Chapter 6. Orthogonality and Least Squares

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6.1. Inner Product, Length, And Orthogonality

Inner Product

- If $\mathbf{u}$ and $\mathbf{v}$ are vector in $\mathbb{R}^n$, then we regard $\mathbf{u}$ and $\mathbf{v}$ as $n \times 1$ matrices. (*i.e. column vector*)
- The transpose $\mathbf{u}^T$ is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which we write as a single real number (a scalar) without brackets.
- The number $\mathbf{u}^T\mathbf{v}$ is called the **inner product** of $\mathbf{u}$ and $\mathbf{v}$, and it is written as $\mathbf{u} \cdot \mathbf{v}$
- This inner product is also referred to as a **dot product**.
- If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then, the inner product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 + v_2 + \cdots + u_n v_n$$

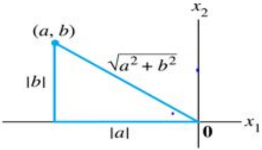
- **Theorem 1**: Let $\mathbf{u}$, $\mathbf{v}$, and W be vectors in $\mathbb{R}^n$, and let c be a scalar. Then
  a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
  b) $(\mathbf{u} + \mathbf{v}) \cdot W = \mathbf{u} \cdot W + \mathbf{v} \cdot W$
  c) $(c \cdot \mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
  d) $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$

- **Remarks**
  - Properties (b) and (c) can be combined several times to produce the following useful property. $(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot W = c_1(\mathbf{u}_1 \cdot W) \cdots + c_p(\mathbf{u}_p \cdot W)$
  - If $\mathbf{v}$ is in $\mathbb{R}^n$, with entries $v_1, \cdots v_n$, then the square root of $\mathbf{v} \cdot \mathbf{v}$ is defined, because $\mathbf{v} \cdot \mathbf{v}$ is nonnegative.

The length of Vector

- **Definition**: The **length** or **norm** of $\mathbf{v}$ is the nonnegative scalar
  - $\|\mathbf{v}\| := \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$ and $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$

- **Remarks**
  - Suppose $\mathbf{v}$ is in $\mathbb{R}^2$, say $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. If we identify $\mathbf{v}$ with a geometric point in the plane, as usual, $\|\mathbf{v}\|$ coincides with the standard notion of the length of the line segment from the origin to $\mathbf{v}$.
  - This follows from the Pythagorean Theorem applied to a triangle such as the one shown in the following figure.



Interpretation of $\|\mathbf{v}\|$ as length.

- **Remarks**
  - For any scalar c, the length $c\mathbf{v}$ is $|c|$ times the length of $\mathbf{v}$. That is, $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$
  - A vector whose length equal to 1 is called a **unit vector.**
  - If we *divide* a nonzero vector $\mathbf{v}$ by its length—that is, multiply by $1/\|\mathbf{v}\|$ — we obtain a unit vector $\mathbf{u}$ because the length of $\mathbf{u}$ is $(1/\|v\|)\|v\|$
  - The process of creating $\mathbf{u}$ from $\mathbf{v}$ is sometimes called **normalizing** $\mathbf{v}$, and we say that $\mathbf{u}$ is in the same direction as $\mathbf{v}$.

- **Example 2:** Let $\mathbf{v} = (1, -2, 2, 0)$. Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{u}$.

- **Solution:**
  - First, compute the length of $\mathbf{v}$: $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = (1)^2 + (-2)^2 + (2)^2 + (0)^2 = 9$. $\|\mathbf{v}\| = \sqrt{9} = 3$.
  - Then, multiply $\mathbf{v}$ by $1/\|\mathbf{v}\|$ to obtain

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

  - To check that $\|\mathbf{u}\| = 1$, it suffices to show that $\|\mathbf{u}\|^2 = 1$

$$\begin{aligned} \|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} &= (\frac{1}{3})^2 + (-\frac{2}{3})^2 + (\frac{2}{3})^2 + (0)^2 \\ &= \frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 0 = 1 \end{aligned}$$

Distance in $\mathbb{R}^n$

- **Definition:** For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between** $\mathbf{u}$ and $\mathbf{v}$, written as $dist(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$dist(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

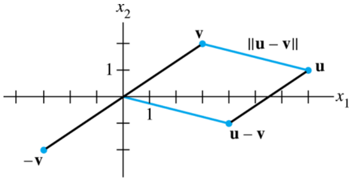
- **Example 4:** Compute the distance between the vectors $\mathbf{u} = (7, 1)$ and $\mathbf{v} = (3, 2)$

- **Solution:**
  - Calculate

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

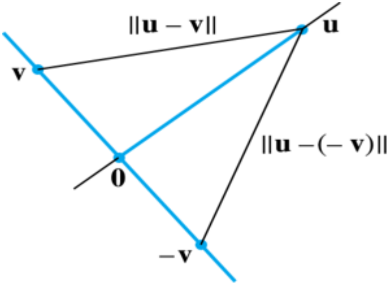
  - The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ are shown in the figure on the next slide.
  - When the vector $\mathbf{u} - \mathbf{v}$ is added to $\mathbf{v}$, the result is $\mathbf{u}$.
  - Notice that the parallelogram in the figure below shows that the distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u}$ to $\mathbf{u}$ is the same as the distance $\mathbf{u} - \mathbf{v}$ to 0.



The distance between $\mathbf{u}$ and $\mathbf{v}$ is the length of $\mathbf{u} - \mathbf{v}$.

Orthogonal Vector

- Consider $\mathbb{R}^2$ or $\mathbb{R}^3$ and two lines through the origin determined by vectors $\mathbf{u}$ and $\mathbf{v}$.
- See the figure below. The two lines shown in the figure are geometrically perpendicular if and only if the distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u}$ to $-\mathbf{v}$



- This is the same as requiring the squares of the distances to be the same.

- Now,

$$\begin{aligned}[dist(\mathbf{u}, -\mathbf{v})]^2 &= \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\| \\ &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}\end{aligned}$$

- The same calculations with $\mathbf{v}$ and $-\mathbf{v}$ interchanged show that

$$\begin{aligned}[dist(\mathbf{u}, \mathbf{v})]^2 &= \|\mathbf{u}\|^2 + \| - \mathbf{v}\|^2 + 2\mathbf{u} \cdot (-\mathbf{v}) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{u}\|^2 - 2\mathbf{u} \cdot \mathbf{v}\end{aligned}$$

- The two squared distances are equal if and only if $2\mathbf{u} \cdot \mathbf{v} = -2\mathbf{u} \cdot \mathbf{v}$, which happens if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.
- This calculation shows that when vectors $\mathbf{u}$ and $\mathbf{v}$ are identified with geometric points, the corresponding lines through the points and the origin are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

- **Definition:** Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.
- **Remark:** The zero vector is orthogonal to every vector $\mathbb{R}^n$ because $0^T\mathbf{v} = 0$ for all $\mathbf{v}$
- **Theorem 2 (The pythogorean Theorem):** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if
$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Orthogonal Complements

- **Definition**
  - If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be **orthogonal** to W.
  - The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the **orthogonal complement** of W and is denoted by $W^\perp$ (and read as "W perpendicular" or simply "W perp")
- **Theorems**
  - A vector $\mathbf{x}$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
  - $W^\perp$ is a subspace of $\mathbb{R}^n$.

- **Theorem 3:** Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(RowA)^{\perp} = NulA \text{ and } (ColA)^{\perp} = NulA^T$$

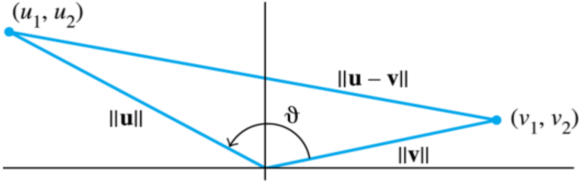
- **Proof :**
  - The row-column rule for computing $A\mathbf{x}$ shows that if $\mathbf{x}$ is in $Nul\ A$, then $\mathbf{x}$ is orthogonal to each row of A (with the rows treated as vectors in $\mathbb{R}^n$).
  - Since the rows of A span the row space, $\mathbf{x}$ is orthogonal to $Row\ A$.
  - Conversely, if $\mathbf{x}$ is orthogonal to $Row\ A$, then $\mathbf{x}$ is certainly orthogonal to each row of A, and hence $A\mathbf{x} = 0$.
  - This proves the first statement of the theorem.
  - Since this statement is true for any matrix, it is true for A^T
  - That is, the orthogonal complement of the row space of A^T is the null space of A^T.
  - This proves the second statement, because $RowA^T = ColA$.

Angles in $\mathbb{R}^2$ *and* $\mathbb{R}^3$

- If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$, then there is a nice connection between their inner product and the angle θ between the two line segments from the origin to the points identified with $\mathbf{u}$ and $\mathbf{v}$.
- The formula is

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| cos\theta \tag{2}$$

- To verify this formula for vectors in $\mathbb{R}^2$ consider the triangle shown in the next figure with sides of lengths, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u} - \mathbf{v}\|$.



The angle between two vectors.

- By the law of cosines,

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|cos\theta$$

which can be rearranged to produce the next equations.

$$\begin{aligned}\|\mathbf{u}\|\|\mathbf{v}\|cos\,\theta &= \frac{1}{2}\left[\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}||^2\right] \\ &= \frac{1}{2}\left[u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 = v_2)^2\right] \\ &= u_1 v_1 + u_2 v_2 \\ &= \mathbf{u}\cdot\mathbf{v}\end{aligned}$$

- The verification for $\mathbb{R}^3$ is similar. When $n > 3$, formula (2) may be used to *define* the angle between two vectors in $\mathbb{R}^n$
- In statistics, the value of $cos\theta$ defined by (2) for suitable vectors **u** and **v** is called a *correlation coefficient*.

Suggested Excercises

- 6.1.11

6.2. Orthogonal Sets

Orthogonal Sets

- **Definition:** A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

- **Theorem 4:** If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

- **Proof:**
  - If $0 = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$ for some scalars $c_1, \ldots c_p$, then

$$\begin{aligned} 0 &= 0 \cdot \mathbf{u}_1 = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{u}_1 \\ &= (c_1\mathbf{u}_1) \cdot \mathbf{u}_1 + (c_2\mathbf{u}_2) \cdot \mathbf{u}_1 + \cdots + (c_p\mathbf{u}_p) \cdot \mathbf{u}_1 \\ &= c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) + c_2(\mathbf{u}_2 \cdot \mathbf{u}_1) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{u}_1) \\ &= c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) \end{aligned}$$

    because $\mathbf{u}_1$ is orthogonal to $\mathbf{u}_2, \ldots, \mathbf{u}_p$
  - Since $\mathbf{u}_1$ is nonzero, $\mathbf{u}_1 \cdot \mathbf{u}_1$ not zero and so $c_1 = 0$. Likewise, $c_2, \ldots, c_p$ must be zero.
  - Thus S is linearly independent.

- **Definition** : An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

- **Theorem 5:** Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination $\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$ are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \ (j = 1, \ldots, p)$$

- **Proof:**
  - The orthogonality of $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ shows that

$$\mathbf{y} \cdot \mathbf{u}_1 = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{u}_1 = c_1(\mathbf{u}_1 \cdot \mathbf{u}_1)$$

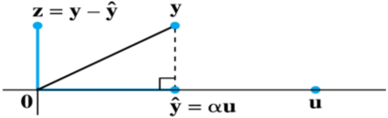
  - Since $\mathbf{u}_1 \cdot \mathbf{u}_1$ is not zero, the equation above can be solved for c_1.
  - To find c_j for $j = 2, \ldots, p$, compute $\mathbf{y} \cdot \mathbf{u}_j$ and solve for c_j

An Orthogonal Projection

- Given a nonzero vector $\mathbf{u}$ in $\mathbb{R}^n$, consider the problem of decomposing a vector $\mathbf{y}$ in $\mathbb{R}^n$ into the sum of two vectors, one a multiple of $\mathbf{u}$ and the other orthogonal to $\mathbf{u}$.
- We wish to write

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \tag{1}$$

where $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α and $\mathbf{z}$ is some vector orthogonal to $\mathbf{u}$. See the following figure.



Finding α to make $\mathbf{y} - \hat{\mathbf{y}}$ orthogonal to $\mathbf{u}$.

- Given any scalar α, let $\mathbf{z} = \mathbf{y} - \alpha\mathbf{u}$, so that (1) is satisfied. Then, $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to $\mathbf{u}$ if and only if

$$0 = (\mathbf{y} - \alpha\mathbf{u}) \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{u} - (\alpha\mathbf{u}) \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{u} - \alpha(\mathbf{u} \cdot \mathbf{u})$$

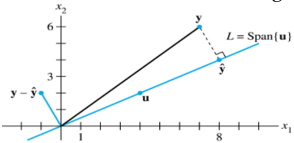
- That is, (1) is satisfied with $\mathbf{z}$ orthogonal to $\mathbf{u}$ if and only if $\alpha = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$ and $\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$
- The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$**, and the vector $\mathbf{z}$ is called the **component of $\mathbf{y}$ orthogonal to $\mathbf{u}$**.

- If c is any nonzero scalar and if $\mathbf{u}$ is replaced by $c\mathbf{u}$ in the definition of $\hat{\mathbf{y}}$, then the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{u}$ is exactly the same as the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$.
- Hence this projection is determined by the *subspace* L spanned by $\mathbf{u}$ (the line through $\mathbf{u}$ and 0).
- Sometimes $\hat{\mathbf{y}}$ is denoted by $proj_L\,\mathbf{y}$ and is called the **orthogonal projection** of $\mathbf{y}$ onto L.
- That is,

$$\hat{\mathbf{y}} = proj_L\,\mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} \tag{2}$$

- **Example 3:** Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$. Then write $\mathbf{y}$ as the sum of two orthogonal vectors, one in $Span\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

- **Solution:**
  - Compute $\mathbf{y} \cdot \mathbf{u} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 40$ and $\mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 20$
  - The orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ is $\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{40}{20} \mathbf{u} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ and the component of $\mathbf{y}$ orthogonal to $\mathbf{u}$ is $\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
  - The sum of these two vector is $\mathbf{y}$, i.e. $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}})$. That is, $\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
  - The decomposition of $\mathbf{y}$ is illustrated in the following figure:



The orthogonal projection of $\mathbf{y}$ onto a line L through the origin.

- (*solution continued*)
  - If the calculation above are correct, then $\{\hat{\mathbf{y}}, \mathbf{y} - \hat{\mathbf{y}}\}$ will be an orthogonal set.
  - As a check, compute

$$\hat{\mathbf{y}} \cdot (\mathbf{y} - \hat{\mathbf{y}}) = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -8 + 8 = 0$$

- **Remark**: Since the line segment in the figure on the previous slide between $\mathbf{y}$ and $\hat{\mathbf{y}}$ is perpendicular to L, by construction of $\hat{\mathbf{y}}$, the point identified with $\hat{\mathbf{y}}$ is the closest point of L to $\mathbf{y}$.

Orthonormal Sets

- **Definitions**
  - A set $\{\mathbf{u}_1, \ldots \mathbf{u}_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors.
  - If W is the subspace spanned by such a set, then $\{\mathbf{u}_1, \ldots \mathbf{u}_p\}$ is an **orthonormal basis** for W, since the set is automatically linearly independent, by Theorem 4.

- **Remark**
  - The simplest example of an orthonormal set is the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^n$.
  - Any nonempty subset of $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is orthonormal, too.

- **Example 2:** Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$. where

$$v_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, v_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, v_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

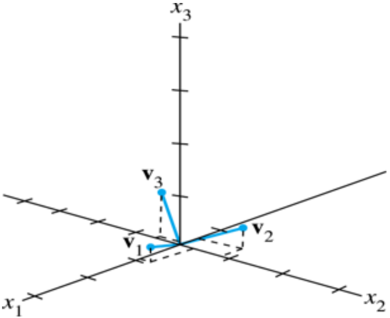
- **Solution:**
  - $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set because

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= -3/\sqrt{66} + 2/\sqrt{66} + 1/\sqrt{66} = 0 \\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= -3/\sqrt{726} - 4/\sqrt{726} + 7/\sqrt{726} = 0 \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= 1/\sqrt{396} - 8/\sqrt{396} + 7/\sqrt{396} = 0 \end{aligned}$$

  - $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are unit vectors because

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_1 &= 9/11 + 1/11 + 1/11 = 1 \\ \mathbf{v}_2 \cdot \mathbf{v}_2 &= 1/6 + 4/6 + 1/6 = 1 \\ \mathbf{v}_3 \cdot \mathbf{v}_3 &= 1/66 + 16/66 + 49/66 = 1 \end{aligned}$$

- Since the set is linearly independent, its three vectors form a basis for $\mathbb{R}^3$. See the following figure.



- When the vectors in an orthogonal set of nonzero vectors are *normalized* to have unit length, the new vectors will still be orthogonal, and hence the new set will be an orthonormal set.

- **Theorem 6:** An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.
- **Proof:**
  - To simplify notation, we suppose that U has only three columns, each a vector in $\mathbb{R}^m$.
  - Let $U = \begin{bmatrix} \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \end{bmatrix}$ and compute

$$U^T U = \begin{bmatrix} \mathbf{u}_1^T \\ \\ \mathbf{u}_2^T \\ \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1^T\mathbf{u}_1 & \mathbf{u}_1^T\mathbf{u}_2 & \mathbf{u}_1^T\mathbf{u}_3 \\ \\ \mathbf{u}_2^T\mathbf{u}_1 & \mathbf{u}_2^T\mathbf{u}_2 & \mathbf{u}_2^T\mathbf{u}_3 \\ \\ \mathbf{u}_3^T\mathbf{u}_1 & \mathbf{u}_3^T\mathbf{u}_2 & \mathbf{u}_3^T\mathbf{u}_3 \end{bmatrix} \tag{4}$$

  - The entries in the matrix at the right are inner products, using transpose notation.
  - The columns of U are orthogonal if and only if

$$\mathbf{u}_1^T\mathbf{u}_2 = \mathbf{u}_2^T\mathbf{u}_1 = 0,\ \mathbf{u}_1^T\mathbf{u}_3 = \mathbf{u}_3^T\mathbf{u}_1 = 0,\ \mathbf{u}_2^T\mathbf{u}_3 = \mathbf{u}_3^T\mathbf{u}_2 = 0 \tag{5}$$

  - The columns of U all have unit length if and only if

$$\mathbf{u}_1^T\mathbf{u}_1 = 1,\ \mathbf{u}_2^T\mathbf{u}_2 = 1,\ \mathbf{u}_3^T\mathbf{u}_3 = 1 \tag{6}$$

  - The theorem follows immediately from (4)–(6).

- **Theorem 7:** Let U be an $m \times n$ matrix with orthonormal columns, and let x and y be in $\mathbb{R}^3$ Then,
  a. $\|Ux\| = \|x\|$
  b. $(Ux) \cdot (Uy) = x \cdot y$
  c. $(Ux) \cdot (Uy) = 0\$$ if and only if $\$x \cdot y = 0$

- **Remark**: Properties a and c say that the linear mapping $x \to Ux$ preserves lengths and orthogonality.

Suggested Excercises

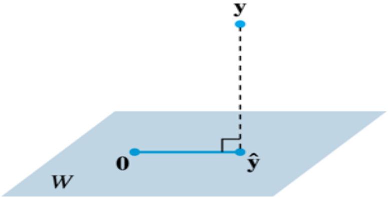
- 6.2.2

6.3. Orthogonal Projections

Orthogonal Projections

- The orthogonal projection of a point in $\mathbb{R}^2$ onto a line through the origin has an important analogue in $\mathbb{R}^n$

- Given a vector $\mathbf{y}$ and a subspace W in $\mathbb{R}^n$, there is a vector in $\hat{\mathbf{y}}$ in W such that
  (1) $\hat{\mathbf{y}}$ is the unique vector in W for which $\mathbf{y}$ - $\hat{\mathbf{y}}$ is orthogonal to W, and
  (2) $\hat{\mathbf{y}}$ is the unique vector in W closest to $\mathbf{y}$.
- See the following figure.
- These two properties of $\hat{\mathbf{y}}$ provide the key to finding the least-squares solutions of linear systems.



The Orthogonal Decomposition Theorem

- **Theorem 8:** Let W be a subspace of $\mathbb{R}^n$.
  - Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

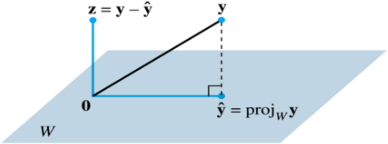
$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}, \tag{1}$$

    where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.
  - In fact, if $\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ is any orthogonal basis of W then,

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \mathbf{u_1} + \cdots + \frac{\mathbf{y} \cdot \mathbf{u_p}}{\mathbf{u_p} \cdot \mathbf{u_p}} \mathbf{u_p} \tag{2}$$

    and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$

- The vector $\hat{\mathbf{y}}$ in (1) is called the **orthogonal projection of y onto** W and often is written as $proj_W\,\mathbf{y}$. See the following figure:



The orthogonal projection of **y** onto W.

- **Proof for $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$:**
  - Let $\{\mathbf{u_1}, \dots, \mathbf{u_p}\}$ be any orthogonal basis for W, and define $\hat{\mathbf{y}}$ by (2). Then $\hat{\mathbf{y}}$ is in W because $\hat{\mathbf{y}}$ is a linear combination of the basis $\mathbf{u_1}, \dots, \mathbf{u_p}$
  - Let $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$. Since $\mathbf{u_1}$ is orthogonal to $\mathbf{u_1}, \dots, \mathbf{u_p}$, it follows from (2) that

$$\mathbf{z} \cdot \mathbf{u_1} = (\mathbf{y} - \hat{\mathbf{y}}) \cdot \mathbf{u_1} = \mathbf{y} \cdot \mathbf{u_1} - (\frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}})\mathbf{u_1} \cdot u_1 = \mathbf{y} \cdot \mathbf{u_1} - \mathbf{y} \cdot \mathbf{u_1} = 0$$

  - Thus, $\mathbf{z}$ is orthogonal to $\mathbf{u_1}$. Similarly, $\mathbf{z}$ is orthogonal to each $\mathbf{u_j}$ in the basis for W.
  - Hence $\mathbf{z}$ is orthogonal to every vector in W. That is, $\mathbf{z}$ is in $W^{\perp}$

- **Proof for unique representation:**
  - To show that the decomposition in (1) is unique, suppose $\mathbf{y}$ can also be written as $\mathbf{y}$ be also written as $\mathbf{y} = \mathbf{y_1} + \mathbf{z_1}$, with $\mathbf{y_1}$ in W and $\mathbf{z_1}$ in $W^{\perp}$
  - Then, $\mathbf{y} + \mathbf{z} = \mathbf{y_1} + \mathbf{z_1}$ (since both sides equal $\mathbf{y}$), and so $\mathbf{y} - \mathbf{y_1} = \mathbf{z_1} - \mathbf{z}$
  - This equality shows that the vector $\mathbf{v} = \mathbf{y} - \mathbf{y_1}$ is in W and in $W^{\perp}$ (because $\mathbf{z_1}$ and $\mathbf{z}$ are both in $W^{\perp}$, and $W^{\perp}$ is a subspace).
  - Hence $\mathbf{v} \cdot \mathbf{v} = 0$ which shows that $\mathbf{v} = 0$.
  - This proves that $\mathbf{y} = \mathbf{y_1}$ and also $\mathbf{z_1} = \mathbf{z}$
  - The uniqueness of the decomposition (1) shows that the orthogonal projection $\hat{\mathbf{y_1}}$ depends only on W and not on the particular basis used in (2).

- **Example 1:**
  - Let $\mathbf{u_1} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
  - Observe that $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthogonal basis for $W = Span\{\mathbf{u_1}, \mathbf{u_2}\}$.
  - Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

- **Solution:**
  - The orthogonal projection of $\mathbf{y}$ onto W is

$$\begin{aligned} \hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \mathbf{u_1} + \frac{\mathbf{y} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}} \mathbf{u_2} \\ &= \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{15}{30} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} \end{aligned}$$

  - Also, $\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$
  - Theorem 8 ensures that $\mathbf{y} - \hat{\mathbf{y}}$ is in $W^{\perp}$

- (*solution continued:*)
  - To check the calculations, verify that $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to both $\mathbf{u_1}$ and $\mathbf{u_2}$ and hence to all of W
  - The desired decomposition of $\mathbf{y}$ is $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$

Properties of Orthogonal Projections

- If $\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$ be any orthogonal basis for W and if $\mathbf{y}$ happens to in W, then the formula for $proj_L\,\mathbf{y}$ is exactly the same as the representation of $\mathbf{y}$ given in Theorem 5 in Section 6.2. In this case, $proj_W\,\mathbf{y}$ =$\mathbf{y}$
- If $\mathbf{y}$ in $W = Span\{\mathbf{u_1}, \ldots, \mathbf{u_p}\}$, then $proj_W\,\mathbf{y} = \mathbf{y}$

The Best Approximation Theorem

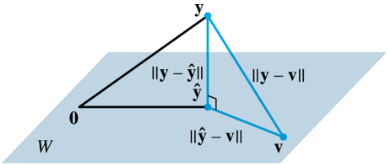
- **Theorem 9:** Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \tag{3}$$

  for all $\mathbf{v}$ in W distinct from $\hat{\mathbf{y}}$

- Remarks
  - The vector $\hat{\mathbf{y}}$ in Theorem 9 is called the best approximation to $\mathbf{y}$ by elements of W.
  - The distance from $\mathbf{y}$ to $\mathbf{v}$, given by $\|\mathbf{y} - \mathbf{v}\|$, can be regarded as the "error' ' of using $\mathbf{v}$ in place of $\mathbf{y}$.
  - Theorem 9 says that this error is minimized when $\mathbf{v} = \hat{\mathbf{y}}$
  - Inequality (3) leads to a new proof that $\hat{\mathbf{y}}$ does not depend on the particular orthogonal basis used to compute it.
  - If a different orthogonal basis for W were used to construct an orthogonal projection of $\mathbf{y}$, then this projection would also be the closest point in W to $\mathbf{y}$, namely, $\hat{\mathbf{y}}$.

- **Proof for the theorem:**
  - Take $\mathbf{v}$ in W distinct from $\hat{\mathbf{y}}$. See the following figure:



The orthogonal projection of **y** onto W is the closest point in W to **y**.

-
  - Then, $\hat{\mathbf{y}} - \mathbf{v}$ is in W. By the Orthogonal Decomposition Theorem, $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to W. In particular, $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to $\hat{\mathbf{y}} - \mathbf{v}$ (which is in W).
  - Since

$$\mathbf{y} - \mathbf{v} = (\mathbf{y} - \mathbf{y}) + (\mathbf{y} - \mathbf{v}),$$

the Pythagorean Theorem gives

$$\|\mathbf{y} - \mathbf{v}\|^2 = \|\mathbf{y} - \mathbf{y}\|^2 + \|\mathbf{y} - \mathbf{v}\|^2$$

  - (See the colored right triangle in the figure. The length of each side is labeled.)
  - Now $\|\mathbf{y} - \mathbf{v}\|^2 > 0$ because $\mathbf{y} - \mathbf{v} \neq 0$, and so inequality (3) follows immediately.

- **Example 4:** The distance from a point $\mathbf{y}$ in $\mathbb{R}^n$ to a subspace W is defined as the distance from $\mathbf{y}$ to the nearest point in W. Find the distance from $\mathbf{y}$ to $Span\{\mathbf{u_1}, \mathbf{u_2}\}$, where

$$y = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \mathbf{u_1} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

- **Solution :**
  - By the Best Approximation Theorem, the distance from $\mathbf{y}$ to W is $\|\mathbf{y} - \hat{\mathbf{y}}\|$, where $\hat{\mathbf{y}} = proj_W\, \mathbf{y}$.
  - Since $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthogonal basis for W,

$$\begin{aligned}
\hat{\mathbf{y}} &= \frac{15}{30}\mathbf{u_1} + \frac{-21}{6}\mathbf{u_2} = \frac{1}{2}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + -\frac{7}{2}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} \\
\mathbf{y} - \hat{\mathbf{y}} &= \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \\
\|\mathbf{y} - \hat{\mathbf{y}}\|^2 &= 3^2 + 6^2 = 45
\end{aligned}$$

  - The distance from $\mathbf{y}$ to W is $\sqrt{45} = 3\sqrt{5}$

- **Theorem 10:**
  - If $\{\mathbf{u_1} \ldots, \mathbf{u_p}\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$proj_W \, \mathbf{y} = (\mathbf{y} \cdot \mathbf{u_1})\mathbf{u_1} + (\mathbf{y} \cdot \mathbf{u_2})\mathbf{u_2} + \cdots + (\mathbf{y} \cdot \mathbf{u_p})\mathbf{u_p} \tag{4}$$

  - If $U = [\mathbf{u_1} \, \mathbf{u_2} \ldots \mathbf{u_p}]$, then for all $\mathbf{y}$ in $\mathbb{R}^n$,

$$proj_W \, \mathbf{y} = \mathbf{U}\mathbf{U}^T \mathbf{y} \tag{5}$$

- **Proof:**
  - Formula (4) follows immediately from (2) in Theorem 8.
  - Also, (4) shows that $proj_W \; \mathbf{y}$ is a linear combination of the columns of U using the weights $\mathbf{y} \cdot \mathbf{u_1}, \mathbf{y} \cdot \mathbf{u_2}, \ldots, \mathbf{y} \cdot \mathbf{u_p}$.
  - The weights can be written as $\mathbf{u_1}^T \mathbf{y}, \mathbf{u_2}^T \mathbf{y}, \ldots, \mathbf{u_p}^T \mathbf{y}$, showing that they are the entries in $U^T \mathbf{y}$ and justifying (5).

6.4 Gram-Schmidt process

Gram-Schmidt process

- **Theorem 11: The Gram-Schmidt Process**
  - Given a basis $\{\mathbf{x_1}, \dots, \mathbf{x_p}\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$
\begin{aligned}
\mathbf{v_1} &= \mathbf{x_1} \\
\mathbf{v_2} &= \mathbf{x_2} - \frac{\mathbf{x_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} \\
\mathbf{v_3} &= \mathbf{x_3} - \frac{\mathbf{x_3} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{x_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\mathbf{v_2} \\
\mathbf{v_p} &= \mathbf{x_p} - \frac{\mathbf{x_p} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} - \frac{\mathbf{x_p} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\mathbf{v_2} - \dots - \frac{\mathbf{x_p} \cdot \mathbf{v_{p-1}}}{\mathbf{v_{p-1}} \cdot \mathbf{v_{p-1}}}\mathbf{v_{p-1}}
\end{aligned}
$$

  - Then $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ is an orthogonal basis for W.
  - In addition **Span** $\{\mathbf{v_1}, \dots, \mathbf{v_k}\}$ = **Span** $\{\mathbf{x_1}, \dots, \mathbf{x_k}\}$ for $1 \le k \le p$.

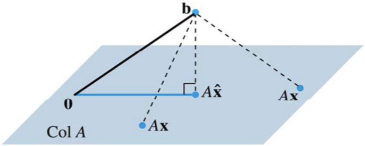
6.5 Least-Squares Problems

Least-Squares Problems

- **Definition:** If A is $m \times n$ and $\mathbf{b}$ is in $\mathbb{R}^m$, a **least-squares solution** of $A\mathbf{x} = \mathbf{b}$ is an $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$$

for all $\mathbf{x}$ in $\mathbb{R}^n$.



The vector $\mathbf{b}$ is closer to $A\hat{\mathbf{x}}$ than to $A\mathbf{x}$ for other $\mathbf{x}$.

- **Example 1**

- **Theorem 14:** Let A be an $m \times n$ matrix. The following statements are logically equivalent:
  - *a.* The equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for each $\mathbf{b}$ in $\mathbb{R}^m$.
  - *b.* The columns of A are linearly independent.
  - *c.* The matrix $A^T A$ is invertible.
  - *d.* The normal equation has a unique solution.

- **Example 4**

Suggested Exercises

- 6.5.2
- 6.8.1

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