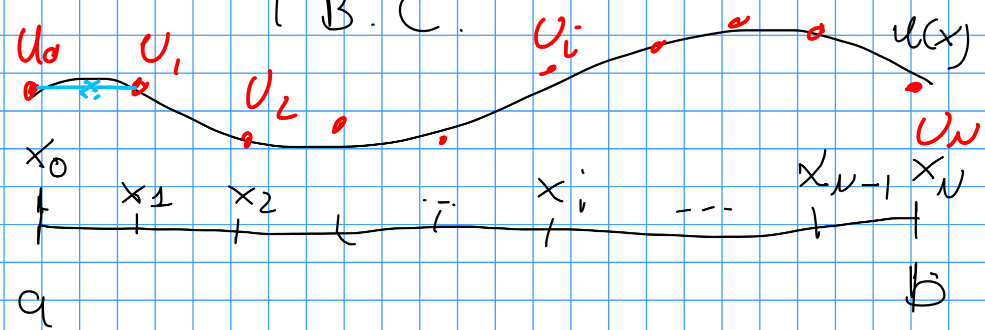
APPROSSIMARE LE SOLUZIONI DELLE PDE

$\boxed{1D}$ $\Omega = [a,b]$

PDE $\begin{cases} F(x, u, \partial_x u, \dots, \partial_x^{(k)} u) = 0 \\ B.C. \end{cases}$

$u(x_i) \approx u_i$



$u \in V$

$\rightsquigarrow \quad \underline{u} = \begin{pmatrix} u_0 \\ \vdots \\ u_N \end{pmatrix}$ INCOGNITA DEL NOSTRO PROBLEMA DISCRETO

$x_{i+1} - x_i = \boxed{h} > 0 \quad \forall\, i = 0, \dots, N-1$

DIFFERENZE DIVISE

$u(x_i) \approx u_i$

$\boxed{\partial_x u(x_i)} \approx ?$

$$\delta_{h,+} u(x) = \frac{u(x+h) - u(x)}{h}$$

D.D. IN AVANTI

$$\lim_{h \to 0} \delta_{h,+} u(x) = \partial_x u^+(x)$$

D.D. ALL'INDIETRO

$$\delta_{h,-} u(x) = \frac{u(x) - u(x-h)}{h} \xrightarrow[h \to 0]{} \partial_x u^{-}(x)$$

D.D. CENTRATA

$$\delta_h u(x) = \frac{u(x+h/2) - u(x-h/2)}{h} \xrightarrow[h \to 0]{} \partial_x u(x)$$

• ACCURATEZZA

TAYLOR EXP. IN x

$$\delta_{h,-} u(x) - u'(x) = \frac{u(x) - u(x-h)}{h} - u'(x) =$$

$$= \frac{\cancel{u(x)} - \left(\cancel{u(x)} - h\,\cancel{u'(x)} + \frac{h^2}{2} u''(\xi)\right)}{h} - \cancel{u'(x)} =$$

$$= -\frac{h}{2} u''(\xi) \qquad \xi \in [x-h, x]$$

$$\left|\delta_{h,-} u(x) - u'(x)\right| \le \frac{h}{2} \max_{\xi \in [a,b]} |u''(\xi)|$$

$$\left|\delta_h u(x) - u'(x)\right| = \left|\frac{u(x+h/2) - u(x-h/2)}{h} - u'(x)\right| =$$

$$= \left| \left\{ \cancel{u(x)} + \underbrace{\frac{h}{2} u'(x)} + \cancel{\frac{h^2}{4} \cdot \frac{1}{2} u''(x)} + \frac{h^3}{8} \cdot \frac{1}{6} u'''(\xi) - \left[\cancel{u(x)} - \underbrace{\frac{h}{2} u'(x)} + \cancel{\frac{h^2}{4} \frac{1}{2} u''(x)} - \frac{h^3}{8} \frac{1}{6} u'''(\xi)\right] \right\} \frac{1}{h} - u' \right.$$

$$\cancel{h}\, u'(x)$$

$$= \frac{h^2}{48}\left|u'''(\xi)+u'''(\eta)\right| \le \frac{h^2}{24}\max_{\xi\in[a,b]}\left|u'''(\xi)\right|$$

$$-u^n \Rightarrow \frac{u^{n+1}-u^{n-1}}{2\Delta t} = F(u^*)$$

$$-u^{n-1}$$

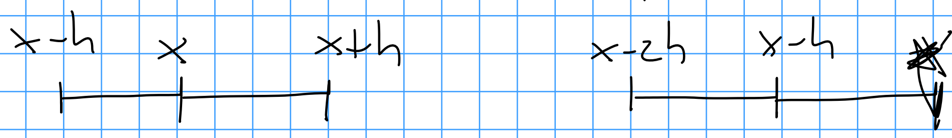
$f:[a,b]\to\mathbb{R}$ $\quad f_h(x)$ $\quad x\in(a,b)$ $\quad h>0$

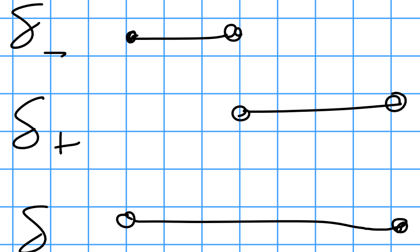
$f_h(x) \to f(x)$ se $\lim_{h\to 0}|f_h(x)-f(x)|=0$

è un'approssimazione di ordine p se

$$|f_h(x)-f(x)| = O(h^p)$$

$$\lim_{h\to 0}\frac{|f_h(x)-f(x)|}{h^p} \le C < \infty$$





$$f'(x) \simeq a_0 f(x) + a_1 f(x-h) + a_2 f(x-2h)$$

TAYLOR 3 ORDINE

$$= O(h^4) + a_0 f(x) + a_1\left(f(x) - h f'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x)\right)$$
$$+ a_2\left(f(x) - 2h f'(x) + 2h^2 f''(x) - \frac{4h^3}{3} f'''(x)\right)$$

$$= O(h^4) + \boxed{(a_0 + a_1 + a_2)} f(x) + \boxed{(-a_1 - 2a_2) h} f'(x)$$
$$+ \boxed{\left(\frac{a_1}{2} + 2a_2\right)} f''(x) h^2 + \underbrace{\boxed{\left(-\frac{a_1}{6} - \frac{4}{3} a_2\right)} h^3 f'''(x)}$$

$$\begin{cases} a_0 + a_1 + a_2 = 0 & \Rightarrow \quad a_0 = -a_1 - a_2 \quad \Rightarrow a_0 = \frac{3}{2} \cdot \frac{1}{h} \\ (-a_1 - 2a_2) h = 1 & \quad 4a_2 - 2a_2 = \frac{1}{h} \Rightarrow a_2 = \frac{1}{2h} \\ \frac{a_1}{2} + 2a_2 = 0 & \Rightarrow \quad a_1 = -4a_2 \quad \Rightarrow a_1 = -\frac{2}{h} \end{cases}$$

$$\underbrace{\left[\frac{3}{2h} f(x) - \frac{2}{h} f(x-h) + \frac{1}{2h} f(x-2h)\right]}_{\delta^* f(x)} \simeq f'(x) +$$
$$\left(-\frac{1}{3 \cdot h} + \frac{2}{3 h}\right) h^3 \cdot f'''(x)$$

$$\left| \delta^* f(x) - f'(x) \right| \leq h^2 \cdot \frac{1}{3} \max_{\xi \in [a,b]} \left| f'''(\xi) \right|$$

GENERALIZZIAMO

STENCIL

$$f' \simeq \frac{1}{h} \left(a_{-p} \cdot f(x-ph) + \dots a_0 f(x) + \dots a_q f(x+qh) \right)$$
$$= \frac{1}{h} \sum_{\ell=-p}^{q} a_\ell f(x + \ell h)$$

ORDINE $= p + q$

DERIVATA ORDINE K CON STENCIL $p+q+1$ PT I

$\rightsquigarrow$ APPROSSIMAZIONE

ORDINE $p+q+1\underbrace{-K}$

$$\frac{1}{h^K}\left(\frac{h^K}{K!}f^{(K)}\right)$$

- DERIVATA $\partial_{xx} f$

$$\delta_h^2 f(x) = \delta_h(\delta_h f)(x) = \delta_h\left(\frac{f(x+h/2) - f(x-h/2)}{h}\right)$$

$$= \frac{\frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h}}{h} = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

- TAYLOR

$$\delta_h^2 f \simeq f'' + \frac{h^2}{24}\left(f^{(4)}(\xi) + f^{(4)}(\eta)\right)$$

$$|\delta_h^2 f - f''| \leq \frac{h^2}{12}\max|f''|$$

$u:[a,b]\to\mathbb{R}$ $\qquad -u''(x)=f(x)$ $\qquad f(x)\in L^2$

$u(a)=u(b)=0$

$0=u_0$ $\qquad$ u_i $\qquad$ $u_N=0$

x_0 $\qquad$ x_i $\qquad$ x_N

a $\qquad\qquad\qquad$ b

$$h = x_{i+1}-x_i \quad \forall i$$

$$-u''(x_i) \approx \underbrace{-\frac{u(x_{i+1})-2u(x_i)+u(x_{i-1})}{h^2} = f(x_i)}$$

$\forall i = 1, \ldots, N-1$

$$\underline{\underline{A}}\,\underline{u} = \underline{f} \qquad f_i = f(x_i)$$

$$u_j = U_j$$

$\Downarrow$

$$a_{ij} = \begin{cases} \frac{2}{h^2} & i=j \\ -\frac{1}{h^2} & j=i+1 \text{ opp } j=i-1 \\ 0 & \text{ALTRIMENTI} \end{cases}$$

$$A = -\frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & & 0 \\ & & & & & & 1 \\ 0 & \cdots & & & 0 & 1 & -2 \end{bmatrix}$$

$$u(a) = \alpha = u_0 \qquad\qquad u(b) = \beta = u_N$$

- $$-\frac{u_0-2u_1+u_2}{h^2} = f_1 \Rightarrow -\frac{-2u_1+u_2}{h^2} = f_1 + \frac{\alpha}{h^2}$$

- $u_0 = \alpha$
- $u_N = \beta$

$$
\underbrace{-\frac{1}{h^2}\begin{bmatrix} -h^2 & 0 & \cdots & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & \cdots & & 0 & -h^2 \end{bmatrix}}_{A}
\begin{bmatrix} u_0 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} \alpha \\ f_1 \\ \vdots \\ f_{N-1} \\ \beta \end{bmatrix}
$$

$\rightsquigarrow \frac{1}{h^2}(1 \; -2 \; 1) \simeq \partial_{xx} + O(h^2)$

- NEUMANN (DISOMOGENEO)

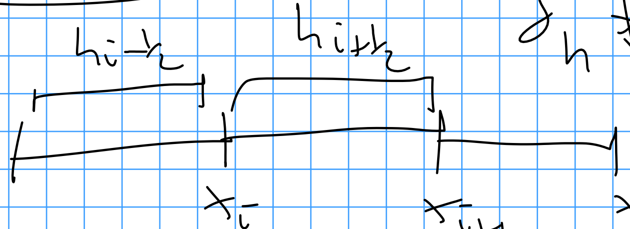
- $\partial_x u(a) = \alpha$ $\qquad u(b) = \beta$

$\approx$

$$\frac{u_1 - u_0}{h} = \alpha \;\to\; \begin{bmatrix} -\frac{1}{h} & \frac{1}{h} & 0 & \cdots & 0 \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} \alpha \\ f_1 \\ \vdots \\ f_N \end{bmatrix}$$

$\simeq \partial_x u + O(h)$

$$\boxed{\frac{-3u_0 + 4u_1 - u_2}{2h}} \simeq \partial_x u + O(h^2)$$



$$\delta_h f = \frac{f(x_i + h_{i+\frac12}) - f(x_i - h_{i-\frac12})}{h_{i+\frac12} + h_{i-\frac12}}$$

$h_{i+\frac12} = x_{i+1} - x_i$

• DISCRETE MAXIMUM PRINCIPLE

• POISSON + DIFFERENZE FINITE + BC DIRI OMOGENEE

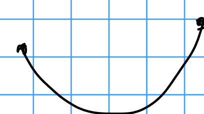
$$\Rightarrow A \in \mathbb{R}^{(N+1)\times(N+1)}$$

$v \in \mathbb{R}^{N+1}$ se $(Av)_i \le 0$

$$\Rightarrow \max_{i=1,\dots,N-1} v_i \le \max\{v_0, v_N\}$$

A LIVELLO CONTINUO $\quad -\partial_{xx} v = f \le 0$

$$\partial_{xx} v \ge 0$$



DIM p. ASSURDO $\exists\, m \quad v_m \quad 1 \le m \le N-1$

$v_m > v_0 \quad v_m > v_N \quad \wedge \quad v_m \ge v_i \quad \forall\, i=1,\dots,N-1$

$$0 \ge (Av)_m = -\frac{1}{\cancel{h^2}}\left(v_{m-1} - 2v_m + v_{m+1}\right)$$

$$v_m \le \frac{v_{m-1}+v_{m+1}}{2} \Rightarrow v_m = v_{m-1} = v_{m+1} = \dots$$

$$v_{m-1} = \dots = v_0$$

$$\dots = v_N$$

↯

ASSURDO

• TEOREMA

FD SU POISSON DIRICHLET $\underline{\underline{A}} \in \mathbb{R}^{(N+1)\times(N+1)}$

U è $\underline{\underline{A}}\,\underline{u} = f$ $\qquad \Omega = [a,b]$

$$\Rightarrow |u(x_i) - u_i| \leq \frac{h^2}{24}(b-a)^2 \max_{a\leq\xi\leq b} |u^{(4)}(\xi)|$$

$\boxed{\text{DIM}}$ $\quad -u'' = f \qquad -u'' - f = 0 \qquad u'' + f = 0$

$$T_h(x) := f(x) + \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

$$= \underbrace{f(x) + u''(x)}_{=0} + \frac{h^2}{24}\left(u^{(4)}(\xi) + u^{(4)}(\eta)\right)$$

$$M = \max_{\xi\in[a,b]} |T(\xi)| \qquad\qquad A u = f$$

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i = f(x_i)$$

$$T(x_i) = \underbrace{f(x_i)}\ + \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} =$$

$$= -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}$$

$$\boxed{h^2 T(x_i) = e_{i+1} - 2e_i + e_{i-1}} \qquad \forall\, i = 1, \dots, N-1$$

$e_i := u(x_i) - u_i \qquad e_0 = 0 \qquad e_N = 0$

$$\phi(x) = \frac{M}{2}\left(x - \frac{a+b}{2}\right)^2$$

$$V \in \mathbb{R}^{N+1} \qquad V_i := e_i + \phi(x_i) \qquad i = 0, \dots, N$$

$$(Av)_i = -\frac{1}{h^2}\left(V_{i+1} - 2V_i + V_{i-1}\right) = -\frac{1}{h^2}\left(e_{i+1} - 2e_i + e_{i-1} + \phi(x_{i+1}) - 2\phi(x_i) + \phi(x_{i-1})\right)$$

$$= -T(x_i) - \frac{M}{2h^2}\left[\left(x_i + h - \frac{a+b}{2}\right)^2 - 2\left(x_i - \frac{a+b}{2}\right)^2 + \left(x_i - h - \frac{a+b}{2}\right)^2\right]$$

$$\left(x_i - \frac{a+b}{2}\right) \quad \begin{matrix} +h \\ +0 \\ -h \end{matrix}$$

$$= -T(x_i) - \frac{M}{2h^2}\cdot\left(2h^2\right) = -T(x_i) - M \leq 0 \qquad \forall i$$

$$\Uparrow$$

$$M = \max_j T(x_j)$$

$$\boxed{\max_{1 \leq i \leq N-1} V_i \leq \max\{V_0, V_N\}}$$

$$V_0 = e_0 + \phi(x_0) = 0 + \frac{M}{2}\left(a - \frac{a+b}{2}\right)^2 = \frac{M}{2}\left(\frac{a-b}{2}\right)^2 =$$

$$= \frac{M}{8}(a-b)^2 = V_N$$

$$V_i \leq \frac{M}{8}(b-a)^2 \quad \forall i \qquad V_i = e_i + \phi(x_i)$$

ERRORE DI TRONCA.

$$\underline{e_i} = u(x_i) - u_i = V_i - \overbrace{\phi(x_i)}^{\geq 0} \leq V_i \leq \frac{M}{8}(b-a)^2$$

$$\leq \frac{h^2}{24}(b-a)^2 \cdot \max_{\xi \in [a,b]} |u^{(4)}(\xi)|$$

$$\rightsquigarrow \quad v_i = -e_i + \phi(x_i)$$

$$-e_i \leq \frac{h^2}{8}(b-a)^2 . \qquad |e_i| \leq \frac{h^2}{8}(b-a)^2 . \ \boxtimes$$