Parabolic Linear Differential Equations

Heat equation

Given a domain $\Omega \in \mathbb{R}$ we look for a solution $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ solution of

$$\partial_t u(t, x) - a\partial_{xx} u(t, x) = f(t, x),$$

with $a > 0$.

Physical applications

- Heat conduction (u temperature),
- Elastic membrane subject to a body force f (u is the displacement),
- Electric potential distribution (u) due to a charge f.

Difference with Elliptic

- Variation in time

Cauchy problem

We couple the PDE with initial conditions (IC) at time $t = 0$ AND boundary conditions (either Nuemann or Dirichlet) for all times $t \in \mathbb{R}^+$.

$$\begin{cases} \partial_t u(t,x) - a\partial_{xx} u(t,x) = f(t,x), & t > 0, x \in \Omega \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = u_D(t,x), & \forall t \in \mathbb{R}^+, x \in \Gamma_D \subset \partial\Omega, \\ \partial_x u(t,x) \cdot \mathbf{n} = u_N(t,x), & \forall t \in \mathbb{R}^+, x \in \Gamma_N \subset \partial\Omega. \end{cases}$$

Periodic boundary conditions

Alternatively, for boundary conditions one can impose periodic conditions, i.e., if $\Omega = [a,b]$, then

$$u(t,a) = u(t,b)$$

for all $t \in \mathbb{R}^+$.

Exact solutions for periodic boundary conditions (Fourier) (1/n)

Eigenfunctions of the differential operator

First of all, let's notice that the trigonometric functions are special functions for the differential operator

$$
\begin{aligned}
\partial_x e^{ixk} &= ik e^{ixk}, & \partial_{xx} e^{ixk} &= -k^2 e^{ixk},\\
\partial_x \sin(kx) &= k\cos(kx), & \partial_{xx} \sin(kx) &= -k^2 \sin(kx),\\
\partial_x \cos(kx) &= -k\sin(kx), & \partial_{xx} \cos(kx) &= -k^2 \cos(kx).
\end{aligned}
$$

Recall:

$$
\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}.
$$

So we focus on the trigonometric functions of the type e^{ixk}.

Exact solutions for periodic boundary conditions (Fourier) (2/n)

Fourier series

For simplicity let's consider $\Omega = [-\pi, \pi]$ with periodic boundary conditions. We can decompose the initial condtion in Fourier series if $u_0 \in L^2(\Omega)$.

$$u_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}, \qquad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(x) e^{-ikx}\mathrm{d}x.$$

Parseval theorem

$$\|\mathbf{c}\|_2^2 = \sum_{k\in\mathbb{Z}} |c_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u_0(x)|^2 \mathrm{d}x = \frac{1}{2\pi}\|u_0\|_2^2.$$

Wikipedia page on Fourier series
Youtube playlist of 3Blue1Brown on Fourier series
Youtube video on solving heat equations with Fourier

Exact solutions for periodic boundary conditions (Fourier) (3/n)

Exploiting linearity for heat equation

Let's us use the ansatz $u(t,x) = \sum_{k\in\mathbb{Z}} c_k(t)e^{ikx}$, where $c_k(t)$ are the Fourier coefficients of the solution at time t.

$$
\begin{aligned}
&\partial_t u(t,x) - a\partial_{xx}u(t,x) = 0\\
&\sum_{k\in\mathbb{Z}} \partial_t c_k(t)e^{ikx} - a\sum_{k\in\mathbb{Z}} c_k(t)\partial_{xx}e^{ikx} = 0\\
&\sum_{k\in\mathbb{Z}} \partial_t c_k(t)e^{ikx} + a\sum_{k\in\mathbb{Z}} k^2 c_k(t)e^{ikx} = 0\\
&\partial_t c_k(t) + ak^2 c_k(t) = 0, \quad \forall k\in\mathbb{Z},\\
&c_k(t) = c_k(0)e^{-ak^2t}, \quad \forall k\in\mathbb{Z}.
\end{aligned}
$$

Discretization of $\partial_t u - \partial_{xx} u = 0$

- Domain in space $\Omega = [a, b]$ and time $[0, T]$
- Grid in space $a = x_0 < x_1 < \cdots < x_i < \cdots < x_{N_x} = b$
- Grid in time $0 = t^0 < t^1 < \cdots < t^n < \cdots < t^{N_t} = T$

Explicit Euler

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = 0$$

Implicit Euler

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} = 0$$

Crank-Nicolson

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{2\Delta x^2} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2\Delta x^2} = 0$$

Numerical solutions

Explicit Euler

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = 0$$

- Explicit -> no systems

Implicit Euler

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} = 0$$

- Linear system

$$LHS = I - \frac{\Delta t}{\Delta x^2} D^2 = \begin{pmatrix} 1 + 2\frac{\Delta t}{\Delta x^2} & -\frac{\Delta t}{\Delta x^2} & 0 & \dots & \dots \\ -\frac{\Delta t}{\Delta x^2} & 1 + 2\frac{\Delta t}{\Delta x^2} & -\frac{\Delta t}{\Delta x^2} & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & -\frac{\Delta t}{\Delta x^2} & 1 + 2\frac{\Delta t}{\Delta x^2} \end{pmatrix} \qquad RHS = u^n$$

Crank-Nicolson

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{2\Delta x^2} - \frac{u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}}{2\Delta x^2} = 0$$

- Linear system

$$LHS = I - \frac{1}{2}\frac{\Delta t}{\Delta x^2}D^2 = \begin{pmatrix} 1+\frac{\Delta t}{\Delta x^2} & -\frac{\Delta t}{2\Delta x^2} & 0 & \dots & \dots \\ -\frac{\Delta t}{2\Delta x^2} & 1+\frac{\Delta t}{\Delta x^2} & -\frac{\Delta t}{2\Delta x^2} & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & -\frac{\Delta t}{2\Delta x^2} & 1+\frac{\Delta t}{\Delta x^2} \end{pmatrix}$$

$$RHS = u^n + \frac{1}{2}\frac{\Delta t}{\Delta x^2}D^2u^n$$

Consistency

Explicit Euler

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = 0$$

$$\begin{aligned}
e_{\Delta t,\Delta x}^{EE} =& \frac{u(t^{n+1}, x_i) - u(t^n, x_i)}{\Delta t} - \frac{u(t^n, x_{i+1}) - 2u(t^n, x_i) + u(t^n, x_{i-1})}{\Delta x^2} \\
=& \partial_t u(t^n, x_i) + \frac{\Delta t}{2}\partial_{tt} u(t^n, x_i) - \partial_{xx} u(t^n, x_i) - \frac{\Delta x^2}{12}\partial_{xxxx} u(t^n, x_i) + O(\Delta t^2) + O(\Delta x^3) \\
=& \frac{\Delta t}{2}\partial_{tt} u(t^n, x_i) - \frac{\Delta x^2}{12}\partial_{xxxx} u(t^n, x_i) + O(\Delta t^2) + O(\Delta x^3) = O(\Delta t) + O(\Delta x^2)
\end{aligned}$$

Second order in space and first order in time

Consistency

Crank-Nicolson

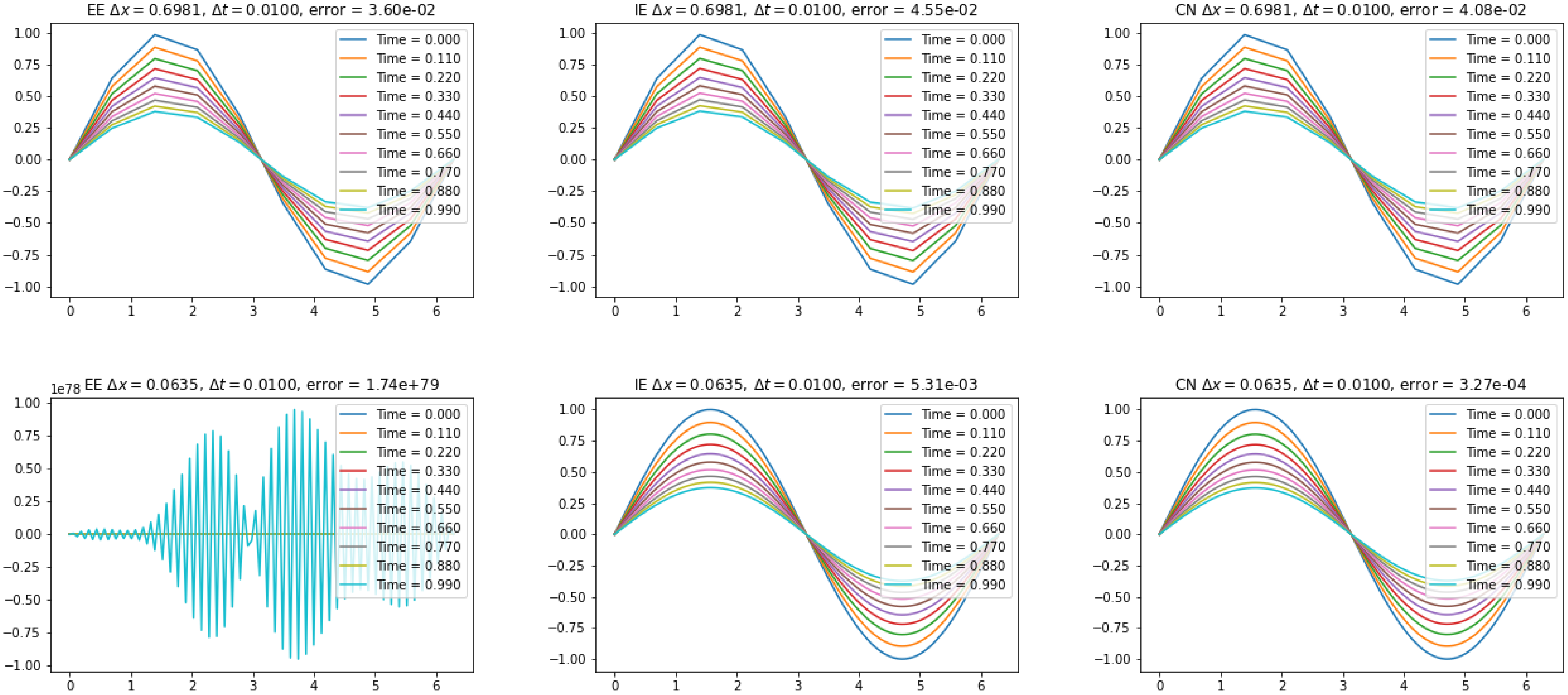
$$\frac{u_i^{n+1}-u_i^n}{\Delta t}-\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{2\Delta x^2}-\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{2\Delta x^2}=0$$

$$\begin{aligned}
e_{\Delta t,\Delta x}^{EE}=&\frac{u(t^{n+1},x_i)-u(t^n,x_i)}{\Delta t}-\frac{u(t^n,x_{i+1})-2u(t^n,x_i)+u(t^n,x_{i-1})}{\Delta x^2}\\
=&\partial_t u(t^n,x_i)+\frac{\Delta t}{2}\partial_{tt}u(t^n,x_i)-\partial_{xx}u(t^n,x_i)-\frac{\Delta x^2}{12}\partial_{xxxx}u(t^n,x_i)\\
&-\frac{\Delta t}{2}\underbrace{\partial_{txx}u(t^n,x_i)}_{=\partial_{tt}u}-\frac{\Delta t}{2}\frac{\Delta x^2}{12}\partial_{xxxxt}u(t^n,x_i)+O(\Delta t^2)+O(\Delta x^4)\\
=&\frac{\Delta t}{2}\partial_{tt}u(t^n,x_i)-\frac{\Delta t}{2}\partial_{tt}u(t^n,x_i)+O(\Delta t^2)+O(\Delta x^2)=O(\Delta t^2)+O(\Delta x^2)
\end{aligned}$$

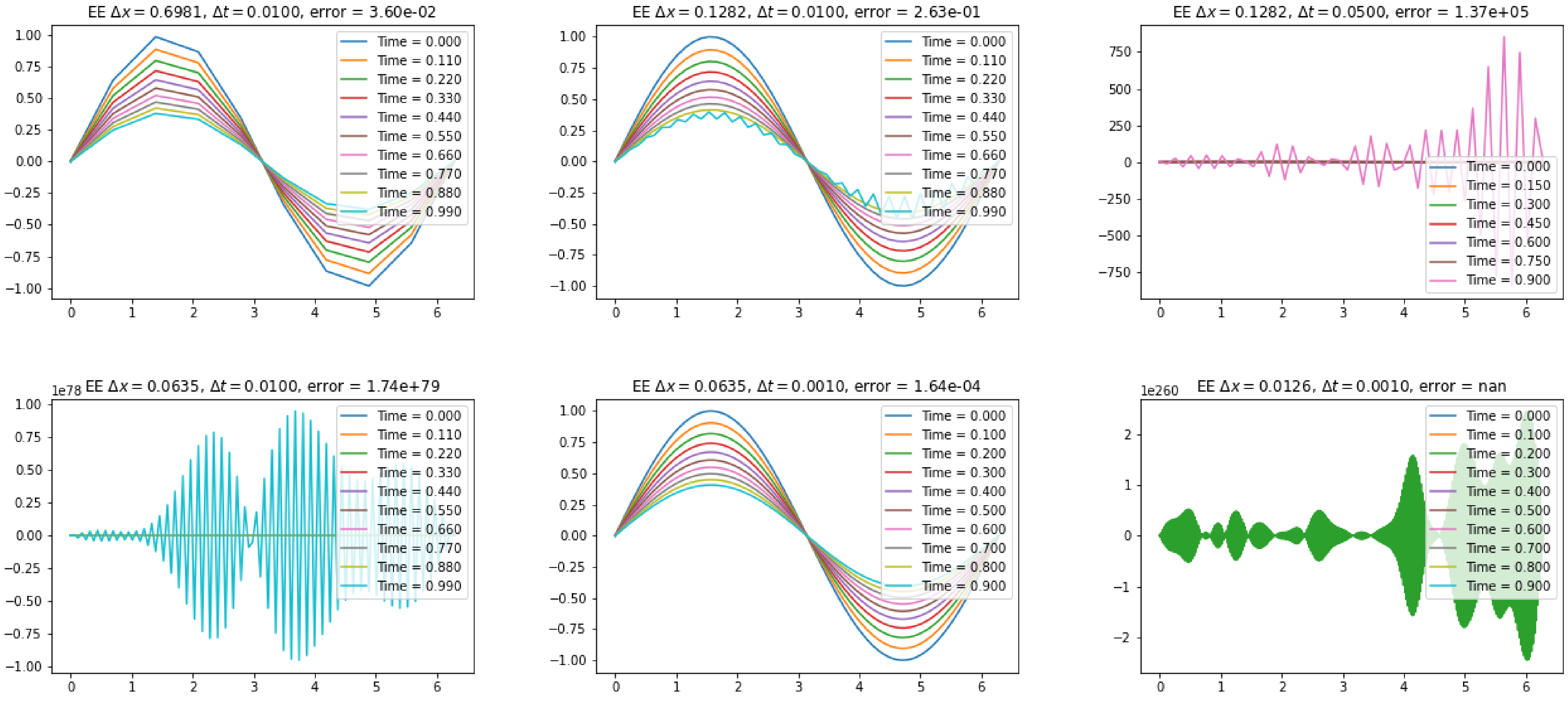
Second order in space and time

Example

$$\begin{cases} \partial_t u - \partial_{xx} u = 0 u_0(x) = \sin(x) \qquad x \in [0, 2\pi], \\ u(t, 0) = 0. \qquad t \in \mathbb{R}^+, \\ u(t, 2\pi) = 0. \qquad t \in \mathbb{R}^+, \end{cases} \qquad u(t,x) = e^{-t}\sin(x) \qquad x \in [0, 2\pi], \qquad t \in \mathbb{R}^+.$$



Explicit Euler



Semidiscretization / Method of lines

We have seen how to discretize the spatial derivatives, we can write a system of ODEs for that discretization.

$$u_i'(t) = \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}}{\Delta x^2} \qquad \forall i = 1, \ldots, N_x.$$

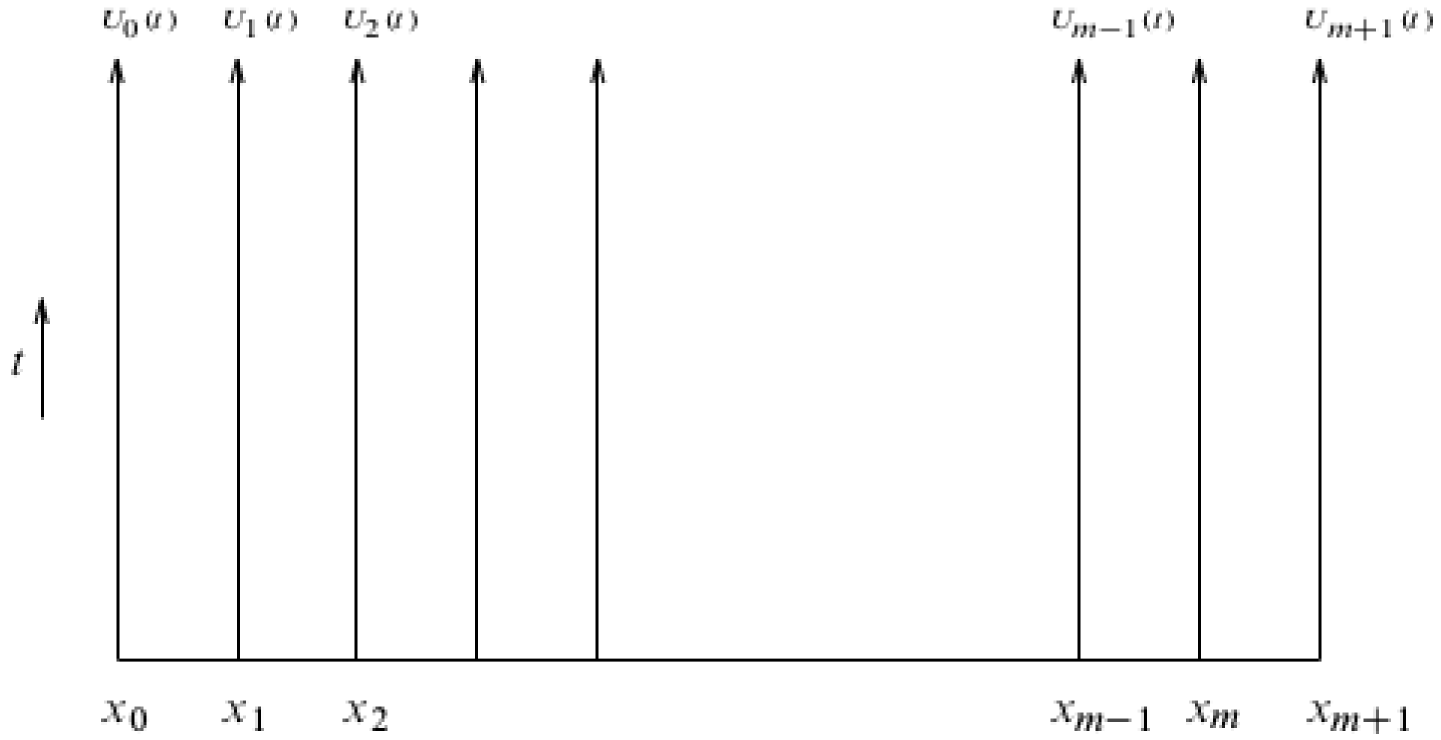
Then, we apply a time discretization method (e.g. explicit Euler, implicit Euler, Runge-Kutta, etc.)

$$U'(t) = AU(t) + g(t) = f(U, t)$$

where g contains boundary conditions and

$$A := \frac{1}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & \ldots & 0 \\ 0 & 1 & -2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -2 \end{bmatrix}$$

Method of lines (MOL) interpretation



Advantage of MOL

We can study the stability of the numerical problem, splitting the spatial and temporal discretization.

Stability region of a RK method

A Runge-Kutta method for a linear problem $u'(t) = \lambda u(t)$ can be written as

$$y^{n+1} = R(z) y^n, \qquad \text{with } z = \lambda \Delta t,$$

and we define the stability region as $\mathcal{S} := \{z \in \mathbb{C} : |R(z)| \leq 1\}$.

Connection with semidiscretized PDE

In our case, we have that the linear system

$$U'(t) = AU(t),$$

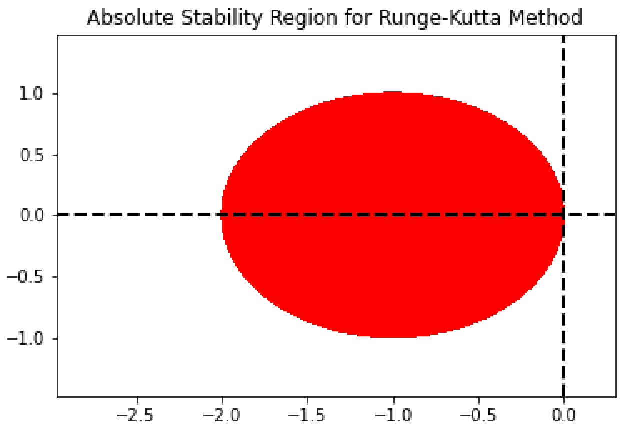
can be diagonalized with an orthogonal transformation Z (i.e. $ZZ^T = I$) such that $Z^T A Z = D$ with D diagonal matrix with the values of the **eigenvalues** of A. So, if we define $Y(t) = Z^T U(t)$ we can study many decoupled equations, instead of one system

$$Y'(t) = Z^T U'(t) = Z^T A U(t) = Z^T A Z Z^T U(t) = D Z^T U(t) = DY(t).$$

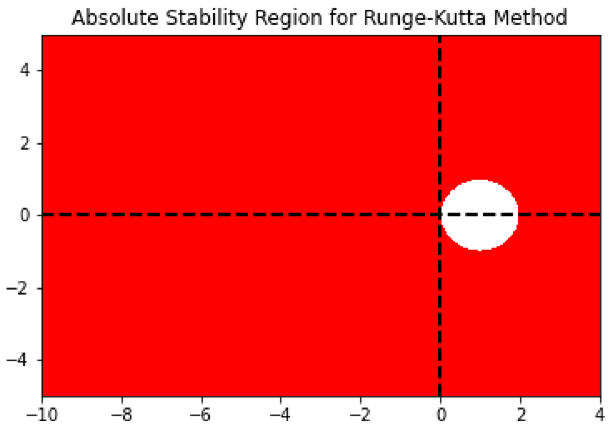
If $\lambda_i \in \mathcal{S}$ for all λ_i eigenvalues of A, then the method is stable.

Stability regions of RK methods

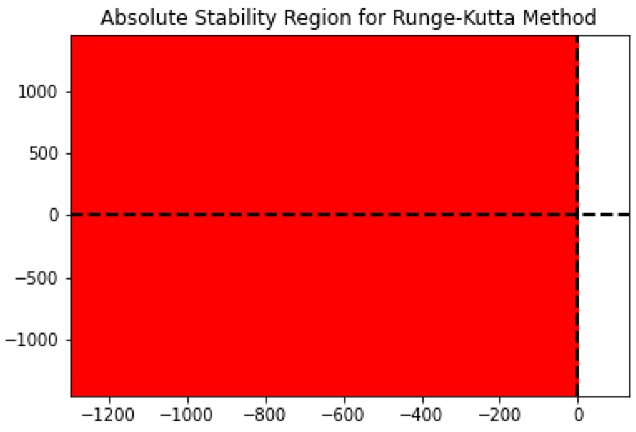
Explicit Euler



Implicit Euler



Crank-Nicolson



Eigenvalues of the spatial semidiscretization

$$A := \frac{1}{\Delta x^2} \underbrace{\begin{bmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & \dots & 0 \\ 0 & 1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -2 \end{bmatrix}}_{=:\tilde{A}}$$

- A is negative definite and symmetric
- A has real non-positive **real** eigenvalues
- The eigenvalues of A scale as $\frac{1}{\Delta x^2}$
- For explicit Euler we need $\Delta t < 2\frac{\Delta x^2}{\max_i \tilde{\lambda}_i}$ where $\tilde{\lambda}_i$ are te eigenvalues of $\tilde{A}$ independent of Δx and Δt.

  Very expensive!
- For implicit Euler and Crank-Nicolson, we are unconditionally (for every Δt) stable!

Von Neumann stability analysis

Lax-Richtmyer stability

Lax equivalence theorem