# Partial Differential Equations

## PDE

Given a domain $\Omega \in \mathbb{R}^d$ where $d > 1$, we seek for $u : \Omega \to \mathbb{R}^s$ where $s \in \mathbb{N}_0$, solution of a **stationary PDE** of order $k$:

$$F(x, t, u, \nabla u \ldots, \nabla^{(k-1)} u, \nabla^{(k)} u, g) = 0$$

with $g : \Omega \to \mathbb{R}^s$ a given function. Or, more explicitely as

$$\mathcal{P}(u, g) \equiv F(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \ldots, \frac{\partial^{p_1 + \cdots + p_d} u}{\partial^{p_1} x_1 \, \partial^{p_2} x_2 \ldots \partial^{p_d} x_d}, g) = 0,$$

where $p_1 + \cdots + p_d \leq k$.

A non stationary PDE of order $k$ reads: find $u : \Omega \times [0, T] \to \mathbb{R}^s$

$$\mathcal{P}(u, g) \equiv F(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \ldots, \frac{\partial^{p_0 + p_1 + \cdots + p_d} u}{\partial^{p_0} t \, \partial^{p_1} x_1 \ldots \partial^{p_d} x_d}, g) = 0,$$

where $p_0 + \cdots + p_d \leq k$.

A classical solution of a PDE is a function $u \in \mathcal{C}^k(\Omega \times [0, T])$ that solves the previous equation.

**Definition**

If the PDE can be written in the form

$$\mathcal{P}(u,g) = a(x)u + b_0(x)\partial_t u + b_1(x)\partial_{x_1} u + \cdots + b_d(x)\partial_{x_d} u + c_{\alpha(2,0,\ldots,0)}\partial_{tt} u + \cdots + \gamma_{\alpha(p_0,\ldots,p_d)} \frac{\partial^{p_0+\cdots+p_d} u}{\partial^{p_0} t\, \partial^{p_1} x_1 \ldots \partial^{p_d} x_d} + \cdots - g = 0,$$

i.e., if the coefficitents of the unknown $u$ and of its derivatives depend only on the independent variables $(t,x)$, then the PDE is **linear**. Else, it is **nonlinear**.

**Definitions**

Consider a nonlinear PDE of order $k$

- if the coefficients of the derivatives of order $k$ depend only on the indepenedent variables $(t,x)$, then the PDE is **semilinear**;
- if the coefficients of the derivates of order $k$ depend on the independent variables $(t,x)$ and on the partial derivatives of $u$ of order at most $k-1$, then the PDE is **quasi-linear**;
- if it's not quasi-linear, its **fully nonlinear**.

## Examples

- Reaction-advection-diffusion equation

$$\partial_t u = u_{xx} + c u_x + u^2,$$

  is semilinear.
- Inviscid Burgers' equation

$$\partial_t u + u u_x = 0,$$

  is quasi-linear but not semilinear.
- The Korteweg-de Vries (KdV) equation

$$\partial_t u + u \partial_x u + \partial_{xxx} u = 0,$$

  is semilinear.
- The Monge-Ampère equation

$$u_{xx} u_{yy} - (u_{xy})^2 = 0$$

  is fully nonlinear.

# First order linear PDE , a.k.a. transport equation

$$u_t + u_x = 0$$

How do I found a general solution?

Let's try this change of variables

$$(x,t) \to (\xi,\eta), \qquad \xi(x,t) = x+t,\, \eta(x,t) = x-t$$

with inverse

$$x = \frac{\xi+\eta}{2}, t = \frac{\xi-\eta}{2}.$$

I substitute the new variables: $v(\xi,\eta) := u(x(\xi,\eta), t(\xi,\eta))$

$$u_x = v_\xi \xi_x + v_\eta \eta_x = v_\xi + v_\eta$$
$$u_t = v_\xi \xi_t + v_\eta \eta_t = v_\xi - v_\eta$$

obtaining a new PDE

$$0 = u_t + u_x = 2v_\xi \iff v_\xi = 0.$$

Implica che $v(\xi,\eta) = f(\eta)$ with $f \in \mathcal{C}^1(\mathbb{R})$. Going back to the original variables

$$u(x,t) = v(\xi(x,t), \eta(x,t)) = f(\xi(x,t)) = f(x-t)$$

# Characteristic lines

$$u_t + u_x = 0, \qquad u(x,t) = v(\xi(x,t), \eta(x,t)) = f(\xi(x,t)) = f(x-t).$$

$$X_{x_0}(t) = x_0 + t$$

# Generalization to different coefficients

$$a(t,x)u_t + b(t,x)u_x + cu(t,x) = g(t,x), (t,x) \in \Omega \subset \mathbb{R}^2.$$

Well defined (non-singular and $\mathcal{C}^1$) transformation $(t,x) \Leftrightarrow (\xi,\eta)$, i.e.,

$$\left|\frac{\partial(\xi,\eta)}{\partial(t,x)}\right| := \left|\begin{pmatrix} \xi_t & \xi_x \\ \eta_t & \eta_x \end{pmatrix}\right| = \xi_t\eta_x - \xi_x\eta_t \neq 0.$$

Change of variables: $u_t = v_\xi \xi_t + v_\eta \eta_t,\ u_x = v_\xi \xi_x + v_\eta \eta_x$, giving

$$(a\xi_t + b\xi_x)v_\xi + (a\eta_t + b\eta_x)v_\eta + cv = g(t(\xi,\eta), x(\xi,\eta))$$

Goal: simplify the previous equation, we choose $\eta$ such that

$$a\eta_t + b\eta_x = 0,$$

so that we obtain an ODE for every $\eta$

$$v_\xi + \frac{c}{a\xi_t + b\xi_x} v = \frac{g(t(\xi,\eta), x(\xi,\eta))}{a\xi_t + b\xi_x}.$$

# Generalization to different coefficients

To obtain $a\eta_t + b\eta_x = 0$, one should notice that, w.l.o.g., we are looking for a curve $x(t)$ such that $\eta(t, x(t)) = \eta_0$ constant for every $t$.

$$0 = \frac{d\eta(t, x(t))}{dt} = \eta_t + \eta_x \frac{\partial x}{\partial t} \implies \frac{\eta_t}{\eta_x} = -\partial_t x(t)$$

Hence, we have

$$\frac{\eta_t}{\eta_x} = -\frac{b}{a} \iff \partial_t x(t) = \frac{b}{a}.$$

Integrating this equation, one obtains the curve $x(t)$, leading to the definition of $\eta(t, x)$ solving for the constant $\eta_0$.

# Example

$$xu_t - tu_x = 1$$

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$$xu_t - tu_x = 1$$

$$(x\xi_t - t\xi_x)v_\xi + (x\eta_t - t\eta_x)v_\eta = 1$$

$$(x\eta_t - t\eta_x) = 0$$

$$\frac{dx}{dt} = -\frac{t}{x}$$

$$\int x\,dx = \int -t\,dt$$

$$x = \sqrt{\eta_0^2 - t^2}$$

$$\eta(t,x) := \sqrt{t^2 + x^2} \qquad \eta_t = \frac{t}{\sqrt{t^2+x^2}}, \quad \eta_x = \frac{x}{\sqrt{t^2+x^2}},$$

$$\xi(t,x) = \arctan(x/t) \qquad \xi_t = \frac{t^2}{x^2+t^2}\frac{-x}{t^2} = -\frac{x}{x^2+t^2}, \quad \xi_x = \frac{t}{x^2+t^2},$$

$$t = \eta\cos(\xi),\ x = \eta\sin(\xi),$$

$$\underbrace{(-\eta\sin(\xi)\frac{\eta\sin(\xi)}{\eta^2} - \eta\cos(\xi)\frac{\eta\cos(\xi)}{\eta^2})}_{=-1}v_\xi = 1, \qquad v = -\xi + f(\eta) \qquad u = -\arctan(x/t) + f(x^2+t^2).$$

## Homework

- Solve $u_x - 2u_y = 0$
- Solve $yu_x - xu_y + uy = xy$

# Second order linear PDE in 2D

Consider the PDE on $\Omega \subset \mathbb{R}^2$

$$\mathcal{P}(u, g) = A\partial_{xx}u + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu - g = 0 \quad \forall (x, y) \in \Omega$$

where $u \in \mathcal{C}^2(\Omega)$ and $A, B, C \in \mathcal{C}^2(\Omega)$ and they do not vanish simultaneously. Let's **classify** the PDE depending on the *discriminant*

$$\Delta := B^2 - 4AC.$$

## Definition

- If $\Delta > 0$ the PDE is said to be hyperbolic (at a point $(x, y)$)
- If $\Delta = 0$ the PDE is said to be parabolic (at a point $(x, y)$)
- If $\Delta < 0$ the PDE is said to be elliptic (at a point $(x, y)$)

- **Hyperbolic example: wave equation**

$$\partial_{tt}u - c\partial_{xx}u = 0 \text{ with } c > 0$$

Indeed, $\Delta = 4c > 0$.

- **Parabolic example: heat equation**

$$\partial_{t}u - c\partial_{xx}u = 0 \text{ with } c > 0$$

Indeed, $\Delta = 0$.

- **Elliptic example: Poisson equation**

$$-c\partial_{xx}u - c\partial_{yy}u = -c\Delta u = f \text{ with } c > 0$$

Indeed, $\Delta = -4c^2 < 0$.

- **Changing sign example: Tricomi equation**

$$yu_{xx} + u_{yy} = 0$$

$\Delta = -4y$.

### Theorem

The sign of the discriminant $\Delta$ is invariant under smooth non-singular transformation of coordinates (i.e. under a change of variables).

### Proof 1/2

We focus only on the second order terms as the first order ones do not contribute to the discriminant. Suppose we perform a smooth change of variables $(x, y) \mapsto (\xi, \eta)$, given by a diffeomorphism. Under this transformation, the second-order derivatives transform as follows:

$$u_{xx} = \alpha^2 u_{\xi\xi} + 2\alpha\beta u_{\xi\eta} + \beta^2 u_{\eta\eta},$$

$$u_{xy} = \alpha\gamma u_{\xi\xi} + (\alpha\delta + \beta\gamma) u_{\xi\eta} + \beta\delta u_{\eta\eta},$$

$$u_{yy} = \gamma^2 u_{\xi\xi} + 2\gamma\delta u_{\xi\eta} + \delta^2 u_{\eta\eta},$$

where

$$\alpha = \frac{\partial x}{\partial \xi}, \quad \beta = \frac{\partial x}{\partial \eta}, \quad \gamma = \frac{\partial y}{\partial \xi}, \quad \delta = \frac{\partial y}{\partial \eta}.$$

**Proof 2/2**

Rewriting the PDE in the new coordinates, the transformed coefficients $A', B', C'$ are given by

$$A' = A\alpha^2 + B\alpha\gamma + C\gamma^2,$$
$$B' = 2A\alpha\beta + B(\alpha\delta + \beta\gamma) + 2C\gamma\delta,$$
$$C' = A\beta^2 + B\beta\delta + C\delta^2.$$

Now, computing the transformed discriminant:

$$\begin{aligned}\Delta' &= B'^2 - 4A'C' \\ &= (2A\alpha\beta + B(\alpha\delta + \beta\gamma) + 2C\gamma\delta)^2 \\ &\quad -4(A\alpha^2 + B\alpha\gamma + C\gamma^2)(A\beta^2 + B\beta\delta + C\delta^2).\end{aligned}$$

Expanding both terms and simplifying, we find that

$$\Delta' = (B^2 - 4AC)(\alpha\delta - \beta\gamma)^2 = \Delta\,\det(J)^2,$$

where $J$ is the Jacobian matrix of the transformation. Since $\det(J)^2 \geq 0$, the sign of $\Delta$ remains unchanged. This proves the invariance of the discriminant sign under a change of variables.

## Hyperbolic canonical form

Consider the wave equation

$$\partial_{tt}u - c\partial_{xx}u = 0$$

with $c > 0$. We can find a change of variables $(x,t) \mapsto (\xi,\eta)$ such that the PDE simplifies to

$$\partial_{\xi\eta}v = 0.$$

The map is defined by

$$\eta = x + t, \quad \xi = x - t.$$

This is the canonical form of a hyperbolic PDE. The general solution is given integrating in $\xi$ and then in $\eta$, i.e.,

$$v(\xi,\eta) = \int^{\xi}\int^{\eta} \partial_{wz}v(w,z)\,dz\,dw = \int^{\xi} f(w)dw = F(\xi) + G(\eta),$$

where $\partial_{\xi}F(\xi) = f(\xi)$.
So, the general solution of the wave equation is

$$u(x,t) = F(x-t) + G(x+t).$$

# Hyperbolic canonical form: can we always get it?

Consider just the second order terms of the hyperbolic PDE $\Delta = B^2 - 4AC > 0$.

$$A\partial_{xx}u + Bu_{xy} + Cu_{yy} = 0.$$

We look for a change of variables $(x, y) \mapsto (\xi, \eta)$ such that the PDE simplifies to

$$\partial_{\xi\eta}v = 0.$$

The transformation can be applied noting that

$$\begin{aligned} u_{xx} &= v_{\xi\xi}(\xi_x)^2 + 2v_{\xi\eta}\xi_x\eta_x + v_{\eta\eta}(\eta_x)^2, \\ u_{xy} &= v_{\xi\xi}\xi_x\xi_y + v_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + v_{\eta\eta}\eta_x\eta_y, \\ u_{yy} &= v_{\xi\xi}(\xi_y)^2 + 2v_{\xi\eta}\xi_y\eta_y + v_{\eta\eta}(\eta_y)^2, \end{aligned}$$

The transformed PDE reads

$$(A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2)v_{\xi\xi} + (2A\xi_x\eta_y + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_x)v_{\xi\eta} + (A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2)v_{\eta\eta} = 0.$$

## Computation space

$$(A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2)v_{\xi\xi} + (2A\xi_x\eta_y + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_x)v_{\xi\eta} + (A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2)v_{\eta\eta} = 0.$$

We want to find the change of variables such that

$$\begin{cases} A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\ A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \end{cases}$$

These are first order PDE, so we are looking for characteristics curves such that $\xi(x, y) = \mathrm{const}$, if we find a curve, for example $y(x)$ such that $\xi(x, y(x)) = \mathrm{const}$, then

$$\frac{d\xi}{dx} = \frac{\partial\xi}{\partial x} + \frac{\partial\xi}{\partial y}\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{\partial_x\xi}{\partial_y\xi}.$$

From the first PDE, we then get

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0,$$

which is called the characteristic equation for the original PDE. This is quadratic equation in $\frac{dy}{dx}$ with $\Delta = B^2 - 4AC > 0$. The two distinct solutions are

$$\frac{dy}{dx} = \frac{+B \pm \sqrt{\Delta}}{2A}.$$

From this we can get the transformation $(x, y) \mapsto (\xi, \eta)$ as we did in the linear PDE.

Computation space

## Example

$$u_{tt} + u_{tx} = 0$$

$$(\xi_t^2 + \xi_t\xi_x)u_{\xi\xi} + (2\xi_t\eta_t + \xi_t\eta_x + \xi_x\eta_t)u_{\xi\eta} + (\eta_t^2 + \eta_t\eta_x)u_{\eta\eta} = 0$$

The equations for $\xi$ and $\eta$ are the same equations.

We look for a curve $y(x)$ such that $\xi(x, y(x)) = \mathrm{const}$, i.e., $\xi(x, y(x)) = x + y(x) = \mathrm{const}$ and that

$$\begin{aligned}
\xi_t^2 + \xi_t\xi_x &= 0 \\
\frac{\xi_t^2}{\xi_x^2} + \frac{\xi_t}{\xi_x} &= 0 \\
\left(\frac{dx}{dt}\right)^2 - \frac{dx}{dt} &= 0 \\
\frac{dx}{dt} &= \begin{cases} 0 \\ 1 \end{cases} \Longrightarrow x(t) = \begin{cases} \xi_0 \\ \eta_0 + t \end{cases} \\
\Longrightarrow \eta = x - t, &\quad \xi = x.
\end{aligned}$$

**What if we try to do the same with a parabolic PDE?**

$$\frac{dx}{dt} = -\frac{B \pm \sqrt{\Delta}}{2A} = -\frac{B}{2A}.$$

There is only one characteristic curve. So, choosing $\xi = 2Ax + Bt$ and $\eta = x$ we get the canonical form

$$A\frac{\partial^2 v}{\partial \xi^2} = 0,$$

with the general solution $v(\xi, \eta) = F(\eta) + \xi G(\eta)$.

**What if we try to do the same with an elliptic PDE?**

There is no characteristics that is conserved. But, one can instead eliminate the coefficient of $u_{\xi\eta}$ to obtain the canonical form for the elliptic PDE. Using $\eta = t$ and $\xi = \frac{2Ax - Bt}{\sqrt{\Delta}}$, we get

$$A\left(\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2}\right) = 0.$$

# Existence, uniqueness and well-posedness

For the PDEs above we have found classes of solutions. How can we find unique solutions to specific problems? What should we need to specify?

## Definition (Cauchy problem)

Consider a PDE of order $k$ in $\Omega \subset \mathbb{R}^d$ and let $S$ be a given smooth surface on $\mathbb{R}^d$. Let also $n = n(x)$ denote the unit normal vector to the surface $S$ at a point $x = (x_1, x_2, \ldots, x_d) \in S$. Suppose that on any point $x$ of the surface $S$ the values of the solution u and of all its directional derivatives up to order $k-1$ in the direction of $n$ are given, i.e., we are given functions $f_0, f_1, \ldots, f_{k-1} : S \to \mathbb{R}$ such that

$$u(x) = f_0(x), \text{ and } \frac{\partial u}{\partial n}(x) = f_1(x), \text{ and } \frac{\partial^2 u}{\partial n^2}(x) = f_2(x), \ldots, \text{ and } \frac{\partial^{k-1} u}{\partial n^{k-1}}(x) = f_{k-1}(x).$$

The **Cauchy problem** consists of finding the unknown function(s) $u$ that satisfy simultaneously the PDE and the conditions above, which are called the **initial conditions** (ICs) and the given functions $f_0, f_1, \ldots, f_{k-1}$, will be referred to as the initial data.
According to the role of the ICs they can be called also **boundary conditions** (BCs).

## Examples (Cauchy problem for transport equation)

$$\begin{cases} u_t + u_x = 0, & (x,t) \in \mathbb{R}^2, \\ u(0,x) = \sin(x), & x \in \mathbb{R}. \end{cases}$$

Here, $S = \{(t,x) \in \mathbb{R}^2 : t = 0\}$.

The general solution of the transport equation is $u(x,t) = f(x-t)$, so that the initial condition reads $f(x) = \sin(x)$, i.e., $u(x,t) = \sin(x-t)$.

## Examples (Cauchy problem for wave equation)

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in \mathbb{R}^2, \\ u(t,0) = \sin(t), & t \in \mathbb{R}, \\ u_x(t,0) = 0, & t \in \mathbb{R}. \end{cases}$$

In this case, $S = \{(t,x) \in \mathbb{R}^2 : x = 0\}$ and $n = (n_t, n_x) = (0, -1)$. The general solution of the wave equation is $u(t,x) = f(x-t) + g(x+t)$, so that the initial (boundary) conditions read $f_0(t) = \sin(t)$ and $f_1(t) = 0$, so

$$\begin{cases} f(-t) + g(t) = \sin(t), \\ f'(-t) + g'(t) = 0, \end{cases} \implies f(\xi) = \frac{1}{2}\sin(-\xi), \quad g(\eta) = \frac{1}{2}\sin(\eta),$$

so, $u(t,x) = \frac{1}{2}(\sin(x+t) + \sin(-x+t))$.

# Theorem (Cauchy-Kovalesvskaya Theorem)

Consider a Cauchy problem for a linear PDE, let $x^0$ be a point of the initial surface $S$, which is assumed to be analytic (very regular). Suppose that $S$ is not a characteristic surface at the point $x^0$. Assume that all the coefficients of the linear PDE, the right-hand side $g$, and all the initial data $f_0, f_1, \ldots, f_{k-1}$ are analytic functions on a neighbourhood of the point $x^0$. Then, the Cauchy problem has a solution $u$, defined in the neighbourhood of $x^0$. Moreover, the solution u is analytic in a neighbourhood of $x^0$ and it is unique in the class of analytic functions.

- Assumptions: Regularity
- Outcome: Existence
- Outcome: Uniqueness
- Outcome: Regularity of the solution

- Is this enough? No, the solution might still mis-behave

# Well-posedness

## Definition

A PDE problem is well-posed if:

1. The PDE has a solution
2. The solution is unique
3. The solution depends continuously on the PDE coefficients and on the problem data (IC/BC)

If the PDE problem is not well-posed, we say it is ill-posed.

## Exercise

Show that the solution of the Cauchy problem for the wave equation

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = 0, \\ u(t,0) = f(t), \\ u_x(t,0) = g(t) \end{cases}$$

for some known BCs $f$ and $g$ is given by the d'Alembert's formula

$$u(t,x) = \frac{1}{2}(f(t-x) + f(t+x)) + \frac{1}{2}\int_{t-x}^{t+x} g(s)\mathrm{d}s.$$

Show that the Cauchy problem is well-posed (skipping the uniqueness). *

# Exercise * Dubrovin's notes

1. Find the solution of the Laplace equation on $\Omega = [0, 2\pi]$ for various $k$

$$\begin{cases} \partial_{tt} u + \partial_{xx} u = 0, \\ u(0, x) = 0, \\ u_t(0, x) = \frac{\sin(kx)}{k}, \\ u(t, 0) = u(t, 2\pi). \end{cases}$$

Steps:

- $u_k = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} [a_n(t)\cos(nx) + b_n(t)\sin(nx)]$
- Substitute in the equation and find the general solution using the method of separation of variables
- $\partial_{tt} a_n(t) = n^2 a_n(t)$ for all $n$ with $a_n(0) = 0, \partial_t a_n(0) = 0$
- $\partial_{tt} b_n(t) = n^2 b_n(t)$ for all $n$ with $b_n(0) = 0, \partial_t b_n(0) = 0$ for $n \neq k$, $\partial_t b_k(0) = 1/k$.
- $u_k(t, x) = \frac{1}{k^2} \sin(kx) \sinh(kt)$

2. Even if $\sup_x |u_k(0, x)| + |\partial_t u_k(0, x)|$ is small, we can find large enough $k$ so that for any time $t_0 > 0$ $\sup_x |u_k(t_0, x)| + |\partial_t u_k(t_0, x)|$ is large.

**Theorem ***

Let $u_k(t,x) = \frac{1}{k^2}\sin(kx)\sinh(kt)$. For any positive $\varepsilon, M, t_0$ there exists an integer $K$ such that for any $k > K$ the initial data satisfies $\sup_x |u_k(0,x)| + |\partial_t u_k(0,x)| < \varepsilon$ but the solution at the time $t_0$ satisfies $\sup_x |u_k(t_0,x)| + |\partial_t u_k(t_0,x)| > M$.

**Proof:** Choosing an integer $K_1$ satisfying $K_1 > \frac{1}{\epsilon}$ we will have the initial condition inequality for any $k \geq K_1$. In order to obtain a lower estimate of the second form at time $t_0$ let us first observe that

$$\sup_{x\in[0,2\pi]} (|u_k(x,t)| + |\partial_t u_k(x,t)|) = \frac{1}{k^2}\sinh(kt) + \frac{1}{k}\cosh(kt) > \frac{1}{k^2}e^{kt}$$

where we have used an obvious inequality $\frac{1}{k} > \frac{1}{k^2}$ $\text{for } k > 1.$

The function $y = \frac{e^x}{x^2}$ is monotone increasing for $x > 2$ and $\lim_{x\to+\infty} \frac{e^x}{x^2} = +\infty$.

Hence for any $t_0 > 0$ there exists $x_0$ such that $\frac{e^x}{x^2} > \frac{M}{t_0^2}$ $\text{for } x > x_0.$

Let $K_2$ be a positive integer satisfying $K_2 > \frac{x_0}{t_0}$.

Then for any $k > K_2$

$$\frac{e^{kt_0}}{k^2} = t_0^2\frac{e^{kt_0}}{k^2t_0^2} > t_0^2\frac{e^{x_0}}{x_0^2} > M.$$

Choosing $K = \max(K_1, K_2)$ we complete the proof of the Theorem.

## Take home message

Not all boundary conditions are suitable for having a well-posed problem.