Notes on Pre-college Linear Algebra

Sim, Min Kyu, Ph.D.

Preface

This note[1] is intended to bridge the gap between high school math and college-level linear algebra. Intended readers are college students who have not learned vector algebra or matrix algebra in their high school years. But I expect readers to be comfortable with solving a system of linear equations such as the following:

Go ahead and please find the value of x and y that solve the following system.

연립방정식

$$
\begin{cases}
2x + 3y - 13 = 0 & (1) \\
4x + 2y - 14 = 0 & (2)
\end{cases}
$$

이원 일차

$$4x + 6y - 26 = 0 \quad (1)'$$

$$-\underline{\left|\, 4x + 2y - 14 = 0 \right.}$$

$$4y - 12 = 0$$

$$\begin{pmatrix} x = 2 \\ y = 3 \end{pmatrix}$$

I believe most of college students regardless of their major can find the solution. Even more, no matter what approach or method one used to solve the system of equations, she or he must be aware of another alternative approach or method to find the same solution. Linear algebra begins with solving such a simple system of linear equations with other introduced concepts (that you might not be familiar with yet) such as vector, space, matrix, determinant, spanning, and so on.

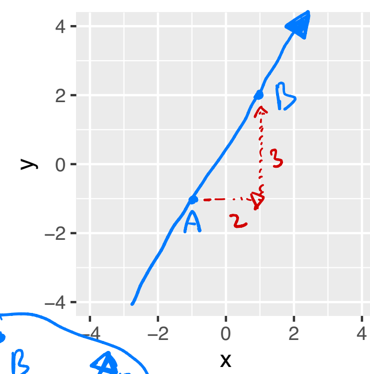
The beauty of mathematics, in my humble opinion, lies in (but not limited to) 1) expressing things in a simple and logical way, 2) then seeing the same thing in a different way, and 3) then finding new things in the world on one's own. Linear algebra is very important subject in proceeding many quantitative disciplines and applications including (in a random order) linear regression, multivariate calculus, machine learning, deep learning, probability theory, mathematical statistics, and so on. When it comes to writing a simpler and faster codes, between programmers who possess the vector/matrix perspective and programmers who do not, the difference is day and night - to this I am not exaggerating.

[1] This note is first written in August 2019 for ITM426-Engineering Math. This note is lastly updated in August 2020.

This note introduces basic notions of linear/matrix algebra so that students who finished studying this note should be well prepared for more serious version of linear algebra. In writing this note, Korean high school textbook "Advanced Mathematics I (고급수학)" is heavily referenced.

For any typos, error, and suggestions, feel free to email me at `mksim@seoultech.ac.kr`.

1. Vector Space



We shall start with some definitions on vectors. Using the grid on the right side, follow the instruction.

- Mark points of $A = (-1, -1)$ and $B = (1, 2)$.
- Then, draw line passing through A and B.
- The line is called a *vector*, and expressed as $\overrightarrow{AB}$.

B A B A

$\overrightarrow{AB}$ $\overrightarrow{BA}$

- Q. Are $\overrightarrow{AB}$ and $\overrightarrow{BA}$ different things?
- A. Yes, we treat them differently. A vector contains a directional information.

- Q. How would you quantify the vector $\overrightarrow{AB}$?
- A. $\overrightarrow{AB} =$ (2, 3) = B − A = (1, 2) − (−1, −1) = (2, 3)

  ending starting

- Q. Is it possible to define the single point A as a vector?
- A. Yes, it is possible by using the origin $(0, 0)$ as a starting point.

Y X A

(vector A) = (vector OA) where O is origin

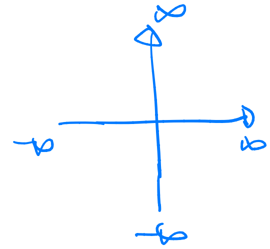
- Q. $\boxed{A = (-1, 1) \text{ is called a } \textit{two-dimensional} \text{ vector}}$ for an obvious reason, then what would you call $C = (-3, 2, 1)$?
- A. A *three dimensional* vector, or, *a vector with three dimension*.

Thus, a vector (x_1, x_2) is called a two-dimensional vector, and a vector (x_1, x_2, x_3) is called a three-dimensional vector. In order to specify a vector having several components, standard notation for a vector uses a bold-faced letter such as **x**, **y**, or **z**. For example, $\mathbf{x} = (x_1, x_2)$ is a two-dimensional vector, where x_1 and x_2 are numbers.

bold face $\mathbf{x}$ vector

$\mathbf{x}$ vs x

bold (vector) regular (numbers)



For the most of the time, we are concerned with real-numbered (실수) vector. That is, above mentioned x_1, x_2, and x_3 are all real numbers. That's why the 2D plane was a good space to express a vector such as $A = (-1, 1)$. The 2D plane is called the *2-dimensional vector space* where each dimension represents a space of real number from $-\infty$ to ∞.

Problem 1 *Define the 3-dimensional vector space (Hint: Use the above statement).*

Problem 2 *For a 3-dimensional vector* $\boxed{\mathbf{x} = (1, 1, 3)}$ *and* $\boxed{\mathbf{y} = (-1, 0, 2)}$, *find the followings:*

1) $2\mathbf{x} + \mathbf{y}$ $= 2 \cdot (1, 1, 3) + 1 \cdot (-1, 0, 2) = (2, 2, 6) + (-1, 0, 2) = (1, 2, 8).$

2) $3\mathbf{y} + 2\mathbf{x}$

Problem 3 *For a 3-dimensional vector* $\mathbf{x} = (2, 1, 3)$ *and* $\mathbf{y} = (1, 0, 1)$, *find a vector* $\mathbf{z}$ *that satisfies the equation:* $3\mathbf{x} + 2\mathbf{z} = \mathbf{x} + 3\mathbf{y}$

2. Linear Independence and Dependence

Let $\mathbf{e_1} = (1,0)$ and $\mathbf{e_2} = (0,1)$, then it is obvious that $\boxed{\mathbf{x} = (3,2)}$ can be also written as $\mathbf{x} = 3\mathbf{e_1} + 2\mathbf{e_2}$. Since the vector $\mathbf{x}$ was expressed as a linear formula containing $\mathbf{e_1}$ and $\mathbf{e_2}$, we formally say the following:

$(3.2) = 3(1.0) + 2(0.1)$

$x = 3e_1 + 2e_2$

linear combination.

The vector $\mathbf{x}$ can be expressed as a linear combination of $\mathbf{e_1}$ and $\mathbf{e_2}$

Would you agree that the vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ in a 2-dimensional vector space are worth enough to have specialized name for them? The $\mathbf{e_1} = (1,0)$ and $\mathbf{e_2} = (0,1)$ are called *unit vectors* in 2-dimensional vector space. Likewise, $\mathbf{e_1} = (1,0,0)$, $\mathbf{e_2} = (0,1,0)$, and $\mathbf{e_3} = (0,0,1)$ are called *unit vectors* in 3-dimensional vector space.

Problem 4 *Let* $\mathbf{x} = (3,4)$, $\mathbf{y} = (1,1)$, *and* $\mathbf{z} = (2,7)$. *Find real numbers* a_1 *and* a_2 *that solves the following equation.* $\boxed{\mathbf{z} = a_1\mathbf{x} + a_2\mathbf{y}.}$

"The vector z can be expressed as a linear combination of vector x and vector y"

In the above problem, were you able to identify the real numbers a_1 and a_2? If so, then we say "$\mathbf{z}$ can be expressed as a linear combination of $\mathbf{x}$ and $\mathbf{y}$". Linear combination is formally defined below.

You will hear "expressed as a linear combination of···" a lot of times in this course.

Definition 1 *For a set of k vectors* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ *and a set of real numbers* $a_1, a_2, \ldots, a_k$, $\boxed{a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_k\mathbf{v_k}}$ *is called a* ***linear combination*** *of vectors* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$.

Problem 5 *In the above problem, 1) Can* $\mathbf{x}$ *be expressed as a linear combination of* $\mathbf{y}$ *and* $\mathbf{z}$*? 2) Can* $\mathbf{y}$ *can be expressed as a linear combination of* $\mathbf{z}$ *and* $\mathbf{x}$*?*

$x = a_1 y + a_2 z$ and find a_1 and a_2 if a_1 and a_2 are real numbers. then TRUE.

The answers are both yes. Thus, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors with relationship such that any one of them can be expressed as a linear combination of the other two vectors. We call this property as $\boxed{\textit{linear dependence}}$. In this case, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are *linearly dependent*.

Problem 6 *Let's consider vectors of* $\mathbf{x} = (1, 1)$ *and* $\mathbf{y} = (1, 2)$. *Are they linearly dependent?*

$x = a_1 y \qquad y = a_1 x$

The answer is no. Since 1) $\mathbf{x}$ cannot be expressed as linear combination of $\mathbf{y}$ and 2) $\mathbf{y}$ cannot be expressed as linear combination of $\mathbf{x}$, $\mathbf{x}$ and $\mathbf{y}$ are *linearly independent*. Formal definition is given as below.

Definition 2 *Consider a set of k vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$. If any vector cannot be expressed as a linear combination of the other $k-1$ vectors, then we say the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are* ***linearly independent***.

Definition 3 *If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are* ***not linearly independent***, *we say they are* ***linearly dependent***. *That is, some vector among the k vectors can be expressed as a linear combination of the other vectors.*

Problem 7 *Investigate whether the following sets of vectors are linearly independent or dependent.*

Consider $\boxed{(2,-1) = a_1(2,5) + (3,1)}$

1) $(1, 2)$, $(2, 5)$
2) $\boxed{(2, -1), (2, 5), (3, 1)}$
3) $(1, 1, 0)$, $(2, 3, 4)$
4) $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$
5) $(1, 1, 0)$, $(2, 3, 4)$, $(3, 0, 2)$

If you find a_1, a_2 that are real numbers, lin dep.
" cannot find a_1, a_2 " " , lin. indep.

(LHS : 3 by 2, RHS : 2 by 1)

necessary & sufficient

To answer the above question, the following theorem may be helpful.

Theorem 1 *Consider a set of k vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$, the vectors are linearly independent $\boxed{\text{if and only if}}$ the only real numbers $a_1, a_2, \ldots, a_k$ that satisfies $a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_k\mathbf{v_k} = 0$ is $a_1 = a_2 = \ldots = a_k = 0$*

That is, to investigate the linear indepdence/dependence of set of vectors, you set the linear combination of the vectors as zero. If the only solution to the equation is all zeros, then the vectors are linearly independent. Otherwise, they are linearly dependent. This theorem is a handy way to tell linear dependence/independence and this theorem is the one of the most important theorems throughout this course.

$a_1(2,-1) + a_2(2,5) + a_3(3,1) = (0,0)$

$\Downarrow$

$2a_1 + 2a_2 + 3a_3 = 0$ — ①

$-a_1 + 5a_2 + a_3 = 0$ — ②

$-2a_1 + 10a_2 + 2a_3 = 0$ — ② × 2

$0 + 12a_2 + 5a_3 = 0$ ① + ② × 2

$\boxed{a_2 = -\frac{5}{12}a_3}$

$-2a_1 - \frac{25}{6}a_3 + 2a_3 = 0$

$-2a_1 - \frac{13}{6}a_3 = 0$

$\boxed{a_1 = -\frac{13}{12}a_3}$

$-\frac{26}{12}a_3 - \frac{10}{12}a_3 + 3a_3 = 0$

$a_3 = 1$, $a_1 = -\frac{13}{12}$, $a_2 = -\frac{5}{12}$

$-\frac{13}{12}(2,1) - \frac{5}{12}(2,5) + 1(3,1) = 0$

$\therefore$ (2,1), (2,5), (3,1) are lin. dep.

Problem 8 *Investigate whether the following sets of vectors are linearly independent or dependent.*

there are real numbers

1) $(2,0,1)$, $(1,3,-1)$
2) $(2,0,1)$, $(4,0,2)$
3) $(4,1,0)$, $(0,2,-1)$, $(3,2,0)$
4) $(1,1,0)$, $(2,3,4)$, $(0,0,0)$
5) $(2,3,0)$, $(0,2,-1)$, $(4,8,-1)$

P9. pf) Let the arbitrary vector (x,y). there exist real numbers a_1 and a_2 such that

$(x,y) = a_1(1,2) + a_2(2,5)$

$\begin{cases} x = a_1 + 2a_2 \\ y = 2a_1 + 5a_2 \end{cases}$

$\Rightarrow \begin{cases} 2x = 2a_1 + 4a_2 \\ y = 2a_1 + 5a_2 \end{cases}$

$\Rightarrow \boxed{y - 2x = a_2}$, $x = a_1 + 2y - 4x$, $\boxed{a_1 = 5x - 2y}$

$(x,y) = \underbrace{\overbrace{(5x-2y)}^{\text{a real number}}(1,2) + \overbrace{(y-2x)}^{\text{a real number}}(2,5)}_{\text{lin. com. of } (1,2) \,\&\, (2,5)}$

Problem 9 *Consider two vectors $v_1 = (1,2)$ and $v_2 = (2,5)$ in a 2-dimensional space. See if the following statement is true.*

$(3, 9) = a_1(1,2) + a_2(2,5)$

$(—, —) =$

> *Any arbitrary 2-dimensional vector can be expressed as a linear combination of v_1 and v_2.*

The statement is indeed true. By linearly combining v_1 and v_2, you can express any arbitrary vector in a 2-dimensional space. Here comes other definitions regarding this. Consider three vectors $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,1,1)$ in a 3-dimensional space. Following statement is again true.

prove.

> *Any 3-dimensional vector can be expressed as a linear combination of v_1, v_2, and v_3.*

Definition 4 *If any vector v in two-dimensional space can be expressed as a linear combination of two-dimensional vectors v_1 and v_2, we call the set of vectors $\{v_1, v_2\}$ as a **basis** of two-dimensional vector space. Also, we call v_1 and v_2 as **basis vectors.** Also, we say v_1 and v_2 **span** the two-dimensional space.*

Problem 10 *Empedocles (B.C. 494-434) was a Greek philosopher who established the four elements theory. The theory*[2] *claims that all the structures in the world are made of four elements, "roots" - fire, air, water, earth. Find the analogy between the four elements theory and the statement in the above problem and fill in the table below.*

[2] https://en.wikipedia.org/wiki/Empedocles

Four elements theory	Vector algebra
roots	v_1 and v_2
any structure	any arbitrary vector
forming	expressing as a linear combination

Now, let's expand the above definition to n-dimensional vector space.

Definition 5 *If any vector v in n-dimensional space can be expressed as a linear combination of n-dimensional vectors v_1, v_2, ..., and v_n, we call the set of vectors $\{v_1, v_2, ..., v_n\}$*

*as a **basis** of n-dimensional vector space. Also, we call each vector as a **basis vector**. Also, we say v_1, v_2, ..., and v_n **span** the n-dimensional space.*

Problem 11 *Suppose $\{v_1, v_2, v_3\}$ is a basis of 3-dimensional vector space. Show that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is also a basis of 3-dimensional vector space.*

Problem 12 *Show that $\{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of 3-dimensional vector space.*

P11 Because $\{v_1, v_2, v_3\}$ is a basis vector, an arbitrary vector v can be expressed as a lin. com. of v_1, v_2, v_3

$$v = a_1v_1 + a_2v_2 + a_3v_3, \quad a_1, a_2, a_3 \in \mathbb{R}$$

$$= b_1v_1 + b_2(v_1+v_2) + b_3(v_1+v_2+v_3)$$

$$\begin{cases} a_1 = b_1 + b_2 + b_3 \\ a_2 = b_2 + b_3 \\ a_3 = b_3 \end{cases} \rightarrow \quad \begin{aligned} b_3 &= a_3 \\ b_2 &= a_2 - a_3 \\ b_1 &= a_1 - (a_2 - a_3) - a_3 \\ &= a_1 - a_2 \end{aligned}$$

the arbitrary vector $v = b_1v_1 + b_2(v_1+v_2) + b_3(v_1+v_2+v_3)$. where $b_1, b_2, b_3 \in \mathbb{R}$

thus, $\{v_1, v_1+v_2, v_1+v_2+v_3\}$ is a basis.

$$v = (a_1 - a_2)v_1 + (a_2 - a_3)(v_1+v_2) + a_3(v_1+v_2+v_3)$$

$\boxed{\text{End of Proof}}$

3. Matrix

$W = (2, 3)$

$W = (\cancel{2, 3, 4})$

We have seen that a vector is a collection of multiple numbers in an 1-dimensional way. Then, would there be an entity that collect multiple vectors? If you collect multiple vectors, what would it look like? With a set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, one can think of collection of these vector such as following.

$(2,3,4,5)$ $(1,1,1,1)$ $(0,0,1,1)$

a number 0D

a vector 1D

$\boxed{\text{a matrix 2D (2D array)}}$

3D

"nD array"

4D

⋮

nD

"tensor"

$$A = \begin{bmatrix} — v_1 — \\ — v_2 — \\ — v_3 — \end{bmatrix}$$

$$A = \begin{bmatrix} | & | & | \\ \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \\ | & | & | \end{bmatrix}$$

A *matrix* is defined as a collection of vectors, shaped as a rectangular. The matrix A above has three column vectors, $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$. Remind that the numbers of elements in $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_3}$ must be same in order to form a rectangular shape. All column vectors in a matrix must have the same number of elements. It is a convention that a matrix is primarily understood as a collection of column vectors (not as a collection of row vectors). A vector is generally written as a column vector, so to speak.

Problem 13 *For a 3-dimensional vector* $\mathbf{v_1} = (2, 1, 3)$ *and* $\mathbf{v_2} = (1, 0, 1)$, *construct a matrix* A *that has* $\mathbf{v_1}$ *and* $\mathbf{v_2}$ *as its column vectors.*

$W = [2, 1, 3]$

$$W = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

The answer is following:

$$A = \begin{array}{c} \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 3 & 1 \end{bmatrix} \end{array}$$

Followings can be said about the matrix A: (You should be very familiar with these throughout this course.)

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$$

- It has two columns.
- It has two column vectors: $(2, 1, 3)$ and $(1, 0, 1)$.
- It has three rows.
- It has three row vector: $(2, 1)$, $(1, 0)$, and $(3, 1)$.
- It has six elements. (Why six?) 3×2
- Its dimension is 3 by 2 i.e. *(number of rows)* by *(number of columns)*
- A is a 3×2 matrix. (read as *three by two matrix*)

"3 by 2"

Let's see how to describe the components (row, column, and element) of a matrix. A matrix is notated with an upper-case letter, such as A in the above case. Its element is notated with its lower-case counterpart, a. To indicate an element at i-th row and j-th column, subscripts are used as a_{ij}

a_{ij}

For example, a 2×3 matrix A can be denoted as:

row index, column index

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 2 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

Problem 14 *Find a 3×3 matrix A such that $a_{ij} = i + 2j - 3$*

$a_{\bullet 1}$

The above matrix A can be denoted in the perspective of rows and columns as following.

$$A = \begin{bmatrix} | & | & | \\ A_{\bullet 1} & A_{\bullet 2} & A_{\bullet 3} \\ | & | & | \end{bmatrix} = \begin{bmatrix} — & A_{1\bullet} & — \\ — & A_{2\bullet} & — \end{bmatrix}$$

, where $A_{\bullet 1}$ is the first column vector and $A_{2\bullet}$ is the second row vector. Not surprisingly, If [two matrices A and B have the same number of row and column] and [all elements are same], then A and B are same. That is, $A{=}B$.

the same number

upper diagonal

diagonal

lower diagonal

Problem 15 *Calculate the following.*

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+3 & 0-2 \\ 2-1 & \cdot & \cdot \end{bmatrix}$$

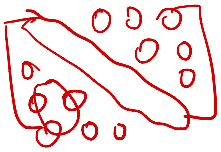
If all elements in a matrix are equal to zero, then we call the matrix as *zero-matrix*. It is a convention that a zero matrix is notated as O. If a $n \times n$ matrix has

"square matrix is $n \times n$ matrix"

uppercase O (대문자).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

diagonal of square matrix



all elements equal to zero except $a_{ii} \neq 0$ for some i, then we call this matrix is a *diagonal matrix*. Followings are examples of diagonal matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -3 & \\ & & & -2 \end{bmatrix}$$

Notice that the 4 by 4 matrix in the above example is still a diagonal matrix, since it is conventionally fine to omit zero elements. Followings hold for same dimensional matrices A, B, C, and a zero-matrix O.

with respect to

- $A + B = B + A$ (commutative)
- $(A + B) + C = A + (B + C)$ (associated)
- $A + O = O + A = A$ (O is the identity element w.r.t. matrix addition)
- $A + (-A) = (-A) + A = O$ ($-A$ is the additive inverse of A)

Corresponding Korean expressions are following.

- (교환법칙)
- (결합법칙)
- (O는 행렬의 덧셈에 대한 항등원)
- ($-A$는 행렬의 덧셈에 대한 A의 역원)

⊕ a lower case letter : a number

⊕ a bold-faced low case letter : a vector

⊕ an upper case letter : a matrix

If one wants to multiply a constant k to a matrix A, then it is written as kA. Elements of kA is nothing but k times the corresponding element of the matrix A.

a number

Followings hold for same dimensional matrices A and B with a constant k.

- $1A = A, (-1)A = -A$
- $0A = O, kO = O$
- $(kl)A = k(lA)$ (associated)
- $(k + l)A = kA + lA$ (distribution)
- $k(A + B) = kA + kB$ (distribution)

Problem 16 *Calculate* $2(A - B) + 3(2A - B)$, *where*

$= 2A - 2B + 6A - 3B$

$= 8A - 5B$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

For A times B to defined,
(# of cols in A) = (# of rows in B)

How would multiplication of matrices are defined? For the matrix multiplication of two 2×2 matrices,

$$\text{For } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$A_{1\cdot} = (a_{11} \quad a_{12})$

$B_{\cdot 1} = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$

$A_{1\cdot} \bullet B_{\cdot 1} = a_{11} \cdot b_{11} + a_{12}b_{21}$

More generally, multiplication of a $m \times n$ matrix A and a $n \times l$ matrix B can be expressed as

$(m \times n) \times (n \times l) \to (m \times l)$

$AB = \begin{bmatrix} A_{1\cdot} \bullet B_{\cdot 1} & A_{1\cdot} \cdot B_{\cdot 2} \\ A_{2\cdot} \cdot B_{\cdot 1} & A_{2\cdot} \cdot B_{\cdot 2} \end{bmatrix}$

$$C = AB = \begin{bmatrix} A_{1\bullet} \cdot B_{\bullet 1} & A_{1\bullet} \cdot B_{\bullet 2} & \cdots & & A_{1\bullet} \cdot B_{\bullet l} \\ A_{2\bullet} \cdot B_{\bullet 1} & A_{2\bullet} \cdot B_{\bullet 2} & \cdots & & A_{2\bullet} \cdot B_{\bullet l} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{m\bullet} \cdot B_{\bullet 1} & A_{m\bullet} \cdot B_{\bullet 2} & \cdots & & A_{m\bullet} \cdot B_{\bullet l} \end{bmatrix}$$

Regarding the operator "$\cdot$", this operator is called a $\boxed{\text{\textit{dot-product} or \textit{inner-product}.}}$ For the same length vector $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x} \cdot \mathbf{y} := \sum_{\forall i} \mathbf{x}_i \mathbf{y}_i$ is called a *inner-product* of vector $\mathbf{x}$ and $\mathbf{y}$.

The resulting matrix's element can be expressed as $C_{ij} = A_{i\bullet} \cdot B_{\bullet j}$. That is, C_{ij} is the inner-product of i-th row vector of A and j-th column vector of B.

$A_{i\cdot} \quad B_{\cdot j}$

Problem 17 *Matrix multiplication is probably a new thing for you. Let's practice them.*

(a)

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 4 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$$

For $C = AB$ where A is $a \times b$ and B is $c \times d$, the condition $b = c$ must be true. That is, the number of columns in A should be same as the number of rows in B. Otherwise, the multiplication is not properly defined. As a result of the multiplication, C is a $a \times d$ matrix.

거듭제곱

✓ **Problem 18** *What is the condition of matrix A that matrix powers (e.g. $A^2 = AA$ and $A^3 = AAA$) are properly defined?*

$3 \times 2 = 2 \times 3$

$0 \times 5 = 5 \times 0$

✓ **Problem 19** *If matrices A and B are both square matrix (A square matrix has same number of row and column), then we know* that *both $\boxed{AB \text{ and } BA}$ are well defined. Then, is $AB = BA$ always true? Provide a counter-example.*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

1) always same

2) some times same and sometimes not same

3) always not same

For matrices A, B, C, and a constant k, where addition and multiplication are properly defined, a few properties for matrix multiplication is as follows:

$ABC \neq BAC$ in general

- $(AB)C = A(BC)$ (associated)
- $A(B + C) = AB + AC$, $(A + B)C = AC + BC$ (distribution)
- $k(AB) = (kA)B = A(kB)$ (constant multiplication)

Again, from your work in the above problem, mind that $AB \neq BA$ in general. In other words, the commutative law does not apply to matrix multiplication.

교환법칙

Problem 20 *Carefully distribute the followings.*

(a) $(A + B)^2$ $= \cancel{A^2 + 2AB + B^2}$

$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B)$

$= A^2 + AB + BA + B^2$ ✓

✓ (b) $(A + B)(A - B)$

Problem 21 *With*

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 4 \\ 0 & 1 \end{bmatrix}$$

$3 \times 1 = 1 \times 3 = 3$

I

, Find the followings.

(a) $(A - B)^2$

(b) $A^2 - 2AB$

A $n \times n$ matrix is called as a *square matrix*, since it looks like square. It has the same number of columns and rows. For a square matrix, we have seen that zero-matrix serves as an *identity matrix for matrix addition*. There is also an *identity matrix for matrix multiplication*, and it is more generally called simply as an *identity matrix*, notated as I. This is equivalent to a number 1, so to speak. The identity matrix is defined as a square matrix whose diagonal elements are all equal to 1 and non-diagonal elements are all zeros. Followings are the identity matrices.

$A + O = O + A = A$

multiplicative identity

identity

$A \times ? = ? \times A = A$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Check

$$\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

	addition	multiplicat
to be defined	same dim.	
identity	O (0)	I (1)
inverse	$-A$	X

In this section, we have defined matrix and its components (row, column, and element). Then, we have defined addition and multiplication of matrices. For addition to be properly defined, the two matrices must have same dimension. For multiplication to be properly defined, the number of columns in the left matrix must be identical to the number of rows in the right matrix. There are identity (항등원) and inverse (역원) for matrix addition, of which zero matrix, O, is additive identity. There is identity (항등원) for matrix multiplication, I, which is analogous to number 1. We call this matrix simply as identity matrix. What is the missing element? Inverse for matrix multiplication!. The upcoming section 5 will discuss the inverse for matrix multiplication.

$AX = XA = I$

X is inverse element of A wrt matrix multiplication!

Ch. 5!

4. Systems of linear equation and singularity

$\begin{pmatrix} 2 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 13 \\ 14 \end{pmatrix}$

$2x_1 + 3x_2 = 13$
$4x_1 + 2x_2 = 14$

$\exists$: there exists
! : unique

We shall go back to the problem at preface.

$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} = A$ "lin indep."

P1
$$
\begin{cases} 2x_1 + 3x_2 - 13 = 0 & (3) \\ 4x_1 + 2x_2 - 14 = 0 & (4) \end{cases}
$$

$\boxed{\exists!\ \text{sol'n}}$

It had a unique solution $x_1 = 2$ and $x_2 = 3$. A unique solution means there is one solution and only one solution. In other words, this problem is solvable ~~with~~ by only one solution. In other words, you can't think of any other combination of x_1 and x_2 that satisfy the linear equations simultaneously. How about the solution to the following problem? How many solutions are there?

$0 \cdot x_1 + 0x_2 + 1 = 0$ (5)×2−(6)
$\Rightarrow 1 = 0$

$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = A$

P2
$$
\begin{cases} 2x_1 + 3x_2 - 13 = 0 & (5) \\ 4x_1 + 6x_2 - 14 = 0 & (6) \end{cases}
$$

: inconsistent (there does not exist a sol'n)

How about the solution to the following problem? How many solutions are there?

$\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ "lin dep."

P3
$$
\begin{cases} \boxed{2x_1 + 3x_2 - 13 = 0} & (7) \\ 4x_1 + 6x_2 - 26 = 0 & (8) \end{cases}
$$

there exist infinite number of sol'n.

(7)×2−(8) $\Rightarrow 0 = 0$

$\begin{pmatrix} x_1 = 2 \\ x_2 = 3 \end{pmatrix}$ $\begin{pmatrix} x_1 = 5 \\ x_2 = 1 \end{pmatrix}$ ⋮

From the above discussion so far, can you generalize the number of solution in the following set of linear equations? By the way, we call such set of linear equation as *a system of linear equations* (연립일차방정식).

"number" ↓ # of sol'n

$$
\begin{cases} a_1x_1 + b_1x_2 + c_1 = 0 & (9) \\ a_2x_1 + b_2x_2 + c_2 = 0 & (10) \end{cases}
$$

i) unique (P1)
ii) 0 (P2)
iii) ∞ (P3)

$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -c_1 \\ -c_2 \end{pmatrix}$, $A \cdot x = b$

$a_1x_1 + b_1x_2 + c_1 = 0$
$a_2x_1 + b_2x_2 + c_2 = 0$

$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -d_1 \\ -d_2 \\ -d_3 \end{pmatrix}$

$a_1x_1 + b_1x_2 + c_1x_3 + d_1 = 0$
$a_2x_1 + b_2x_2 + c_2x_3 + d_2 = 0$
$a_3x_1 + b_3x_2 + c_3x_3 + d_3 = 0$

The system of linear equation has···

- A unique solution if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$
- Infinitely many solutions if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- No solution (inconsistent) if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

What happens in the bigger systems? That is, can you generalize the number of solutions in the case of three linear equations with three unknown variables? The first half of linear algebra course is concerned with solving such a system of linear equations in a more systematic ways. The above can be written with matrix notation as follows.

$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ → if this matrix is invertible

P1. P2. P3

Problem 22 *For the systems of linear equations in the problem 1, 2, and 3 above, express each as the matrix notation of* $A\mathbf{x} = \mathbf{b}$.

With the matrix notation, it is clear that A determines whether the solution will be a unique or not. If A indicates that the solution x is not unique, then there may be infinitely many solutions or no solution depending on b. Let's discuss further regarding the condition of uniqueness that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ under the matrix A written as the following.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

∃! sol'n. Q. ∃! sol'n ?

A. I need to check whether
$ad - bc = 0$ 0 or ∞ (A is singular)
or $ad - bc \neq 0$ ∃! sol'n (A is nonsingular)

To reconcile with our previous finding regarding uniqueness, it can be said that $\frac{a}{c} \neq \frac{b}{d}$ is the condition for a unique solution. Or, $ad - bc \neq 0$. Is this formula $ad - bc$ worth to have its own name? Yes it is, because it tells about the matrix and gives big clue on the number of solutions. This formula is called a *determinant*. For a 2×2 square matrix A, a determinant is written as $|A| = ad - bc$. If $|A| \neq 0$ (A has a non-zero determinant), then $A\mathbf{x} = \mathbf{b}$ has a unique solution. In this case, A is said to be a *non-singular* matrix. If $|A| = 0$, then $A\mathbf{x} = \mathbf{b}$ may have no solution or infinitely many solution depending on $\mathbf{b}$. In this case, A is said to be a *singular* matrix.

$Ax = b$

5. Inverse matrix

The first half of linear algebra course is concerned with solving such a system of linear equations, formed as $A\mathbf{x} = \mathbf{b}$. We shall discuss this further with focus on the matrix A. In the previous chapter, we have seen that the problem at the preface can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A\mathbf{x} = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \end{bmatrix} = \mathbf{b}$$

Remind that a simple linear equation such as $\boxed{2x = 3}$ is solved in the following steps.

$0 \cdot x = 3$
$0 \cdot x = 0$
"std procedure" is not applicable
$\because$ 0 does not have a multiplicative inverse

multiplicative inverse w.r.t "2"

$$\begin{aligned} 2x &= 3 && (11) \\ 2^{-1}2x &= 2^{-1}3 && (12) \\ 1x &= \boxed{\frac{3}{2}} && (13) \end{aligned}$$

multiplicative identity

This was possible because 2 had its multiplicative inverse, 2^{-1}. In other words, $0x = 3$ cannot be solved in a such way since 0^{-1} does not exist. It cannot be solved any way. It has no solution. On the other hand, $0x = 0$ works out for all x. Thus, it has infinitely many number of solutions. In a similar way, system of linear equations, with the aid from matrix notation, can be viewed as follows.

multiplicative inverse w.r.t "A"

$$\begin{aligned} \boxed{A\mathbf{x} = \mathbf{b}} && (14) \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} && (15) \\ I\mathbf{x} &= \boxed{A^{-1}\mathbf{b}} && (16) \end{aligned}$$

std procedure

multiplicative identity

$\exists!$ unique sol'n.

inverse matrix of A
"A inverse"

Analagously to a simple linear equation, $A\mathbf{x} = \mathbf{b}$ has a unique solution $\boxed{\mathbf{x} = A^{-1}\mathbf{b}}$ as long as A^{-1} exists. If A^{-1} does not exist, then the number of solution may be zero or ∞. $\boxed{\text{Not surprisingly}}$, for a 2×2 matrix A, A^{-1} does exist if and only ~~of~~ if $|A| = ad - bc \neq 0$.

Problem 23 *Why is it not surprising?*

A^{-1} exists $\Leftrightarrow$ $\exists!$ sol'n $\Leftrightarrow$ $|A| = ad - bc \neq 0$
Sec5 Sec4 non zero $\boxed{\text{determinant}}$.

A^{-1} is called as an inverse matrix of A. The product of the original matrix and its inverse becomes an identity matrix. Though matrix multiplication is not commutative in general, it is so between the original matrix and its inverse. That is, $\boxed{AA^{-1} = A^{-1}A = I}$. For a 2×2 matrix, an inverse matrix is defined as follows.

$AA^{-1} = I$

$AB \neq BA$ in general

$AA^{-1} = A^{-1}A$

$A^{-1} = \frac{1}{|A|} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\boxed{A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Do you see the familiar $ad - bc$ at the denumerator? What happens if $ad - bc = 0$? In the case, an inverse of A is not defined. That's right. A determinant determines whether a matrix has an inverse or not. So far, we have found that the followings equivalence.

Theorem 2 *For a* $n \times n$ *matrix* A*, the followings are all equivalent. (TFAE)*

2x2

- $A\mathbf{x} = \mathbf{b}$ *has a unique solution.*
- non-zero determinant, i.e. $|A| \neq 0$ (For 2x2, $|A| = ad - bc \neq 0$)
- A^{-1} exists, or we say A is invertible
- non-singular
- a set of column vectors in A is lin indep.

Theorem. TFAE. for nxn matrix

- $A\mathbf{x} = \mathbf{b}$ does not have a unique sol'n
- zero determinant, i.e. $|A| = 0$ (For 2x2, $|A| = ad - bc = 0$)
- A^{-1} does not exist, or we say A is NOT invertible
- A is singular
- a set column vectors in A is lin. dep.

Problem 24 *1) Express the following system of linear equations into matrix form. 2) Determine if there exists a unique solution using determinant. ~~3)~~ If there exists a unique solution, then find the inverse matrix and 4) confirm that your solution is right by checking* $AA^{-1} = A^{-1}A = I$. *5) Then, identify the solution using the inverse matrix. 6) Then, make sure your solution solves the problem.*

3) (inverse mat)

(a)

$$x - y = 3 \quad (17)$$

$$2x + 3y = 7 \quad (18)$$

(a) i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

ii) $|A| = 1\cdot 3 - (-1)\cdot 2 = 5 \neq 0$

$\exists!$ sol'n

iii) $A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

iv) $A\cdot A^{-1} = A^{-1}A = I$

$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\cdot\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$(1, 2) = -1(-1, -2)$

$$x - y = 3 \quad (19)$$

$$2x - 2y = 7 \quad (20)$$

$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$

$Ax = b$

v) $x = A^{-1}\cdot b$

$= \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 16 \\ 1 \end{bmatrix} = \begin{bmatrix} 16/5 \\ 1/5 \end{bmatrix}$

vi) $\frac{16}{5} - \frac{1}{5} = 3$ (o)

$\frac{32}{5} + \frac{3}{5} = 7$ (o)

Problem 25 *From (a) in the above problem, identify the two column vectors. Are they linearly dependent or independent? Identify the two row vectors. Are they linearly dependent or independent? Do the same for (b) in the above problem.*

Problem 26 *What is the relationship between linear dependence and zeroness of determinants?*

Problem 27 *What do you think you will learn in this linear algebra course?*

- How to obtain the inverse matrix of a 3×3 matrix?
-
-
-
-

*...they do not go to the course lectures, even to the first one in a course, as **tabulae rasae**[3]. They have thought beforehand about the problems the lectures will be dealing with and have in mind certain questions and problems of their own. They have been occupied with the topic and it interests them. Instead of being passive receptacles of words and ideas, they listen, they **hear**, and most important, they **receive** and they **respond** in an active, productive way. What they listen to stimulates their own thinking processes. New questions, new ideas, new perspectives arise in their minds. Their listening is an alive process. They listen with interest, hear what the lecturer says, and spontaneously come to life in response to what they hear. They do not simply acquire knowledge that they can take home and memorize. Each student has been affected and has changed: each is different after the lecture than he or she was before it. Of course, this mode of learning can prevail only if the lecture offers stimulating material. - from "To have or to be" (1976) by Erich Fromm*

[3] 백지상태

소유냐 존재냐.

Solution to Selected Problems

Problem 1.

Three-dimensional vector space is the space for a three-dimensional vector (x_1, x_2, x_3) such that $-\infty < x_1, x_2, x_3 < \infty$.

Problem 5.

1) $\mathbf{x} = \frac{13}{5}\mathbf{y} + \frac{1}{5}\mathbf{z}$.

Problem 6.

There is no constant a that satisfies $\mathbf{x} = a\mathbf{y}$, and there is no constant b that satisfies $\mathbf{y} = b\mathbf{x}$. Thus, they are not linearly dependent.

Problem 7.

2) $(2, -1) = \frac{-5}{13}(2, 5) + \frac{12}{13}(3, 1)$. $\therefore$ linearly dependent.

Problem 8.

3) Let $a_1(4, 1, 0) + a_2(0, 2, -1) + a_3(3, 2, 0) = (0, 0, 0)$. It follows $4a_1 + 3a_3 = 0, a_1 + 2a_2 + 2a_3 = 0, -a_2 = 0$. From the third equation, $a_2 = 0$ and plug this into the first two equations easily verify that $a_1 = a_3 = 0$ as well. Thus, the three vectors are linearly independent.

4) Let $a_1(1, 1, 0) + a_2(2, 3, 4) + a_3(0, 0, 0) = (0, 0, 0)$. Then, $a_1 = a_2 = 0, a_3 = 1$ solves. Since such a non-zero solution exists, they are linearly dependent.

Problem 11.

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for a 3D vector space, any 3D vector **x** can be expressed as a linear combination of them such as

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3,$$

where a_1, a_2, a_3 are real numbers.

In order to prove that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is a basis for a 3D space, we need to show that the same **x** in the above equation can be expressed as a linear combination of vectors: $\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$

That is, we should set

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = b_1\mathbf{v}_1 + b_2(\mathbf{v}_1 + \mathbf{v}_2) + b_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$$

and solve for b_1, b_2, b_3 in terms of a_1, a_2, a_3.

That is, we need solve the following.

$$\begin{aligned} a_1 &= b_1 + b_2 + b_3 \\ a_2 &= b_2 + b_3 \\ a_3 &= b_3 \end{aligned}$$

With some calculation, you shall find $b_3 = a_3$, $b_2 = a_2 - a_3$, and $b_1 = a_1 - (a_2 - a_3) - a_3 = a_1 - a_2$. Finally, it follows that

$$\mathbf{x} = (a_1 - a_2)\mathbf{v}_1 + (a_2 - a_3)(\mathbf{v}_1 + \mathbf{v}_2) + a_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$$

Since a_1, a_2, a_3 are real numbers, so are $a_1 - a_2, a_2 - a_3, a_3$. Thus, the above equation tells us that the any 3D vector **x** can be expressed as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$. This proves the claim. $\square$

Problem 21.

(a) $(A - B)^2 = (A - B)(A - B) = AA - AB - BA - BB = A^2 - AB - BA - B^2$.

Problem 11.

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for a 3D vector space, any 3D vector **x** can be expressed as a linear combination of them such as

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3,$$

where a_1, a_2, a_3 are real numbers.

In order to prove that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is a basis for a 3D space, we need to show that the same **x** in the above equation can be expressed as a linear combination of vectors: $\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$

That is, we should set

$$\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = b_1\mathbf{v}_1 + b_2(\mathbf{v}_1 + \mathbf{v}_2) + b_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$$

and solve for b_1, b_2, b_3 in terms of a_1, a_2, a_3.

That is, we need solve the following.

$$\begin{aligned} a_1 &= b_1 + b_2 + b_3 \\ a_2 &= b_2 + b_3 \\ a_3 &= b_3 \end{aligned}$$

With some calculation, you shall find $b_3 = a_3$, $b_2 = a_2 - a_3$, and $b_1 = a_1 - (a_2 - a_3) - a_3 = a_1 - a_2$. Finally, it follows that

$$\mathbf{x} = (a_1 - a_2)\mathbf{v}_1 + (a_2 - a_3)(\mathbf{v}_1 + \mathbf{v}_2) + a_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$$

Since a_1, a_2, a_3 are real numbers, so are $a_1 - a_2, a_2 - a_3, a_3$. Thus, the above equation tells us that the any 3D vector **x** can be expressed as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$. This proves the claim. $\square$

Problem 21.

(a) $(A - B)^2 = (A - B)(A - B) = AA - AB - BA - BB = A^2 - AB - BA - B^2$.