- EQUAZIONE DEL CALORE

$$\overbrace{\partial_t u(t,x)} - \overbrace{a\,\partial_{xx} u(t,x)} = f(t,x)$$

$a>0$

$$u: \underset{\substack{\Uparrow \\ \mathbb{R}}}{\Omega} \times \mathbb{R}^+$$

$\Omega=[a,b]$

$x\in\Omega \quad t\in\mathbb{R}^+$

$3D \to \underset{x}{2D} \times \underset{t}{1D}$

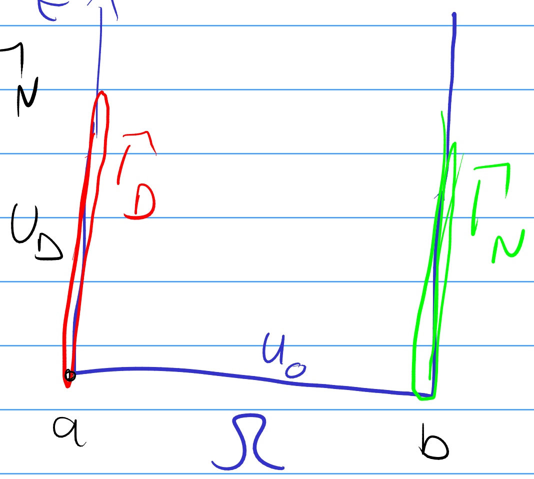
$4D \to \underset{x}{3D} \times \underset{t}{1D}$

$$\nabla\cdot\underbrace{(a\nabla u),}\ \overset{\nabla}{\varphi}$$

$$\begin{cases} \partial_t u - a\,\partial_{xx} u = f & \forall\, t>0 \quad \forall x\in\Omega \\ u(0,x) = u_0(x) & \forall x\in\Omega \\ u(t,x) = u_D(t,x) & \forall \underset{t}{x} \in \Gamma_D \\ a\,\partial_x u(t,x) \overset{\cdot n}{=} U_N(t,x) & \forall x\in\Gamma_N \end{cases}$$

- PERIODICHE

$u(a,t) = u(b,t) \quad \forall t$



EQ ELLITTICHE HANNO BISOGNO DI DIRICHLET

EQ PARABOLICHE HANNO BIS. CONDIZIONI INIZIALI

• SOLUZ ESATTE PER CONDIZ. BORDO PERIODIC

$$\bullet \ \sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \qquad \bullet \ \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\rightarrow \boxed{e^{ikx}}, \ \sin(kx), \ \cos(kx) \qquad k \in \mathbb{Z}$$

$$x \in [-\pi, \pi] \qquad \bullet \ e^{ix} = \cos(x) + i\sin(x)$$

$$\partial_x \sin(kx) = k\cos(kx)$$

$$\partial_{xx} \underbrace{\sin(kx)} = \underbrace{-k^2} \underbrace{\sin(kx)},$$

$$\partial_x \underbrace{e^{ikx}}_{\substack{\downarrow \\ \text{AUTOFUNZIONE} \\ \partial_x}} = \underbrace{ik}_{\text{AUTOVALORE DELL'OPERATORE}} e^{ikx}$$

$$\underbrace{u_0(x)} = \sum_{k \in \mathbb{Z}} c_k e^{ikx}$$

FORMULA UTILE

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} \cdot e^{-iwx} = \delta_{kw} \quad \forall k,w \in \mathbb{Z}$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(x) \cdot e^{-ikx} dx$$

$$k = w \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} \cdot e^{-ikx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 = 1$$

$$k \neq w \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx} \cdot e^{-iwx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(k-w)\cdot x} dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{i(k-w)}\cdot\underbrace{e^{i(k-w)\cdot x}}_{\text{PERIODICA}}\right]_{-\pi}^{\pi} = 0$$

PERIODO 2π PERCHÉ $k \neq w$

$$u_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}$$

CONVERGE se $u_0 \in L^2$

e CONVERGE A u_0 IN L^2

$$\|c\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u_0(x)|^2\,dx = \frac{1}{2\pi}\|u_0\|_{L^2}^2$$

$$\parallel$$

$$\sum_{k\in\mathbb{Z}} c_k^2$$

$$\partial_t u - a\,\partial_{xx} u = 0$$

$u_0 \in L^2$

$$u_0(x) = \sum c_k^0 e^{ikx}$$

$$u(t,x) = \overbrace{\sum_{k\in\mathbb{Z}} c_k(t)\cdot e^{ikx}}\Big]\ \text{ANSATZ}$$

$$\partial_t u - a\,\partial_{xx} u = 0$$

$$\partial_t \sum_{k\in\mathbb{Z}} c_k(t)\,e^{ikx} - a\,\partial_{xx}\sum_{k\in\mathbb{Z}} c_k(t)\,e^{ikx} = 0$$

$$\sum_{k\in\mathbb{Z}} \partial_t c_k(t)\cdot e^{ikx} - a\sum_{k\in\mathbb{Z}} c_k(t)\,\underbrace{\partial_{xx} e^{ikx}}_{-k^2 e^{ikx}} = 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iwx}\cdot\sum_{k\in\mathbb{Z}} e^{ikx}\underbrace{\left[\partial_t c_k(t) + a k^2 c_k(t)\right]}_{\text{ODE}} = 0$$

$\forall w \in \mathbb{Z}$

$$\to \sum_k \delta_{wk} \left[\partial_t c_k(t) + a k^2 c_k(t) \right] = 0$$

$$\partial_t c_w(t) + a w^2 c_w(t) = 0 \quad \forall w$$

$$c_w(t) = c_w^0 \cdot e^{-a w^2 t} \quad \forall w \in \mathbb{Z}$$

$$u(t,x) = \sum_{k \in \mathbb{Z}} c_k(t)\, e^{ixk} \quad \forall x \in [-\pi, \pi] \quad \forall t \in \mathbb{R}^+$$

$$(\partial_{xx} u)_i \approx \frac{U_{i+1} - 2U_i + U_{i-1}}{\Delta x^2} \quad \|$$

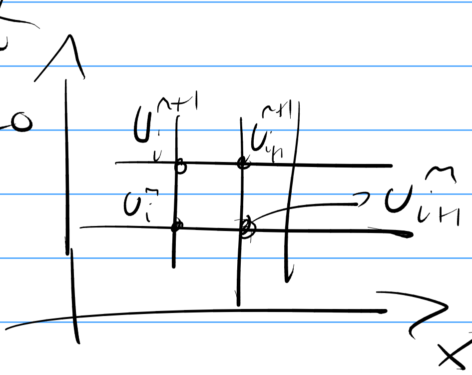
$\boxed{\partial_t}$ → EULERO ESPLICITO } 1° ORDINE

→ EULERO IMPLICITO } IN TEMPO

→ CRANK-NICOLSON } 2° ORDINE

EE.

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} - \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2} = 0$$



IE.

$$\downarrow$$
$$\partial_{xx}^{n+1}$$

C.N.

$$\frac{1}{2}\left(\partial_{xx}^{n} + \partial_{xx}^{n+1}\right)$$

$$U_i^{n+1} - U_i^n - \frac{\Delta t}{\Delta x^2}\left(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}\right) = 0$$

$$I \cdot \underline{u}^{n+1} - \underline{u}^n - \frac{\Delta t}{\Delta x^2} \underbrace{\begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}}_{D^2} \underline{u}^{n+1} = \underline{0}$$

$$\left[\underline{I} - \underbrace{\overbrace{\frac{\Delta t}{\Delta x^2}}}\, D^2\right] \underline{u}^{n+1} = \underline{u}^n$$

C.N. $$\underline{I}\, \underline{u}^{n+1} - \underline{u}^n - \frac{1}{2}\frac{\Delta t}{\Delta x^2} D^2 \underline{u}^{n+1} - \frac{1}{2}\frac{\Delta t}{\Delta x^2} D^2 \underline{u}^n = 0$$

$$\left[\underline{I} - \frac{1}{2}\frac{\Delta t}{\Delta x^2} D^2\right] \underline{u}^{n+1} = \underline{u}^n + \frac{1}{2}\frac{\Delta t}{\Delta x^2} D^2 \underline{u}^n$$

- ERRORE DI CONSISTENZA $\quad u_t - u_{xx} = 0$

$$e^{ET}_{\Delta t, \Delta x} = \underbrace{\frac{u(t^{n+1}, x_i) - u(t^n, x_i)}{\Delta t}} - \frac{u(t^n, x_{i+1}) + u(t^n, x_{i-1}) - 2u(t^n, x_i)}{\Delta x^2}$$

TAYLOR IN (t^n, x_i)

$$= \frac{\cancel{u(t^n, x_i)} + \Delta t\, \partial_t u(t^n, x_i) + \frac{\Delta t^2}{2} \partial_{tt} u(t^n, x_i) + O(\Delta t^3) - \cancel{u(t^n, x_i)}}{\Delta t}$$

$$- \frac{\partial_{xx} u(t^n, x_i) \cdot \Delta x^2 + \frac{\Delta x^4}{12} \partial_{xxxx} u(t^n, x_i) + O(\Delta x^6)}{\Delta x^2}$$

$$= \frac{\Delta t}{2} \overline{\partial_{tt} u} + \frac{\Delta x^2}{12} \partial_{xxxx} u + O(\_)$$

$$= \underbrace{O(\Delta t)}_{\text{PRIM'ORDINE NEL TEMPO}} + \underbrace{O(\Delta x^2)} \rightarrow \text{SECOND'ORDINE SPAZIO}$$

$$\partial_t u = \partial_{xx} u$$

$$\partial_{txx} u = \partial_t(\partial_{xx} u) = \partial_{tt} u$$

$$\underline{U}'(t) = A \cdot U(t) + g(t) = f(U,t)$$

$$A = \frac{1}{\Delta x^2}\begin{bmatrix} -2 & 1 & \\ 1 & \ddots & \ddots \\ & \ddots & \ddots \end{bmatrix} = \frac{1}{\Delta x^2} D^2$$

$$\lambda \in \mathbb{C}$$

$u'(t) = \lambda u(t)$ $\quad (Re(\lambda) \le 0)$

$y^{n+1} = R(z) \cdot y^n$ $\quad S := \{z : |R(z)| \le 1\}$

$z = \Delta t \cdot \lambda \in \mathbb{C}$ $\quad \Downarrow$

REGIONE DI STABILITÀ

SISTEMA $\nwarrow$ $U'(t) = A U(t)$ $\rightsquigarrow \underbrace{u' = \lambda u}$ $\nearrow$ SCALARE

$A = Z D Z^T$ $\quad Z Z^T = I$

$\Updownarrow$

$Z^T A Z = D$ $\quad$ D diagonale

D sono gli autovalori di A

(A bella, per esempio sym def pos/neg)

$$Y = Z^T U$$

$$Y' = Z^T U' = Z^T A U$$

$$= \underbrace{Z^T A Z Z^T U}_{D \quad I \quad Y} = DY \Rightarrow Y' = DY$$

$Y' = DY$ È SYS ODE DIAGONALE

⇒ DISACCOPPIARE LE EQUAZIONI

$y_i'(t) = \underbrace{d_{ii}}\, y_i(t)$ → EQUAZ. DI DAHLQUIST

$y' = \lambda y$

↓

AUTOVALORI DI A

↓

SE $\Delta t \cdot d_{ii} \in S \quad \forall i$

⇒ STABILE

- SLIDE 16 ETE IN STAB REG ⇒ ETE · Δt IN STAB REG

- $d_{ii} \approx \frac{1}{\Delta x^2}$

$\Delta t \cdot d_{ii} > -2$

$\Delta t \cdot \frac{1}{\Delta x^2} \cdot (-?) > -2$

$\Delta t < \Delta x^2 \cdot 2 \cdot (?)$

- ERRORE DI CONSISTENZA
- STABILITÀ

CONVERGONO? $\Delta x, \Delta t \to 0$?

$\Delta t = c\Delta x$ $\Delta t = c \cdot \Delta x^2$

$h = \Delta x$

$U_h \xrightarrow[h \to 0]{?} u^{ex}$