Chapter 2. Matrix Algebra (2/2)

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2.5. Matrix Factorizations

Matrix Factorizations

- A *factorization* of a matrix A is an equation that expresses A as a product of two or more matrices.
- Whereas matrix multiplication involves a *synthesis* of data (combining the effects of two or more linear transformations into a single matrix), matrix factorization is an *analysis* of data.

The LU Factorization

- The LU factorization, described on the next few slides, is motivated by the fairly common industrial and business problem of solving a sequence of equations, all with the same coefficient matrix:

$$A\mathbf{x} = b_1, \quad A\mathbf{x} = b_2, \quad ..., A\mathbf{x} = \mathbf{b}_p \tag{5}$$

- When A is invertible, one could compute A^{-1} and then compute $A^{-1}b_1$, $A^{-1}b_2$, and so on.
- However, it is more efficient to solve the first equation in the sequence (5) by row reduction and obtain the LU factorization of A at the same time. Thereafter, the remaining equations in sequence (5) are solved with the LU factorization.

The LU Factorization

- At first, assume that A is an $m \times n$ matrix that can be row reduced to echelon form, *without row interchanges*.
- Then A can be written in the form $A = LU$, were L is an $m \times n$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A.
- For instance, see Fig. 1 below. Such a factorization is called an **LU factorization** of A. The matrix L is invertible and is called a unit lower triangular matrix.

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

The LU Factorization

- Before studying how to construct L and U, we should look at why they are so useful. When $A = LU$, the equation $A\mathbf{x} = \mathbf{b}$ can be written as $L(U\mathbf{x}) = \mathbf{b}$.
- Writing $\mathbf{y}$ for $U\mathbf{x}$, we can find $\mathbf{x}$ by solving the pair of equations

$$Ly = \mathbf{b}$$
$$U\mathbf{x} = y$$

- First solve $Ly = \mathbf{b}$ for y, and then solve $U\mathbf{x} = y$ for $\mathbf{x}$. See Fig. 2 on the next slide. Each equation is easy to solve because L and U are triangular.

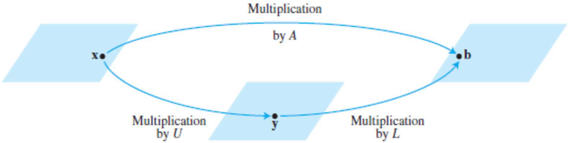


FIGURE 2 Factorization of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

The LU Factorization

- **Example 1** It can be verified that

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU$$

- Use this factorization of A to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$

- **Solution**
  - The solution of $L\mathbf{y} = \mathbf{b}$ needs only 6 multiplications and 6 additions, because the arithmetic takes place only in column 5.

$$[L \mid \mathbf{b}] = \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] = [I \mid \mathbf{y}]$$

  - Then, for $U\mathbf{x} = \mathbf{y}$, the "backward" phase of row reduction requires 4 divisions, 6 multiplications, and 6 additions.
  - For instance, creating the zeros in column 4 of $[U \mid \mathbf{y}]$ requires 1 division in row 4 and 3 multiplication - addition pairs to add multiples of row 4 to the rows above.

$$[U|\mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

  - To find $\mathbf{x}$ requires 28 arithmetic operations, or "*flops*' ' (floating point operations), excluding the cost of finding L and U. In contrast, row reduction of $[A|\mathbf{b}]$ to $[I|\mathbf{x}]$ takes 62 operations.

An LU Factorization Algorithm

- Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another below it.
- In this case, there exist unit lower triangular elementary matrices $E_1, ..., E_p$ such that $E_p ... E_1 A = U$. Then,

$$A = (E_p ... E_1)^{-1} U = LU \qquad (3)$$

where

$$L = (E_p ... E_1)^{-1} \qquad (4)$$

- It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular. Thus L is unit lower triangular.
- Note that row operations in equation (3), which reduce A to U, also reduce the L in equation (4) to I, because $E_p ... E_1 L = (E_p ... E_1)(E_p ... E_1)^{-1} = I$. This observation is the key to *constructing L*.

An LU Factorization Algorithm

Algorithm for an LU Factorization

- Step 1) Reduce A to an echelon form U by a sequence of row replacement operations, if possible. e.g.) $(R2 \leftarrow R2 - 3R1)$
- Step 2) Place entries in L such that the same sequence of row operations reduces L to I.
- Step 1 is not always possible, but when it is, the argument above shows that an LU factorization exists.
- Example 2 on the followings slides will show how to implement Step 2. By construction, L will satisfy

$$(E_p ... E_1)L = I$$

using the same $E_p ... E_1$ as in equation (3). Thus L will be invertible, by the Invertible Matrix Theorem, with $(E_p ... E_1) = L^{-1}$. From (3), $L^{-1}A = U$, and $A = LU$. So Step 2 will produce an acceptable L.

An LU Factorization Algorithm

- **Example 2** Find an LU factorization of

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

- **Solution:**
  - Since A has four rows, L should be 4×4. The first column of L is the first column of A divided by the top pivot entry:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix}$$

- **(Solution continued:)**
  - Compare the first columns of A and L. The row operations that create zeros in the first column of A will also create zeros in the first column of L.
  - To make this same correspondence of row operations on A hold for the rest of L, watch a row reduction of A to an echelon form U. That is, *highlight the entries* in each matrix that are used to determine the sequence of row operations that transform A onto U.

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

- **(Solution continued:)**
  - The highlighted entries above determine the row reduction of A to U. At each pivot column, divide the highlighted entries by the pivot and place the result onto L:
  - **An easy calculation verifies that this L and U satisfy $LU = A$.**

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix}$$

$$\div 2 \quad \div 3 \quad \div 2 \quad \div 5$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

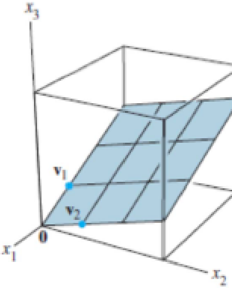
Suggested Exercises

- 2.5.7
- 2.5.9
- 2.5.11

2.8. Subspaces of $\mathbb{R}^n$

Subspaces of $\mathbb{R}^n$

- **Definition:** A **subspace** of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:
  a) The zero vector is in H.
  b) For each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H. (*closed under vector addition*)
  c) For each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H. (*closed under scalar multiplication*)
- A plane through the origin is the standard way to visualize the subspace in Example 1 on the next slide. See Fig. 1 below:



Subspaces of $\mathbb{R}^n$

- **Example 1** If v_1 and v_2 are in $\mathbb{R}^n$ and $H = Span\{v_1, v_2\}$, prove that H is a subspace of $\mathbb{R}^n$.

- **Proof**
  1. To verify this statement, note that the zero vector is in H (because $v_1 + v_2$ is a linear combination of v_1 and v_2).
  2. Now take two arbitrary vectors in H, say

     $$u = s_1 v_1 + s_2 v_2 \quad and \quad v = t_1 v_1 + t_2 v_2$$

     Then,

     $$u + v = (s_1 + t_1)v_1 + (s_2 + t_2)v_2,$$

     which shows that $u + v$ is a linear combination of v_1 and v_2 and hence is in H.
  3. Also, for any scalar c, the vector cu is in H, because

     $$cu = c(s_1 v_1 + s_2 v_2) = cs_1(v_1) + cs_2(v_2)$$

Column space and Null space of a matrix

- **Definition:** The **column space** of a matrix A is the set $Col\ A$ of all linear combinations of the columns of A.
- If $A = [\mathbf{a}_1, ..., \mathbf{a}_n]$ with the columns of $\mathbb{R}^m$, then $Col\ A$ is the same as $Span\{a_1...a_n\}$.
- Example 4 shows that the column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

- **Example 4** Determine whether $\mathbf{b}$ is in the column space of A, where

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}.$$

- **Solution:**
  - The vector $\mathbf{b}$ is a linear combination of the columns of A if and only if $\mathbf{b}$ can be written as $A\mathbf{x}$ for some $\mathbf{x}$. That is, if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
  - Row reducing the augmented matrix $[A \mid \mathbf{b}]$,

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right] \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & 6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & 6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  - We conclude that
    - $A\mathbf{x} = \mathbf{b}$ is **consistent**
    - $\mathbf{b}$ is in $Col\ A$.

Column space and Null space of a matrix

- **Definition:** The **null space** of a matrix A is the set $Nul\ A$ of all solutions of the homogeneous equation $A\mathbf{x} = 0$.

- **Theorem 12:** The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system $A\mathbf{x} = 0$ of m homogenous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

- **Proof:**
  1. The zero vector is in $NulA$ (because $A0 = 0$).
  2. To show that $NulA$ satisfies that other two properties required for a subspace, take any $\mathbf{u}$ and $\mathbf{v}$ in $Nul\ A$. That is, suppose $A\mathbf{u} = 0$ and $A\mathbf{v} = 0$. Then, by a property of matrix multiplication, $A(u + v) = Au + Av = 0 + 0 = 0$. Thus, $\mathbf{u} + \mathbf{v}$ satisfies $A = 0$, and so $\mathbf{u} + \mathbf{v}$ is in $NulA$.
  3. Also, if $\mathbf{u} \in Nul\ A$, then for any scalar c, $A(c\mathbf{u}) = c(A\mathbf{u}) = c(0) = 0$, which shows that $c\mathbf{u}$ is in $Nul\ A$.

Basis for a subspace

- **Definition:** A **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.
- **Example 5**
  - The columns of an invertible $n \times n$ matrix form a basis because they are linearly independent and span $\mathbb{R}^n$, by the Invertible Matrix Theorem.
  - One such matrix is the $n \times n$ identity matrix. Its columns are denoted by $e_1, ..., e_n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, e_1 = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \ e_n = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

  - The set $\{e_1, ..., e_n\}$ is called the **standard basis** for $\mathbb{R}^n$. See Fig. 3 on the next slide.

Basis for a subspace

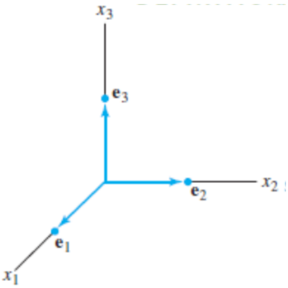


FIGURE 3
The standard basis for $\mathbb{R}^3$.

- **Theorem 13:** The pivot columns of a matrix A form a basis for the column space of A.

Suggested Exercises

- 2.8.11
- 2.8.12

2.9. Dimension and rank

The dimension of a subspace

- **Definition:** The **dimension** of a nonzero subspace H, denoted by $dim\ H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{0\}$ is defined to be zero.

- **Definition:** The **rank** of a matrix A, denoted by $rank\ A$, is the dimension of the column space of A.

The dimension of a subspace

- **Example 3** Determine the rank of the matrix

$$A \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$$

- **Solution:**
  - Reduce A to echelon form:

$$\mathrm{A} \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & -6 & 4 & 14 & -20 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

- The matrix A has 3 pivot columns, so $rank\ A = 3$.

The dimension of a subspace

- **Theorem 14** If a matrix A has n columns, then $rank\, A + dim\, Nul\, A = n$.
- **Theorem 15** Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

Recap - The invertible matrix theorem

- **Theorem 8:** Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.
  - **a)** A is an invertible matrix.
  - **b)** A is row equivalent to the $n \times n$ identity matrix.
  - **c)** A has n pivot positions.
  - **d)** The equation $A\mathbf{x} = 0$ has only the trivial solution.
  - **e)** The columns of A form a linearly independent set.
  - **f)** The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
  - **g)** The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
  - **h)** The columns of A span $\mathbb{R}^n$.
  - **i)** The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
  - **j)** There is an $n \times n$ matrix C such that $CA = I$.
  - **k)** There is an $n \times n$ matrix D such that $AD = I$.
  - **l)** A^T is an invertible matrix.

Addendum on the Invertible Theorem

- **The Invertible Theorem (continued)**

  m) The columns of A form a basis of $\mathbb{R}^n$.
  n) $Col\ A = \mathbb{R}^n$
  o) $dim\, Col\ A = n$
  p) $rank\ A = n$
  q) $Nul\ A = \{0\}$
  r) $dim\ Nul\ A = 0$

The proof for the added statements

- Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning.
- The other five statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:
  $(g) \Rightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (r) \Rightarrow (q) \Rightarrow (d)$
- Statement (g), which says that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, implies statement (n), because $Col\ A$ is precisely the set of all $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent.
- The implications $(n) \Rightarrow (o) \Rightarrow (p)$ follow from the definitions of $dimension$ and $rank$.
- If the rank of A is n, the number of columns of A, then $dim\ Nul\ A = 0$, by the Rank Theorem, and so $Nul\ A = \{0\}$. Thus $(p) \Rightarrow (r) \Rightarrow (q)$
- Also, statement (q) implies that the equation $A\mathbf{x} = 0$ has only the trivial solution, which is statement (d).
- Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete.

Suggested Exercises

- Supplementary Exercises
  - (At the end of the chapter, p.178-179)
  - 2.1 (all subproblems)

Acknowledgement

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