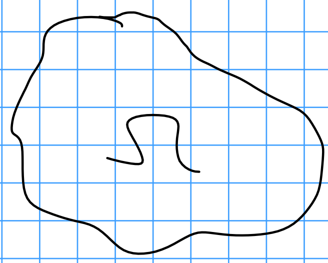
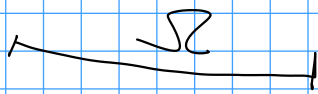
$\Omega \subset \mathbb{R}^d$




$\subset \mathbb{R}^2$

$u : \Omega \to \mathbb{R}^s$




$\subset \mathbb{R}^1$

$$F(\nabla^{(k)} u, \nabla^{(k-1)} u, \dots, \nabla u, u) = 0$$

$$u, \quad \nabla u = \begin{pmatrix} \partial_{x_1} u \\ \partial_{x_2} u \\ \partial_{x_3} u \\ \vdots \\ \partial_{x_d} u \end{pmatrix} \in \mathbb{R}^{d \times s}$$

$$(\nabla^{(2)} u^\ell)_{ij} = \partial_{x_i} \partial_{x_j} u^\ell \qquad \ell = 1, \dots, s$$

$$\nabla^{(2)} u \in \mathbb{R}^{d \times d \times s}$$

$$\nabla \begin{pmatrix} \partial_{x_1}\partial_{x_1} u & \partial_{x_1}\partial_{x_2} u & \dots & \partial_{x_1}\partial_{x_d} u \\ \partial_{x_2}\partial_{x_1} u & \dots & & \\ \vdots & & & \\ \partial_{x_d}\partial_{x_1} u & \dots & & \partial_{x_d}\partial_{x_d} u \end{pmatrix}$$

$$\nabla^{(k)} u \in \mathbb{R}^{\overbrace{d \times \dots \times d}^{} \times s}_{\text{k-volte}}$$

$$(\nabla^{(k)} u^\ell)_{i_1, \dots, i_k} = \partial_{x_{i_1}} \partial_{x_{i_2}} \dots \partial_{x_{i_k}} u^\ell$$

---

- DERIVATE PARZIALI

$$\partial_t u(t, x) = \frac{\partial u(t,x)}{\partial t} = u_t(t,x)$$

- DERIVATA TOTALE

$$\frac{d}{dt} u(t, x(t)) = \underbrace{\frac{\partial u}{\partial t}}(t, x(t)) + \frac{\partial}{\partial x} u(t, x(t)) \cdot \frac{dx(t)}{dt}$$

- $\nabla u = \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}$ GRADIENTE

$u(x,y)$

- LAPLACIANO $u(x,y)$

$$\Delta u = \nabla^2 u = \partial_{xx} u + \partial_{yy} u$$

- DIVERGENZA

$$\mathrm{div} \begin{pmatrix} u \\ v \end{pmatrix} = \nabla \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \partial_x u + \partial_y v$$

---

$$\mathrm{div}(u) = \sum_{i=1}^{d} \partial_{x_i} u^i = \partial_{x_i} u^i$$

$u \in \mathbb{R}^d$ $\quad u = \begin{pmatrix} u^1 \\ u^2 \\ \vdots \\ u^d \end{pmatrix}$

$\underbrace{u \cdot \nabla u}$ $\quad u \in \mathbb{R}^d$

$\nabla u \in \mathbb{R}^{d \times d}$

$$u^i \cdot \partial_{x_i} u^J = \underbrace{\sum_{i=1}^{d}} u^i \partial_{x_i} u^J = (u \cdot \nabla u)^J \in \mathbb{R}^d$$

# RIPASSO DI ODE

$$\begin{cases} \dfrac{dy(t)}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases} \qquad y : [0, t_{end}] \to \mathbb{R}^S$$

ODE DI ORDINE $p$

$$\partial_t^{(p)} y = f(t, y(t), \dots, y^{(p-1)}(t)) \qquad y : [0, t_{end}] \to \mathbb{R}^S$$

$$\begin{cases} y'(t) = z_1(t) \\ z_1'(t) = z_2(t) \quad \cancel{(= y''(t))} \\ \vdots \\ z_{p-2}'(t) = z_{p-1}(t) \\ z_{p-1}'(t) = \cancel{y^{(p)}(t)} = f(t, y, z_1, \dots, z_{p-2}) \end{cases} \qquad (1)$$

$$w = \begin{pmatrix} y \\ z_1 \\ \vdots \\ z_{p-1} \end{pmatrix} \in \mathbb{R}^{p \times S}$$

(1) SISTEMA DI ODE PRIMO ORDINE

$$\begin{cases} \dfrac{dy(t)}{dt} = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

FORMA INTEGRALE $\Rightarrow \displaystyle\int_0^T \frac{dy(t)}{dt}\, dt = \int_0^T F(t, y(t))\, dt$

$$y(T) - y(0) = \int_0^T F(t, y(t))\, dt$$

$$\begin{cases} y'(t) = -\lambda\, y(t) \\ y(t_0) = y_0 \end{cases} \qquad y : [0, T] \to \mathbb{R}$$

$$\underbrace{\int_0^t \frac{y'(s)}{y(s)}\, ds}_{} = \int_0^t -\lambda\, ds \qquad \log(y(s))' = \frac{y'(s)}{y(s)}$$

$$\Big[\log(y(s))\Big]_0^t = -\lambda(t - 0)$$

$$\log(y(t)) - \log(y(0)) = -\lambda t$$

$$\log\left(\frac{y(t)}{y(0)}\right) = -\lambda t \qquad \frac{y(t)}{y(0)} = e^{-\lambda t}$$

$$y(t) = y(0)\, e^{-\lambda t} \qquad \lambda > 0$$

$$\underline{y}(t) = e^{At}\, \underline{y}(0)$$

$$e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^k}{k!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{At} = I + A \cdot t + \frac{A^2}{2} t^2 + \dots \frac{A^k t^k}{k!}$$

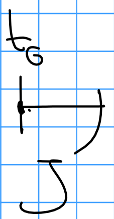
$$\begin{cases} \dfrac{d}{dt} y(t) = F(t, y(t)) \\ y(0) = y_0 \end{cases}$$

$y : [0,T] \to \mathbb{R}^s$

$F : [0,T] \times \mathbb{R}^s \to \mathbb{R}^s$

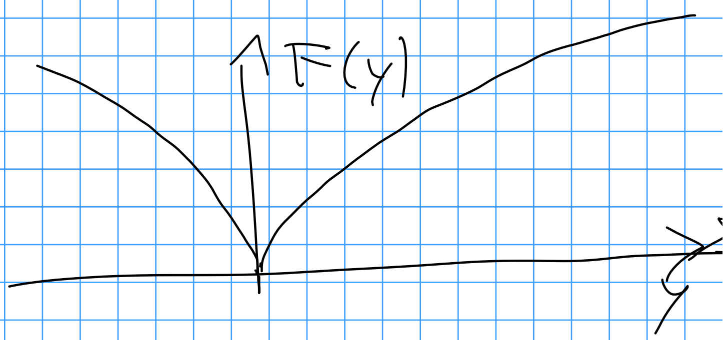
F CONTINUA

SE F CONTINUA IN $(t_0 = 0, y_0)$, $\Rightarrow \exists$ UN INTERVALLO $J \ni t_0$ $\exists\, y(t) : y' = F(t,y)$ $\forall\, t \in J$




ESISTENZA

UNICITÀ




$$\begin{cases} y' = \sqrt{|y|} = F(t, y) \\ y(0) = 0 \end{cases}$$

- $y(t) \equiv 0$ $\quad y' = 0$ ✓ $\quad F(y) = \sqrt{|0|} = 0$ ✓
- $y(t) = \dfrac{t^2}{4}$ $\quad y' = \dfrac{2t}{4} = \dfrac{t}{2}$ $\quad \boxed{t \geq 0}$

$$\| \quad F(y) = \sqrt{\frac{t^2}{4}} = \frac{|t|}{2}$$

SE F è LIPSCHITZ CONTINUA SU $[0,T] \times \mathbb{R}^s$ CON COSTANTE $L \geq 0$ : $\forall\, (t,y), (t,z)$

$$\boxed{\|F(t,y) - F(t,z)\| \le L\,\|y - z\|}$$

$\Rightarrow$ UNICITÀ SOLUZIONE. $\quad y'(t) = F(t, y(t))$

$y(0) = y_0$

- REGOLARITÀ

Se $F \in C^p \Rightarrow y \in C^{p+1}$

$$y(t) = y(0) + \int_0^t F(t, y(t))\, dt \,.$$

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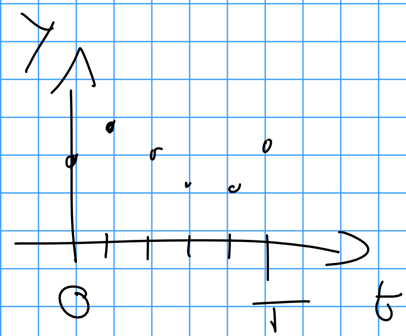
$\boxed{\text{ODE}}$

$$\begin{cases} \dfrac{dy(t)}{dt} = F(t, y(t)) \\ y(0) = \boxed{y_0} \end{cases} \qquad y : [0, T] \to \mathbb{R}^d$$



$$t^0 = 0 < t^1 < t^2 < \dots < t^N = T$$

$$y(t^n) \approx y^n$$

$$y^0 = y_0 \rightsquigarrow y^1 \to y^2 \dots$$

$$y' = F \rightsquigarrow y(t) = y(s) + \int_s^t F(\xi, y(\xi))\, d\xi$$

FOCUS ON $[t^n, t^{n+1}]$

$$y(t^{n+1}) = y(t^n) + \int_{t^n}^{t^{n+1}} F(\xi, y(\xi))\, d\xi$$

$$y^{n+1} := y^n + \int_{t^n}^{t^{n+1}} \underbrace{F(t^n, y^n)}_{\text{APPROX } F(\xi, y(\xi))} \, d\xi$$

$$y^{n+1} := y^n + \Delta t^n \cdot F(t^n, y^n)$$

EULERO ESPLICITO

$$\Delta t^n := t^{n+1} - t^n$$

EULERO IN AVANTI

$$c' = \begin{pmatrix} -5 & 1 \\ 5 & -1 \end{pmatrix} c = \begin{matrix} -5c_1 + c_2 \\ 5c_1 - c_2 \end{matrix}$$

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ERRORE

$$e_n = y(t^n) - y^n$$

ERRORE TOTALE

$$\varepsilon_n = \underbrace{y(t^{n+1}) - y(t^n)} - \Delta t \, F(t^n, y(t^n))$$

ERRORE DI CONSISTENZA

$$= \int_{t^n}^{t^{n+1}} y'(t) \, dt - \int_{t^n}^{t^{n+1}} \underbrace{F(t^n, y(t^n))} \, dt$$

$$F(t^n, y(t^n)) = y'(t^n)$$

NON DIPENDE DA $t$

$$= \int_{t^n}^{t^{n+1}} y'(t) - y'(t^n) \, dt = *$$

$$\omega(g, \Delta t) := \max_{t, t' : |t - t'| < \Delta t} |g(t) - g(t')|$$

$$|\varepsilon_n| = |*| \le \Delta t \cdot \omega(y', \Delta t)$$

$$e_{n+1} = y(t^{n+1}) - y^{n+1} = \overbrace{y(t^{n+1}) - y(t^n) - \Delta t\, F(t^n, y(t^n))}^{\varepsilon_n}$$

$$+ \underbrace{y(t^n)} + \Delta t\, F(t^n, y(t^n)) - (\underbrace{y^n} + \Delta t\, F(t^n, y^n))$$

$$= \varepsilon_n + e_n + \Delta t\, \underbrace{\left(F(t^n, \underbrace{y(t^n)}) - F(t^n, \underbrace{y^n})\right)}$$

$$\underline{|e_{n+1}|} \leq |\varepsilon_n| + |e_n| + \Delta t\, L \cdot \underbrace{|y(t^n) - y^n|}_{e_n}$$

LIPSCHITZ CONSTANT DI $F$

$$|e_{n+1}| \leq |\varepsilon_n| + |e_n| \underbrace{(1 + \Delta t L)}, \quad \forall n$$

$$\Downarrow \qquad (1+\Delta t L)\,|e_{n-1}| + |\varepsilon_{n-1}|$$

$$|e_n| \leq e^{L|t^n - t^0|} \underbrace{|e_0|}_{\simeq 0} + \sum_{i=0}^{n-1} e^{L(t^n - t^{i+1})} \underbrace{|\varepsilon_i|}_{\leq \Delta t\, \omega(y', \Delta t)}$$

$$|e_n| \leq e^{L|t^n - t^0|} |e_0| + \Delta t\, \omega(y', \Delta t) \cdot \underbrace{\sum_{i=0}^{n-1} e^{L(t^n - t^{i+1})}}$$

$$\leq e^{L|t^n - t^0|} |e_0| + \underbrace{\omega(y', \Delta t)} \frac{e^{L(t^n - t^0)} - 1}{L}$$

Se $\Delta t \to 0$ $\quad \omega(y', \Delta t) \to 0$ $\quad$ Se $y'$ CONTINUA

$$\omega = \max_{t,t' : |t-t'| < \Delta t} |y'(t) - y'(t')| \qquad y' = F$$

$$e_1 = |y^1 - y(t^1)| = \qquad t^1 - t^0 = \Delta t$$

$$= \Big| \underbrace{y^0 + \Delta t\, F(y^0)}_{\text{DEFINIZIONE DI EULERO ESPLICITO}} - \Big( \underbrace{y_0 + \Delta t\, y'(t^0) + \frac{\Delta t^2}{2} y''(t^0) + O(\Delta t^3)}_{y(t^1) \to \text{ espansione di Taylor in } t^0} \Big) \Big|$$

$$= \Big| y_0 + \Delta t\, y'(t^0) - y_0 - \Delta t\, y'(t^0) - \frac{\Delta t^2}{2} y''(t^0) + O(\Delta t^3) \Big|$$

$$\leq \frac{\Delta t^2}{2} |y''(t^0)| + O(\Delta t^3)$$

$$e_1 \xrightarrow[\Delta t \to 0]{} 0$$

errore di consistenza

$$e_N \approx \sum_{i=1}^{N} |y(t^i) - y^i| \approx N \cdot \left( \frac{\Delta t^2}{2} \max_{t \in [0,T]} |y''| + O(\Delta t^3) \right)$$

$$N \longleftrightarrow \Delta t$$

$$[0,T] \qquad \Delta t = \frac{T}{N} \Rightarrow N = \frac{T}{\Delta t}$$

$$e_N \lessapprox \frac{T}{\Delta t} \cdot \left( \frac{\Delta t^2}{2} \max |y''| + O(\Delta t^3) \right) =$$

$$= T \cdot \frac{\boxed{\Delta t}}{2} \max |y''| + O(\Delta t^2)$$

ORDINE DI ACCURATEZZA DI UN METODO È IL MASSIMO $p \in \mathbb{N} : |e_N| \leq C \cdot \Delta t^p \quad \Delta t \in \mathbb{R}^+$

$$\varepsilon_i = \underline{O(\Delta t^{p+1})} \qquad e_N = O(\Delta t^p).$$

$$e_N \simeq \sum_{i=1}^{N} |\varepsilon_i| \le N \cdot O(\Delta t^{p+1}) = \underbrace{\frac{T}{\Delta t}}_{N} \cdot O(\Delta t^{p+1})$$

$$= O(\Delta t^p)$$

$$C = O(\Delta t)$$

$$\frac{C(\Delta t)}{\Delta t} \le D \in \mathbb{R}$$

$$\lim_{\Delta t \to 0} \frac{C(\Delta t)}{\Delta t} \le D \in \mathbb{R}$$

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$$y'(t) = M y(t) \qquad y : [0, T] \to \mathbb{R}^s$$

$$y^{n+1} = y^n + \Delta t M \cdot y^n \qquad M \in \mathbb{R}^{s \times s}$$

$$= (I + \Delta t M)\, y^n$$

$$= (I + \Delta t M)(I + \Delta t M) \cdot y^{n-1}$$

$$= \dots$$

$$= \underline{(I + \Delta t M)^n} \quad y^0$$

$S$ INVERTIBILE : $\hat{M} = S^{-1} M S$

È NELLA FORMA CANONICA DI JORDAN

$$\begin{bmatrix} * & * & 0 & 0 & \text{———} \\ 0 & * & 0 & 0 & \text{———} \\ | & 0 & * & 0 & \text{———} \\ | & | & 0 & * & 0 \text{———} \\ & | & | & & \ddots \end{bmatrix} \in \mathbb{C}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 \qquad \lambda = \pm i$$

$$S \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} S^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$M = S \hat{M} S^{-1}$$

$$y^{n+1} = y^n + \Delta t \, M y^n$$

$$\underbrace{S^{-1} y^{n+1}}_{=: \hat{y}^{n+1}} = S^{-1} y^n + \Delta t \, \cancel{S}^{-1} (\cancel{S} \hat{M} \underbrace{S^{-1}) \cdot y^n}_{\hat{y}^n}$$

$$\hat{y}^{n+1} = \hat{y}^n + \Delta t \, \hat{M} \hat{y}^n$$

$i$ : quando $M_i : [0 \; 0 \; * \; 0 \; 0]$

$$\Rightarrow \hat{y}_i^{n+1} = \hat{y}_i^n + \Delta t \, \hat{M}_{ii} \cdot \hat{y}_i^n$$

quand $M_i : [0 \; 0 \; \cdots \; * \; * \; 0 \; 0]$ ✓

$$\left( \hat{y}_i^{n+1} = \hat{y}_i^n + \Delta t \, \hat{M}_{ii} \, \hat{y}_i^n \right) + \Delta t \, \hat{M}_{i,i+1} \underbrace{\overbrace{\hat{y}_{i+1}^n}}$$

ORA SOLO EQUAZIONI SCALARI

$$y' = q\,y \qquad y : [0, T] \to \mathbb{R}$$

$$q \in \mathbb{C}$$

$$y^{n+1} = y^n + \Delta t\, q\, y^n$$
$$= (1+\Delta t q)\, y^n = \ldots = (1+\Delta t q)^n \cdot y^0$$

$|y^{n+1}| \leq |y^n|$ se $|q| \leq 1$

SCHEMA È STABILE SE $|y^{n+1}| \leq |y^n|$

$$|y^{n+1}| \leq |y^0|$$

$$|y^{n+1}| = |(1+\Delta t q)^n \cdot y^0| \leq |(1+\Delta t q)^n|\,|y^0|$$
$$= \underbrace{|1+\Delta t q|^n}_{\leq 1}\,|y^0|$$

$$|1+\Delta t q|^n \leq 1$$

$$\iff \underbrace{|1+\Delta t q|}\leq 1 \qquad q \in \mathbb{C}$$

$$|1+\Delta t \operatorname{Re}(q) + \Delta t\, i \operatorname{Im}(q)| \leq 1$$

$$\overbrace{|1+\Delta t \underbrace{\operatorname{Re}(q)}_{\in \mathbb{R}}|}^{\leq} \leq 1 \qquad y' = -\lambda y$$

$$\underbrace{-2 \leq \Delta t \cdot \operatorname{Re}(q) \leq 0} \qquad \operatorname{Re}(q) \leq 0$$

PER ESSERE BEN POSTO

$$\Delta t \le -\frac{2}{Re(q)} > 0$$

$$1 + \underbrace{\Delta t q}_{z \in \mathbb{C}}$$

FUNZIONE DI STABILITÀ DI EULERO ESPLICITO

$$R(z) := 1 + z$$

$$|R(\Delta t q)| \le 1$$

$$|R(z)| \le 1$$

$$|1 + z| \le 1$$

$$|1 + Re(z) + i\,Im(z)| = \underset{D}{\sqrt{(1 + \underset{x}{Re(z)})^2 + \underset{y}{Im(z)}^2}} \le 1$$

CERCHIO CENTRO IN $(-1, 0) = -1$
CON RAGGIO 1

$Re(q) = 0 \quad Im(q) \ne 0 \Rightarrow$ EULERO ESPLICITO NON È STABILE MAI

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

EULERO IMPLICITO

$$y^{n+1} = y^n + \Delta t\, F(t^{n+1}, y^{n+1})$$

$$y' = q y$$

$$\Downarrow$$

$$y^{n+1} = y^n + \Delta t\, q\, y^{n+1} \leftarrow$$

$$(1 - \Delta t q)\, y^{n+1} = y^n$$

$$y^{n+1} = \frac{1}{1 - \Delta t \cdot q} \cdot y^n$$

$$R(z) = \frac{1}{1-z}$$ FUNZIONE DI STABILITÀ PER EULERO IMPLICITO

$$|R(z)| \leq 1$$

$$\left|\frac{1}{1-z}\right| \leq 1 \qquad |1 - z| \geq 1$$

$$\Downarrow$$

$$\sqrt{(1 - \mathrm{Re}(z))^2 + \mathrm{Im}(z)^2} \geq 1$$

$\mathrm{Re}(z) \Rightarrow x$, $\mathrm{Im}(z) \Rightarrow y$

$(1, 0)$

TUTTI I PUNTI FUORI DAL CERCHIO CENTRATO IN 1 $R = 1$

$$\varepsilon_n = O(\Delta t^2) \qquad y' = F(y) \simeq y^0 + (y - y^0) \cdot \frac{\partial F}{\partial y}$$

$$e_N = O(\Delta t) \quad \text{PRIM'ORDINE}$$

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