}{3}A_1') - A_{1/2}' A_{2/1}'^T$$

$$B_{22} = \frac{2}{3}\lambda^2 I_K + \lambda(\frac{2}{3}A_1' - \frac{1}{3}A_2') - A_2' A_2'^T$$

Problems: convection parameter

Then, if we restart from the following condition

$$\sum_{i,j=1}^{2} \langle B_{ij}\xi_i, \xi_j \rangle \geq 0 \qquad \forall \xi_j \in \mathbb{R}^K,$$

Different from scalar case $K = 1$. Scalar case:

$$\sum_{i,j=1}^{2} \langle B_{ij}\xi_i, \xi_j \rangle \geq 0 \qquad \forall \xi_j \in \mathbb{R},$$

you can get something solvable, but in our case, what we get is:

$$\frac{2}{3}\sum_{i,j=1}^{2} \langle \xi_i, \xi_j \rangle \lambda^2 + \frac{\lambda}{3}\big(\langle (2A_2' - A_1')\xi_1, \xi_1 \rangle +$$
$$+ \langle (-A_2' + 2A_1')\xi_2, \xi_2 \rangle + \langle (A_2' + A_1' + (A_2' + A_1')^T)\xi_1, \xi_2 \rangle\big) +$$
$$+ \sum_{j,i=1}^{2} \langle A_i' A_j'^T \xi_i, \xi_j \rangle \geq 0, \qquad \forall \xi_1, \xi_2 \in \mathbb{R}^K.$$

Problems: convection parameter

How they saw this was in the sense of

$$\underline{\xi}^T B \underline{\xi} \geq 0.$$

So doing spectral analysis, finding the eigenvalues of B and imposing the positivity of both of them for *scalar* case. Finally, they got this condition from a 4th degree equation

$$\lambda \geq \max\left(-A_1' - A_2', 2A_1' - A_2', -A_1' + 2A_2'\right).$$

But for general case B is a $2K \times 2K$ matrix and I have no clue how to find the $2K$ eigenvalues.

Problems: changing the convection parameter

If we change the convection parameter from timestep to timestep, we get big oscillations.
Where should this come from?
Back to IMEX 1

Residual distribution - Choice of the scheme

How to split into $\phi_\sigma^K \Rightarrow$ choice of the scheme. For example, we can rewrite SUPG in this way:

$$\phi_\sigma^K(U_h) = \int_K \varphi_\sigma(\nabla \cdot A(U_h) - S(U_h))dx + \tag{15}$$

$$+h_K \int_K (\nabla \cdot A(U_h) \cdot \nabla \cdot \varphi_\sigma)\, \tau\, (\nabla \cdot A(U_h) \cdot \nabla \cdot U_h)\,. \tag{16}$$

Furthermore, we can write the Galerkin FEM scheme with jump stabilization by **burman**:

$$\phi_\sigma^K = \int_K \varphi_\sigma(\nabla \cdot A(U_h) - S(U_h))dx + \sum_{e|\text{edge of } K} \theta h_e^2 \int_e [\nabla U_h] \cdot [\nabla \varphi_\sigma] d\Gamma, \tag{17}$$

Residual Distribution - Choice of the scheme

$$\phi_\sigma^{K,LxF}(U_h) = \int_K \varphi_\sigma \left(\nabla \cdot A(U_h) - S(U_h)\right) dx + \alpha_K(U_\sigma - \overline{U}_h^K), \qquad (18)$$

where $\overline{U}_h^K$ is the average of U_h over the cell K and α_K is defined as

$$\alpha_K = \max_{e \text{ edge } \in K} \left(\rho_S\left(\nabla A(U_h) \cdot \mathbf{n}_e\right)\right), \qquad (19)$$

ρ_S is the spectral radius.
For monotonicity near strong discontinuities, PSI limiter:

$$\beta_\sigma^K(U_h) = \max\left(\frac{\Phi_\sigma^{K,LxF}}{\Phi^K}, 0\right)\left(\sum_{j\in K}\max\left(\frac{\Phi_j^{K,LxF}}{\Phi^K}, 0\right)\right)^{-1} \qquad (20)$$

Residual Distribution - Choice of the scheme

Blending between LxF and PSI:

$$\begin{aligned}\phi_\sigma^{*,K} &= (1-\Theta)\beta_\sigma^K \phi_\sigma^K + \Theta \Phi_\sigma^{K,LxF},\\ \Theta &= \frac{|\Phi^K|}{\sum_{j\in K} |\Phi_j^{K,LxF}|}.\end{aligned} \tag{21}$$

Nodal residual is finally given by

$$\phi_\sigma^K = \phi_\sigma^{*,K} + \sum_{e|\text{edge of } K} \theta h_e^2 \int_e [\nabla U_h]\cdot[\nabla \varphi_\sigma] d\Gamma. \tag{22}$$

DeC – Proof

Proof.

Let U^* be the solution of $\mathcal{L}^2(U^*)=0$. We know that $\mathcal{L}^1(U^*)=\mathcal{L}^1(U^*)-\mathcal{L}^2(U^*)$, so that

$$\begin{aligned}
\mathcal{L}^1(U^{(k+1)})-\mathcal{L}^1(U^*)=&\left(\mathcal{L}^1(U^{(k)})-\mathcal{L}^2(U^{(k)})\right)-\left(\mathcal{L}^1(U^*)-\mathcal{L}^2(U^*)\right)\\
=&\left(\mathcal{L}^1(U^{(k)})-\mathcal{L}^1(U^*)\right)-\left(\mathcal{L}^2(U^{(k)})-\mathcal{L}^2(U^*)\right)\\
\alpha_1||U^{(k+1)}-U^*||\leq&||\mathcal{L}^1(U^{(k+1)})-\mathcal{L}^1(U^*)||=\\
=&||\mathcal{L}^1(U^{(k)})-\mathcal{L}^2(U^{(k)})-(\mathcal{L}^1(U^*)-\mathcal{L}^2(U^*))||\leq\\
\leq&\alpha_2\Delta||U^{(k)}-U^*||.\\
||U^{(k+1)}-U^*||\leq&\left(\frac{\alpha_2}{\alpha_1}\Delta\right)||U^{(k)}-U^*||\leq\left(\frac{\alpha_2}{\alpha_1}\Delta\right)^{k+1}||U^{(0)}-U^*||.
\end{aligned}$$

After K iteration we have an error at most of $\eta^K\cdot||U^{(0)}-U^*||$. $\square$