Lecture 06 - More Distributions

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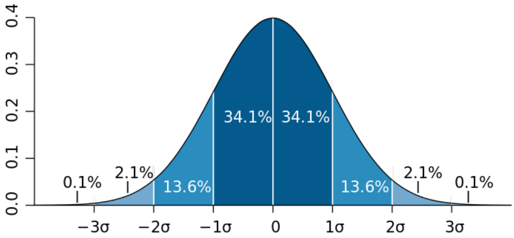
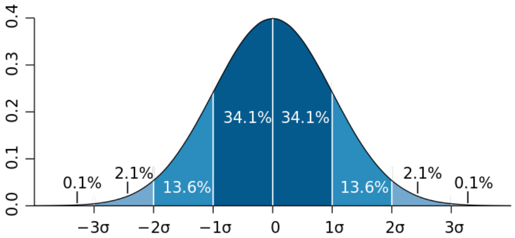
I. Normal Distribution

정규 분포

- 평균과 분산에 의해서 결정되는 bell-shaped pdf를 가진 분포
- 확률변수 X가 정규분포를 따르고 평균과 분산이 각각 μ와 σ^2라고 한다면 $X \sim N(\mu, \sigma^2)$으로 표기한다.



- The values less than one standard deviation away from the mean account for 68.27% of the set.
- Two standard deviations from the mean account for 95.45%.
- Three standard deviations account for 99.73%.



- For $X \sim N(\mu, \sigma^2)$,
  - $\mathbb{P}(\mu - \sigma \leq X \leq \mu + \sigma) = 68.3\%$
  - $\mathbb{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 95.5\%$
  - $\mathbb{P}(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = 99.73\%$

정규 분포의 모수 (parameters)

- 평균: μ, location parameter
- 표준편차: σ: scale parameter

- normal 분포를 따르는 변수는
  - 상수를 더해도 normal: $X \sim N(\mu, \sigma^2) \Rightarrow X + b \sim N(\mu + b, \sigma^2)$
  - 상수를 곱해도 normal: $X \sim N(\mu, \sigma^2) \Rightarrow aX \sim N(a\mu, a^2\sigma^2)$
  - 종합하면 $X \sim N(\mu, \sigma) \Rightarrow aX + b \sim N(a\mu + b, a^2\sigma^2)$

Exercise 1

For a random variable $X \sim N(5, 3^2)$, identify the distribution for the following random variables.

- $2X \sim$
- $X + 3 \sim$
- $2X + 7 \sim$

표준 정규 분포 (Standard···)

- 정규분포 중에서 평균이 0, 분산이 1인 경우를 표준 정규 분포라고 한다. $(N(0,1))$
- 표준정규분포를 따르는 변수를 Z라고 표기한다. 즉, $Z \sim N(0,1)$.
- 앞에서 다룬 정규분포의 성질 $X \sim N(\mu, \sigma^2) \Rightarrow aX + b \sim N(a\mu + b, a^2\sigma^2)$을 이용하여
  - 표준정규분포를 이용해서 다른 정규분포를 얼마든지 만들수 있다. (Standard normal distribution is the building block for all normal distributions)
  - 마찬가지로 표준이 아닌 정규분포를 이용해서 표준정규분포를 얼마든지 만들수 있다.

Exercise 2

For a random variable $X \sim N(5, 3^2)$, $aX + b$ is a standard normal distribution. What is a and b?

Exercise 3

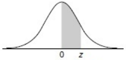
With a standard normal variable Z, we have $aZ + b \sim N(10, 3^2)$. What is a and b?

표준 정규 분포표

Table AIV.2 Standard Norms Table

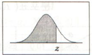
Area between 0 and z

P(0<Z<1.55)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998

3. 확률분포표



표준정규분포표

$(\Pr(Z \le z) = \Phi(z),\ Z \sim N(0,1))$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Exercise 4

앞 페이지의 표준정규분포표를 사용하여 다음의 수치들을 계산하라.

1. $\mathbb{P}(Z \leq 2.0)$
2. $\mathbb{P}(Z \leq 1.96)$
3. $\mathbb{P}(Z \leq 2.33)$
4. $X \sim N(50, 10^2)$, $\mathbb{P}(X \leq 66.5)$
5. $X \sim N(50, 10^2)$, $\mathbb{P}(30 \leq X)$
6. $X \sim N(70, 10^2)$, $\mathbb{P}(X \geq 65)$
7. $X \sim N(\mu, \sigma^2)$, $\mathbb{P}(\mu - 3\sigma \leq X \leq \mu + 3\sigma)$

Exercise 5

초등학생의 몸무게는 정규분포를 따르고 평균이 42, 표준 편차가 5라고 한다. 무작위로 뽑은 학생의 몸무게가 40과 54사이일 확률은 얼마인가?

II. Binomial Approximation

Review

- 베르누이 분포 (Bernouli dist.)
  - Binary outcome의 random 시행을 1회 실시한것
  - $X \sim Ber(p)$, where p is the probability of success.
- 이항 분포 (Binomial dist.)
  - Binary outcome의 random 시행을 n회 실시한것중 성공한 횟수
  - $X \sim bin(n, p)$, where n is the number of trials and p is the success prob.
  - If $B_i \sim Ber(p)$, then $X = \sum_{i=1}^{n} B_i \sim bin(n, p)$.
  - $bin(n, p)$의 평균은 np, 분산은 $np(1-p)$이다.

Experiment in R

- 동전을 4번 던지고, $p = 0.6$일 때, 앞면이$\cdots$
  1. 0번 나올 확률은? $X \sim bin(4, 0.6)$, $\mathbb{P}(X = 0) = {}_4C_0(0.6)^0(0.4)^4$
  2. 1번 나올 확률은? $X \sim bin(4, 0.6)$, $\mathbb{P}(X = 1) = {}_4C_1(0.6)^1(0.4)^3$
  3. 2번 나올 확률은? $X \sim bin(4, 0.6)$, $\mathbb{P}(X = 2) = {}_4C_2(0.6)^2(0.4)^2$
  4. 3번 나올 확률은? $X \sim bin(4, 0.6)$, $\mathbb{P}(X = 3) = {}_4C_3(0.6)^3(0.4)^1$
  5. 4번 나올 확률은? $X \sim bin(4, 0.6)$, $\mathbb{P}(X = 4) = {}_4C_4(0.6)^4(0.4)^0$

- 하나씩 해보려면$\cdots$

```r
dbinom(0, 4, 0.6) # 1
```

```
## [1] 0.0256
```

```r
dbinom(1, 4, 0.6) # 2
```

```
## [1] 0.1536
```

```r
dbinom(2, 4, 0.6) # 3
```

```
## [1] 0.3456
```

```r
dbinom(3, 4, 0.6) # 4
```

```
## [1] 0.3456
```

```r
dbinom(4, 4, 0.6) # 5
```

```
## [1] 0.1296
```

- 한 번에 하려면?!

```r
dbinom(0:4, 4, 0.6) # pmf
```

```
## [1] 0.0256 0.1536 0.3456 0.3456 0.1296
```

```r
pbinom(0:4, 4, 0.6) # cdf
```

```
## [1] 0.0256 0.1792 0.5248 0.8704 1.0000
```

R documentation on binomial (1)

- `dbinom`은 binomial distribution의 pmf를 알려주는 R의 함수이다. 아래 명령 중 하나를 시행하면 다음의 스크린 샷이 등장한다.

```
help(dbinom)
? dbinom
```

Binomial

From stats v3.3
by R-core R-core@R-project.org

99.99th
Percentile

The Binomial Distribution

Density, distribution function, quantile function and random generation for the binomial distribution with parameters `size` and `prob` .

This is conventionally interpreted as the number of successes in `size` trials.

Keywords distribution

Usage

```
dbinom(x, size, prob, log = FALSE)
pbinom(q, size, prob, lower.tail = TRUE, log.p = FALSE)
qbinom(p, size, prob, lower.tail = TRUE, log.p = FALSE)
rbinom(n, size, prob)
```

R documentation on binomial (2)

Arguments

x, q	vector of quantiles.
p	vector of probabilities.
n	number of observations. If `length(n) > 1`, the length is taken to be the number required.
size	number of trials (zero or more).
prob	probability of success on each trial.
log, log.p	logical; if TRUE, probabilities p are given as log(p).
lower.tail	logical; if TRUE (default), probabilities are $P[X \le x]$, otherwise, $P[X > x]$.

R documentation on binomial (3)

Details

The binomial distribution with `size` $= n$ and `prob` $= p$ has density

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, \ldots, n$. Note that binomial *coefficients* can be computed by `choose` in R.

If an element of `x` is not integer, the result of `dbinom` is zero, with a warning.

$p(x)$ is computed using Loader's algorithm, see the reference below.

The quantile is defined as the smallest value x such that $F(x) \geq p$, where F is the distribution function.

Value

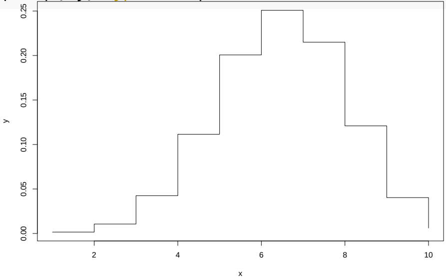
`dbinom` gives the density, `pbinom` gives the distribution function, `qbinom` gives the quantile function and `rbinom` generates random deviates.

If `size` is not an integer, `NaN` is returned. The length of the result is determined by `n` for `rbinom`, and is the maximum of the lengths of the numerical arguments for the other functions. The numerical arguments other than `n` are recycled to the length of the result. Only the first elements of the logical arguments are used.

R을 이용한 이항분포의 실험 - 반복횟수를 늘리면서

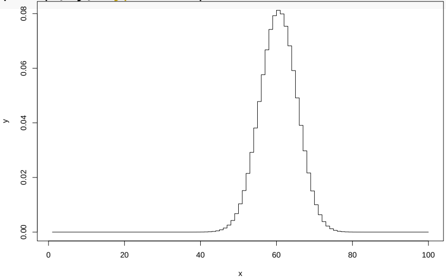
- 10회 시행

```
x <- 1:10
y <- dbinom(x, size=10, prob=0.6)
plot(x, y, type = "s")
```



- 100회 시행

```
x <- 1:100
y <- dbinom(x, size=100, prob=0.6)
plot(x, y, type = "s")
```



- Looks similar to something⋯?

- 이항분포가 정규분포와 비슷한지 알아보기 위해 정규분포의 pdf를 같이 그려보자.
- 이항분포 $bin(100, 0.6)$과 비교해야할 정규분포의 파라미터는 무엇인가?

- Google에서 "R normal distribution"로 검색하면 아래 문서를 확인할 수 있다. 이를 이용해 정규분포의 그래프도 그려본다.

Normal {stats} R Documentation

The Normal Distribution

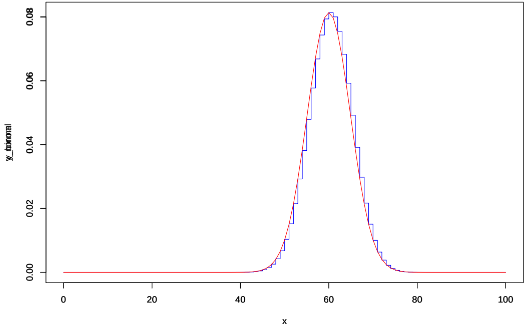
Description

Density, distribution function, quantile function and random generation for the normal distribution with mean equal to `mean` and standard deviation equal to `sd`.

Usage

```
dnorm(x, mean = 0, sd = 1, log = FALSE)
pnorm(q, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE)
qnorm(p, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE)
rnorm(n, mean = 0, sd = 1)
```

```
x <- 0:100
y_binom <- dbinom(x, size=100, prob=0.6)
y_normal <- dnorm(x, mean=60, sd=sqrt(24))
plot(x, y_binom, type = "s", col="blue") # draw in stair step
par(new=TRUE)
plot(x, y_normal, type = "l", col="red") # draw in line
```



- 이항분포와 정규분포가 매우 비슷하다!

Binomial Approximation (to Normal)

- 일반적으로 $bin(n, p)$는 다음 조건을 만족하면 정규분포 $N(np, npq)$에 근사함이 알려져있다.
  - 조건 1: n이 클 수록
  - 조건 2: p가 0.5에 가까울 수록

Exercise 6

Fair coin ($p = 0.5$)를 100번 던져서 60회 이상의 앞면이 나올 확률을 정규분포를 이용하여 답하라

Exercise 7

이항분포의 정규분포에의 근사를 이용하여 아래의 질문에 답하라.

1. Fair coin ($p = 0.5$)를 1000번 던져서 600회 이상의 앞면이 나올 확률은?
2. Fair coin ($p = 0.5$)를 1000번 던져서 몇 번의 앞면이 나와야 95%의 확신을 가지고 “이 동전은 fair한 동전이 아니다!”라고 이야기 할 수 있는가?

Some preview

- Fair coin ($p = 0.5$)를 1000번 던졌을 때에 앞면이 나오는 횟수에 대한 95% **신뢰구간**은 무엇인가?

(Optional) Binomial Approximation to Poisson

- [large n and p close to $1/2$]이면, 정규분포에 근사한다.
- [large n and p very small] 이면 어떻게 되는가?
- Example
  - 콜센터로 전화오는 횟수
  - 가게에 손님이 찾아오는 횟수
- 파라미터는 어떻게 되는가?
- Poisson distribution의 특징
  - 평균과 분산이 같다! (λ)
  - Heavy-tailed
  - Queuing model