HYPERBOLIC CONSERVATION LAWS
NONLINEAR PART 2

$$\partial_t u + \partial_x f(u) = 0$$

$$f(u) = \frac{u^2}{2}$$

$$\eta(u) = \frac{u^2}{2} \qquad \eta'(u) = u \qquad \eta''(u) = 1 > 0$$

$$\partial_t \eta(u) = \eta'(u) \cdot \partial_t u = \eta'(u) \cdot (-\partial_x f(u))$$

$$= -u\, \partial_x \frac{u^2}{2} = -u \cdot u\, \partial_x u = -u^2 \partial_x u = -\partial_x \frac{u^3}{3}$$

DIFFERENZIABILE TUTTO

$$g(u) = \frac{u^3}{3} \qquad f'(u) \cdot \eta'(u) = g'(u)$$

$$u \cdot u = u^2 \quad \checkmark$$

• SOLUZIONE DEBOLE DEL RIEMANN PROBLEM

$$u_0(x) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases} \qquad u_L < u_R$$

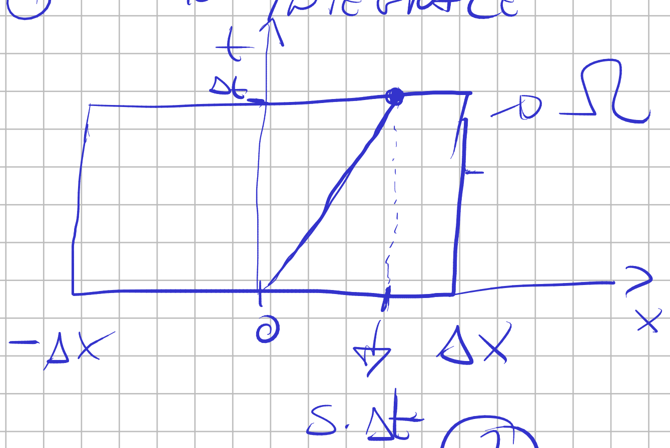
$$u(x,t) = \begin{cases} u_L & x < st \\ u_R & x > st \end{cases}$$

$$S = \frac{f(u_L)-f(u_R)}{u_L-u_R} = \frac{\frac{u_L^2}{2}-\frac{u_R^2}{2}}{u_L-u_R} = \frac{\cancel{(u_L-u_R)}(u_L+u_R)}{2\cancel{(u_L-u_R)}}$$

$$= \frac{u_L+u_R}{2}$$

- ES. $u(x,t)$ È SOLUZIONE DEBOLE DI RIEMANN PROBLEM.
- $u(x,t)$ NON È UNA SOLUZIONE ENTROPICA

$$\partial_t \eta(u) + \partial_x g(u) \le 0 \to \text{INTEGRALE}$$



$$\Delta t < |S|^{-1} \cdot \Delta x$$

$$\Omega = [-\Delta x, \Delta x] \times [0, \Delta t]$$

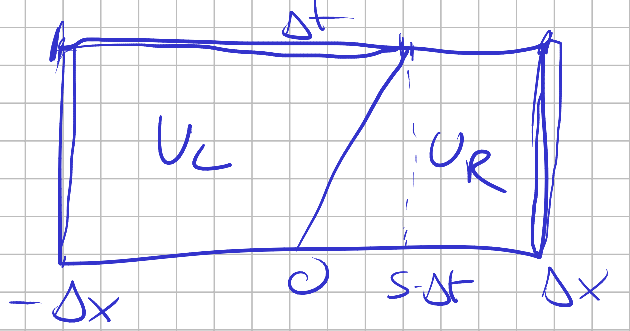
$$\int_{-\Delta x}^{\Delta x} \int_0^{\Delta t} \partial_t \eta(u) + \partial_x g(u) \, dx\, dt \quad \overset{?}{\substack{\le \\ \ge \\ =}} \; 0$$

$$\int_{-\Delta x}^{\Delta x} \eta(u(\Delta t, x)) - \eta(u(0,x)) \, dx$$

$$+ \int_0^{\Delta t} g(u(t,\Delta x)) - g(u(t,-\Delta x)) \, dt =$$

$$\int_{-\Delta x}^{\Delta x} \frac{u^2(\Delta t, x)}{2} - \frac{u^2(0,x)}{2} \, dx + \int_0^{\Delta t} \frac{u^3(t,\Delta x) - u^3(t,-\Delta x)}{3} \, dt$$

$$= \int_{-\Delta x}^{s\Delta t} \frac{U_L^2}{2}\, dx + \int_{s\cdot\Delta t}^{\Delta x} \frac{U_R^2}{2}\, dx$$



$$- \int_{-\Delta x}^{0} \frac{U_L^2}{2}\, dx - \int_{0}^{\Delta x} \frac{U_R^2}{2}\, dx$$

$$+ \int_{0}^{\Delta t} \frac{U_R^3 - U_L^3}{3}\, dt =$$

$$= s\cdot\Delta t\, \frac{U_L^2}{2} - s\cdot\Delta t\, \frac{U_R^2}{2} + \Delta t\, \frac{U_R^3 - U_L^3}{3}$$

$$= \Delta t \left[\frac{U_L+U_R}{2} \cdot \frac{U_L^2 - U_R^2}{2} + \frac{U_R^3 - U_L^3}{3} \cdot \frac{U_L^2 - U_R^2}{U_L^2 - U_R^2} \right]$$

$$= \Delta t\, \frac{U_L^2 - U_R^2}{2} \left[\frac{U_L+U_R}{2} + \frac{2}{3}\, \frac{U_R^3 - U_L^3}{U_L^2 - U_R^2} \right]$$

$$= \Delta t\, \frac{U_L^2 - U_R^2}{2} \left[\frac{U_L+U_R}{2} + \frac{2}{3}\, \frac{(-1)\cancel{(U_R - U_L)}\,(U_R^2 + U_R U_L + U_L^2)}{\cancel{(U_L - U_R)}\,(U_L + U_R)} \right]$$

$$= \Delta t\, \frac{U_L^2 - U_R^2}{2} \left[\frac{3U_L^2 + 6U_L U_R + 3U_R^2 - 4U_R^2 - 4U_R U_L - 4U_L^2}{6(U_L + U_R)} \right]$$

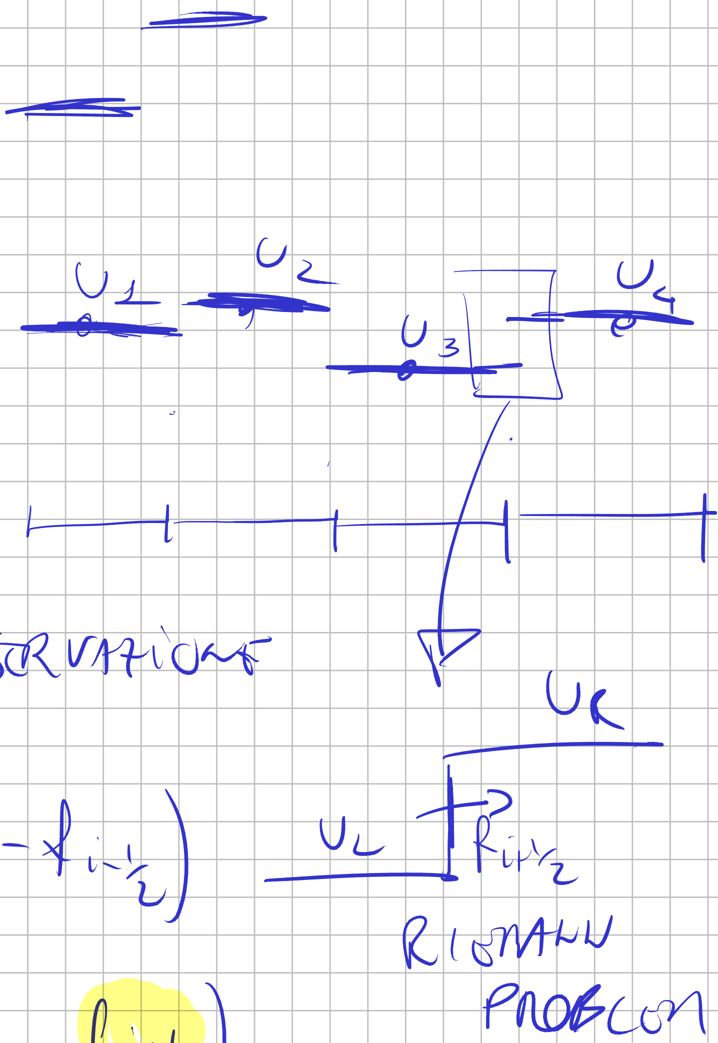
$$= \Delta t\, \frac{U_L^2 - U_R^2}{2} \left[\frac{-U_L^2 + 2U_L U_R - U_R^2}{6(U_L + U_R)} \right] =$$

$U_L < U_R$

$$= -\Delta t\, \frac{(U_L - U_R)\cancel{(U_L + U_R)}}{2}\, \frac{(U_L - U_R)^2}{6\cancel{(U_L + U_R)}} = -\frac{\Delta t}{12} \underbrace{(U_L - U_R)^3}_{<0} > 0$$

NUMERICA

1. I RIEMANN PROBLEM CI TROVIAMO A LIVELLO DISCRETO



2. PRESERVARE CONSERVAZIONE

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2} - f_{i-1/2}\right)$$

$$u_{i+1}^{n+1} = u_{i+1}^n - \frac{\Delta t}{\Delta x}\left(f_{i+3/2} - f_{i+1/2}\right)$$

CIÓ CHE TOLGO DA UNA PARTE LO METTO NELL'ALTRA $\Rightarrow$ CONSERVO MASSA TOTALE

$$\sum_{i=1}^{N} u_i^n \overset{\checkmark}{=} \sum_{i=1}^{N} u_i^{n+1} = \sum u_i^n + \underbrace{\frac{\Delta t}{\Delta x}\sum\left(f_{i+1/2} - f_{i-1/2}\right)}_{=0}$$

3. $f_{i+1/2}$ = FLUSSI NUMERICI CHE DOBBIAMO DEFINIRE

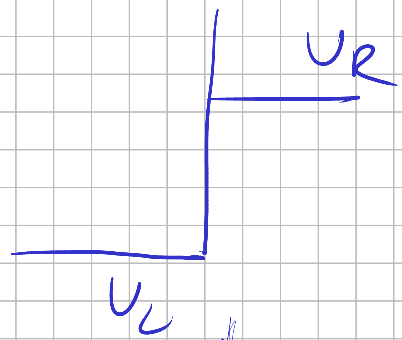
$$\int^x \partial_t u + \int^x \partial_x f = 0$$

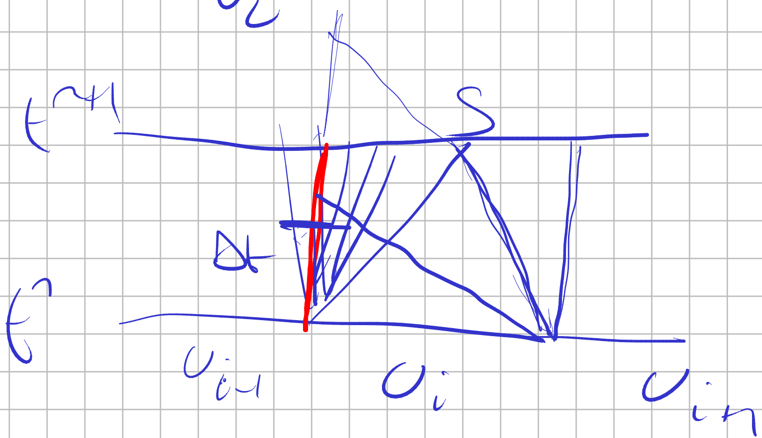
$$\partial_t \int_{x_{i-1/2}}^{x_{i+1/2}} u + f_{i+1/2}(t) - f_{i-1/2}(t) = 0$$

GODUNOV

$$\underbrace{\int_{t^n}^{t^{n+1}} \partial_t \underbrace{\int_{x_{i-1/2}}^{x_{i+1/2}} u}_{\Delta x \cdot u_i}}_{\Delta x\left(u_i^{n+1} - u_i^n\right)} + \int_{t^n}^{t^{n+1}} \underbrace{\int_{x_{i-1/2}}^{x_{i+1/2}} \partial_x f(u)}_{\int_{t^n}^{t^{n+1}} f(u(t,x_{i+1/2})) - f(u(t,x_{i-1/2}))} = 0$$

$$u_i^{n+1} = u_i^n - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} f_{i+1/2}(t) - f_{i-1/2}(t) \, dt$$





SE SI RIESCE ESATTA BENE

SE NO SI APPROSSIMA

$$\hat{f}(u_i, u_{i+1}) \approx f(u_{i+1/2})$$

- LAX-FRIEDRICHS

$$\hat{f}(U_L, U_R) = \frac{f(U_L)+f(U_R)}{2} - \frac{\Delta x}{2\Delta t}(U_R - U_L)$$

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\hat{f}(u_i, u_{i+1}) - \hat{f}(u_{i-1}, u_i)\right)$$

$$= u_i^n - \frac{\Delta t}{\Delta x}\left(\frac{f(u_{i+1}) - f(u_{i-1})}{2} - \frac{\Delta x}{\Delta t}\frac{u_{i+1} - 2u_i + u_{i-1}}{2}\right)$$

$$= u_i^n - \Delta t\left(\frac{f(u_{i+1}) - f(u_{i-1})}{2\Delta x}\right) + \underbrace{\frac{u_{i+1} - 2u_i + u_{i-1}}{2}}_{\simeq \Delta x^2 \partial_{xx} u}$$

È UNA BUONA IDEA

PERCHÉ VANISHING-VISCOSITY SOLUTION

CI INTERESSA PER $\Delta x \to 0$

- RUSANOV = LOCAL-LAX FRIEDRICHS

1° ORDINE

$$\partial_t u + \underbrace{a}_{\text{VELOCITÀ}} \partial_x u = 0 \qquad \partial_t u + \partial_x f(u) = 0$$

$$\partial_t u + \underbrace{f'(u)}_{\text{VELOCITÀ}} \cdot u_x = 0$$

$$\Delta t \leq \frac{\Delta x}{|a|}$$

$$\Delta t \leq \min_i \frac{\Delta x}{|f'(u_i)|} = \frac{\Delta x}{\max_i |f'(u_i)|}$$

$$\Delta t := 0.9 \cdot \frac{\Delta x}{\max_i |f'(u_i)|}$$