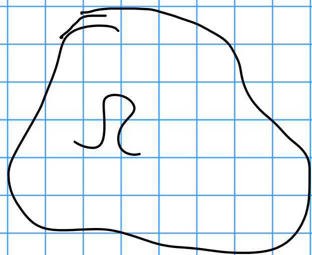


$\Omega \subset \mathbb{R}^d$

$u : \Omega \to \mathbb{R}^s \qquad s \in \mathbb{N}$

PDE STAZIONARIA DI ORDINE K

$$F(x, u, \nabla u, \nabla^{(2)} u, \dots, \nabla^{(k)} u, \underbrace{g}_{\text{DATI ESTERNI}}) = 0$$

$$\nabla u = \begin{pmatrix} \partial_{x_1} u \\ \partial_{x_2} u \\ \vdots \\ \partial_{x_d} u \end{pmatrix} \qquad \nabla^{(2)} u = \begin{pmatrix} \partial_{x_1}\partial_{x_1} u & \partial_{x_1}\partial_{x_2} u & \dots & \partial_{x_1}\partial_{x_d} u \\ \vdots & & & \\ \partial_{x_d}\partial_{x_1} u & , \dots & & \partial_{x_d}\partial_{x_d} u \end{pmatrix}$$

$$P(u, g) = F\left(x, u, \frac{\partial}{\partial x_1} u, \dots \frac{\partial}{\partial x_d} u, \frac{\partial^2}{\partial x_1 \partial x_1} u, \dots\right.$$

$$\frac{\partial^{p_1 + \dots + p_d}}{\partial^{p_1}_{x_1} \dots \partial^{p_d}_{x_d}} u \; : \; p_1 + \dots + p_d \le K$$

$$u \in C^K(\Omega)$$

PDE NONSTAZIONARIE t

$$P(u, g) = F\left(x, \underline{t}, u, \frac{\partial u}{\partial t}, \dots, \frac{\partial^2 u}{\partial t \partial x}, \dots\right.$$

$$\frac{\partial^{p_0 + p_1 \dots + p_d}}{\partial^{p_0} t \, \partial^{p_1} x_1 \dots \partial^{p_d} x_d} u \qquad p_0 + \dots p_d \le K$$

SOLUZIONE PDE $P(u, g)$ $\quad u \in C^K(\Omega \times [0, T])$

PRIMO ORDINE LINEARE

EQUAZIONE DEL TRASPORTO

? • $\partial_t u + \partial_x u = 0$

$\Omega = \mathbb{R}$

$t \in [0, T]$

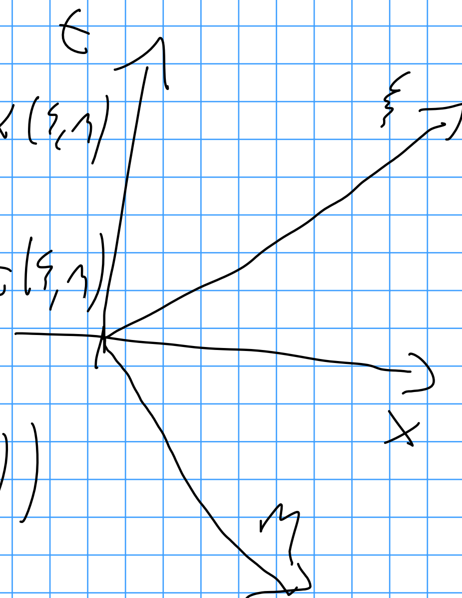
$u : \mathbb{R} \times [0,T] \to \mathbb{R}$

$(x, t) \to (\xi, \eta)$

$\xi(x,t) = \underbrace{x+t}$ $\qquad \eta(x,t) = x - t$

$$\frac{\xi + \eta}{2} = \frac{x+t+x-t}{2} = \frac{2x}{2} = x(\xi,\eta)$$

$$\frac{\xi - \eta}{2} = \frac{x+t-x+t}{2} = \frac{2t}{2} = t(\xi,\eta)$$



$$V(\xi,\eta) := u\left(x(\xi,\eta), t(\xi,\eta)\right)$$

$$u_x = V_\xi \cdot \xi_x + V_\eta \cdot \eta_x$$

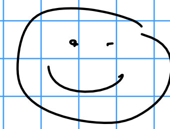
$$u_t = V_\xi \cdot \xi_t + V_\eta \cdot \eta_t$$

$$\begin{cases} \xi_x = \dfrac{\partial(x+t)}{\partial x} = 1 \\ \xi_t = 1 \\ \eta_x = \dfrac{\partial(x-t)}{\partial x} = 1 \\ \eta_t = \dfrac{\partial(x-t)}{\partial t} = -1 \end{cases}$$

$$\begin{cases} u_x = V_\xi + V_\eta \\ u_t = V_\xi - V_\eta \end{cases}$$

$$u_t + u_x = 0$$

$$V_\xi + \cancel{V_\eta} + V_\xi - \cancel{V_\eta} = 0$$

$2V_\xi = 0 \Rightarrow$ ODE! 

$\Downarrow$

$V =$ COSTANTE DIPENDE DA η

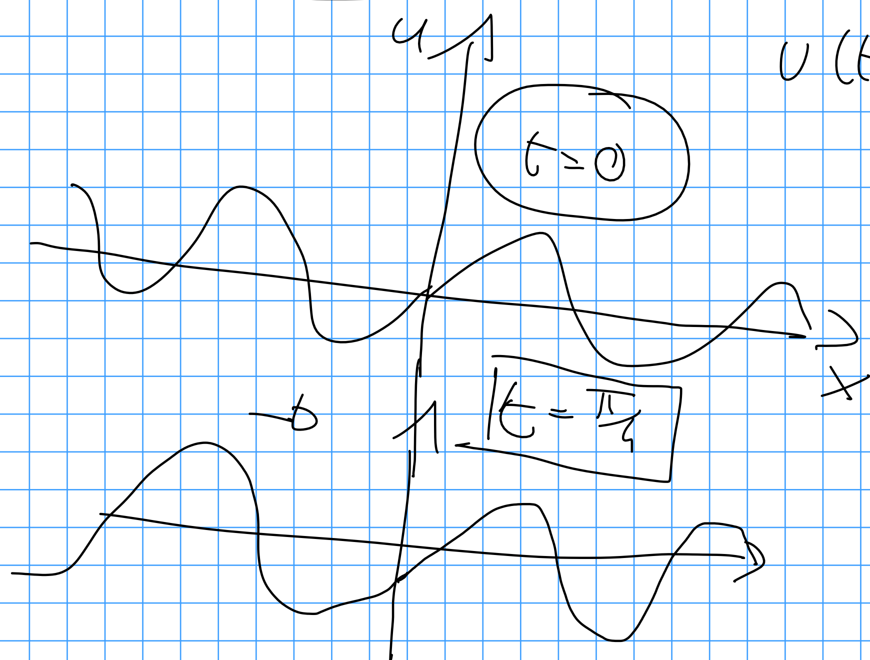
$V(\xi,\eta) = f(\eta) \qquad f \in C^1(\mathbb{R})$

$u(x,t) = V(\xi(x,t), \eta(x,t)) = f(\eta) = f(x-t)$

$\boxed{u_t + u_x = 0}$

$u(t=0, x) = \sin(x) = f(x-0)$
$= f(x)$

$u(t=\frac{\pi}{4}, x) = \sin(x - \frac{\pi}{4})$

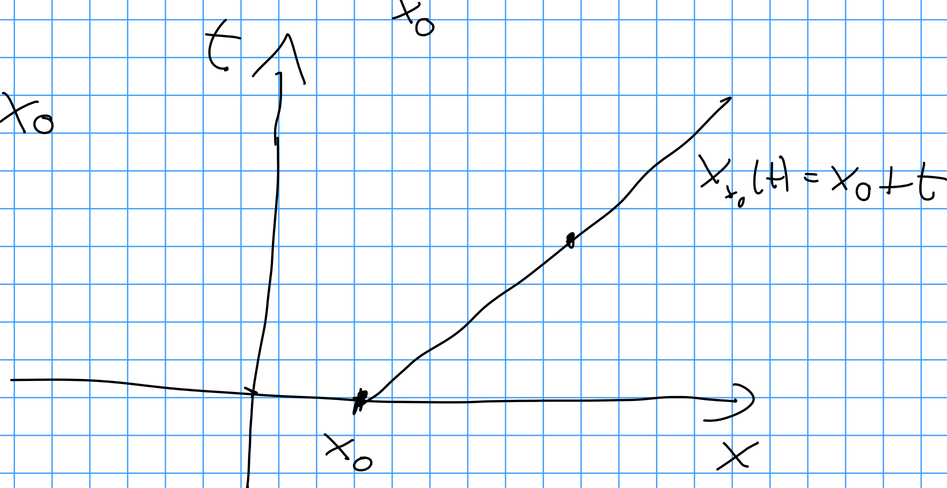


$\boxed{X_{x_0}(t) = x_0 + t}$

$\Downarrow$

$X_{x_0}(t) - t = x_0$

$f(\underbrace{x-t}_{x_0}) = f(x_0)$



$$u_t + u_x = 0 \to v_\xi = 0$$

$$a(t,x)\, u_t + b(t,x)\, u_x + c(x,t)\, u = g(t,x)$$

$$u : \mathbb{R} \times [0,T] \to \mathbb{R}$$

$$(t,x) \to (\xi, \eta)$$

$$\left| \frac{\partial(\xi,\eta)}{\partial(t,x)} \right| = \begin{vmatrix} \xi_t & \xi_x \\ \eta_t & \eta_x \end{vmatrix} = \xi_t \eta_x - \xi_x \eta_t \neq 0$$

$$u_t = v_\xi \xi_t + v_\eta \eta_t \ ; \quad u_x = v_\xi \xi_x + v_\eta \eta_x$$

$$a(t,x)\left(v_\xi \xi_t + \overline{v_\eta \eta_t} \right) + b(t,x) \left(v_\xi \xi_x + \overline{v_\eta \eta_x} \right)$$

$$+ c(t,x) \cdot v = g(t,x)$$

$$t(\xi,\eta)$$

$$x(\xi,\eta)$$

$$a(t,x)\, v_\eta \eta_t + b(t,x)\, v_\eta \eta_x = 0$$

$$a v_\xi \cdot \xi_t + b v_\xi \xi_x + c \cdot v = g$$

$$v_\xi (a \xi_t + b \xi_x) + c v = g$$

$$v_\xi + \frac{c \cdot v}{a \xi_t + b \xi_x} = \frac{g}{a \xi_t + b \xi_x}$$

$$\boxed{a \eta_t + b \eta_x = 0} \quad \text{A}$$

$$\eta(t, x(t)) = \eta_0$$

$$0 = \frac{d \eta(t, x(t))}{dt} = \eta_t + \eta_x \cdot \frac{dx}{dt} = 0$$

$$\boxed{\frac{\eta_t}{\eta_x}} = -\frac{dx}{dt} \qquad \text{A} \quad a\frac{\eta_t}{\eta_x} = -b$$

$$-\frac{dx}{dt} = -\frac{b}{a}$$

$$\frac{dx}{dt} = \frac{b}{a} \Rightarrow \underbrace{x(t)} = t\frac{b}{a} + \eta_0$$

$$\Rightarrow \eta_0 \Rightarrow \underbrace{\eta = x - t\frac{b}{a}}$$

- MATHTALK 13-15 MARTIN FLOURLEL
  GIT + COMMAND LINE

$$a\eta_t + b\eta_x = 0 \leftarrow \frac{\eta_t}{\eta_x} = -\partial_t x(t)$$

$$\frac{\eta_t}{\eta_x} = -\frac{b}{a} \Rightarrow -\partial_t x(t) = -\frac{b}{a}$$

$$\partial_t x(t) = \frac{b}{a}$$

$$\eta(t, x(t)) = \eta_0$$

ESEMPIO $\rightarrow x u_t - t u_x = 1$

$$(\xi, \eta) \leftarrow (t, x)$$

$$v(\eta, \xi) = u(t(\eta, \xi), x(\eta, \xi))$$

$$u(t, x) = v(\eta(t, x), \xi(t, x)) \leftarrow$$

$$u_t = v_\eta \cdot \eta_t + v_\xi \xi_t$$

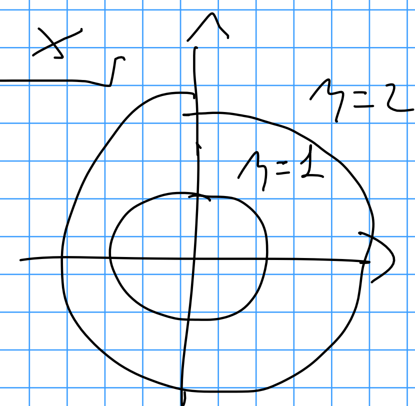
$$u_x = v_\eta \cdot \eta_x + v_\xi \xi_x$$

$$x \left(\eta_t \cdot v_\eta + \xi_t v_\xi\right) - t \cdot \left(\eta_x v_\eta + \xi_x v_\xi\right) = 1$$

$$\underbrace{(x \eta_t - t \eta_x)}_{=0} \cdot v_\eta + (x \xi_t - t \xi_x) v_\xi = 1$$

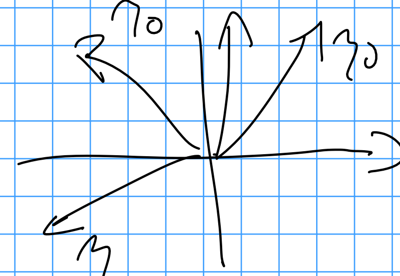
$$x \eta_t - t \eta_x = 0 \qquad \eta_0 = \eta(t, x(t))$$

$$\underbrace{-\frac{\eta_t}{\eta_x} = + \frac{dx}{dt} = -\frac{t}{x}}\qquad \underbrace{\frac{dx}{dt} = -\frac{t}{x}}$$



$$\int_{x_0}^{x(t)} x \, dx = \int_{t_0}^{t} -t \, dt$$

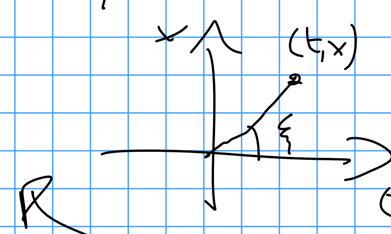
$$\frac{x(t)^2}{2} - \frac{x_0^2}{2} = -\frac{t^2}{2} + \frac{t_0^2}{2}$$



$$x(t)^2 + t^2 = \eta_0^2$$

$$\eta(t, x) = t^2 + x^2$$

$$\frac{d}{dt}\left(\frac{x}{t}\right)$$



$$\xi(t, x) = \arctan\left(\frac{x}{t}\right)$$

$$\eta_t = 2t \qquad \eta_x = 2x \qquad \xi_t = \frac{1}{1 + \left(\frac{x}{t}\right)^2} \cdot \left(-\frac{x}{t^2}\right) = -\frac{x}{x^2+t^2}$$

$$\xi_x = \frac{t}{x^2+t^2}$$

$$(x\xi_t - t\xi_x)\, V_\xi = 1$$

$$\left(x\left(-\frac{x}{x^2+t^2}\right) - t \cdot \frac{t}{x^2+t^2}\right) V_\xi = 1$$

$$-V_\xi = 1 \qquad V(\xi,\eta) = -\xi + f(\eta) \qquad f \in C^1$$

$$u(t,x) = V(\xi(t,x), \eta(t,x)) = -\arctan\left(\frac{x}{t}\right) + f(x^2+t^2)$$

- SLIDE 20 $y \to t$ CORREGGI

PDE LINEARI SECOND'ORDINE IN 2D

$$P(u,g) = A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u - g = 0$$

$$u \in C^2(\mathbb{R}^2) \qquad A(x,y) \qquad A \in C^2(\Omega)$$

$$\Delta = B^2 - 4AC$$

- $\Delta > 0 \Rightarrow$ PDE IPERBOLICHE
- $\Delta = 0 \Rightarrow$ PDE PARABOLICHE
- $\Delta < 0 \Rightarrow$ PDE ELLITTICHE

- EQUAZIONE DELLE ONDE

$$\partial_{tt} u - c^2 \partial_{xx} u = 0 \qquad c^2 \text{ VELOCITÀ DEL SUONO}$$

$$\Delta = 0 + 4c^2 = 4c^2 > 0 \Rightarrow \text{IPERBOLICA}$$

- EQUAZIONE DEL CALORE

$$u_t - c\,u_{xx} = 0 \qquad c > 0$$

$$\Delta = 0 + 0 = 0 \Rightarrow \text{PARABOLICA}$$

- POISSON

$$\underbrace{-c\Delta u}_{\text{LAPLACIANO}} = -c\,\partial_{xx}u - c\,\partial_{yy}u = f$$

$$\underbrace{\Delta}_{\text{DISCRIMINANTE}} = 0 - 4c^2 = -4c^2 < 0 \Rightarrow \text{ELLITTICA}$$

- TRICOMI

$$y \cdot u_{xx} + u_{yy} = 0$$

$$\Delta = 0 - 4y \Rightarrow \begin{cases} y > 0 \Rightarrow \text{ELLITTICA} \\ y = 0 \Rightarrow \text{PARABOLICO} \\ y < 0 \Rightarrow \text{IPERBOLICA} \end{cases}$$

$\boxed{\text{TEO}}$ IL SEGNO DI Δ NON CAMBIA AL CAMBIO DI VARIABILI.

$\boxed{\text{DIM}}$ $(x,y) \mapsto (\xi,\eta)$ $\quad \det\begin{pmatrix} \xi_t & \xi_x \\ \eta_t & \eta_x \end{pmatrix} \neq 0$

$$u_{xx} = (u_x)_x = (\underbrace{u_\eta \eta_x}_{\checkmark} + \underbrace{u_\xi \xi_x}_{})_x$$

$$= ((u_{\eta\eta}\eta_x + u_{\eta\xi}\xi_x)\cdot \eta_x + u_\eta \eta_{xx})$$

$$+ \left(u_{\xi\eta}\eta_x + u_{\xi\xi}\xi_x\right)\xi_x +$$ PRIM'ORDINE

$$= u_{\eta\eta}\eta_x^2 + 2u_{\eta\xi}\left(\xi_x \eta_x\right) + u_{\xi\xi}\xi_x^2$$

$$u_{xy} = \alpha\gamma\, u_{\xi\xi} + (\alpha\delta + \beta\gamma)\, u_{\xi\eta} + \beta\delta\, u_{\eta\eta}$$

$$u_{yy} = \gamma^2 u_{\xi\xi} + 2\gamma\delta\, u_{\xi\eta} + \delta^2 u_{\eta\eta}$$

$$\alpha = \xi_x \qquad \beta = \eta_x \qquad \gamma = \xi_y \qquad \delta = \eta_y$$

$$A\left(\alpha^2 u_{\xi\xi} + 2\alpha\beta\, u_{\xi\eta} + \beta^2 u_{\eta\eta}\right) +$$
$$+ B\left(\alpha\gamma\, u_{\xi\xi} + (\alpha\delta+\beta\gamma)\, u_{\xi\eta} + \beta\delta\, u_{\eta\eta}\right) +$$
$$+ C\left(\gamma^2 u_{\xi\xi} + 2\gamma\delta\, u_{\xi\eta} + \delta^2 u_{\eta\eta}\right) = 0$$

$$A' u_{\xi\xi} + B' u_{\xi\eta} + C' u_{\eta\eta} +$$ PRIM'ORDINE $= 0$

$\Rightarrow A' = A\alpha^2 + 2B\alpha\gamma + C\gamma^2$

$\Rightarrow B' = A\, 2\alpha\beta + B(\alpha\delta + \beta\gamma) + C\, 2\gamma\delta$

$\Rightarrow C' = A\beta^2 + 2B\beta\delta + C\delta^2$

$B'^2 - 4A'C'$

$$\Delta' = \cancel{4\alpha^2\beta^2 A^2} + B^2(\alpha\delta + \beta\gamma)^2 \cancel{+4C^2\, 4\gamma^2\delta^2}$$
$$+ 4\alpha\beta AB(\alpha\delta + \beta\cancel{\gamma}) + 8\alpha\beta AC\gamma\delta$$
$$+ 4BC(\alpha\delta + \beta\gamma)\gamma\delta$$
$$- \cancel{4A^2\alpha^2\beta^2} - 8AB\alpha^2\beta\delta - 4AC\alpha^2\delta^2$$

$$- 4 \cancel{AB} \beta^2 \alpha\gamma - 8 B^2 \alpha\gamma\beta\delta - 4 BC \alpha\gamma\delta^2$$

$$- 4 AC \underline{\beta^2\gamma^2} - 8 BC \beta\delta\gamma^2 - \cancel{4 C^2 \delta^2} \gamma^2$$

• SLIDE 15 CORNEGGI

$$= \underbrace{(B^2 - 4AC)}_{\Delta} \overbrace{\underbrace{(\alpha\delta - \beta\gamma)^2}_{\neq}}^{> 0}$$

$$\det\left(\frac{\partial(\xi,\eta)}{\partial(x,y)}\right) = (\eta_x \xi_y - \eta_y \xi_x) \neq 0$$

$$\Delta > 0 \longrightarrow \Delta' > 0$$

$$\Delta = 0 \longrightarrow \Delta' = 0$$

$$\Delta < 0 \longrightarrow \Delta' < 0$$

EQUAZIONE DELLE ONDE (IPERBOLICA)

$$\partial_{tt} u - c\, \partial_{xx} u = 0 \qquad c > 0$$

$\hookrightarrow$ GOAL $\rightsquigarrow$ $\partial_{\xi\eta} v = 0$

$$v(\eta, \xi) = u(x(\eta,\xi), t(\eta,\xi))$$

$\boxed{c=1}$ $\to$ $\eta = x + t \qquad \xi = x - t$

$$\underbrace{v_{\xi\eta} = 0}$$

$$v(\xi,\eta) = \int^{\xi}\int^{\eta} \overbrace{\partial_{wz} v(w,z)}^{=0} \, dz\, dw =$$

$$= \int^{\xi} f(w)\,dw = \underbrace{F(\xi)} + G(\eta)$$

$$\partial_\xi F(\xi) = f(\xi)$$

$$u = \underbrace{F(x-t)} + G(x+t)$$

POSSIAMO RISCRIVERE UN equazione IPERBOLICA IN FORMA CANONICA $V_{\xi\eta} = 0$? (SEMPRE)

$$A\,\xi_x^2 + B\,\xi_x \xi_y + C\,\xi_y^2 = 0$$

$$A\,\frac{\xi_x^2}{\xi_y^2} + B\,\frac{\xi_x}{\xi_y} + C = 0$$

- IPERBOLICHE $\Delta > 0 \Rightarrow$ 2 LINEE CARATTERISTICHE

  $\Rightarrow$ CAMBIO DI VARIABILI : $\rightarrow$ PDE CANONICA

  $$B' \cdot V_{\xi\eta} = 0$$

- PARABOLICHE $\Delta = 0 \Rightarrow$ 1 SOLUZIONE

  $\Rightarrow$ 1 LINEA CARATTERISTICA

$$x(t) = -\frac{B}{2A} t + \xi_0$$

$$\xi(x,t) = x + \frac{B}{2A} t \qquad \eta = x$$

⇒ PARABOLICA CANONICA

$$A\, v_{\zeta\zeta} = 0$$

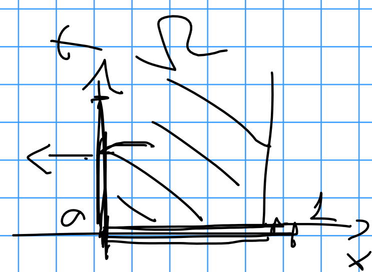
$$v(\xi,\eta) = \int^{\zeta}\int^{\xi} \overbrace{v_{\zeta\zeta}}^{=0}\, d\xi\, d\zeta = \int^{\zeta} f(\eta)\, d\zeta =$$

$$\zeta \cdot \underbrace{f(\eta)} + \underbrace{g(\eta)}$$

• $\Delta < 0 \Rightarrow 0$ LINEE CARATTERISTICHE ELLITTICHE

→ CANONICA $\quad A\left(\frac{\partial^2 v}{\partial \zeta^2} + \frac{\partial^2 v}{\partial \eta^2}\right) = 0.$

PROBLEMA DI CAUCHY



• PDE ORDINE k $\qquad u : \Omega \to \mathbb{R} \qquad \Omega \subseteq \mathbb{R}^d$

S, $\qquad x \in S$

$S = \{x=0\} \cup \{t=0\}$

$n = n(x) \qquad n \perp S$

$f_0, f_1, \dots, f_{k-1} : S \to \mathbb{R}$

$u(x) = f_0(x) \quad \forall x \in S$

$$\underbrace{\frac{\partial u}{\partial n}}_{= \nabla u \cdot \vec{n}}(x) = f_1(x) \quad \forall x \in S \quad \dots \quad \frac{\partial^{k-1} u}{\partial n^{k-1}}(x) = f_{k-1}(x)$$

$\forall x \in S$

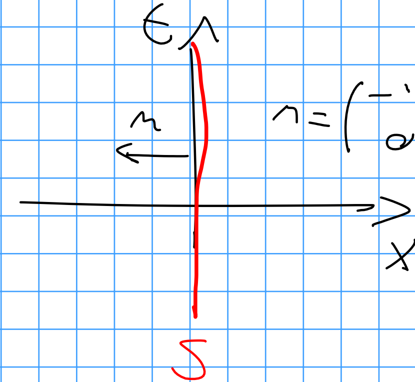
$$\begin{cases} u_t + u_x = 0 & \forall (x,t) \in \mathbb{R}^2 \\ u(0,x) = \sin(x) & \forall x \in \mathbb{R} \end{cases}$$

$$S = \{ (t,x) \in \mathbb{R}^2 : t = 0 \}$$

$$u(t,x) = f(x-t) \qquad u(0,x) = f(x) = \sin(x)$$

$$= \sin(x-t)$$

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in \mathbb{R}^2 \\ u(t,0) = \sin(t) & \forall t \in \mathbb{R} \\ u_x(t,0) = 0 & \forall t \in \mathbb{R} \end{cases}$$



$$n = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$S = \{ (t,x) : x = 0 \}$$

$$\frac{\partial u}{\partial n} = \begin{pmatrix} u_x \\ u_t \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -u_x$$

$$u(x,t) = f(\overset{\xi}{x-t}) + g(\overset{\eta}{x+t})$$

$$\begin{cases} u(t,0) = \sin(t) \\ u_x(t,0) = 0 \end{cases} \Rightarrow \begin{cases} f(-t) + g(t) = \sin(t) \\ f'(-t) + g'(t) = 0 \end{cases}$$

$$f(\xi) = \frac{1}{2}\sin(-\xi) \qquad g(\eta) = \frac{1}{2}\sin(\eta) \qquad \boxtimes$$

UNICITÀ $\qquad u(t,x) = \frac{1}{2}\left(\sin(x+t) + \sin(-x+t)\right)$