Review of Functional analysis concepts

Linear and bilinear functionals

Given a functional space V, a linear functional is a map $L : V \to \mathbb{R}$ that satisfies linearity: $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$ for all $u, v \in V$ and scalars $\alpha, \beta \in \mathbb{R}$.

A bilinear functional is a map $B : V \times V \to \mathbb{R}$ that is linear in each argument.

Boundedness and Continuity

A functional L is bounded if there exists a constant C such that $|L(u)| \leq C||u||_V$ for all $u \in V$. If V is a Banach space (normed and complete), then a linear bounded functional is also continuous.

Dual Space

The dual space $V^* = V'$ is the space of all bounded linear functionals on V.

$$V^* := \{F : V \to \mathbb{R} : \ F \text{ is linear and bounded}\}.$$

Norm

The norm of a functional $L \in V^*$ is defined as $||L||_{V^*} = \sup_{||u||_V \leq 1} |L(u)| = \sup_{||u||_V \neq 0} \frac{|L(u)|}{||u||_V}$.

Hilbert Space

A Hilbert space H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.

The inner product is a bilinear function $(\cdot,\cdot)_H : V \times V \to \mathbb{R}$ that is symmetric and positive definite. The induced norm is $||u||_H := \sqrt{(u,u)_H}$.

Riesz Representative

The Riesz representation theorem states that for every bounded linear functional L on a Hilbert space H, there exists a unique element $v_L \in H$ such that

$$L(u) = (u, v_L)_H$$

for all $u \in H$. Moreover, $||L||_{H^*} = ||u_L||_H$.
Conversly, for every element $u \in H$ there exists a linear and bounded functional L_u such that

$$L_u(v) = (u, v)_H \text{ for every } v \in H.$$

Moreover, $||L_u||_{H^*} = ||u||_H$.

Hence, there is a bijection between H and H^*.

Bilinear form

Given V a normed functional space, a bilinear form a is a function that maps every two elements of V to a scalar

$$a : V \times V \to \mathbb{R}.$$

A form

- is bilinear if
  - $a(\lambda u + \mu w, v) = \lambda a(u, v) + \mu a(w, v)$ for every $\lambda, \mu \in \mathbb{R}$ and every $v, w, u \in V$, and
  - $a(u, \lambda v + \mu w) = \lambda a(u, v) + \mu a(u, w)$ for every $\lambda, \mu \in \mathbb{R}$ and every $v, w, u \in V$;
- is continuous if there exists an $M > 0$ such that

$$a(u, v) \leq M\|u\|_V \|v\|_V \text{ for every } v, u \in V;$$

- is symmetric if $a(u, v) = a(v, u)$ for every $u, v \in V$;
- is positive if $a(v, v) > 0$ for all $v \in V$ with $v \neq 0$;
- is coercive if there exists $\alpha > 0$ such that $a(v, v) > \alpha\|v\|_V^2$ for all $v \in V$.

Distributions

Let $\Omega \subset \mathbb{R}^d$ be an open set and $f : \Omega \to \mathbb{R}$ a function.

Support of a Function

The support of a function f, denoted by $\operatorname{supp}(f)$, is the closure of the set where f is non-zero.

$$\operatorname{supp}(f) := \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

Compact Support

A function has compact support if its support is a compact set.

C^∞ Compact Support Functions

A function is in $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ if it is infinitely differentiable and has compact support in Ω.

Convergence in $\mathcal{D}(\Omega)$

A sequence of functions $\{f_n\}$ in $\mathcal{D}(\Omega)$ converges to f in $C_c^\infty(\Omega)$ if

- exists a fixed compact set K that contains all supports of f_n
- all derivatives of f_n converge uniformly to the corresponding derivatives of f, i.e. $\partial_{x_1^{p_1}\dots x_d^{p_d}} f_n \to \partial_{x_1^{p_1}\dots x_d^{p_d}} f$ for all $p_1, \dots, p_d$.

Distributions

A **distribution** is a linear functional $T : \mathcal{D}(\Omega) \to \mathbb{R}$ that is continuous, i.e.,

$$\lim_{k\to\infty} T(\varphi_k) = T(\varphi),$$

for all $\varphi_k \to_{\mathcal{D}} \varphi \in \mathcal{D}$.
Hence, the distribution space $\mathcal{D}^*(\Omega)$ is the dual of $\mathcal{D}(\Omega)$.

Notation for distribution $T \in \mathcal{D}^*(\Omega)$ applied to a function $f \in \mathcal{D}(\Omega)$: $T(f) = \langle T, f\rangle$.

Example Dirac Delta

The Dirac delta distribution δ_a with $a \in \Omega$ a point, is defined by $\delta_a(\phi) = \phi(a)$ for all $\phi \in \mathcal{D}(\Omega)$. It is a distribution that "picks out" the value of a function at a point.

Convergence in $\mathcal{D}^*(\Omega)$

A sequence of distributions T_n converges in $\mathcal{D}^*(\Omega)$ to $T \in \mathcal{D}^*(\Omega)$ if

$$\lim_{n\to\infty} T_n(\varphi) = T(\varphi), \qquad \forall\varphi \in \mathcal{D}(\Omega).$$

$L^2(\Omega)$ squared summable functions

$$L^2(\Omega) := \{f : \Omega \to \mathbb{R} \text{ such that } \int_\Omega f(x)^2 \mathrm{d}x < \infty\}.$$

1. $L^2(\Omega)$ is a Hilbert space with scalar product $(f, g) := \int_\Omega f(x)g(x)\mathrm{d}x$.
2. The $L^2(\Omega)$ norm is define through the inner product as $\|f\|_{L^2(\Omega)} := \sqrt{\int_\Omega f(x)^2 \mathrm{d}x}$.
3. To every function $f \in L^2(\Omega)$ is associated a distribution $T_f \in \mathcal{D}^*(\Omega)$ defined by
$$T_f(\varphi) := \int_\Omega f(x)\varphi(x)\mathrm{d}x, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$
4. $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, i.e., for every $f \in L^2(\Omega)$ there exists a sequence of $\varphi_k \in \mathcal{D}(\Omega)$ such that
$$\|\varphi_k - f\|_{L^2(\Omega)} \to 0.$$
5. $\mathcal{D}(\Omega) \subset L^2(\Omega) \implies (L^2(\Omega))^* = L^2(\Omega) \subset \mathcal{D}^*(\Omega)$.

Example: convergence to Dirac Delta

Let $\chi_{[a,b]}$ be the characteristic function on the interval $[a,b] \subset \mathbb{R}$ defined as

$$\chi_{[a,b]}(x) = \begin{cases} 0 & \text{if } x \notin [a,b], \\ 1 & \text{if } x \in [a,b]. \end{cases}$$

Let us build the sequence of functions in $L^2(\mathbb{R})$ $f_n(x) := \frac{n}{2}\chi_{[-1/n,1/n]}(x)$. Clearly, we have that

1. $\int_{\mathbb{R}} f_n(x)\mathrm{d}x = 1$
2. $T_{f_n}(\varphi) = \int_{\mathbb{R}} f_n(x)\varphi(x)\mathrm{d}x = \frac{n}{2}\int_{-1/n}^{1/n} \varphi(x)\mathrm{d}x = \frac{n}{2}(\Phi(1/n) - \Phi(-1/n))$ where $\frac{d}{dx}\Phi(x) = \varphi(x)$.
3. Let $h_n = 1/n$, $T_{f_n}(\varphi) = \frac{\Phi(h)-\Phi(-h)}{2h}$
4. $\lim_{n\to\infty} T_{f_n}(\varphi) = \lim_{n\to\infty} \frac{\Phi(h)-\Phi(-h)}{2h} = \frac{d}{dx}\Phi(0) = \varphi(0)$.
5. $T_{f_n}(\varphi) \to \varphi(0) = \delta_0(\varphi)$.

Derivation in distributional sense

Let $T \in \mathcal{D}^*(\Omega)$, with $\Omega \subset \mathbb{R}^d$. We can define the derivative of T using the integration by parts.

$$\partial_{x_i} T(\varphi) = \langle \partial_{x_i} T, \varphi \rangle := -\langle T, \partial_{x_i} \varphi \rangle, \qquad \forall \varphi \in \mathcal{D}(\Omega) = C_c^\infty(\Omega).$$

If T is a T_f with $f \in \mathcal{C}^1(\Omega)$, it is clearly the classical derivative. Let's see in 1D with $\Omega = [a, b]$.

$$\partial_x T_f(\varphi) = \langle \partial_x T_f, \varphi \rangle = \int_a^b \partial_x f(x) \varphi(x) \mathrm{d}x = \underbrace{[f(x)\varphi(x)]_a^b}_{=0} - \int_a^b f(x) \partial_x \varphi(x) \mathrm{d}x, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Higher derivatives

$$\left\langle \frac{\partial^{p_1 + \cdots + p_d} T}{\partial x_1^{p_1} \ldots \partial x_d^{p_d}}, \varphi \right\rangle := (-1)^{p_1 + \cdots + p_d} \left\langle T, \frac{\partial^{p_1 + \cdots + p_d} \varphi}{\partial x_1^{p_1} \ldots \partial x_d^{p_d}} \right\rangle, \qquad \forall \varphi \in \mathcal{D}(\Omega) = C_c^\infty(\Omega).$$

Example: Derivative of Heaviside function

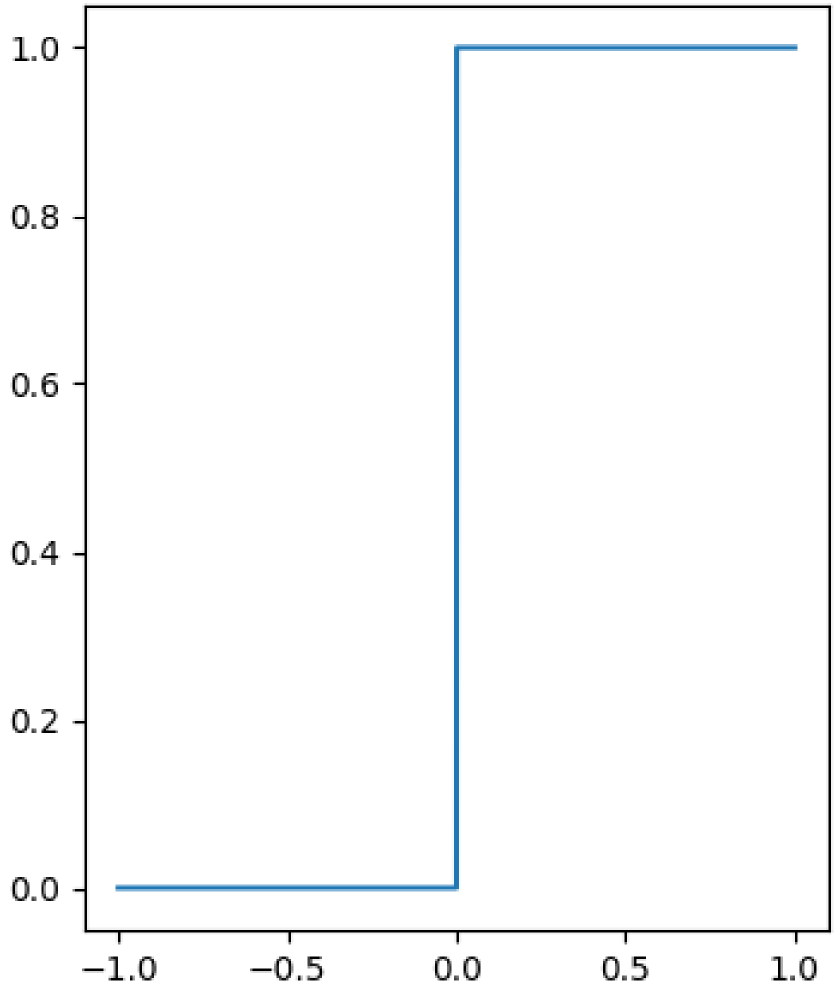
$$H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

- $H \in L^2((-1,1))$
- $H \notin C((-1,1))$
- $T_H \in \mathcal{D}^*((-1,1))$

$$\langle \partial_x T_H, \varphi \rangle = -\int_{-1}^{1} H(x) \partial_x \varphi(x) \mathrm{d}x$$

$$= -\int_0^1 \partial_x \varphi(x) \mathrm{d}x = -[\varphi]_0^1 = \varphi(0)$$

$$\implies \partial_x H = \delta_0.$$



Sobolev Spaces

As we have seen $L^2(\Omega) \subset \mathcal{D}^*(\Omega)$. This does not imply that their distributional derivatives are still in L^2. The Heaviside function is in L^2 but its derivative it's not.

We need to introduce other spaces!

Sobolev spaces

Let $\Omega \subset \mathbb{R}^d$ and $k \in \mathbb{N}_0$. We define the Sobolev space of order k on Ω the space of the functions in $L^2(\Omega)$ with distributional derivatives up to order k in $L^2(\Omega)$.

$$H^k(\Omega) := \{f \in L^2(\Omega) : \partial_{x_1^{p_1}\dots x_d^{p_d}} f \in L^2(\Omega),\ \text{for all } p_1, \dots, p_d : p_1 + \dots + p_d \leq k\}.$$

- $H^{k+1}(\Omega) \subset H^k(\Omega)$
- $L^2(\Omega) = H^0(\Omega)$
- Heaviside $H \in H^0((-1,1))$, but $H \notin H^1((-1,1))$

Examples

- Example of $H^\infty(\Omega)$ but not $C(\Omega)$

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0, \\ 3 & \text{if } x = 0. \end{cases}$$

- Example of $H^1(\Omega)$ but not $H^2(\Omega)$

$$f(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \qquad f'(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Norms and inner products of Sobolev spaces

- Sobolev spaces $H^k(\Omega)$ are Hilbert space with respect to the following scalar product

$$(f,g)_k = (f,g)_{H^k(\Omega)} := \sum_{p_1+\cdots+p_d\le k} \int_\Omega \partial_{x_1^{p_1}\ldots x_d^{p_d}} f \cdot \partial_{x_1^{p_1}\ldots x_d^{p_d}} g \,\mathrm{d}x,$$

  with the norms

$$\|f\|_k = \|f\|_{H^k(\Omega)} := \sqrt{\sum_{p_1+\cdots+p_d\le k} \int_\Omega (\partial_{x_1^{p_1}\ldots x_d^{p_d}} f)^2 \,\mathrm{d}x},$$

- Seminorms

$$|f|_k = |f|_{H^k(\Omega)} := \sqrt{\sum_{p_1+\cdots+p_d= k} \int_\Omega (\partial_{x_1^{p_1}\ldots x_d^{p_d}} f)^2 \,\mathrm{d}x},$$

- $\|f\|_k = \sqrt{\sum_{m=0}^k |f|_m^2}$

Examples for $k = 1$

$$(f, g)_1 = (f, g)_{H^1(\Omega)} = \int_\Omega f(x)\, g(x)\, \mathrm{d}x + \int_\Omega f'(x)\, g'(x)\, \mathrm{d}x$$

$$\|f\|_1 = \sqrt{\int_\Omega f^2(x)\, \mathrm{d}x + \int_\Omega (f'(x))^2\, \mathrm{d}x} = \sqrt{\|f\|^2_{L^2(\Omega)} + \|f'\|^2_{L^2(\Omega)}}$$

$$|f|_1 = \sqrt{\int_\Omega (f'(x))^2\, \mathrm{d}x} = \|f'\|_{L^2(\Omega)}$$

Boundary for bounded domains

Property

If $\Omega \subset \mathbb{R}^d$ is open with a *smooth enough* boundary, then $H^k(\Omega) \subset C^m(\bar{\Omega})$ if $m < k - \frac{d}{2}$.

Careful, in this case we mean that there is a representative of the function in H^k such that it also belongs to C^m. In the previous example where

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0, \\ 3 & \text{if } x = 0. \end{cases}$$

there exists a continuous representative of this function $f(x) = x^2$ which is the same function in $L^2(\Omega)$.

$H_0^1(\Omega)$

Let Ω be a bounded domain. We denote with $H_0^1(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. (*morally zero on the boundary*)

Poincarè inequality

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary. There exists a constant $C = C(\Omega) > 0$ such that for all $u \in H_0^1(\Omega)$,

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)} = C|u|_1.$$

Proof

Since $\Omega \subset \mathbb{R}^d$ is bounded there exists a ball $S_R = \{x : |x - x_0| < R\}$ that contains Ω. Since, $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, we can prove the inequality for $u \in \mathcal{D}(\Omega)$ and pass to the limit to get it for H_0^1. Notice that $\mathrm{div}(x - x_0) = d$. So,

$$\|u\|_{L^2(\Omega)}^2 = d^{-1}\int_\Omega d\,|u(x)|^2 \mathrm{d}x = d^{-1}\int_\Omega \mathrm{div}(x - x_0)|u(x)|^2 \mathrm{d}x = -d^{-1}\int_\Omega (x - x_0)\nabla(|u(x)|^2)\mathrm{d}x =$$

$$-2d^{-1}\int_\Omega (x - x_0)u(x)\nabla(u(x))\mathrm{d}x \leq 2d^{-1}\|x - x_0\|_\infty \|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} = 2d^{-1}R\|u\|_{L^2(\Omega)}|u|_1.$$

Proposition

On $H_0^1(\Omega)$ the seminorm $|\cdot|_1$ is actually a norm and it is equivalent to $\|\cdot\|_1$.

Proof

$$\|u\|_1^2 = |u|_1^2 + \|u\|_{L^2}^2 \leq (1 + C^2)|u|_1^2.$$

On the other hand

$$|u|_1^2 \leq |u|_1^2 + \|u\|_{L^2}^2 = \|u\|_1^2.$$