Finite Element Method for Elliptic Differential Equations

Pro and cons of the Finite Differences methods

Pro

- Easy to setup
- Simple implementation
- Easy high order stencils
- Easy truncation error analysis with Taylor expansion

Cons

- Difficult to generlize to more complex geometries
- Difficult to deal with boundaries + high order
- No general analysis of stability, existence, uniqueness

Can we write a discrete formulation of the type $a(u, v) = F(v)\,\forall v \in V$?

Goals:

1. find a discrete Hilbert space $V_h \subset V \subset H^1(\Omega)$
2. find a discrete bilinear form $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ continuous and coercive that approximates a
3. find a discrete linear form $F_h : V_h \to \mathbb{R}$ bounded that approximates F.

$h > 0$ is a parameter that describes the discretization scale of the discrete space (e.g. the minimum of the Δx in the mesh).

So, let's suppose that we have $V_h \subset V : \dim(V_h) = N_h < \infty\,\forall h > 0$.
Now, we can simply take a_h as the restriction of a on V_h and F_h as the restriction on V_h as well. This means that we can simply look for a solution $u_h \in V_h$ such that for every $v_h \in V_h$

$$a(u_h, v_h) = F(v_h).$$

This is called **Galerkin problem**.

Let's move to a basis of V_h

Let's consider a basis for V_h given by $\{\varphi_i\}_{j=1}^{N_h}$, since we are talking about linear and bilinear operators, we can instead look for the approximation $u_h \in V_h$ such that

$$a(u_h, \varphi_i) = F(\varphi_i), \qquad \forall i = 1, \ldots, N_h.$$

Moreover, also $u_h(x) = \sum_{j=1}^{N_h} u_j \varphi_j(x)$, we have

$$\sum_{j=1}^{N_h} a(\varphi_j, \varphi_i) u_j = F(\varphi_i), \qquad \forall i = 1, \ldots, N_h.$$

We can denote with A the *stiffness* matrix and with $\mathbf{f} \in \mathbb{R}^{N_h}$ the right-hand-side vector defined as

$$a_{ij} = a(\varphi_j, \varphi_i), \qquad f_i = F(\varphi_i).$$

The Galerkin problem can be written as a linear system for the vector $\mathbf{u} \in \mathbb{R}^{N_h}$

$$A\mathbf{u} = \mathbf{f}.$$

A is positive definite

A associated to the elliptic problem $a(u, v) = F(v)\, \forall v \in V$ where $a(\cdot, \cdot)$ is bilinear and coercive, then A is positive definite.

Proof

Recall that B is positive definite if $\mathbf{v}^\top B \mathbf{v} \geq 0 \quad \forall \mathbf{v} \in \mathbb{R}^n$ and $\mathbf{v}^\top B \mathbf{v} = 0 \Leftrightarrow \mathbf{v} = 0$.
The map

$$\mathbf{v} = (v_i) \in \mathbb{R}^{N_h} \leftrightarrow v_h(x) = \sum_{j=1}^{N_h} v_j \varphi_j(x) \in V_h$$

is a bijection between $\mathbb{R}^{N_h}$ and V_h. For any vector $\mathbf{v} \in \mathbb{R}^{N_h}$ we have

$$\begin{aligned} \mathbf{v}^\top A \mathbf{v} &= \sum_{j=1}^{N_h} \sum_{i=1}^{N_h} v_i a_{ij} v_j = \sum_{j=1}^{N_h} \sum_{i=1}^{N_h} v_i a(\varphi_j, \varphi_i) v_j \\ &= a\left(\sum_{j=1}^{N_h} v_j \varphi_j, \sum_{i=1}^{N_h} v_i \varphi_i \right) = a(v_h, v_h) \geq \alpha \|v_h\|_V^2 \geq 0. \end{aligned}$$

Moreover, if $\mathbf{v}^\top A \mathbf{v} = 0$ then $\|v_h\|_V = 0$, i.e. $v_h = 0$ and $\mathbf{v} = \mathbf{0}$.

Exercise

A is symmetric if and only if a is symmetric.

Analysis of Galerkin method

- Existence and uniqueness of a discrete solution u_h;
- Stability of the discrete solution u_h;
- Convergence of u_h towards the exact solution u for $h \to 0$.

Existence and uniqueness

Lax-Milgram lemma holds for any Hilbert space, so, also for V_h!!!

Moreover, $a(\cdot, \cdot)$ and $F(\cdot)$ are the same of the weak formulation.

Corollary

There exists one unique solution $u_h \in V_h$ of the Galerkin problem $a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$.

Alternative proof (as for FD)

A is positive definite, so invertible.

Stability

Following the Corollary of Lax-Milgram, we can say for the Galerkin method that

Corollary

Galkerin method is stable uniformly with respect to h since it holds

$$\|u_h\|_V \leq \frac{1}{\alpha}\|F\|_{V^*}.$$

Indeed,

$$\|u_h\|_V^2 = \|u_h\|_{V_h}^2 \leq \frac{1}{\alpha}a(u_h, u_h) = F(u_h) \leq \|F\|_{V^*}\,\|u_h\|_V.$$

Continuity on data

Let u_h be the solution for the Galerkin problem with F rhs and let w_h be the solution of the Galerkin problem with G rhs, then

$$\|u_h - w_h\|_V \leq \frac{1}{\alpha}\|F - G\|_{V^*}.$$

Convergence (1/n)

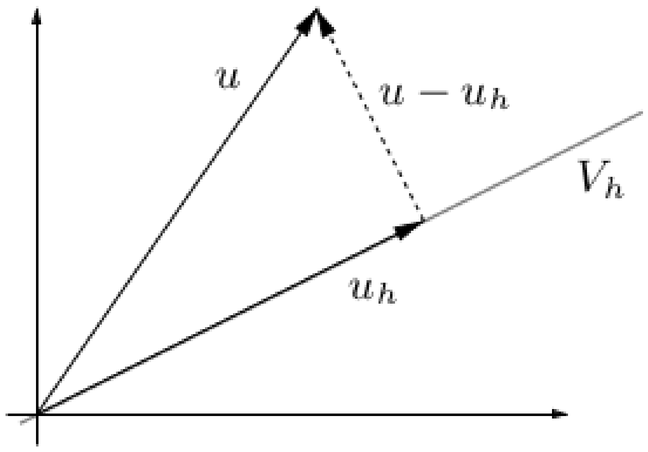
Goal: check that $u_h \to u$ for $h \to 0$ in V.

Galerkin Orthogonality

$$a(u - u_h, v_h) = 0 \qquad \forall v_h \in V_h.$$

Proof Bilinearity and

$$a(u, v_h) = F(v_h) = a(u_h, v_h), \qquad \forall v_h \in V_h \subset V.$$



Why orthogonality?

$a(\cdot, \cdot)$ is a scalar product in V if it's symmetric (since it's coercive). The associated norm is called **energy norm** and it is defined as

$$\|v_h\|_a = \sqrt{a(v_h, v_h)}.$$

u_h is the orthogonal projection of u onto V_h with the scalar product $a(\cdot, \cdot)$.
In particular, u_h is the minimizer of the error in energy norm

$$u_h = \arg\min_{v_h \in V_h} \|v_h - u\|_a.$$

Convergence (2/n) (Céa's Lemma)

Let $v_h \in V_h$, compute

$$a(u-u_h, u-u_h) = a(u-u_h, u-v_h) + \underbrace{a(u-u_h, v_h-u_h)}_{=0 \text{ since } v_h-u_h \in V_h}$$

Moreover, using the continuity constant C of $a(\cdot,\cdot)$, we have

$$|a(u-u_h, u-v_h)| \leq C\|u-u_h\|_V \|u-v_h\|_V.$$

On the other side, using the coercivity constant α, we have that

$$\|u-u_h\|_V^2 \leq \frac{1}{\alpha} a(u-u_h, u-u_h) \leq \frac{C}{\alpha}\|u-u_h\|_V \|u-v_h\|_V$$

So,

$$\|u-u_h\|_V \leq \frac{C}{\alpha}\|u-v_h\|_V \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u-v_h\|_V.$$

Convergence (3/3)

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V.$$

Céa's lemma tells us that even if u_h is not the best approximation for the V norm in V_h, its error will decrease as the best approximation error will decrease.
So we can just enlarge the space V_h, i.e., let $h \to 0$ so that the discrete space saturates the space V. i.e.,

$$\lim_{h \to 0} \inf_{v_h \in V_h} \|v - v_h\|_V = 0, \qquad \forall v \in V.$$

Then, we will have convergence of the Galerkin method also in the $\|\cdot\|_V$ norm!

Order of convergence

$$\inf_{v_h \in V_h} \|v - v_h\|_V = O(h^p) \implies \|u - u_h\|_V = O(h^p).$$

Finite element method (1 dimension)

Take $\Omega = (a, b)$ and we want to approximate $H^1((a, b))$ with a space depending on a scale h. Consider a partition of (a, b) called $\mathcal{T}_h$ composed of N intervals $K_j := (x_{j-1}, x_j)$ with $j = 1, \dots, N$ with size $h_j = x_j - x_{j-1}$ for $j = 1, \dots, N$ with

$$a = x_0 < x_1 < \dots < x_N = b$$

and we set $h = \max_{j=1,\dots,N} h_j$.

Motivational: Since $H^1((a, b)) \subset C^0([a, b])$ (slide 16 of lesson_012), we can look for a continuous functions in V_h (not really necessary).

$$X_h^r = \left\{ v_h \in C^0(\bar{\Omega}) : v_h|_{K_j} \in \mathbb{P}^r(K_j) \text{ for every } K_j \in \mathcal{T} \right\} \subset H^1((a, b)).$$

Let's choose a basis

$V_h = X_h^R = \langle \varphi_1, \dots, \varphi_{N_h} \rangle$
Choices to have simple life and sparse A:

- Lagrangian
- As local as possible

X_h^1

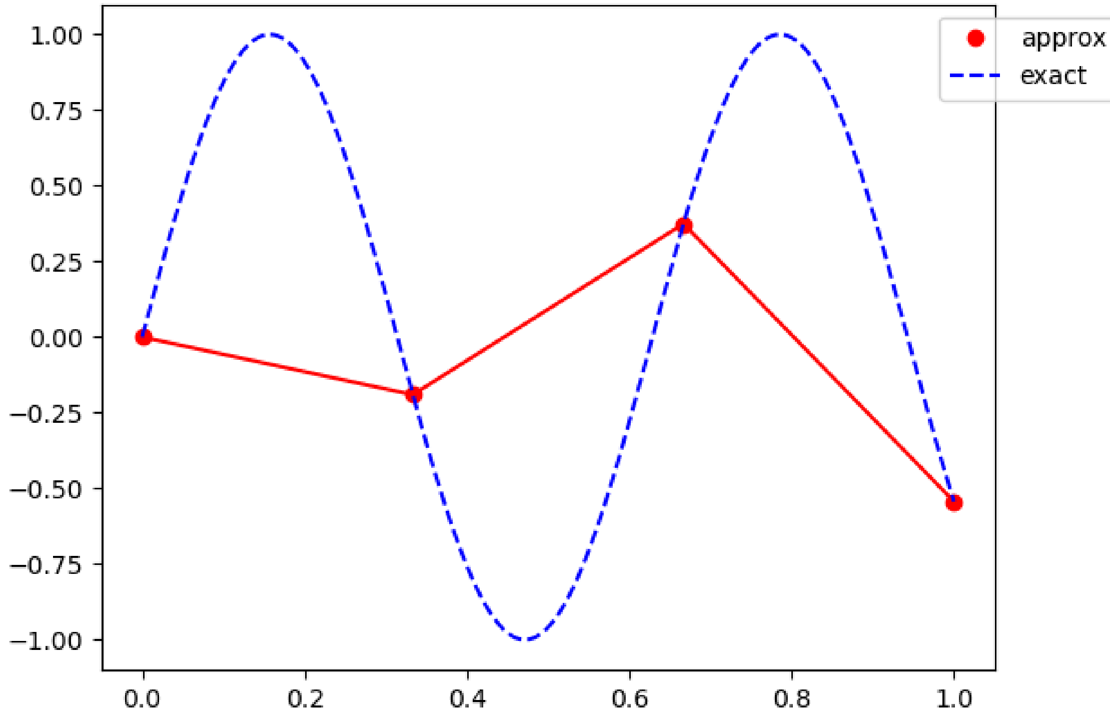
Piece-wise linear functions!

How many degrees of freedom do we have? For every cell K_j with $j = 1, \dots, N$ there are two coefficients to choose (to define a line): $2N$ possibilities, in every vertex x_j for $j = 1, \dots, N-1$ we have to impose continuity: $N-1$ constraints. Total $N+1$ degrees of freedom.

Can we find a practical way to define such degrees of freedom?

A line can be defined through two points values, if we choose exactly the values of the function in the points x_j for $j = 0, \dots, N$ then we have

- exactly $N+1$ degrees of freedom
- lines in each cell
- continuity.



Lagrangian basis functions for each cell

So, we take $\varphi_i \in X_h^1$ such that

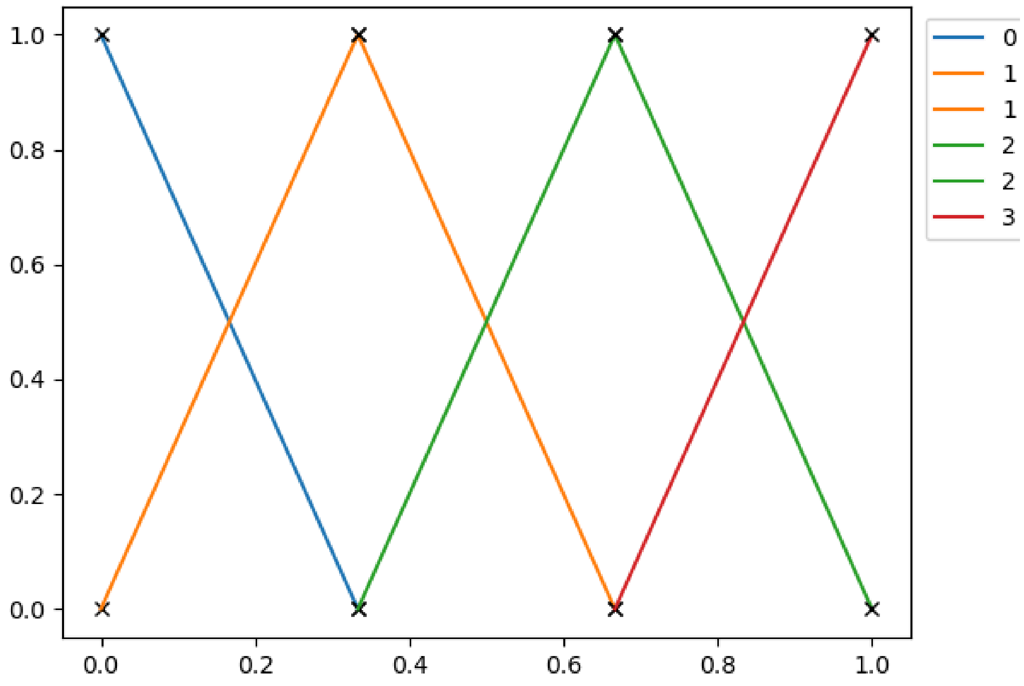
$$\varphi_i(x_i) = \delta_{ij}, \qquad \forall i, j = 0, \ldots, N.$$

More specifically,

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} \leq x < x_i, \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x_i \leq x < x_{i+1}, \\ 0 & \text{else.} \end{cases}$$

$$\operatorname{supp}(\varphi_i) = (x_{i-1}, x_{i+1})$$

$$a_{ij} \neq 0 \iff j \in \{i-1, i, i+1\}.$$



Reference element

It is useful to define every basis function onto a reference element $[0, 1]$ and then transform the basis functions onto the physical element $[x_{i-1}, x_i]$.
We use a linear transformations $T_{i+1} : [0, 1] \to [x_i, x_{i+1}]$ defined as

$$x = T_{i+1}(\xi) = x_i + \xi(x_{i+1} - x_i), \qquad \xi = T_{i+1}^{-1}(x) = \frac{x - x_i}{x_{i+1} - x_i}.$$

So the two basis functions in the reference element can be defined as

$$\hat{\varphi}_0(\xi) = 1 - \xi, \qquad \hat{\varphi}_1(\xi) = \xi,$$

this means that

$$\varphi_i(x) = \hat{\varphi}_0(T_{i+1}^{-1}(\xi)) = \hat{\varphi}_0\left(\frac{x - x_i}{x_{i+1} - x_i}\right), \qquad \varphi_{i+1}(x) = \hat{\varphi}_1(T_{i+1}^{-1}(\xi)) = \hat{\varphi}_1\left(\frac{x - x_i}{x_{i+1} - x_i}\right).$$

X_h^2

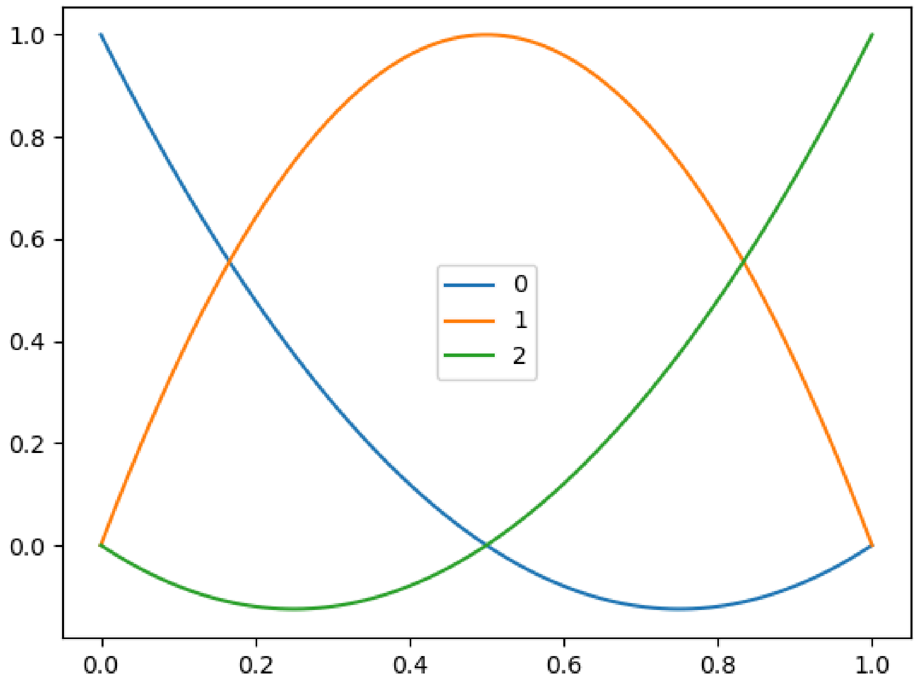
Let's work on the reference element.

Consider 3 equispaced points $\{0, \frac{1}{2}, 1\} \subset [0, 1]$ and the corresponding Lagrangian basis functions

$$\hat{\varphi}_0(\xi) = 2(\xi - \frac{1}{2})(\xi - 1),$$
$$\hat{\varphi}_1(\xi) = 4\xi(1 - \xi),$$
$$\hat{\varphi}_2(\xi) = 2\xi(\xi - \frac{1}{2}).$$

Alternatively, something gerarchical

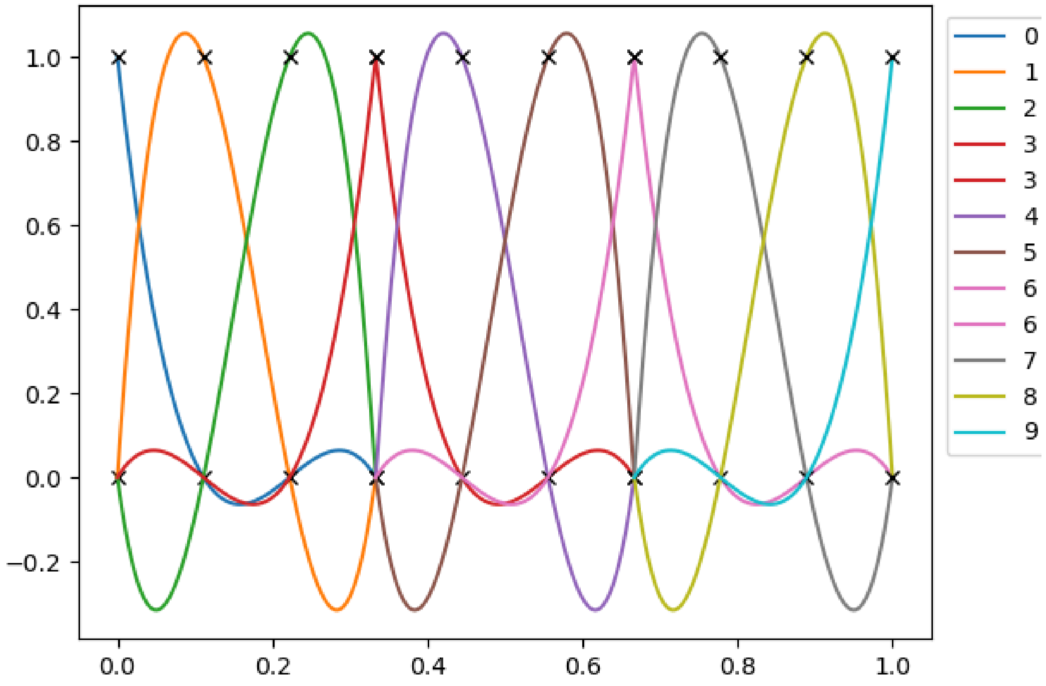
$$\hat{\varphi}_0(\xi) = 1 - \xi,$$
$$\hat{\varphi}_1(\xi) = \xi,$$
$$\hat{\varphi}_2(\xi) = \xi(1 - \xi).$$

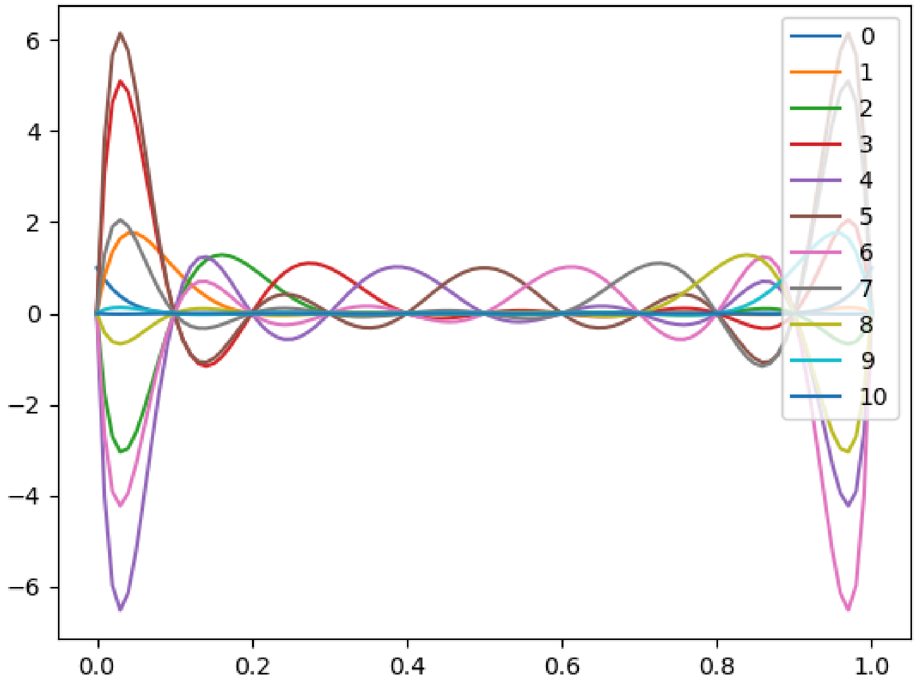


X_h^r

One can proceed with higher orders similarly. Using a reference element will help the construction.

For Lagrangian basis functions: careful with the choice of nodes inside the reference element! Equispaced might lead to Gibbs' phenomena!





Discrete problem!

General Poisson-reaction problem

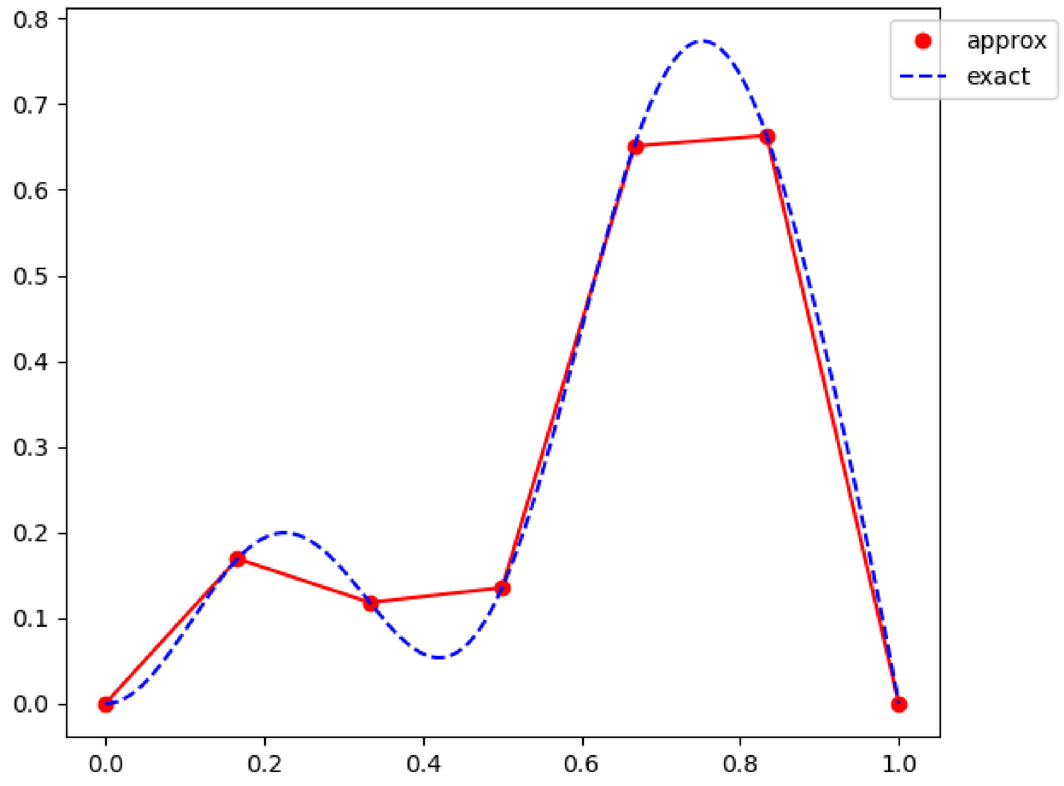
$$\begin{cases} -u'' + \sigma u = f, \quad x \in (a,b) \\ u(a) = 0, \\ u(b) = 0. \end{cases}$$

The weak formulation reads: find $u \in H_0^1((a,b))$ such that

$$\int_a^b u'v'\mathrm{d}x + \int_a^b \sigma uv\mathrm{d}x = \int_a^b fv\mathrm{d}x \qquad \forall v \in H_0^1((a,b)).$$

Discretely, choose
$V_h := \{v_h \in X_h^1 : v_h(a) = v(b) = 0\} \subset H_0^1((a,b))$,
so the discrete problem reads: find $u_h \in V_h$ such that

$$\int_a^b u_h'v_h'\mathrm{d}x + \int_a^b \sigma u_h v_h\mathrm{d}x = \int_a^b fv_h\mathrm{d}x \qquad \forall v_h \in V_h.$$



Assemble the problem!

We check for all basis functions $\varphi_i \in V_h$ instead of all $v_h \in V_h$ and we can expand $u_h(x) = \sum_{j=1}^{N_h} u_j \varphi_j(x)$, to obtain a system

$$\sum_{j=1}^{N_h} \int_a^b u_j \varphi_j'(x)\varphi_i'(x)\mathrm{d}x + \sum_{j=1}^{N_h} \int_a^b \sigma u_j \varphi_j(x)\varphi_i(x)\mathrm{d}x = \int_a^b f\varphi_i(x)\mathrm{d}x \qquad \forall i = 1, \dots, N_h.$$

So we get the linear system

$$A\mathbf{u} = \mathbf{f},$$

with

$$A = [a_{ij}], \quad a_{ij} = \int_a^b \varphi_j'(x)\varphi_i'(x)\mathrm{d}x + \int_a^b \sigma\varphi_j(x)\varphi_i(x)\mathrm{d}x, \qquad \mathbf{f} = [f_i],\; f_i = \int_a^b f\varphi_i(x)\mathrm{d}x$$

and $\mathbf{u} = [u_j]$ the unknown of our system.

Assemble the matrix!

We have seen that $\operatorname{supp}(\varphi_i) \subset [x_{i-1}, x_{i+1}]$, so the integrals

$$a_{ij} = \int_a^b \varphi_j'(x)\varphi_i'(x)\mathrm{d}x + \int_a^b \sigma\varphi_j(x)\varphi_i(x)\mathrm{d}x = 0 \qquad \text{for } |i-j| > 1.$$

So, we just need to compute the terms $a_{i,i-1}, a_{i,i}, a_{i,i+1}$.

Example

$$\begin{aligned} a_{i,i-1} &= \int_a^b \varphi_i'\varphi_{i-1}' + \sigma\varphi_i\varphi_{i-1}\,\mathrm{d}x = \int_{x_{i-1}}^{x_i} \varphi_i'\varphi_{i-1}' + \sigma\varphi_i\varphi_{i-1}\,\mathrm{d}x \\ a_{i,i} &= \int_a^b \varphi_i'\varphi_i' + \sigma\varphi_i\varphi_i\,\mathrm{d}x = \int_{x_{i-1}}^{x_i} \varphi_i'\varphi_i' + \sigma\varphi_i\varphi_i\,\mathrm{d}x + \int_{x_i}^{x_{i+1}} \varphi_i'\varphi_i' + \sigma\varphi_i\varphi_i\,\mathrm{d}x \end{aligned}$$

Focus on one integral

Change of variables into the reference domain! (Recall $\xi = T_i^{-1}(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}$)

$$\begin{aligned}
&\int_{x_{i-1}}^{x_i} \partial_x \varphi_i(x) \partial_x \varphi_{i-1}(x) + \sigma \varphi_i(x) \varphi_{i-1}(x) \, \mathrm{d}x \\
=&\int_0^1 \frac{\partial \xi}{\partial x} \partial_\xi \hat{\varphi}_1(\xi) \frac{\partial \xi}{\partial x} \partial_\xi \hat{\varphi}_0(\xi) + \sigma \hat{\varphi}_1(\xi) \hat{\varphi}_0(\xi) \frac{dT_i(\xi)}{d\xi} \mathrm{d}\xi \\
=&\frac{1}{h_i} \int_0^1 \partial_\xi \hat{\varphi}_1(\xi) \partial_\xi \hat{\varphi}_0(\xi) \mathrm{d}\xi + h_i \int_0^1 \sigma \hat{\varphi}_1(\xi) \hat{\varphi}_0(\xi) \mathrm{d}\xi
\end{aligned}$$

with $h_i = (x_i - x_{i-1})$.

If coefficients are constant, the integrals can be computed just for the reference element and then be multiplied by coefficients when assembling the bigger matrix!

Matrix structure for high order X_h^r

First of all, let's reorder the DoFs indexes: $K_i = [x_{i-1}, x - i]$, and we put inside some points that on the reference element we denote by $0 = \hat{y}_0 < \hat{y}_1 < \cdots < \hat{y}_r = 1$

$$y_\alpha = y_{(i,s)} = x_{i-1} + (x_i - x_{i-1})\hat{y}_s \qquad \text{for } i = 1, \ldots, N,\ s = 0, \ldots, r,$$

with the equivalence $y_{(i,0)} = y_{(i-1,r)}$.
So, we can map with a bijection the indexes $(i, s) \leftrightarrow \alpha = (i-1) \cdot r + s$ for $\alpha = 0, \ldots, rN$.

Draw a matrix example!

Error estimation (1/2)

Goal: move from error of Galerkin approximation to interpolation error.

Interpolation error

For a function $v \in C^0((a,b))$, take the interpolant of v in X_h^r as

$$\Pi_h^r v(x_i) = v(x_i), \quad \forall i = 0, \ldots, N_h.$$

Theorem (see Quarteroni for proof)

Let $v \in H^{r+1}((a,b))$ for $r \geq 1$ and let $\Pi_h^r v \in X_h^r$ its interpolant. It holds that

$$|v - \Pi_h^r v|_{H^k((a,b))} \leq C_{k,r} h^{r+1-k} |v|_{H^{r+1}((a,b))}, \qquad \text{for } k = 0, 1.$$

The constants $C_{k,r}$ are independent of v and h.

Error estimation (2/2)

Let $u \in V$ be the exact solution of the variational problem and u_h its Finite Element approximation with polynomials of degree r, with $u_h \in V_h = V \cap X_h^r$. Let $u \in H^{p+1}((a,b))$ for a $p \geq r$. Then, it holds

$$\|u - u_h\|_V \leq \frac{M}{\alpha} C h^r |u|_{H^{r+1}((a,b))}$$

with C independent of u and h.

Proof

It's trivial from the previous result and Céa's Lemma, i.e.,

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v \in V_h} \|u - v_h\|_V.$$

Boundary conditions

Dirichlet

As for finite differences, we can simply exclude the Dirichlet boundary DoFs from the system and solve for these DoFs the equation $u_0 = u(a)$ or $u_{N_h} = u(b)$ by setting

$$a_{11} = 1, \qquad f_1 = u(a), \qquad \text{or } a_{N_h N_h} = 1, \quad f_{N_h} = u(b).$$

Neumann

Recall the weak formulation for Neumann, for example for $u'(a) = g$ and $u(b) = \beta$, for all φ_i for $i = 1, \ldots, N_h - 1$

$$\int_a^b f\varphi_i \mathrm{d}x = \int_a^b -u''\varphi_i \mathrm{d}x = \int_a^b u'\varphi_i' \mathrm{d}x - [u'\varphi_i]_a^b = \int_a^b u'\varphi_i' \mathrm{d}x + u'(a)\varphi_i(a)$$

The only nonzero Neumann term is the one for $\varphi_0(a) = 1$. So,

$$\int_a^b u'\varphi_0' \mathrm{d}x = \int_a^b f\varphi_0 \mathrm{d}x - g$$

for the equation $i = 0$ we change only the right hand side adding the Neumann contribution.

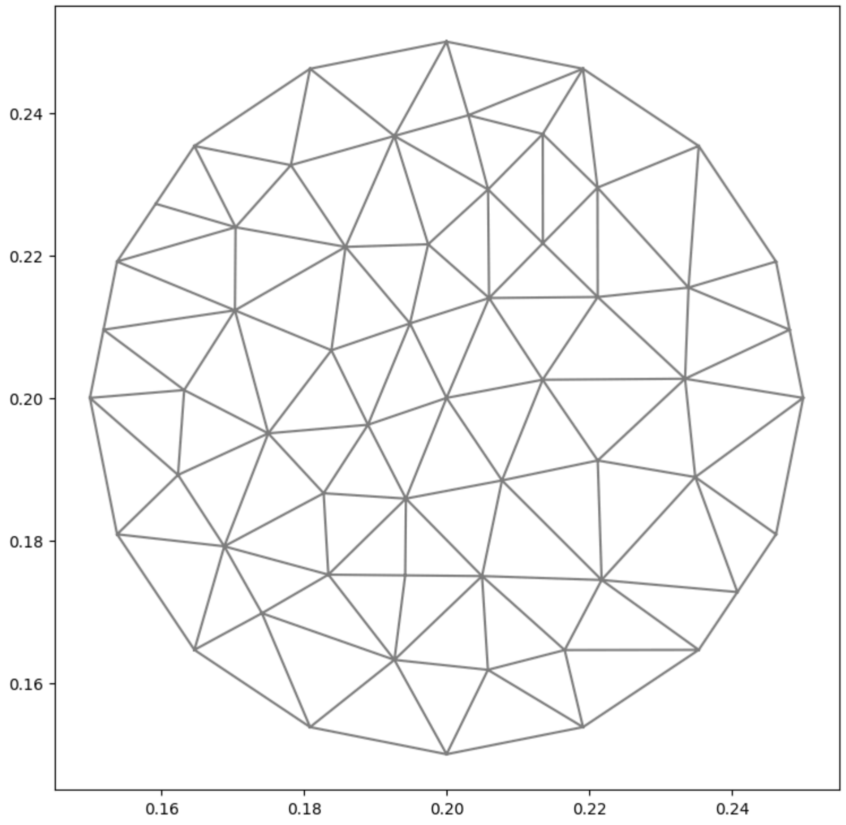
CODE IT!

2D Finite Elements

1. Discretize geometry
2. Define a reference element
3. Define structures on reference element
4. Assemble global matrices
5. Solve linear system

Discretize the geometry

1. Approximation of the shape (the circle is not anymore a circle)
2. $\Omega \subset \mathbb{R}^{d=2}$ approximated with $\Omega_h = \mathrm{int}\left(\cup_{K\in\mathcal{T}_h} K\right)$
3. $h = \max_{K\in\mathcal{T}_h} \mathrm{diam}(K)$ where $\mathrm{diam}(K) = \max_{x,y\in K} \|x-y\|$
4. Choice on the shape of the 2D basic elements K
    i. Quadrilaterals are easy to deal with, but are not as flexible as
    ii. Triangles/tetrahedrons are simpler to build given a geometry, simple also to use as fundamental object
    iii. Exahedron ...
    iv. Polyhedron ...
5. Regularity of the elements, ρ_K is the diameter of the circle/sphere inscribed in the triangle/tetrahedron etc. $\frac{h_K}{\rho_K} \leq \delta, \quad \forall K \in \mathcal{T}_h, \forall h > 0.$
6. There are some algorithms to create a mesh, we do not go into details, e.g. Delaunay.



Reference element = Triangle!

Goal: build polynomial space on 2D reference element.

Reference element: we choose the tringle with vertices $\mathbf{x}_1 = (0,0)$, $\mathbf{x}_2 = (1,0)$, $\mathbf{x}_3 = (0,1)$.

Polynomial space: not clear. Degree p

1. $\mathbb{P}^p = \{f(x,y) = \sum_{ij:i+j\leq p} a_{ij}x^i y^j,\ a_{ij} \in \mathbb{R}\}$, e.g. $\mathbb{P}^1 = \{a + bx + cy : a,b,c \in \mathbb{R}\}$;
2. $\mathbb{Q}^p = \{f(x,y) = \sum_{ij:i,j\leq p} a_{ij}x^i y^j,\ a_{ij} \in \mathbb{R}\}$, e.g. $\mathbb{Q}^1 = \{a + bx + cy + d\,xy : a,b,c,d \in \mathbb{R}\}$.

For triangles $\mathbb{P}^p$ is more natural, for quadrilaterals $\mathbb{Q}^p$ is more natural.

Indeed, if we look for Lagrangian basis functions, in a triangle it might be useful to use the three vertices as Lagrangian points, hence, 3 linear basis function ($p=1$). In a quadrilateral, the four corner will lead to 4 basis functions.

$\mathbb{P}^1$

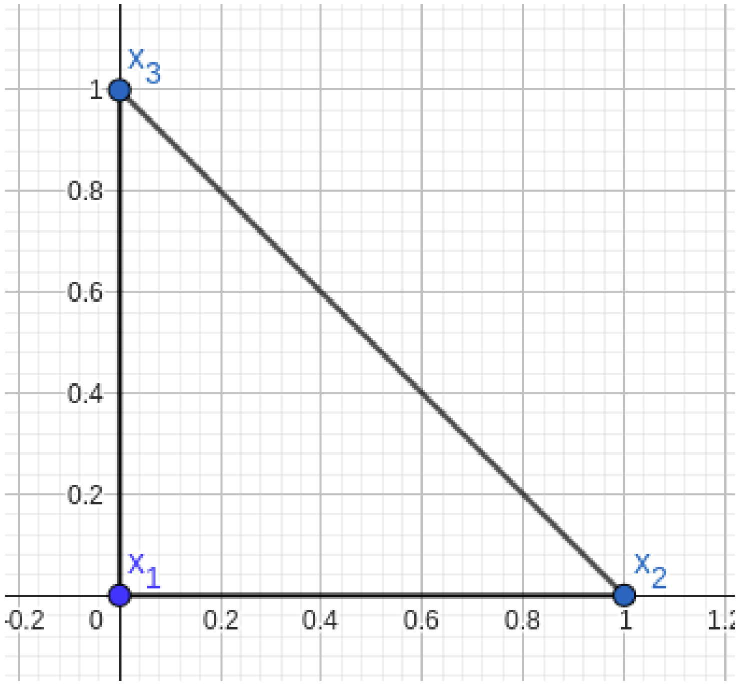
We are looking for 3 basis functions $\hat{\varphi}_i$ with $i = 1, 2, 3$ such that

- $\hat{\varphi}_i(\mathbf{x}_j) = \delta_{ij}$ for all $i = 1, 2, 3$
- $\hat{\varphi}_i(\mathbf{x}) = a_i + b_i x + c_i y$ for all $i = 1, 2, 3$

Let's solve this linear system and we get

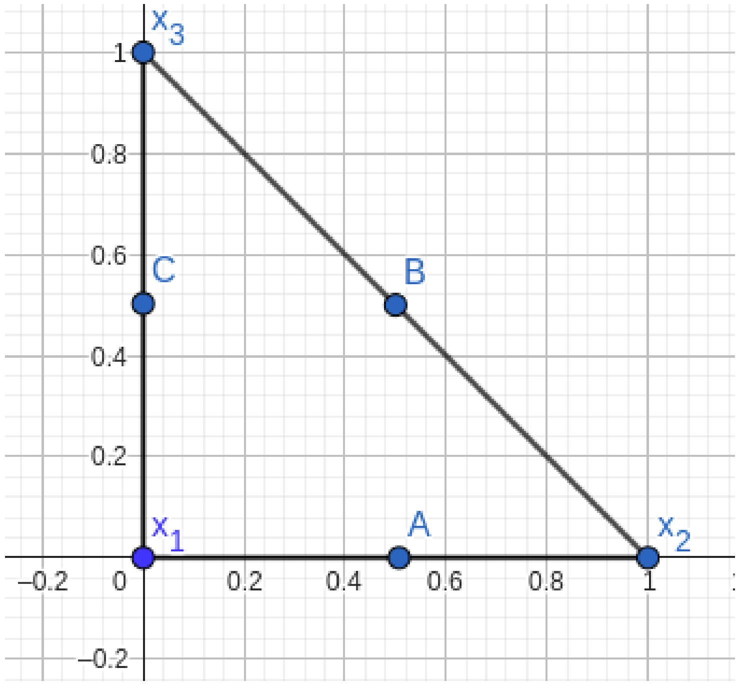
- $\varphi_1(\mathbf{x}) = 1 - x - y$
- $\varphi_2(\mathbf{x}) = x$
- $\varphi_3(\mathbf{x}) = y$

Draw a $\mathbb{P}^1$ basis function in 3D



$\mathbb{P}^2$

- How many coefficients? $1, x, y, xy, x^2, y^2$ are 6
- How many cofficients in general? $\frac{(p+1)(p+2)}{2}$ (check with induction noting that $\#DOF(p+1) = \#DOF(p) + p + 2$)
- Choice of points? Following the triangular pattern $x_4 = (0.5, 0), x_5 = (0.5, 0.5), x_6 = (0, 0.5)$.
  Basis functions must be
- $\varphi_1(\mathbf{x}) = 2(1 - x - y)(\frac{1}{2} - x - y)$
- $\varphi_2(\mathbf{x}) = 2x(x - \frac{1}{2})$
- $\varphi_3(\mathbf{x}) = 2y(y - \frac{1}{2})$
- $\varphi_4(\mathbf{x}) = 4x(1 - x - y)$
- $\varphi_5(\mathbf{x}) = 4xy$
- $\varphi_6(\mathbf{x}) = 4y(1 - x - y)$

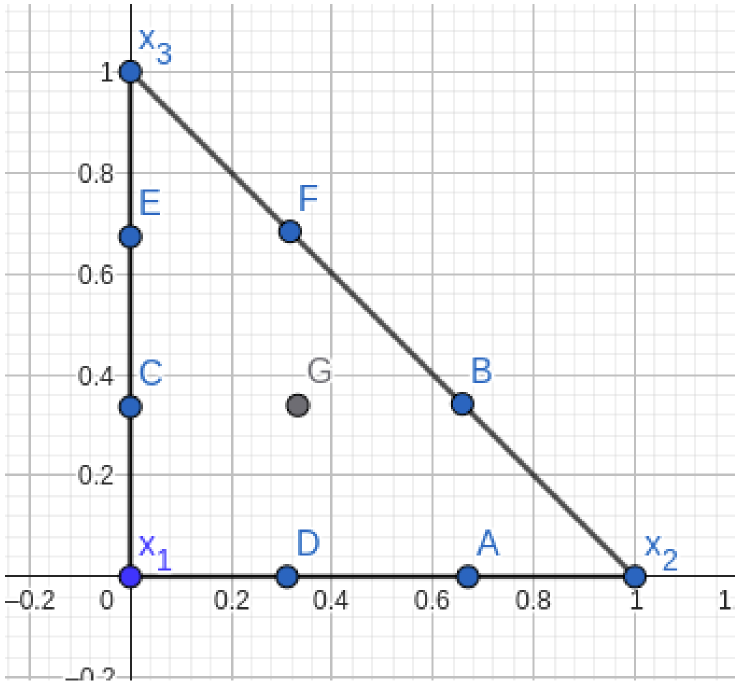


Exercise

Check that $\varphi_i(\mathbb{x}_j) = \delta_{ij}$, check that $\sum_{\varphi_i} \equiv 1$.

$\mathbb{P}^p$

- Once can generalize and obtain all the Lagrangian basis functions for all orders
- Chosen the Lagrangian points, solve for the Lagrangian basis functions
- Use them!



Assemble integrals in the reference element

As an example, in $\hat{K}$ the reference element, we might need to compute

$$\int_{\hat{K}} \nabla\hat{\varphi}_i(\mathbf{x}) \cdot \nabla\hat{\varphi}_j(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_0^1 \int_0^{1-x} \nabla\hat{\varphi}_i(\mathbf{x}) \cdot \nabla\hat{\varphi}_j(\mathbf{x}) \mathrm{d}y \, \mathrm{d}x$$

- Polynomials integrals -> small quadrature rule is enough for exact quadrature

Pull back on the physical elements

If we have a triangle K with vertices x_1, x_2, x_3, there exists a unique affine map that transform K into $\hat{K}$ that we can define as

$$T : \hat{K} \to K, \qquad \mathbf{x} = T(\hat{\mathbf{x}}) = A\hat{\mathbf{x}} + b, \quad A \in \mathbb{R}^{2\times 2}, \, b \in \mathbb{R}^2.$$

To find A and b there are some closed formula, or you can solve the linear system for the 6 coefficients with the 3 vertices (6 equations).

So, we can define the basis functions on K as the pull back of the reference basis functions: $\varphi_i(\mathbf{x}) = \hat{\varphi}_i(T^{-1}(\mathbf{x}))$.

Then, let's compute integrals on the physical domain

$$\begin{aligned}\int_K \nabla_{\mathbf{x}}\varphi_i(\mathbf{x}) \cdot \nabla_{\mathbf{x}}\varphi_i(\mathbf{x}) \mathrm{d}\mathbf{x} &= \int_{\hat{K}} \nabla_{\mathbf{x}} T^{-1}(\mathbf{x}) \nabla_{\hat{\mathbf{x}}}\hat{\varphi}_i(\hat{\mathbf{x}}) \cdot \nabla_{\mathbf{x}} T^{-1}(\mathbf{x}) \nabla_{\hat{\mathbf{x}}}\hat{\varphi}_i(\hat{\mathbf{x}}) \det(\nabla_{\hat{\mathbf{x}}} T(\hat{\mathbf{x}})) \mathrm{d}\hat{\mathbf{x}} \\ &= \int_{\hat{K}} A^{-1}\nabla_{\hat{\mathbf{x}}}\hat{\varphi}_i(\hat{\mathbf{x}}) \cdot A^{-1}\nabla_{\hat{\mathbf{x}}}\hat{\varphi}_i(\hat{\mathbf{x}}) \det(A) \mathrm{d}\hat{\mathbf{x}}\end{aligned}$$

or some quadrature rules for triangles that are mapped onto the physical one.

Assemble the whole problem

Many DoFs will have support on more triangles, so to gather all the information, we need to loop over the triangles, and sum all the contributions to various DoFs.

- Apply BC
- Solve the linear system