Linear Elliptic Differential Equations

Poisson equation

Given a domain $\Omega \in \mathbb{R}^d$ where $d > 1$, we seek for $u : \Omega \to \mathbb{R}$ solution of

$$-\Delta u = f \text{ in } \Omega,$$

where Δ is the Laplacian operator, i.e., $\partial^2_{x_1} + \partial^2_{x_2} + \cdots + \partial^2_{x_d}$.

We say the equation is homogeneous if $f \equiv 0$. The homogeneous Poisson equation is called Laplace equation.

Physical applications

- Heat conduction (u temperature),
- Elastic membrane subject to a body force f (u is the displacement),
- Electric potential distribution (u) due to a charge f.

Physical derivation

u is a concentration, at equilibrium there will be zero net flux $\mathbf{F}$ through the boundary of any regular subdomain $S \subset \Omega$, so

$$\int_{\partial S} \mathbf{F} \cdot \mathbf{n} \mathrm{d}\Gamma = 0,$$

using the Gauss-Green theorem, we have that

$$\int_S \mathrm{div} \mathbf{F} \mathrm{d}x = \int_{\partial S} \mathbf{F} \cdot \mathbf{n} \mathrm{d}\Gamma = 0,$$

so $\mathrm{div}\mathbf{F} = 0$. In many applications it is reasonable to assume that the flux is proportional to the gradient of the concentration (chemical concentration, heat), from the higher concentration region to the lower ones $\mathbf{F} = -a\nabla u$.

Substituting we get

$$\mathrm{div}\mathbf{F} = -\mathrm{div}(a\nabla u) = -a\Delta u = 0.$$

Boundary conditions

To obtain uniqueness of the solution, we need to enforce some extra constraints.

Dirichlet boundary conditions

$$u = g \text{ on } \partial\Omega.$$

If $g \equiv 0$, then they are called homogeneous Dirichlet BC.

Elastic application -> imposing a given displacement.

Neumann boundary conditions

$$\nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial \mathbf{n}} = h \text{ on } \partial\Omega.$$

$\mathbf{n}$ is the normal vector going out of the domain Ω in each point of the boundary $\partial\Omega$.

Elastic application -> prescribed surface traction or stress on the boundary.

Boundary conditions

Geometric Combinations

$$\partial\Omega = \Gamma_D \cup \Gamma_N, \qquad \Gamma_D^\circ \cap \Gamma_N^\circ = \emptyset$$

$$\begin{cases} u = g \text{ on } \Gamma_D \\ \nabla u \cdot \mathbf{n} = g \text{ on } \Gamma_N. \end{cases}$$

Physical Combinations: Robin boundary

$$\nabla u \cdot \mathbf{n} + \gamma u = r \text{ on } \Gamma_R.$$

Regularity of the solution

It is not always possible to find the strong solution of the equation.
Consider the problem $-\Delta u = 1$ on $\Omega = [0,1]^2$ with homogeneous Dirichlet boundary conditions. Clearly, in $(0,0)$ the solution at the boundaries is such that

$$-\Delta u(0,0) = -\partial_{xx} u(0,0) - \partial_{yy} u(0,0) = 0,$$

as the BC impose that $u(x,0) = 0 = u(0,y)$ for all $x, y \in [0,1]$.

So, even if $f \in C^0(\bar{\Omega})$ it does not makes sense to look for a solution in $\mathcal{C}^2(\bar{\Omega})$.
What we are actually looking for, in this case, is a solution in the space $C^2(\Omega) \cap C^0(\bar{\Omega}) \supset C^2(\bar{\Omega})$.

Towards a weak formulation

Let's go back to 1D and to the homogeneous Dirichlet BCs.

$$\begin{cases} -u''(x) = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

This describes the displacement of a string under a transversal force with intensity $f(x)$ in each point. The total force acting on a portion of the domain $(0, x)$ is given by $F(x) = \int_0^x f(s)\mathrm{d}s$.

It is not possible to use this formulation to describe, for example, the case where the force is applied in only a point, for example $x_0 = \frac{1}{2}$. There $f = -\delta_{x_0}$ would be what describes the force. A physical solution of course exists, it is continuous but not C^1 on the whole domain, in particular it will have a discontinuity in the derivative in x_0.

$$u(x) = \begin{cases} -\frac{1}{2}x, & x < \frac{1}{2}, \\ \frac{1}{2}x - \frac{1}{2}, & x < \frac{1}{2}. \end{cases}$$

Even if we take $f \in L^2((0,1))$, but not continuous we might have similar problems: take $f = -\chi_{[0.4,0.6]}$, then the solution (physically) is

$$u(x) = \begin{cases} -\frac{1}{10}x, & x < 0.4, \\ \frac{1}{2}x^2 - \frac{1}{2}x + \frac{2}{25}, & x \in [0.4, 0.6], \\ -\frac{1}{10}(1-x), & x > 0.6. \end{cases}$$

Clearly, $u \in C^1$ but $u \notin C^2$. Still, we would like our problem to have a meaning also in these contexts. Goal: get rid of the second derivative! Consider a smooth test function $v \in \mathcal{D}((0,1))$

$$-u'' = f \implies -u''v = fv \implies \int_0^1 -u''(x)v(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x.$$

Integration by parts

$$\int_0^1 -u''(x)v(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x \implies \int_0^1 u'(x)v'(x)\,\mathrm{d}x - \left[u'v\right]_0^1 = \int_0^1 f(x)v(x)\,\mathrm{d}x$$

since $v \in \mathcal{D}((0,1))$ it is zero on the boundary, so,

$$\int_0^1 u'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x.$$

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We can consider a test function in $\mathcal{D}$ with zero boundaries because we are imposing the Dirichlet boundary conditions on u itself, so, we already know u at the boundaries.

Hence, one could think that looking for a solution and a test function in

$$V = \{v \in C^1([0,1]) : v(0) = v(1) = 0\}$$

is a possibility. Unfortunately, looking for something so regular is still too much as the regularity of the solution will depends on f. (Lesson 011)

The problem is that the space V with the norm $|\cdot|_1$ is not complete.

Let's enlarge the space, to get a complete functional space.

Take $u, v \in V := H_0^1((0,1))$, now all the integrals are meaningful if we take $f \in L^2((0,1))$, and the space normed with $|\cdot|_1$ is complete.

Variational equivalent problem

Weak problem is find $u \in H_0^1((0,1))$ such that for every $v \in H_0^1((0,1))$

$$\int_0^1 u'(x)v'(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x.$$

Equivalent **variational problem** is find $u \in V = H_0^1((0,1))$ such that

$$\begin{cases} J(u) = \min_{v\in V} J(v) \text{ with} \\ J(v) := \frac{1}{2}\int_0^1 (v')^2 \mathrm{d}x - \int_0^1 fv\mathrm{d}x. \end{cases}$$

Sketch of the proof

Define for every $w \in V$ the function $\psi(\delta) = J(u+\delta w)$.

$$\psi(\delta) = \frac{1}{2}\int_0^1 (u')^2 + \delta\int_0^1 u'w' + \frac{\delta^2}{2}\int_0^1 (w')^2 - \int_0^1 fu - \delta\int_0^1 fw.$$

This is a quadratic function in δ, it's a parabola, and the minimum is at

$$\delta = -\frac{\int_0^1 u'w' - \int_0^1 fw}{\int_0^1 (w')^2} = 0.$$

Hence, for every w and every δ, $\psi(0) = J(u) \leq J(u+\delta w) = \psi(\delta)$.

Nonhomogeneous Poisson problem

If we have nonhomogeneous Dirichlet BC, e.g.

$$\begin{cases} -u'' = f \\ u(0) = u_L, \quad u(1) = u_R, \end{cases}$$

we can consider the *lifting* $u_{lift} := [(1-x)u_L + xu_R]$ that solves $u''_{lift} = 0$, $u_{lift}(0) = u_L$ and $u_{lift}(1) = u_R$. Defining
$\tilde{u} := u - u_{lift}$, we have that $\tilde{u}$ solves the homogeneous problem

$$\begin{cases} -\tilde{u}'' = -\tilde{u}'' = f \\ \tilde{u}(0) = u(0) - u_{lift}(0) = 0, \qquad \tilde{u}(1) = u(1) - u_{lift}(1) = 0, \end{cases}$$

and we are back to the previous case!

Neumann boundary conditions

$$\begin{cases} -u'' = f \\ u'(0) = h_0, \quad u'(1) = h_1, \end{cases}$$

is clearly not well defined, as if u is a solution, then also $\tilde{u}(x) = u(x) + c$ for every $c \in \mathbb{R}$ is a solution! Non uniqueness!

Possibilities:

- Change the problem into something like $-u'' + \sigma u = f$.
- Change BC into a mixed BC: one boundary Dirichlet, one boundary Neumann.

Mixed homogeneous conditions (homogeneous in Dirichlet)

$$\begin{cases} -u'' = f \\ u(0) = 0, \quad u'(1) = g_1. \end{cases}$$

As for Dirichlet problem, I need the test function to be 0 on the left boundary, while, since we have no information on the right boundary, I have to let them free on the right.

$$V = \{v \in H^1((0,1)) : v(0) = 0\}$$

Let's write the weak formulation for every $v \in V$, again using integration by parts, we have

$$0 = \int_0^1 -u''v - fv \, \mathrm{d}x = \int_0^1 u'v' - fv \, \mathrm{d}x - \underbrace{u'(1)}_{=g_1}v(1) + u'(0)\underbrace{v(0)}_{=0} = \int_0^1 u'v' - fv \, \mathrm{d}x - g_1 v(1).$$

Weak formulation for 2D problems

Let $\Omega \subset \mathbb{R}^2$ a bounded domain with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$. The Poisson problem with Dirichlet and Neumann BC reads

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u(x) = u_D(x), & \text{in } \Gamma_D, \\ \nabla u(x) \cdot \mathbf{n} = g_N(x), & \text{in } \Gamma_N. \end{cases}$$

- Test function $v \in V = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$

$$\int_\Omega -\Delta u v - f v \mathrm{d}x = 0 + BCs.$$

Instead of integration by parts, we use the divergence theorem: $\int_\Omega \mathrm{div}(\mathbf{a})\mathrm{d}x = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n}\, \mathrm{d}\gamma$ and we notice that

$$\int_\Omega \mathrm{div}(v\nabla u)\mathrm{d}x = \int_{\partial\Omega} v\nabla u \cdot \mathbf{n}\, \mathrm{d}\gamma$$

$$\int_\Omega \mathrm{div}(v\nabla u)\, \mathrm{d}x = \int_\Omega \sum_{i=1}^d \partial_{x_i}(v \partial_{x_i} u)\, \mathrm{d}x = \int_\Omega \sum_{i=1}^d \partial_{x_i} v \cdot \partial_{x_i} u \, \mathrm{d}x + \int_\Omega \sum_{i=1}^d \partial_{x_i x_i} u \cdot v \, \mathrm{d}x =$$

$$= \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega v \Delta u \, \mathrm{d}x$$

Laplacian Green's formula

$$-\int_\Omega v\Delta u \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Omega \mathrm{div}(v\nabla u)\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_{\partial\Omega} v\nabla u \cdot \mathbf{n} \,\mathrm{d}\gamma.$$

Weak formulation of 2D problem

Let's go back to our problem, we can use the fact that $v = 0$ on Γ_D and that $\nabla u \cdot \mathbf{n} = g_N$ on Γ_N to write the weak formulation of Poisson problem as find $u \in V_D := \{v \in H_1(\Omega) : v|_{\Gamma_D} = u_D\}$

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \int_{\partial\Omega} v\, g_N(\gamma)\,\mathrm{d}\gamma = 0 \qquad \forall v \in V = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}.$$

To have symmetry between the space of u and the space of v, we can use again the lifting $u_{lift} \in V_D$ such that $-\Delta u_{lift} = 0$ and solve for $\tilde{u} = u - u_{lift} \in V$.

Proposition [Check Quarteroni]

The weak formulation is equivalent to the strong formulation where the differential operators are meant in a distributional sense (take as a test function $v \in \mathcal{D}$).

General problem

Let $\Omega \subset \mathbb{R}^d$, $\Gamma_N \cup \Gamma_D = \partial\Omega$, $\Gamma_N^\circ \cap \Gamma_D^\circ = \emptyset$, $f \in L^2(\Omega)$, $\mu, \sigma \in L^\infty(\Omega)$, $u_D \in H^1(\Omega)$ and $g \in L^2(\Gamma_N)$. Find $u \in V_D = H^1(\Omega) \cap \{v : v|_{\Gamma_D} = u_D\}$ such that

$$\begin{cases} -\mathrm{div}(\mu \nabla u) + \sigma u = f, & \text{in } \Omega, \\ u = u_D, & \text{in } \Gamma_D, \\ \mu \nabla u \cdot \mathbf{n} = g, & \text{in } \Gamma_N. \end{cases}$$

Weak formulation

Find $u \in V_D$ such that for every $v \in V$

$$\int_\Omega \mu \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega \sigma\, u\, v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x + \int_{\Gamma_N} g v \,\mathrm{d}\gamma.$$

Let's symmetrize the spaces using a known lifting $u_{lift} \in V_D$, so that we look for a $\tilde{u} = u - u_{lift} \in V$ such that

$$\int_\Omega \mu \nabla \tilde{u} \cdot \nabla v \,\mathrm{d}x + \int_\Omega \sigma\, \tilde{u}\, v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x + \int_{\Gamma_N} g v \,\mathrm{d}\gamma - \int_\Omega \mu \nabla u_{lift} \cdot \nabla v \,\mathrm{d}x - \int_\Omega \sigma\, u_{lift}\, v \,\mathrm{d}x.$$

Bilinear form

We can define now the bilinear form $a : V \times V \to \mathbb{R}$ and the linear form $F : V \to \mathbb{R}$ defined as

$$\begin{cases} a(u,v) := \int_\Omega \mu \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega \sigma\, u\, v \,\mathrm{d}x, \\ F(v) := \int_\Omega f\, v \,\mathrm{d}x + \int_{\Gamma_N} g v \,\mathrm{d}\gamma - \int_\Omega \mu \nabla u_{lift} \cdot \nabla v \,\mathrm{d}x - \int_\Omega \sigma\, u_{lift}\, v \,\mathrm{d}x. \end{cases}$$

The previous problem is now: find $\tilde{u} \in V$ such that for all $v \in V$ $a(u,v) = F(v)$.

Exercise

- F is linear and bounded
- a is symmetric: $a(u,v) = a(v,u)$
- a is continuous: $|a(u,v)| \leq C\|u\|_V \|v\|_V$
- a is coercive: $|a(u,u)| \geq \alpha \|u\|_V^2$

Existance and uniqueness

Lax-Milgram Lemma

Let V a Hilbert space, $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ a bilinear **continuous** and **coercive** form, $F : V \to \mathbb{R}$ a bounded linear functional. Then, there exists and it is unique the solution of the problem: find $u \in V$ such that for every $v \in V$

$$a(u, v) = F(v).$$

Proof [Evans]

Corollary

The solution u is bounded by boundary and right hand side data, i.e., $\|u\|_V \leq \frac{1}{\alpha}\|F\|_{V^*}$.

Proof

$$\alpha\|u\|_V^2 \leq a(u, u) = F(u) \leq \|F\|_{V^*}\|u\|_V.$$

Equivalent variational problem

If, in addition, a is symmetric, then the problem is equivalent to the following variational problem: find u such that

$$\begin{cases} J(u) = \min_{v \in V} J(v), \text{ with} \\ J(v) := \frac{1}{2} a(v, v) - F(v). \end{cases}$$

Exercise: Proof

Summary: Poisson on Hilbert spaces!

If we can write the Poisson problem in the form

$$a(u,v) = F(v)\ \forall v \in V,\ \text{with}\ a(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v \,\mathrm{d}x + \int_\Omega \sigma\, u\, v \,\mathrm{d}x,$$

and we look for a solution $u \in V$ with V a Hilbert space, then we can use the Lax-Milgram Lemma to prove existence and uniqueness of the solution.

- continuity $a(u,v) \leq \max(\mu,\sigma)\|u\|_1\|v\|_1$
- coercivity $a(u,u) \geq \min(\mu,\sigma)\|u\|_1^2$ or $a(u,u) \geq \mu C(\Omega)\|u\|_1^2$ (if $\sigma = 0$ and Dirichlet BC, we use Poincarè inequality)
- $F(v) = \int_\Omega f v \,\mathrm{d}x$ is a bounded linear functional if $f \in L^2(\Omega)$.