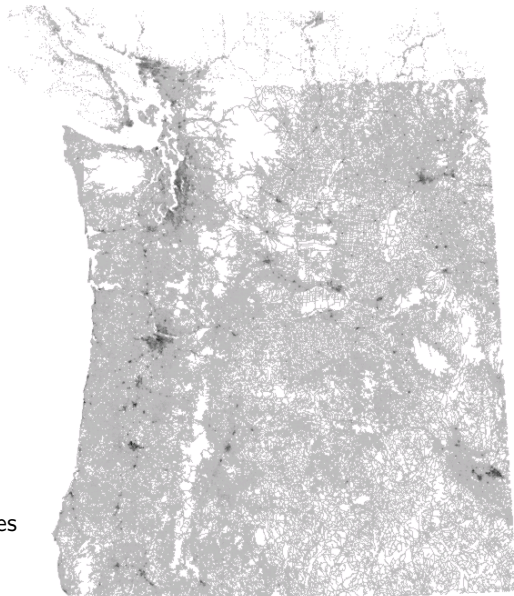


Efficient Point-to-Point Shortest Path Algorithms

Andrew V. Goldberg (Microsoft Research)
Chris Harrelson (Google)
Haim Kaplan (Tel Aviv University)
Renato F. Werneck (Princeton University)

Example Graph



Northwest
 $n = 1.6\text{M}$ vertices
 $m = 3.8\text{M}$ arcs

Shortest Paths

- Point-to-point shortest path problem (P2P):
 - Given:
 - * directed graph with nonnegative arc lengths $\ell(v, w)$;
 - * source vertex s ;
 - * target vertex t .
 - Goal: find shortest path from s to t .
- Our study:
 - Large road networks:
 - * 330K (Bay Area) to 30M (North America) vertices.
 - Algorithms work in **two stages**:
 - * preprocessing: may take hours, outputs linear amount of data;
 - * query: should take milliseconds, uses the preprocessed data.

Obvious Algorithm

- Precompute all shortest paths and store distance matrix.
- Will not work on large graphs ($n = 30\text{M}$).
 - $O(n^2)$ space: ~ 26 PB.
 - $\tilde{O}(nm)$ time: years (single Dijkstra takes $\sim 10\text{s}$).

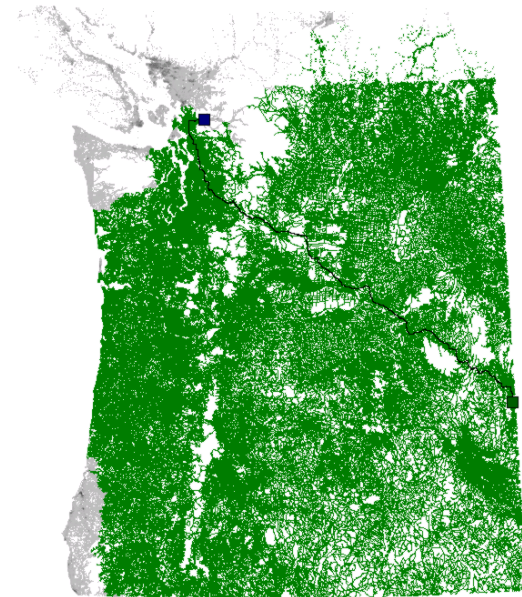
(All times on a 2.4 GHz AMD Opteron with 16 GB of RAM.)

Dijkstra's Algorithm

- Vertices processed in increasing order of distance:
 - maintains a **distance label** $d(v)$ for each vertex:
 - * upper bound on $\text{dist}(s, v)$;
 - * initially, $d(s) = 0$ and $d(v) = \infty$ for all other vertices.
 - In each iteration:
 - * Pick unscanned vertex v with smallest $d(\cdot)$ (use heap).
 - * Scan v :
 - For each edge (v, w) , check if $d(w) > d(v) + \ell(v, w)$.
 - If it is, set $d(w) \leftarrow d(v) + \ell(v, w)$.
 - Stop when the target t is about to be scanned.
 - [Dijkstra'59, Dantzig'63].
- Intuition:
 - grow a ball around s and stop when t is scanned.

5

Dijkstra's Algorithm



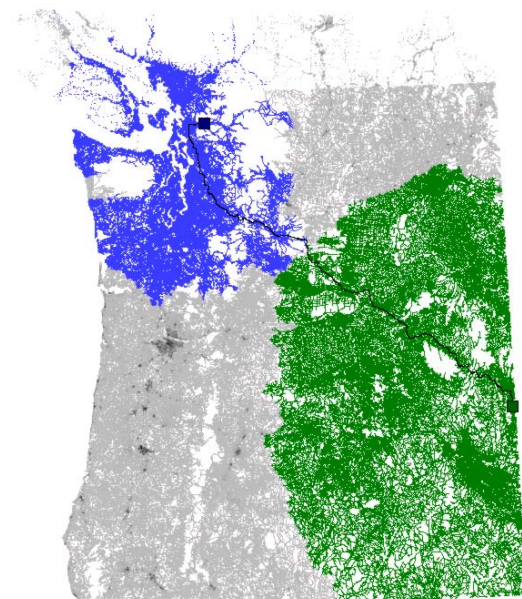
6

Bidirectional Dijkstra's Algorithm

- Bidirectional Dijkstra's algorithm:
 - **forward** search from s with labels d_f :
 - * performed on the **original graph**.
 - **reverse** search from t with labels d_r :
 - * performed on the **reverse graph**;
 - * same set of vertices, each arc (v, w) becomes (w, v) .
 - alternate in any way.
- Intuition: grow a ball around each end (s and t) until they "meet".

7

Bidirectional Dijkstra's Algorithm



8

Bidirectional Dijkstra's Algorithm

- Possible stopping criterion:
 - a vertex v is about to be scanned a second time:
 - * once in each direction;
 - v may not be on the shortest path.
- We must maintain the length μ of the best path seen so far:
 - initially, $\mu = \infty$;
 - when scanning an arc (v, w) in the forward search and w is scanned in the reverse search, update μ if $d_f(v) + \ell(v, w) + d_r(w) < \mu$.
 - similar procedure if scanning an arc in the reverse search.

9

Bidirectional Dijkstra's Algorithm

- Stronger stopping condition:
 - Let top_f and top_r be the top heap values (forward and reverse).
 - Stop when $\text{top}_f + \text{top}_r \geq \mu$.
 - Previous stopping criterion is a special case.
- Why does it work?
 - Suppose there exists an s - t path P with length less than μ .
 - There must be an arc (v, w) on this path such that:
 - * $\text{dist}(s, v) < \text{top}_f$ and
 - * $\text{dist}(w, t) < \text{top}_r$.
 - Both v and w have been scanned already.
 - When the second of these was scanned, it would have found the P .
 - * Contradiction: P cannot exist.

10

Part I: A* Search

11

A* Search

- Define **potential function** $\pi(v)$ and modify lengths:
 - $\ell_\pi(v, w) = \ell(v, w) - \pi(v) + \pi(w)$
 - $\ell_\pi(v, w)$: **reduced cost** of arc (v, w) .
- All s - t paths change by same amount: $\pi(t) - \pi(s)$.
- A* search:
 - Equivalent to Dijkstra on the modified graph:
 - * correct if $\ell_\pi(v, w) \geq 0$ (π **feasible**).
 - Vertices scanned in increasing order of $k(v) = d(v) + \pi(v)$:
 - * $\pi(v)$: estimate on $\text{dist}(v, t)$;
 - * $k(v)$: estimated length of shortest s - t path through v .
 - If $\pi(t) = 0$ and π feasible, $\pi(v)$ is a **lower bound** on $\text{dist}(v, t)$.
- All we need are good feasible lower bounds (e.g., Euclidean).

12

A* Search

- Why is A* equivalent to Dijkstra on the modified graph?
 - Dijkstra picks vertices with increasing (modified) distance from s :
 - * $\text{dist}_\pi(s, v) = \text{dist}(s, v) - \pi(s) + \pi(v)$
 - A* search picks vertices with increasing key:
 - * $k(v) = \text{dist}(s, v) + \pi(v)$
 - $\pi(s)$ is constant: these orders are the same.
- Why is $\pi(v)$ a lower bound on $\text{dist}(v, t)$ when π is feasible and $\pi(t) = 0$?
 - Take the shortest path from v to t .
 - Two ways of computing its reduced cost:
 1. $\text{dist}(v, t) - \pi(v) + \pi(t) = \text{dist}(v, t) - \pi(v)$ (since $\pi(t) = 0$);
 2. sum of the reduced costs of all arcs:
 - * must be nonnegative, since π is feasible.
 - Combining them: $\text{dist}(v, t) - \pi(v) \geq 0 \Rightarrow \pi(v) \leq \text{dist}(v, t)$.

13

Bidirectional A* Search

- Bidirectional search needs two potential functions:
 - $\pi_f(v)$: estimate on $\text{dist}(v, t)$.
 - $\pi_r(v)$: estimate on $\text{dist}(s, v)$.
- Reduced cost of arc (v, w) :
 - Forward: $\ell_f(v, w) = \ell(v, w) - \pi_f(v) + \pi_f(w)$.
 - Reverse: $\ell_r(w, v) = \ell(v, w) - \pi_r(w) + \pi_r(v)$.
 - * the arc appears as (w, v) in the reverse graph.
- These values must be **consistent**:

$$\begin{aligned} \ell_f(v, w) &= \ell_r(w, v) \\ \ell(v, w) - \pi_f(v) + \pi_f(w) &= \ell(v, w) - \pi_r(w) + \pi_r(v) \\ \pi_f(w) + \pi_r(w) &= \pi_f(v) + \pi_r(v) \end{aligned}$$
- This must be true for all pairs (v, w) , i.e., $(\pi_f + \pi_r) = \text{constant}$.

14

Bidirectional A* Search

- Must use consistent potential functions.
- In general, two arbitrary feasible functions π_f and π_r are **not** consistent.
- Their **average** is both feasible and consistent [Ilkeda et al. 94]:
 - $p_f(v) = \frac{1}{2}(\pi_f(v) - \pi_r(v))$
 - $p_r(v) = \frac{1}{2}(\pi_r(v) - \pi_f(v)) = -p_f(v)$
- To make the algorithm more intuitive, we make:
 - $p_f(v) = \frac{1}{2}(\pi_f(v) - \pi_r(v)) + \frac{\pi_r(t)}{2}$
 - $p_r(v) = \frac{1}{2}(\pi_r(v) - \pi_f(v)) + \frac{\pi_f(s)}{2}$
 - Added terms are constant: functions still feasible and consistent.
 - When π_f and π_r are lower bounds, $p_f(t) = 0$ and $p_r(s) = 0$.
- p usually provides worse bounds than π :
 - still worth it in practice.

15

Bidirectional A* Search

- Standard bidirectional Dijkstra:
 - stop when $\text{top}_f + \text{top}_r \geq \mu$.
 - * top_f : length of the path from s to top element of forward heap.
 - * top_r : length of (reverse) path from t to top element of reverse heap.
 - * μ : best s - t path seen so far.
- Bidirectional A* search: same, but on the **modified graph**:
 - Let v_f and v_r be the top elements in each heap;
 - Length of path s - v_f is $d_f(v_f) + p_f(v_f) - p_f(s) = \text{top}_f - p_f(s)$.
 - Length of reverse path t - v_r is $d_r(v_r) + p_r(v_r) - p_r(t) = \text{top}_r - p_r(t)$.
 - Stopping criterion:

$$[\text{top}_f - p_f(s)] + [\text{top}_r - p_r(t)] \geq [\mu - p_f(s) + p_f(t)]$$
 - Simplifying and using $p_f(t) = 0$:

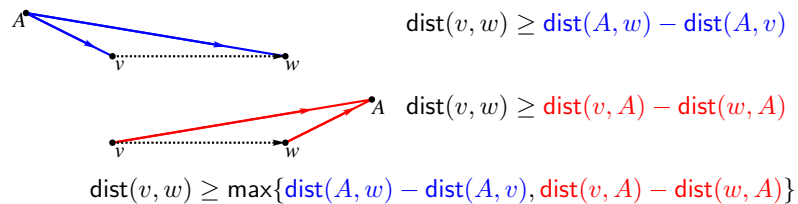
$$\text{top}_f + \text{top}_r \geq \mu + p_r(t).$$

16

Lower Bounds

- Preprocessing:
 - select a constant number of **landmarks** (we use 16);
 - for each landmark, precompute distance to and from every vertex.

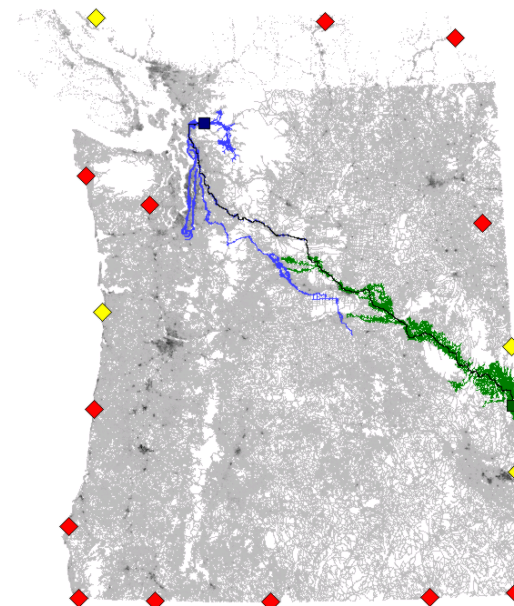
- Lower bounds use the **triangle inequality**:



- A good landmark appears “before” v or “after” w .
- More than one landmark: pick maximum (still feasible).

17

Query with Landmarks



18

Experimental Results

- Northwest (1 649 045 vertices), 1000 random pairs:

METHOD	PREPROCESSING		QUERY		
	minutes	MB	avgscan	maxscan	ms
Bidirectional Dijkstra	—	28	518 723	1 197 607	340.74
Landmarks	4	132	16 276	150 389	12.05

- Vertices scanned: $\sim 1\%$ on average, $\sim 10\%$ on bad cases.

19

Landmark Selection

- Landmark selection happens in two stages.
- Preprocessing:
 - Pick a small number of landmarks (we use **16**).
 - * more landmarks: better queries, more space.
 - Store on disk distances to and from each landmark.
- Query (s and t known):
 - using all available landmarks is expensive;
 - pick a small subset (**2 to 6**) that is good for the search.

20

Landmark Selection during Preprocessing

- Ultimate goal:
 - There should be a landmark “behind” every s - t pair.
 - Graphs are big, cannot evaluate this exactly: use heuristics.
 - * All methods are quasi-linear.
- Algorithms:
 - Simple methods: **random**, **farthest**, **planar**;
 - **avoid**: adds landmarks “behind” regions not currently covered;
 - **maxcover**: avoid + local search:
 - * goal: maximize #arcs with zero reduced cost.
- Best in practice is **maxcover**:
 - queries ~ 3 times as fast as **random**;
 - preprocessing ~ 15 times slower.

21

Landmark Selection at Query Time

- Use only an **active** subset:
 - prefer landmarks that give the best lower bound on $\text{dist}(s, t)$.
- We use **dynamic selection**:
 - start with two landmarks (best forward + best reverse);
 - periodically check if a new landmark would help;
 - heaps rebuilt when landmarks added.
- Performance in practice:
 - picks only ~ 3 landmarks;
 - fewer nodes visited than with any fixed number of landmarks.

22

Part II: Reach

23

Reaches

- Let v be a vertex on the **shortest** path P between s and t .



- **Reach** of v with respect to P :

$$\text{reach}(v, P) = \min\{\text{dist}(s, v), \text{dist}(t, v)\}$$

- Reach of v with respect to the whole graph:

$$\text{reach}(v) = \max_P \{\text{reach}(v, P)\},$$

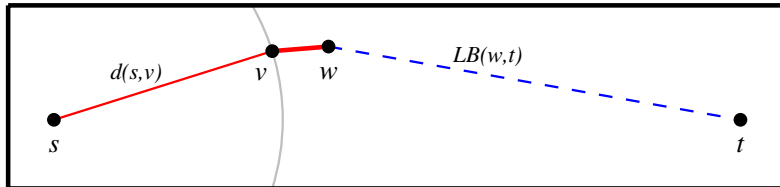
over all shortest paths P that contain v [Gutman'04].

- Intuition:
 - vertices on highways have high reach;
 - vertices on local roads have low reach.

24

Using Reaches

- Reaches can be used to prune the search during an s - t query.
- While scanning an edge (v, w) :
 - If $\text{reach}(w) < \min\{d(s, v) + \ell(v, w), \text{LB}(w, t)\}$, then w can be pruned.

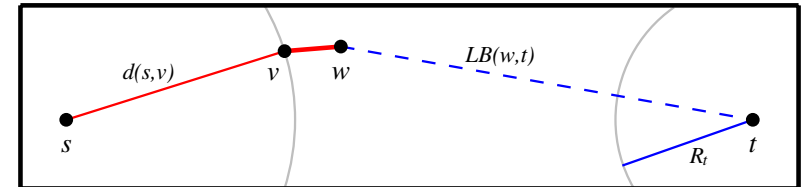


- How do we obtain lower bounds?
 - Explicitly: Euclidean distances (Gutman's suggestion), landmarks.
 - Implicitly: make the search bidirectional.

25

Implicit Bounds: Bidirectional Search

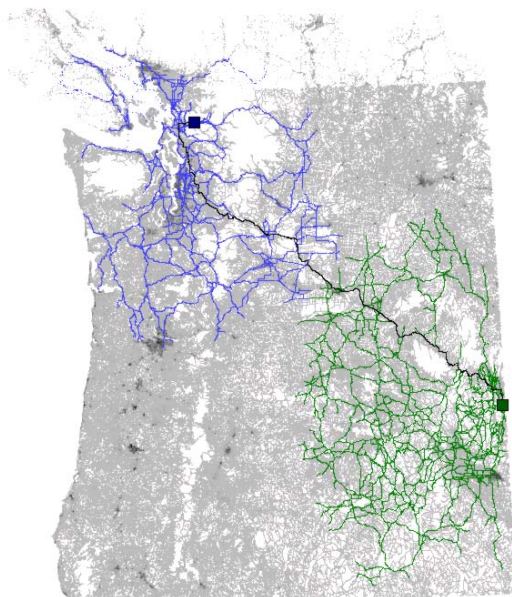
- Let R_t be the radius of the reverse search:
 - R_t is the value of the top element in the reverse heap;
 - if w not labeled in the reverse direction, then $d(w, t) \geq R_t$.



- Pruning test: $\text{reach}(w) < \min\{d(s, v) + \ell(v, w), R_t\}$
 - for best results, balance the forward and reverse searches by radius.

26

Queries with Reaches



27

Experimental Results

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METHOD	PREPROCESSING		QUERY		
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Landmarks	4	132	16 276	150 389	12.05
Reaches	1100	34	53 888	106 288	30.61

28

Computing Reaches

- Trivial algorithm:
 - compute every $s-t$ path;
 - determine reach of each vertex on each path.
- Implementation:
 - Build shortest path tree T_r from each vertex r ;
 - Determine reach of each vertex v within the tree:

$$\text{reach}(v, T_r) = \min\{\text{depth}(v), \text{height}(v)\}$$
 - Take maximum over all r .
- Runs in $\tilde{O}(nm)$ time:
 - overnight on Bay Area, years on North America.

29

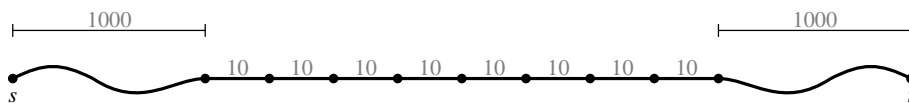
Computing Reaches

- Query still correct with **upper bounds** on reaches.
- We use iterative algorithm:
 1. find vertices with reach at most ϵ ;
 - look only at **partial** shortest path trees (depth $\sim 2\epsilon$).
 2. eliminate vertices with small reach;
 - if no vertices remain, stop;
 - otherwise, increase ϵ and start another iteration.
- Use **penalties** to account for vertices already eliminated:
 - reaches no longer exact, but valid upper bounds
- Works well if many vertices are eliminated between iterations.

30

Shortcuts

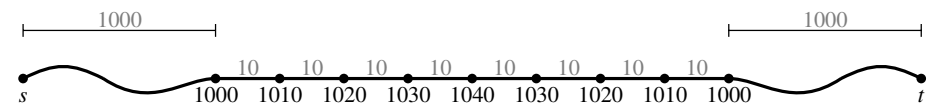
- Consider a sequence of vertices of degree two on the path below:



31

Shortcuts

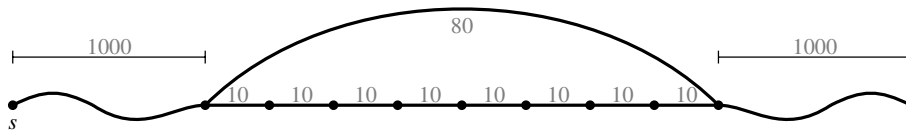
- Consider a sequence of vertices of degree two on the path below:
 - they all have high reach;



32

Shortcuts

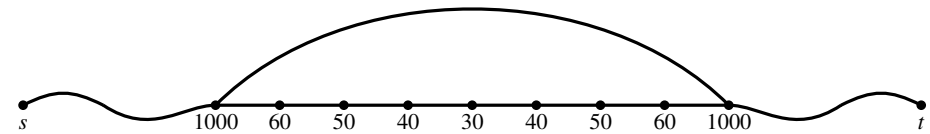
- Consider a sequence of vertices of degree two on the path below:
 - they all have high reach.
- Add a **shortcut**:
 - single edge bypassing a path (with same length).
 - assume ties are broken by taking path with fewer nodes.



33

Shortcuts

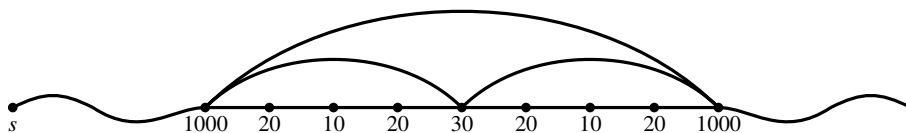
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34

Shortcuts

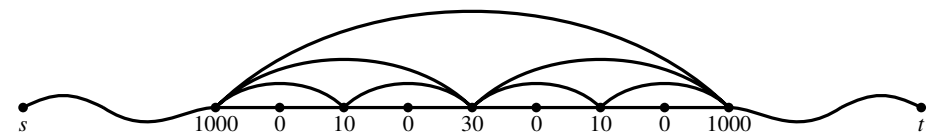
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 - assume ties are broken by taking path with fewer nodes.
- More shortcuts can be added recursively.



35

Shortcuts

- Consider a sequence of vertices of degree two on the path below:
 - they all have high reach.
- Add a **shortcut**:
 - single edge bypassing a path (with same length).
 - assume ties are broken by taking path with fewer nodes.
- More shortcuts can be added recursively.



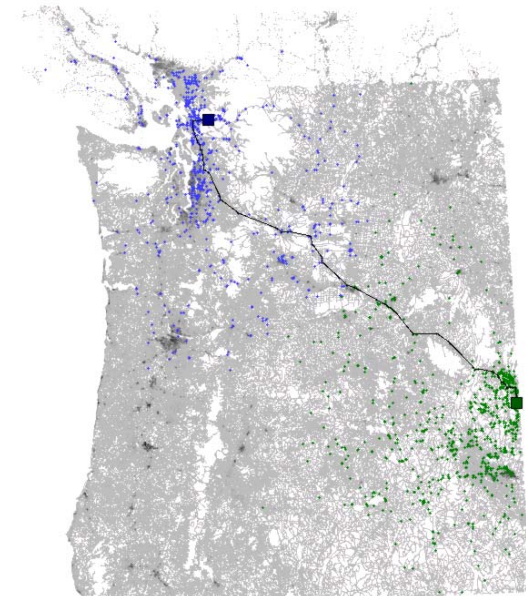
36

Shortcuts

- Adding shortcuts during preprocessing:
 - speeds up queries (pruning more effective);
 - speeds up preprocessing (graph shrinks faster);
 - requires slightly more space (graph has more arcs).
- Shortcuts bypass vertices of degree two:
 - some have degree two in the original graph;
 - some acquire degree two as other vertices are eliminated.
- Sanders and Schultes [ESA'05]:
 - similar idea for hierarchy-based algorithm.

37

Reaches with Shortcuts



38

Experimental Results

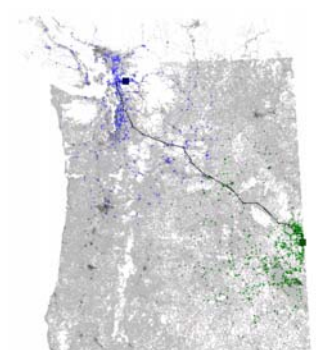
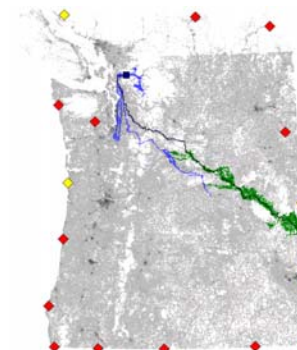
- Northwest (1 649 045 vertices), 1000 random pairs:

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	minutes	MB	avgscan	maxscan	ms
Bidirectional Dijkstra	—	28	518 723	1 197 607	340.74
Landmarks	4	132	16 276	150 389	12.05
Reaches	1100	34	53 888	106 288	30.61
Reaches+Shortcuts	17	100	2 804	5 877	2.39

39

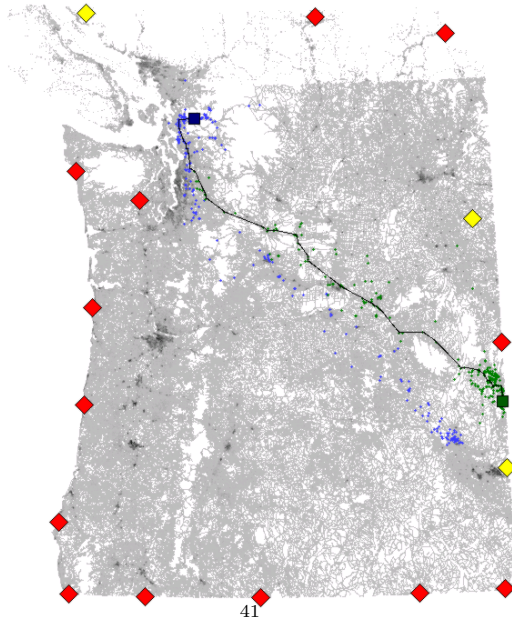
Reaches and Landmarks

- A* search with landmarks can use reaches:
 - A* gives the search a sense of direction.
 - Reaches make the search sparser.
- Landmarks have dual purpose:
 1. guide the search;
 2. provide lower bounds for reach-based pruning.



40

Reaches and Landmarks (with Shortcuts)



41

Experimental Results

- Northwest (1649 045 vertices), 1000 random pairs:

METHOD	PREPROCESSING		QUERY		
	minutes	MB	avgscan	maxscan	ms
Bidirectional Dijkstra	—	28	518 723	1 197 607	340.74
Landmarks	4	132	16 276	150 389	12.05
Reaches	1100	34	53 888	106 288	30.61
Reaches+Shortcuts	17	100	2 804	5 877	2.39
Reaches+Shortcuts+Landmarks	21	204	367	1 513	0.73

42

Summary of Results

- North America (29 883 886 vertices), 1000 random pairs:

METHOD	PREPROCESS		QUERY		
	hours	GB	avgscan	maxscan	ms
Bidirectional Dijkstra	—	0.5	10 255 356	27 166 866	7 633.9
Landmarks	1.6	2.3	250 381	3 584 377	393.4
Reaches+Shortcuts	11.3	1.8	14 684	24 618	17.4
Reaches+Shortcuts+Landmarks	12.9	3.6	1 595	7 450	3.7

43

Future Directions

- Theory:
 - For which classes of graphs does each algorithm work?
 - How to find a good set of landmarks?
 - What is the best set of shortcuts for a given graph?
 - Is there a faster algorithm for computing exact reaches?
 - Is there a better algorithm for computing approximate reaches?
- Practice:
 - Reduce size of preprocessed data.
 - Make queries more cache-efficient.

44

References

- Goldberg, Harrelson, and Werneck (in preparation):
 - Goldberg and Harrelson (SODA'05):
 - * “ALT algorithm” (A* search + Landmarks + Triangle inequality).
 - Goldberg and Werneck (Alenex'05):
 - * improved preprocessing and queries;
 - * Pocket PC implementation.
- Goldberg, Kaplan, and Werneck (2005):
 - reach with shortcuts + A* search.

<http://www.cs.princeton.edu/~rwerneck/public.htm>