

A new efficient explicit Deferred Correction framework:
analysis and applications to hyperbolic PDEs and adaptivity



Davide Torlo*, Lorenzo Micalizzi

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School for Advanced Studies, Trieste, Italy
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Essentially hyperbolic problems:
unconventional numerics, and applications
Ascona - October 2022

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2016

- PhD in hyperbolic PDE field with Rémi



History of residual distribution and DeC

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Spatial Discretizations

- Finite Volume

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Residual Di

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Journal of Computational Physics
Volume 229, Issue 16, 10 August 2010, Pages 5653-5691

Explicit Runge–Kutta residual distribution
schemes for time dependent problems: Second
order case

M. Ricchiuto & R. Abgrall



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Residual Distribution

Physics

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Numerical Mathematics and Advanced Applications ENUMATH 2015 pp 75–86 | Cite as
How to Avoid Mass Matrix for Linear Hyperbolic Problems
[Rémi Abgrall](#) , [Paola Bacigaluppi](#) & [Svetlana Tokareva](#)
Conference paper | [First Online: 11 November 2016](#)
Explicit schemes for order case
[M. Ricchiuto](#) , [R. Abgrall](#) 

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Published: 18 July 2017

High Order Schemes for Hyperbolic Problems Using Globally Continuous Approximation and Avoiding Mass Matrices

R. Abgrall

Journal of Scientific Computing 73, 461–494 (2017) | [Cite this article](#)

643 Accesses | 24 Citations | [Metrics](#)

M. Ricchiuto & R. Abgrall

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Deferred Correction (January 2017)

- Arbitrarily high order

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- Strong Stability Preserving RK

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Residual Distribution

Journal of Computational Physics

Using Globally

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Computers & Mathematics with Applications
Volume 78, Issue 2, 15 July 2019, Pages 274-297
High-order residual distribution scheme for the time-dependent Euler equations of fluid dynamics

Rémi Abgrall , Paola Bacigaluppi , Svetlana Tokareva 

R. Abgrall 

Journal of Scientific Computing

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2016

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Residual Distribution



Computers & Mathematics with Applications
Volume 78, Issue 2, 15 July 2018

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Computational Methods in Science and Engineering

High Order Asymptotic Preserving Deferred Correction Implicit-Explicit Schemes for Kinetic Models

Rémi Abgrall and Davide Torto

<https://doi.org/10.1137/19M128973X>

R. Abgrall

Journal of Scientific Computing

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Residual Distribution

Computers & Mathematics
Volume 78, Issue 2, 15
High-order
Computational Methods in Science and Engineering
High Order Accuracy
[Submitted on 9 Jun 2021]
Relaxation Deferred Correction Methods and their Applications to Residual Distribution Schemes
Rémi Abgrall, Elise Le Mélede, Philipp Öffner, Davide Torlo

Journal of Computational Physics
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Computers & Mathematics

Published: 21 September 2021

Spectral Analysis of Continuous FEM for Hyperbolic PDEs:
Influence of Approximation, Stabilization, and Time-Stepping

Sixtine Michel , Davide Torlo, Mario Ricchiuto & Rémi Abgrall
Journal of Scientific Computing 89, Article number: 31 (2021) | [Cite this article](#)

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Application

Rémi Abgrall, Elise Le Méh
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Computers & Math

Published: 21 September 2021

Spectral Analysis
Influence

Published: 17 February 2021

DeC and ADER: Similarities, Differences and a Unified Framework

Maria Han Veiga , Philipp Öffner & Davide Torlo

[Journal of Scientific Computing](#) 87, Article number: 2 (2021) | [Cite this article](#)

643 Accesses

M. Ricchiuto , R. Abgrall 

- IV
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Table of contents

- ① Introduction to DeC
- ② An efficient Deferred Correction
- ③ Application to PDEs
- ④ Conclusions

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① Introduction to DeC

② An efficient Deferred Correction

③ Application to PDEs

④ Conclusions

Deferred Correction (DeC)

History of DeC

- Original framework for solution of **nonlinear equations**

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- Iterative method for **ODEs** with Taylor expansion
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Explicit, implicit, stability
Dutt, Greengard, Rokhlin (2000)

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- **IMEX** DeC: *Minion (2003)*
- **Operators** based DeC, generalization to many problems: *Abgrall (2017)*

DeC iterations

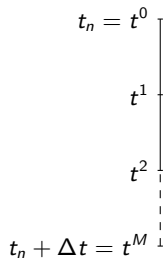
$$\frac{d}{dt}\mathbf{u}(t) = \mathbf{G}(t, \mathbf{u}(t)),$$

DeC iterations

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$$\frac{d}{dt} \mathbf{u}(t) = \mathbf{G}(t, \mathbf{u}(t)), \quad \mathbf{u}_n \approx \mathbf{u}(t_n), \quad \mathbf{u}^m \approx \mathbf{u}(t^m)$$

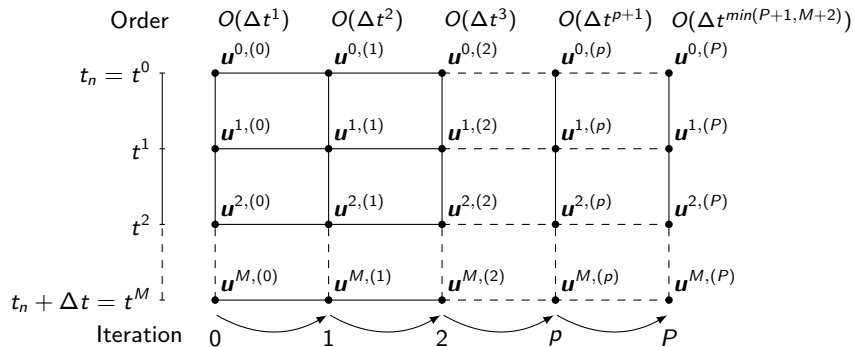


Which sub time nodes?

Equispaced, Gauss-Lobatto

DeC iterations

$$\frac{d}{dt} \mathbf{u}(t) = \mathbf{G}(t, \mathbf{u}(t)), \quad \mathbf{u}_n \approx \mathbf{u}(t_n), \quad \mathbf{u}^m \approx \mathbf{u}(t^m)$$



Which sub time nodes?

Equispaced, Gauss-Lobatto

t^0
 t^1
 t^2
 t^{m-1}
 t^m
 t^M

\mathcal{L}_{Δ}^2 operator

$$\mathcal{L}_{\Delta}^2(\mathbf{u}^0, \dots, \mathbf{u}^M) = \mathcal{L}_{\Delta}^2(\underline{\mathbf{u}}) := \begin{cases} \mathbf{u}^M - \mathbf{u}^0 - \int_{t^0}^{t^M} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 - \int_{t^0}^{t^1} \mathbf{G}(\mathbf{u}(s)) ds \end{cases}$$

- Implicit RK
- Order of accuracy $\geq M + 1$
- Difficult to solve directly

t^0
 t^1
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\mathcal{L}_{Δ}^1 operator

$$\mathcal{L}_{\Delta}^1(\mathbf{u}^0, \dots, \mathbf{u}^M) = \mathcal{L}_{\Delta}^1(\underline{\mathbf{u}}) := \begin{cases} \mathbf{u}^M - \mathbf{u}^0 - \Delta t \beta^M \mathbf{G}(\mathbf{u}^0) \\ \dots \\ \mathbf{u}^1 - \mathbf{u}^0 - \Delta t \beta^1 \mathbf{G}(\mathbf{u}^0) \end{cases}$$

- First order accurate
- Explicit or easy to solve

Deferred Correction

How to combine two methods keeping the accuracy of the second and the stability and simplicity of the first one?

$$\underline{u}^{0,(p)} := \underline{u}(t_n), \quad p = 0, \dots, P,$$

$$\underline{u}^{m,(0)} := \underline{u}(t_n), \quad m = 1, \dots, M$$

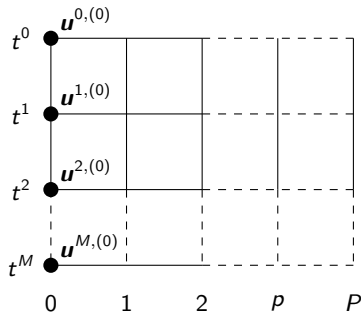
$$\mathcal{L}_\Delta^1(\underline{u}^{(p)}) = \mathcal{L}_\Delta^1(\underline{u}^{(p-1)}) - \mathcal{L}_\Delta^2(\underline{u}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

DeC Theorem

- \mathcal{L}_Δ^1 coercive
- $\mathcal{L}_\Delta^1 - \mathcal{L}_\Delta^2$ Lipschitz

DeC converges and $\min(P, M + 1)$ is the order of accuracy.

- $\mathcal{L}^1(\underline{u}) = 0$, first order accuracy, easily invertible.
- $\mathcal{L}^2(\underline{u}) = 0$, high order $M + 1$.



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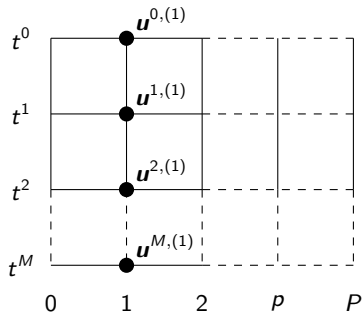
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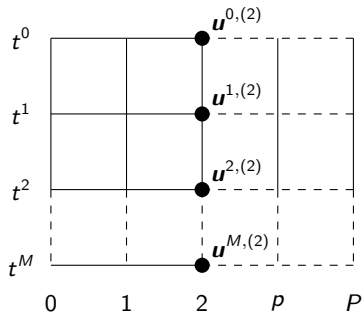
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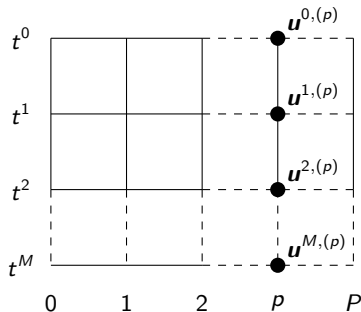
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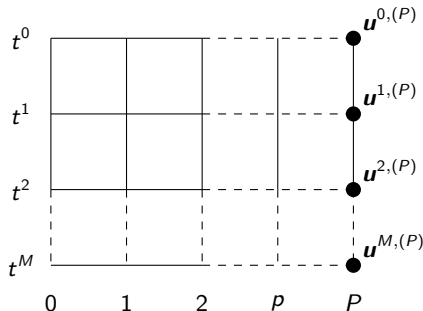
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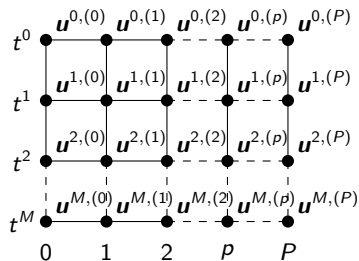


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$$\mathbf{u}^{m,(p)} = \mathbf{u}^0 + \sum_{r=0}^M \theta_r^m \mathbf{G}(t^r, \mathbf{u}^{r,(p-1)}), \quad \forall m = 1, \dots, M, \quad p = 1, \dots, P$$

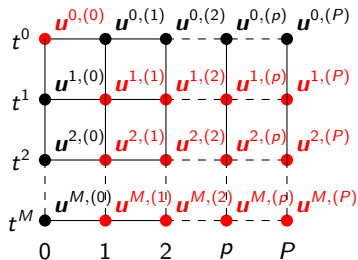
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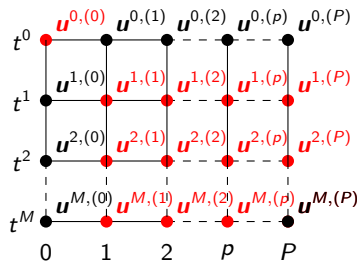
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DeC as RK for ODEs

$$\mathcal{L}_{\Delta}^1(\underline{u}^{(p)}) = \mathcal{L}_{\Delta}^1(\underline{u}^{(p-1)}) - \mathcal{L}_{\Delta}^2(\underline{u}^{(p-1)}) \text{ with } p = 1, \dots, P.$$

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c	\underline{u}^0	$\underline{u}^{(1)}$	$\underline{u}^{(2)}$	$\underline{u}^{(3)}$	\dots	$\underline{u}^{(M-1)}$	$\underline{u}^{(M)}$	A
0	0							\underline{u}^0
$\underline{\beta}_{1:}$	$\underline{\beta}_{1:}$	$\underline{0}$						$\underline{u}^{(1)}$
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\Theta_{1:,1:}$	$\underline{0}$					$\underline{u}^{(2)}$
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{0}$	$\Theta_{1:,1:}$	$\underline{0}$				$\underline{u}^{(3)}$
	\vdots	\vdots		\ddots	\ddots			\vdots
	\vdots	\vdots			\ddots	\ddots		\vdots
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{0}$	\dots	\dots	$\underline{0}$	$\Theta_{1:,1:}$	$\underline{0}$	$\underline{u}^{(M)}$
\underline{b}	$\Theta_{M,0}$	$\underline{0}$	\dots	\dots	\dots	$\underline{0}$	$\Theta_{M,1:}$	$\underline{u}^{M,(M+1)}$

Large costs!

Large costs!

- DeC $S = M \cdot (P - 1) + 1$
 - DeC equi $S = (P - 1)^2 + 1$
 - DeC GLB $S = \left\lceil \frac{P}{2} \right\rceil (P - 1) + 1$

Equispaced

P	M	DeC
2	1	2
3	2	5
4	3	10
5	4	17
6	5	26
7	6	37
8	7	50
9	8	65
10	9	82

Gauss-Lobatto

P	M	DeC
2	1	2
3	2	5
4	2	7
5	3	13
6	3	16
7	4	25
8	4	29
9	5	41
10	5	46

Large costs!

- DeC $S = M \cdot (P - 1) + 1$
 - DeC equi $S = (P - 1)^2 + 1$
 - DeC GLB $S = \left\lceil \frac{P}{2} \right\rceil (P - 1) + 1$

Equispaced

P	M	DeC
2	1	2
3	2	5
4	3	10
5	4	17
6	5	26
7	6	37
8	7	50
9	8	65
10	9	82

Gauss-Lobatto

P	M	DeC
2	1	2
3	2	5
4	2	7
5	3	13
6	3	16
7	4	25
8	4	29
9	5	41
10	5	46

How can we save computational time?

Table of contents

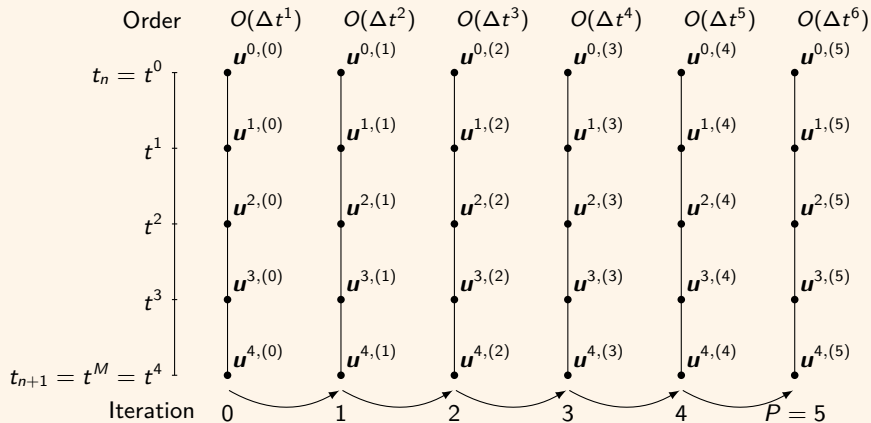
① Introduction to DeC

② An efficient Deferred Correction

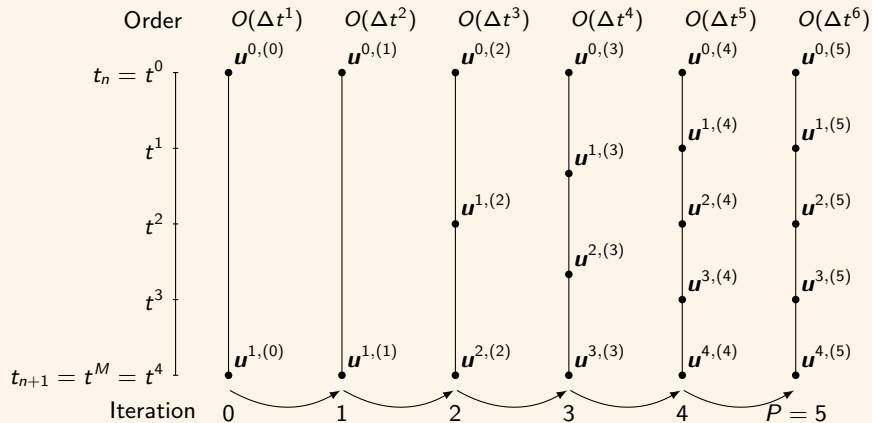
③ Application to PDEs

④ Conclusions

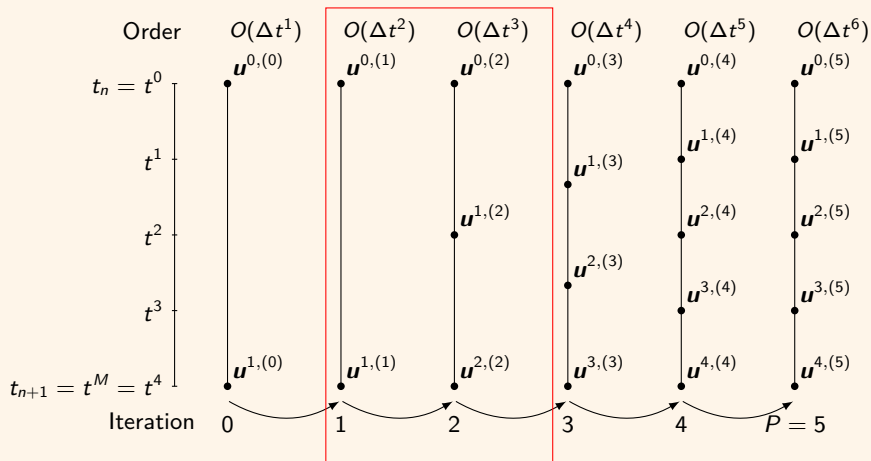
Idea for reduction of stages



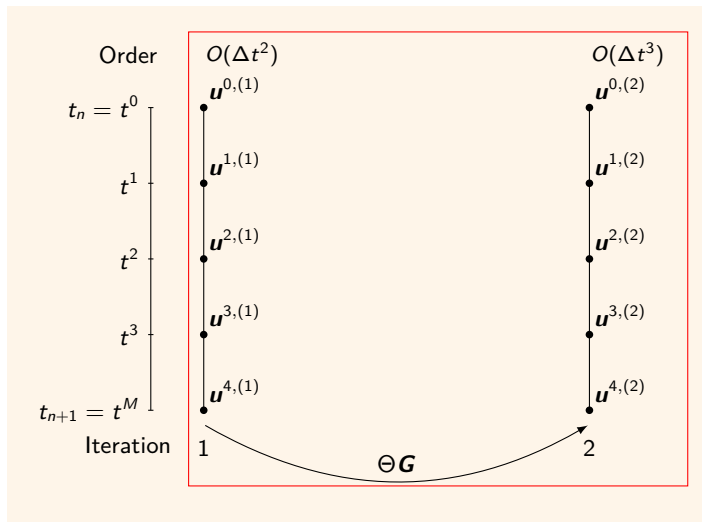
Idea for reduction of stages



Idea for reduction of stages



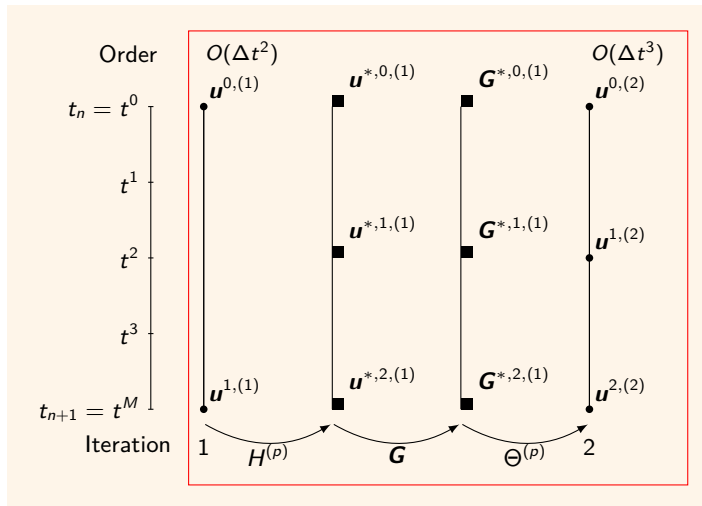
How to communicate between iterations?



DeC

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta G(\underline{u}^{(p-1)})$$

How to communicate between iterations?



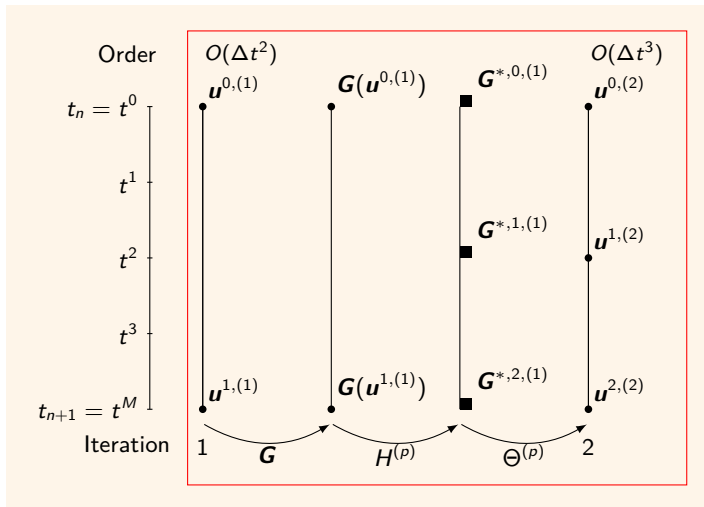
DeC

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta \underline{G}(\underline{u}^{(p-1)})$$

DeCu

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta^{(p)} \underline{G}(H^{(p)} \underline{u}^{(p-1)})$$

How to communicate between iterations?



DeC

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta G(\underline{u}^{(p-1)})$$

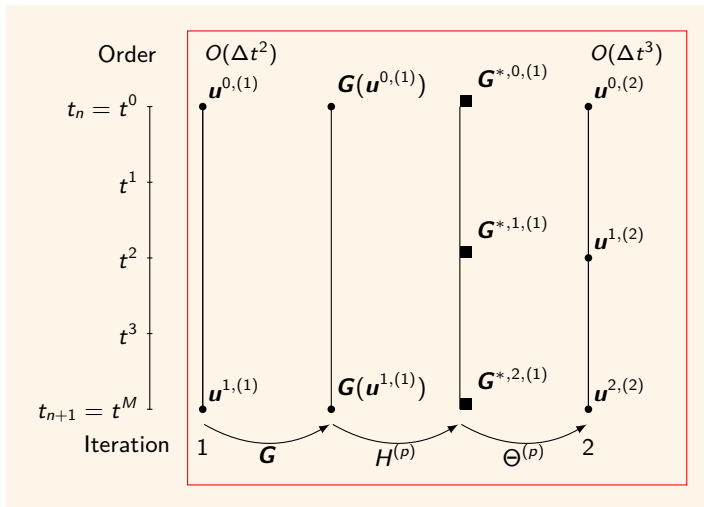
DeCu

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta^{(p)} G(H^{(p)} \underline{u}^{(p-1)})$$

DeCdu

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta^{(p)} H^{(p)} G(\underline{u}^{(p-1)})$$

How to communicate between iterations?



DeC

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta G(\underline{u}^{(p-1)})$$

DeCu

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta^{(p)} G(H^{(p)} \underline{u}^{(p-1)})$$

$$\underline{u}^{*(p)} = \underline{u}^0 + \Delta t H^{(p)} \Theta^{*(p-1)} G(\underline{u}^{*(p-1)})$$

DeCdu

$$\underline{u}^{(p)} = \underline{u}^0 + \Delta t \Theta^{(p)} H^{(p)} G(\underline{u}^{(p-1)})$$

Efficient DeC into RK framework

$$\text{DeC} \quad S = M \cdot (P - 1) + 1$$

\mathbf{c}	\mathbf{u}^0	$\mathbf{u}^{(1)}$	$\mathbf{u}^{(2)}$	$\mathbf{u}^{(3)}$	\dots	$\mathbf{u}^{(M-1)}$	$\mathbf{u}^{(M)}$	A	dim
0	0							\mathbf{u}^0	1
$\underline{\beta}_{1:}$	$\underline{\beta}_{1:}$	$\underline{0}$						$\mathbf{u}^{(1)}$	M
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\Theta_{1:,1:}$	$\underline{0}$					$\mathbf{u}^{(2)}$	M
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{0}$	$\Theta_{1:,1:}$	$\underline{0}$				$\mathbf{u}^{(3)}$	M
	\vdots	\vdots		\ddots	\ddots			\vdots	M
	\vdots	\vdots			\ddots	\ddots		\vdots	M
$\underline{\beta}_{1:}$	$\Theta_{1:,0}$	$\underline{0}$	\dots	\dots	$\underline{0}$	$\Theta_{1:,1:}$	$\underline{0}$	$\mathbf{u}^{(M)}$	M
\mathbf{b}	$\Theta_{M,0}$	$\underline{0}$	\dots	\dots	\dots	$\underline{0}$	$\Theta_{M,1:}$	$\mathbf{u}^{M,(M+1)}$	

Efficient DeC into RK framework

DeCu $S = M \cdot (P - 1) + 1 - \frac{(M-1)(M-2)}{2}$

\mathbf{c}	\mathbf{u}^0	$\underline{\mathbf{u}}^{*(1)}$	$\underline{\mathbf{u}}^{*(2)}$	$\underline{\mathbf{u}}^{*(3)}$	\dots	$\underline{\mathbf{u}}^{*(M-2)}$	$\underline{\mathbf{u}}^{*(M-1)}$	$\underline{\mathbf{u}}^{(M)}$	A	dim
0	0								\mathbf{u}^0	1
$\beta_{\underline{1:}}^{(2)}$	$\beta_{\underline{1:}}^{(2)}$	$\underline{\underline{0}}$							$\underline{\mathbf{u}}^{*(1)}$	2
$\beta_{\underline{1:}}^{(3)}$	$\mathbf{W}_{1:,0}^{(2)}$	$\mathbf{W}_{1:,1:}^{(2)}$	$\underline{\underline{0}}$						$\underline{\mathbf{u}}^{*(2)}$	3
$\beta_{\underline{1:}}^{(4)}$	$\mathbf{W}_{1:,0}^{(3)}$	$\underline{\underline{0}}$	$\mathbf{W}_{1:,1:}^{(3)}$	$\underline{\underline{0}}$					$\underline{\mathbf{u}}^{*(3)}$	4
	\vdots	\vdots		\ddots	\ddots				\vdots	\vdots
	\vdots	\vdots			\ddots	\ddots			\vdots	\vdots
$\beta_{\underline{1:}}^{(M)}$	$\mathbf{W}_{1:,0}^{(M-1)}$	$\underline{\underline{0}}$	\dots	\dots	$\underline{\underline{0}}$	$\mathbf{W}_{1:,1:}^{(M-1)}$	$\underline{\underline{0}}$	$\underline{\underline{0}}$	$\underline{\mathbf{u}}^{*(M-1)}$	M
$\beta_{\underline{1:}}^{(M)}$	$\mathbf{W}_{1:,0}^{(M)}$	$\underline{\underline{0}}$	\dots	\dots	\dots	$\underline{\underline{0}}$	$\mathbf{W}_{1:,1:}^{(M)}$	$\underline{\underline{0}}$	$\underline{\mathbf{u}}^{(M)}$	M
\mathbf{b}	$\mathbf{W}_{M,0}^{(M+1)}$	$\underline{\underline{0}}$	\dots	\dots	\dots	\dots	$\underline{\underline{0}}$	$\mathbf{W}_{M,1:}^{(M+1)}$	$\underline{\mathbf{u}}^{M,(M+1)}$	

$$\mathbf{W}^{(p)} := \begin{cases} \mathbf{H}^{(p)} \Theta^{(p)} \in \mathbb{R}^{(p+2) \times (p+1)}, & \text{if } p = 2, \dots, M-1, \\ \Theta^{(M)} \in \mathbb{R}^{(M+1) \times (M+1)}, & \text{if } p \geq M. \end{cases}$$

Efficient DeC into RK framework

DeCdu $S = M \cdot (P - 1) + 1 - \frac{M(M-1)}{2}$

\mathbf{c}	\mathbf{u}^0	$\underline{\mathbf{u}}^{(1)}$	$\underline{\mathbf{u}}^{(2)}$	$\underline{\mathbf{u}}^{(3)}$	\dots	$\underline{\mathbf{u}}^{(M-2)}$	$\underline{\mathbf{u}}^{(M-1)}$	$\underline{\mathbf{u}}^{(M)}$	A	dim
0	0								\mathbf{u}^0	1
$\beta_{1:}^{(1)}$	$\beta_{1:}^{(1)}$	$\underline{\underline{0}}$							$\underline{\mathbf{u}}^{(1)}$	1
$\beta_{1:}^{(2)}$	$Z_{1:,0}^{(2)}$	$Z_{1:,1:}^{(2)}$	$\underline{\underline{0}}$						$\underline{\mathbf{u}}^{(2)}$	2
$\beta_{1:}^{(3)}$	$Z_{1:,0}^{(3)}$	$\underline{\underline{0}}$	$Z_{1:,1:}^{(3)}$	$\underline{\underline{0}}$					$\underline{\mathbf{u}}^{(3)}$	3
\vdots	\vdots	\vdots		\ddots	\ddots				\vdots	\vdots
\vdots	\vdots	\vdots			\ddots	\ddots			\vdots	\vdots
$\beta_{1:}^{(M-1)}$	$Z_{1:,0}^{(M-1)}$	$\underline{\underline{0}}$	\dots	\dots	$\underline{\underline{0}}$	$Z_{1:,1:}^{(M-1)}$	$\underline{\underline{0}}$	$\underline{\underline{0}}$	$\underline{\mathbf{u}}^{(M-1)}$	$M - 1$
$\beta_{1:}^{(M)}$	$Z_{1:,0}^{(M)}$	$\underline{\underline{0}}$	\dots	\dots	\dots	$\underline{\underline{0}}$	$Z_{1:,1:}^{(M)}$	$\underline{\underline{0}}$	$\underline{\mathbf{u}}^{(M)}$	M
\mathbf{b}	$Z_{M,0}^{(M+1)}$	$\underline{\underline{0}}$	\dots	\dots	\dots	\dots	$\underline{\underline{0}}$	$Z_{M,1:}^{(M+1)}$	$\underline{\mathbf{u}}^{M,(M+1)}$	

$$Z^{(p)} := \begin{cases} \Theta^{(p)} H^{(p-1)} \in \mathbb{R}^{(p+1) \times p}, & \text{if } p = 1, \dots, M, \\ \Theta^{(M)} \in \mathbb{R}^{(M+1) \times (M+1)}, & \text{if } p > M. \end{cases}$$

Computational costs reduction: RK stages

Equispaced

P	M	DeC	DeCu	DeCdu
2	1	2	2	2
3	2	5	5	4
4	3	10	9	7
5	4	17	14	11
6	5	26	20	16
7	6	37	27	22
8	7	50	35	29
9	8	65	44	37
10	9	82	54	46
11	10	101	65	56
12	11	122	77	67
13	12	145	90	79

Gauss-Lobatto

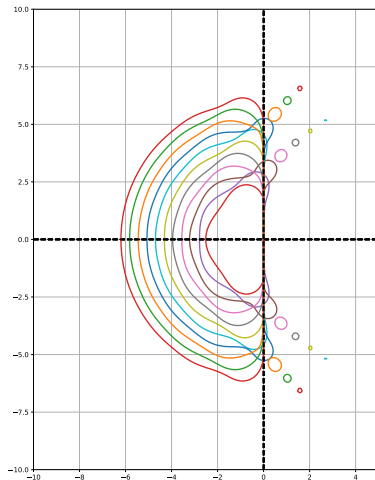
P	M	DeC	DeCu	DeCdu
2	1	2	2	2
3	2	5	5	4
4	2	7	7	6
5	3	13	12	10
6	3	16	15	13
7	4	25	22	19
8	4	29	26	23
9	5	41	35	31
10	5	46	40	36
11	6	61	51	46
12	6	67	57	52
13	7	85	70	64

DeC-DeCu-DeCdu

The **stability function** of DeC, DeCu, DeCdu of order P for any nodes distribution is

$$R(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^P}{P!}.$$

DeC, DeCu, DeCdu

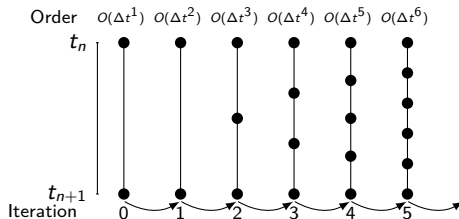


How can we exploit the increasing order of accuracy?

How can we exploit the increasing order of accuracy?

Adaptive order DeC

- Set tolerance ε
- Check at each iteration if $\|\underline{\mathbf{u}}^{(p)} - \underline{\mathbf{u}}^{(p-1)}\| < \varepsilon$
- Stop at a certain order when tolerance is reached



How can we exploit the increasing order of accuracy?

Adaptive order DeC

- Set tolerance ε
- Check at each iteration if $\|\underline{u}^{(p)} - \underline{u}^{(p-1)}\| < \varepsilon$
- Stop at a certain order when tolerance is reached



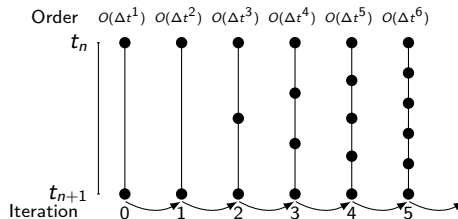
Saving on useless iterations



Reach the needed order for tolerance



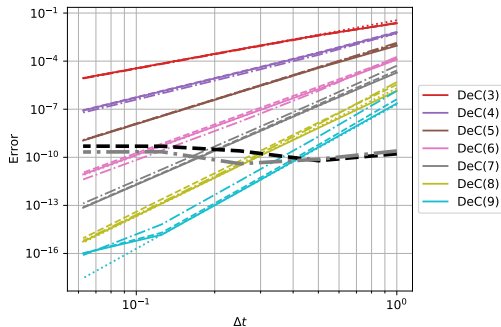
Sub-optimal (waste of few stages)



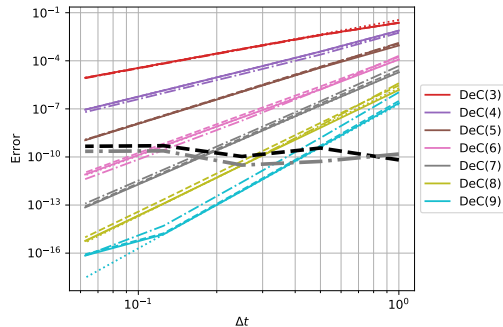
ODE test: Vibrating system

$$my'' + ry' + ky = F \cos(\Omega t + \varphi), \quad y(0) = A, \quad y'(0) = B.$$

Equispaced



Gauss-Lobatto

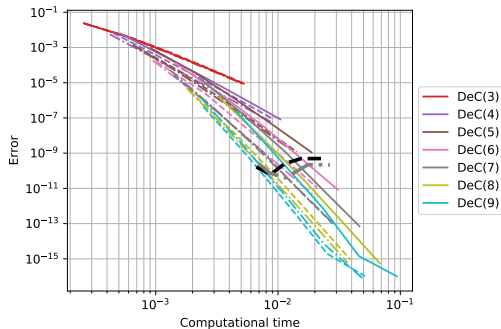


DeC —, DeCu — —, DeCdu — · —, adaptive in grey/black

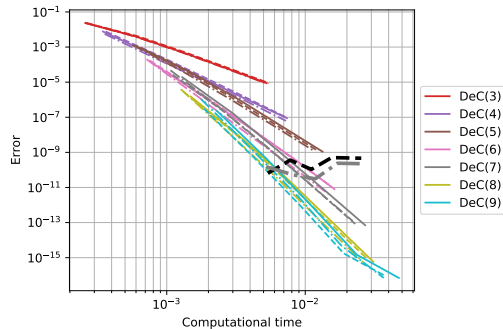
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$$my'' + ry' + ky = F \cos(\Omega t + \varphi), \quad y(0) = A, \quad y'(0) = B.$$

Equispaced



Gauss-Lobatto

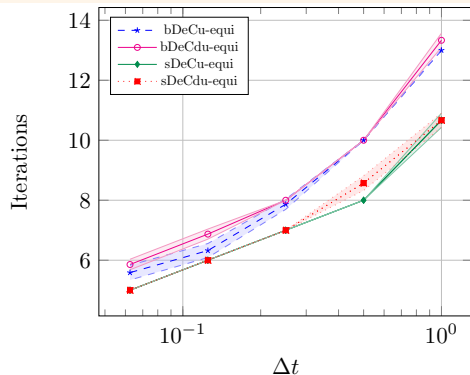


DeC —, DeCu — —, DeCdu — · —, adaptive in grey/black

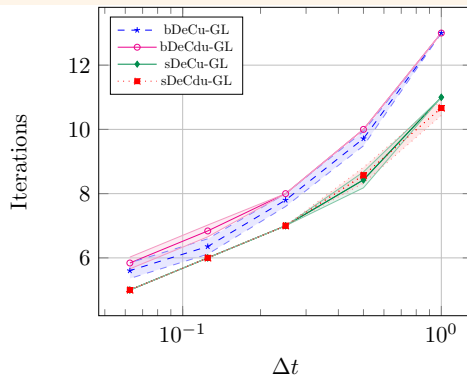
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Equispaced



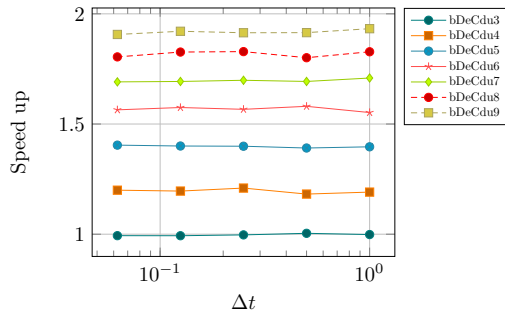
Gauss-Lobatto



ODE test: Vibrating system

$$my'' + ry' + ky = F \cos(\Omega t + \varphi), \quad y(0) = A, \quad y'(0) = B.$$

Equispaced



Gauss-Lobatto

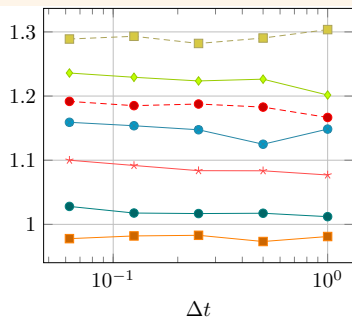


Table of contents

① Introduction to DeC

② An efficient Deferred Correction

③ Application to PDEs

④ Conclusions

Residual Distribution (RD)

- Originally somehow Finite Volume
- **Finite Element**
- Runge Kutta + Mass matrix correction (Rémi + Mario)
- DeC + RD (Rémi 2017)

$$\mathcal{L}_{\Delta}^2$$

$$\mathcal{L}_{\Delta,i}^{2,m}(\mathbf{u}) := \int_{\Omega} \varphi_i \varphi_j dx (u_j^m - u_j^0) + \Delta t \sum_{r=0}^M \theta_r^m \sum_K \Phi_K^i(u^r)$$

RD setting

- $\partial_t u + \nabla \cdot F(u) = 0$
- $V_h = \{u \in \mathcal{C}(\Omega) : u|_K \in \mathbb{P}_M\}$
- $\Phi_K(u) = \int_K \nabla \cdot F(u) dx$
- $\Phi_K^i(u) = \int_K \varphi_i(x) \nabla \cdot F(u) dx + \text{ST}_i(u)$
- NOT method of lines

$$\mathcal{L}_{\Delta}^1$$

$$\mathcal{L}_{\Delta,i}^{1,m}(\mathbf{u}) := \int_{\Omega} \varphi_i dx (u_i^m - u_i^0) + \Delta t \beta^m \sum_K \Phi_K^i(u^0)$$

$$\underbrace{\int_K \varphi_i dx (u_i^{m,(p)} - u_i^{m,(p-1)})}_{\mathcal{L}_{\Delta,i}^{1,m}(u^{(p)}) - \mathcal{L}_{\Delta,i}^{1,m}(u^{(p-1)})} = \underbrace{\int_K \varphi_i \varphi_j dx (u_j^{m,(p)} - u_j^0) + \Delta t \sum_{r=0}^M \theta_r^m \sum_K \Phi_K^i(u^{r,(p-1)})}_{\mathcal{L}_{\Delta,i}^{2,m}(u^{(p-1)})}$$

DeCu for RD

DeC for RD

$$\underbrace{\int_K \varphi_i dx (u_i^{m,(p)} - u_i^{m,(p-1)})}_{\mathcal{L}_{\Delta,i}^{1,m}(u^{(p)}) - \mathcal{L}_{\Delta,i}^{1,m}(u^{(p-1)})} = \underbrace{\int_K \varphi_i \varphi_j dx (u_j^{m,(p)} - u_j^0) + \Delta t \sum_{r=0}^M \theta_r^m \sum_K \Phi_K^i(u^{r,(p-1)})}_{\mathcal{L}_{\Delta,i}^{2,m}(u^{(p-1)})}$$

DeCu for RD

$$\underbrace{\int_K \varphi_i dx (u_i^{m,(p)} - \mathbf{u}_i^{*,m,(p-1)})}_{\mathcal{L}_{\Delta,i}^{1,m}(u^{(p)}) - \mathcal{L}_{\Delta,i}^{1,m}(\mathbf{u}^{*,(p-1)})} = \underbrace{\int_K \varphi_i \varphi_j dx (\mathbf{u}_j^{*,m,(p-1)} - u_j^0) + \Delta t \sum_{r=0}^M \theta_r^m \sum_K \Phi_K^i(\mathbf{u}^{*,r,(p-1)})}_{\mathcal{L}_{\Delta,i}^{2,m}(\mathbf{u}^{*,(p-1)})}$$

DeCu for RD

DeC for RD

$$\underbrace{\int_K \varphi_i dx (u_i^{m,(p)} - u_i^{m,(p-1)})}_{\mathcal{L}_{\Delta,i}^{1,m}(u^{(p)}) - \mathcal{L}_{\Delta,i}^{1,m}(u^{(p-1)})}} = \underbrace{\int_K \varphi_i \varphi_j dx (u_j^{m,(p)} - u_j^0) + \Delta t \sum_{r=0}^M \theta_r^m \sum_K \Phi_K^i(u^{r,(p-1)})}_{\mathcal{L}_{\Delta,i}^{2,m}(u^{(p-1)})}$$

DeCu for RD

$$\underbrace{\int_K \varphi_i dx (u_i^{m,(p)} - u_i^{*,m,(p-1)})}_{\mathcal{L}_{\Delta,i}^{1,m}(u^{(p)}) - \mathcal{L}_{\Delta,i}^{1,m}(u^{*,(p-1)})}} = \underbrace{\int_K \varphi_i \varphi_j dx (u_j^{*,m,(p-1)} - u_j^0) + \Delta t \sum_{r=0}^M \theta_r^m \sum_K \Phi_K^i(u^{*,r,(p-1)})}_{\mathcal{L}_{\Delta,i}^{2,m}(u^{*,(p-1)})}$$

Computational cost

- Depends on update evaluation, less on flux evaluations
- DeC $C \approx (P-1)M + 1$
- DeCu $C \approx (P-1)M + 1 - \frac{M(M-1)}{2}$

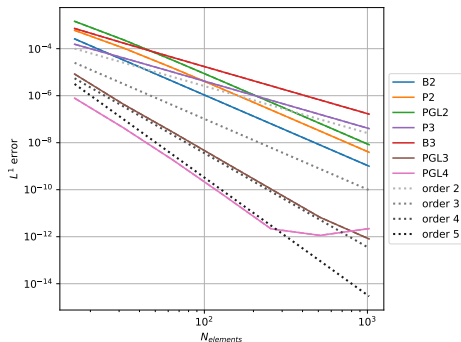
Test PDE: linear advection equation

$$\begin{cases} \partial_t u + \partial_x u = 0 \\ u(0, x) = \cos(2\pi x) \end{cases}$$

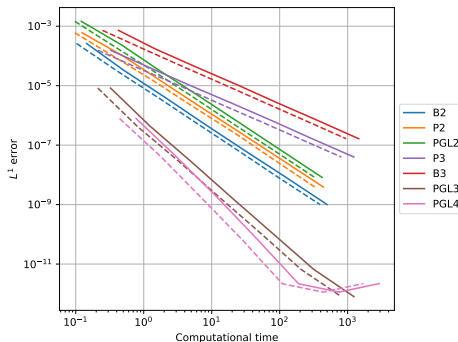
DeC —

DeCu ---

Convergence



Computational Time



Test PDE: linear advection equation

$$\begin{cases} \partial_t u + \partial_x u = 0 \\ u(0, x) = \cos(2\pi x) \end{cases}$$

Speed up

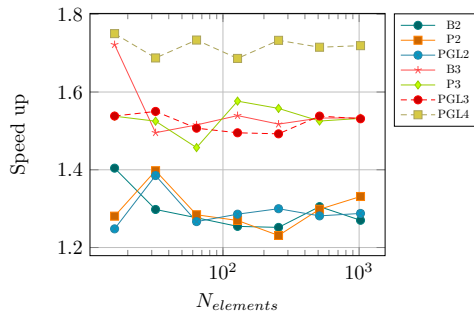


Table of contents

- ① Introduction to DeC
- ② An efficient Deferred Correction
- ③ Application to PDEs
- ④ Conclusions

Summary

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- Increasing order of reconstruction with iterations
- Great Speed up
- Not too big implementation in a DeC code (Remi's birthday present)
- Adaptive with tolerance
- DeC and RD for PDEs

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Perspectives

- Increasing spatial discretizations order
- IMEX
- ADER

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Perspectives

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THANK YOU!

Preprint

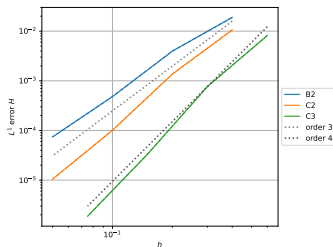
L. Micalizzi, D. Torlo. A new efficient explicit Deferred Correction framework: analysis and applications to hyperbolic PDEs and adaptivity. [arXiv:2210.02976](https://arxiv.org/abs/2210.02976).

Test PDE: shallow water equations

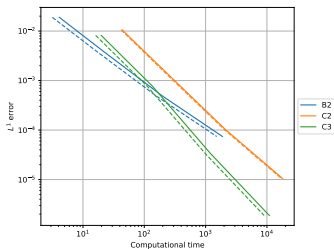
$$\begin{cases} \partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + \frac{g}{2} h^2 \end{pmatrix} = 0 \\ \text{IC} = \text{moving vortex} \end{cases}$$

bDeC —
bDeCu —

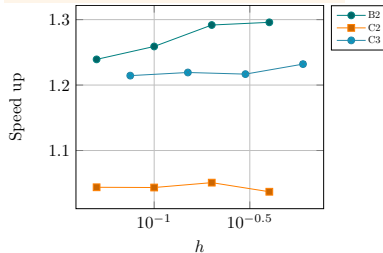
Convergence



Computational Time




Speed Up



Which sub-time-interval?

Big DeC (bDeC)

$$\mathcal{L}_{\Delta}^2(\mathbf{u}^0, \dots, \mathbf{u}^M) = \mathcal{L}_{\Delta}^2(\underline{\mathbf{u}}) :=$$
$$\begin{cases} \mathbf{u}^M - \mathbf{u}^0 - \int_{t^0}^{t^M} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^m - \mathbf{u}^0 - \int_{t^0}^{t^m} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 - \int_{t^0}^{t^1} \mathbf{G}(\mathbf{u}(s)) ds \end{cases}$$

 Parallel subtimesteps

Many DeCs

Which sub-time-interval?

Small DeC (sDeC)

t^0

t^1

t^2

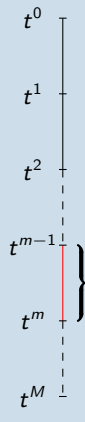
t^{m-1}


t^m

t^M

$\mathcal{L}_{\Delta}^2(\mathbf{u}^0, \dots, \mathbf{u}^M) = \mathcal{L}_{\Delta}^2(\underline{\mathbf{u}}) :=$

$$\begin{cases} \mathbf{u}^M - \mathbf{u}^{M-1} - \int_{t^{M-1}}^{t^M} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^m - \mathbf{u}^{m-1} - \int_{t^{m-1}}^{t^m} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 - \int_{t^0}^{t^1} \mathbf{G}(\mathbf{u}(s)) ds \end{cases}$$

 Serial subtimesteps

 More accurate

Many DeCs



Which sub-time-interval?

Small DeC (sDeC)

Diagram illustrating the Small DeC (sDeC) approach. A vertical timeline shows time intervals from t^0 to t^M . The interval $[t^{m-1}, t^m]$ is highlighted in red and labeled with a brace. The corresponding equation for this interval is:

$$\mathcal{L}_{\Delta}^2(\mathbf{u}^0, \dots, \mathbf{u}^M) = \mathcal{L}_{\Delta}^2(\underline{\mathbf{u}}) := \begin{cases} \mathbf{u}^M - \mathbf{u}^{M-1} - \int_{t^{M-1}}^{t^M} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^m - \mathbf{u}^{m-1} - \int_{t^{m-1}}^{t^m} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 - \int_{t^0}^{t^1} \mathbf{G}(\mathbf{u}(s)) ds \end{cases}$$

Legend:

-  Serial subtimesteps
-  More accurate



Which sub-time-nodes?

Equispaced

Diagram illustrating the Equispaced approach. A vertical timeline shows time intervals from t^0 to t^9 . The corresponding equation for this interval is:

$$\mathcal{L}_{\Delta}^2(\mathbf{u}^0, \dots, \mathbf{u}^M) = \mathcal{L}_{\Delta}^2(\underline{\mathbf{u}}) := \begin{cases} \mathbf{u}^M - \mathbf{u}^{M-1} - \int_{t^{M-1}}^{t^M} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^m - \mathbf{u}^{m-1} - \int_{t^{m-1}}^{t^m} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 - \int_{t^0}^{t^1} \mathbf{G}(\mathbf{u}(s)) ds \end{cases}$$

Legend:

-  Easy to implement
-  Accuracy $M + 1$

Many DeCs

Which sub-time-interval?

Small DeC (sDeC)

Diagram illustrating the sub-time-interval for the Small DeC (sDeC) method. A vertical axis shows time steps from t^0 to t^M . A red bracket highlights the interval from t^{m-1} to t^m , indicating serial subtime steps.

$$\mathcal{L}_{\Delta}^2(\mathbf{u}^0, \dots, \mathbf{u}^M) = \mathcal{L}_{\Delta}^2(\underline{\mathbf{u}}) := \begin{cases} \mathbf{u}^M - \mathbf{u}^{M-1} - \int_{t^{M-1}}^{t^M} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^m - \mathbf{u}^{m-1} - \int_{t^{m-1}}^{t^m} \mathbf{G}(\mathbf{u}(s)) ds \\ \vdots \\ \mathbf{u}^1 - \mathbf{u}^0 - \int_{t^0}^{t^1} \mathbf{G}(\mathbf{u}(s)) ds \end{cases}$$

Serial subtime steps (Red sad face icon)

More accurate (Green happy face icon)

Which sub-time-nodes?

Equispaced

Diagram illustrating sub-time-nodes for the Equispaced method. A vertical axis shows time steps from t^0 to t^9 . The nodes are equispaced.

Easy to implement (Green happy face icon)

Accuracy $M + 1$ (Red sad face icon)

Gauss-Lobatto

Diagram illustrating sub-time-nodes for the Gauss-Lobatto method. A vertical axis shows time steps from t^0 to t^9 . The nodes are Gauss-Lobatto nodes.

Accuracy $2M$ (Green happy face icon)

Less standard polynomials in time (Red sad face icon)