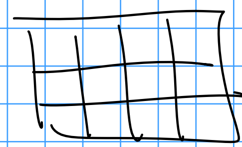
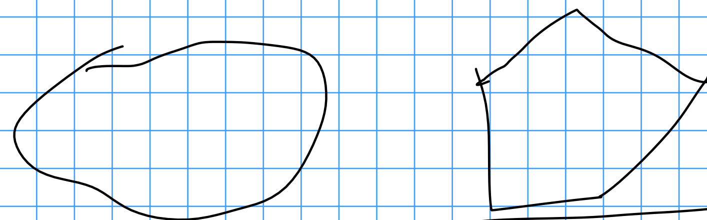
• ELEMENTI FINITI





$$\boxed{a(u,v) = F(v)} \qquad \forall v \in \boxed{V}$$

TROVA $u \in V$

⟹ APPROSSIMAZIONE DI $V \to V_h$

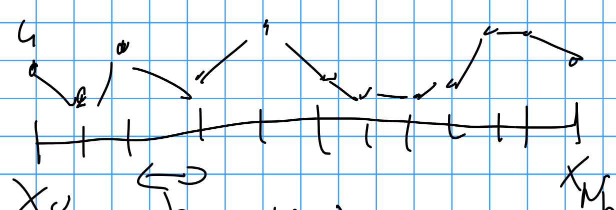
$h \to 0 \qquad V_h \to V$ $\qquad$ h = SIZE ELEMENTI DELLA DISCRETI. SPAZIALE

1. V_h HILBERT $\subset V \subset H^1(\Omega)$
2. $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ $\qquad$ CONT, COERC, (SYM)
3. $F_h(\cdot) : V_h \to \mathbb{R}$ $\qquad$ CONT, LINEARE

• $V_h \subset V \qquad \dim(V_h) = N_h < \infty \qquad \forall h > 0$

• $a_h = a\big|_{V_h \times V_h}$

• $F_h = F$



TROVA $u_h \in V_h$:

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h \subset V \qquad \underline{u} = \begin{pmatrix} u_0 \\ \vdots \\ u_{N_h} \end{pmatrix} \in \mathbb{R}^{N_h}$$

PROBLEMA DI <u>GALERKIN</u>

$$V_h = \mathrm{Span}\left(\{\varphi_i\}_{i=1}^{N_h}\right) = \left\langle \{\varphi_i\}_{i=1}^{N_h} \right\rangle$$

$$V_h = \left\{ u \in V : u = \sum_{i=1}^{N_h} u_i \varphi_i , \; u_i \in \mathbb{R} \; \forall i = 1, \dots, N_h \right\}$$

$$\dim(V_h) = N_h$$

$$\underset{V_h}{\overset{\Uparrow}{u}} \longleftrightarrow \underset{\mathbb{R}^{N_h}}{\overset{\Uparrow}{\underline{u}}} = \begin{pmatrix} u_1 \\ \vdots \\ u_{N_h} \end{pmatrix}$$

TROVA $u_h \in V_h$:

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

$$\Updownarrow$$

$$a(u_h, \varphi_i) = F(\varphi_i) \quad \forall i = 1, \dots, N_h$$

$$u_h = \sum_{j=1}^{N_h} u_j \varphi_j$$

$$\sum_{j=1}^{N_h} a(\varphi_j, \varphi_i)\, u_j = F(\varphi_i)$$

$$\underline{\underline{A}} \cdot \underline{u} = \underline{F} \qquad \underline{u} = [u_j]$$

INCOGNITE

$$u_h = \sum_{j=1}^{N_h} u_j \varphi_j(x)$$

$$\underline{\underline{A}} = [a_{ij}] \qquad a_{ij} = a(\varphi_j, \varphi_i)$$

$$\underline{F} = [F_i] \qquad F_i = F(\varphi_i)$$

Se $a(\cdot,\cdot)$ FUNZ. BILIN ASS AL PROBLEMA ELLITTICO $a(u,v) = F(v) \; \forall v \in V$ è a BILINEARE COERCIVO

$\Rightarrow A$ è DEFINITA POSITIVA.

A è DEF POS $\Leftrightarrow$ $v^T A v \geq 0 \wedge v^T A v = 0 \Leftrightarrow v = \underline{0}$

<u>DIM</u> $u_h \Leftrightarrow \underline{u}$

$$v^T A v = \sum_{i,j=1}^{N_h} v_i a_{ij} v_j = \sum_{i,j} v_i a(\varphi_j, \varphi_i) v_j$$

$$= a\left(\sum_{j=1}^{N_h} v_j \varphi_j, \sum_{i=1}^{N_h} v_i \varphi_i\right) =$$

$$= a(v_h, v_h) \geq \alpha \|v_h\|_V^2 \geq 0$$

$$= 0 \Leftrightarrow \|v_h\|_V^2 = 0$$

$$\Updownarrow$$

$$\underline{v} = 0 \qquad \boxtimes$$

1. ESISTENZA e UNICITÀ di u_h?
2. STABILE LA SOLUZIONE?
3. $u_h \to u$ PER $h \to 0$

$$\boxed{-\Delta u = f}$$

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1. LEMMA LAX-MILGRAM $\forall V_h$ <u>HILBERT</u> $\boxed{F(v) = \int f \cdot v}$

$\Rightarrow a(\cdot, \cdot) = F(\cdot)$ $\boxed{a = \int \nabla u \cdot \nabla v \, dx}$

$\Rightarrow \exists! \, u_h \in V_h : a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

- 1.BIS $\Rightarrow$ $\boxed{\underline{\underline{A}}\, \underline{u} = \underline{F}}$ A DEF POS $\Rightarrow \exists A^{-1}$

• 2 STABILITÁ

α È COEFFICIENTE COERCIVITÁ di a
$\to$ $a(u,u) \geq \alpha \|u\|^2 \quad \forall u$

• $\|u_h\|_V \leq \frac{1}{\alpha}\|F\|_{V^*}$

$F : V \to \mathbb{R}$

$a : V \times V \to \mathbb{R}$

DIM

$\boxed{a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h}$

$$\|u_h\|_V^2 = \|u_h\|_{V_h}^2 \leq \frac{1}{\alpha} a(u_h, u_h) = \frac{1}{\alpha} F(u_h)$$

$$\leq \frac{1}{\alpha}\|F\|_{V^*} \cdot \|u_h\|_V$$

$$\|u_h\|_V \leq \frac{1}{\alpha}\|F\|_{V^*} \qquad \boxtimes$$

COROLLARIO $\quad u_h : a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

$w_h : a(w_h, v_h) = G(v_h) \quad \forall v_h \in V_h$

$$\Rightarrow \|u_h - w_h\|_V \leq \frac{1}{\alpha}\|F - G\|_{V^*}$$

es. $F(v) = \int_\Omega f \cdot v - \int_{\partial\Omega} g \cdot v$

3. CONVERGENZA $\quad$ OBIETTIVO : $u_h \to u \quad h \to 0$

• PROIEZIONE DI GALERKIN / ORTOGONALITÁ DI GALERKIN

$$a(u_h - u, v_h) = 0 \qquad \forall v_h \in V_h$$

u_h è SOLUZIONE DI $a(u_h, v_h) \overset{(1)}{=} F(v_h) \quad \forall v_h \in V_h$

u è SOLUZ. DI $a(u, v) \overset{(2)}{=} F(v) \quad \forall v \in V$

$$V_h \subset V$$

$$a(u_h, v_h) \underset{(1)}{=} F(v_h) \overset{(2)}{=} a(u, v_h)$$

$$v_h \in V_h \subset V$$

$$\Rightarrow a(u_h - u, v_h) = 0 \qquad \forall v_h \in V_h$$

- $a(\cdot,\cdot)$ è UN PRODOTTO SCALARE se a SYM

$$\|v_h\|_a = \sqrt{a(v_h, v_h)}$$

$$\|u_h - u\|_a = \underbrace{a(u_h - u, u_h - u)}$$

$$u_h = \underset{v_h \in V_h}{\operatorname{argmin}} \|v_h - u\|_a$$

- LEMMA DI CÉA

AGGIUNGI / TOGLI v_h

$$a(u - u_h, u - u_h) \overset{(1)}{=} a(u - u_h, \underbrace{u - v_h}_{\notin V_h}) + \underbrace{a(u - u_h, v_h - u_h)}_{= 0}$$

CONTINUITÀ di a

$$|a(u - u_h, u - v_h)| \underset{(2)}{\leq} C \cdot \|u - u_h\|_V \cdot \|u - v_h\|_V$$

$$\|u - u_h\|_V^2 \leq \frac{1}{\alpha} a(u - u_h, u - u_h) \overset{(1+2)}{\leq} \frac{C}{\alpha} \|u - u_h\|_V \cdot \|u - v_h\|_V \qquad \forall v_h \in V_h$$

$$\|u-u_h\|_V \leq \frac{C}{\alpha}\|u-v_h\|_V \quad \forall\, v_h \in V_h$$

$$\|u-u_h\|_V \leq \inf_{v_h\in V_h} \frac{C}{\alpha}\|u-v_h\|_V \, . \qquad \boxtimes$$

u_h POTREBBE NON ESSERE LA MIGLIORE APPROSSIMAZIONE IN $\|\cdot\|_V$

PERÒ $\|u-u_h\|_V$ DIMINUIRÀ COME LA MIGLIORE APPROSSIMAZIONE DI u IN V_h

- SE SCEGLIAMO V_h SE

$$\lim_{h\to 0} \inf_{v_h\in V_h} \|v-v_h\|_V = 0 \quad \forall\, v \in V$$

$$\Rightarrow \|u-u_h\| \xrightarrow[h\to 0]{} 0$$

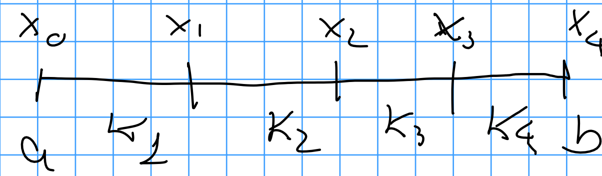
$$\inf_{v_h\in V_h}\|v-v_h\|_V = O(h^p) \Rightarrow \|u-u_h\|_V = O(h^p)$$

- ELEMENTI FINITI:

__1D__ $\quad \Omega = (a,b) \qquad V = H^1((a,b)) \qquad V_h \subset V$

$$\mathcal{T}_h = \{K_j\}_{j=1}^{N_h} \qquad K_j = [x_{j-1}, x_j] \quad \forall\, j = 1, \dots, N_h$$

$$\boxed{|x_j - x_{j-1}| = h}$$



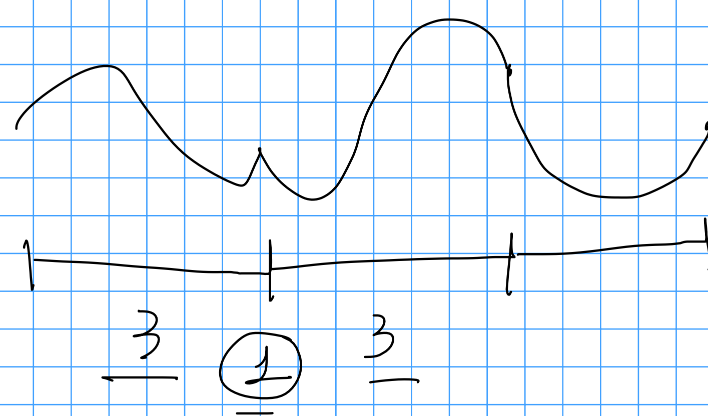
$$h = \max_j |x_j - x_{j-1}|$$

- $H^1((a,b)) \subset C^0([a,b])$ $\quad V_h$ CONTINUO

$$V_h = X_h^r = \{ v_h \in C^0(\bar{\Omega}) : v_h|_{K_j} \in \mathbb{P}^r(K_j) \quad \forall K_j \in \mathcal{T}_h \}$$

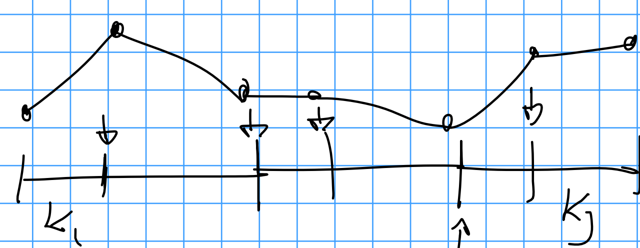
$$\subset H^1((a,b))$$

$$V_h = X_h^r = \langle \varphi_1, \dots, \varphi_{N_h} \rangle$$



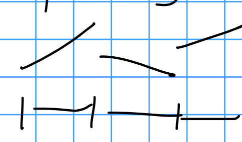
$$a_{ij} = a(\varphi_i, \varphi_j) = \int \nabla \varphi_i \nabla \varphi_j$$

- $r = 1$ $\quad \mathbb{P}^1(K_j)$ $\Rightarrow$



$$y = \boxed{ax + b}$$

2 DOF (GRADI DI LIBERTÀ) $\quad N \cdot 2$



MA VINCOLI CONTINUITÀ

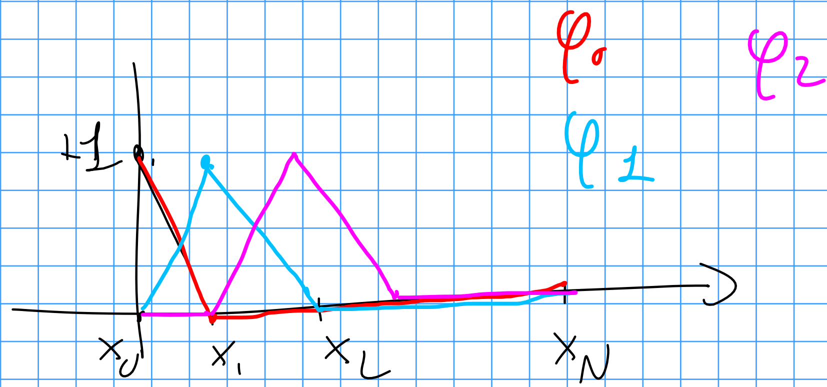
$N - 1$ VINCOLI $\Rightarrow$ $\quad 2N - (N-1) = N + 1$ DOF

$\Rightarrow$ SE HO IL VALORE ALLE INTERFACCE

$\Rightarrow \exists! \ u_h \in X_h^1 : u_h(x_j) = u_j \quad j = 0, \dots, N$

$u_0 = 1$
$u_1 = 0$
$u_2 = 0$
$u_3 = 0$
$\vdots$

$= u_h = \sum u_i \varphi_i = \underline{\underline{\varphi_0}}$



$u_0 = 1 \Rightarrow u(x_0) = 1$

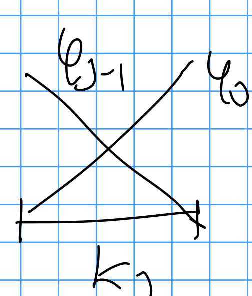
$u(x_i) = u_i \qquad \varphi_i(x_j) = \delta_{ij}$

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{se } x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{se } x \in [x_i, x_{i+1}] \\ 0 & \text{altrimenti} \end{cases}$$

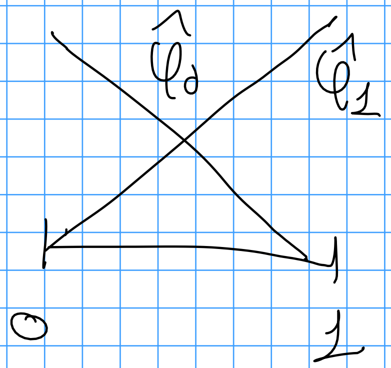
$\text{supp}(\varphi_i) = (x_{i-1}, x_{i+1})$

$$a_{ij} = \int_\Omega \nabla \varphi_i \nabla \varphi_j \, dx = \overbrace{\int_{K_j} \nabla\varphi_i \nabla \varphi_j}^{\substack{\neq 0 \\ \Leftarrow \\ i = j-1, j}} + \overbrace{\int_{K_{j+1}} \nabla\varphi_i \nabla\varphi_j}^{\substack{\neq 0 \\ \Uparrow \\ i = j, j+1}}$$

$\Rightarrow \quad a_{ij} = 0 \quad \text{se} \quad |i - j| > 1$



$\int_{K_J} \partial_x \varphi_i \, \partial_x \varphi_k$ $\qquad i = J-1, J \qquad k = J-1, J$



$\mathbb{P}^1$

$$\hat\varphi_0(\xi) = 1 - \xi$$

$$\hat\varphi_1(\xi) = \xi$$

$$T_i : [0,1] \to [x_{i-1}, x_i]$$

$$T_i(\xi) = \xi(x_i - x_{i-1}) + x_{i-1}$$

$$T_i^{-1}(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}$$

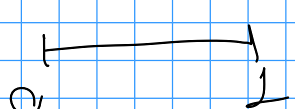
$$\varphi_i(x) = \hat\varphi_1\left(T_i^{-1}(x)\right) = \hat\varphi_1\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)$$

$\forall x \in K_i$

$$\varphi_i(x) = \hat\varphi_0\left(T_{i+1}^{-1}(x)\right) = \hat\varphi_0\left(\frac{x - x_i}{x_{i+1} - x_i}\right)$$

$\forall x \in K_{i+1}$

- $X_h^2 = \{ u \in C^0([a,b]) : u|_{K_J} \in \mathbb{P}^2(K_J) \ \forall J \}$



$\mathbb{P}^2([0,1])$

LAGRANGIANA $\Rightarrow$

$\hat{x}_0 = 0$

$\hat{x}_1 = \frac{1}{2}$

$\hat{x}_2 = 1$