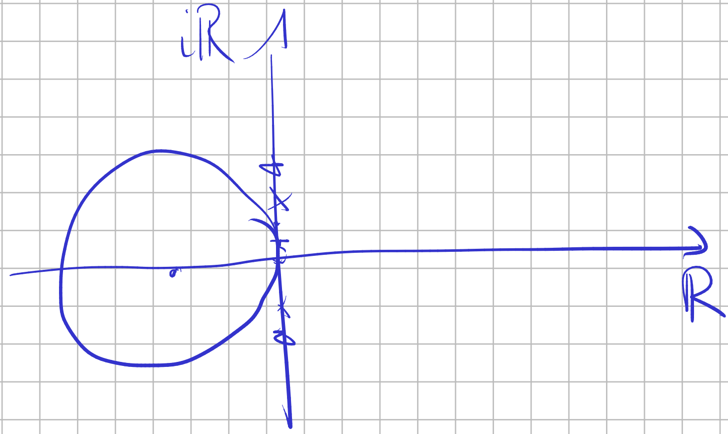
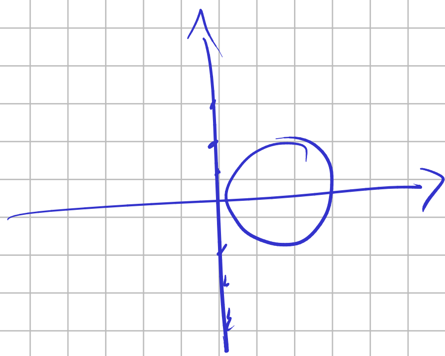
- LINEAR ADVECTION PARTE 2

$$|R(z)| < 1$$



- EULERO IMPLICITO



$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} = 0$$

$$0 = \frac{c_k^{n+1} e^{ikj\Delta x} - c_k^n e^{ikj\Delta x}}{\Delta t} + \frac{c_k^{n+1} e^{ik(j+1)\Delta x} - c_k^{n+1} e^{ik(j-1)\Delta x}}{2\Delta x}$$

$$c_k^{n+1}\left(1 + \frac{\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)\right) = c_k^n$$

$$|g(k)| = \left|\left(1 + \frac{\Delta t}{\Delta x} i \sin(k\Delta x)\right)^{-1}\right|$$

$$= |\,\#\,|^{-1}$$

$$= \Bigg|\underbrace{\sqrt{1 + \left(\frac{\Delta t}{\Delta x}\right)^2 \sin(k\Delta x)^2}}_{\geq 1}\Bigg|^{-1} \leq 1 \qquad \forall k, \Delta x, \Delta t$$

$$\|U^n\|_2 \leq \|U^{n-1}\|_2$$

$$\|U^n\|_2 < \|U^{n-1}\|$$

$\Downarrow$

se $k\Delta x \neq 0 + z\pi \quad z \in \mathbb{Z}$

$$U_j^{n+1} + \frac{\Delta t}{2\Delta x}\left(U_{j+1}^{n+1} - U_{j-1}^{n+1}\right) = U_j^n$$

$$\left(I + \frac{\Delta t}{2\Delta x} \cdot D\right) \underline{U}^{n+1} = \underline{U}^n$$

$$D = \begin{pmatrix} 0 & 1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{pmatrix}$$

$\Rightarrow$ NON È SYMMETRICA

$\Rightarrow$ NON È DEFINITA POSITIVA

ANCHE RISOLVERE SYS LINEARI NUMERICAMENTE ~~PUÒ~~ ESSERE + DIFFICILE

- TORNIAMO ALL'ESPLICITO!

$$U_j^{n+1} = U_j^n - a\frac{\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$U_j^{n+1} = \frac{U_{j+1}^n + U_{j-1}^n}{2} - a\frac{\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$= U_j^n + \underbrace{\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{2}} - a\frac{\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$C_k^{n+1} e^{ijk\Delta x} = C_k^n \left[e^{ikj\Delta x} + \frac{e^{ik(j+1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j-1)\Delta x}}{2} + \right.$$

$$- a \frac{\Delta t}{\Delta x} \frac{e^{ik(j+1)\Delta x} - e^{ik(j-1)\Delta x}}{2} \Big]$$

$$g(k) = 1 + \Big(\underbrace{\frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2}}_{\cos(k\Delta x)} - 1 \Big) - a \frac{\Delta t}{\Delta x} i \sin(k\Delta x)$$

$$= 1 + \underbrace{(\cos(k\Delta x) - 1)}_{\leq 0} - a \frac{\Delta t}{\Delta x} i \sin(k\Delta x)$$

$$\theta = k\Delta x$$

$$|g(k)| = \sqrt{\cos(\theta)^2 + a^2 \frac{\Delta t^2}{\Delta x^2} \sin^2(\theta)} \leq 1$$

$$\Leftarrow \text{se} \quad a^2 \frac{\Delta t^2}{\Delta x^2} \leq 1 \implies$$

$$\boxed{\Delta t \leq \frac{\Delta x}{|a|}} \implies \text{STABILE}$$

$$\Delta t \sim \Delta x$$

CFL CONDITION $\quad \Delta t \leq \frac{\Delta x}{|a|} \implies$ STABILITÀ

$$\Delta t := CFL \cdot \frac{\Delta x}{|a|} \qquad CFL \in \mathbb{R}^+$$

$$CFL \leq 1$$

CONSIGLIO PRATICO

$CFL \sim 0.9$

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{2} = \frac{1}{2}\underbrace{\Delta x^2}\cdot U_{xx} + O(\Delta x^4)$$

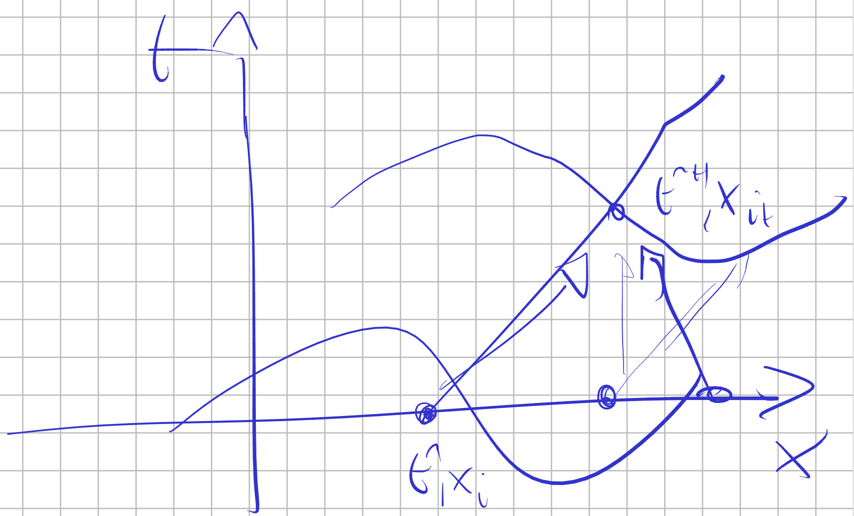
$$u(t+\Delta t, x) \approx u(t, x) + \frac{1}{2}\Delta x^2 \cdot U_{xx} \underbrace{- a\,\Delta t\, U_x}$$

$$\Delta t \sim \Delta x$$

$$\partial_t u + a\, u_x - \underbrace{\Delta x}\,\frac{1}{2}\cdot U_{xx} = 0$$

LAX-FRIEDRICHS è CONSISTENTE CON

$$\Delta x \to 0 \qquad \xrightarrow[\Delta x \to 0]{} \qquad U_t + a U_x = 0$$



$$\frac{U_{j+1} - U_{j-1}}{2\Delta x}$$

se $a > 0$

$$U_i^{n+1} = U_i^n - a\,\Delta t\,\frac{U_i - U_{i-1}}{\Delta x}$$

se $a < 0$

$$= U_i^n - a\,\frac{\Delta t}{\Delta x}\left(U_{i+1} - U_i\right)$$

$$u_j^{n+1} = U_j^n - a\,\Delta t\,\frac{\left(U_j^n - U_{j-1}^n\right)}{\Delta x} = U_j^n - \frac{a\,\Delta t}{2\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)$$

$$+ a\cdot\Delta t\,\frac{U_{j+1}^n - U_{j-1}^n - 2U_j^n + 2U_{j-1}}{2\Delta x}$$

$$= u_j^n - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) + a\frac{\Delta t}{2}\underbrace{\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x}}_{\Delta x\, \partial_{xx} u}$$

$$\frac{a}{2}\Delta t \Delta x \; u_{xx}$$

EULERO ESPL $\quad u' = \lambda u$

$$\boxed{u^{n+1} = u^n + \Delta t \lambda\, u^n}$$

$$= R(\underbrace{\Delta t \lambda}_{}) \cdot u^n$$

$$S = \{ z : |R(\underbrace{z}_{})| \leq 1 \}$$

$$C_k^{n+1} = C_k^n + (1+\cos$$

$$\varepsilon = C \cdot \Delta x$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \varepsilon\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\Delta x^2} = 0$$

$$\Downarrow$$

$$\underbrace{\partial_t u_j^n}_{} + \frac{\Delta t}{2}\partial_{tt} u_j^n + \underbrace{a\,\partial_x u_j^n}_{} + a\frac{\Delta t^2}{6}\partial_{xxx} u$$

$$- C\cdot\frac{\Delta x}{2}\partial_{xx} u = O(\Delta x^3) + O(\Delta t^2)$$

$$\underbrace{\frac{\Delta t}{2}}_{}\partial_{tt} u - C\frac{\Delta x}{2}\partial_{xx} u + O(\Delta x^2) + O(\Delta t^2)$$

PRIM'ORDINE TEMPO $\quad$ PRIM'ORDINE IN SPAZIO

$$\frac{\Delta t}{2}\partial_{tt}u - C\frac{\Delta x}{2}\partial_{xx}u = 0 \Rightarrow$$ SECOND'ORD. IN SP. e TEMP!

$$\partial_t u = -a\partial_x u$$

$$\partial_{tt}u = -a\partial_{xt}u = -a\,\partial_x\partial_t u = a^2\partial_{xx}u$$

$$\frac{\Delta t}{2}a^2\partial_{xx}u - c\frac{\Delta x}{2}\partial_{xx}u = 0$$

$$c\frac{\Delta x}{2} = \frac{\Delta t}{2}a^2 \qquad c = \frac{\Delta t}{\Delta x}\cdot a^2$$

$\Rightarrow$ SECOND'ORDINE.

- LAX WENDROFF

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \frac{\Delta t}{2}a^2\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = 0$$

- SECONDA STRATEGIA $\qquad u_t + a u_x = 0 \qquad u_{tt} = a^2 u_{xx}$

$$u_j^{n+1} \simeq u_j^n + \Delta t\,\partial_t u_j + \frac{\Delta t^2}{2}\partial_{tt}u_j$$

$$\simeq u_j^n - a\Delta t\,\partial_x u_j + \frac{\Delta t^2}{2}a^2\partial_{xx}u_j$$

$$= u_j^n - \underbrace{a\,\Delta t\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + a^2\frac{\Delta t^2}{2}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}}$$

$$g(k) = 1 - a\frac{\Delta t}{\Delta x}i\sin(\theta) + \frac{a^2\Delta t^2}{\Delta x^2}(\cos(\theta)-1)$$

$\theta = k\Delta x \qquad CFL = \frac{\Delta t}{\Delta x} \cdot a \qquad a > 0$

$$1 - CFL\, i \sin(\theta) + CFL^2 (\cos\theta - 1)$$

Se $CFL \leq 1 \Rightarrow$ STABILI

- UPWIND $\qquad \partial_t u_j = -a \dfrac{u_{j-2} - 4u_{j-1} + 3u_j}{2\Delta x}$

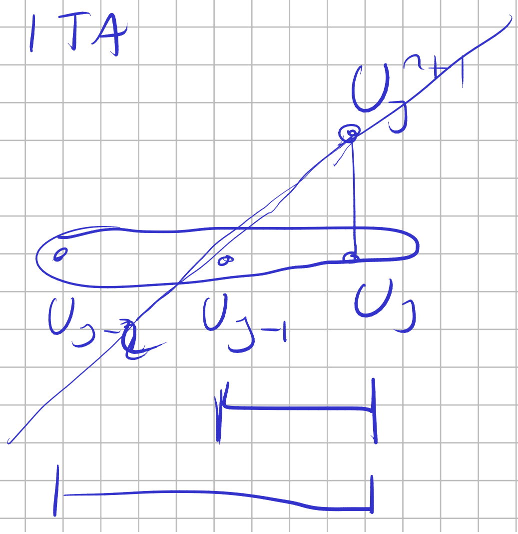
2^{nd} ORDINE IN SPAZIO
(NO INFO SUL TEMPO)

EULERO ESPLICITO $\rightarrow$ 1^o ORD. TEMPO ☹

AGGIUNGO IL TERMINE DI SECOND'ORDINE DELL'ESPANSIONE DI TAYLOR IN TEMPO

$$u_j^{n+1} = u_j^n - a\Delta t \frac{u_{j-2}^n - 4u_{j-1}^n + 3u_j}{2\Delta x} + a^2 \Delta t^2 \frac{u_{j-2}^n - 2u_{j-1}^n + u_j^n}{2\Delta x^2}$$

$CFL \leq 2 \Rightarrow$ STABILITÀ



- BORDI

PERIODICI

$x_0 = x_N$

↳ 1. $x_0, \_\_\_, x_{N-1}$ + ATTENZIONE ALLE DERIVATE

$$\partial_x u_0 = \frac{u_1 - u_{N-1}}{2\Delta x}$$

$u(x_N)$ NON FA PARTE DELLA DISCR.

↳ 2. $x_0, \_\_\_, x_N$ + MOLTA ATTENZ.

$$\partial_x u_0 = \frac{u_1 - u_{N-1}}{2\Delta x}$$

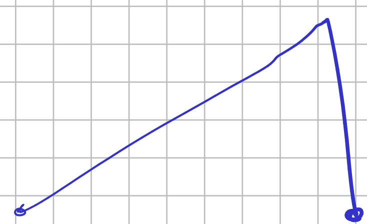
⇓ IN PYTHON

$$\frac{u[1] - u[-2]}{2\Delta x}$$

$$\partial_x u_N = \frac{u[1] - u[-2]}{2\Delta x}$$

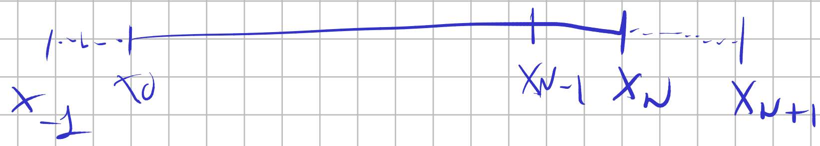
$\partial_x u[-1] =$

- DIRICHLET $u[0] = u_L$



- NEUMANN OMOGENEO ⟺ NON MI INTERESSA COSA SUCCEDE

DOVE ESCE L'INFORMAZIONE



PRIMA DI INIZIARE A CALCOLARE LE DERIVATE

- PERIODICHE

$$U_{-1} := U_{N-1}$$

$$U_{N+1} := U_1$$

- DIRICHLET → $\begin{cases} U_{-1} = U_L \\ U_0 = U_L \end{cases}$ (NON CONSIGLIATO)

- NEUMANN → $U_{N+1} := U_N$