PROBLEMI IPERBOLICI

$$\partial_{tt} u - \partial_{xx} u = 0 \qquad \text{EQUAZIONE}$$

v : • $\partial_t v = \partial_x u$

$$\partial_{tt} u - \partial_t (\partial_x v) = 0$$

$$\boxed{\bullet\ \partial_t u - \partial_x v = 0}$$

• $\partial_t v - \partial_x u = 0$

MOTIVAZIONE DALLA FLUIDO DINAMICA

$\rho(t,x)$ DENSITÁ DEL FLUIDO

$v(t,x)$ VELOCITÁ DEL FLUIDO

$$\underbrace{\int_{x_1}^{x_2} \rho(t,x)\,dx}$$ COME VARIA LA MASSA IN QUESTO DOMINIO? $(x_1;x_2)$

FLUSSO DI MASSA   e.g. (x_1, x_2)

CAMBIA $\approx$ $\underbrace{\rho(t,x_1)\cdot v(t,x_1)}$ $- \rho(t,x_2)\cdot v(t,x_2)$

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(t,x)\,dx = \rho(t,x_1)\cdot v(t,x_1) - \rho(t,x_2)\cdot v(t,x_2)$$

LEGGE DI CONSERVAZIONE (DELLA MASSA)

IN FORMA INTEGRALE.

$\forall\, x_1 < x_2$

DERIV. IN x_2

$$\frac{\partial}{\partial t}\rho(t,x) = -\partial_x(\rho \cdot v)$$

$$\partial_t \rho + \partial_x(\rho\cdot v) = 0$$

LEGGE DI CONSERVAZIONE (MASSA) (IN FORMA DIFFERENZIALE)

v = COSTANTE

$$\partial_t \rho + \underset{}{\overset{\in\mathbb{R}}{v}}\,\partial_x \rho = 0$$

LEGGE DI CONSERVAZIONE (GENERALE)

$$\partial_t u + \partial_x f(u) = 0 \qquad u\in\mathbb{R}^d \qquad u:\mathbb{R}\to\mathbb{R}^d$$

MULTI-D $\qquad \partial_t u + \nabla\cdot F(u) = 0 \qquad u:\mathbb{R}^D\to\mathbb{R}^d$

- MOMENTO $\qquad \partial_t(\rho v) + \partial_x(\rho v^2 + P) = 0$

es. $P = P(\rho)$

- ENERGIA $\qquad \partial_t(\rho E) + \partial_x(\rho v E + P v) = 0$

GAS IDEALI EQUAZIONE DI STATO

$$\underline{P = (\gamma-1)\,\rho\cdot\left(E - \frac{v^2}{2}\right)} \iff E = \overbrace{\frac{P}{\rho(\gamma-1)}}^{\text{ENERGIA INT}} + \overbrace{\frac{v^2}{2}}^{\text{CINETICA}}$$

$$u = \begin{pmatrix} \rho \\ \rho v \\ \rho E \end{pmatrix} \qquad \partial_t \begin{pmatrix} \rho \\ \rho v \\ \rho E \end{pmatrix} + \partial_x \begin{pmatrix} \rho v \\ \rho v^2 + p \\ \rho v E + p v \end{pmatrix} = 0$$

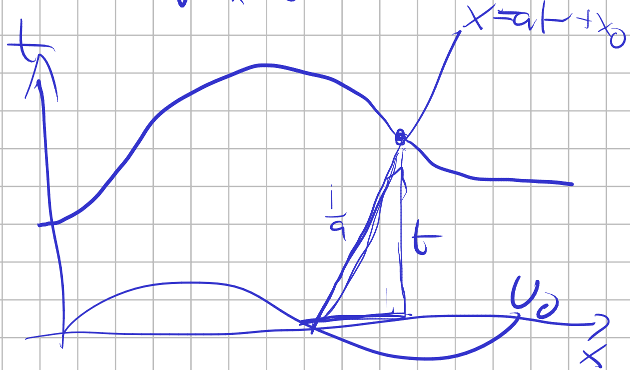
EQUAZ. DEL TRASPORTO $\quad a \in \mathbb{R}$

$\Omega \subseteq \mathbb{R}$ 1D $\quad u : \mathbb{R}^+ \times \Omega \to \mathbb{R}$

$$\begin{cases} \partial_t u(t,x) + a \cdot \partial_x u(t,x) = 0 & \forall x \in \Omega \quad t \in \mathbb{R}^+ \\ u(0,x) = u_0(x) & \forall x \in \Omega \end{cases}$$

PROBLEMA DI CAUCHY

$$u(t,x) = u_0(x - at)$$



RETTE CARATTERISTICHE $\quad x = at + x_0$

$$\partial_t u + a \partial_x u = u_0'(x-at) \cdot (-a) + a \cdot u_0'(x-at) = 0$$

$x(t) \underset{\text{CHE}}{\overset{\text{TALI}}{\Rightarrow}} 0 = \frac{d}{dt} u(t, x(t)) =$

$$= \partial_t u(t, x(t)) + \partial_x u(t,x) \cdot \boxed{\partial_t x(t)} = 0$$

$\to$ SOLUZIONE DI $\partial_t u = -a \partial_x u$

$$\partial_t x(t) = - \frac{\partial_t u(t,x)}{\partial_x u(t,x)} = a$$

$$x(t) = a \cdot t + x_0$$

• $u_t + a(x) \cdot u_x = 0$ "TRASPORTO CON VELOCITÀ NON COSTANTE"

CARATTERISTICHE FACILI

$$\begin{cases} \partial_t x(t) = a(x) \\ x(0) = x_0 \end{cases}$$

PER TROVARE $x(t)$

BISOGNA RISOLVERE UN ODE

$$\int_{x_1}^{x_2} \partial_t u = -\int_{x_1}^{x_2} a(x) \cdot \partial_x u \, dx$$

NON SI PUÒ FARE!

$$\ldots \Rightarrow F(x_2) - F(x_1)$$

• TRASPORTO NON COSTANTE

$$\partial_t u + (a(x) \cdot u)_x = 0 \qquad \circledast$$

$$\int_{x_1}^{x_2} \partial_t u = -\int_{x_1}^{x_2} \partial_x (a(x) \, u(t,x)) \, dx$$

$$= a(x_1) \cdot u(t,x_1) - a(x_2) \cdot u(t,x_2)$$

È UNA LEGGE DI CONSERVAZIONE! ☺

$\Rightarrow$ PIÙ COMPLICATO TROVARE CARATTERISTICHE

$$\partial_t x(t) = -\frac{\partial_t u}{\partial_x u} = ?$$

$\circledast$ $\partial_t u + a \cdot u_x + a_x \cdot u = 0$

$$\partial_t u + a \cdot u_x = -a_x \cdot u \qquad (2)$$

CARATTERISTICHE $\underbrace{\partial_t x(t)} = a(x(t))$

$u(t, x(t))$

$$\frac{d}{dt} u(t,x(t)) = \partial_t u + \partial_x u \cdot \overbrace{\partial_t x(t)}^{a(x(t))}$$

$$= \partial_t u + a(x(t)) \cdot \partial_x u \overset{(2)}{=} - a_x(x,t) \cdot u(t,x)$$

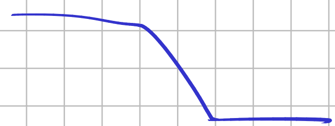
$$\frac{d}{dt} u(t,x(t)) = - a_x(x(t),t) \cdot u(t,x(t))$$

ODE IN TEMPO

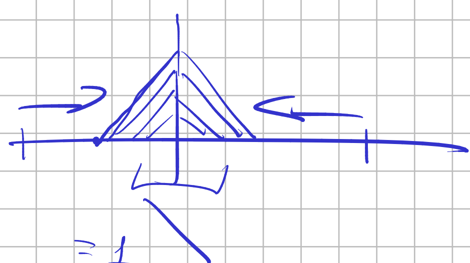
⇝ PDE → 1 ODE CARATTERISTICHE

1 ODE TROVARE SOL SULLE CARATTERISTICHE.

$$\partial_t u + a(x)\, \partial_x u = 0$$



$$a(x) = \begin{cases} +1 & x<0 \\ -1 & x<0 \end{cases} \quad \Bigg| \quad \begin{cases} 1 & x<-1 \\ -x & -1<x<1 \\ -1 & x>1 \end{cases}$$



$u_0(x) = 1$

$u(x,t) = 1$

$$\frac{d}{dt} u = \underbrace{\overbrace{-a_x}^{=1} \; u}$$

$$\frac{d}{dt} u = u \qquad u(t) = e^{t} \cdot u_0(t=0)$$

$\boxed{\text{(• INFOSTUD)}}$

$\Omega = [x_L, x_R]$

$\boxed{\text{DEBOLE}}$ $\varphi \in C^\infty[\mathbb{R}^+ \times \Omega]$

$$\int_0^T \int_{x_L}^{x_R} \varphi \cdot \left(\partial_t u + (a(x) \cdot u)_x\right) dx\, dt = 0$$

$$0 = \left[\int_{x_L}^{x_R} \varphi \cdot u \, dx\right]_0^T - \int_0^T \int_{x_L}^{x_R} \partial_t \varphi \cdot u(t,x)\, dx\, dt$$

$$+ \left[\int_0^T \varphi \cdot a \cdot u \, dt\right]_{x_L}^{x_R} - \int_0^T \int_{x_L}^{x_R} \partial_x \varphi \cdot a(x)\, u(t,x)\, dx\, dt$$

$$\int_{x_L}^{x_R} \varphi(x,T)\, u(x,T)\, dx - \int_{x_L}^{x_R} \varphi(x,0) \cdot u(x,0)\, dx$$

$$- \int_0^T \int_{x_L}^{x_R} \partial_t \varphi \cdot u \, dx\, dt - \int_0^T \int_{x_L}^{x_R} \partial_x \varphi \cdot a(x)\, u(t,x)\, dx\, dt$$

$$+ \int_0^T \varphi(x_R,t)\, a(x_R)\, u(x_R,t)\, dt - \int_0^T \varphi(x_L,t)\, a(x_L)\, u(x_L,t)\, dt$$

$$= 0$$

$\rightsquigarrow$ FORMA INTEGRALE $\quad \varphi \equiv 1$

$$\int_{x_L}^{x_R} u(x,T)\, dx - \int_{x_L}^{x_R} u(x,0)\, dx +$$

$$+ \int_0^T a(x_R)\, u(x_R,t)\, dt - \int_0^T a(x_L)\, u(x_L,t)\, dt = 0$$

CONSERVAZIONE ENERGIA $\quad \dfrac{u^2}{2}$

$\quad \hookrightarrow a \in \mathbb{R}$

$\varphi = u$

$$\int_{t^0}^{t'} \int_{x_L}^{x_R} \overline{u\, \partial_t u} + \int_{t^0}^{t'} \int_{x_L}^{x_R} u \cdot \partial_x (a \cdot u)\, dx\, dt = 0$$

$$u\,\partial_t u = \partial_t \frac{u^2}{2} \qquad a \cdot u\,\partial_x u = a \cdot \partial_x \frac{u^2}{2}$$

$$\int_{t^0}^{t^1}\int_{x_L}^{x_R} \partial_t \frac{u^2}{2}\, dx\,dt + \int_{t^0}^{t^1}\int_{x_L}^{x_R} \underbrace{\partial_x\left(a\,\frac{u^2}{2}\right)}_{\partial_x\left(a \frac{u^2}{2}\right)} dx\,dt = 0$$

$$\int_{x_L}^{x_R} \frac{u^2(t^1,x)}{2}\,dx - \int_{x_L}^{x_R} \frac{u^2(t^0,x)}{2}\,dx + \int_{t^0}^{t^1} a(x_L)\,\frac{u^2(x_R,t)}{2}\,dt - \int_{t^0}^{t^1} a(x_L)\,\frac{u^2(x_L,t)}{2}\,dt = 0$$

VANISHING VISCOSITY SOLUTIONS

$$\partial_t u^\varepsilon + a\,\partial_x u^\varepsilon = \varepsilon \cdot u^\varepsilon_{xx} \Rightarrow u^\varepsilon \in C^\infty((0,T) \times (x_L, x_R))$$

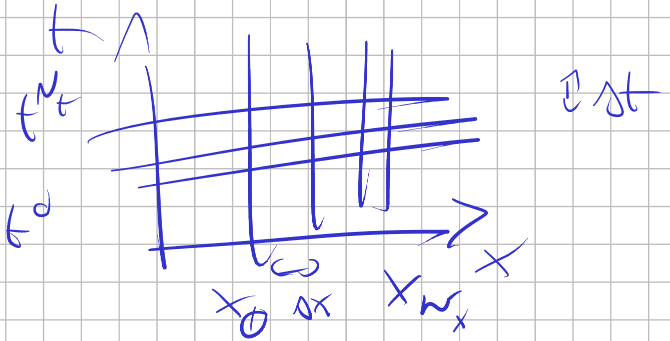
$$u^\varepsilon \xrightarrow[\varepsilon \to 0]{} u \qquad \partial_t u + a\,\partial_x u = 0$$

$$\lim_{\varepsilon \to 0} u^\varepsilon(t,x) = u(t,x)$$ IN quasi OGNI (x,t)

DIFFERENZE FINITE PER EQ. TRASPORTO

$$\partial_t u + a \cdot \partial_x u = 0 \qquad a \in \mathbb{R}$$

$$a = 1$$



- DIFFERENZE CENTRATE (SPAZIO)

$$(\partial_x u)_i \simeq \frac{U_{i+1} - U_{i-1}}{2\Delta x}$$

- EULERO ESPLICITO

$$\partial_x e^{ikx} = \underbrace{ik}_{\in i\mathbb{R}} \partial_x e^{ikx}$$

STABILE PER VON NEUMANN

$$U_J^n = c_k^n e^{ik(\Delta x \cdot J)} \qquad c_k^{n+1} = \overbrace{g(k)}^{\text{COEFF. DI AMPL.}} c_k^n$$

$$U_J^{n+1} = c_k^{n+1} e^{ik(\Delta x \cdot J)}$$

$$\frac{U_J^{n+1} - U_J^n}{\Delta t} + \frac{U_{J+1}^n - U_{J-1}^n}{2\Delta x} = 0$$

$$c_k^{n+1} \cdot e^{ik(\Delta x \cdot J)} = c_k^n e^{ik(\Delta x J)} +$$

$$- \frac{\Delta t}{\Delta x} \frac{e^{ik(\Delta x(J+1))} - e^{ik(\Delta x(J-1))}}{2} \cdot c_k^n$$

$$g(k) = 1 - \frac{\Delta t}{\Delta x} i \underbrace{\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2i}}_{\sin(k\Delta x)}$$

$$= 1 - i \frac{\Delta t}{\Delta x} \cdot \sin(k\Delta x)$$

$$|g(k)| = \sqrt{1 + \underbrace{\frac{\Delta t^2}{\Delta x^2}} \cdot \sin^2(k\Delta x)}$$

SE $\Delta t \sim \Delta x$ $\quad \frac{\Delta t^2}{\Delta x^2} \sim C \Rightarrow |g(k)| \cancel{\leq} 1 + c\Delta t$

$\Rightarrow \quad \Delta t \sim \Delta x^2 \qquad \frac{\Delta t^2}{\Delta x^2} \approx \frac{\Delta t^2}{\Delta t} \sim \Delta t$

$$\sqrt{1 + \Delta t \sin(\,) \cdot c} \leq 1 + \Delta t \cdot a \, .$$

LAX RICHTMYER STABLE SE $\quad \Delta t \sim \Delta x^2$