Convex optimization overview

Mathématiques Informatique et Statistique Appliquées (MISA)
Université d'Antananarivo

Mathematical optimization problem

$$\begin{aligned} \text{minimize } & f_0(x) \\ \text{s.t } & f_i(x) \leq 0, \ i = 1, ..., m \\ & g_i(x) = 0, \ i = 1, ..., p \end{aligned}$$

- $x \in \mathbb{R}^n$ is the optimization variable
- f_0 is the **objective or cost function**
- f_i are the **inequality constraint** functions
- h_i are the **equality constraint** functions

- x represents some action like:
  - trades in a portfolio
  - airplane control surface deflections
  - schedule or assignment
  - resource allocation
  - transmitted signal
- Constraints limit actions or impose conditions on outcome.
- The smaller the objective $f_0(x)$, the better. It might be
  - total cost (or negative profit)
  - deviation from desired or target outcome
  - fuel use
  - risk

Examples

Engineering design

- x represents a design (of a circuit, device, structure,...).
- Constraints come from manufacturing process and performance requirements.
- The objective $f_0(x)$ is combination of cost, weight, power.

Statistics/machine learning

- x represents the parameters in a model.
- Constraints impose requirements on model parameters(e.g., nonnegativity).
- The objective $f_0(x)$ is the prediction error on some observed data(and possibly a term that penalizes model complexity).

Inversion

- x is something we want to estimate/reconstruct, given some measurement y.
- Constraints come from prior knowledge about x.
- Objective $f_0(x)$ measures deviation between predicted and actual measurements.

Worst-case analysis (pessimization)

- x represents actions or parameters out of our control(and possibly under the control of an adversary).
- Constraints limit the possible values of the parameters.
- Minimizing $-f_0(x)$ finds worst possible parameter values.

- Optimization problems are everywhere.
- But most of them are **intractable** (cannot be solved).
- there is an exception : convex optimization problems which can (generally) be solved.

Convex optimization problem

$$\begin{aligned} \text{minimize } & f_0(x) \\ \text{s.t } & f_i(x) \leq 0, \; i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

- $x \in \mathbb{R}^n$ is the optimization variable
- $f_i, \; i = 0, \ldots, m$ are convex functions.
- The equality constraints are affine.

Motivations

- Beautiful, nearly complete theory (duality, optimality conditions...)
- Effective algorithms and methods (in theory and in practice)
  - get global solution
  - polynomial complexity
- Conceptual unification of many methods
- Lots of applications
  - machine learning, statisticsl
  - finance
  - supply chain, revenue management, advertising
  - control
  - signal processing (image, sound ...)
  - networking
  - circuit design
  - combinatorial optimization
  - quantum mechanics
  - flux-based analysis

The approach

- Recognize/formulate problems as a convex optimization problem
- Then, you can (usually) solve it (numrically).

Some tricks:

- change of variables
- approximation of true objective, constraints
- relaxation: ignore terms or constraints you can't handle

Solvers

Medium-scale

- 1000s–10000s variables, constraints
- reliably solved by interior-point methods on single machine(especially for problems in standard cone form), exploit problem sparsity
- not quite a technology, but getting there
- used in control, finance, engineering design, ...

Large-scale

- 100k – 1B variables, constraints
- solved using custom often problem specific methods (limited memory BFGS, stochastic subgradient, block coordinate descent, operator splitting methods)
- require custom implementation, tuning for each problem
- used in machine learning, image processing,

Modeling languages

- high level language support for convex optimization
  - describe problem in high level language
  - description automatically transformed to a standard form
  - solved by standard solver, transformed back to original form
- implementations:
  - YALMIP, CVX (Matlab)
  - CVXPY (Python)
  - Convex.jl (Julia)
  - CVXR (R)