

# Partial Differential Equations

# PDE

Given a domain  $\Omega \in \mathbb{R}^d$  where  $d > 1$ , we seek for  $u : \Omega \rightarrow \mathbb{R}^s$  where  $s \in \mathbb{N}_0$ , solution of a **stationary PDE** of order  $k$ :

$$F(x, t, u, \nabla u \dots, \nabla^{(k-1)} u, \nabla^{(k)} u, g) = 0$$

with  $g : \Omega \rightarrow \mathbb{R}^s$  a given function. Or, more explicitly as

$$\mathcal{P}(u, g) \equiv F(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \frac{\partial^{p_1 + \dots + p_d} u}{\partial^{p_1} x_1 \partial^{p_2} x_2 \dots \partial^{p_d} x_d}, g) = 0,$$

where  $p_1 + \dots + p_d \leq k$ .

A non stationary PDE of order  $k$  reads: find  $u : \Omega \times [0, T] \rightarrow \mathbb{R}^s$

$$\mathcal{P}(u, g) \equiv F(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \frac{\partial^{p_0 + p_1 + \dots + p_d} u}{\partial^{p_0} t \partial^{p_1} x_1 \dots \partial^{p_d} x_d}, g) = 0,$$

where  $p_0 + \dots + p_d \leq k$ .

A classical solution of a PDE is a function  $u \in \mathcal{C}^k(\Omega \times [0, T])$  that solves the previous equation.

## Definition

If the PDE can be written in the form

$$\mathcal{P}(u, g) = a(x)u + b_0(x)\partial_t u + b_1(x)\partial_{x_1} u + \cdots + b_d(x)\partial_{x_d} u + c_{\alpha(2,0,\dots,0)}\partial_{tt} u + \cdots + \gamma_{\alpha(p_0,\dots,p_d)} \frac{\partial^{p_0+\dots+p_d} u}{\partial^{p_0} t \partial^{p_1} x_1 \dots \partial^{p_d} x_d} + \cdots - g = 0,$$

i.e., if the coefficients of the unknown  $u$  and of its derivatives depend only on the independent variables  $(t, x)$ , then the PDE is **linear**. Else, it is **nonlinear**.

## Definitions

Consider a nonlinear PDE of order  $k$

- if the coefficients of the derivatives of order  $k$  depend only on the independent variables  $(t, x)$ , then the PDE is **semilinear**;
- if the coefficients of the derivatives of order  $k$  depend on the independent variables  $(t, x)$  and on the partial derivatives of  $u$  of order at most  $k - 1$ , then the PDE is **quasi-linear**;
- if it's not quasi-linear, it's **fully nonlinear**.

## Examples

- Reaction-advection-diffusion equation

$$\partial_t u = u_{xx} + cu_x + u^2,$$

is semilinear.

- Inviscid Burgers' equation

$$\partial_t u + uu_x = 0,$$

is quasi-linear but not semilinear.

- The Korteweg-de Vries (KdV) equation

$$\partial_t u + u\partial_x u + \partial_{xxx} u = 0,$$

is semilinear.

- The Monge-Ampère equation

$$u_{xx}u_{yy} - (u_{xy})^2 = 0$$

is fully nonlinear.

# First order linear PDE , a.k.a. transport equation

$$u_t + u_x = 0$$

How do I found a general solution?

Let's try this change of variables

$$(x, t) \rightarrow (\xi, \eta), \quad \xi(x, t) = x + t, \eta(x, t) = x - t$$

with inverse

$$x = \frac{\xi + \eta}{2}, t = \frac{\xi - \eta}{2}.$$

I substitute the new variables:  $v(\xi, \eta) := u(x(\xi, \eta), t(\xi, \eta))$

$$u_x = v_\xi \xi_x + v_\eta \eta_x = v_\xi + v_\eta$$

$$u_t = v_\xi \xi_t + v_\eta \eta_t = v_\xi - v_\eta$$

obtaining a new PDE

$$0 = u_t + u_x = 2v_\xi \iff v_\xi = 0.$$

Implica che  $v(\xi, \eta) = f(\eta)$  with  $f \in \mathcal{C}^1(\mathbb{R})$ . Going back to the original variables

$$u(x, t) = v(\xi(x, t), \eta(x, t)) = f(\xi(x, t)) = f(x - t)$$

# Characteristic lines

$$u_t + u_x = 0, \quad u(x, t) = v(\xi(x, t), \eta(x, t)) = f(\xi(x, t)) = f(x - t).$$
$$X_{x_0}(t) = x_0 + t$$

# Generalization to different coefficients

$$a(t, x)u_t + b(t, x)u_x = g(t, x), (t, x) \in \Omega \subset \mathbb{R}^2.$$

Well defined (non-singular and  $\mathcal{C}^1$ ) transformation  $(t, x) \Leftrightarrow (\xi, \eta)$ , i.e.,

$$\left| \frac{\partial(\xi, \eta)}{\partial(t, x)} \right| := \left| \begin{pmatrix} \xi_t & \xi_x \\ \eta_t & \eta_x \end{pmatrix} \right| = \xi_t \eta_x - \xi_x \eta_t \neq 0.$$

Change of variables:  $u_t = v_\xi \xi_t + v_\eta \eta_t$ ,  $u_x = v_\xi \xi_x + v_\eta \eta_x$ , giving

$$(a\xi_t + b\xi_x)v_\xi + (a\eta_t + b\eta_x)v_\eta + cv = g(t(\xi, \eta), x(\xi, \eta))$$

Goal: simplify the previous equation, we choose  $\eta$  such that

$$a\eta_t + b\eta_x = 0,$$

so that we obtain an ODE for every  $\eta$

$$v_\xi + \frac{c}{a\xi_t + b\xi_x} v = \frac{g(t(\xi, \eta), x(\xi, \eta))}{a\xi_t + b\xi_x}.$$

# Generalization to different coefficients

To obtain  $a\eta_t + b\eta_x = 0$ , one should notice that, w.l.o.g., we are looking for a curve  $x(t)$  such that  $\eta(t, x(t)) = \eta_0$  constant for every  $t$ .

$$0 = \frac{d\eta(t, x(t))}{dt} = \eta_t + \eta_x \frac{\partial x}{\partial t} \implies \frac{\eta_t}{\eta_x} = -\partial_t x(t)$$

Hence, we have

$$\frac{\eta_t}{\eta_x} = -\frac{b}{a} \iff \partial_t x(t) = -\frac{b}{a}.$$

Integrating this equation, one obtains the curve  $x(t)$ , leading to the definition of  $\eta(t, x)$  solving for the constant  $\eta_0$ .



# Example

$$xu_t - tu_x = 1$$

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$$(x\xi_t - t\xi_x)v_\xi + (x\eta_t - t\eta_x)v_\eta = 1$$

$$(x\eta_t - t\eta_x) = 0$$

$$\frac{dx}{dt} = -\frac{t}{x}$$

$$\int x dx = \int -t dt$$

$$x = \sqrt{\eta_0^2 - t^2}$$

$$\eta(t, x) := \sqrt{t^2 + x^2}$$

$$\xi(t, x) = \arctan(x/t)$$

$$t = \eta \cos(\xi), \quad x = \eta \sin(\xi),$$

$$\underbrace{\left( -\eta \sin(\xi) \frac{\eta \sin(\xi)}{\eta^2} - \eta \cos(\xi) \frac{\eta \cos(\xi)}{\eta^2} \right)}_{=-1} v_\xi = 1,$$

$$v = -\xi + f(\eta) \quad u = -\arctan(x/t) + f(x^2 + y^2).$$

$$\eta_t = \frac{t}{\sqrt{t^2 + x^2}}, \quad \eta_x = \frac{x}{\sqrt{t^2 + x^2}},$$

$$\xi_t = \frac{t^2}{x^2 + t^2} \frac{-x}{t^2} = -\frac{x}{x^2 + t^2}, \quad \xi_x = \frac{t}{x^2 + y^2},$$

## Homework

- Solve  $u_x - 2u_y = 0$
- Solve  $yu_x - xu_y + uy = xy$

# Second order linear PDE in 2D

Consider the PDE on  $\Omega \subset \mathbb{R}^2$

$$\mathcal{P}(u, g) = A\partial_{xx}u + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu - g = 0 \quad \forall (x, y) \in \Omega$$

where  $u \in \mathcal{C}^2(\Omega)$  and  $A, B, C \in \mathcal{C}^2$  and they do not vanish simultaneously. Let's **classify** the PDE depending on the *discriminant*

$$\Delta := B^2 - 4AC.$$

## Definition

- If  $\Delta > 0$  the PDE is said to be hyperbolic (at a point  $(x, y)$ )
- If  $\Delta = 0$  the PDE is said to be parabolic (at a point  $(x, y)$ )
- If  $\Delta < 0$  the PDE is said to be elliptic (at a point  $(x, y)$ )

### Hyperbolic example: wave equation

$$\partial_{tt}u - c\partial_{xx}u = 0 \text{ with } c > 0$$

Indeed,  $\Delta = 4c > 0$ .

### Parabolic example: heat equation

$$\partial_t u - c\partial_{xx}u = 0 \text{ with } c > 0$$

Indeed,  $\Delta = 0$ .

### Elliptic example: Poisson equation

$$-c\partial_{xx}u - c\partial_{yy}u = -c\Delta u = f \text{ with } c > 0$$

Indeed,  $\Delta = -4c^2 < 0$ .

### Changing sign example: Tricomi equation

$$yu_{xx} + u_{yy} = 0$$

$\Delta = -4y$ .

## Theorem

The sign of the discriminant  $\Delta$  is invariant under smooth non-singular transformation of coordinates (i.e. under a change of variables).

### Proof 1/2

We focus only on the second order terms as the first order ones do not contribute to the discriminant.

Suppose we perform a smooth change of variables  $(x, y) \mapsto (\xi, \eta)$ , given by a diffeomorphism.

Under this transformation, the second-order derivatives transform as follows:

$$\begin{aligned}u_{xx} &= \alpha^2 u_{\xi\xi} + 2\alpha\beta u_{\xi\eta} + \beta^2 u_{\eta\eta}, \\u_{xy} &= \alpha\gamma u_{\xi\xi} + (\alpha\delta + \beta\gamma) u_{\xi\eta} + \beta\delta u_{\eta\eta}, \\u_{yy} &= \gamma^2 u_{\xi\xi} + 2\gamma\delta u_{\xi\eta} + \delta^2 u_{\eta\eta},\end{aligned}$$

where

$$\alpha = \frac{\partial x}{\partial \xi}, \quad \beta = \frac{\partial x}{\partial \eta}, \quad \gamma = \frac{\partial y}{\partial \xi}, \quad \delta = \frac{\partial y}{\partial \eta}.$$

## Proof 2/2

Rewriting the PDE in the new coordinates, the transformed coefficients  $A'$ ,  $B'$ ,  $C'$  are given by

$$\begin{aligned}A' &= A\alpha^2 + 2B\alpha\gamma + C\gamma^2, \\B' &= A\alpha\beta + B(\alpha\delta + \beta\gamma) + C\gamma\delta, \\C' &= A\beta^2 + 2B\beta\delta + C\delta^2.\end{aligned}$$

Now, computing the transformed discriminant:

$$\begin{aligned}\Delta' &= B'^2 - A'C' \\&= (A\alpha\beta + B(\alpha\delta + \beta\gamma) + C\gamma\delta)^2 \\&\quad - (A\alpha^2 + 2B\alpha\gamma + C\gamma^2)(A\beta^2 + 2B\beta\delta + C\delta^2).\end{aligned}$$

Expanding both terms and simplifying, we find that

$$\Delta' = (B^2 - AC)(\alpha\delta - \beta\gamma)^2 = \Delta \det(J)^2,$$

where  $J$  is the Jacobian matrix of the transformation. Since  $\det(J)^2 \geq 0$ , the sign of  $\Delta$  remains unchanged. This proves the invariance of the discriminant sign under a change of variables.

# Hyperbolic canonical form

Consider the wave equation

$$\partial_{tt}u - c\partial_{xx}u = 0$$

with  $c > 0$ . We can find a change of variables  $(x, t) \mapsto (\xi, \eta)$  such that the PDE simplifies to

$$\partial_{\xi\eta}v = 0.$$

The map is defined by

$$\eta = x + t, \quad \xi = x - t.$$

This is the canonical form of a hyperbolic PDE. The general solution is given integrating in  $\xi$  and then in  $\eta$ , i.e.,

$$v(\xi, \eta) = \int^{\xi} \int^{\eta} \partial_{wz}v(w, z) dz dw = \int^{\xi} f(w)dw = F(\xi) + G(\eta),$$

where  $\partial_{\xi}F(\xi) = f(\xi)$ .

So, the general solution of the wave equation is

$$u(x, t) = F(x - t) + G(x + t).$$



## Hyperbolic canonical form: can we always get it?

Consider the just the second order terms of the hyperbolic PDE  $\Delta = B^2 - 4AC > 0$ .

$$A\partial_{xx}u + Bu_{xy} + Cu_{yy} = 0.$$

We look for a change of variables  $(x, y) \mapsto (\xi, \eta)$  such that the PDE simplifies to

$$\partial_{\xi\eta}v = 0.$$

The transformation can be applied noting that

$$u_{xx} = v_{\xi\xi}(\xi_x)^2 + 2v_{\xi\eta}\xi_x\eta_x + v_{\eta\eta}(\eta_x)^2,$$

$$u_{xy} = v_{\xi\xi}\xi_x\xi_y + v_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + v_{\eta\eta}\eta_x\eta_y,$$

$$u_{yy} = v_{\xi\xi}(\xi_y)^2 + 2v_{\xi\eta}\xi_y\eta_y + v_{\eta\eta}(\eta_y)^2,$$

The transformed PDE reads

$$(A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2)v_{\xi\xi} + (2A\xi_x\eta_y + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_x)v_{\xi\eta} + (A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2)v_{\eta\eta} = 0.$$

## Computation space

$$(A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2)v_{\xi\xi} + (2A\xi_x\eta_y + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_x)v_{\xi\eta} + (A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2)v_{\eta\eta} = 0.$$

We want to find the change of variables such that

$$\begin{cases} A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\ A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \end{cases}$$

These are first order PDE, so we are looking for characteristics curves such that  $\xi(x, y) = \text{const}$ , if we find a curve, for example  $y(x)$  such that  $\xi(x, y(x)) = \text{const}$ , then

$$\frac{d\xi}{dx} = \frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial y} \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{\partial_x \xi}{\partial_y \xi}.$$

From the first PDE, we then get

$$A\left(\frac{dy}{dx}\right)^2 + B\frac{dy}{dx} + C = 0,$$

which is called the characteristic equation for the original PDE. This is quadratic equation in  $\frac{dy}{dx}$  with  $\Delta = B^2 - 4AC > 0$ . The two distinct solutions are

$$\frac{dy}{dx} = \frac{-B \pm \sqrt{\Delta}}{2A}.$$

From this we can get the transformation  $(x, y) \mapsto (\xi, \eta)$  as we did in the linear PDE.

## Computation space

## Example

$$u_{tt} + u_{tx} = 0$$

$$(\xi_t^2 + \xi_t \xi_x) u_{\xi\xi} + (2\xi_t \eta_t + \xi_t \eta_x + \xi_x \eta_t) u_{\xi\eta} + (\eta_t^2 + \eta_t \eta_x) u_{\eta\eta} = 0$$

The equations for  $\xi$  and  $\eta$  are the same equations.

We look for a curve  $y(x)$  such that  $\xi(x, y(x)) = \text{const}$ , i.e.,  $\xi(x, y(x)) = x + y(x) = \text{const}$  and that

$$\xi_t^2 + \xi_t \xi_x = 0$$

$$\frac{\xi_t^2}{\xi_x^2} + \frac{\xi_t}{\xi_x} = 0$$

$$\left( \frac{dx}{dt} \right)^2 - \frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = \begin{cases} 0 \\ 1 \end{cases} \implies x(t) = \begin{cases} \xi_0 \\ \eta_0 + t \end{cases}$$

$$\implies \eta = x - t, \quad \xi = x.$$

### What if we try to do the same with a parabolic PDE?

$$\frac{dx}{dt} = -\frac{B \pm \sqrt{\Delta}}{2A} = -\frac{B}{2A}.$$

There is only one characteristic curve. So, choosing  $\xi = 2Ax + Bt$  and  $\eta = x$  we get the canonical form

$$A \frac{\partial^2 v}{\partial \xi^2} = 0,$$

with the general solution  $v(\xi, \eta) = F(\eta) + \xi G(\eta)$ .

### What if we try to do the same with an elliptic PDE?

There is no characteristics that is conserved. But, one can instead eliminate the coefficient of  $u_{\xi\eta}$  to obtain the canonical form for the elliptic PDE. Using  $\eta = t$  and  $\xi = \frac{2Ax - Bt}{\sqrt{\Delta}}$ , we get

$$A \left( \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} \right) = 0.$$

# Existence, uniqueness and well-posedness

For the PDEs above we have found classes of solutions. How can we find unique solutions to specific problems? What should we need to specify?

## Definition (Cauchy problem)

Consider a PDE of order  $k$  in  $\Omega \subset \mathbb{R}^d$  and let  $S$  be a given smooth surface on  $\mathbb{R}^d$ . Let also  $n = n(x)$  denote the unit normal vector to the surface  $S$  at a point  $x = (x_1, x_2, \dots, x_d) \in S$ . Suppose that on any point  $x$  of the surface  $S$  the values of the solution  $u$  and of all its directional derivatives up to order  $k - 1$  in the direction of  $n$  are given, i.e., we are given functions  $f_0, f_1, \dots, f_{k-1} : S \rightarrow \mathbb{R}$  such that

$$u(x) = f_0(x), \text{ and } \frac{\partial u}{\partial n}(x) = f_1(x), \text{ and } \frac{\partial^2 u}{\partial n^2}(x) = f_2(x), \dots, \text{ and } \frac{\partial^{k-1} u}{\partial n^{k-1}}(x) = f_{k-1}(x).$$

The **Cauchy problem** consists of finding the unknown function(s)  $u$  that satisfy simultaneously the PDE and the conditions above, which are called the **initial conditions** (ICs) and the given functions  $f_0, f_1, \dots, f_{k-1}$ , will be referred to as the initial data.

According to the role of the ICs they can be called also **boundary conditions** (BCs).

## Examples (Cauchy problem for transport equation)

$$\begin{cases} u_t + u_x = 0, & (x, t) \in \mathbb{R}^2, \\ u(0, x) = \sin(x), & x \in \mathbb{R}. \end{cases}$$

Here,  $S = \{(t, x) \in \mathbb{R}^2 : t = 0\}$ .

The general solution of the transport equation is  $u(x, t) = f(x - t)$ , so that the initial condition reads  $f(x) = \sin(x)$ , i.e.,  $u(x, t) = \sin(x - t)$ .



## Examples (Cauchy problem for wave equation)

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x, t) \in \mathbb{R}^2, \\ u(t, 0) = \sin(x), & x \in \mathbb{R}, \\ u_x(t, 0) = 0, & x \in \mathbb{R}. \end{cases}$$

In this case,  $S = \{(t, x) \in \mathbb{R}^2 : x = 0\}$  and  $n = (n_t, n_x) = (0, -1)$ . The general solution of the wave equation is  $u(t, x) = f(x - t) + g(x + t)$ , so that the initial (boundary) conditions read  $f_0(x) = \sin(x)$  and  $f_1(x) = 0$ , so

$$\begin{cases} f(-t) + g(t) = \sin(t), \\ f'(-t) + g'(t) = 0, \end{cases} \implies f(\xi) = \frac{1}{2}\sin(-\xi), \quad g(\eta) = \frac{1}{2}\sin(\eta),$$

so,  $u(t, x) = \frac{1}{2}(\sin(x + t) + \sin(-x + t))$ .

# Theorem (Cauchy-Kovalevskaya Theorem)

Consider a Cauchy problem for a linear PDE, let  $x^0$  be a point of the initial surface  $S$ , which is assumed to be analytic (very regular). Suppose that  $S$  is not a characteristic surface at the point  $x^0$ . Assume that all the coefficients of the linear PDE, the right-hand side  $g$ , and all the initial data  $f_0, f_1, \dots, f_{k-1}$  are analytic functions on a neighbourhood of the point  $x^0$ . Then, the Cauchy problem has a solution  $u$ , defined in the neighbourhood of  $x^0$ . Moreover, the solution  $u$  is analytic in a neighbourhood of  $x^0$  and it is unique in the class of analytic functions.

- Assumptions: Regularity
  - Outcome: Existence
  - Outcome: Uniqueness
  - Outcome: Regularity of the solution
- 
- Is this enough? No, the solution might still mis-behave

# Well-posedness

## Definition

A PDE problem is well-posed if:

1. The PDE has a solution
2. The solution is unique
3. The solution depends continuously on the PDE coefficients and on the problem data (IC/BC)

If the PDE problem is not well-posed, we say it is ill-posed.

We will see in the next section, how to impose properly IC and BC.

## Exercise

Show that the solution of the Cauchy problem for the wave equation

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = 0, \\ u(t, 0) = f(t), \\ u_x(t, 0) = g(t) \end{cases}$$

for some known BCs  $f$  and  $g$  is given by the d'Alembert's formula

$$u(t, x) = \frac{1}{2}(f(t-x) + f(t+x)) + \frac{1}{2} \int_{t-x}^{t+x} g(s) ds.$$

Show that the Cauchy problem is well-posed (skipping the uniqueness).

## Exercise

1. Find the solution of the Laplace equation on  $\Omega = [0, 2\pi]$  for various  $k$

$$\begin{cases} \partial_{tt}u + \partial_{xx}u = 0, \\ u(0, x) = 0, \\ u_t(0, x) = \frac{\sin(kx)}{k}, \\ u(0, t) = u(2\pi, t). \end{cases}$$

Steps:

- $u_k = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} [a_n(t) \cos(nx) + b_n(t) \sin(nx)]$
  - Substitute in the equation and find the general solution using the method of separation of variables
  - $\partial_{tt}a_n(t) = n^2a_n(t)$  for all  $n$  with  $a_n(0) = 0, \partial_t a_n(0) = 0$
  - $\partial_{tt}b_n(t) = n^2b_n(t)$  for all  $n$  with  $b_n(0) = 0, \partial_t b_n(0) = 0$  for  $n \neq k, \partial_t b_k(0) = 1/k$ .
  - $u_k(t, x) = \frac{1}{k^2} \sin(kx) \sinh(kt)$
2. Even if  $\sup_x |u_k(0, x)| + |\partial_t u_k(0, x)|$  is small, we can find large enough  $k$  so that for any time  $t_0 > 0$   $\sup_x |u_k(t_0, x)| + |\partial_t u_k(t_0, x)|$  is large.

## Theorem

Let  $u_k(t, x) = \frac{1}{k^2} \sin(kx) \sinh(kt)$ . For any positive  $\varepsilon, M, t_0$  there exists an integer  $K$  such that for any  $k > K$  the initial data satisfies  $\sup_x |u_k(0, x)| + |\partial_t u_k(0, x)| < \varepsilon$  but the solution at the time  $t_0$  satisfies  $\sup_x |u_k(t_0, x)| + |\partial_t u_k(t_0, x)| > M$ .

**Proof:** Choosing an integer  $K_1$  satisfying  $K_1 > \frac{1}{\varepsilon}$  we will have the initial condition inequality for any  $k \geq K_1$ . In order to obtain a lower estimate of the second form at time  $t_0$  let us first observe that

$$\sup_{x \in [0, 2\pi]} (|u_k(x, t)| + |\partial_t u_k(x, t)|) = \frac{1}{k^2} \sinh(kt) + \frac{1}{k} \cosh(kt) > \frac{1}{k^2} e^{kt}$$

where we have used an obvious inequality  $\frac{1}{k} > \frac{1}{k^2}$  for  $k > 1$ .

The function  $y = \frac{e^x}{x^2}$  is monotone increasing for  $x > 2$  and  $\lim_{x \rightarrow +\infty} \frac{e^x}{x^2} = +\infty$ .

Hence for any  $t_0 > 0$  there exists  $x_0$  such that  $\frac{e^x}{x^2} > \frac{M}{t_0^2}$  for  $x > x_0$ .

Let  $K_2$  be a positive integer satisfying  $K_2 > \frac{x_0}{t_0}$ .

Then for any  $k > K_2$

$$\frac{e^{kt_0}}{k^2} = t_0^2 \frac{e^{kt_0}}{k^2 t_0^2} > t_0^2 \frac{e^{x_0}}{x_0^2} > M.$$

Choosing  $K = \max(K_1, K_2)$  we complete the proof of the Theorem.

## Take home message

Not all boundary conditions are suitable for having a well-posed problem.