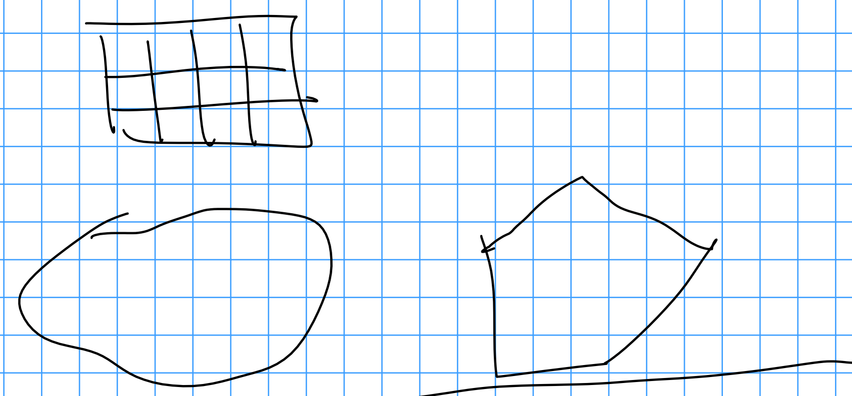
• ELEMENTI FINITI



$$\boxed{a(u,v) = F(v)} \qquad \forall v \in \boxed{V}$$

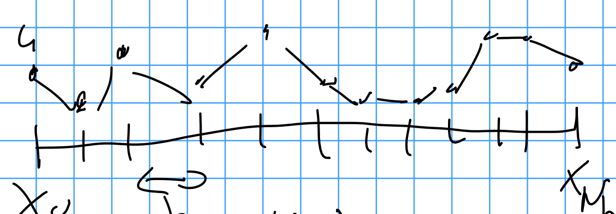
TROVA $u \in V$

⟶ APPROSSIMAZIONE DI $V \to V_h$

$h \to 0 \qquad V_h \to V$ $\qquad h$ = SIZE ELEMENTI DELLA DISCRETI. SPAZIALE

1. V_h HILBERT $\subset V \subset H^1(\Omega)$
2. $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ CONT, COERC, (SYM)
3. $F_h(\cdot) : V_h \to \mathbb{R}$ CONT, LINEARE

- $V_h \subset V \qquad \dim(V_h) = N_h < \infty \qquad \forall h > 0$
- $a_h = a\big|_{V_h \times V_h}$
- $F_h = F$



TROVA $u_h \in V_h$:

$$a(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h \subset V$$

$$\underline{u} = \begin{pmatrix} u_0 \\ \vdots \\ u_{N_h} \end{pmatrix} \in \mathbb{R}^{N_h}$$

PROBLEMA DI GALERKIN

$$V_h = \mathrm{span}(\{\varphi_i\}_{i=1}^{N_h}) = \langle \{\varphi_i\}_{i=1}^{N_h} \rangle$$

$$V_h = \{ u \in V : u = \sum_{i=1}^{N_h} u_i \varphi_i , \ u_i \in \mathbb{R} \ \forall i = 1, \dots, N_h \}$$

$$\dim(V_h) = N_h$$

$$\underset{V_h}{\overset{\Uparrow}{u}} \longleftrightarrow \underset{\mathbb{R}^{N_h}}{\overset{\Uparrow}{\underline{u}}} = \begin{pmatrix} u_1 \\ \vdots \\ u_{N_h} \end{pmatrix}$$

TROVA $u_h \in V_h$:

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

$$\Updownarrow$$

$$a(u_h, \varphi_i) = F(\varphi_i) \quad \forall i = 1, \dots, N_h$$

$$u_h = \sum_{j=1}^{N_h} u_j \varphi_j$$

$$\sum_{j=1}^{N_h} a(\varphi_j, \varphi_i)\, u_j = F(\varphi_i)$$

$$\underline{\underline{A}} \cdot \underline{u} = \underline{F} \qquad \underline{u} = [u_j] \rightarrow \text{INCOGNITE}$$

$$u_h = \sum_{j=1}^{N_h} u_j \varphi_j(x)$$

$$\underline{\underline{A}} = [a_{ij}] \qquad a_{ij} = a(\varphi_j, \varphi_i)$$

$$\underline{F} = [F_i] \qquad F_i = F(\varphi_i)$$

Se $a(\cdot,\cdot)$ FUNZ. BILIN ASS AL PROBLEMA ELLITTICO $a(u,v) = F(v) \ \forall v \in V$ e a BILINEARE COERCIVO

$\Rightarrow A$ è DEFINITA POSITIVA.

A è DEF POS $\Leftrightarrow$ $v^T A v \geq 0 \wedge v^T A v = 0 \Leftrightarrow v = \underline{0}$

<u>DIM</u> $u_h \Leftrightarrow \underline{u}$

$$v^T A v = \sum_{i,j=1}^{N_h} v_i a_{ij} v_j = \sum_{i,j} v_i a(\varphi_j, \varphi_i) v_j$$

$$= a\left(\sum_{j=1}^{N_h} v_j \varphi_j, \sum_{i=1}^{N_h} v_i \varphi_i\right) =$$

$$= a(v_h, v_h) \geq \alpha \|v_h\|_V^2 \geq 0$$

$$= 0 \Leftrightarrow \|v_h\|_V^2 = 0$$

$$\Updownarrow$$

$$\underline{v} = 0 \qquad \boxtimes$$

1. ESISTENZA e UNICITÀ di u_h?
2. STABILE LA SOLUZIONE?
3. $u_h \to u$ PER $h \to 0$!!!

$\boxed{-\Delta u = f}$

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1. LEMMA LAX-MILGRAM $\forall V_h$ HILBERT $\boxed{F(v) = \int f \cdot v}$

$\Rightarrow a(\cdot,\cdot) = F(\cdot)$ $\boxed{a = \int \nabla u \cdot \nabla v \, dx}$

$\Rightarrow \exists! \, u_h \in V_h : a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

- 1.BIS $\Rightarrow$ $\boxed{\underline{\underline{A}} \underline{u} = \underline{F}}$ A DEF POS $\Rightarrow \exists A^{-1}$

2. STABILITÀ

α è coefficiente coercività di a

$a(u,u) \geq \alpha \|u\|^2 \quad \forall u$

- $\|u_h\|_V \leq \frac{1}{\alpha} \|F\|_{V^*}$

$F: V \to \mathbb{R}$

$a: V \times V \to \mathbb{R}$

$\boxed{a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h}$

DIM

$$\|u_h\|_V^2 = \|u_h\|_{V_h}^2 \leq \frac{1}{\alpha} a(u_h, u_h) = \frac{1}{\alpha} F(u_h)$$

$$\leq \frac{1}{\alpha} \|F\|_{V^*} \cdot \|u_h\|_V$$

$$\|u_h\|_V \leq \frac{1}{\alpha} \|F\|_{V^*} \qquad \boxtimes$$

COROLLARIO $\quad u_h: \; a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

$w_h: \; a(w_h, v_h) = G(v_h) \quad \forall v_h \in V_h$

$$\Rightarrow \|u_h - w_h\|_V \leq \frac{1}{\alpha} \|F - G\|_{V^*}$$

es. $F(v) = \int_\Omega f \cdot v - \int_{\partial\Omega} g \cdot v$

3. CONVERGENZA $\quad$ OBIETTIVO: $u_h \to u \quad h \to 0$

- PROIEZIONE DI GALERKIN / ORTOGONALITÀ DI GALERKIN

$a(u_h - u, v_h) = 0 \quad \forall v_h \in V_h$

u_h è SOLUZIONE DI $a(u_h, v_h) \overset{(1)}{=} F(v_h) \quad \forall v_h \in V_h$

u è SOLUZ. DI $a(u, v) \overset{(2)}{=} F(v) \quad \forall v \in V$

$V_h \subset V$

$a(u_h, v_h) \underset{(1)}{=} F(v_h) \overset{(2)}{=} a(u, v_h)$

$v_h \in V_h \subset V$

$\Rightarrow a(u_h - u, v_h) = 0 \quad \forall v_h \in V_h$

- $a(\cdot,\cdot)$ è UN PRODOTTO SCALARE se a SYM

$\|v_h\|_a = \sqrt{a(v_h, v_h)}$

$u_h = \underset{v_h \in V_h}{\operatorname{argmin}} \|v_h - u\|_a \qquad \|u_h - u\|_a = \underbrace{a(u_h - u, u_h - u)}$

- LEMMA DI CÉA

AGGIUNGI/TOGLI V_h

$a(u - u_h, u - u_h) \overset{(1)}{=} a(u - u_h, \underbrace{u - v_h}_{\notin V_h}) + \underbrace{a(u - u_h, v_h - u_h)}_{= 0}$

CONTINUITÀ di a

$|a(u - u_h, u - v_h)| \leq C \cdot \|u - u_h\|_V \cdot \|u - v_h\|_V$

$\|u - u_h\|_V^2 \underset{(2)}{\leq} \frac{1}{\alpha} a(u - u_h, u - u_h) \overset{(1+2)}{\leq} \frac{C}{\alpha} \|u - u_h\|_V \cdot \|u - v_h\|_V \quad \forall v_h \in V_h$

$$\|u-u_h\|_V \leq \frac{C}{\alpha}\|u-v_h\|_V \quad \forall\, v_h \in V_h$$

$$\|u-u_h\|_V \leq \inf_{v_h \in V_h} \boxed{\frac{C}{\alpha}\|u-v_h\|_V} . \qquad \boxtimes$$

u_h POTREBBE NON ESSERE LA MIGLIORE APPROSSIMAZIONE IN $\|\cdot\|_V$

PERÒ $\|u-u_h\|_V$ DIMINUIRÀ COME LA MIGLIORE APPROSSIMAZIONE DI u IN V_h

- Se SCEGLIAMO V_h Se

$$\lim_{h\to 0} \inf_{v_h \in V_h} \|v-v_h\|_V = 0 \quad \forall\, v \in V$$

$$\Rightarrow \|u-u_h\| \xrightarrow[h\to 0]{} 0$$

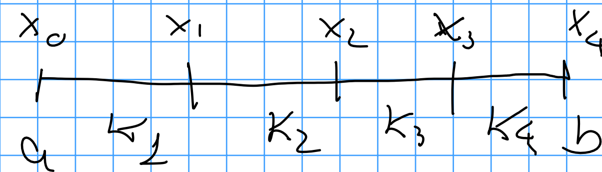
$$\inf_{v_h \in V_h} \|v-v_h\|_V = O(h^p) \Rightarrow \|u-u_h\|_V = O(h^p)$$

- ELEMENTI FINITI:

<u>1D</u> $\Omega = (a,b)$ $\quad V = H^1((a,b))$ $\quad V_h \subset V$

$$\mathcal{T}_h = \{K_j\}_{j=1}^{N_h} \qquad K_j = [x_{j-1}, x_j] \quad \forall\, j = 1, \dots, N_h$$

$$\boxed{|x_j - x_{j-1}| = h}$$



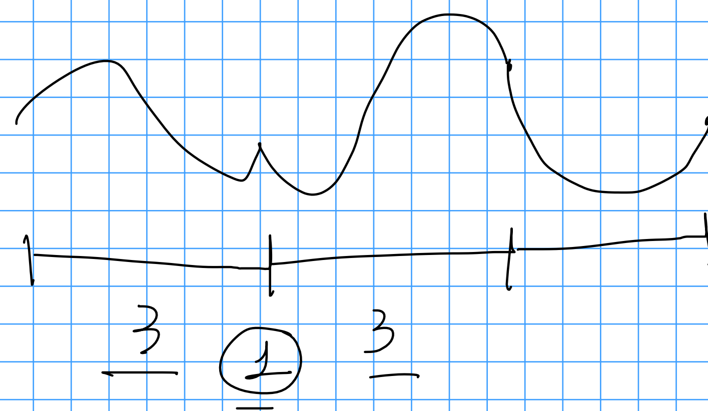
$$h = \max_j |x_j - x_{j-1}|$$

• $H^1((a,b)) \subset C^0([a,b])$ $\quad$ V_h CONTINUO

$$V_h = X_h^r = \{ v_h \in C^0(\bar{\Omega}) : v_h|_{K_j} \in \mathbb{P}^r(K_j) \quad \forall K_j \in \mathcal{T}_h \}$$

$$\subset H^1((a,b))$$

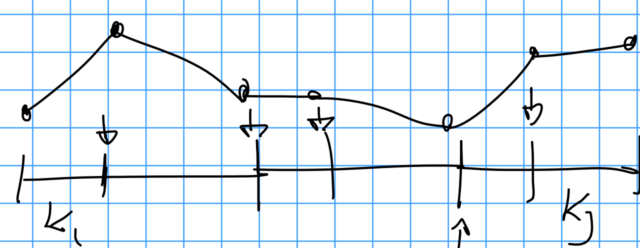
$$V_h = X_h^r = \langle \varphi_1, \dots, \varphi_{N_h} \rangle$$



$$a_{ij} = a(\varphi_i, \varphi_j)$$

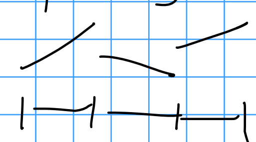
$$= \int \nabla \varphi_i \nabla \varphi_j$$

• $r = 1$ $\quad$ $\mathbb{P}^1(K_j)$ $\quad \Rightarrow$



$$y = \boxed{ax + b}$$

2 DOF (GRADI DI LIBERTÀ) $\quad$ $N \cdot 2$



MA VINCOLI CONTINUITÀ

$N - 1$ VINCOLI $\Rightarrow$ $\quad 2N - (N-1) = N + 1$ DOF

$\Rightarrow$ SE HO IL VALORE ALLE INTERFACCE

$\Rightarrow \exists!\ u_h \in X_h^1 : u_h(x_j) = u_j \quad j = 0, \dots, N$

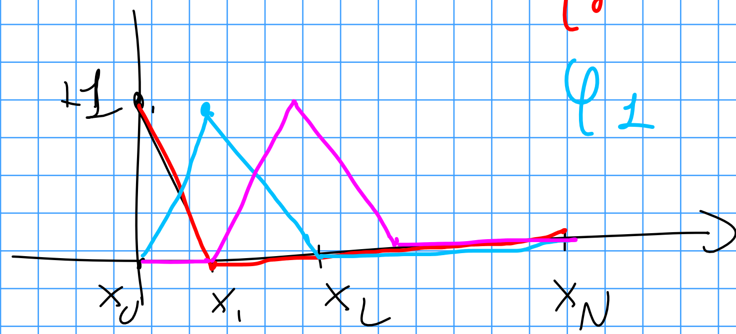
$u_0 = 1 \qquad = u_h = \sum u_i \varphi_i = \underline{\underline{\varphi_0}}$

$u_1 = 0$

$u_2 = 0$

$u_3 = 0$

$|$



$U_0 = 1 \Rightarrow u(x_0) = 1$

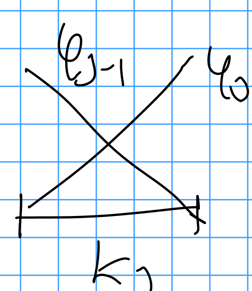
$u(x_i) = u_i \qquad \varphi_i(x_j) = \delta_{ij}$

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{se } x \in [x_{i-1}, x_i] \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{se } x \in [x_i, x_{i+1}] \\[2ex] 0 & \text{altrimenti} \end{cases}$$

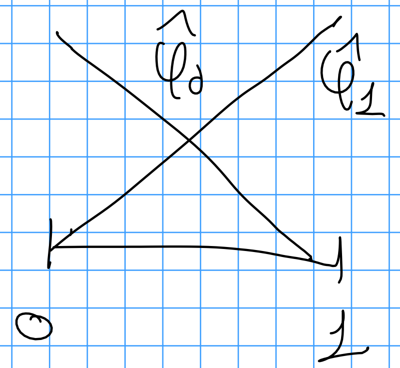
$\text{supp}(\varphi_i) = (x_{i-1}, x_{i+1})$

$$a_{ij} = \int_\Omega \nabla \varphi_i \nabla \varphi_j \, dx = \overbrace{\int_{K_j} \nabla\varphi_i \nabla\varphi_j}^{i = j-1,\, j \;\Rightarrow\; \neq 0} + \overbrace{\int_{K_{j+1}} \nabla\varphi_i \nabla\varphi_j}^{i = j,\, j+1 \;\Rightarrow\; \neq 0}$$

$\Rightarrow \quad a_{ij} = 0 \quad \text{se} \quad |i - j| > 1$



$$\int_{K_J} \partial_x \varphi_i \, \partial_x \varphi_k \qquad i = J-1, J \qquad k = J-1, J$$

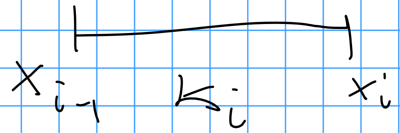


$\mathbb{P}^1$

$$\hat{\varphi}_0(\xi) = 1 - \xi$$

$$\hat{\varphi}_1(\xi) = \xi$$

$$T_i : [0,1] \to [x_{i-1}, x_i]$$



$$T_i(\xi) = \xi(x_i - x_{i-1}) + x_{i-1}$$

$$T_i^{-1}(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}$$

$$\varphi_i(x) = \hat{\varphi}_1\left(T_i^{-1}(x)\right) = \hat{\varphi}_1\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)$$

$\forall x \in K_i$

$$\varphi_i(x) = \hat{\varphi}_0\left(T_{i+1}^{-1}(x)\right) = \hat{\varphi}_0\left(\frac{x - x_i}{x_{i+1} - x_i}\right)$$

$\forall x \in K_{i+1}$

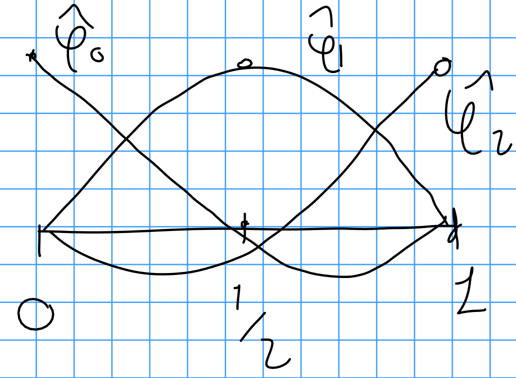
- $X_h^2 = \{ u \in C^0([a,b]) : u|_{K_J} \in \mathbb{P}^2(K_J) \ \forall J \}$

$\mathbb{P}^2([0,1])$

LAGRANGIANA $\Rightarrow$ $\hat{x}_0 = 0$

$\hat{x}_1 = \frac{1}{2}$

$\hat{x}_2 = 1$



RIPASSONE

TROVARE $u_h \in V_h : a(u_h, v_h) = F(v_h) \quad \forall v_h \in V$

$$V_h = \langle \varphi_i \rangle_{i=1}^{N_h}$$

$\Downarrow$

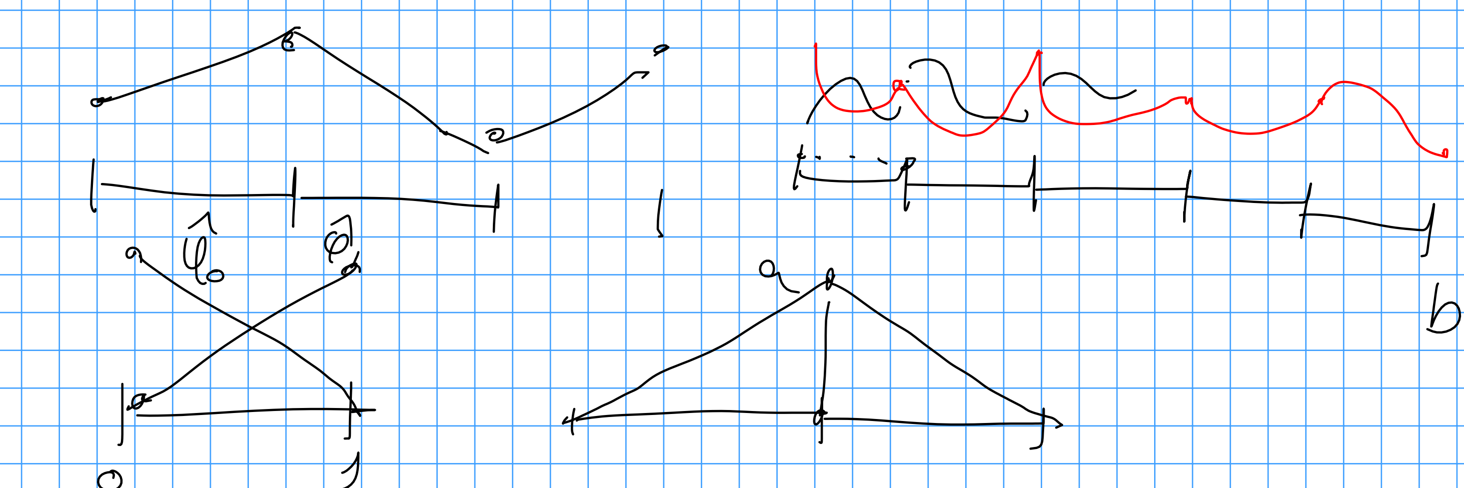
$\mathbb{R}^{N_h \times N_h} \ni A \qquad a_{ij} = a(\varphi_j, \varphi_i) \in \mathbb{R}$

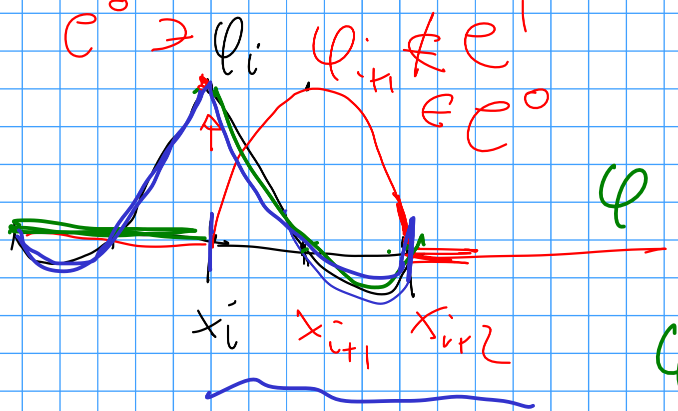
$\mathbb{R}^{N_h} \ni \underline{F} \qquad F_i = F(\varphi_i) \in \mathbb{R}$

V_h BELLO $: \lim_{h \to 0} \inf_{v_h \in V_h} \|v_h - u\|_V = 0 \quad \forall u \in V$

$\Rightarrow u_h \xrightarrow[h \to 0]{} u$

$\boxed{1D}$ POLINOMIALI A TRATTI





$\varphi \notin C^\infty$

$\varphi_{i+1}, \varphi_i \in C^\infty((x_i, x_{i+1}))$

$\varphi_{i+1}, \varphi_i \in C^0((x_i-\varepsilon, x_i+\varepsilon))$

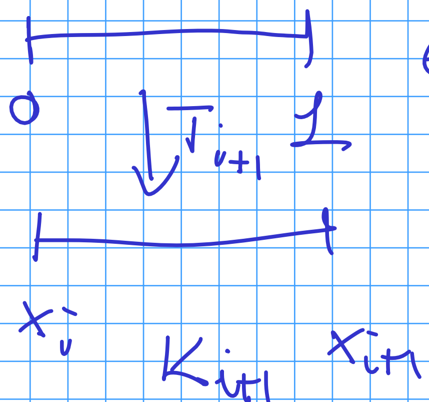
$\notin C^1((x_i-\varepsilon, x_i+\varepsilon))$

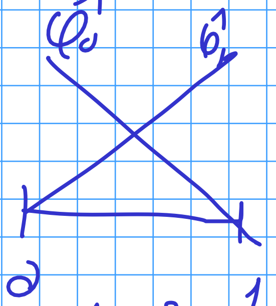
$\varphi_i(x_j) = \delta_{ij}$

$\mathrm{Supp}(\varphi_{i+1}) = [x_i, x_{i+2}]$

$\mathrm{Supp}(\varphi_i) = [x_{i-2}, x_{i+2}]$

ELEMENTO DI RIFERIMENTO





$\mathbb{P}^1$

$\hat a_{00} = \int \partial_x \varphi_0 \, \partial_x \varphi_0$

$\hat a_{10} = \int \partial_x \varphi_0 \, \partial_x \varphi_1$

$\hat a_{01} = \int \partial_x \varphi_1 \, \partial_x \varphi_0$

$\hat a_{11} = \int \partial_x \varphi_1 \, \partial_x \varphi_1$

$\mathbb{P}^r = \langle \hat\varphi_i \rangle_{i=0}^r$

$\hat A \in \mathbb{R}^{(r+1)\times(r+1)}$

$\sigma \in \mathbb{R}$

$$\begin{cases} -u'' + \sigma u = f & x \in (a,b) \\ u(a) = 0 \\ u(b) = 0 \end{cases}$$

$\Longrightarrow$

PROBLEMA DEBOLE

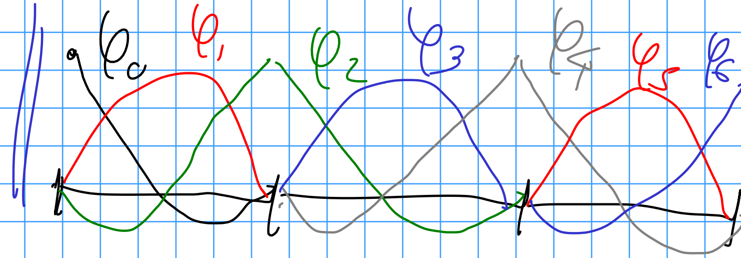
TROVA $u \in V = H^1_0$

$$\int_a^b u'v' + \sigma \int_a^b u \cdot v = \int f v \quad \forall v \in V$$

PROBLEMA DI GALERKIN TROVA $u_h \in V_h$:

$$\int_a^b u_h' v_h' + \sigma \int_a^b u_h v_h = \int_a^b f v_h \quad \forall v_h \in V_h$$

$$u_h(x) = \sum_{i=1}^{N_h} u_i \cdot \varphi_i(x) \qquad N_h = 7$$



$$\forall\, v_h \in V_h \qquad v_h \leftarrow \varphi_i$$

$$\sum_{j=1}^{N_h} \int_a^b \varphi_j'(x) \cdot \varphi_i'(x)\, dx \cdot u_j + \sigma \sum_{j=1}^{N_h} \int_a^b \varphi_j(x)\, \varphi_i(x)\, dx \cdot u_j$$

$$= \int_a^b f(x)\, \varphi_i(x)\, dx \qquad \forall\, i = 1, \dots, N_h$$

TROVARE $u_j \quad j = 1, \dots, N_h$

$$A = [a_{ij}] \qquad a_{ij} = \int_a^b \varphi_j'(x) \cdot \varphi_i'(x)\, dx + \sigma \int_a^b \varphi_j \cdot \varphi_i$$

$$\underline{f} = [f_i] \qquad f_i = \int_a^b f(x)\, \varphi_i\, dx$$

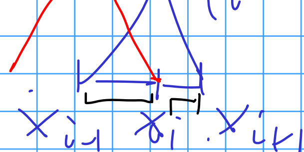
$$a_{ij} = \int_a^b \varphi_j'(x) \cdot \varphi_i'(x)\, dx + \sigma \int_a^b \varphi_j \cdot \varphi_i = 0$$

$$\Longleftrightarrow \quad |i - j| > 1 \qquad (\mathbb{P}^1)$$

$$|i - j| > 2 \qquad (\mathbb{P}^2)$$

$$|i - j| > r \qquad (\mathbb{P}^r)$$

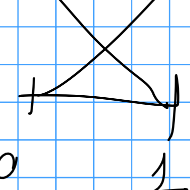
$\mathbb{P}1$

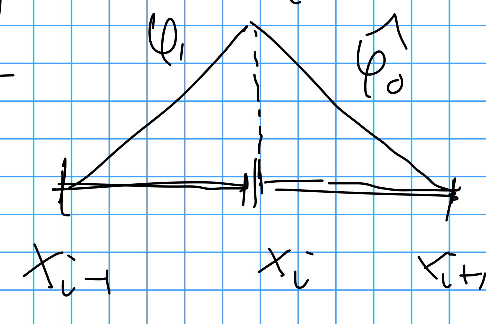


$$0 \neq a_{i,i-1}$$

$$0 \neq a_{i,i}$$

$$0 \neq a_{i,i+1}$$





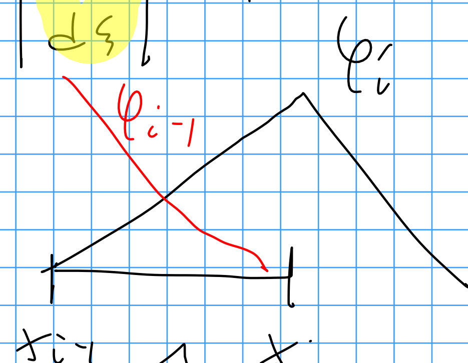
$$a_{i,i-1} = \int_a^b \dots = \int_{x_{i-1}}^{x_i} \varphi_i' \varphi_{i-1}' + \sigma\, \varphi_i \varphi_{i-1} \, dx$$

$$a_{i,i} = \int_a^b \dots = \int_{x_{i-1}}^{x_i} \varphi_i' \varphi_i' + \sigma\, \varphi_i \varphi_i \, dx + \int_{x_i}^{x_{i+1}} \varphi_i' \varphi_i' + \sigma\, \varphi_i \varphi_i \, dx$$

$T_i : [0,1] \to [x_{i-1}, x_i]$

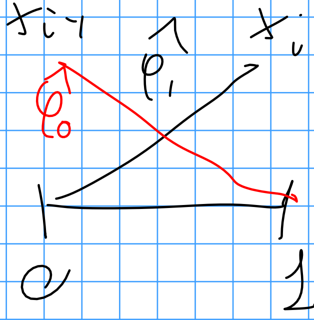
$$a_{i,i-1} = \int_a^b \partial_x \varphi_i(x) \cdot \partial_x \varphi_{i-1}(x) + \sigma\, \varphi_i(x) \cdot \varphi_{i-1}(x) \, dx$$

$$= \int_0^1 \frac{\partial \xi}{\partial x} \cdot \partial_\xi \varphi_i(\overset{x}{\overset{\shortparallel}{T_i(\xi)}}) \cdot \frac{\partial \xi}{\partial x} \cdot \partial_\xi \varphi_{i-1}(T_i(\xi)) + \sigma\, \varphi_i(T_i(\xi)) \cdot \varphi_{i-1}(T_i(\xi)) \left| \frac{dT_i}{d\xi} \right| d\xi$$



$$\to \varphi_i(T_i(\xi)) = \hat{\varphi}_1(\xi)$$

$$\varphi_{i-1}(T_i(\xi)) = \hat{\varphi}_0(\xi)$$

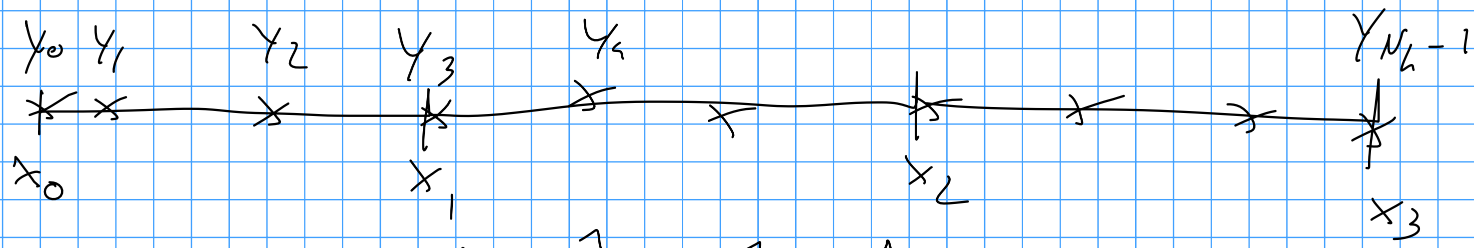


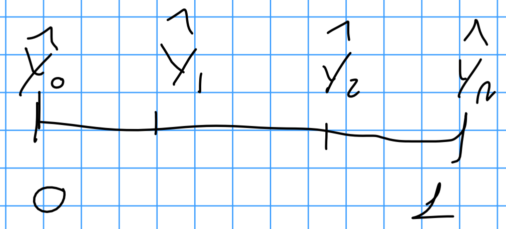
$$\frac{d\xi}{dx} = \frac{d\, T_i^{-1}(x)}{dx} = \frac{1}{x_i - x_{i-1}} = \frac{1}{\Delta x}$$

$$\left| \frac{dT_i}{d\xi} \right| = |x_i - x_{i-1}| = \Delta x$$

$$a_{i,i-1} = \frac{\Delta x}{\Delta x^2} \underbrace{\int_0^1 \hat{\varphi}_1' \cdot \hat{\varphi}_0' \, d\xi}_{\hat{S}_{10}} + \sigma \cdot \Delta x \underbrace{\int_0^1 \hat{\varphi}_1 \cdot \hat{\varphi}_0 \, d\xi}_{\hat{m}_{10}}$$

$$a_{ii} = \frac{1}{\Delta x}\left(\hat S_{11} + \hat S_{00}\right) + \sigma \Delta x \left(\hat m_{00} + \hat m_{11}\right)$$





$$y_\alpha = y_{(i,s)} = x_i + (x_{i+1} - x_i)\cdot \hat y_s$$

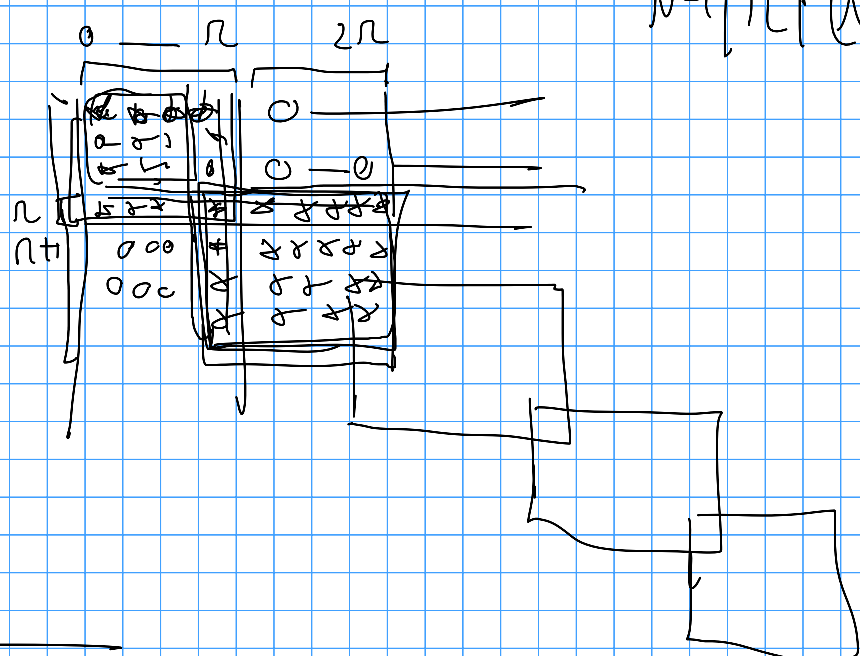
$i = 0, \ldots, N-1$

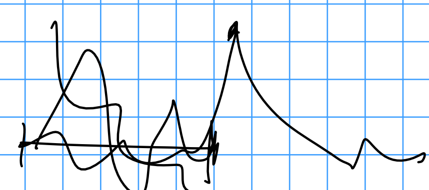
$s = 0, \ldots, r$

$$y_{(i,r)} = y_{(i+1,0)}$$

$(i,s) \to \alpha = i \cdot r + s$

i	s	α
0	0	0
0	r	r
1	0	r
$N-1$	r	$(N-1)r + r = N\cdot r$





ERRORE

$\boxed{h = \Delta x}$

$v \in C^0([a,b])$ PRENDO L'INTERPOLANTE DI $v \in X_h^r$

$$\Pi_h^r(v)(y_\alpha) = v(y_\alpha) \qquad \forall\ \alpha = 0, \_\_, N_h - 1$$

<u>TEOREMA</u>

$v \in H^{r+1}((a,b)) \quad r \geq 1 \quad \Pi_h^r v \in X_h^r$

$$\Rightarrow \quad |v - \Pi_h^r v|_{H^k((a,b))} \leq C_{k,r} \cdot h^{r+1-k} |v|_{H^{r+1}((a,b))}$$

PER $k = 0, 1$

$k = 0 \qquad |\ \ |_{H^0} = \|\ \ \|_{L^2}$

- $\| v - \Pi^r v \|_{L^2} \leq C_{0,r}\, h^{r+1} \cdot |v|_{H^{r+1}}$

$$\Downarrow$$

$$\text{se } e^{\infty} \quad \sqrt{\int (\partial_x^{r+1} v)^2}$$

- $|v - \Pi^r v|_{H^1} \leq C_{1,r}\, h^r \cdot |v|_{r+1}$

$$\| u - u_h \|_V \leq \frac{\Gamma}{\alpha} \| u - v_h \|_V$$

$V = H^1 \qquad\qquad \forall\ v_h$

$$\Downarrow$$

$$\leq \frac{\Gamma}{\alpha} \| u - \Pi_h^r u \| \leq \frac{\Gamma}{\alpha} C \cdot h^r |u|_{r+1}$$

- <u>DIRICHLET</u>

$$u_0 = u(a) \qquad u_{N_h-1} = u(b)$$

$$a_{00} = 1 \qquad a_{0j} = 0 \qquad \forall j > 0$$

$$f_0 = u(a)$$

- <u>NEUMANN</u> $\qquad u'(a) = g \qquad u(b) = \beta$

$$\int_a^b -u'' \cdot \varphi_i \, dx = \int_a^b f \cdot \varphi_i \, dx$$

$$\varphi_i \in H^1 \cap \overbrace{\{\varphi_i(b)=0\}}^{u}$$

$$\Downarrow$$

$$\int_a^b u' \cdot \varphi_i' - \left[u' \cdot \varphi_i \right]_a^b =$$

$$\varphi \qquad \Downarrow \qquad \varphi_{N_h-1}$$

<u>NON È CONSIDERATA</u>

$$= \int_a^b u' \varphi_i' - \Big(u'(b) \cdot \underbrace{\varphi_i(b)}_{=0} \Big)$$

$$+ \; u'(a) \cdot \varphi_i(a)$$

$$= \int_a^b u' \cdot \varphi_i' \, dx + \underbrace{g \cdot \varphi_i(a)}\; = \int f \cdot \varphi_i \, dx$$

$$\int_a^b u' \cdot \varphi_i' \, dx = \int f \cdot \varphi_i \, d\cancel{x} - \underbrace{g \cdot \varphi_i(a)}$$

$$\underbrace{\varphi_i(a)} = 0 \qquad i > 0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\varphi_0(a) \neq 0$$

$$f_0 = \overbrace{\int f \varphi_0 \, dx} - g \qquad = 1$$

$$\tilde{v}(a) = 0 \Rightarrow V = H^1_0$$

$$a(u,v) = F(u) \qquad u(a) = \alpha \Rightarrow \tilde{u} = u - u^{LIFT}$$

$$u^{LIFT} \qquad u(a) = \alpha$$