ITM426, Final Exam, 2022 Fall

Solution and Grading

- ITM 426 Engineering Mathematics
- Dec 16, 2022
- Duration: 90 minutes
- Weights: 30%
- 5 Questions

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- Write legibly.
- Justification is necessary unless stated otherwise.
- Partial points are given only sparingly for the most problems because you are expected to 1) carry out proper sanity check and 2) correct your mistake by doing so.

1	25
2	10
3	20
4	15
5	30
Total	100

#1. Mark True or False. No justification is necessary. `[Each 5pt]`

- For a square matrix A, vectors in $Col\, A$ are orthogonal to vectors in $Nul\, A$.
- For a $m \times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space of A.
- Cholesky decomposition is always possible for a covariance matrix.
- Users must be able to evaluate the value of derivative function in order to use the Newton's method.
- In general, fixed point algorithm exhibits the faster convergence compared to Newton's method.

Difficulty: Easy
Amount of work: 25%
Suggested answer:

- False. Counterexample: $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
- True. See Theorem 6.1.3.
- True. Because a proper covariance matrix is always psd, which allows Cholesky decomposition as defined in the class note.
- True. See the lecture note.
- False. In general, Newton's method is faster.

#2. Suppose a vector $\mathbf{y}$ is orthogonal to vectors $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to the vector $\mathbf{u}+\mathbf{v}$. `[10pt]`

Difficulty: Easy
Amount of work: 10%
Suggested answer: $\mathbf{y} \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{y} \cdot \mathbf{u} + \mathbf{y} \cdot \mathbf{v} = 0 + 0 = 0$. Thus, $\mathbf{y}$ is orthogonal to $\mathbf{u}+\mathbf{v}$.

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#3. What is the distance between the point $\mathbf{y}$ and the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$? `[20pt]`

$$\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Difficulty: Medium
Amount of work: 20%
Suggested answer:

Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. The Best Approximation Theorem says that $\hat{\mathbf{y}}$, which is the orthogonal projection of $\mathbf{y}$ onto $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, is the closest point to $\mathbf{y}$ in W. This vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{1}{2}\mathbf{v}_1 + \frac{3}{2}\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Thus, the distance between the point and the plane is $\sqrt{2^2 + 4^2 + 2^2} = \sqrt{24}$.

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#4. Find an orthogonal basis for the column space of the following matrix. [15pt]

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

Difficulty: Medium
Amount of work: 20%
Suggested answer:

Call the columns of the matrix $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ and perform the Gram-Schmidt process on these vectors:

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - (-2)\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{x}_3 - \frac{3}{2}\mathbf{v}_1 - \left(-\frac{1}{2}\right)\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$.

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#5. From an experiment, four observations of two variables x and y are collected: $(2,3),(3,2),(5,1)$, and $(6,0)$, answer the following.

(a) Construct a sample covariance matrix. [15pt]

(b) Find the equation $y = \beta_0 + \beta_1 x$ of the least-square line. [15pt]

Difficulty: Medium-Hard
Amount of work: 30%
Suggested answer:
(a) $X = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 3 & 2 & 1 & 0 \end{bmatrix}$, $M = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 1.5 & 1.5 & 1.5 & 1.5 \end{bmatrix}$, $S = (X-M)(X-M)^t/(4-1) = (1/3)\begin{bmatrix} 10 & -7 \\ -7 & 5 \end{bmatrix}$.
(b)

The design matrix X and the observation vector $\mathbf{y}$ are $X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, and one can compute

$X^T X = \begin{bmatrix} 4 & 16 \\ 16 & 74 \end{bmatrix}, X^T\mathbf{y} = \begin{bmatrix} 6 \\ 17 \end{bmatrix}, \hat{\boldsymbol{\beta}} = (X^T X)^{-1} X^T \mathbf{y} = \begin{bmatrix} 4.3 \\ -.7 \end{bmatrix}$. The least-squares line $y = \beta_0 + \beta_1 x$ is thus $y = 4.3 - .7x$.

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Write your name before detaching this page. Your Name: ______________________________