Arbitrary High–Order Positivity–Preserving Finite–Volume Shallow–Water scheme without Restrictions on the CFL

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Shallow Water equations

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathcal{F}(\mathbf{u}) = \mathcal{S}(\mathbf{u}, x, y) \quad \text{on} \quad \mathbf{\Omega}_T = \mathbf{\Omega} \times [0, T] \subset \mathbb{R}^2 \times \mathbb{R}^+ \tag{1}$$

with conserved variables, flux and source terms given by

$$\mathbf{u} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} hu & hv \\ hu^2 + g\frac{h^2}{2} & huv \\ huv & hv^2 + g\frac{h^2}{2} \end{bmatrix}, \quad \mathcal{S} = -gh \begin{bmatrix} 0 \\ \frac{\partial b}{\partial x}(x, y) \\ \frac{\partial b}{\partial y}(x, y) \end{bmatrix} \tag{2}$$

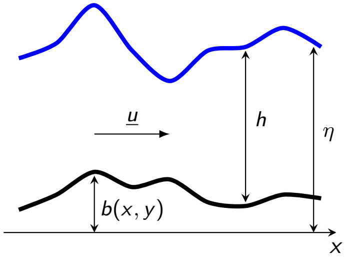
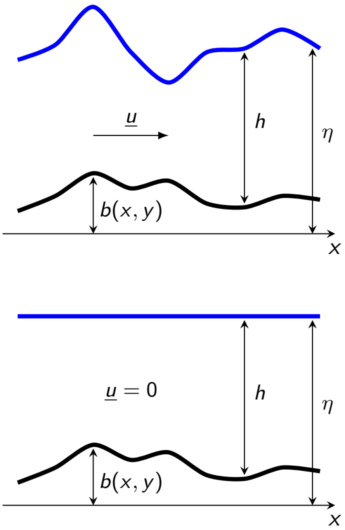


Figure: Shallow Water Equations: definition of the variables.

Numerical Method

- Positivity preserving
- Accurate approximation of the solution, i.e., use of high order methods (WENO, DeC)
- Conservation of the total mass
- Conservation of naturally balanced steady states (ex. lake at rest)



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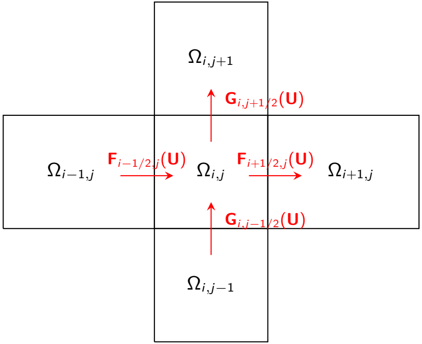
Finite Volume method

$$\frac{\mathrm{d}\mathbf{U}_{i,j}}{\mathrm{d}t} + \frac{\mathbf{F}_{i+1/2,j} - \mathbf{F}_{i-1/2,j}}{\Delta x} + \frac{\mathbf{G}_{i,j+1/2} - \mathbf{G}_{i,j-1/2}}{\Delta y} = \mathbf{S}_{i,j}$$

$$\mathbf{U}_{i,j}(t) := \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \mathbf{u}(x,y,t)\,\mathrm{d}x\mathrm{d}y.$$



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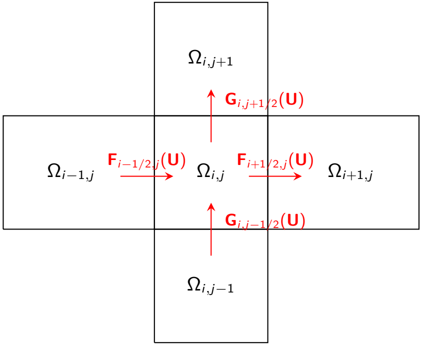
$$\mathbf{S}_{i,j}(t) := \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} \mathcal{S}(x,y,t)\,\mathrm{d}x\mathrm{d}y$$

$$\mathbf{F}_{i+1/2,j} = \frac{1}{\Delta y} \int_{y_{j-1/2}}^{y_{j+1/2}} \mathbf{F}(\mathbf{U}(x_{i+1/2},y,t))\,\mathrm{d}y,$$

$$\mathbf{G}_{i,j+1/2} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{G}(\mathbf{U}(x,y_{j+1/2},t))\,\mathrm{d}x.$$

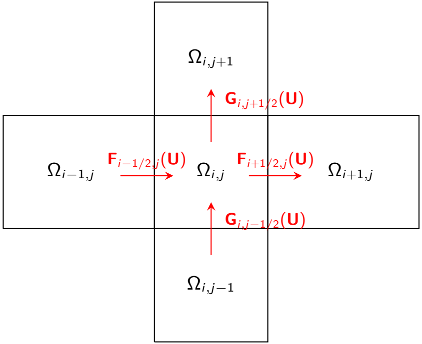


$$\frac{\mathrm{d}\mathbf{U}_{i,j}}{\mathrm{d}t} + \frac{\mathbf{F}_{i+1/2,j} - \mathbf{F}_{i-1/2,j}}{\Delta x} + \frac{\mathbf{G}_{i,j+1/2} - \mathbf{G}_{i,j-1/2}}{\Delta y} = \mathbf{S}_{i,j}$$

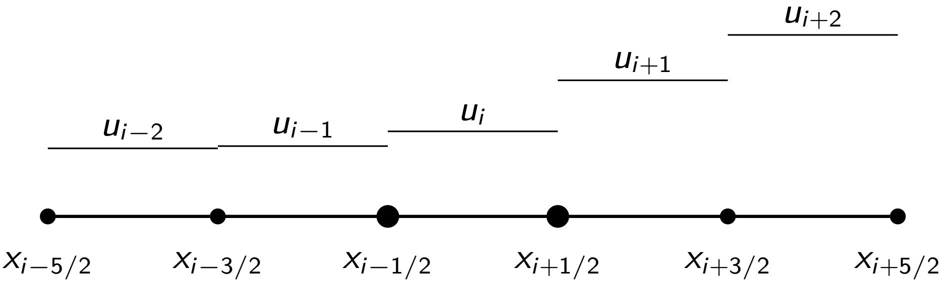
$$\mathbf{U}^L_{i+1/2,\theta} = \mathbf{U}(x^L_{i+1/2}, y_\theta) \,, \quad \mathbf{U}^R_{i+1/2,\theta} = \mathbf{U}(x^R_{i+1/2}, y_\theta).$$

$$\hat{\mathbf{F}}(\mathbf{U}^L, \mathbf{U}^R) = \frac{1}{2}\left(\mathbf{F}(\mathbf{U}^R) + \mathbf{F}(\mathbf{U}^L)\right) - \frac{1}{2} s_{max}\left(\mathbf{U}^R - \mathbf{U}^L\right),$$

$$\mathbf{F}_{i+1/2,j} = \frac{1}{\Delta y} \sum_{\theta=1}^{N_\theta} w_\theta \hat{\mathbf{F}}(\mathbf{U}^L_{i+1/2,\theta}, \mathbf{U}^R_{i+1/2,\theta}).$$



High order reconstruction: WENO[1]



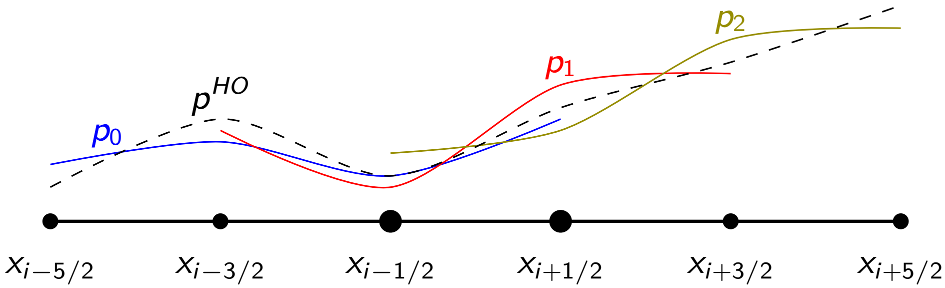
- u_i cell averages
- p^{HO} high order reconstruction polynomials
- p_j low order reconstruction polynomials
- β_j smoothness indicator

Consider a (interface, quadrature) point $\xi \in [x_{i-1/2}, x_{i+1/2}]$

- Optimal weights d_j^ξ: $\sum_j d_j^\xi p_j(\xi) = p^{HO}(\xi)$
- Nonlinear weights $\omega_j^\xi = \frac{d_j^\xi}{(\beta_j+\varepsilon)^2}$

[1] C.-W. Shu. In Advanced numerical approximation of nonlinear hyperbolic equations. Springer, 1998.

High order reconstruction: WENO[1]



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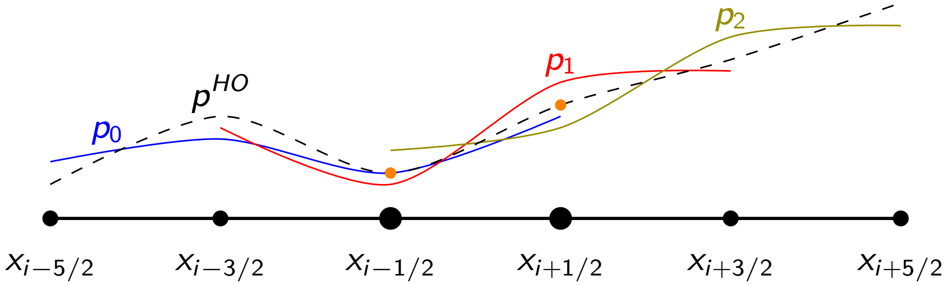

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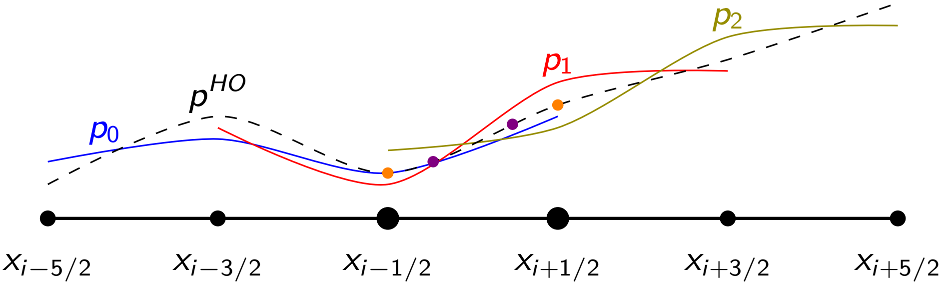

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Positivity Limiter[2]

$$\alpha = \sum_{\theta=1}^{N_\theta} w_\theta h^L_{i+1/2,j+\theta}$$

$$\beta = \sum_{\theta=1}^{N_\theta} w_\theta h^R_{i-1/2,j+\theta}$$

$$\xi = \frac{h_{i,j} - w_1^{Lobatto}\alpha - w_1^{Lobatto}\beta}{1-2w_1^{Lobatto}}$$

$$m_\theta := \min(\xi, h^L_{i+1/2,j+\theta}, h^R_{i-1/2,j+\theta})$$

$$\omega = \begin{cases} 1 & \text{if } h_{i,j} = m_\theta \\ \min\left(1, \left|\frac{h_{i,j}-\varepsilon}{h_{i,j}-m_\theta}\right|\right) & \text{else} \end{cases}$$

$$h^L_{i+1/2,j+\theta} := h_{i,j} + \omega(h^L_{i+1/2,j+\theta} - h_{i,j})$$

$$h^R_{i-1/2,j+\theta} := h_{i,j} + \omega(h^R_{i-1/2,j+\theta} - h_{i,j})$$

Pro

- Provable positive
- Easy to implement
- Local in cell
- Explicit

Cons

- Proof relies on Lobatto weights
- **CFL constraint for explicit Euler of $w^{Lobatto}$**
  - WENO3 $w_1^{Lobatto} = 1/6$
  - WENO5 $w_1^{Lobatto} = 1/12$
  - WENO7 $w_1^{Lobatto} = 1/20$
- Usable only with explicit Euler and SSPRK
- There are no SSPRK (with positive coefficients) with order higher than 4 (S. Gottlieb, D. I. Ketcheson, and C.-W. Shu. World Scientific, 2011.)

[2] B. Perthame and C.-W. Shu. Numerische Mathematik, 1996.

Well balanced techniques

Steady states

Lake at rest

$$\begin{cases} q(x,y) \equiv 0 \\ h(x,y) + b(x,y) \equiv \eta_0 \end{cases}$$

1D moving water equilibrium

$$\begin{cases} h(x,y)u(x,y) \equiv q_0 \\ v(x,y) \equiv 0 \\ \partial_x \left(\frac{q_0^2}{h} + g\frac{h^2}{2} \right) + gh\partial_x b = 0 \end{cases}$$

Vortexes

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^\infty - \tilde{y}\omega(r) \\ v^\infty + \tilde{x}\omega(r) \end{pmatrix} \\ h(x,y) = h(r),\ h'(r) = \frac{r\omega(r)^2}{g} \\ (\tilde{x},\tilde{y}) = (x,y) + t(u^\infty, v^\infty) \\ r^2 = \tilde{x}^2 + \tilde{y}^2 \end{cases}$$

[3] J. P. Berberich, P. Chandrashekar, and C. Klingenberg. Computers & Fluids, 2021.

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Remove the residual of the known analytical solution U^* an its discretization operators $\mathbf{F}^*$, $\mathbf{G}^*$, $\mathbf{S}^*$[3]

$$\frac{\partial(\mathbf{U}_{i,j} \qquad)}{\partial t} + \frac{\mathbf{F}_{i+1/2,j} - \mathbf{F}_{i-1/2,j}}{\Delta x} \qquad + \frac{\mathbf{G}_{i,j+1/2} - \mathbf{G}_{i,j-1/2}}{\Delta y} \qquad = \mathbf{S}_{i,j}$$

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- Fast to implement
- Suited for lake at rest and vortexes

[3] J. P. Berberich, P. Chandrashekar, and C. Klingenberg. Computers & Fluids, 2021.

Finite volume WENO scheme

- Arbitrary high order
- Provably positive
- CFL $= 1$
- Well balanced

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Production destruction systems

Production Destruction systems (PDS)

$$\begin{cases} d_t c_i = P_i(\boldsymbol{c}) - D_i(\boldsymbol{c}), \quad i = 1, \dots, I, & P_i(\boldsymbol{c}) = \sum_{j=1}^I p_{i,j}(\boldsymbol{c}), \\ \boldsymbol{c}(t=0) = \boldsymbol{c}_0, & D_i(\boldsymbol{c}) = \sum_{j=1}^I d_{i,j}(\boldsymbol{c}), \end{cases}$$

$$p_{i,j}(\boldsymbol{c}), d_{i,j}(\boldsymbol{c}) \geq 0, \qquad \forall i, j \in I, \quad \forall \boldsymbol{c} \in \mathbb{R}^{+,I}.$$

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Conservation

$$\sum_{i=1}^I c_i(0) = \sum_{i=1}^I c_i(t), \ \forall t$$

$$\Longleftrightarrow$$

$$p_{i,j}(\mathbf{c}) = d_{j,i}(\mathbf{c}),$$

$$\forall i, j \in \boldsymbol{I}, \ \forall \mathbf{c} \in \mathbb{R}^{+,I}.$$

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$$\iff$$

$$p_{i,j}(\mathbf{c}) = d_{j,i}(\mathbf{c}),$$

$$\forall i,j \in I, \; \forall \mathbf{c} \in \mathbb{R}^{+,I}.$$

Positivity

If P_i, D_i Lipschitz, and if when

$$c_i \to 0 \Rightarrow D_i(\mathbf{c}) \to 0$$

$$\Longrightarrow$$

$$c_i(0) > 0 \, \forall i \in I \Rightarrow c_i(t) > 0$$

$$\forall i \in I \; \forall t > 0.$$

Production destruction systems

Production Destruction systems (PDS)

$$\begin{cases} d_t c_i = P_i(\mathbf{c}) - D_i(\mathbf{c}), \quad i = 1, \dots, I, & P_i(\mathbf{c}) = \sum_{j=1}^{I} p_{i,j}(\mathbf{c}), \\ \mathbf{c}(t=0) = \mathbf{c}_0, & D_i(\mathbf{c}) = \sum_{j=1}^{I} d_{i,j}(\mathbf{c}), \end{cases}$$

$$p_{i,j}(\mathbf{c}), d_{i,j}(\mathbf{c}) \geq 0, \qquad \forall i,j \in I, \quad \forall \mathbf{c} \in \mathbb{R}^{+,I}.$$

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Applications

- Chemical reactions
- Biological systems
- Population evolution
- PDEs

Modified Patankar methods

Explicit Euler

$$c_i^{n+1} = c_i^n + \Delta t \left(\sum_j p_{i,j}(\boldsymbol{c}^n) \qquad - \sum_j d_{i,j}(\boldsymbol{c}^n) \qquad \right)$$

- **Conservative**
- First order
- **Conditionally** Positive
- Explicit

Modified Patankar methods

Patankar Euler[4]

$$c_i^{n+1} = c_i^n + \Delta t \left(\sum_j p_{i,j}(\mathbf{c}^n) \qquad - \sum_j d_{i,j}(\mathbf{c}^n) \frac{c_i^{n+1}}{c_i^n} \right)$$

- **Not Conservative**
- First order
- **Unconditionally Positive**
- Implicit, but easy

$$\left(1 + \Delta t \frac{\sum_j d_{i,j}(\mathbf{c}^n)}{c_i^n} \right) c_i^{n+1} = c_i^n + \Delta t \sum_j p_{i,j}(\mathbf{c}^n)$$

[4] S. Patankar. CRC press, 1980.

Modified Patankar methods

Modified Patankar Euler[4]

$$c_i^{n+1} = c_i^n + \Delta t \left(\sum_j p_{i,j}(\mathbf{c}^n) \frac{c_j^{n+1}}{c_j^n} - \sum_j d_{i,j}(\mathbf{c}^n) \frac{c_i^{n+1}}{c_i^n} \right)$$

- **Conservative**
- First order
- **Unconditionally Positive**
- Linearly implicit

$$\underline{\underline{\mathrm{M}}}(\mathbf{c}^n)\mathbf{c}^{n+1} = \mathbf{c}^n$$

$$\underline{\underline{\mathrm{M}}}(\mathbf{c}^n)_{i,j} = \begin{cases} m_{i,i}(\mathbf{c}^n) = 1 + \Delta t \sum_{j=1}^{I} \frac{d_{i,j}(\mathbf{c}^n)}{c_i^n}, & i = 1, \dots, I, \\ m_{i,j}(\mathbf{c}^n) = -\Delta t \frac{p_{i,j}(\mathbf{c}^n)}{c_j^n}, & i,j = 1, \dots, I,\ i \neq j. \end{cases}$$

[4] H. Burchard, E. Deleersnijder, and A. Meister. Applied Numerical Mathematics, 2003.

Modified Patankar methods

Modified Patankar Runge Kutta[4][5][6]

$$c_i^{(s)} = c_i^n + \Delta t \sum_k a_{s,k} \left(\sum_j p_{i,j}(\mathbf{c}^{(k)}) \frac{c_j^{(s)}}{\sigma_j^{(s)}} - \sum_j d_{i,j}(\mathbf{c}^{(s)}) \frac{c_i^{(s)}}{\sigma_i^{(s)}} \right)$$

- **Conservative**
- High order
- **Unconditionally Positive**
- Linearly implicit

$$\underline{\underline{\mathrm{M}}}(\{\mathbf{c}^{(k)}\}_{k=0}^{s-1})\mathbf{c}^{(s)} = \mathbf{c}^n$$

$$\underline{\underline{\mathrm{M}}}(\{\mathbf{c}^{(k)}\}_{k=0}^{s-1})_{i,j} = \begin{cases} m_{i,i}(\mathbf{c}^n) = 1 + \Delta t \sum_k a_{s,k} \sum_{j=1}^{I} \frac{d_{i,j}(\mathbf{c}^{(k)})}{\sigma_i^{(s)}}, & i = 1,\dots,I, \\ m_{i,j}(\mathbf{c}^n) = -\Delta t \sum_k a_{s,k} \frac{p_{i,j}(\mathbf{c}^{(k)})}{\sigma_j^{(s)}}, & i,j = 1,\dots,I,\ i \neq j. \end{cases}$$

[4] H. Burchard, E. Deleersnijder, and A. Meister. Applied Numerical Mathematics, 2003.
[5] S. Kopecz and A. Meister. BIT Numerical Mathematics, 2018.
[6] J. Huang, W. Zhao, and C.-W. Shu. Journal of Scientific Computing, 2018.

Modified Patankar methods

Modified Patankar Deferred Correction[4]

$$c_i^{m,(k+1)} = c_i^0 + \Delta t \sum_{r,j} \theta_r^m \left(p_{i,j}(\mathbf{c}^{r,(k)}) \frac{c_{\gamma(j,i,\theta_r^m)}^{m,(k+1)}}{c_{\gamma(j,i,\theta_r^m)}^{m,(k)}} - d_{i,j}(\mathbf{c}^{r,(k)}) \frac{c_{\gamma(i,j,\theta_r^m)}^{m,(k+1)}}{c_{\gamma(i,j,\theta_r^m)}^{m,(k)}} \right)$$

- **Conservative**
- **Arbitrary high** order
- **Unconditionally Positive**
- Linearly implicit

$$\underline{\underline{\mathrm{M}}}(\underline{\mathbf{c}}^{(k-1)}, m)\mathbf{c}^{m,(k)} = \mathbf{c}^n$$

$$\underline{\underline{\mathrm{M}}}(\underline{\mathbf{c}}^{(k-1)}, m)_{ij} = \begin{cases} 1 + \Delta t \sum_{r=0}^{M} \sum_{l=1}^{I} \frac{\theta_r^m}{c_i^{m,(k-1)}} \left(d_{i,l}(\mathbf{c}^{r,(k-1)}) \chi_{\{\theta_r^m > 0\}} - p_{i,l}(\mathbf{c}^{r,(k-1)}) \chi_{\{\theta_r^m < 0\}} \right) & \text{for } i = j \\ -\Delta t \sum_{r=0}^{M} \frac{\theta_r^m}{c_j^{m,(k-1)}} \left(p_{i,j}(\mathbf{c}^{r,(k-1)}) \chi_{\{\theta_r^m > 0\}} - d_{i,j}(\mathbf{c}^{r,(k-1)}) \chi_{\{\theta_r^m < 0\}} \right) & \text{for } i \neq j \end{cases}$$

[4] P. Öffner and D. Torlo. Applied Numerical Mathematics, 2020.

Positivity of Modified Patankar methods

$$\underline{\underline{\mathrm{M}}}(\boldsymbol{c}^n)\boldsymbol{c}^{n+1} = \boldsymbol{c}^n$$

$$\underline{\underline{\mathrm{M}}}(\boldsymbol{c}^n)_{i,j} = \begin{cases} m_{i,i}(\boldsymbol{c}^n) = 1 + \Delta t \sum_{j=1}^{I} \frac{d_{i,j}(\boldsymbol{c}^n)}{c_i^n}, & i = 1, \dots, I, \\ m_{i,j}(\boldsymbol{c}^n) = -\Delta t \frac{p_{i,j}(\boldsymbol{c}^n)}{c_j^n}, & i,j = 1, \dots, I,\ i \neq j. \end{cases}$$

Unconditionally positivity: $M^{-1} > 0$

- $\underline{\underline{\mathrm{M}}} = D - A$ with $D > 0$ and $A > 0$
- $\underline{\underline{\mathrm{M}}}$ Diagonally dominant by columns:

$$D_{ii} > \sum_j A_{ji}$$

- **Jacobi Iterations** to solve $\underline{\underline{\mathrm{M}}}x = b$

$$x^{(p+1)} = D^{-1}(Ax^{(p)} + b) > 0$$

- Converges because diagonally dominant

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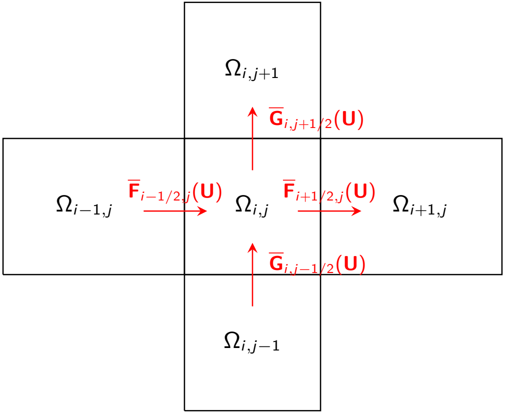
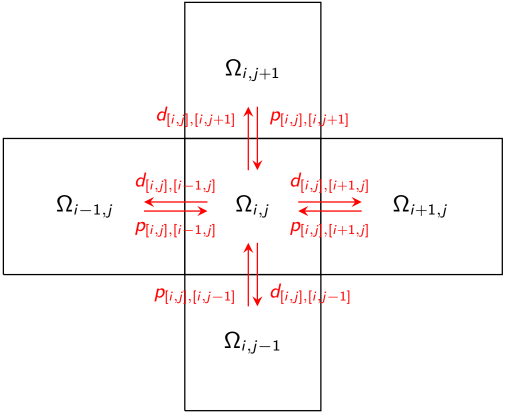
Water height equation as a PDS





$$\frac{\mathrm{d}h_{i,j}}{\mathrm{d}t}+\frac{\bar{\mathbf{F}}_{i+1/2,j}-\bar{\mathbf{F}}_{i-1/2,j}}{\Delta x}+\frac{\bar{\mathbf{G}}_{i,j+1/2}-\bar{\mathbf{G}}_{i,j-1/2}}{\Delta y}=0$$

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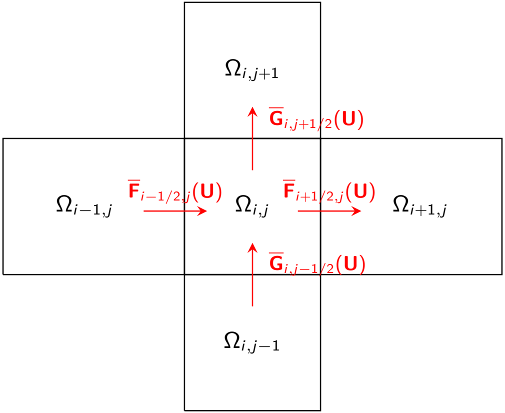

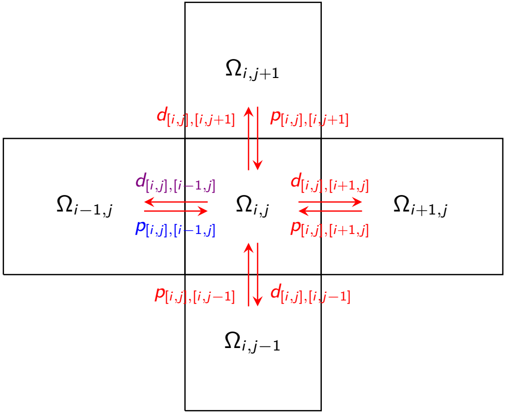

$$\frac{\mathrm{d}h_{i,j}}{\mathrm{d}t}+\frac{\bar{\mathbf{F}}_{i+1/2,j}-\bar{\mathbf{F}}_{i-1/2,j}}{\Delta x}+\frac{\bar{\mathbf{G}}_{i,j+1/2}-\bar{\mathbf{G}}_{i,j-1/2}}{\Delta y}=0$$

$$p_{[i,j],[i-1,j]}(\mathbf{U}) := +\frac{1}{\Delta x}\overline{\mathbf{F}}_{i-1/2,j}(\mathbf{U})^{+},$$

$$d_{[i,j],[i-1,j]}(\mathbf{U}) := -\frac{1}{\Delta x}\overline{\mathbf{F}}_{i-1/2,j}(\mathbf{U})^{-},$$

Final Numerical Method: one stage of mPDeC

- **Well balanced** step with analytical solution $\mathbf{F}^*, \mathbf{G}^*, \mathbf{S}^*$ (optional)

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- Update discharges hu, hv with classical finite volume

Properties and setup

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- Unconditionally positive

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Parameters

- WENO5
- Rusanov numerical flux
- CFL $= 0.9$
- mPDeC order 5 with 3 Gauss–Lobatto subtimesteps (13 stages)
- Periodic boundary conditions

Well balanced

Only for lake at rest steady state

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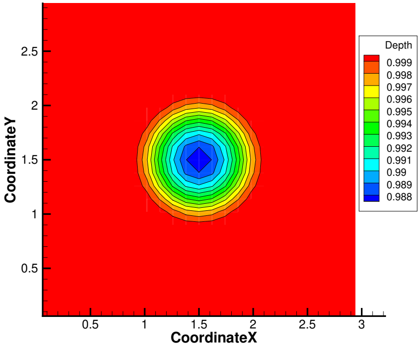
Order of accuracy: unsteady vortex

Initial Condition



Solution at all times

$$\Omega = [0,3] \times [0,3]$$

$$(h_0, u_0, v_0) = (1, 2, 3)$$

$$h(r) = h_0 - \delta h(r) = h_0 - \gamma \begin{cases} e^{-\frac{1}{\arctan^3(1-r^2)}}, & \text{if } r < 1, \\ 0, & \text{else}, \end{cases}$$

$$r^2 = (x - u_0 t - 1.5)^2 + (y - u_0 t - 1.5)^2$$

$$\gamma = 0.1$$

$$\begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = \sqrt{2\, g\, \partial_r h} \begin{pmatrix} (y-1.5) \\ -(x-1.5) \end{pmatrix},$$

$$CFL = 0.7, \quad T = 0.1$$

$$Nx = Ny \in \{25, 50, 100, 200, 300, 400, 500, 600\}$$

Order of accuracy: unsteady vortex

Error decay

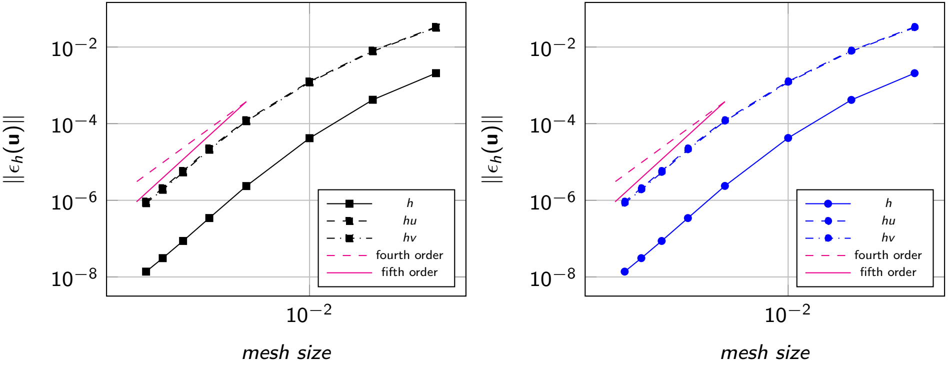


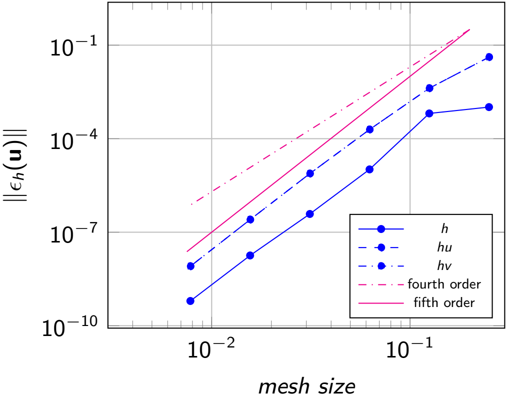
Figure: Unsteady vortex: convergence tests, left WENO5-DeC, right WENO5-mPDeC.

Order of accuracy: lake at rest

Lake at rest not well balanced

$$
\begin{aligned}
&\Omega = [0,1]\times[0,1]\\
&b(x,y) = 0.1\,\sin(2\pi x)\,\cos(2\pi y),\\
&h(x,y) = 1 - b(x,y),\\
&u = v = 0\\
&CFL = 0.9, \quad T = 0.1\\
&Nx = Ny \in \{4, 8, 16, 32, 64, 128\}
\end{aligned}
$$

Error Decay



Well balancedness: perturbation of lake at rest wet and dry

Perturbation of lake at rest

$$\Omega = [-5,5] \times [-2,2]$$

$$b(x,y) = b(x,y) = \begin{cases} e^{1-\frac{1}{1-r^2}}, & \text{if } r^2 < 1, \\ 0, & \text{else }, \end{cases}$$

$$r^2 = x^2 + y^2$$

$$h(x,y) = \max\{0.7 - b(x,y), 10^{-6}\},$$

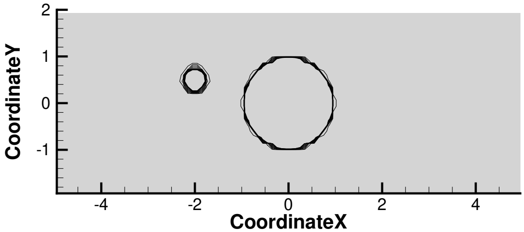
$$\tilde{h}(x,y) = h(x,y) + \begin{cases} 0.05\, e^{1-\frac{1}{(1-\rho^2)^2}}, & \text{if } \rho^2 < 1, \\ 0, & \text{else }, \end{cases}$$

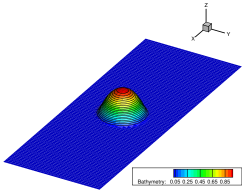
$$\rho^2 = 9((x+2)^2 + (x-0.5)^2)$$

$$u = v = 0$$

$$CFL = 0.9, \quad Nx = 100, \quad Ny = 30$$

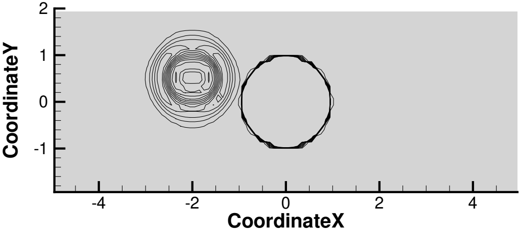
Initial Condition and Bathymetry



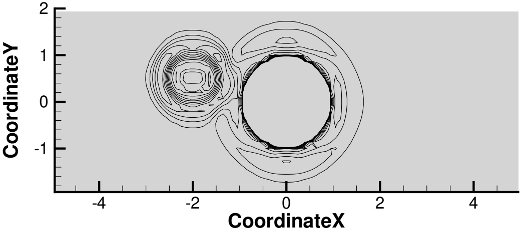


Well balancedness: perturbation of lake at rest wet and dry

Well Balanced T=0.25

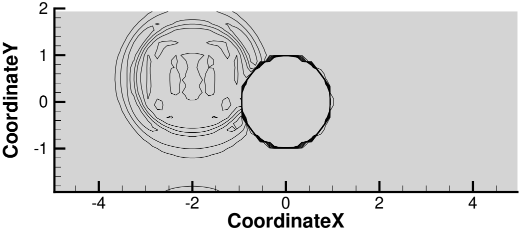


Not Well Balanced T=0.25

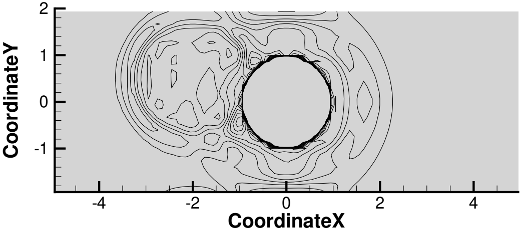


Well balancedness: perturbation of lake at rest wet and dry

Well Balanced T=0.5

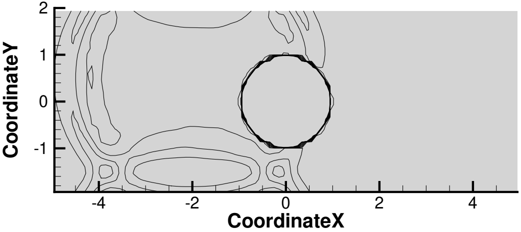


Not Well Balanced T=0.5

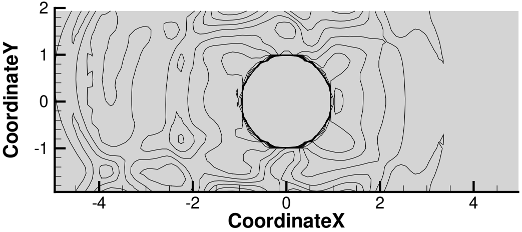


Well balancedness: perturbation of lake at rest wet and dry

Well Balanced T=1



Not Well Balanced T=1

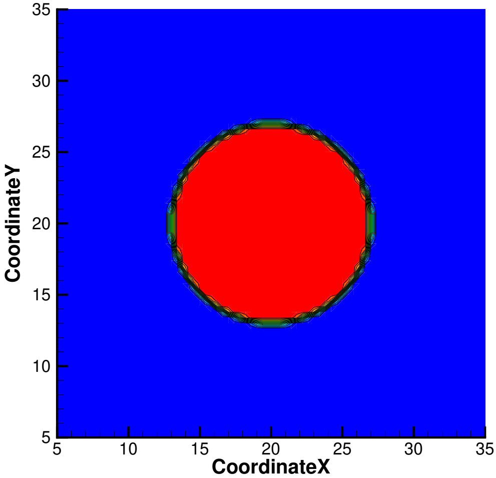


Circular Dry Dam Break

Dry Dam Break

- $\Omega = [0, 40]x[0, 40]$
- $h^0 = \begin{cases} 2.5 & \text{if } r < 7, \\ 10^{-6} & \text{else} \end{cases}$
- $r^2 = (x-20)^2 + (y-20)^2$
- $b = 0$
- $(u^0, v^0) = (0, 0)$
- $T = 0.9$
- CFL=0.9
- $N_x = N_y = 100$

Simulations $T = 0$



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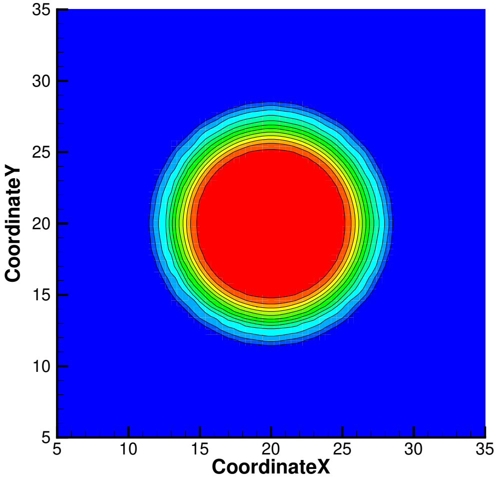
- $b = 0$
- $(u^0, v^0) = (0, 0)$
- $T = 0.9$
- CFL=0.9
- $N_x = N_y = 100$

Simulations $T = 0.3$



Circular Dry Dam Break

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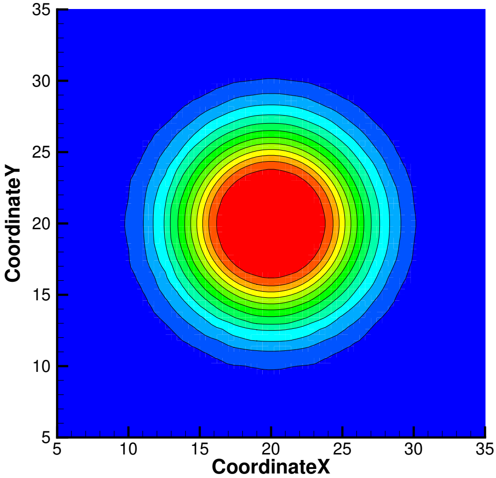
- $b = 0$
- $(u^0, v^0) = (0,0)$
- $T = 0.9$
- CFL=0.9
- $N_x = N_y = 100$

Simulations $T = 0.6$



Circular Dry Dam Break

Dry Dam Break

- $\Omega = [0, 40]x[0, 40]$
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- $r^2 = (x-20)^2 + (y-20)^2$
- $b = 0$
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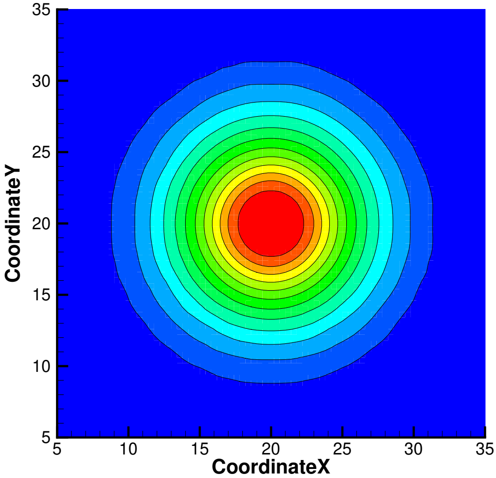
- $T = 0.9$
- CFL=0.9
- $N_x = N_y = 100$

Simulations $T = 0.9$

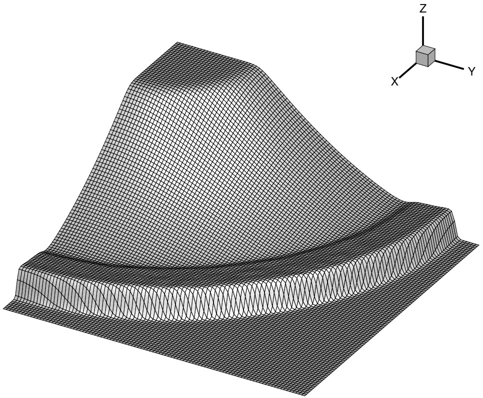


Circular Wet Dam Break

Wet Dam Break

- $\Omega = [0, 40]x[0, 40]$
- $h^0 = \begin{cases} 10 & \text{if } r < 7, \\ 0.5 & \text{else} \end{cases}$
- $r^2 = (x - 25)^2 + (y - 20)^2$
- $b = 0$
- $(u^0, v^0) = (0, 0)$
- $T = 0.8$
- CFL=1
- $N_x = N_y = 200$

Simulations $T = 0.8$



Circular Wet Dam Break

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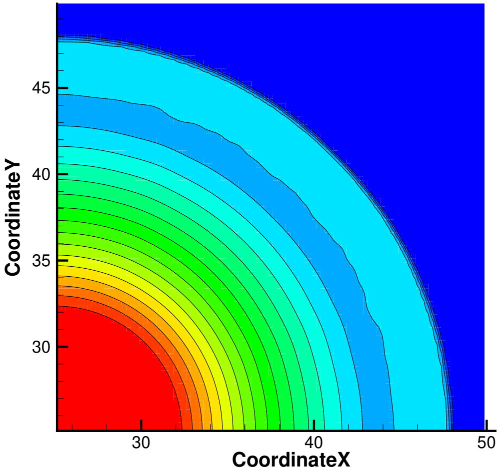

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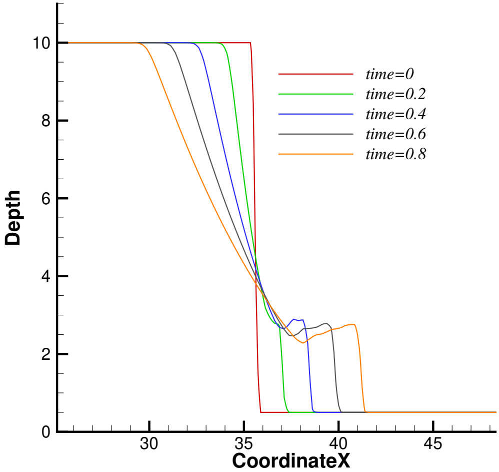
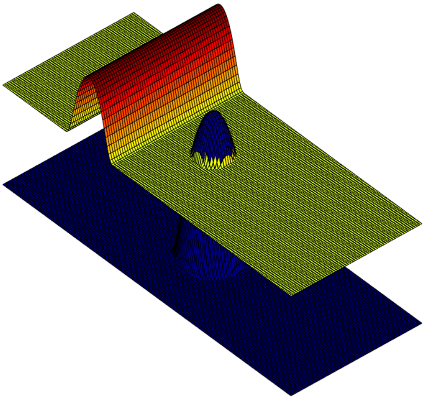


Simulations: Wave over dry island



- $\Omega = [-5, 5] \times [-2, 2]$
- $N_x = 400, \; N_y = 120$
- $b(x,y) =$
  $$\begin{cases} e^{1-\frac{1}{1-r^2}}, & \text{if } r^2 < 1, \\ 0, & \text{else} \end{cases},$$
  where $r^2 = x^2 + y^2$.

  $h(x,y) = 0.7 - b(x,y) +$
  $$\begin{cases} 0.5\, e^{1-\frac{1}{(1-\rho^2)^2}}, & \text{if } \rho^2 < 1, \\ 0, & \text{else}, \end{cases}$$
  where $\rho^2 = (x+2)^2$,

  $(u, v) = (1, 0)$,
- $T = 1, \; CFL = 0.9$

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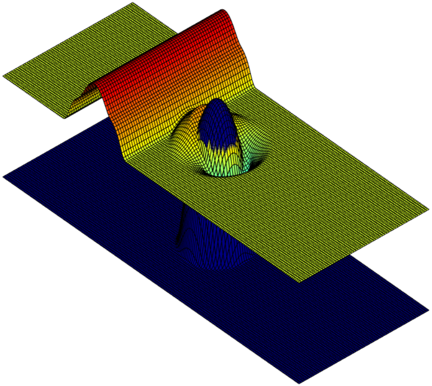

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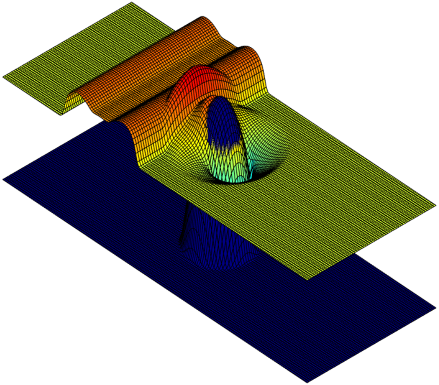

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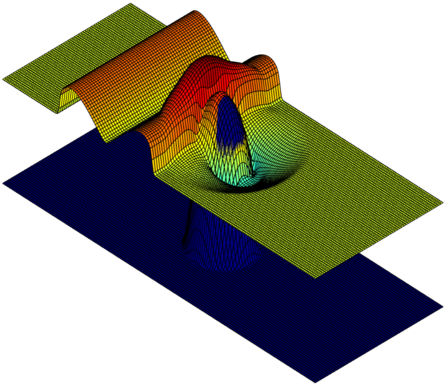

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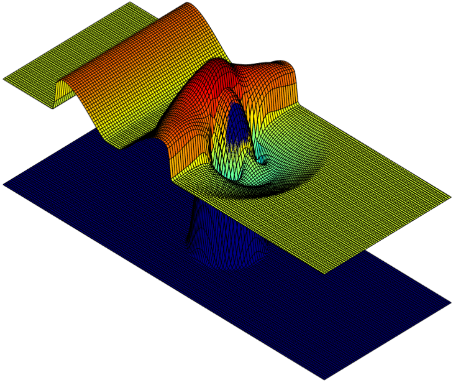

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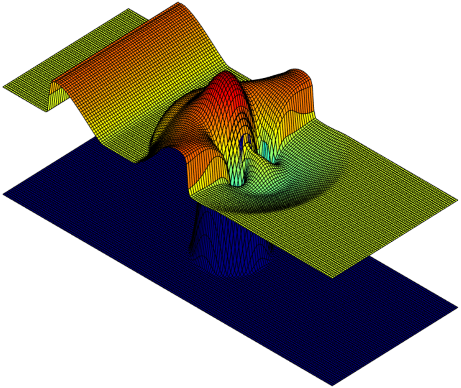

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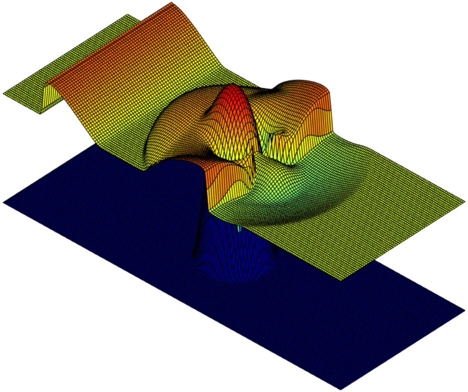
Simulations: Wave over dry island



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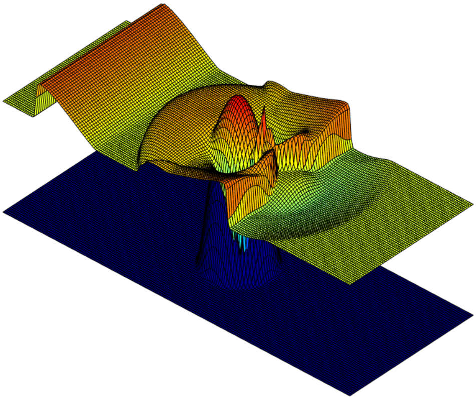

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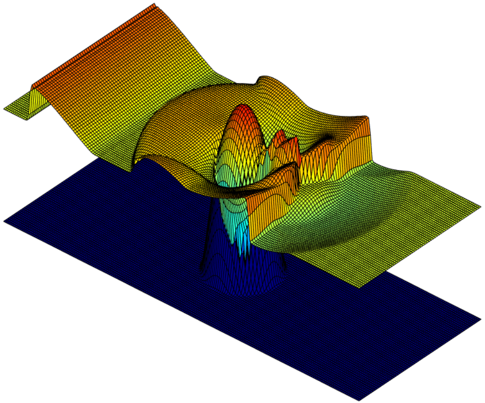

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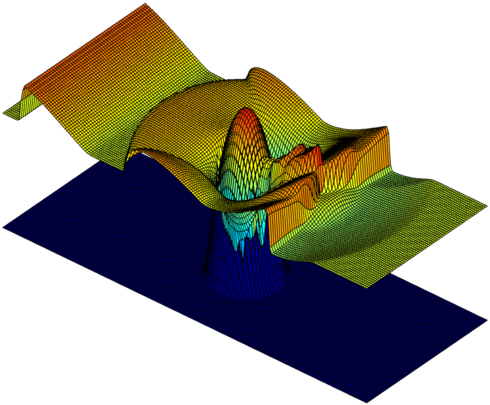

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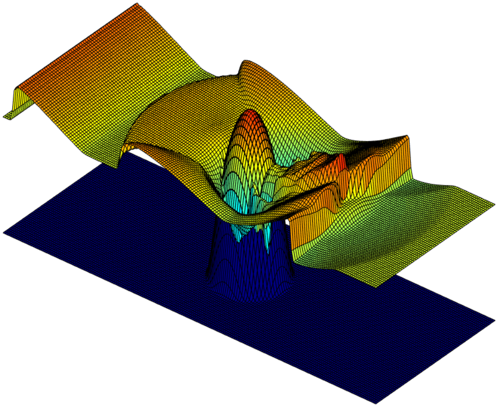

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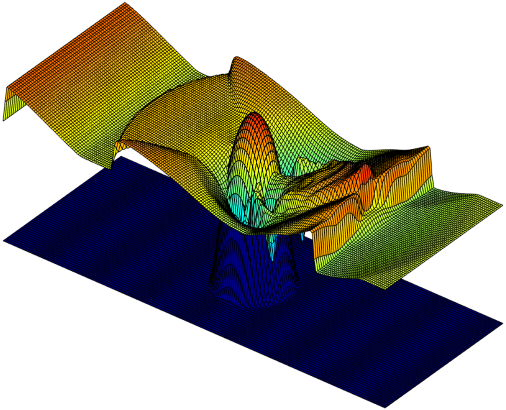

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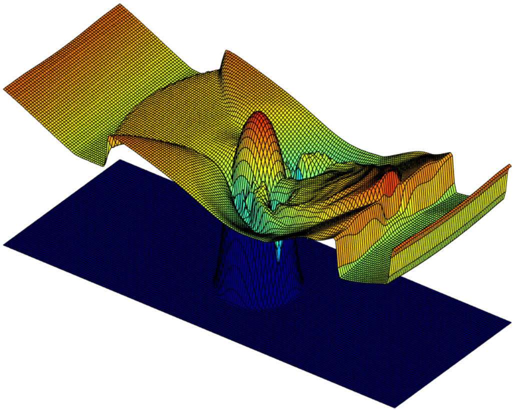

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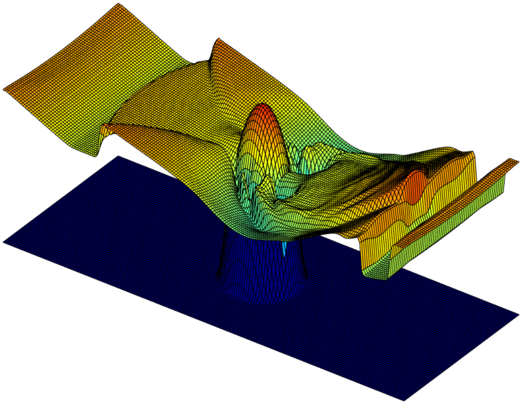

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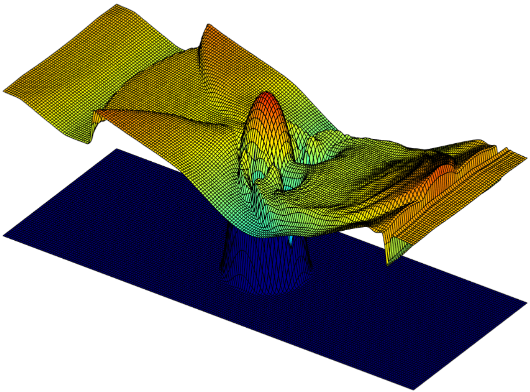

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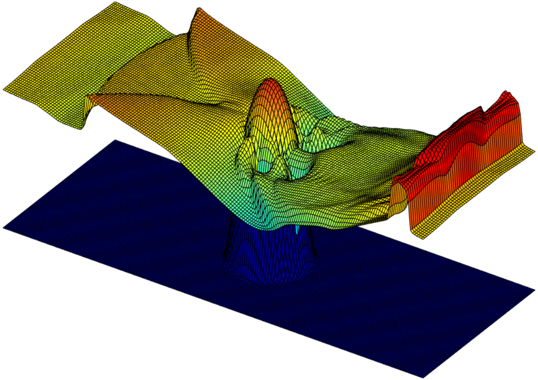

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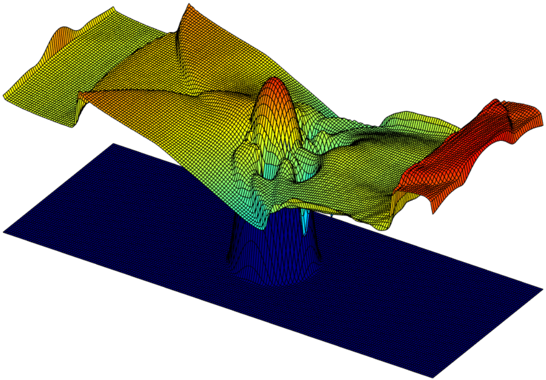

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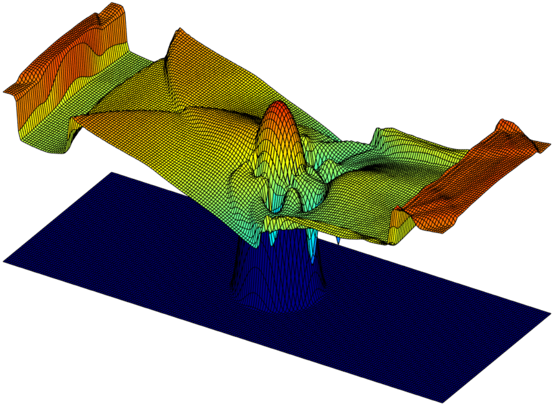

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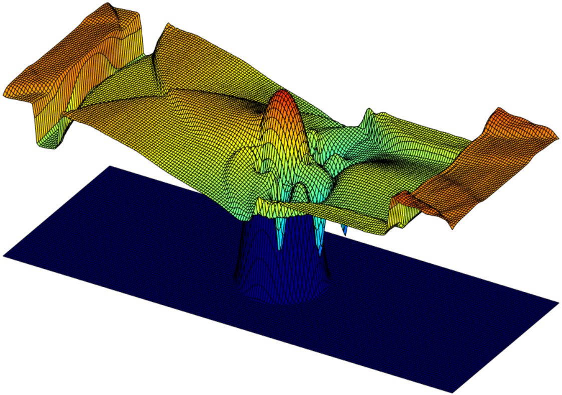

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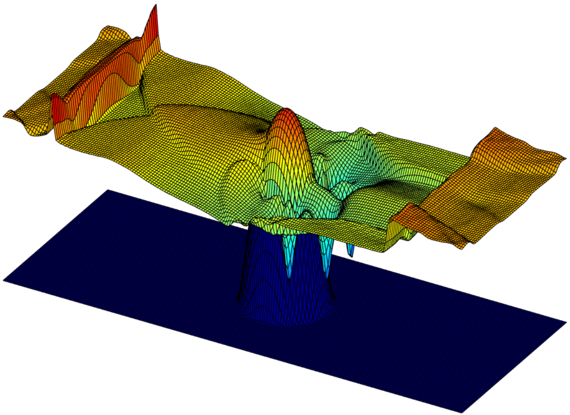

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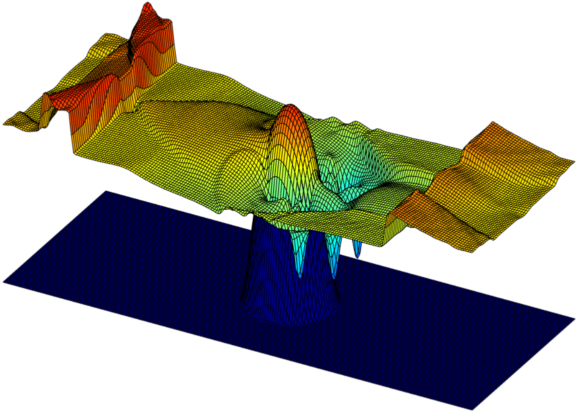

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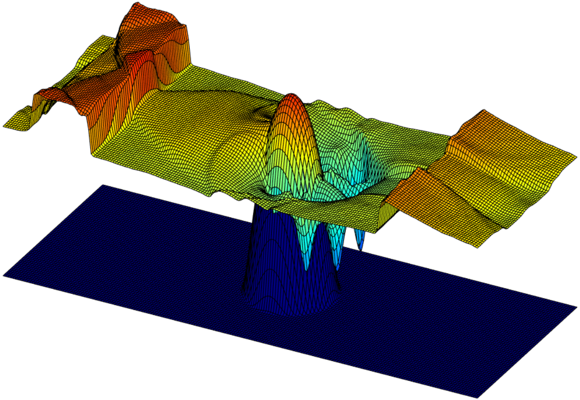

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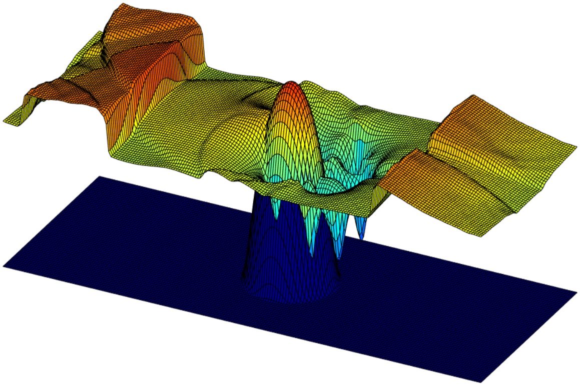

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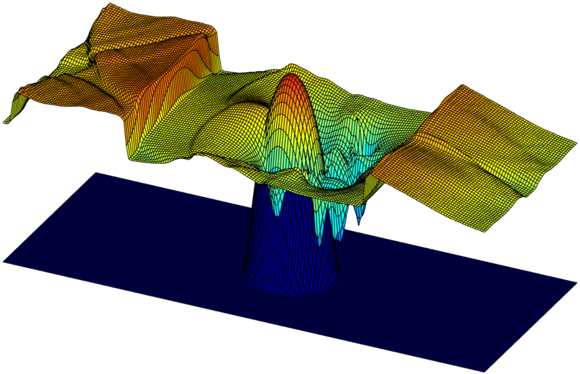

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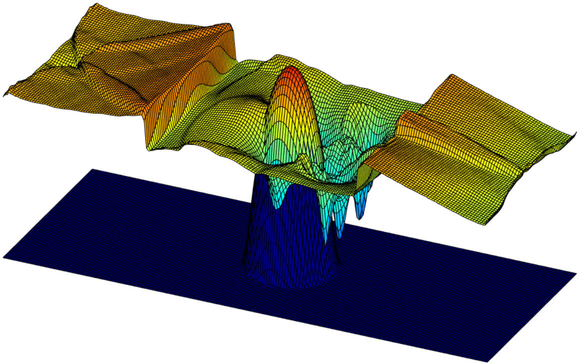

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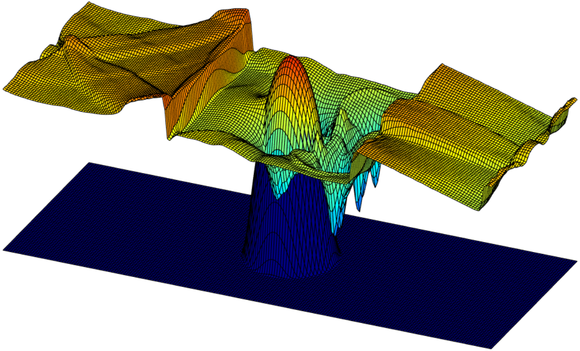

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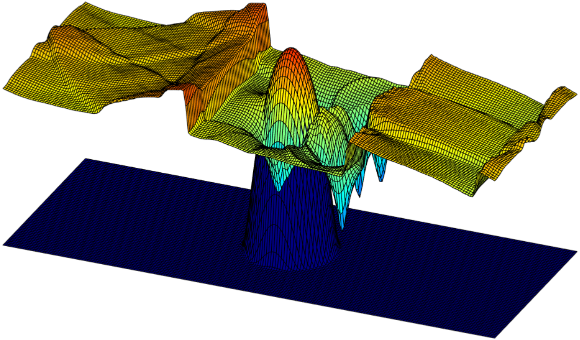

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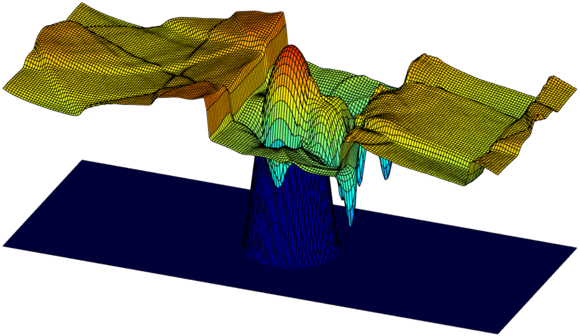

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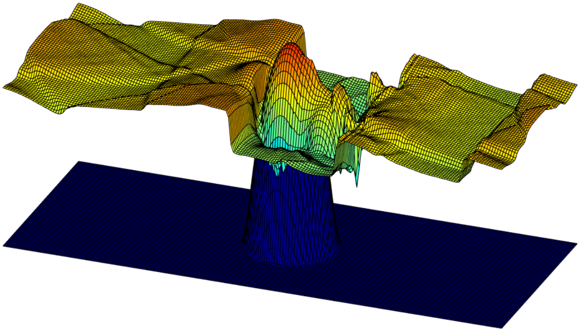

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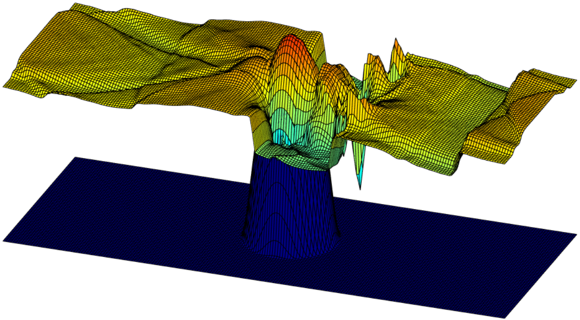

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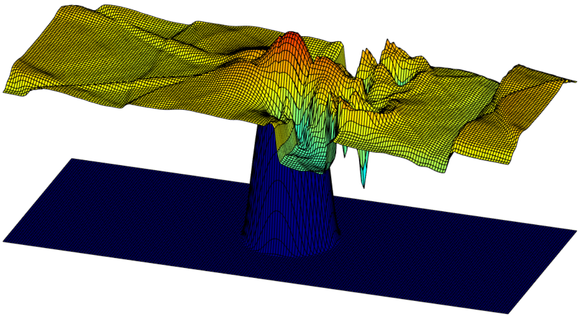

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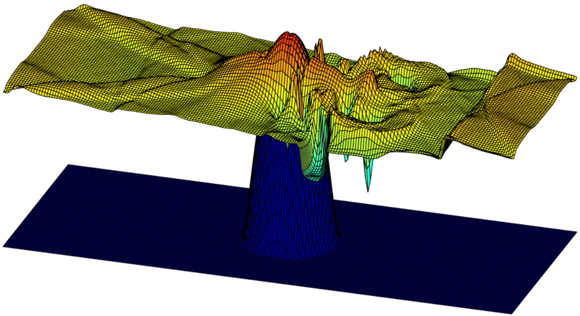

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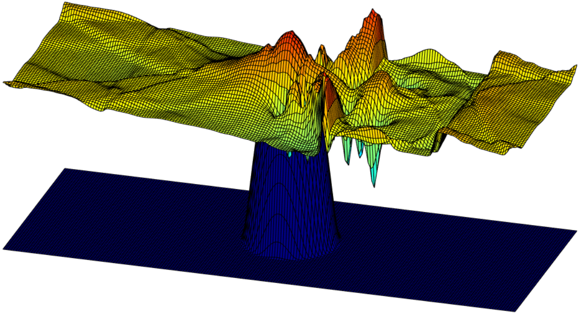

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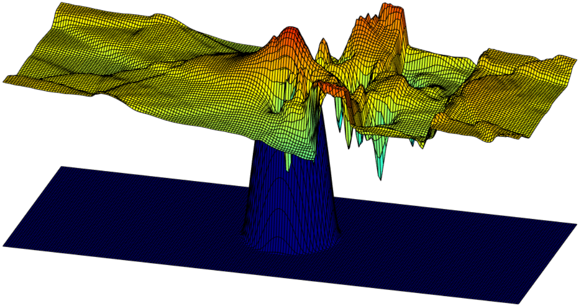

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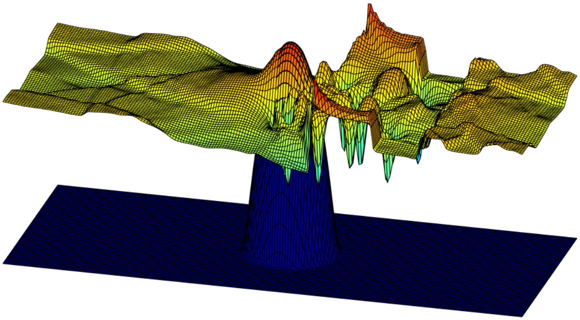

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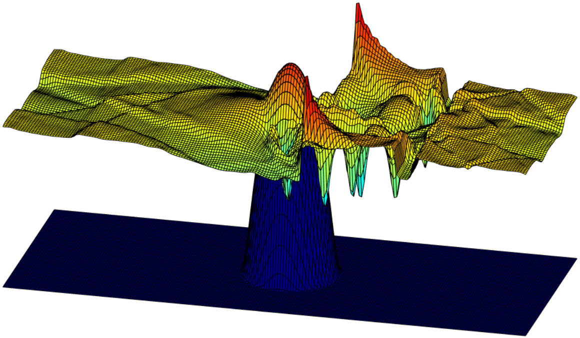

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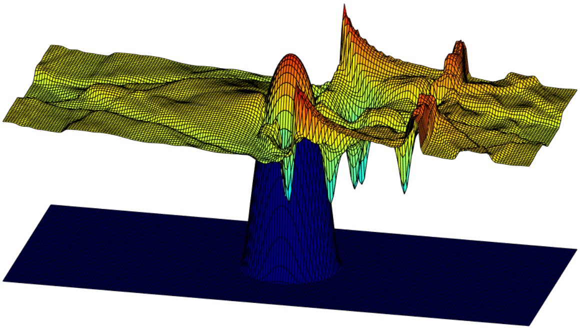

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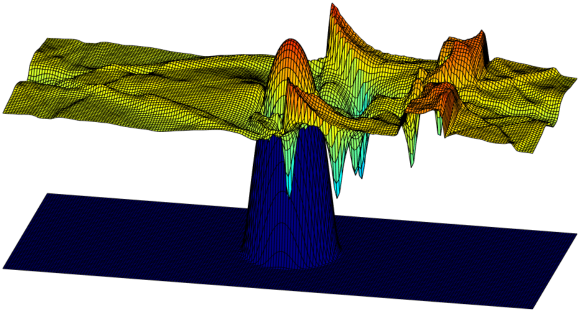

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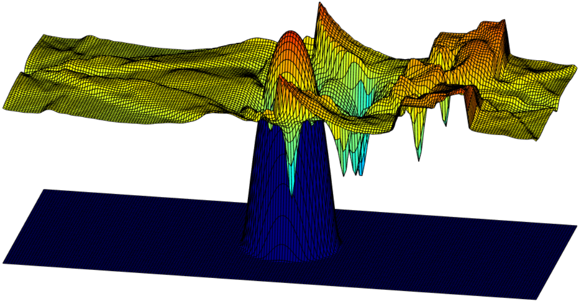

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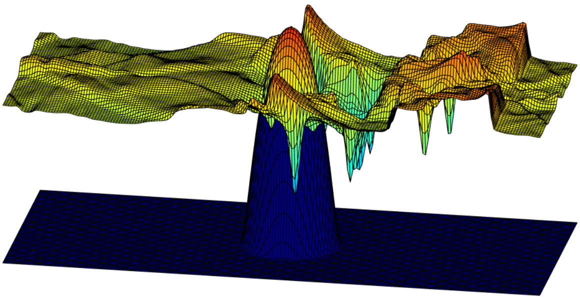

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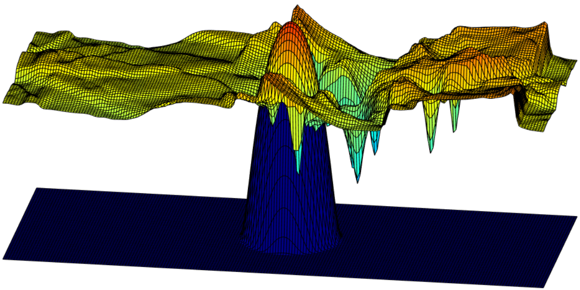

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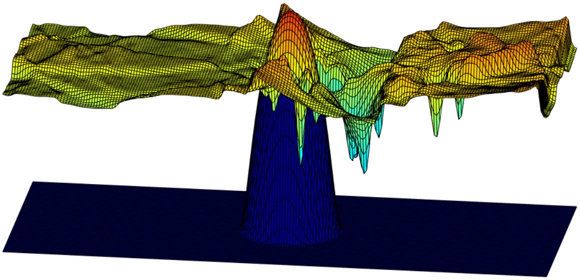

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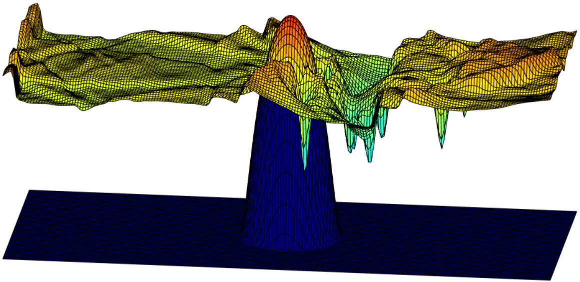

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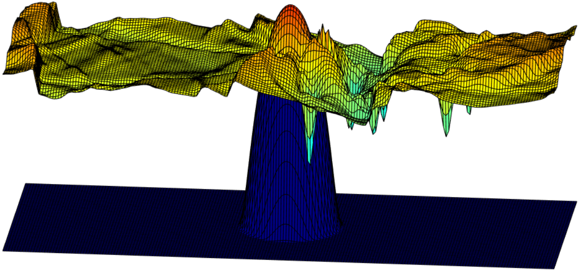

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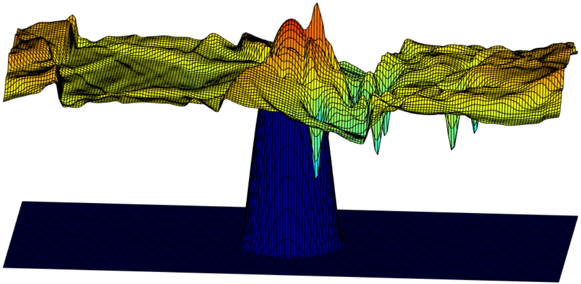

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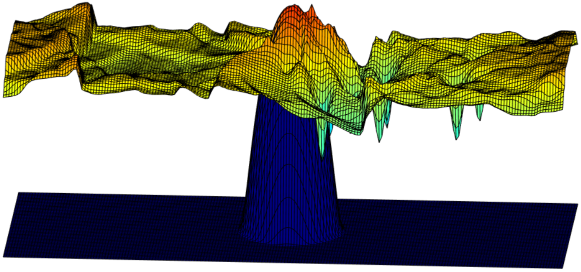

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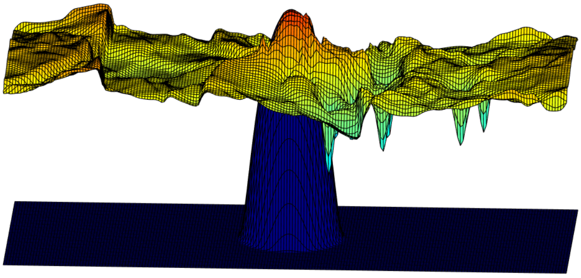

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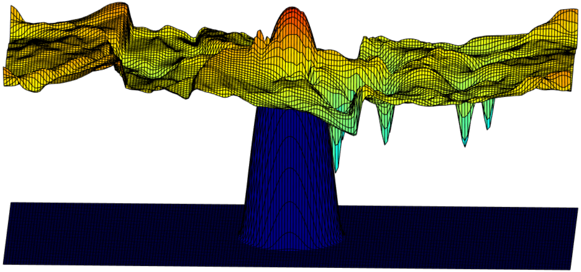

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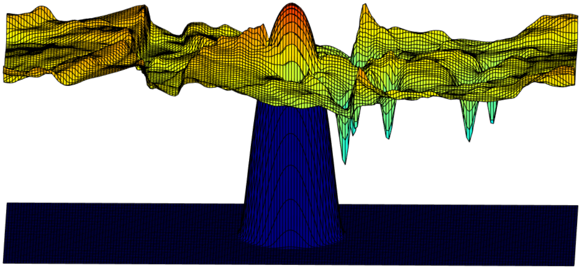

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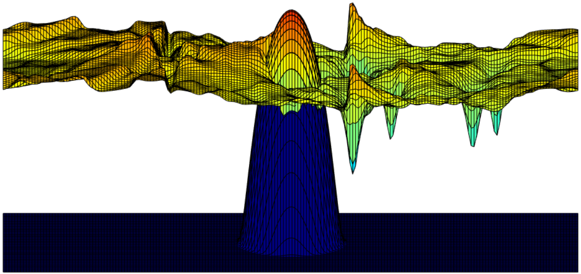

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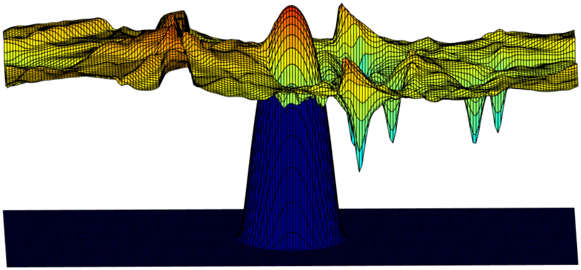

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Summary and perspectives

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• Finite Volume • WENO5 • Positivity Limiter • Production–Destruction System • Modified Patankar DeC • Very Sparse Linear System • Well–Balanced Technique • CFL=1	• Other Systems (Euler) • Other Well–Balancing Techniques • Preservation of Other Equilibria • Preservation of Energy for Smooth Flows • Stability of Modified Patankar

THANK YOU!

Preprint
M. Ciallella, L. Micalizzi, P. Öffner, D. Torlo. An Arbitrary High Order and Positivity Preserving Method for the Shallow Water Equations. arXiv:2110.13509.

Code: github.com/accdavlo/sw-mpdec