- PROBLEMI IPERBOLICI NON LINEARI

- $\partial_t u + \partial_x f(u) = 0$ $\qquad f$ CONVESSO

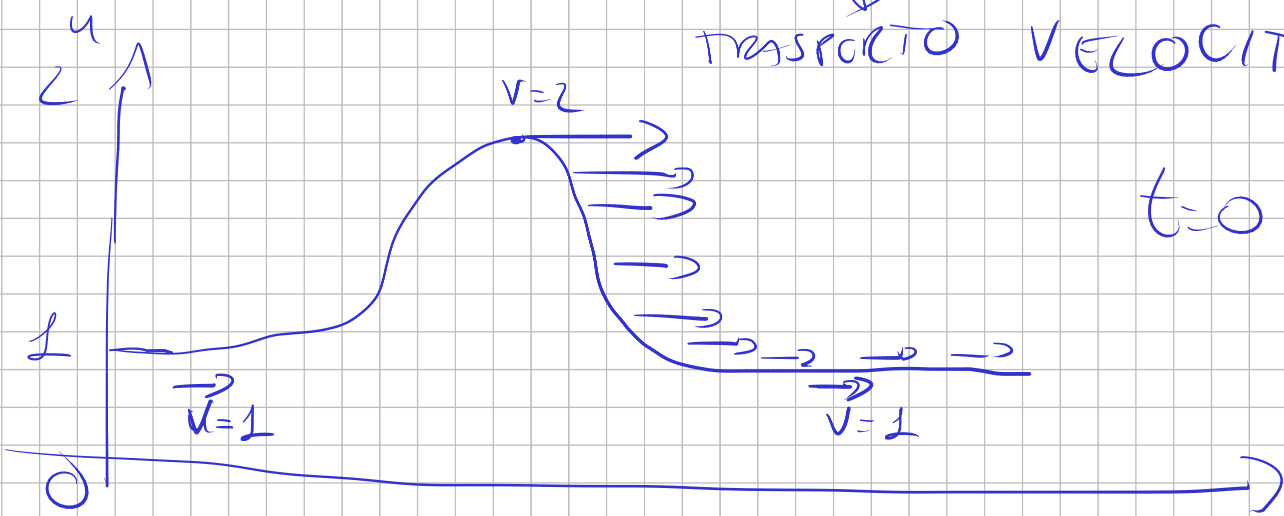
$u : \mathbb{R} \to \mathbb{R}$ $\qquad f'' > 0$

- $f(u) = \dfrac{u^2}{2}$ $\qquad$ EQUAZIONE DI BURGERS

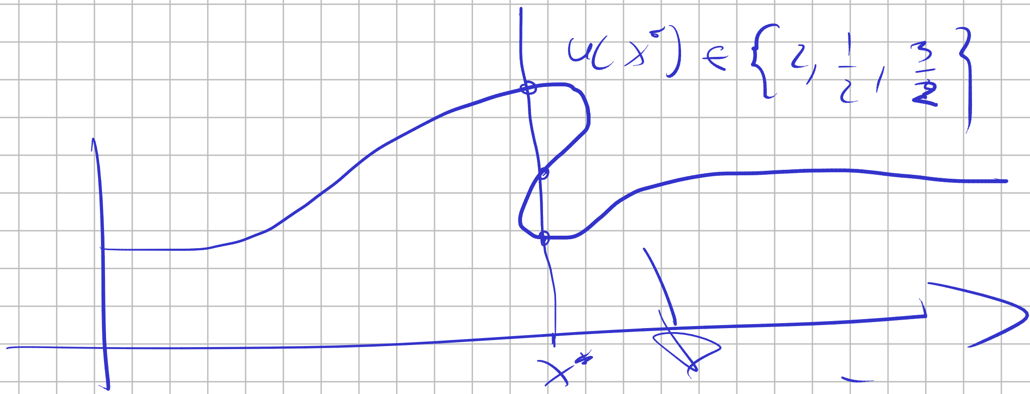
$$\partial_t (u) + \partial_x \frac{u^2}{2} = 0$$

$\rightsquigarrow$ APPROX. DI ONDE (ACQUA)
DI NAVIER-STOKES

Su $u \in C^1$ $\qquad \partial_t u + \underbrace{u}_{\text{TRASPORTO VELOCITÀ} = u} \partial_x u = 0$

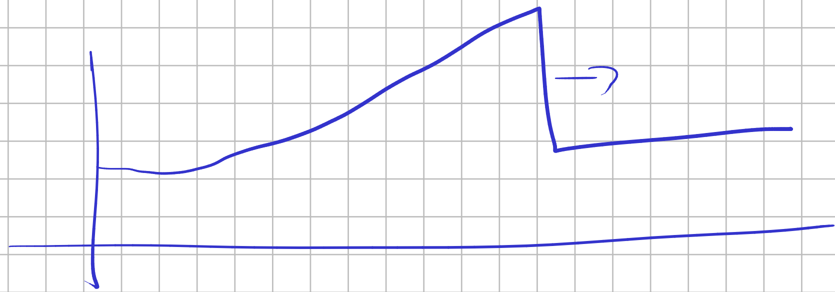


$t = 0$



$u(x^*) \in \left\{2, \frac{1}{2}, \frac{3}{2}\right\}$ $\qquad t = 1$

NON È UNA FUNZIONE



DISCONTINUITÀ

=> CASO REGOLARE AGGIUNGIAMO VISCOSITÀ

$$\partial_t u^\nu + u \partial_x u^\nu = \underbrace{\nu}_{\in \mathbb{R}^+} \partial_{xx} u^\nu$$

$u^\nu \in C^\infty$ $\quad$ u^ν è MOLTO REGOLARE

$\nu > 0$ $\quad$ $\exists !$ SOLUZIONE

NON VISCOSO

$$\partial_t u + u \partial_x u = 0$$

u costante sulle caratteristiche $\underline{x(t)}$

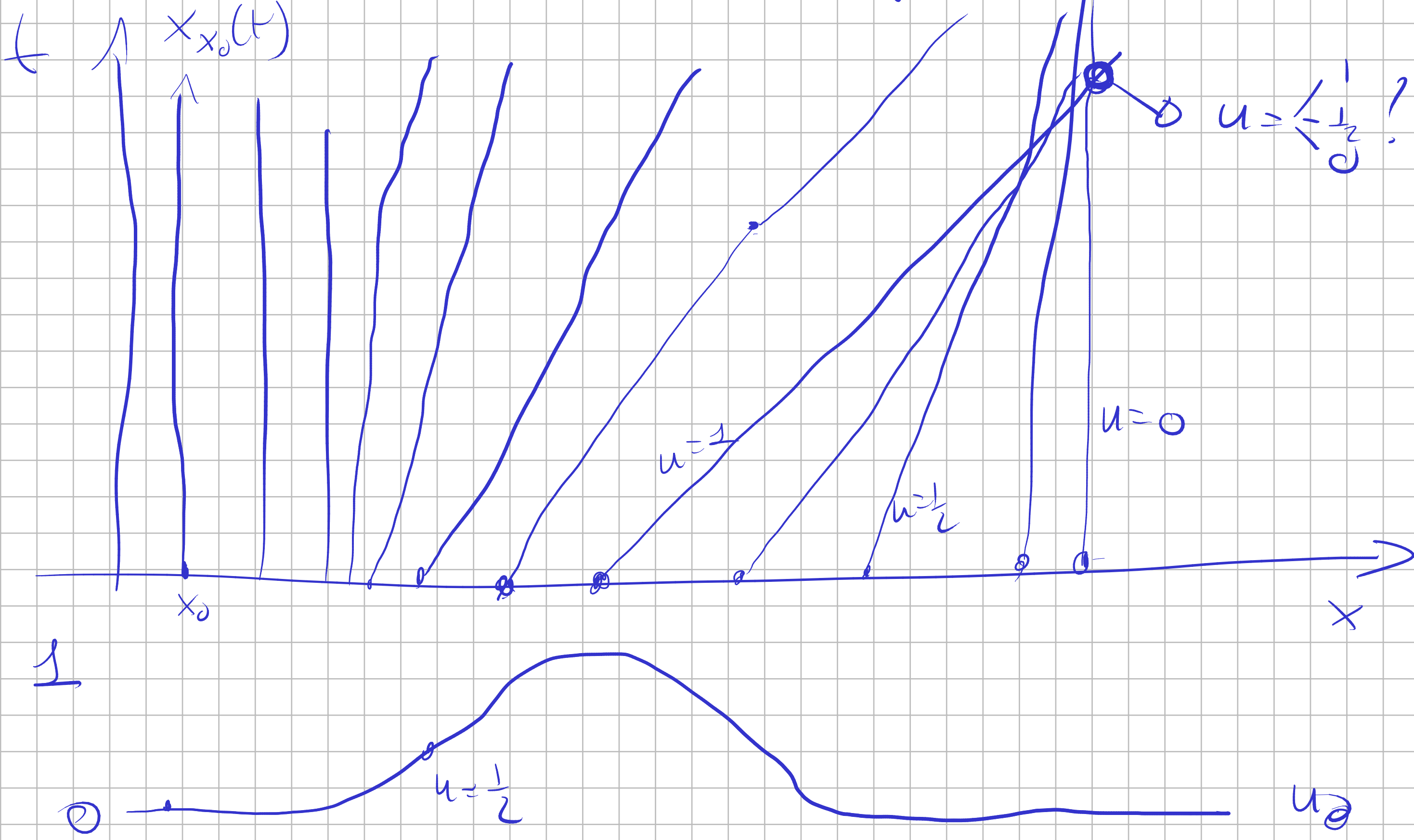
$$\frac{d}{dt} u(t, x(t)) = 0$$

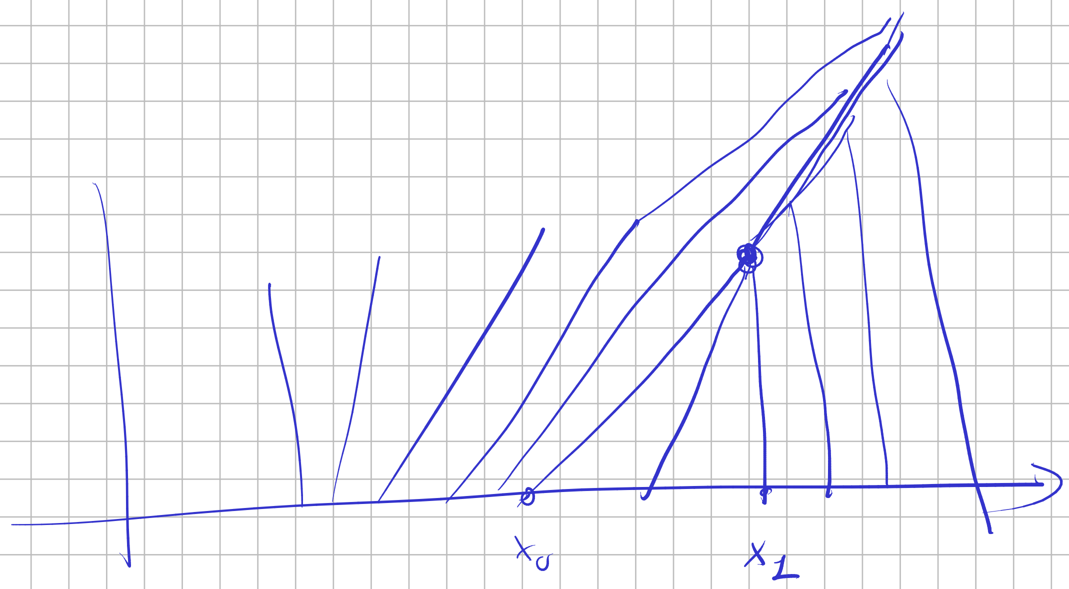
$$\to \underline{\partial_t x(t)} = \underline{u}(t, x(t))$$

$$u(t_0, x_0) = u_0(x_0) \qquad t_0 = 0$$

$$x(t) = u_0(x_0) \cdot t + x_0$$

CARATTERISTICHE



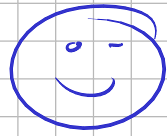


ESERCIZIO

$$1)\begin{cases} x_{x_0}(t) = u_0(x_0)\,t + x_0 \\ x_{x_1}(t) = u_0(x_1)\,t + x_1 \end{cases}$$

$$t^* = \frac{-1}{\min_x u_0'(x)} \longrightarrow 2) \text{ calcola quando } \partial_x u(t,x) \to \infty$$

SOLUZIONE VISCOUS BURGERS



$$\lim_{\nu \to 0} u^\nu = u$$ SOLUZIONE DEL PROBLEMA NON VISCOSO

$\Downarrow$

VANISHING-VISCOSITY SOLUZIONE

VISCOSITÀ EVANESCENTE

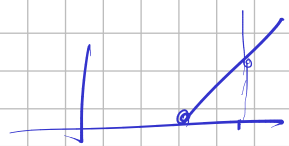
- SOLUZIONE ~~DEBOLE~~ DELLA FORMA INTEGRALE

$$\partial_t u + \partial_x f(u) = 0$$

$$\int_{x_1}^{x_2} \underbrace{\partial_t u}_{} + f(u(t,x_2)) - f(u(t,x_1)) = 0 \qquad \forall x_1, x_2$$

$\Downarrow$ DISCONTINUA IN TEMPO?

$$\partial_t \int_{x_1}^{x_2} u$$



- FORMA DEBOLE

$$\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R})$$
$$u \in \mathbb{R}\times\mathbb{R} \to \mathbb{R}$$

$$0 = \int_0^\infty \int_{-\infty}^{+\infty} \varphi(t,x)\cdot \partial_t u(t,x) + \varphi \cdot \partial_x f(u(t,x))\, dx\, dt$$

INTEGRALI PER PARTE

$$0 = \int_0^\infty \int_{-\infty}^{+\infty} \partial_t \varphi \cdot u \, dx\, dt + \int_{-\infty}^{\infty} \varphi(t=0,x)\, u(0,x)\, dx$$
$$+ \int_0^\infty \int_{-\infty}^{\infty} \partial_x \varphi \cdot f(u)\, dx\, dt$$

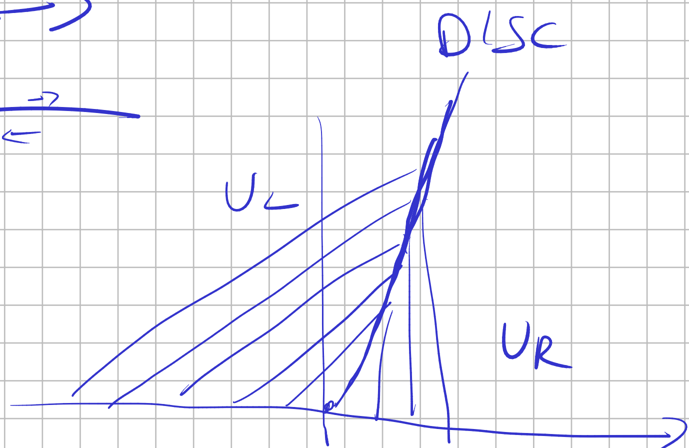
- VANISHING-VISCOSITY SOL È SOL DEBOLE ☺
- LA SOLUZIONE DEBOLE NON È UNICA.
  - ↳ $\Rightarrow$
  - ↳ VINCOLI SU ENTROPIA

PROBLEMI DI RIEMANN

$$u(0,x) = \begin{cases} u_L & x<0 \\ u_R & x>0 \end{cases} \qquad u_L, u_R \in \mathbb{R}$$

$$f(u) = \frac{u^2}{2} \quad \text{BURGERS!} \qquad f'' > 0$$

$u_L > u_R$

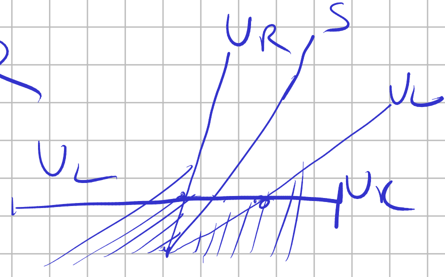


$x = st$

$\Downarrow$

s = VELOCITÀ DELLA DISCONTINUITÀ

$S \rightarrow [-L, L] \quad L \in \mathbb{R}$



$$\int_{-L}^{L} \partial_t u(x,t)\,dx = (*)$$

$$u(x,t) = \begin{cases} u_L & x<st \\ u_R & x>st \end{cases}$$

1) $\circledast = \int_{-L}^{L} -\partial_x f = f(u(t,-L)) - f(u(t,L))$

$= f(u_L) - f(u_R)$

2) $\circledast$ $\int_{-L}^{L} \partial_t u(x,t)\, dx = \partial_t \int_{-L}^{L} u(x,t)\, dx$

$= \partial_t \left[\int_{-L}^{st} u_L\, dx + \int_{st}^{L} u_R\, dx \right]$

$= \partial_t \left[(st+L)\, u_L + (L-st)\, u_R \right]$

$= s\, u_L - s\, u_R$

1+2 $\quad s(u_L - u_R) = f(u_L) - f(u_R)$

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R}$$

CONDIZIONE DI RANKINE-HUGONIOT

$\Uparrow$

CONSERVAZIONE

es. BURGERS

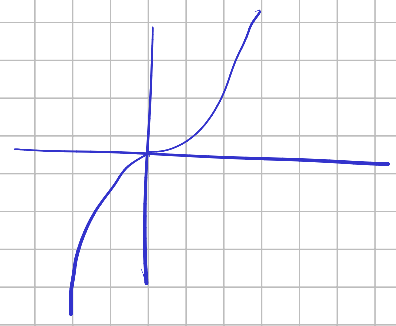
$$S = \frac{\frac{u_L^2}{2} - \frac{u_R^2}{2}}{u_L - u_R} = \frac{\frac{(u_L - u_R)(u_L + u_R)}{2}}{u_L - u_R}$$

$$= \frac{u_L + u_R}{2}$$

$$u_0 = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

$$s = \frac{1}{2} \Rightarrow u(x,t) = \begin{cases} 1 & x < \frac{1}{2}t \\ 0 & x > \frac{1}{2}t \end{cases}$$

$y = u^{1/3} \qquad u = y^3$



$$\partial_t u + u\,\partial_x u = 0$$

$$= \partial_t y^3 + y^3 \partial_x y^3 = 0$$

$$= 3y^2 \partial_t y + 3y^3 \cdot y^2 \partial_x y = 3y^2 \underbrace{\left[\partial_t y + y^3 \partial_x y\right]}_{=0} = 0$$

$$\partial_t y + \frac{1}{4}\partial_x\left(y^4\right) = 0$$

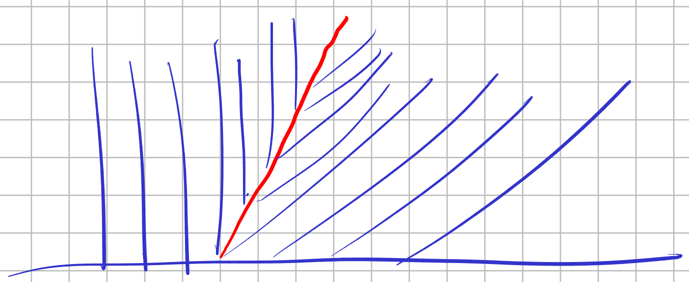
$$f(y) = \frac{y^4}{4}$$

$$S^y = \frac{f(y_L) - f(y_R)}{y_L - y_R} = \frac{1}{4}\,\frac{y_L^4 - y_R^4}{y_L - y_R} = \frac{1}{4}(y_L + y_R)(y_L^2 + y_R^2)$$

$$= \frac{1}{4}\left(u_L^{1/3} + u_R^{1/3}\right)\left(u_L^{2/3} + u_R^{2/3}\right) \neq S^u = \frac{u_L + u_R}{2}$$

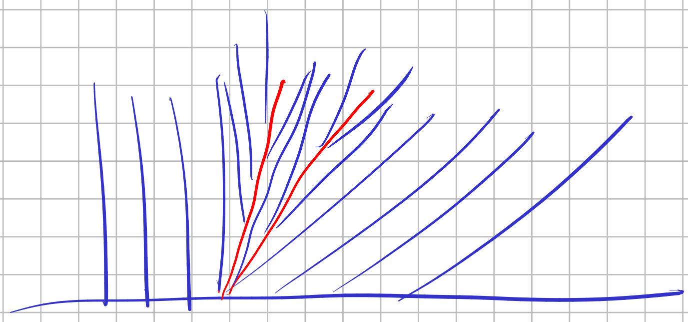
$u_L = 1 \qquad u_R = 0 \qquad \Downarrow \quad S^y = \frac{1}{4} \qquad S^u = \frac{1}{2}$

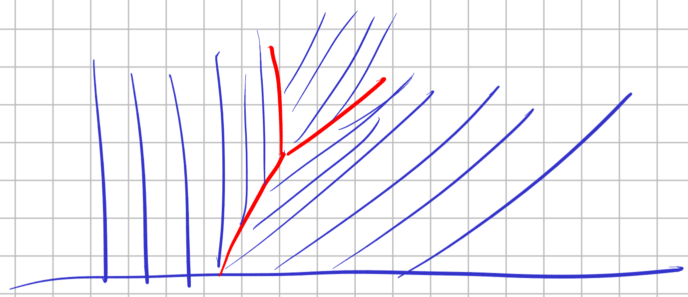
$u_L < u_R$



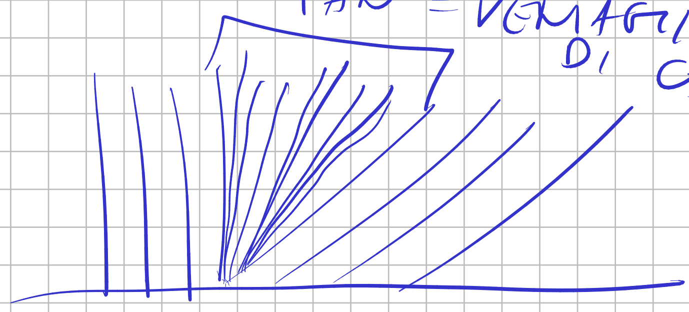
$\exists$ soluz. debole $\Leftarrow$ $u(x,t) = \begin{cases} u_L & x < st \\ u_R & x > st \end{cases}$

SOLUZ. DEBOLE





• RAREFAZIONE



$$\partial_t u(x,t) + \partial_x f(u(t,x)) = 0$$

$$x, t \leftarrow \lambda x, \lambda t \qquad \lambda \in \mathbb{R}^+$$

$$\frac{\partial \lambda t}{\partial t} \frac{\partial u}{\partial \lambda t}(x,t) + \frac{\partial \lambda x}{\partial x} \frac{\partial f}{\partial \lambda x}(u,t) = 0$$

$$\cancel{\lambda}\, \partial_{\lambda t} u(x,t) + \cancel{\lambda} \cdot \partial_{\lambda x} f(u(x,t)) = 0 \quad \star$$

$$\rightsquigarrow \quad \underbrace{\xi = \frac{x}{t}} \qquad u(t,x) = z(\xi) = z\left(\frac{x}{t}\right) \quad t > 0$$

$$\| \qquad\qquad$$

$$u\left(1, \frac{x}{t}\right) = u\left(\frac{t}{t}, \frac{x}{t}\right)$$

$$\uparrow_{\lambda = t}$$

* $\tilde{x} = \lambda x \qquad \tilde{t} = \lambda t$

$$\to \partial_{\tilde{t}}\, u\left(\frac{\tilde{x}}{\lambda}, \frac{\tilde{t}}{\lambda}\right) + \partial_{\tilde{x}} f\left(u\left(\frac{\tilde{x}}{\lambda}, \frac{\tilde{t}}{\lambda}\right)\right) = 0$$

$$u_0\left(\frac{\tilde{x}}{\lambda}, \frac{\tilde{t}}{\lambda}\right) = u_0(\tilde{x}, \tilde{t})$$

$$\to \tilde{u}(\tilde{x}, \tilde{t}) := u\left(\frac{\tilde{x}}{\lambda}, \frac{\tilde{t}}{\lambda}\right)$$

$$\partial_{\tilde{t}}\, \tilde{u}(\tilde{x}, \tilde{t}) + \partial_{\tilde{x}} f(\tilde{u}(\tilde{x}, \tilde{t})) = 0$$

$$\tilde{u}(x, t) = u(x, t)$$

$$u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right) = u(x,t) \implies \lambda = t$$

$$u(x,t) = u\left(\frac{x}{t}, 1\right) = z\left(\frac{x}{t}\right)$$

$$z(\xi) = z\left(\frac{x}{t}\right) \qquad \xi = \frac{x}{t} \qquad \partial_t \xi = -\frac{x}{t^2}$$

$$0 = \partial_t\, z(\xi) + \partial_x f(z(\xi)) =$$

$$= z'(\xi) \cdot \partial_t \xi + f'(z(\xi)) \cdot z'(\xi) \cdot \partial_x \xi$$

$$= z'(\xi) \cdot \left(-\frac{x}{t^2}\right) + f'(z(\xi)) \cdot z'(\xi)\, \frac{1}{t} = 0$$

$$\frac{1}{t} z'(\xi) \underbrace{\left[-\xi + f'(z(\xi))\right]}_{=0} = 0$$

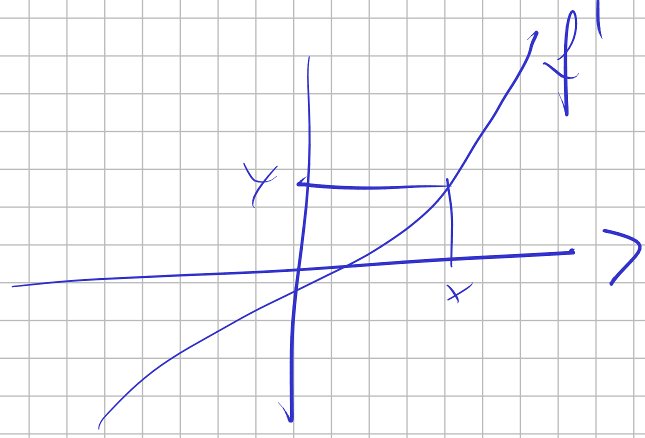
$$f'(z(\xi)) = \xi \qquad\qquad f'' > 0$$

$$\Downarrow$$

f' è invertibile $\Leftarrow$ f' monotona crescente

$$\Downarrow$$

$$z(\xi) = (f')^{-1}(\xi)$$



es. BURGERS

$$f(u) = \frac{u^2}{2}$$

$$f' = u$$

$$z(\xi) = \xi$$

$$\parallel$$

$$u(x,t) = \frac{x}{t}$$

RAREFACTION RIEMANN PROBLEM

$$u_L < u_R$$

$$u(x,t) = \begin{cases} u_L & x < u_L \cdot t \\ (f')^{-1}\left(\frac{x}{t}\right) & u_L t < x < u_R \cdot t \\ u_R & x > u_R \cdot t \end{cases}$$

$\Rightarrow$ LA SOLUZ. DEBOLE CHE CI INTERESSA

ALTRO MODO PER TROVARE LA SOLUZIONE RILEVANTE DAL PUNTO DI VISTA FISICO

- ENTROPIA

$$\partial_t u^\nu + \partial_x f(u^\nu) = \nu u_{xx}$$

$$\lim_{\nu \to 0} u^\nu = u$$

ENTROPIA $\eta(u)$ $\quad \eta'' \geq 0 \quad$ $g(u)$ FLUSSO ENTROPICO

$$\longrightarrow \quad \eta'(u) \cdot f'(u) = g'(u)$$

(η, g) ENTROPY PAIR

$$\eta'(u)\,\partial_t u^\nu + \eta'(u)\cdot\partial_x f(u^\nu) = \eta'(u)\cdot\nu\cdot u^\nu_{xx}$$

$$\partial_t(\eta(u(t,x))) + \underbrace{\eta'(u)\cdot f'(u^\nu)}_{g'(u^\nu)}\cdot\partial_x u^\nu = \underbrace{\eta'(u)\cdot\nu\cdot u^\nu_{xx}}$$

$$\partial_t \eta(u) + \partial_x g(u^\nu) = \nu\cdot\partial_{xx}\eta(u) \underbrace{- \underbrace{\nu}_{>0}\cdot\underbrace{\eta''}_{>0}\cdot\underbrace{(\partial_x u)^2}_{\geq 0}}_{\leq 0}$$

$$\partial_{xx}\eta(u^\nu) = \partial_x\left(\eta'(u^\nu)\cdot\partial_x u^\nu\right) =$$

$$= \underbrace{\eta''(u^\nu)\cdot(\partial_x u)^2} + \underbrace{\eta'(u)\cdot\partial_{xx}u^\nu}$$

$$\partial_t \eta(u) + \partial_x g(u^\nu) - \nu\cdot\partial_{xx}\eta(u) \leq 0$$

$\nu \to 0 \quad \to u \to$ DISCONTI

$\rightarrow$ DEBOLE $\qquad \varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}) \qquad \varphi \geq 0$

$$\int^x\int^t \varphi\,(\partial_t \eta + \partial_x g - \nu \cdot \partial_{xx} \eta) \leq 0$$

$$\int^x\int^t - \varphi_t \cdot \eta - \varphi_x \cdot g - \nu\, \partial_{xx} \varphi \cdot \eta \leq 0$$

$$\int^x\int^t \varphi_t\, \eta(u^\nu) + \varphi_x\, g(u^\nu) + \underbrace{\nu\, \varphi_{xx} \cdot \eta(u^\nu)}_{\to 0} \geq 0$$

$\lim_{\nu \to 0}$

$\Downarrow$

$$\int^x\int^t \varphi_t\, \eta(u) + \varphi_x\, g(u) \geq 0$$

$$\rightarrow \boxed{\eta_t(u) + g_x(u) \leq 0}$$