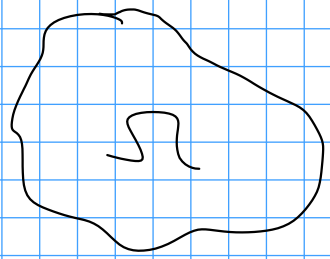
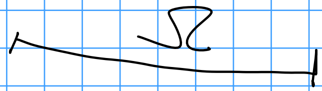
$\Omega \subset \mathbb{R}^d$



$\subset \mathbb{R}^2$

$u : \Omega \to \mathbb{R}^s$



$\subset \mathbb{R}^1$

$$F(\nabla^{(k)}u, \nabla^{(k-1)}u, \dots, \nabla u, u) = 0$$

$$u, \qquad \nabla u = \begin{pmatrix} \partial_{x_1} u \\ \partial_{x_2} u \\ \partial_{x_3} u \\ \vdots \\ \partial_{x_d} u \end{pmatrix} \in \mathbb{R}^{d \times s}$$

$$(\nabla^{(2)} u^\ell)_{ij} = \partial_{x_i} \partial_{x_j} u^\ell \qquad \ell = 1, \dots, s$$

$$\nabla^{(2)} u \in \mathbb{R}^{d \times d \times s} \qquad \xrightarrow{\nabla} \begin{pmatrix} \partial_{x_1}\partial_{x_1} u & \partial_{x_1}\partial_{x_2} u & \dots & \partial_{x_1}\partial_{x_d} u \\ \partial_{x_2}\partial_{x_1} u & - & - & - \\ \vdots & & & \\ \partial_{x_d}\partial_{x_1} u & - & - & \partial_{x_d}\partial_{x_d} u \end{pmatrix}$$

$$\nabla^{(k)} u \in \mathbb{R}^{\underbrace{d \times \dots \times d}_{k\text{-volte}} \times s}$$

$$(\nabla^{(k)} u^\ell)_{i_1 \dots i_k} = \partial_{x_{i_1}} \partial_{x_{i_2}} \cdots \partial_{x_{i_k}} u^\ell$$

- DERIVATE PARZIALI

$$\partial_t u(t, x) = \frac{\partial u(t,x)}{\partial t} = u_t(t,x)$$

- DERIVATA TOTALE

$$\frac{d}{dt} u(t, x(t)) = \underbrace{\frac{\partial u}{\partial t}}(t, x(t)) + \frac{\partial}{\partial x} u(t, x(t)) \cdot \frac{d x(t)}{dt}$$

- $\nabla u = \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}$ GRADIENTE

$u(x,y)$

- LAPLACIANO $u(x,y)$

$$\Delta u = \nabla^2 u = \partial_{xx} u + \partial_{yy} u$$

- DIVERGENZA

$$\mathrm{div} \begin{pmatrix} u \\ v \end{pmatrix} = \nabla \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \partial_x u + \partial_y v$$

$$\mathrm{div}(u) = \sum_{i=1}^{d} \partial_{x_i} u^i = \partial_{x_i} u^i$$

$u \in \mathbb{R}^d$ $u = \begin{pmatrix} u^1 \\ u^2 \\ \vdots \\ u^d \end{pmatrix}$

$\underbrace{u \cdot \nabla u}$ $u \in \mathbb{R}^d$

$\nabla u \in \mathbb{R}^{d \times d}$

$$u^i \cdot \partial_{x_i} u^J = \underbrace{\sum_{i=1}^{d}} u^i \partial_{x_i} u^J = (u \cdot \nabla u)^J \in \mathbb{R}^d$$

RIPASSO DI ODE

$$\begin{cases} \dfrac{dy(t)}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases} \qquad y : [0, t_{end}] \to \mathbb{R}^S$$

ODE DI ORDINE p

$$\partial_t^{(p)} y = f(t, y(t), \dots, y^{(p-1)}(t)) \qquad y : [0, t_{end}] \to \mathbb{R}^S$$

$$\begin{cases} y'(t) = z_1(t) \\ z_1'(t) = z_2(t) \quad \cancel{(= y''(t))} \\ \vdots \\ z_{p-2}'(t) = z_{p-1}(t) \\ z_{p-1}'(t) = \cancel{y^{(p)}(t)} = f(t, y, z_1, \dots, z_{p-2}) \end{cases} \qquad (1)$$

$$w = \begin{pmatrix} y \\ z_1 \\ \vdots \\ z_{p-1} \end{pmatrix} \in \mathbb{R}^{p \times S}$$

(1) SISTEMA DI ODE PRIMO ORDINE

$$\begin{cases} \dfrac{dy(t)}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

FORMA INTEGRALE $\Rightarrow \displaystyle\int_0^T \frac{dy(t)}{dt}\, dt = \int_0^T F(t, y(t))\, dt$

$$y(T) - y(0) = \int_0^T F(t, y(t))\, dt$$

$$\begin{cases} y'(t) = -\lambda\, y(t) \\ y(t_0) = y_0 \end{cases} \qquad y : [0,T] \to \mathbb{R}$$

$$\underbrace{\int_0^t \frac{y'(s)}{y(s)}\, ds}_{} = \int_0^t -\lambda\, ds \qquad \log(y(s))' = \frac{y'(s)}{y(s)}$$

$$\Big[\log(y(s))\Big]_0^t = -\lambda\,(t-0)$$

$$\log(y(t)) - \log(y(0)) = -\lambda t$$

$$\log\left(\frac{y(t)}{y(0)}\right) = -\lambda t \qquad \frac{y(t)}{y(0)} = e^{-\lambda t}$$

$$y(t) = y(0)\, e^{-\lambda t} \qquad \lambda > 0$$

$$\underline{y}(t) = e^{At}\, \underline{y}(0)$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^k}{k!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{At} = I + A\cdot t + \frac{A^2}{2} t^2 + \dots \frac{A^k t^k}{k!}$$

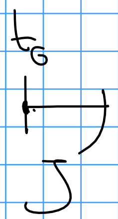
$$\begin{cases} \frac{d}{dt} y(t) = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

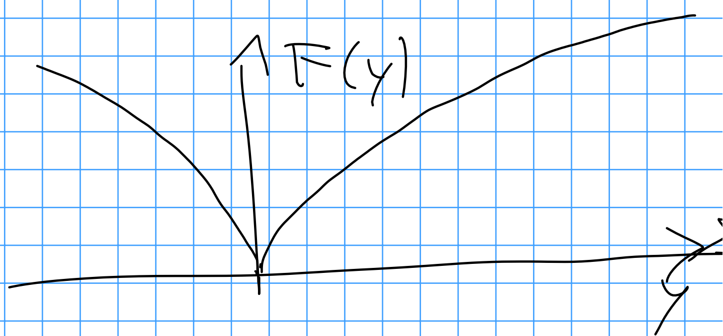
$y : [0, T] \to \mathbb{R}^S$

$F : [0, T] \times \mathbb{R}^S \to \mathbb{R}^S$

F CONTINUA

Se F CONTINUA IN $(t_0 = 0, y_0)$, $\Rightarrow \exists$ UN INTERVALLO $J \ni t_0$ $\quad \exists\, y(t) : y' = F(t, y)$ $\forall t \in J$





ESISTENZA

UNICITÀ

$$\begin{cases} y' = \sqrt{|y|} = F(t, y) \\ y(0) = 0 \end{cases}$$

- $y(t) \equiv 0 \qquad y' = 0 \ \checkmark \qquad F(y) = \sqrt{|0|} = 0 \ \checkmark$
- $y(t) = \frac{t^2}{4} \qquad y' = \frac{2t}{4} = \frac{t}{2} \qquad \boxed{t \geq 0}$

$$\| $$

$$F(y) = \sqrt{\frac{t^2}{4}} = \frac{|t|}{2}$$

SE F è LIPSCHITZ CONTINUA SU $[0, T] \times \mathbb{R}^S$ CON COSTANTE $L \geq 0$: $\forall (t, y), (t, \cancel{z})$

$$\boxed{\| F(t,y) - F(t,z) \| \leq L \, \| y - z \|}$$

$\Rightarrow$ UNICITÀ SOLUZIONE. $\begin{cases} y'(t) = F(t, y(t)) \\ y(0) = y_0 \end{cases}$

- REGOLARITÀ

Se $F \in C^p \Rightarrow y \in C^{p+1}$

$$y(t) = y(0) + \int_0^t F(t, y(t)) \, dt .$$