# ITM426, Quiz 3, 2024 Fall

Solution and Grading

- ITM 426 Engineering Mathematics 24F
- Nov 22, 2024
- Duration: 90 minutes
- Weights: 25%
- 5 Questions

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- Write legibly.
- Justification is necessary unless stated otherwise.
- Partial points are given only sparingly for the most problems because you are expected to 1) carry out proper sanity check and 2) correct your mistake by doing so.

| | |
|---|---|
| 1 | 10 |
| 2 | 10 |
| 3 | 10 |
| 4 | 10 |
| 5 | 10 |
| Total | 50 |

#1. Mark True or False for each statement. No justification is necessary. `[Each 5pt]`

- A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$. (**TRUE** / **FALSE**)
- The only three-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself. (**TRUE** / **FALSE**)

**Difficulty**: Medium
**Amount of work**: 10%
**Solution**:

- False. The plane must pass through the origin.
- True. See Example 4 in Section 4.5.

#2. Prove that if $A^2$ is the zero matrix, then the only eigenvalue of $A$ is 0.`[10pt]`

**Difficulty**: Medium
**Amount of work**: 15%
**Suggested answer**:
By definition, $\lambda$ is an eigenvalue if a nonzero $\mathbf{x}$ exists for $A\mathbf{x} = \lambda\mathbf{x}$. Consider $0 = A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$. Thus, $\lambda^2\mathbf{x} = 0$. Only possible value of $\lambda$ that works for nonzero $\mathbf{x}$ is 0
**Grading scheme**: No partial point. This problem does not assume $A$ as a 2 by 2 matrix.

#3. This question concerns a series of tri-diagonal matrix[1]. The matrices have 3's on the main diagonal, 2's on the upper diagonal, and 1's on the lower diagonal. That is, the series goes as follows:

$$A_1 = \begin{bmatrix} 3 \end{bmatrix},\ A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix},\ A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix},\ A_4 = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix},\ A_5 = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}, \cdots$$

Let $D_n$ be the determinant of $A_n$. Find the value of $a$ and $b$ that completes the recursive formula for $D_n$ of the following. Use cofactor expansion and justify your answer. `[10pt]`

$$D_n = aD_{n-1} + bD_{n-2}$$

**Difficulty**: Medium
**Amount of work**: 20%
**Solution**: $D_n = (3)D_{n-1} + (-2)D_{n-2}$. Because $A_n$ can be viewed as

$$A_n = \begin{bmatrix} 3 & 2 & 0 & \cdots & \cdots & 0 \\ 1 & & & & & \\ 0 & & & & & \\ \cdots & & & A_{n-1} & & \\ \cdots & & & & & \\ 0 & & & & & \end{bmatrix} \text{ or } A_n = \begin{bmatrix} 3 & 2 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \cdot & 2 & 0 & \cdots & \cdots & 0 \\ 0 & \cdot & & & & & \\ \cdots & \cdot & & & A_{n-2} & & \\ \cdots & \cdot & & & & & \\ 0 & \cdot & & & & & \end{bmatrix},$$

it follows that, by cofactor expansion, $|A_n| = 3|A_{n-1}| - 2\,(1 \cdot |A_{n-2}| + 2 \cdot 0)$.
**Grading scheme**: No partial credit.

[1] A tri-diagonal matrix is a band matrix that has nonzero elements only on the main diagonal, the lower diagonal (the first diagonal below the main diagonal), and the upper diagonal (the first diagonal above the main diagonal)

#4. First, use the diagonalization method. Then, using the result, find $A^{100}$ `[10pt]`

$$A = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}$$

**Difficulty**: Medium
**Amount of work**: 20%
**Solution**: Proper diagonalization will lead to $A = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0.2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -5 \end{bmatrix}^{-1}$.
Considering $0.2^{100} \approx 0$, $A^{100} = \begin{bmatrix} 1 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0.2^{100} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} 5/8 & 3/8 \\ 5/8 & 3/8 \end{bmatrix}$
**Grading scheme**: 5pts for proper diagonalization (sanity check is desired.) 5pts for the rest of the work. If proper diagonalization is not carried, then 0 pts.

#5. Consider a $m$ by $n$ matrix $A$ with the following properties.

- $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has no solutions.
- $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has exactly one solution.

Provide all possible combinations for $m$, $n$, and the rank $r$ of $A$. `[10pt]`

**Difficulty**: Medium-Hard
**Amount of work**: 20%
**Solution**:

- First off, $m = 3$ because $A\mathbf{x}$ is a three dimensional vector.
- From the first property, the column space of $A$ cannot span the whole $\mathbb{R}^3$. In other words, the dimension of column space (which is equal to the rank $r$) is less than 3.
- From the second property, number of pivots (which is equal to rank $r$) is the same as the number of variables in $\mathbf{x}$ (which is equal to the rank $n$). In other words, $r = n$.
- Since we have $r = n < m = 3$, all possible combinations are $(m = 3, n = r = 2)$ and $(m = 3, n = r = 1)$. Examples of $A$ are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, and many more.

**Grading scheme**: No partial credit

(detachable)