EQUAZIONE DEL CALORE - PARTE 2

$$U_{\underset{\downarrow 0}{\Delta x}, \underset{\downarrow 0}{\Delta t}} \longrightarrow U^{ex}$$

<u>CONVERGENZA</u>

$\Delta t, \Delta x \to 0$?

$$\Delta t = c\, \Delta x \qquad c \in \mathbb{R}^+$$

$$\Delta t = c\, \Delta x^2$$

$\Delta x \to 0$

- $U^{n+1} = \underbrace{B(\Delta t)}\cdot U^n + b^n(\Delta t)$

- $U^{n+1} = U^n + \dfrac{\Delta t}{\Delta x^2} D^2 U^n$

$$= \underbrace{(I + \Delta t\, A)}_{B(\Delta t)} U^n \qquad \text{EULERO ESPLICITO}$$

- $U^{n+1} = U^n + \Delta t\, A \cdot U^{n+1}$ EULERO IMPLICITO

$$(I - \Delta t A) \cdot U^{n+1} = U^n$$

$$U^{n+1} = \underbrace{(I - \Delta t A)^{-1}}_{B(\Delta t)} \cdot U^n$$

- $U^{n+1} = U^n + \frac{1}{2} \Delta t A U^n + \frac{1}{2} \Delta t A U^{n+1}$

$$\left(I - \tfrac{1}{2} \Delta t A\right) U^{n+1} = \left(I + \tfrac{1}{2}\Delta t A\right) U^n$$

$$U^{n+1} = \left(I - \tfrac{1}{2}\Delta t A\right)^{-1}\left(I + \tfrac{1}{2}\Delta t A\right) \cdot U^n$$

CRANK-NICOLSON $B(\Delta t)$

- STABILITÀ LAX-RICHTMYER

$$U^{n+1} = B(\Delta t) \cdot U^n + b^n(\Delta t)$$

STABILE SE $\forall\, T$ TEMPO FINALE $\in \mathbb{R}^+$

$\exists\, C_T \in \mathbb{R}^+ :$

$$\| B(\Delta t)^n \| \leq C_T$$

$$\forall\, n : n \cdot \Delta t \leq T.$$

$$U^{n+1} = B(\Delta t)\, U^n = B(\Delta t)\, B(\Delta t)\, U^{n-1} = B^2(\Delta t)\, U^{n-1}$$

$$= \ldots = \underbrace{B(\Delta t)^{n+1}} \cdot U^0$$

TEOREMA ~~LAX~~ DI EQUIVALENZA DI LAX

DATO UN METODO LINEARE e CONSISTENTE

IL METODO È CONVERGENTE $\iff$ STABILE PER LAX-RICHTMYER

STABILITÀ $\|B(\Delta t)\|_2 = \max eig\left((I - \Delta t A)^{-1}\right) \leq 1$

DEF. NEG.

$I - \Delta t A$ } DEF. POS.

e AUTOVALORI ≥ 1

$\|B\| \leq 1$

$\|B^n\| \leq 1$

$$\|B^n\| \le \underbrace{\|B\| \cdot \ldots \cdot \|B\|}_{n\text{-volte}}$$ SEMPRE STABILE

- CRANK-NICOLSON

$$\|B(\Delta t)\|_2 = \max eig\left(\left(I - \frac{\Delta t}{2} A\right)^{-1}\left(I + \frac{\Delta t}{2} A\right)\right)$$

$$\le \max_i \frac{1 + \frac{1}{2}\Delta t \lambda_i}{1 - \frac{1}{2}\Delta t \lambda_i} \le 1$$

$\to \lambda_i < 0$

- $$\|B(\Delta t)\| \le \overbrace{1 + \alpha \Delta t}^{1 + \Delta t A - \frac{1}{\Delta x^2}}$$

$\alpha \in \mathbb{R}$

$$\|B^n(\Delta t)\| \le (1 + \alpha \Delta t)^n \le e^{\alpha T}$$

$n \cdot \Delta t \le T$

VON NEUMANN STABILITY

- SOLO PER PROBLEMI CON PERIODIC BC
- COEFFICIENTI COSTANTI $\quad \partial_t u - a \partial_{xx} u = 0$, $a \in \mathbb{R}$, $a \notin C(\mathbb{R})$
- ALTRE BC MEGLIO MA NON IN ANALISI VON NEUMANN

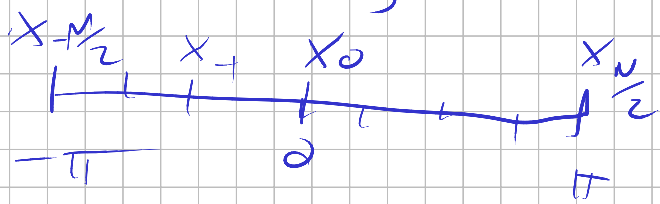
e^{ikx} SONO AUTOFUNZIONI di ∂_x

$$\partial_x e^{ikx} = \underbrace{ik}_{\in \mathbb{C}} \underbrace{e^{ikx}}$$

$$W^K_J = e^{i\underbrace{(J\Delta x)}_{x_J}\cdot K}$$

$J \in \mathbb{Z}$

$-\frac{N}{2} \leq J \leq \frac{N}{2}$



$[-\pi, \pi]$

$$x_J = J \cdot \Delta x \qquad \Delta x = \frac{2\pi}{N}$$

W^K_J AUTOVETTORI

$$(DV)_J := \frac{V_{J+1} - V_{J-1}}{\Delta x}$$

$$(DW^K)_J = \frac{W^K_{J+1} - W^K_{J-1}}{2\Delta x} = \frac{e^{i(J+1)\Delta x \cdot K} - e^{i(J-1)\Delta x K}}{2\Delta x}$$

$$= e^{i(J\Delta x)K}\left(\frac{e^{i\Delta x K} - e^{-i\Delta x K}}{2\Delta x\, i}\right) i$$

$$= e^{i(J\Delta x)K}\,\frac{1}{\Delta x}\, i\sin(\Delta x K)$$

$$= W^K_J \cdot \underbrace{\frac{i\sin(\Delta x K)}{\Delta x}}_{} \in \mathbb{C}$$

- $\|U^{n+1}\|_2 \leq \underbrace{(1+\alpha\Delta t)}_{}\,\|U^n\|_2 \qquad U^{n+1} \in \mathbb{R}^N$

$\|c_K^{n+1}\|_2 \leq (1+\alpha\Delta t)\,\|c_K^n\|_2 \qquad \forall K \in \mathbb{Z}$

$$C_k^{n+1} = \underbrace{g(k)}_{\downarrow} C_k^n$$

COEFFICIENTE DI AMPLIFICAZIONE

SE $|g(k)| \leq (1 + \alpha \Delta t) \quad \forall k \in \mathbb{Z}$

$\Rightarrow$ STABILI RICHTMYER.

EULERO ESPLICITO $\quad U_j^{n+1} = U_j^n + a\frac{\Delta t}{\Delta x^2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$

$$C_k^{n+1} \cdot e^{ij\Delta x k} = C_k^n e^{ij\Delta x k} + a\frac{\Delta t}{\Delta x^2}\left(C_k^n \cdot e^{i(j+1)\Delta x k} - 2C_k^n e^{i(j)\cdot\Delta x k} + e^{i(j-1)\Delta x k} C_k^n\right)$$

$$\frac{C_k^{n+1}}{C_k^n} \cdot \cancel{e^{ij\Delta x k}} = \cancel{e^{ij\Delta x k}}\left[1 + a\frac{\Delta t}{\Delta x^2}\left(e^{i\Delta x k} - 2 + e^{-i\Delta x k}\right)\right]$$

$$g(k) = 1 + a\frac{\Delta t}{\Delta x^2}\left(-2 + \frac{e^{i\Delta x k} + e^{-i\Delta x k}}{2} \cdot 2\right)$$

$$= 1 + \underbrace{2\overbrace{a\frac{\Delta t}{\Delta x^2}}^{\geq 0}(\cos(\Delta x k) - 1)}_{\leq 0}$$

$|g(k)| \leq 1$

$$-1 \leq \cos \leq 1$$

$$-2 \leq \cos - 1 \leq 0$$

$$\underbrace{2a\frac{\Delta t}{\Delta x^2}\overbrace{\underbrace{(\cos(\Delta x k) - 1)}_{}}^{\leq 0}}_{= -2} \geq -2$$

$$\cancel{2a\frac{\Delta t}{\Delta x^2} \le -\frac{2}{\cos(\Delta x k)-1} = \frac{2}{1-\cos(\Delta x k)} \le 1}$$

WORST CASE SCENARIO

$$\cos - 1 = -2$$

$$-4a\frac{\Delta t}{\Delta x^2} \ge -2 \Rightarrow \Delta t \le \frac{1}{2a}\Delta x^2$$

• EULERO IMPLICATO

$$c_k^{n+1} e^{ik(j\Delta x)} = c_k^n e^{i\Delta x k j} + a\frac{\Delta t}{\Delta x^2}\left(c_k^{n+1}e^{i(j+1)\Delta x k} - 2c_k^{n+1}e^{ij\Delta x k} + c_k^{n+1}e^{i(j-1)\Delta x k}\right)$$

$$c_k^{n+1}\cancel{e^{ik(j\Delta x)}}\left(1 - a\frac{\Delta t}{\Delta x^2}\left[e^{i\Delta x k}+e^{-i\Delta x k}-2\right]\right) = c_k^n \cancel{e^{ij\Delta x k}}$$

$$c_k^{n+1}\left[1 - a\frac{\Delta t}{\Delta x^2}\left(-2+2\cos(\Delta x k)\right)\right] = c_k^n$$

$$g(k) = \frac{1}{\underbrace{1\underbrace{-\underbrace{2a\frac{\Delta t}{\Delta x^2}}_{\ge 0}\underbrace{(\cos(\Delta x k)-1)}_{\le 0}}_{\ge 0}}_{\ge 1}} \le 1 \qquad \forall k \ \forall \Delta x$$

$$c_k'(t) = \overbrace{-\frac{a}{\Delta x^2}2(1-\cos(\Delta x k))}^{\lambda \to \in \mathbb{C}} c_k(t)$$

- FORMULAZIONE DEBOLE

$$\begin{cases} \partial_t u - \nabla\cdot(a\nabla u) = f & \text{in } \mathbb{R}^+\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u(t,x) = u_D(x) & \text{su } \Gamma_D \subset \partial\Omega \\ a\nabla u\cdot n = g_N(x) & \text{su } \Gamma_N \subset \partial\Omega \end{cases}$$

$u(t) \in H^1(\Omega)$ $\forall t\in\mathbb{R}^+$: $\forall v \in H^1_{\Gamma_D}(\Omega)$

$$\int_\Omega \partial_t u\cdot v + \underbrace{a\nabla v\cdot\nabla u \, dx}_{\to\, a(u,v)} =$$

$$= \underbrace{\int_\Omega f\cdot v\,dx + \int_{\Gamma_N} g_N\cdot v\,ds}_{\to\, F(v)}$$

$+$ $u(x,t) = u_D(x)$ $\forall x\in\Gamma_D$

- $a(\cdot,\cdot): V\times V \to \mathbb{R}$ BIL, LIMITATA

DEBOLMENTE COERCIVA

$$\exists\lambda\geq 0 \quad \exists\alpha>0 : \quad a(v,v) + \lambda\|v\|^2_{L^2} \geq \alpha\|v\|^2_V$$

$\forall v\in V$

- F LINEARE LIMITATO
- $u_0\in L^2$ $f\in L^2$

$\Rightarrow$ $\exists!$ SOLUZIONE

$$\int_\Omega v\,\partial_t u + a(v,u) = 0 \qquad \forall v\in V$$

SCEGLIAMO $\quad v = u \qquad \int u\,\partial_t u = -a(u,u)$

$$\int_\Omega u\,\partial_t u\,dx = \int_\Omega \partial_t \frac{u^2}{2} = \partial_t \frac{\int_\Omega u^2}{2} = \partial_t \frac{\|u\|_{L^2}^2}{2}$$

$$\|$$

$$-a(u,u) \leq -\alpha \|u\|_V^2 \leq 0$$

$$\partial_t \|u\|_{L^2}^2 \leq 0 \qquad \Rightarrow \qquad \|u(t)\|_2 \leq \|u(0)\|_2$$

$$V \to V_h = \langle \varphi_i \rangle_{i=1}^{N_h} \qquad u_h, v_h \in V_h$$

$$\int \int_\Omega \partial_t u_h \cdot v_h + a(u_h, v_h) = F(v_h)$$

$$u_h(t,x) = g_h(t,x) \qquad \text{su } x \in \Gamma_D \quad \forall t > 0$$

$$\underline{\underline{M}}\,\partial_t \underline{U}(t) + \underline{\underline{A}} \cdot \underline{u}(t) = \underline{F}$$

$$M_{ij} = \int_\Omega \varphi_i \varphi_j$$

IMPLICIT EULER

$$M \frac{U^{n+1} - U^n}{\Delta t} + A\,u^{n+1} = F^{n+1}$$

$$\left(\frac{M}{\Delta t} + A\right) \underline{U}^{n+1} = \frac{M}{\Delta t} U^n + F^{n+1}$$

scelgo $v_h = u_h^{n+1}$

$$\int_\Omega \frac{u_h^{n+1}-u_h^n}{\Delta t} \cdot u_h^{n+1} + a \nabla u_h^{n+1} \cdot \nabla u_h^{n+1} = 0$$

$$\int_\Omega (u_h^{n+1})^2 + \underbrace{\Delta t\, a (\nabla u_h^{n+1})^2}_{>0} = \overbrace{\int_\Omega u_h^n \cdot u_h^{n+1}}^{} \leq *$$

$$\int_\Omega (u_h^{n+1})^2 \leq$$

$$0 \leq (a-b)^2 = a^2+b^2-2ab$$

$$2ab \leq a^2+b^2$$

$$ab \leq \frac{a^2+b^2}{2}$$

$$* \leq \int_\Omega \frac{(u_h^n)^2}{2} + \frac{(u_h^{n+1})^2}{2} = \frac{\|u^n\|_2^2}{2} + \frac{\|u^{n+1}\|_2^2}{2}$$

$$\|u^{n+1}\|_2^2 \leq \frac{\|u^n\|_2^2}{2} + \frac{\|u^{n+1}\|^2}{2}$$

$$\frac{\|u^{n+1}\|_2^2}{2} \leq \frac{\|u^n\|_2^2}{2} \Rightarrow \|u^{n+1}\|_2 \leq \|u^n\|_2$$

$$\underline{u}^T M \underline{u} = \sum_{i,j} \int \overbrace{u_i \varphi_i}^{u_h} \cdot \overbrace{\varphi_j u_j}^{u_h} = \int_\Omega u_h^2$$

$$M = \int \varphi_i \varphi_j$$