ODE ✓ → PDE $\underline{2D}$ $\underline{1^{st}\text{ ORDINE}}$

→ "PDE" $\underline{1D}$ $\underline{2^{nd}\text{ ORDINE}}$

$\partial_{xx} u + \partial_{yy} u = 0$

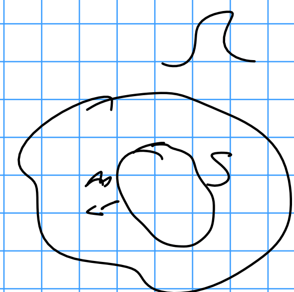
• $\Omega \subset \mathbb{R}^d$ $\quad d \geq 1 \quad u : \Omega \to \mathbb{R}$

$-\Delta u = f$ in Ω EQ. DI POISSON

$\partial^2_{x_1} u + \partial^2_{x_2} u + \dots + \partial^2_{x_d} u = f$

• se $f = 0 \Rightarrow$ OMOGENEA → EQ. DI LAPLACE

• $\int_{\partial S} F \cdot n \, d\Gamma = 0$



$S \subset \Omega$ → GAUSS-GREEN

$\int_S \operatorname{div} F \, dx = \int_{\partial S} F \cdot n \, d\Gamma$

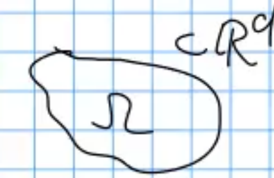
→ $\operatorname{div}(F) = 0$ in Ω

↳ u → $F = -a\nabla u$ $\quad a \in \mathbb{R}$ $a > 0$

$\operatorname{div}(-a\nabla u) = 0 = -a\Delta u = 0$

$$\operatorname{div}(\nabla u) = \operatorname{div}\begin{pmatrix}\partial_{x_1} u\\ \vdots \\ \partial_{x_d} u\end{pmatrix} = \partial_{x_1}\partial_{x_1} u + \dots \partial_{x_d}\partial_{x_d} u = \Delta u$$

BORDI E UNICITÀ DELLA SOLUZIONE



- CONDIZIONE AL BORDO DI DIRICHLET

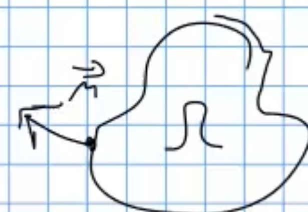
$$u(x) = g(x) \quad \forall x \in \partial\Omega$$

se $g \equiv 0$ CONDIZ. BORDO DI DIRICHLET OMOGENEE

B.C. BOUNDARY CONDITIONS

- CONDIZIONI AL BORDO DI NEUMANN

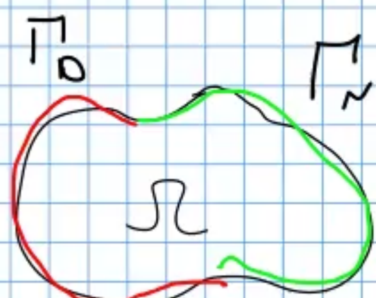
$$\frac{\partial u(x)}{\partial n} = \nabla u(x)\,\vec{n} = h(x) \quad \forall x \in \partial\Omega$$



- NEUMANN OMOGENEO $h \equiv 0$

- MIXED GEOMETRICAL

$$\begin{cases} -\Delta u = f & \text{EQ} \\ u = g \quad \text{su } \Gamma_D \\ \dfrac{\partial u}{\partial n} = h \quad \text{su } \Gamma_N \end{cases} \Bigg\} \text{B.C.}$$

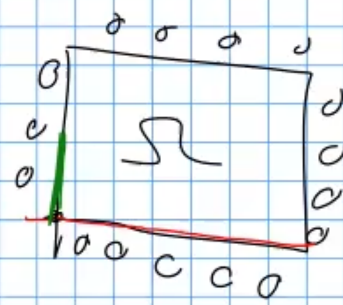


- ROBIN B.C.

$$\nabla u \cdot n + \gamma u = r \quad \forall x \in \Gamma_R$$

$$\begin{cases} \boxed{-\Delta u = 1} & \text{su } \Omega = [0,1]^2 \\ u(x,0)=0 \\ u(x,1)=0 \\ u(0,y)=0 \\ u(1,y)=0 \end{cases}$$

B.C. DIRICHLET OMOGENEE



Se cerchiamo $u \in C^2(\bar{\Omega})$

$$-\Delta u(0,0) = -\underbrace{\partial_{xx} u(0,0)}_{} - \overbrace{\partial_{yy} u(0,0)}^{= 0} \neq 1$$

$$\lim_{h \to 0} \underbrace{\frac{u(x+2h,0) - 2u(x+h,0) + u(x,0)}{h^2}}_{} = 0$$

$$u \in C(\bar{\Omega}) \cap C^2(\underbrace{\mathring{\Omega}}_{}) \supset C^2(\bar{\Omega})$$

INTERNO DI Ω

PERCHÉ $f \in C(\mathring{\Omega})$ $f = -\Delta u$

- f DISCONTINUA $\quad$ <u>1D</u> $\quad \begin{cases} -u'' = f & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$

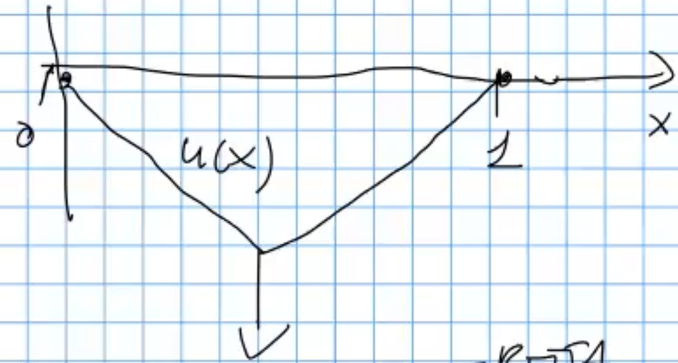
STRINGA ELASTICA

CON FORZA SU INTERVALLO a,b : $F = \int_a^b f(s)\, ds$

FORZA SU $(0,x)$ $\quad F(x) = \int_0^x f(s)\, ds$

- P. ESEMPIO $\quad -\delta_{\frac{1}{2}} = f$

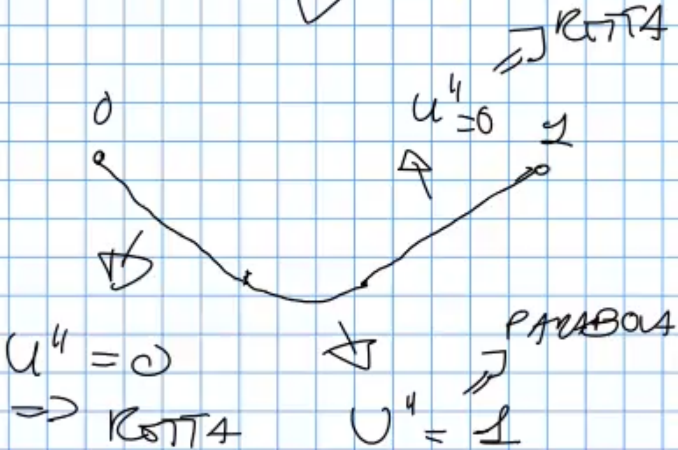
  $f \in \mathcal{D}^* \quad f \notin L^2$



- ES 2 $\quad f = -\chi_{[0.4,\, 0.6]}$

  $f \in L^2$

  $f \notin C$



$$v \in \mathcal{D}(\Omega) = \mathcal{D}((0,1)) = C_0^\infty((0,1))$$

$$-u'' = f \Rightarrow -u'' \cdot v = f \cdot v$$

$$\underbrace{\int_0^1 -u''(x) \cdot v(x)\, dx}_{\text{P. PARTI}} = \int_0^1 f(x)\, v(x)\, dx$$

$$\Big[-u'(x) \cdot \underbrace{v(x)}_{=0}\Big]_0^1 + \int_0^1 u'(x) \cdot v'(x)\, dx = \int_0^1 f(x)\, v(x)\, dx$$

$$\Rightarrow \int_0^1 u' \cdot v' = \int_0^1 f \cdot v \qquad \text{FORMA DEBOLE} \quad \forall v \in \mathcal{D}(\Omega)$$

$$V = \{u : u \in C^1((0,1)) \cap C^0([0,1]) : u(0) = u(1) = 0\}$$

V NON È COMPLETO PER LA NORMA $|\cdot|_1$

$V = H_0^1((0,1))$ $\leftarrow$ È UNO SPAZIO DI HILBERT

$\Rightarrow$ COMPLETO RISPETTO ALLA NORMA $|\cdot|_1$.

PROBLEMA DEBOLE: $$\int_0^1 u' \cdot v' \, dx = \int_0^1 f \cdot v \, dx \qquad (*)$$

$u, v \in H_0^1 \qquad f \in L^2$

PROBLEMA DEBOLE
$\left[\begin{array}{l} \text{TROVA } u \text{ IN } H_0^1 : \text{DATA UNA } f \in L^2 \\ \forall v \in H_0^1 \text{ VALE } (*) \end{array}\right.$

- FORMULAZIONE VARIAZIONALE EQUIVALENTE

TROVA $u \in H_0^1((0,1))$:

$$\left\{ J(u) = \min_{v \in H_0^1} J(v) \right.$$

$$J(v) := \frac{1}{2} \int_0^1 (v')^2 dx - \int_0^1 f \cdot v \, dx$$

<u>DIM</u> $\forall w \in V \qquad \psi(\delta) = J(u + \delta w) \qquad \delta \in \mathbb{R}$

$$\psi(\delta) = \frac{1}{2} \int_0^1 ((u+\delta w)')^2 dx - \int_0^1 f \cdot (u + \delta w) \, dx =$$

$$= \frac{1}{2} \int_0^1 (u' + \delta w')^2 dx - \int_0^1 f u - \delta \int_0^1 f \cdot w$$

$$= \frac{1}{2} \int_0^1 (u')^2 + \delta \int_0^1 u' \cdot w' \, dx + \frac{\delta^2}{2} \cdot \int_0^1 (w')^2$$

$$- \int_0^1 f u - \delta \int_0^1 f w$$

$$= \left[\frac{\int_0^1 (w')^2}{2} \right] \delta^2 + \left[\int_0^1 u'w' - \int_0^1 f w \right] \delta + \dots$$

ψ È UNA PARABOLA IN δ $\Rightarrow$ MINIMO

MINIMIZZARE È $\quad \delta = - \dfrac{\left[\int_0^1 u'w' - \int_0^1 f w \right]}{\int_0^1 (w')^2 dx}$

PROBLEMA DEBOLE

$\int_0^1 u'w' = \int f w \quad \forall w$

$\Downarrow$

SE u È SOLUZ DI PROB DEBOLE $\Rightarrow \delta = 0$

$\Rightarrow$ u È IL MINIMO DI J

SE u È IL MINIMO DI J

$\Rightarrow \delta = 0 \quad \forall w \quad \Rightarrow$ PROBLEMA DEBOLE

$$\psi(0) = J(u) \le J(u + \delta w) = \psi(\delta)$$

- $\begin{cases} -u'' = f \\ u(0) = U_L, \quad u(1) = U_R \end{cases}$

- LIFTING $\quad U_{LIFT}(x) := \left[(1-x) \cdot U_L + x \cdot U_R \right]$

$U_{LIFT}(0) = U_L \qquad U_{LIFT}(1) = U_R$

$u''_{LIFT}(x) = 0$

$\tilde{U} := u - U_{LIFT}$

$$-\tilde{u}'' = -u'' - \cancel{u''_{LIFT}}^{\,0} = f$$

$\tilde{u}(0) = U(0) - U_{LIFT}(0) = U_L - U_L = 0$

$\tilde{u}(1) = u(1) - U_{LIFT}(1) = U_R - U_R = 0$

$\Rightarrow$ PROBLEMA DI POISSON CON BC. DIRICHLET OMO
$\tilde{u}$

- u DIRICHLET NON OMOGENEE $\longrightarrow \tilde{U}$ DIRICHLET OMOGENEE

- NEUMANN BOUNDARY CONDITIONS

$$\begin{cases} -u'' = f \\ u'(0) = h_0 \qquad u'(1) = h_1 \end{cases}$$

- SUPPONIAMO CHE U SIA SOLUZIONE

$\Rightarrow \quad \tilde{U}(x) = u(x) + \underset{\in \mathbb{R}}{C}$ È UNA SOLUZIONE

NON BEN POSTO : NON HA SOLUZIONE UNICA.

- NEUMANN - DIRICHLET

$$\begin{cases} -u'' = f \\ u(0) = 0 \qquad u'(1) = g_1 \end{cases}$$

$$V = \{ v \in H^1((0,1)) : v(0) = 0 \} \qquad \forall v \in V$$

$$0 = \int_0^1 -u'' \cdot v - f \cdot v \, dx = \int_0^1 u' \cdot v' - f \cdot v \, dx - [u' \cdot v]_0^1 = *$$

$$[u' v]_0^1 = \underbrace{u'(1)}_{g_1} \underbrace{v(1)}_{\neq 0} - u'(0) \cdot \underbrace{v(0)}_{=0}$$

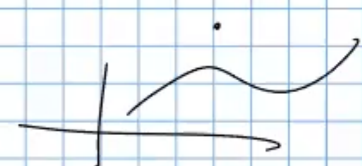
$$* = \int_0^1 u' \cdot v' - \int_0^1 f \cdot v - g_1 \overbrace{v(1)}$$

A LIVELLO di H^1 NON HA SENSO

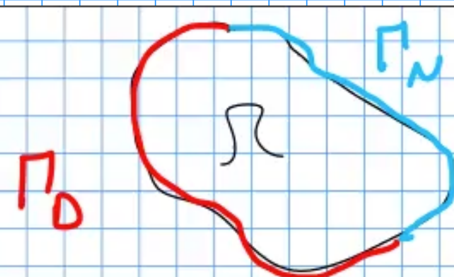
USA LIFTING

• se $g_1 = 0$ (NEUMANN OMOGENEO)

$$\Rightarrow \underbrace{\int_0^1 u' v' = \int_0^1 f v} \qquad \forall v \in V$$



$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d \\ u(x) = u_D(x) & \forall x \in \Gamma_D \\ \nabla u(x) \cdot n = g_N(x) & \forall x \in \Gamma_N \end{cases}$$



$$u, v \in V = \{ v \in H^1(\Omega) : v|_{\Gamma_D} = 0 \}$$

$$\int_0^1 -\Delta u \cdot v - f v \, dx = 0$$

$$\int_\Omega \operatorname{div}(\underset{\mathbb{R}^d}{\overset{}{a}}) \, dx = \int_{\partial\Omega} a \cdot n \, d\gamma$$

$$\int_\Omega \operatorname{div}(\underbrace{\underbrace{v}_{\in\mathbb{R}} \cdot \underbrace{\nabla u}_{\in\mathbb{R}^d}}_{\in\mathbb{R}^d})\, dx = \int_{\partial\Omega} v\, \nabla u \cdot n \, d\gamma$$

$\downarrow$

$$\int_\Omega \sum_{i=1}^d \partial_{x_i}(v \cdot \partial_{x_i} u)\, dx = \int_\Omega \sum_{i=1}^d \partial_{x_i}(v) \cdot \partial_{x_i}(u)$$

$$+ \int_\Omega \sum_i v \cdot \partial_{x_i x_i} u = \int_\Omega \nabla v \cdot \nabla u + \int_\Omega v \cdot \Delta u$$

$$\int \Delta u \cdot v\, dx = \int_{\partial\Omega} v \cdot \nabla u \cdot \vec{n}\, d\gamma - \int_\Omega \nabla u \cdot \nabla v\, dx$$

$$\Rightarrow \int_\Omega -\Delta u \cdot v - f v = \underbrace{\int_\Omega \nabla u \cdot \nabla v\, dx - \int_\Omega f \cdot v\, dx}_{\text{FORMULAZIONE DEBOLE}} \underbrace{- \int_{\partial\Omega} v \cdot \nabla u \cdot \vec{n}\, d\gamma}$$

$$\Rightarrow \int_{\partial\Omega} = \int_{\Gamma_D} + \int_{\Gamma_N}$$

$$\int_{\Gamma_D} \underbrace{v}_{=0} \cdot \nabla u \cdot \vec{n}\, d\gamma = 0$$

$$\int_{\Gamma_N} v \cdot \underbrace{\nabla u \cdot \vec{n}}_{g_N}\, d\gamma$$

TERMINE NOTO

LIFTING E SI PUÒ TOGLIERE

LIFTING PER MULTI-D

$$\begin{cases} u_{LIFT} = h_D(x) & \forall x \in \Gamma_D \\ -\Delta u_{LIFT}(x) = 0 & \forall x \in \Omega \end{cases}$$

$a(u,v) := \int_\Omega \nabla u \nabla v \, dx$

$F(v) := \int_\Omega f \cdot v \, dx$

TROVA $u \in V$: $\forall v \in V$ $\boxed{a(u,v) = F(v)}$

- F è LIMITATO
- a è SIMMETRICO
- a è CONTINUO : $|a(u,v)| \leq C \cdot \|u\|_V \|v\|_V$
- a è COERCIVO : $|a(u,u)| \geq \alpha \|u\|_V^2$

$\alpha > 0$

- LEMMA LAX - MILGRAM
- V HILBERT
- $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ CONTINUA e COERCIVA BILINEARE
- $F : V \to \mathbb{R}$ LIMITATA = CONTINUA LINEARE

$\Rightarrow \exists!$ SOLUZIONE A $\boxed{a(u,v) = F(v) \ \forall v \in V}$

- COROLLARIO u SOLUZIONE DI $\uparrow$ È LIMITATA DAI DATI : $\|u\|_V \leq \frac{1}{\alpha} \|F\|_{V^*}$

α È COST DI COERCIVITÀ DI $a(\cdot,\cdot)$

$\alpha \|u\|_V^2 \leq a(u,u) = F(u) \leq \|F\|_{V^*} \cdot \|u\|_V$

$$\|u\|_V \leq \frac{1}{\alpha}\|F\|_{V'} \qquad \boxtimes$$