A new efficient explicit Deferred Correction framework: analysis and applications to hyperbolic PDEs and adaptivity

Davide Torlo*, **Lorenzo Micalizzi**

*MathLab, Mathematics Area, SISSA International School for Advanced Studies, Trieste, Italy
davidetorlo.it

Essentially hyperbolic problems: unconventional numerics, and applications
Ascona - October 2022

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History of residual distribution and DeC

2016

- PhD in hyperbolic PDE field with Rémi

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Spatial Discretizations

- Finite Volume

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- Nowadays **Continuous Finite Element**

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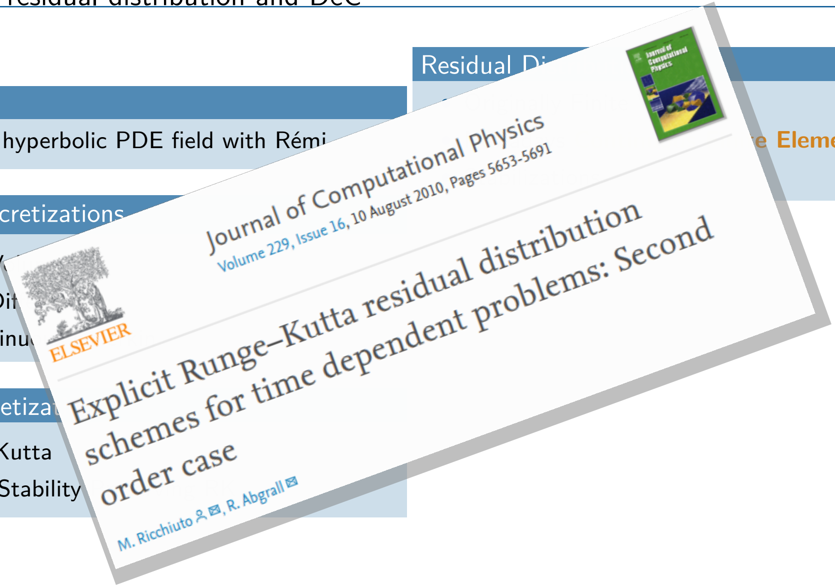

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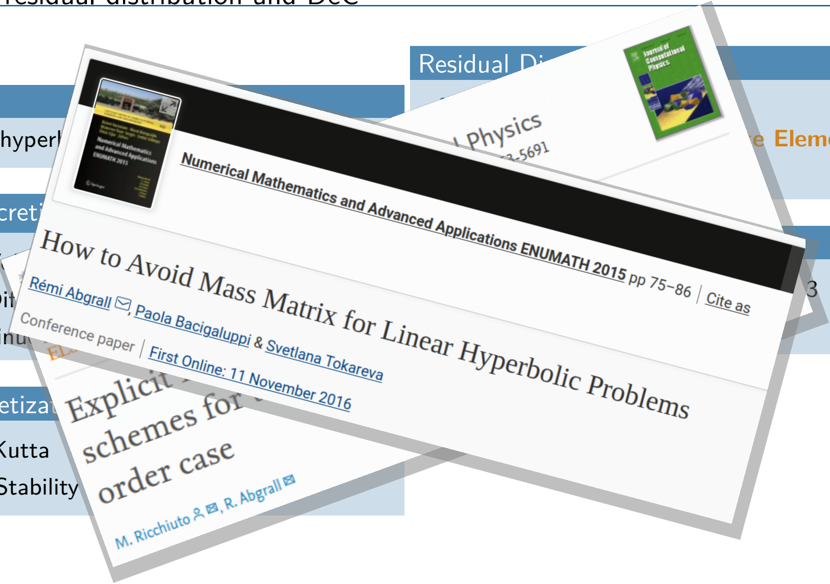
Numerical Mathematics and Advanced Applications ENUMATH 2015 pp 75–86 | Cite as

How to Avoid Mass Matrix for Linear Hyperbolic Problems

Rémi Abgrall, Paola Bacigaluppi & Svetlana Tokareva

Conference paper | First Online: 11 November 2016

Physics

Explicit schemes for order case

M. Ricchiuto, R. Abgrall

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- Arbitrarily high order

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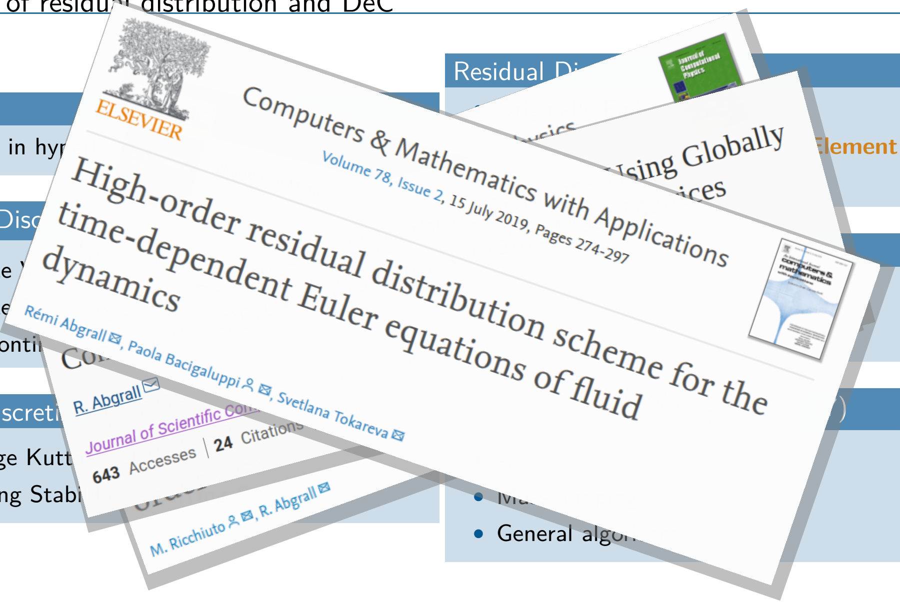
ELSEVIER

Computers & Mathematics with Applications

Volume 78, Issue 2, 15 July 2019, Pages 274-297

High-order residual distribution scheme for the time-dependent Euler equations of fluid dynamics

Rémi Abgrall, Paola Bacigaluppi, Svetlana Tokareva

R. Abgrall

Journal of Scientific Com

643 Accesses | 24 Citations

M. Ricchiuto, R. Abgrall

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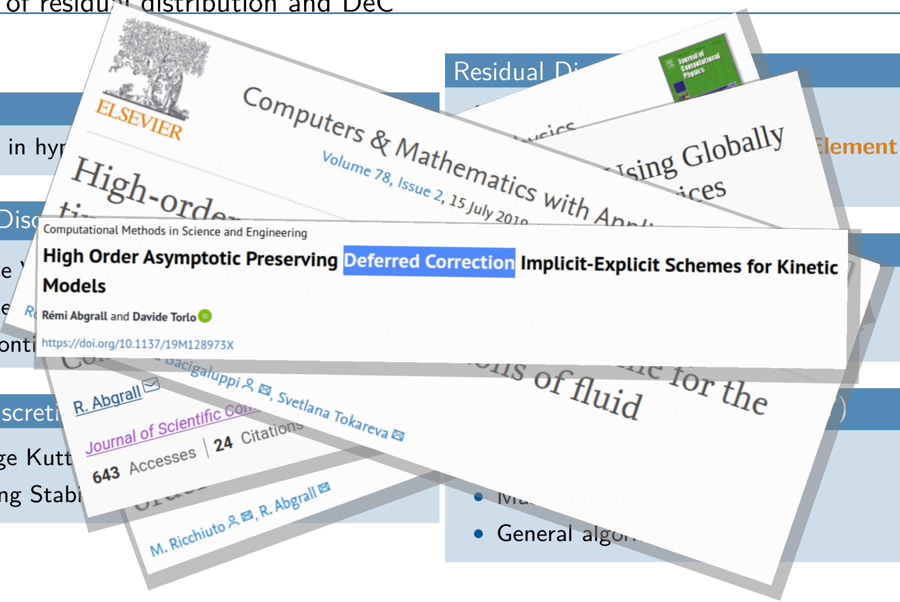

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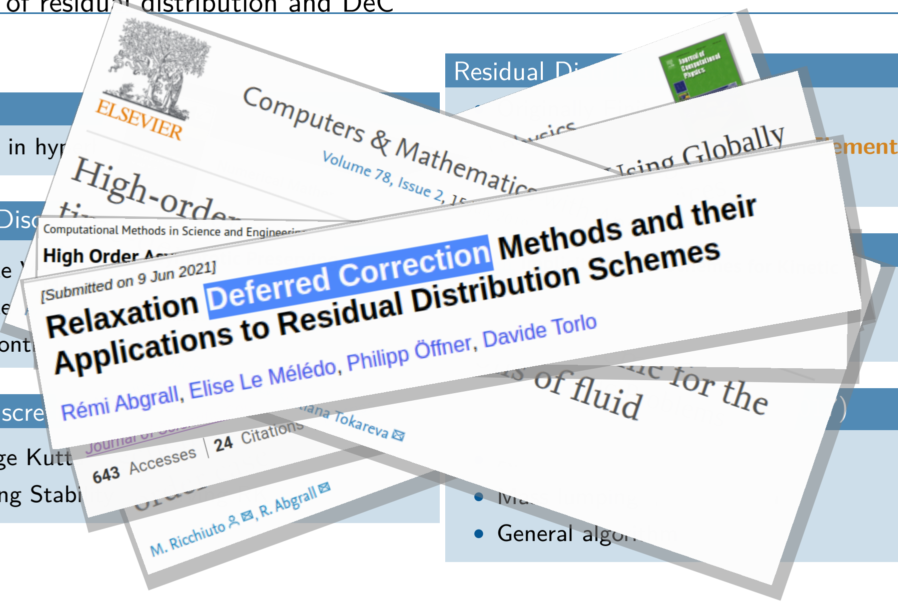

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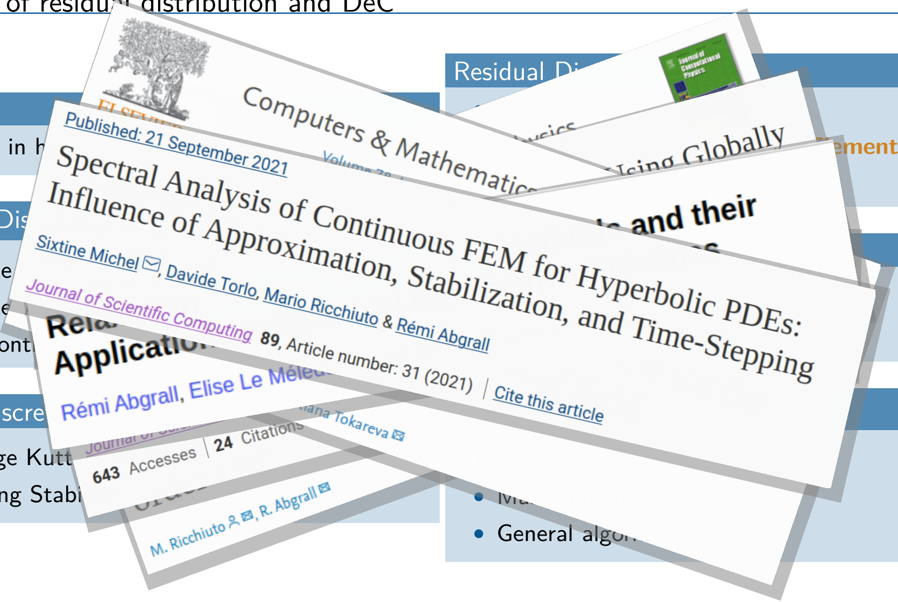

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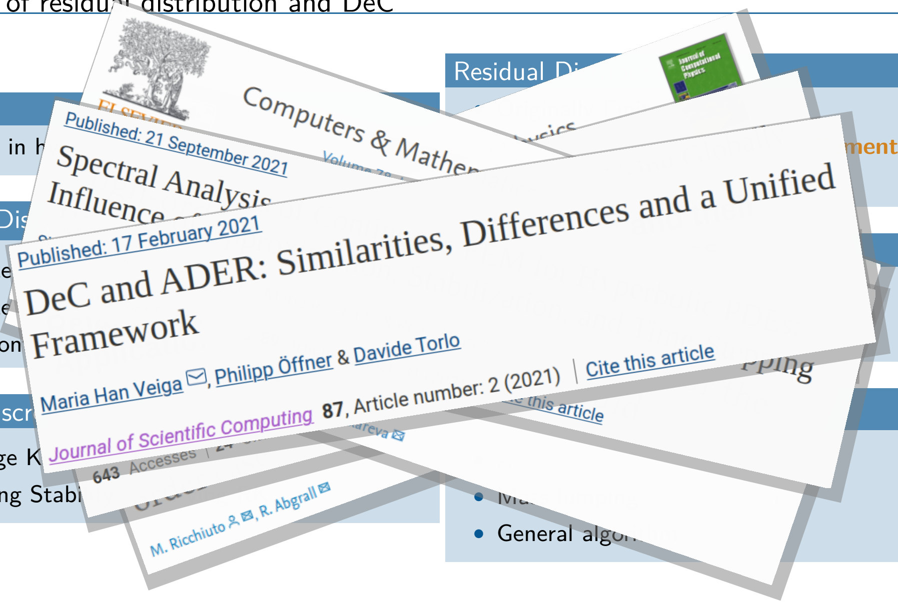

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- **IMEX** DeC: *Minion (2003)*
- **Operators** based DeC, generalization to many problems: *Abgrall (2017)*

DeC iterations

$$\frac{d}{dt}\boldsymbol{u}(t) = \boldsymbol{G}(t, \boldsymbol{u}(t)),$$

DeC iterations

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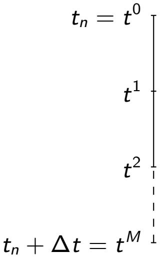

Which sub time nodes? **Equispaced**, **Gauss-Lobatto**

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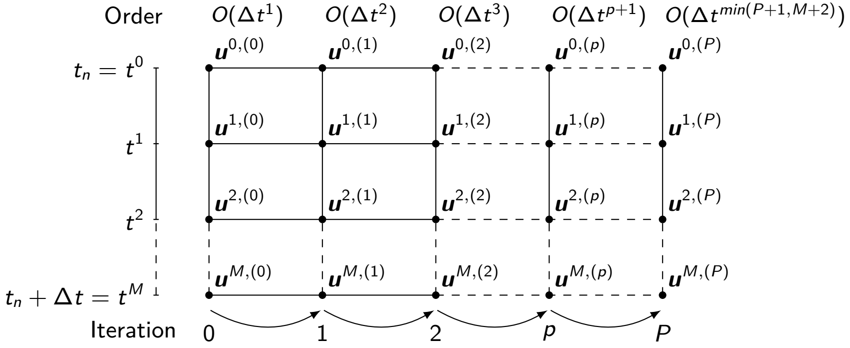
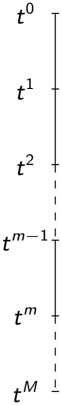


Which sub time nodes? **Equispaced**, **Gauss-Lobatto**

DeC operators

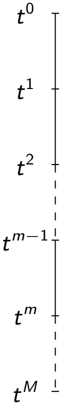


$\mathcal{L}^2_\Delta$ operator

$$\mathcal{L}^2_\Delta(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M)=\mathcal{L}^2_\Delta(\underline{\boldsymbol{u}}):=$$
$$\begin{cases} \boldsymbol{u}^M-\boldsymbol{u}^0-\int_{t^0}^{t^M}\boldsymbol{G}(\boldsymbol{u}(s))ds \\ \vdots \\ \boldsymbol{u}^1-\boldsymbol{u}^0-\int_{t^0}^{t^1}\boldsymbol{G}(\boldsymbol{u}(s))ds \end{cases}$$

- Implicit RK
- Order of accuracy $\geq M+1$
- Difficult to solve directly

DeC operators



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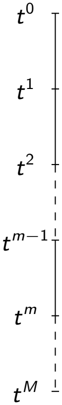

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- First order accurate
- Explicit or easy to solve

Deferred Correction

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

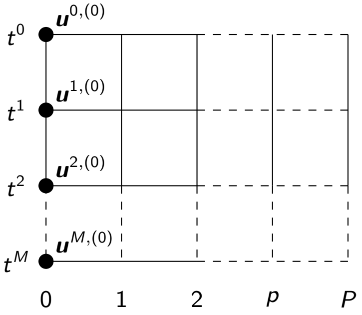
$$\boldsymbol{u}^{0,(p)} := \boldsymbol{u}(t_n), \quad p = 0, \dots, P,$$
$$\boldsymbol{u}^{m,(0)} := \boldsymbol{u}(t_n), \quad m = 1, \dots, M$$
$$\mathcal{L}^1_\Delta(\underline{\boldsymbol{u}}^{(p)}) = \mathcal{L}^1_\Delta(\underline{\boldsymbol{u}}^{(p-1)}) - \mathcal{L}^2_\Delta(\underline{\boldsymbol{u}}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

- $\mathcal{L}^1(\underline{\boldsymbol{u}}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{\boldsymbol{u}}) = 0$, high order $M+1$.

DeC Theorem

- $\mathcal{L}^1_\Delta$ coercive
- $\mathcal{L}^1_\Delta - \mathcal{L}^2_\Delta$ Lipschitz

DeC converges and $\min(P, M+1)$ is the order of accuracy.



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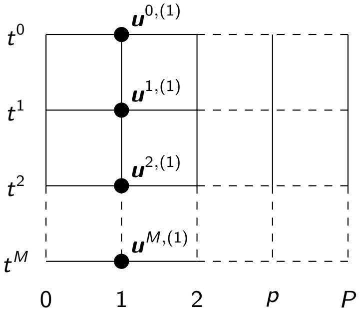

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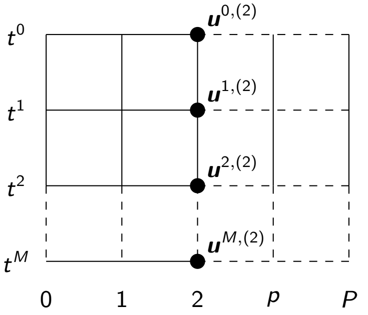

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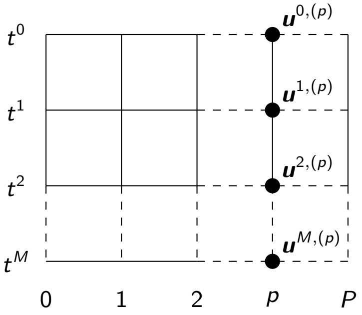

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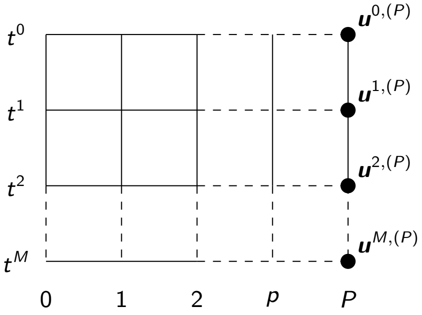

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$$\boldsymbol{u}^{m,(p)} = \boldsymbol{u}^0 + \sum_{r=0}^{M} \theta_r^m \boldsymbol{G}(t^r, \boldsymbol{u}^{r,(p-1)}), \qquad \forall m = 1, \dots, M, \; p = 1, \dots, P$$

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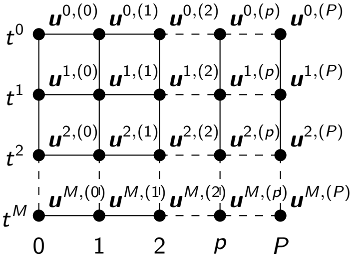

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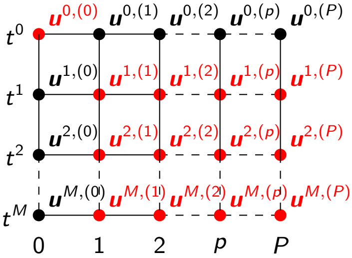

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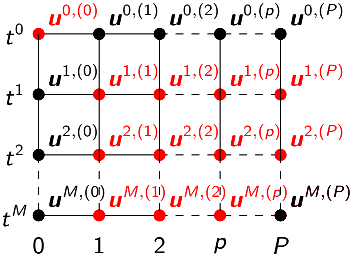

$\boldsymbol{c}$	$\boldsymbol{u}^0$	$\underline{\boldsymbol{u}}^{(1)}$	$\underline{\boldsymbol{u}}^{(2)}$	$\underline{\boldsymbol{u}}^{(3)}$	$\cdots$	$\underline{\boldsymbol{u}}^{(M-1)}$	$\underline{\boldsymbol{u}}^{(M)}$	A
0	0							$\boldsymbol{u}^0$
$\underline{\beta}_{1:}$	$\underline{\beta}_{1:}$	$\underline{\underline{0}}$						$\underline{\boldsymbol{u}}^{(1)}$
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\Theta_{1:,1:}$	$\underline{\underline{0}}$					$\underline{\boldsymbol{u}}^{(2)}$
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{\underline{0}}$	$\Theta_{1:,1:}$	$\underline{\underline{0}}$				$\underline{\boldsymbol{u}}^{(3)}$
	$\vdots$	$\vdots$		$\ddots$	$\ddots$			$\vdots$
	$\vdots$	$\vdots$			$\ddots$	$\ddots$		$\vdots$
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{\underline{0}}$	$\cdots$	$\cdots$	$\underline{\underline{0}}$	$\Theta_{1:,1:}$	$\underline{\underline{0}}$	$\underline{\boldsymbol{u}}^{(M)}$
$\boldsymbol{b}$	$\Theta_{M,0}$	$\underline{0}$	$\cdots$	$\cdots$	$\cdots$	$\underline{0}$	$\Theta_{M,1:}$	$\underline{\boldsymbol{u}}^{M,(M+1)}$

Costs

Large costs!

Costs

Large costs!

- DeC $S = M \cdot (P-1)+1$
  - DeC equi $S=(P-1)^2+1$
  - DeC GLB $S=\left\lceil \frac{P}{2} \right\rceil (P-1)+1$

Equispaced

P	M	DeC
2	1	2
3	2	5
4	3	10
5	4	17
6	5	26
7	6	37
8	7	50
9	8	65
10	9	82

Gauss–Lobatto

P	M	DeC
2	1	2
3	2	5
4	2	7
5	3	13
6	3	16
7	4	25
8	4	29
9	5	41
10	5	46

Costs

Large costs!

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How can we save computational time?

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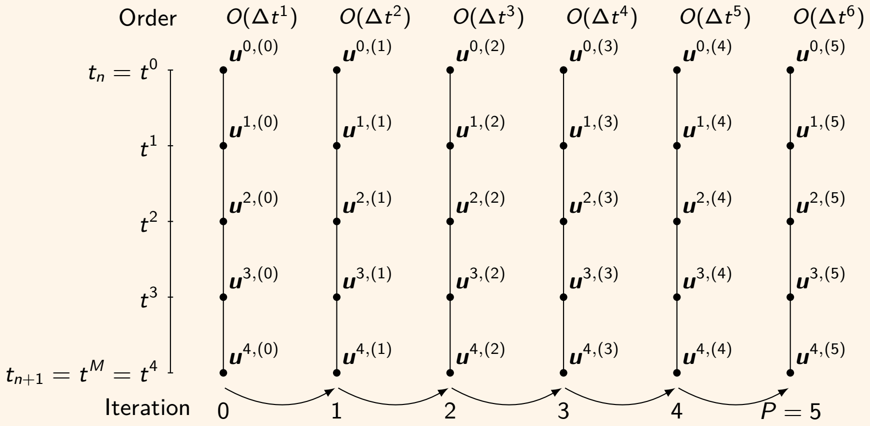
Idea for reduction of stages



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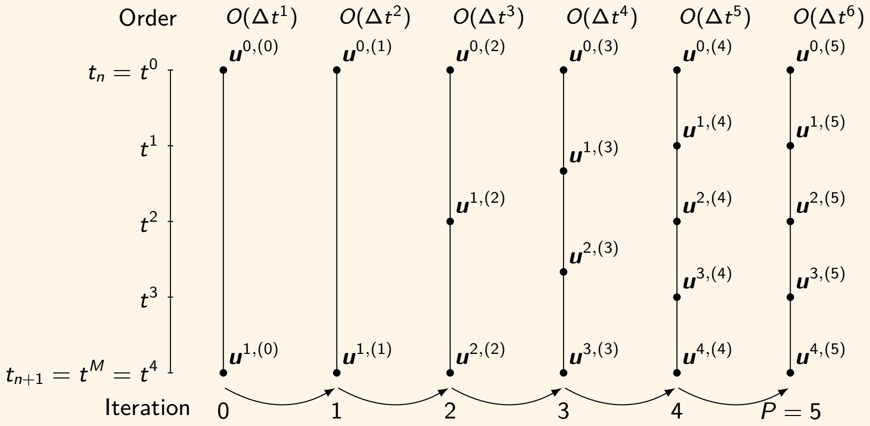

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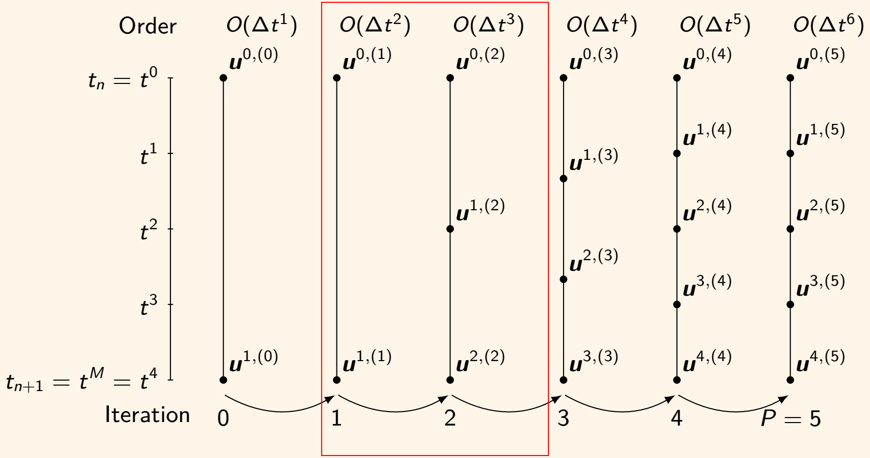
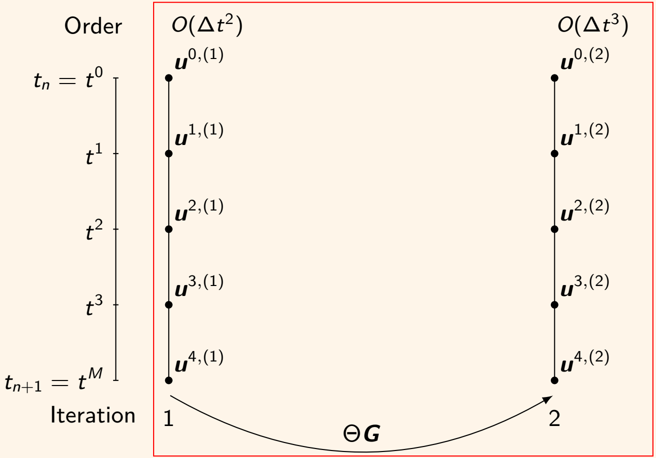


How to communicate between iterations?



DeC

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^0 + \Delta t \Theta \boldsymbol{G}(\underline{\boldsymbol{u}}^{(p-1)})$$

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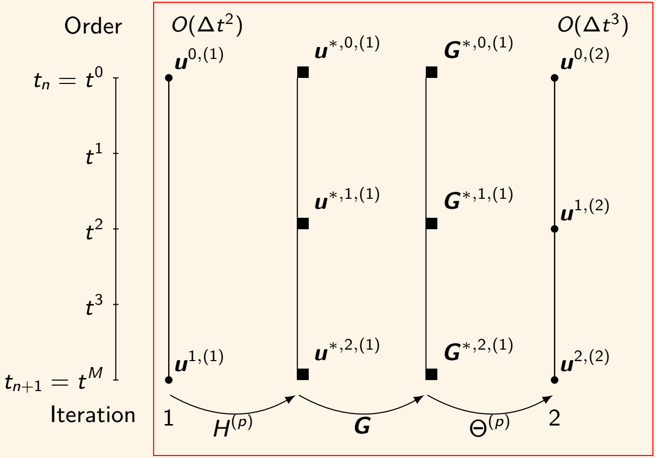

DeC

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t \Theta \boldsymbol{G}(\underline{\boldsymbol{u}}^{(p-1)})$$

DeCu

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t \Theta^{(p)} \boldsymbol{G}(H^{(p)} \underline{\boldsymbol{u}}^{(p-1)})$$

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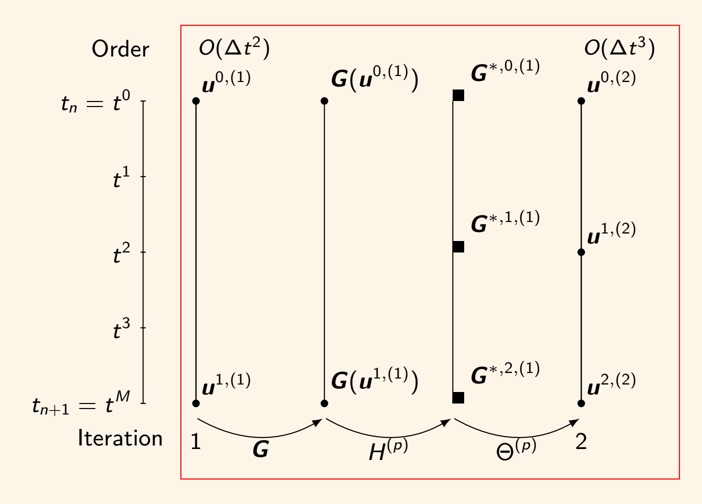

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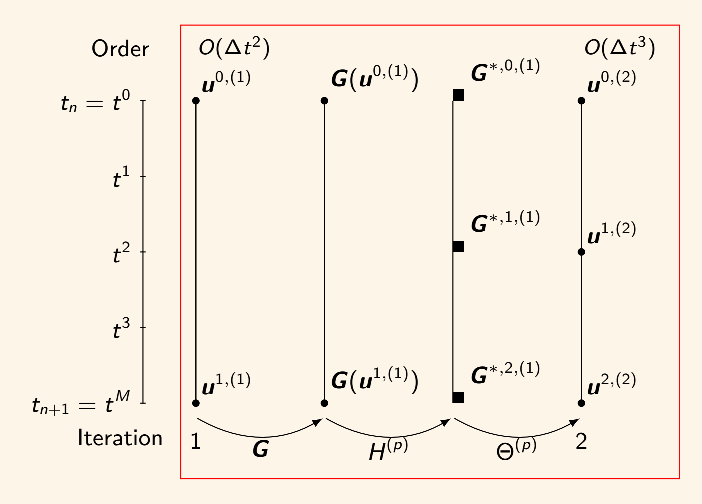

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DeCu

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t \Theta^{(p)} \boldsymbol{G}(H^{(p)} \underline{\boldsymbol{u}}^{(p-1)})$$

$$\underline{\boldsymbol{u}}^{*(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t H^{(p)} \Theta^{*(p-1)} \boldsymbol{G}(\underline{\boldsymbol{u}}^{*(p-1)})$$

DeCdu

$$\underline{\boldsymbol{u}}^{(p)} = \underline{\boldsymbol{u}}^{0} + \Delta t \Theta^{(p)} H^{(p)} \boldsymbol{G}(\underline{\boldsymbol{u}}^{(p-1)})$$

Efficient DeC into RK framework

DeC $\qquad S = M\cdot(P-1)+1$

$\boldsymbol{c}$	$\boldsymbol{u}^0$	$\underline{\boldsymbol{u}}^{(1)}$	$\underline{\boldsymbol{u}}^{(2)}$	$\underline{\boldsymbol{u}}^{(3)}$	$\cdots$	$\underline{\boldsymbol{u}}^{(M-1)}$	$\underline{\boldsymbol{u}}^{(M)}$	A	dim
0	0							$\boldsymbol{u}^0$	1
$\underline{\beta}_{1:}$	$\underline{\beta}_{1:}$	$\underline{\underline{0}}$						$\underline{\boldsymbol{u}}^{(1)}$	M
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\Theta_{1:,1:}$	$\underline{\underline{0}}$					$\underline{\boldsymbol{u}}^{(2)}$	M
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{\underline{0}}$	$\Theta_{1:,1:}$	$\underline{\underline{0}}$				$\underline{\boldsymbol{u}}^{(3)}$	M
	$\vdots$	$\vdots$		$\ddots$	$\ddots$			$\vdots$	M
	$\vdots$	$\vdots$			$\ddots$	$\ddots$		$\vdots$	M
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{\underline{0}}$	$\cdots$	$\cdots$	$\underline{\underline{0}}$	$\Theta_{1:,1:}$	$\underline{\underline{0}}$	$\underline{\boldsymbol{u}}^{(M)}$	M
$\boldsymbol{b}$	$\Theta_{M,0}$	$\underline{0}$	$\cdots$	$\cdots$	$\cdots$	$\underline{0}$	$\Theta_{M,1:}$	$\underline{\boldsymbol{u}}^{M,(M+1)}$	

Efficient DeC into RK framework

DeCu $\quad S = M \cdot (P-1) + 1 - \frac{(M-1)(M-2)}{2}$

$\boldsymbol{c}$	$\underline{\boldsymbol{u}}^0$	$\underline{\boldsymbol{u}}^{*(1)}$	$\underline{\boldsymbol{u}}^{*(2)}$	$\underline{\boldsymbol{u}}^{*(3)}$	$\cdots$	$\underline{\boldsymbol{u}}^{*(M-2)}$	$\underline{\boldsymbol{u}}^{*(M-1)}$	$\underline{\boldsymbol{u}}^{(M)}$	A	dim
0	0								$\underline{\boldsymbol{u}}^0$	1
$\underline{\beta}^{(2)}_{1:}$	$\underline{\beta}^{(2)}_{1:}$	$\underline{\underline{0}}$							$\underline{\boldsymbol{u}}^{*(1)}$	2
$\underline{\beta}^{(3)}_{1:}$	$W^{(2)}_{1:,0}$	$W^{(2)}_{1:,1:}$	$\underline{\underline{0}}$						$\underline{\boldsymbol{u}}^{*(2)}$	3
$\underline{\beta}^{(4)}_{1:}$	$W^{(3)}_{1:,0}$	$\underline{\underline{0}}$	$W^{(3)}_{1:,1:}$	$\underline{\underline{0}}$					$\underline{\boldsymbol{u}}^{*(3)}$	4
	$\vdots$	$\vdots$		$\ddots$	$\ddots$				$\vdots$	$\vdots$
	$\vdots$	$\vdots$			$\ddots$	$\ddots$			$\vdots$	$\vdots$
$\underline{\beta}^{(M)}_{1:}$	$W^{(M-1)}_{1:,0}$	$\underline{\underline{0}}$	$\cdots$	$\cdots$	$\underline{\underline{0}}$	$W^{(M-1)}_{1:,1:}$	$\underline{\underline{0}}$	$\underline{\underline{0}}$	$\underline{\boldsymbol{u}}^{*(M-1)}$	M
$\underline{\beta}^{(M)}_{1:}$	$W^{(M)}_{1:,0}$	$\underline{\underline{0}}$	$\cdots$	$\cdots$	$\cdots$	$\underline{\underline{0}}$	$W^{(M)}_{1:,1:}$	$\underline{\underline{0}}$	$\underline{\boldsymbol{u}}^{(M)}$	M
$\boldsymbol{b}$	$W^{(M+1)}_{M,0}$	$\underline{0}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\underline{0}$	$W^{(M+1)}_{M,1:}$	$\underline{\boldsymbol{u}}^{M,(M+1)}$	

$$W^{(p)} := \begin{cases} H^{(p)}\Theta^{(p)} \in \mathbb{R}^{(p+2)\times(p+1)}, & \text{if } p = 2,\dots,M-1, \\ \Theta^{(M)} \in \mathbb{R}^{(M+1)\times(M+1)}, & \text{if } p \geq M. \end{cases}$$

Efficient DeC into RK framework

DeCdu $\quad S = M\cdot(P-1)+1 - \frac{M(M-1)}{2}$

$\boldsymbol{c}$	$\boldsymbol{u}^0$	$\underline{\boldsymbol{u}}^{(1)}$	$\underline{\boldsymbol{u}}^{(2)}$	$\underline{\boldsymbol{u}}^{(3)}$	$\cdots$	$\underline{\boldsymbol{u}}^{(M-2)}$	$\underline{\boldsymbol{u}}^{(M-1)}$	$\underline{\boldsymbol{u}}^{(M)}$	A	dim
0	0								$\boldsymbol{u}^0$	1
$\underline{\beta}^{(1)}_{1:}$	$\underline{\beta}^{(1)}_{1:}$	$\underline{\underline{0}}$							$\underline{\boldsymbol{u}}^{(1)}$	1
$\underline{\beta}^{(2)}_{1:}$	$Z^{(2)}_{1:,0}$	$Z^{(2)}_{1:,1:}$	$\underline{\underline{0}}$						$\underline{\boldsymbol{u}}^{(2)}$	2
$\underline{\beta}^{(3)}_{1:}$	$Z^{(3)}_{1:,0}$	$\underline{\underline{0}}$	$Z^{(3)}_{1:,1:}$	$\underline{\underline{0}}$					$\underline{\boldsymbol{u}}^{(3)}$	3
	$\vdots$	$\vdots$		$\ddots$	$\ddots$				$\vdots$	$\vdots$
	$\vdots$	$\vdots$			$\ddots$	$\ddots$			$\vdots$	$\vdots$
$\underline{\beta}^{(M-1)}_{1:}$	$Z^{(M-1)}_{1:,0}$	$\underline{\underline{0}}$	$\cdots$	$\cdots$	$\underline{\underline{0}}$	$Z^{(M-1)}_{1:,1:}$	$\underline{\underline{0}}$	$\underline{\underline{0}}$	$\underline{\boldsymbol{u}}^{(M-1)}$	$M-1$
$\underline{\beta}^{(M)}_{1:}$	$Z^{(M)}_{1:,0}$	$\underline{\underline{0}}$	$\cdots$	$\cdots$	$\cdots$	$\underline{\underline{0}}$	$Z^{(M)}_{1:,1:}$	$\underline{\underline{0}}$	$\underline{\boldsymbol{u}}^{(M)}$	M
$\boldsymbol{b}$	$Z^{(M+1)}_{M,0}$	$\underline{0}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\underline{0}$	$Z^{(M+1)}_{M,1:}$	$\underline{\boldsymbol{u}}^{M,(M+1)}$	

$$Z^{(p)} := \begin{cases} \Theta^{(p)} H^{(p-1)} \in \mathbb{R}^{(p+1)\times p}, & \text{if } p = 1,\dots,M, \\ \Theta^{(M)} \in \mathbb{R}^{(M+1)\times(M+1)}, & \text{if } p > M. \end{cases}$$

Computational costs reduction: RK stages

Equispaced

P	M	DeC	DeCu	DeCdu
2	1	2	2	2
3	2	5	5	4
4	3	10	9	7
5	4	17	14	11
6	5	26	20	16
7	6	37	27	22
8	7	50	35	29
9	8	65	44	37
10	9	82	54	46
11	10	101	65	56
12	11	122	77	67
13	12	145	90	79

Gauss-Lobatto

P	M	DeC	DeCu	DeCdu
2	1	2	2	2
3	2	5	5	4
4	2	7	7	6
5	3	13	12	10
6	3	16	15	13
7	4	25	22	19
8	4	29	26	23
9	5	41	35	31
10	5	46	40	36
11	6	61	51	46
12	6	67	57	52
13	7	85	70	64

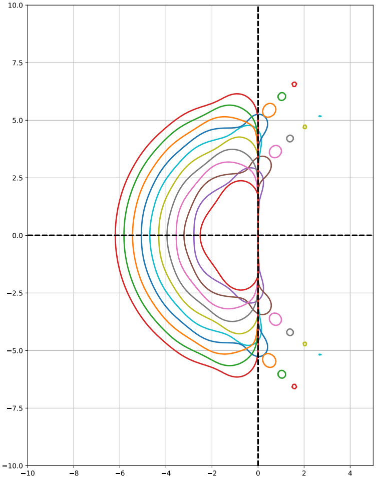
Stability Properties

DeC-DeCu-DeCdu

The **stability function** of DeC, DeCu, DeCdu of order P for any nodes distribution is

$$R(z) = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^P}{P!}.$$

DeC, DeCu, DeCdu



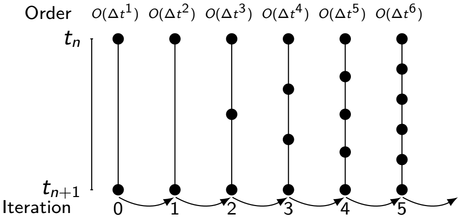
How can we exploit the increasing order of accuracy?

Adaptive DeC

How can we exploit the increasing order of accuracy?

Adaptive order DeC

- Set tolerance ε
- Check at each iteration if $\left\| \underline{\boldsymbol{u}}^{(p)} - \underline{\boldsymbol{u}}^{(p-1)} \right\| < \varepsilon$
- Stop at a certain order when tolerance is reached



Adaptive DeC

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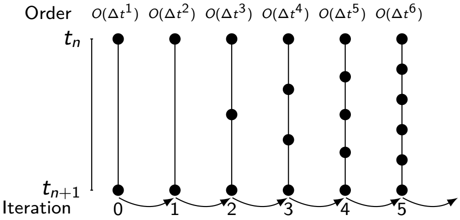
Adaptive order DeC

- Set tolerance ε
- Check at each iteration if $\left\| \underline{\boldsymbol{u}}^{(p)} - \underline{\boldsymbol{u}}^{(p-1)} \right\| < \varepsilon$
- Stop at a certain order when tolerance is reached

☺ Saving on useless iterations

☺ Reach the needed order for tolerance

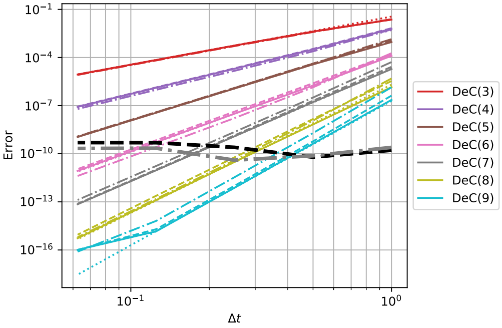
☹ Sub-optimal (waste of few stages)



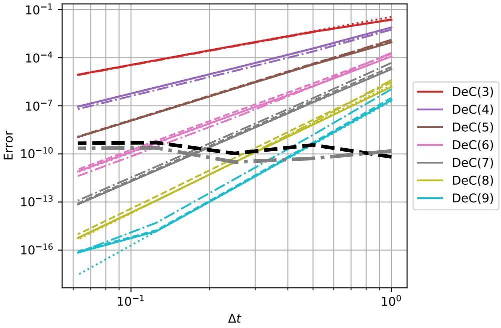
ODE test: Vibrating system

$$my'' + ry' + ky = F\cos(\Omega t + \varphi), \qquad y(0) = A, \qquad y'(0) = B.$$

Equispaced



Gauss-Lobatto



DeC $-$, DeCu $--$, DeCdu $-\cdot-$, adaptive in grey/black

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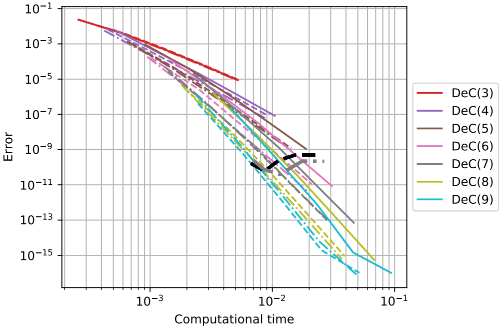

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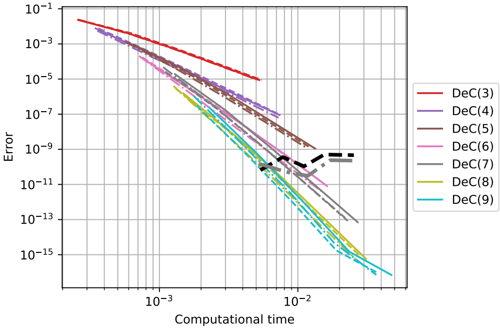

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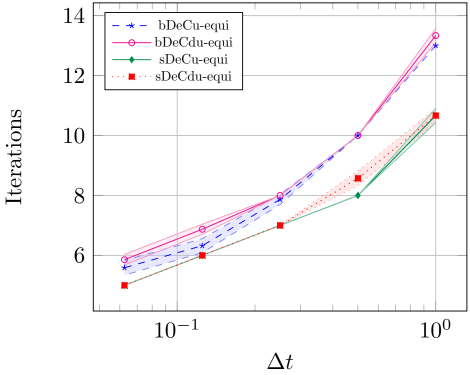

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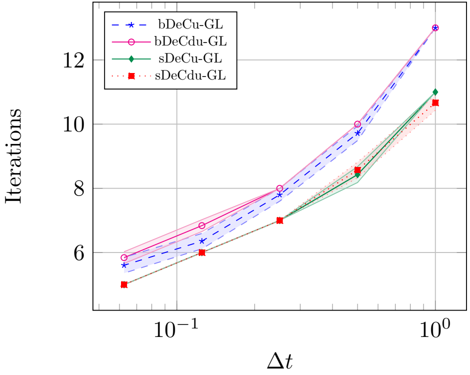

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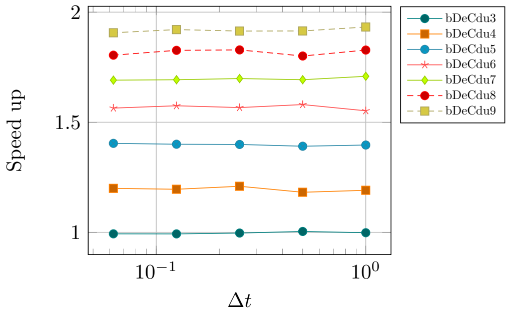

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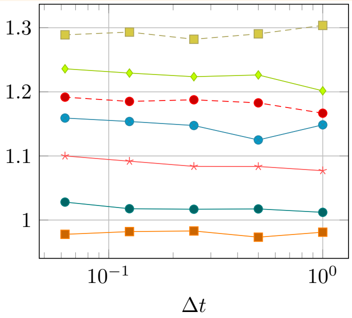

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Residual Distributions and DeC

Residual Distribution (RD)

- Originally somehow Finite Volume
- **Finite Element**
- Runge Kutta + Mass matrix correction (Rémi + Mario)
- DeC + RD (Rémi 2017)

RD setting

- $\partial_t u + \nabla \cdot F(u) = 0$
- $V_h = \{u \in \mathcal{C}(\Omega) : u|_K \in \mathbb{P}_M\}$
- $\Phi_K(u) = \int_K \nabla \cdot F(u) dx$
- $\Phi_K^i(u) = \int_K \varphi_i(x) \nabla \cdot F(u) dx + \mathrm{ST}_i(u)$
- NOT method of lines

$\mathcal{L}^2_\Delta$

$$\mathcal{L}^{2,m}_{\Delta,i}(\boldsymbol{u}) := \int_\Omega \varphi_i \varphi_j dx (u_j^m - u_j^0) + \Delta t \sum_{r=0}^{M} \theta_r^m \sum_K \Phi_K^i(u^r)$$

$\mathcal{L}^1_\Delta$

$$\mathcal{L}^{1,m}_{\Delta,i}(\boldsymbol{u}) := \int_\Omega \varphi_i dx (u_i^m - u_i^0) + \Delta t \beta^m \sum_K \Phi_K^i(u^0)$$

DeCu for RD

DeC for RD

$$\underbrace{\int_K \varphi_i dx(u_i^{m,(p)} - u_i^{m,(p-1)})}_{\mathcal{L}^{1,m}_{\Delta,i}(u^{(p)})-\mathcal{L}^{1,m}_{\Delta,i}(u^{(p-1)})} = \underbrace{\int_K \varphi_i\varphi_j dx(u_j^{m,(p)} - u_j^0) + \Delta t \sum_{r=0}^{M} \theta_r^m \sum_K \Phi_K^i(u^{r,(p-1)})}_{\mathcal{L}^{2,m}_{\Delta,i}(u^{(p-1)})}$$

DeCu for RD

DeC for RD

$$\underbrace{\int_K \varphi_i dx (u_i^{m,(p)} - u_i^{m,(p-1)})}_{\mathcal{L}^{1,m}_{\Delta,i}(u^{(p)}) - \mathcal{L}^{1,m}_{\Delta,i}(u^{(p-1)})} = \underbrace{\int_K \varphi_i \varphi_j dx (u_j^{m,(p)} - u_j^0) + \Delta t \sum_{r=0}^M \theta_r^m \sum_K \Phi_K^i (u^{r,(p-1)})}_{\mathcal{L}^{2,m}_{\Delta,i}(u^{(p-1)})}$$

DeCu for RD

$$\underbrace{\int_K \varphi_i dx (u_i^{m,(p)} - {\color{red}u_i^{*,m,(p-1)}})}_{\mathcal{L}^{1,m}_{\Delta,i}(u^{(p)}) - \mathcal{L}^{1,m}_{\Delta,i}({\color{red}u^{*,(p-1)}})} = \underbrace{\int_K \varphi_i \varphi_j dx ({\color{red}u_j^{*,m,(p-1)}} - u_j^0) + \Delta t \sum_{r=0}^M \theta_r^m \sum_K \Phi_K^i ({\color{red}u^{*,r,(p-1)}})}_{\mathcal{L}^{2,m}_{\Delta,i}({\color{red}u^{*,(p-1)}})}$$

DeCu for RD

DeC for RD

$$\underbrace{\int_K \varphi_i dx(u_i^{m,(p)} - u_i^{m,(p-1)})}_{\mathcal{L}^{1,m}_{\Delta,i}(u^{(p)})-\mathcal{L}^{1,m}_{\Delta,i}(u^{(p-1)})} = \underbrace{\int_K \varphi_i \varphi_j dx(u_j^{m,(p)} - u_j^0) + \Delta t \sum_{r=0}^{M} \theta_r^m \sum_K \Phi_K^i(u^{r,(p-1)})}_{\mathcal{L}^{2,m}_{\Delta,i}(u^{(p-1)})}$$

DeCu for RD

$$\underbrace{\int_K \varphi_i dx(u_i^{m,(p)} - u_i^{*,m,(p-1)})}_{\mathcal{L}^{1,m}_{\Delta,i}(u^{(p)})-\mathcal{L}^{1,m}_{\Delta,i}(u^{*,(p-1)})} = \underbrace{\int_K \varphi_i \varphi_j dx(u_j^{*,m,(p-1)} - u_j^0) + \Delta t \sum_{r=0}^{M} \theta_r^m \sum_K \Phi_K^i(u^{*,r,(p-1)})}_{\mathcal{L}^{2,m}_{\Delta,i}(u^{*,(p-1)})}$$

Computational cost

- Depends on update evaluation, less on flux evaluations

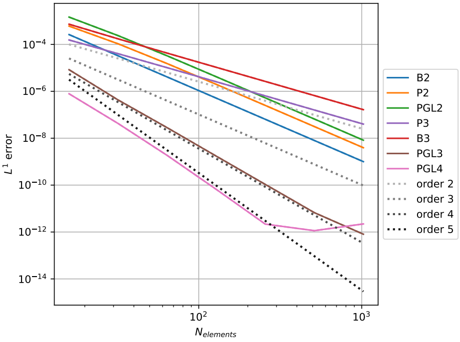
- DeC $C \approx (P-1)M + 1$
- DeCu $C \approx (P-1)M + 1 - \frac{M(M-1)}{2}$

Test PDE: linear advection equation

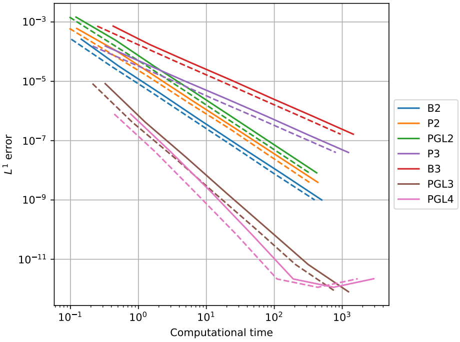
$$\begin{cases} \partial_t u + \partial_x u = 0 \\ u(0,x) = \cos(2\pi x) \end{cases}$$

DeC —
DeCu – –

Convergence



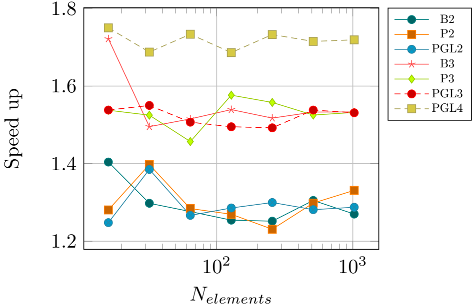
Computational Time



Test PDE: linear advection equation

$$\begin{cases} \partial_t u + \partial_x u = 0 \\ u(0,x) = \cos(2\pi x) \end{cases}$$

Speed up



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Summary and perspectives

Summary

- DeC
- Efficient DeC
- Increasing order of reconstruction with iterations
- Great Speed up
- Not too big implementation in a DeC code (Remi's birthday present)
- Adaptive with tolerance
- DeC and RD for PDEs

Summary

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- Increasing spatial discretizations order
- IMEX
- ADER
- Adaptive with other criteria

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- IMEX
- ADER
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THANK YOU!

Preprint

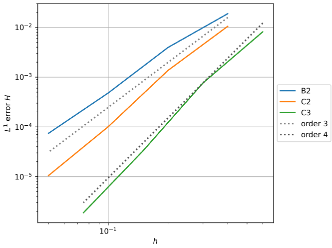
L. Micalizzi, D. Torlo. A new efficient explicit Deferred Correction framework: analysis and applications to hyperbolic PDEs and adaptivity. arXiv:2210.02976.

Test PDE: shallow water equations

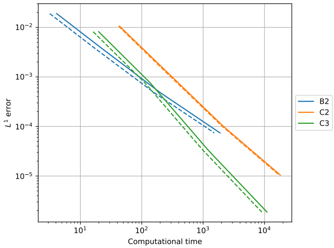
$$\begin{cases} \partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + \frac{g}{2} h^2 \end{pmatrix} = 0 \\ \text{IC} = \text{moving vortex} \end{cases}$$

bDeC —
bDeCu – –

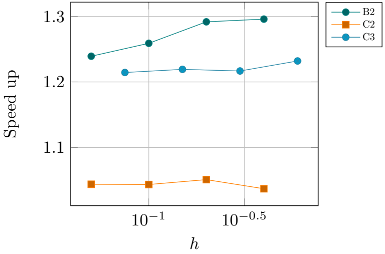
Convergence



Computational Time

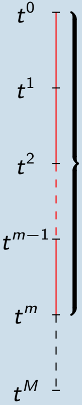


Speed Up



Many DeCs

Which sub-time-interval?



Big DeC (bDeC)

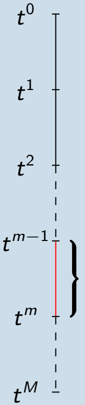
$$\mathcal{L}^2_\Delta(\boldsymbol{u}^0, \dots, \boldsymbol{u}^M) = \mathcal{L}^2_\Delta(\underline{\boldsymbol{u}}) :=$$

$$\begin{cases} \boldsymbol{u}^M - \boldsymbol{u}^0 - \int_{t^0}^{t^M} \boldsymbol{G}(\boldsymbol{u}(s))ds \\ \vdots \\ \boldsymbol{u}^m - \boldsymbol{u}^0 - \int_{t^0}^{t^m} \boldsymbol{G}(\boldsymbol{u}(s))ds \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \int_{t^0}^{t^1} \boldsymbol{G}(\boldsymbol{u}(s))ds \end{cases}$$

Parallel subtimesteps

Many DeCs

Which sub-time-interval?



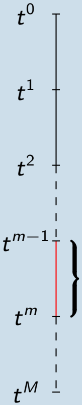
Small DeC (sDeC)

$$\mathcal{L}^2_\Delta(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M)=\mathcal{L}^2_\Delta(\underline{\boldsymbol{u}}):=\begin{cases}\boldsymbol{u}^M-\boldsymbol{u}^{M-1}-\int_{t^{M-1}}^{t^M}\boldsymbol{G}(\boldsymbol{u}(s))ds\\ \vdots\\ \boldsymbol{u}^m-\boldsymbol{u}^{m-1}-\int_{t^{m-1}}^{t^m}\boldsymbol{G}(\boldsymbol{u}(s))ds\\ \vdots\\ \boldsymbol{u}^1-\boldsymbol{u}^0-\int_{t^0}^{t^1}\boldsymbol{G}(\boldsymbol{u}(s))ds\end{cases}$$

Serial subtimesteps

More accurate

Which sub-time-interval?



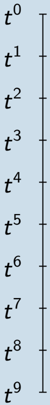
Small DeC (sDeC)

$$\mathcal{L}^2_\Delta(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M) = \mathcal{L}^2_\Delta(\underline{\boldsymbol{u}}) :=$$

$$\begin{cases} \boldsymbol{u}^M - \boldsymbol{u}^{M-1} - \int_{t^{M-1}}^{t^M} \boldsymbol{G}(\boldsymbol{u}(s))ds \\ \vdots \\ \boldsymbol{u}^m - \boldsymbol{u}^{m-1} - \int_{t^{m-1}}^{t^m} \boldsymbol{G}(\boldsymbol{u}(s))ds \\ \vdots \\ \boldsymbol{u}^1 - \boldsymbol{u}^0 - \int_{t^0}^{t^1} \boldsymbol{G}(\boldsymbol{u}(s))ds \end{cases}$$

- Serial subtimesteps
- More accurate

Which sub-time-nodes?

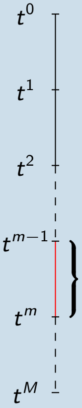


Equispaced

- Easy to implement
- Accuracy $M+1$

Many DeCs

Which sub-time-interval?

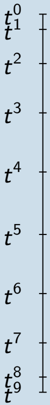


Small DeC (sDeC)

$$\mathcal{L}^2_\Delta(\boldsymbol{u}^0,\dots,\boldsymbol{u}^M)=\mathcal{L}^2_\Delta(\underline{\boldsymbol{u}}):=\begin{cases}\boldsymbol{u}^M-\boldsymbol{u}^{M-1}-\int_{t^{M-1}}^{t^M}\boldsymbol{G}(\boldsymbol{u}(s))ds\\ \vdots\\ \boldsymbol{u}^m-\boldsymbol{u}^{m-1}-\int_{t^{m-1}}^{t^m}\boldsymbol{G}(\boldsymbol{u}(s))ds\\ \vdots\\ \boldsymbol{u}^1-\boldsymbol{u}^0-\int_{t^0}^{t^1}\boldsymbol{G}(\boldsymbol{u}(s))ds\end{cases}$$

- Serial subtimesteps
- More accurate

Which sub-time-nodes?



Equispaced

- Easy to implement
- Accuracy $M+1$

Gauss-Lobatto

- Accuracy $2M$
- Less standard polynomials in time