Nonlinear Hyperbolic PDEs

A nonlinear Conservation law

We have seen the general form for a scalar conservation law

$$\partial_t u + \partial_x f(u) = 0.$$

Some hypothesis on the flux function f are needed to have a well posed problem. For example, we will assume that f is a convex function, i.e., $f''(u) \geq 0$ for all u.
We can think about the first generalization of the advection equation, i.e., **Burgers'** equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0.$$

Here, we have chosen $f(u) = \frac{u^2}{2}$, which is a nonlinear function of u. **Burgers'** equation approximates the Navier-Stokes equations for incompressible fluids in a very simplified version.
In this context, we can write the conservation law in its linearized form

$$\partial_t u + u \partial_x u = 0$$

where we see that this is equivalent to a transport problem where the velocity is u itself.

Viscous Burgers' equation

$$\partial_t u + u\partial_x u = 0$$

is also called **inviscid Burgers' equation**. We can add a diffusion term to the equation, leading to the **viscous Burgers' equation**

$$\partial_t u + u\partial_x u = \nu\partial_{xx} u.$$

Again, this is the simplest model to describe nonlinear and viscous effects in fluids.

Exact solution of inviscid Burgers' equation

The **inviscid Burgers' equation**, when smooth, is a nonlinear PDE that has a solution that uses again the characteristics.
The characteristics satisfy

$$\partial_t x(t) = u(t, x(t)), \qquad x(0) = x_0,$$

so, u is constant along the characteristics, since

$$\frac{d}{dt}u(t, x(t)) = \partial_t u + \partial_x u \, \partial_t x(t) = \partial_t u + u\partial_x u = 0.$$

So, the characteristics are straight lines, since u is constant along them. So the characteristics are straight lines determined by the initial data $u_0(x)$, i.e., $u(t, x(t)) = u_0(x_0)$.

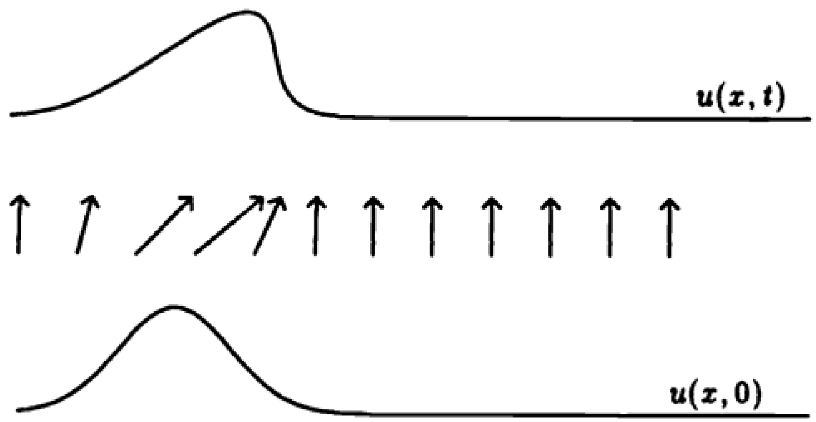

Figure 3.3. Characteristics and solution for Burgers' equation (small t).

Shock formation in Burgers' equation

The solution of the inviscid Burgers' equation is not unique. As soon as the characteristics cross, we have a problem. Indeed, if we have two characteristics that cross, we have two different values of u at the same point (t, x), so the solution is not unique.
When for the first time two characteristics cross, the function has an infinite slope and the wave *breaks* and forms a **shock**. Beyond this point there is no classical solution anymore as the solution is discontinuous.

Exercise

Show that if we solve $\partial_t u + u\partial_x u = 0$ with smooth initial data $u_0(x)$, we have that the solution is smooth until the time t^* when the characteristics cross. The time t^* is given by

$$t^* = \frac{-1}{\min_x u_0'(x)}.$$

After breaking point?

If we want to continue to continue with the characteristics we will end up with solutions that is multivalued in some points. This does not make sense.

We can find the correct physical model using the **viscous Burgers' equation** in the vanishing viscosity limit. The solution of the viscous Burgers' equation is smooth and converges to the solution of the inviscid Burgers' equation as $\nu \to 0$.

Weak solutions

When discontinuities appear, the classical PDE has no longer meaning. So, we either look at the **integral version**

$$\partial_t \int_{x_1}^{x_2} u(t,x)\mathrm{d}x + f(u(t,x_2)) - f(t,x_1) = 0$$

or we can recur to the **weak formulation** of the PDE: find $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that for all $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R})$

$$\int_0^\infty \int_{-\infty}^{+\infty} \varphi(t,x)\partial_t u(t,x) + \varphi(t,x)\partial_x f(u(t,x))\mathrm{d}x\,\mathrm{d}t = 0 \qquad \Longrightarrow$$

$$\int_0^\infty \int_{-\infty}^{+\infty} \partial_t\varphi(t,x)u(t,x) + \partial_x\varphi(t,x)f(u(t,x))\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty} \varphi(0,x)u(0,x)\mathrm{d}x.$$

Here, we have ignored all boundary terms taking support on $\mathbb{R} \times \mathbb{R}$, except the initial condition. One can keep them into the formulation, integrating on compact domains.

The **vanishing viscosity solution** we defined above is a weak solution.
But **weak solutions** are not unique, and we might need to filter out the physically relevant solution, for example, adding entropy constraints, leading to **entropy solutions** (which are also the **vanishing viscosity solution**).

Riemann Problem

The Riemann problem is the simplest discontinuous problem that a conservation law might observe. Once we understand it, we can generalize to all other solutions. Indeed, as we have seen before, discontinuities might arise also from smooth solutions, so, we better be ready to deal with them.

Riemann problem for Burgers' equation $\partial_t u + u \partial_x u = 0$ is given by the following initial condition

$$u(0,x) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$$

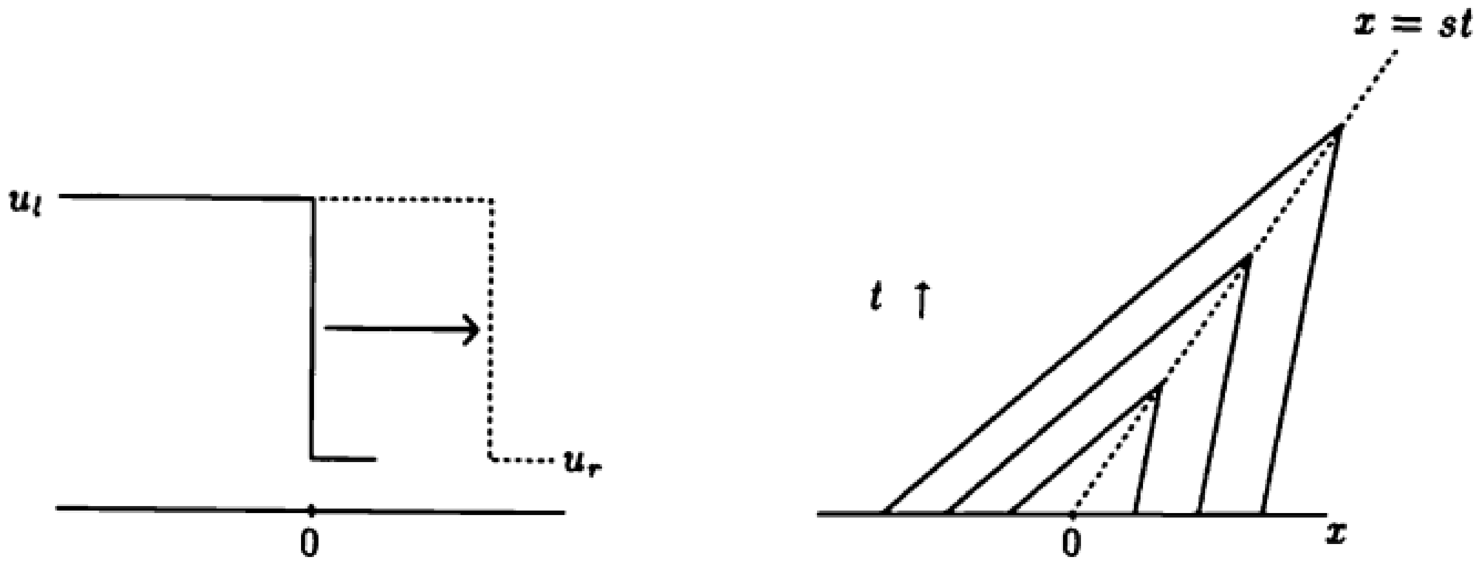

Figure 3.8. Shock wave.

Riemann problem: shock wave

We first analyse the case when $u_L > u_R$ in Burgers'. The solution is given by a shock wave, as all the characteristics on the left of the domain travel faster than the one on the right