

Chapter 6. Orthogonality and Least Squares

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6.1. Inner Product, Length, And Orthogonality

Inner Product

- If \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^n , then we regard \mathbf{u} and \mathbf{v} as $n \times 1$ matrices. (i.e. column vector)
 - The transpose \mathbf{u}^T is a $1 \times n$ matrix, and the matrix product $\mathbf{u}^T \mathbf{v}$ is a 1×1 matrix, which we write as a single real number (a scalar) without brackets.
 - The number $\mathbf{u}^T \mathbf{v}$ is called the **inner product** of \mathbf{u} and \mathbf{v} , and it is written as $\mathbf{u} \cdot \mathbf{v}$
 - This inner product is also referred to as a **dot product**.

- If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then, the inner product of \mathbf{u} and \mathbf{v} is

$$\mathbf{u} \cdot \mathbf{v} = [u_1 \quad u_2 \quad \dots \quad u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

- **Theorem 1:** Let \mathbf{u} , \mathbf{v} , and W be vectors in \mathbb{R}^n , and let c be a scalar. Then

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) \cdot W = \mathbf{u} \cdot W + \mathbf{v} \cdot W$
- $(c \cdot \mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$

• Remarks

- Properties (b) and (c) can be combined several times to produce the following useful property. $(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot W = c_1(\mathbf{u}_1 \cdot W) + \cdots + c_p(\mathbf{u}_p \cdot W)$
- If \mathbf{v} is in \mathbb{R}^n , with entries v_1, \dots, v_n , then the square root of $\mathbf{v} \cdot \mathbf{v}$ is defined, because $\mathbf{v} \cdot \mathbf{v}$ is nonnegative.

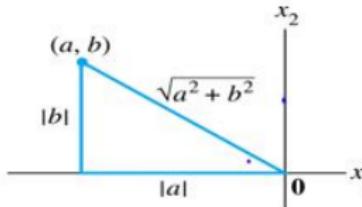
The length of Vector

- **Definition:** The **length** or **norm** of \mathbf{v} is the nonnegative scalar

- $\|\mathbf{v}\| := \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$ and $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$

- **Remarks**

- Suppose \mathbf{v} is in \mathbb{R}^2 , say $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. If we identify \mathbf{v} with a geometric point in the plane, as usual, $\|\mathbf{v}\|$ coincides with the standard notion of the length of the line segment from the origin to \mathbf{v} .
- This follows from the Pythagorean Theorem applied to a triangle such as the one shown in the following figure.



Interpretation of $\|\mathbf{v}\|$ as length.

• Remarks

- For any scalar c , the length cv is $|c|$ times the length of \mathbf{v} . That is, $\|cv\| = |c|\|\mathbf{v}\|$
 - A vector whose length equal to 1 is called a **unit vector**.
 - If we *divide* a nonzero vector \mathbf{v} by its length—that is, multiply by $1/\|\mathbf{v}\|$ —we obtain a unit vector \mathbf{u} because the length of \mathbf{u} is $(1/\|\mathbf{v}\|)\|\mathbf{v}\|$
 - The process of creating \mathbf{u} from \mathbf{v} is sometimes called **normalizing** \mathbf{v} , and we say that \mathbf{u} is in the same direction as \mathbf{v} .

- **Example 2:** Let $\mathbf{v} = (1, -2, 2, 0)$. Find a unit vector \mathbf{u} in the same direction as \mathbf{u} .

• Solution:

- First, compute the length of \mathbf{v} : $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v} = (1)^2 + (-2)^2 + (2)^2 + (0)^2 = 9$.
 $\|\mathbf{v}\| = \sqrt{9} = 3$.
- Then, multiply \mathbf{v} by $1/\|\mathbf{v}\|$ to obtain

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

- To check that $\|\mathbf{u}\| = 1$, it suffices to show that $\|\mathbf{u}\|^2 = 1$

$$\begin{aligned} \|\mathbf{u}\|^2 &= \mathbf{u} \cdot \mathbf{u} = \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + (0)^2 \\ &= \frac{1}{9} + \frac{4}{9} + \frac{4}{9} + 0 = 1 \end{aligned}$$

Distance in \mathbb{R}^n

- **Definition:** For \mathbf{u} and \mathbf{v} in \mathbb{R}^n , the **distance between \mathbf{u} and \mathbf{v}** , written as $dist(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$dist(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

- **Example 4:** Compute the distance between the vectors $\mathbf{u} = (7, 1)$ and $\mathbf{v} = (3, 2)$

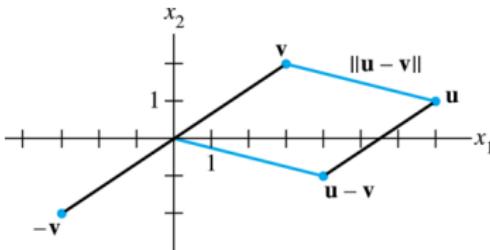
- **Solution:**

- Calculate

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

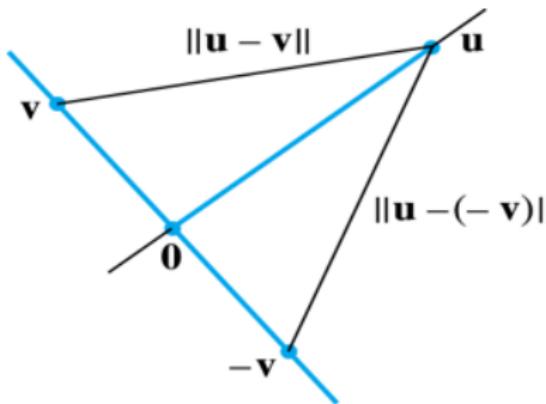
- The vectors \mathbf{u} , \mathbf{v} , and $\mathbf{u} - \mathbf{v}$ are shown in the figure on the next slide.
- When the vector $\mathbf{u} - \mathbf{v}$ is added to \mathbf{v} , the result is \mathbf{u} .
- Notice that the parallelogram in the figure below shows that the distance from \mathbf{u} to \mathbf{v} is the same as the distance from \mathbf{u} to \mathbf{u} is the same as the distance $\mathbf{u} - \mathbf{v}$ to 0.



The distance between \mathbf{u} and \mathbf{v} is
the length of $\mathbf{u} - \mathbf{v}$.

Orthogonal Vector

- Consider \mathbb{R}^2 or \mathbb{R}^3 and two lines through the origin determined by vectors \mathbf{u} and \mathbf{v} .
- See the figure below. The two lines shown in the figure are geometrically perpendicular if and only if the distance from \mathbf{u} to \mathbf{v} is the same as the distance from \mathbf{u} to $-\mathbf{v}$



- This is the same as requiring the squares of the distances to be the same.

- Now,

$$\begin{aligned}
 [dist(\mathbf{u}, -\mathbf{v})]^2 &= \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 \\
 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\
 &= \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) \\
 &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\
 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}
 \end{aligned}$$

- The same calculations with \mathbf{v} and $-\mathbf{v}$ interchanged show that

$$\begin{aligned}
 [dist(\mathbf{u}, \mathbf{v})]^2 &= \|\mathbf{u}\|^2 + \|-\mathbf{v}\|^2 + 2\mathbf{u} \cdot (-\mathbf{v}) \\
 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}
 \end{aligned}$$

- The two squared distances are equal if and only if $2\mathbf{u} \cdot \mathbf{v} = -2\mathbf{u} \cdot \mathbf{v}$, which happens if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.
- This calculation shows that when vectors \mathbf{u} and \mathbf{v} are identified with geometric points, the corresponding lines through the points and the origin are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

- **Definition:** Two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^n are orthogonal (to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.
- **Remark:** The zero vector is orthogonal to every vector \mathbb{R}^n because $0^T \mathbf{v} = 0$ for all \mathbf{v}
- **Theorem 2 (The pythagorean Theorem):** Two vectors \mathbf{u} and \mathbf{v} are orthogonal if and only if
$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Orthogonal Complements

• Definition

- If a vector \mathbf{z} is orthogonal to every vector in a subspace W of \mathbb{R}^n , then \mathbf{z} is said to be **orthogonal** to W .
- The set of all vectors \mathbf{z} that are orthogonal to W is called the **orthogonal complement** of W and is denoted by W^\perp (and read as “ W perpendicular” or simply “ W perp”)

• Theorems

- A vector \mathbf{x} is in W^\perp if and only if \mathbf{x} is orthogonal to every vector in a set that spans W .
- W^\perp is a subspace of \mathbb{R}^n .

- **Theorem 3:** Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A , and the orthogonal complement of the column space of A is the null space of A^T :

$$(RowA)^\perp = Nul A \text{ and } (ColA)^\perp = Nul A^T$$

- **Proof :**

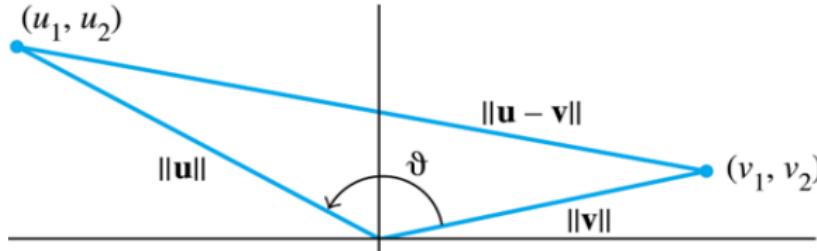
- The row-column rule for computing Ax shows that if x is in $Nul A$, then x is orthogonal to each row of A (with the rows treated as vectors in \mathbb{R}^n).
- Since the rows of A span the row space, x is orthogonal to $Row A$.
- Conversely, if x is orthogonal to $Row A$, then x is certainly orthogonal to each row of A , and hence $Ax = 0$.
- This proves the first statement of the theorem.
- Since this statement is true for any matrix, it is true for A^T .
- That is, the orthogonal complement of the row space of A^T is the null space of A^T .
- This proves the second statement, because $RowA^T = ColA$.

Angles in \mathbb{R}^2 and \mathbb{R}^3

- If \mathbf{u} and \mathbf{v} are nonzero vectors in either \mathbb{R}^2 or \mathbb{R}^3 , then there is a nice connection between their inner product and the angle θ between the two line segments from the origin to the points identified with \mathbf{u} and \mathbf{v} .
- The formula is

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \quad (2)$$

- To verify this formula for vectors in \mathbb{R}^2 consider the triangle shown in the next figure with sides of lengths, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{u} - \mathbf{v}\|$.



The angle between two vectors.

- By the law of cosines,

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$$

which can be rearranged to produce the next equations.

$$\begin{aligned}\|\mathbf{u}\|\|\mathbf{v}\| \cos \theta &= \frac{1}{2} [\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2] \\ &= \frac{1}{2} [u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2] \\ &= u_1 v_1 + u_2 v_2 \\ &= \mathbf{u} \cdot \mathbf{v}\end{aligned}$$

- The verification for \mathbb{R}^3 is similar. When $n > 3$, formula (2) may be used to *define* the angle between two vectors in \mathbb{R}^n
- In statistics, the value of $\cos\theta$ defined by (2) for suitable vectors \mathbf{u} and \mathbf{v} is called a *correlation coefficient*.

Suggested Exercises

- 6.1.11

6.2. Orthogonal Sets

Orthogonal Sets

- **Definition:** A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ in \mathbb{R}^n is said to be an **orthogonal set** if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.
- **Theorem 4:** If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in \mathbb{R}^n , then S is linearly independent and hence is a basis for the subspace spanned by S .
- **Proof:**

- If $0 = c_1\mathbf{u}_1 + \dots + c_p\mathbf{u}_p$ for some scalars c_1, \dots, c_p , then

$$\begin{aligned} 0 &= 0 \cdot \mathbf{u}_1 = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_p\mathbf{u}_p) \cdot \mathbf{u}_1 \\ &= (c_1\mathbf{u}_1) \cdot \mathbf{u}_1 + (c_2\mathbf{u}_2) \cdot \mathbf{u}_1 + \dots + (c_p\mathbf{u}_p) \cdot \mathbf{u}_1 \\ &= c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) + c_2(\mathbf{u}_2 \cdot \mathbf{u}_1) + \dots + c_p(\mathbf{u}_p \cdot \mathbf{u}_1) \\ &= c_1(\mathbf{u}_1 \cdot \mathbf{u}_1) \end{aligned}$$

because \mathbf{u}_1 is orthogonal to $\mathbf{u}_2, \dots, \mathbf{u}_p$

- Since \mathbf{u}_1 is nonzero, $\mathbf{u}_1 \cdot \mathbf{u}_1$ not zero and so $c_1 = 0$. Likewise, c_2, \dots, c_p must be zero.
- Thus S is linearly independent.

- **Definition :** An **orthogonal basis** for a subspace W of \mathbb{R}^n is a basis for W that is also an orthogonal set.
- **Theorem 5:** Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of \mathbb{R}^n . For each \mathbf{y} in W , the weights in the linear combination $\mathbf{y} = c_1\mathbf{u}_1 + \dots + c_p\mathbf{u}_p$ are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \quad (j = 1, \dots, p)$$

- **Proof:**

- The orthogonality of $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ shows that

$$\mathbf{y} \cdot \mathbf{u}_1 = (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \dots + c_p\mathbf{u}_p) \cdot \mathbf{u}_1 = c_1(\mathbf{u}_1 \cdot \mathbf{u}_1)$$

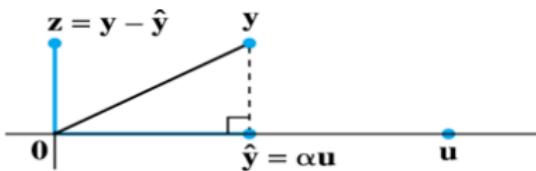
- Since $\mathbf{u}_1 \cdot \mathbf{u}_1$ is not zero, the equation above can be solved for c_1 .
- To find c_j for $j = 2, \dots, p$, compute $\mathbf{y} \cdot \mathbf{u}_j$ and solve for c_j

An Orthogonal Projection

- Given a nonzero vector \mathbf{u} in \mathbb{R}^n , consider the problem of decomposing a vector \mathbf{y} in \mathbb{R}^n into the sum of two vectors, one a multiple of \mathbf{u} and the other orthogonal to \mathbf{u} .
- We wish to write

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \quad (1)$$

where $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α and \mathbf{z} is some vector orthogonal to \mathbf{u} . See the following figure.



Finding α to make $\mathbf{y} - \hat{\mathbf{y}}$ orthogonal to \mathbf{u} .

- Given any scalar α , let $\mathbf{z} = \mathbf{y} - \alpha\mathbf{u}$, so that (1) is satisfied. Then, $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to \mathbf{u} if and only if

$$0 = (\mathbf{y} - \alpha\mathbf{u}) \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{u} - (\alpha\mathbf{u}) \cdot \mathbf{u} = \mathbf{y} \cdot \mathbf{u} - \alpha(\mathbf{u} \cdot \mathbf{u})$$

- That is, (1) is satisfied with \mathbf{z} orthogonal to \mathbf{u} if and only if $\alpha = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$ and $\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}$
- The vector $\hat{\mathbf{y}}$ is called the **orthogonal projection of \mathbf{y} onto \mathbf{u}** , and the vector \mathbf{z} is called the **component of \mathbf{y} orthogonal to \mathbf{u}** .

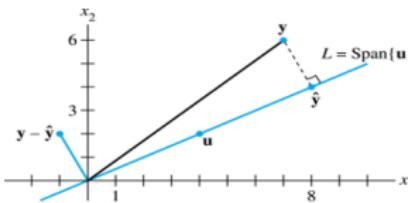
- If c is any nonzero scalar and if \mathbf{u} is replaced by $c\mathbf{u}$ in the definition of $\hat{\mathbf{y}}$, then the orthogonal projection of \mathbf{y} onto $c\mathbf{u}$ is exactly the same as the orthogonal projection of \mathbf{y} onto \mathbf{u} .
- Hence this projection is determined by the *subspace* L spanned by \mathbf{u} (the line through \mathbf{u} and 0).
- Sometimes $\hat{\mathbf{y}}$ is denoted by $\text{proj}_L \mathbf{y}$ and is called the **orthogonal projection** of \mathbf{y} onto L .
- That is,

$$\hat{\mathbf{y}} = \text{proj}_L \mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \quad (2)$$

- **Example 3:** Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of \mathbf{y} onto \mathbf{u} . Then write \mathbf{y} as the sum of two orthogonal vectors, one in $\text{Span}\{\mathbf{u}\}$ and one orthogonal to \mathbf{u} .

- **Solution:**

- Compute $\mathbf{y} \cdot \mathbf{u} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 40$ and $\mathbf{u} \cdot \mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 20$
- The orthogonal projection of \mathbf{y} onto \mathbf{u} is $\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{40}{20} \mathbf{u} = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ and the component of \mathbf{y} orthogonal to \mathbf{u} is $\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
- The sum of these two vector is \mathbf{y} , i.e. $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}})$. That is, $\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
- The decomposition of \mathbf{y} is illustrated in the following figure:



The orthogonal projection of \mathbf{y} onto a line L through the origin.

• (*solution continued*)

- If the calculation above are correct, then $\{\hat{\mathbf{y}}, \mathbf{y} - \hat{\mathbf{y}}\}$ will be an orthogonal set.
- As a check, compute

$$\hat{\mathbf{y}} \cdot (\mathbf{y} - \hat{\mathbf{y}}) = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -8 + 8 = 0$$

- **Remark:** Since the line segment in the figure on the previous slide between \mathbf{y} and $\hat{\mathbf{y}}$ is perpendicular to L , by construction of $\hat{\mathbf{y}}$, the point identified with $\hat{\mathbf{y}}$ is the closest point of L to \mathbf{y} .

Orthonormal Sets

• Definitions

- A set $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors.
- If W is the subspace spanned by such a set, then $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an **orthonormal basis** for W , since the set is automatically linearly independent, by Theorem 4.

• Remark

- The simplest example of an orthonormal set is the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ for \mathbb{R}^n .
- Any nonempty subset of $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is orthonormal, too.

- **Example 2:** Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis of \mathbb{R}^3 . where

$$\mathbf{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

- **Solution:**

- $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set because

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = -3/\sqrt{66} + 2/\sqrt{66} + 1/\sqrt{66} = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = -3/\sqrt{726} - 4/\sqrt{726} + 7/\sqrt{726} = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = 1/\sqrt{396} - 8/\sqrt{396} + 7/\sqrt{396} = 0$$

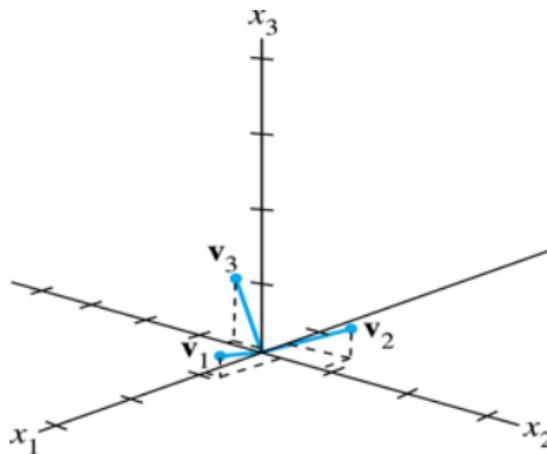
- $\mathbf{v}_1, \mathbf{v}_2$, and \mathbf{v}_3 are unit vectors because

$$\mathbf{v}_1 \cdot \mathbf{v}_1 = 9/11 + 1/11 + 1/11 = 1$$

$$\mathbf{v}_2 \cdot \mathbf{v}_2 = 1/6 + 4/6 + 1/6 = 1$$

$$\mathbf{v}_3 \cdot \mathbf{v}_3 = 1/66 + 16/66 + 49/66 = 1$$

- Since the set is linearly independent, its three vectors form a basis for \mathbb{R}^3 . See the following figure.



- When the vectors in an orthogonal set of nonzero vectors are *normalized* to have unit length, the new vectors will still be orthogonal, and hence the new set will be an orthonormal set.

- **Theorem 6:** An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

- **Proof:**

- To simplify notation, we suppose that U has only three columns, each a vector in \mathbb{R}^m .
- Let $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ and compute

$$U^T U = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1^T \mathbf{u}_1 & \mathbf{u}_1^T \mathbf{u}_2 & \mathbf{u}_1^T \mathbf{u}_3 \\ \mathbf{u}_2^T \mathbf{u}_1 & \mathbf{u}_2^T \mathbf{u}_2 & \mathbf{u}_2^T \mathbf{u}_3 \\ \mathbf{u}_3^T \mathbf{u}_1 & \mathbf{u}_3^T \mathbf{u}_2 & \mathbf{u}_3^T \mathbf{u}_3 \end{bmatrix} \quad (4)$$

- The entries in the matrix at the right are inner products, using transpose notation.
- The columns of U are orthogonal if and only if

$$\mathbf{u}_1^T \mathbf{u}_2 = \mathbf{u}_2^T \mathbf{u}_1 = 0, \quad \mathbf{u}_1^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_1 = 0, \quad \mathbf{u}_2^T \mathbf{u}_3 = \mathbf{u}_3^T \mathbf{u}_2 = 0 \quad (5)$$

- The columns of U all have unit length if and only if

$$\mathbf{u}_1^T \mathbf{u}_1 = 1, \quad \mathbf{u}_2^T \mathbf{u}_2 = 1, \quad \mathbf{u}_3^T \mathbf{u}_3 = 1 \quad (6)$$

- The theorem follows immediately from (4)–(6).

- **Theorem 7:** Let U be an $m \times n$ matrix with orthonormal columns, and let x and y be in \mathbb{R}^3 . Then,
 - a. $\|Ux\| = \|x\|$
 - b. $(Ux) \cdot (Uy) = x \cdot y$
 - c. $(Ux) \cdot (Uy) = 0$ if and only if $x \cdot y = 0$
- **Remark:** Properties a and c say that the linear mapping $x \rightarrow Ux$ preserves lengths and orthogonality.

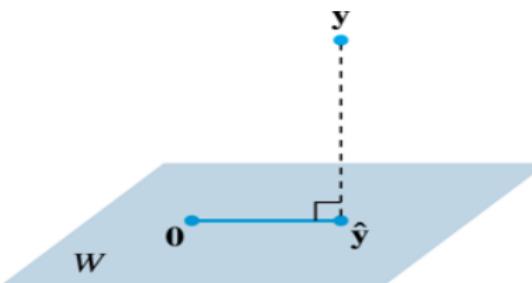
Suggested Exercises

- 6.2.2

6.3. Orthogonal Projections

Orthogonal Projections

- The orthogonal projection of a point in \mathbb{R}^2 onto a line through the origin has an important analogue in \mathbb{R}^n .
- Given a vector \mathbf{y} and a subspace W in \mathbb{R}^n , there is a vector in $\hat{\mathbf{y}}$ in W such that
 - $\hat{\mathbf{y}}$ is the unique vector in W for which $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to W , and
 - $\hat{\mathbf{y}}$ is the unique vector in W closest to \mathbf{y} .
- See the following figure.
- These two properties of $\hat{\mathbf{y}}$ provide the key to finding the least-squares solutions of linear systems.



The Orthogonal Decomposition Theorem

- **Theorem 8:** Let W be a subspace of \mathbb{R}^n .
 - Then each \mathbf{y} in \mathbb{R}^n can be written uniquely in the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}, \quad (1)$$

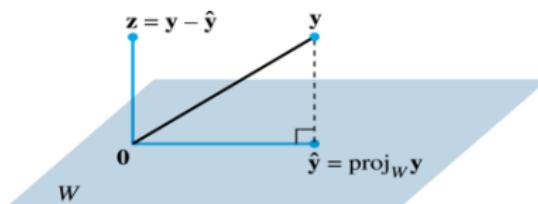
where $\hat{\mathbf{y}}$ is in W and \mathbf{z} is in W^\perp .

- In fact, if $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is any orthogonal basis of W then,

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \cdots + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p} \mathbf{u}_p \quad (2)$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$

- The vector $\hat{\mathbf{y}}$ in (1) is called the **orthogonal projection of \mathbf{y} onto W** and often is written as $\text{proj}_W \mathbf{y}$. See the following figure:



The orthogonal projection of \mathbf{y} onto W .

• Proof for $\hat{y} \in W$ and $z \in W^\perp$:

- Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be any orthogonal basis for W , and define \hat{y} by (2). Then \hat{y} is in W because \hat{y} is a linear combination of the basis $\mathbf{u}_1, \dots, \mathbf{u}_p$
- Let $\mathbf{z} = \mathbf{y} - \hat{y}$. Since \mathbf{u}_1 is orthogonal to $\mathbf{u}_1, \dots, \mathbf{u}_p$, it follows from (2) that

$$\mathbf{z} \cdot \mathbf{u}_1 = (\mathbf{y} - \hat{y}) \cdot \mathbf{u}_1 = \mathbf{y} \cdot \mathbf{u}_1 - \left(\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \right) \mathbf{u}_1 \cdot \mathbf{u}_1 = \mathbf{y} \cdot \mathbf{u}_1 - \mathbf{y} \cdot \mathbf{u}_1 = 0$$

- Thus, \mathbf{z} is orthogonal to \mathbf{u}_1 . Similarly, \mathbf{z} is orthogonal to each \mathbf{u}_j in the basis for W .
- Hence \mathbf{z} is orthogonal to every vector in W . That is, \mathbf{z} is in W^\perp

• Proof for unique representation:

- To show that the decomposition in (1) is unique, suppose \mathbf{y} can also be written as \mathbf{y} be also written as $\mathbf{y} = \mathbf{y}_1 + \mathbf{z}_1$, with \mathbf{y}_1 in W and \mathbf{z}_1 in W^\perp
- Then, $\mathbf{y} + \mathbf{z} = \mathbf{y}_1 + \mathbf{z}_1$ (since both sides equal \mathbf{y}), and so $\mathbf{y} - \mathbf{y}_1 = \mathbf{z}_1 - \mathbf{z}$
- This equality shows that the vector $\mathbf{v} = \mathbf{y} - \mathbf{y}_1$ is in W and in W^\perp (because \mathbf{z}_1 and \mathbf{z} are both in W^\perp , and W^\perp is a subspace).
- Hence $\mathbf{v} \cdot \mathbf{v} = 0$ which shows that $\mathbf{v} = 0$.
- This proves that $\mathbf{y} = \mathbf{y}_1$ and also $\mathbf{z}_1 = \mathbf{z}$
- The uniqueness of the decomposition (1) shows that the orthogonal projection \hat{y}_1 depends only on W and not on the particular basis used in (2).

Example 1:

- Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
- Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.
- Write \mathbf{y} as the sum of a vector in W and a vector orthogonal to W .

Solution:

- The orthogonal projection of \mathbf{y} onto W is

$$\begin{aligned}\hat{\mathbf{y}} &= \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 \\ &= \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \frac{9}{30} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{15}{30} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}\end{aligned}$$

- Also, $\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$
- Theorem 8 ensures that $\mathbf{y} - \hat{\mathbf{y}}$ is in W^\perp

• (*solution continued:*)

- To check the calculations, verify that $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to both \mathbf{u}_1 and \mathbf{u}_2 and hence to all of W

- The desired decomposition of \mathbf{y} is $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$

Properties of Orthogonal Projections

- If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be any orthogonal basis for W and if \mathbf{y} happens to in W , then the formula for $\text{proj}_L \mathbf{y}$ is exactly the same as the representation of \mathbf{y} given in Theorem 5 in Section 6.2. In this case, $\text{proj}_W \mathbf{y} = \mathbf{y}$
- If \mathbf{y} in $W = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$, then $\text{proj}_W \mathbf{y} = \mathbf{y}$

The Best Approximation Theorem

- **Theorem 9:** Let W be a subspace of \mathbb{R}^n , let \mathbf{y} be any vector in \mathbb{R}^n , and let $\hat{\mathbf{y}}$ be the orthogonal projection of \mathbf{y} onto W . Then $\hat{\mathbf{y}}$ is the closest point in W to \mathbf{y} , in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \quad (3)$$

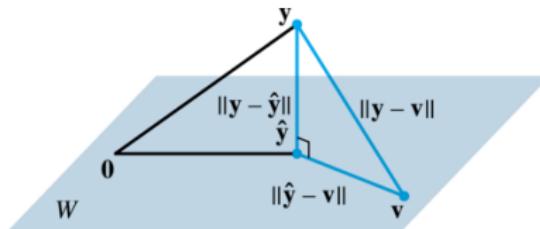
for all \mathbf{v} in W distinct from $\hat{\mathbf{y}}$

• Remarks

- The vector $\hat{\mathbf{y}}$ in Theorem 9 is called the best approximation to \mathbf{y} by elements of W .
- The distance from \mathbf{y} to \mathbf{v} , given by $\|\mathbf{y} - \mathbf{v}\|$, can be regarded as the “error” of using \mathbf{v} in place of \mathbf{y} .
- Theorem 9 says that this error is minimized when $\mathbf{v} = \hat{\mathbf{y}}$
- Inequality (3) leads to a new proof that $\hat{\mathbf{y}}$ does not depend on the particular orthogonal basis used to compute it.
- If a different orthogonal basis for W were used to construct an orthogonal projection of \mathbf{y} , then this projection would also be the closest point in W to \mathbf{y} , namely, $\hat{\mathbf{y}}$.

• Proof for the theorem:

- Take \mathbf{v} in W distinct from $\hat{\mathbf{y}}$. See the following figure:



The orthogonal projection of \mathbf{y} onto W is the closest point in W to \mathbf{y} .

- Then, $\hat{\mathbf{y}} - \mathbf{v}$ is in W . By the Orthogonal Decomposition Theorem, $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to W . In particular, $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to $\hat{\mathbf{y}} - \mathbf{v}$ (which is in W).
- Since

$$\mathbf{y} - \mathbf{v} = (\mathbf{y} - \hat{\mathbf{y}}) + (\hat{\mathbf{y}} - \mathbf{v}),$$

the Pythagorean Theorem gives

$$\|\mathbf{y} - \mathbf{v}\|^2 = \|\mathbf{y} - \hat{\mathbf{y}}\|^2 + \|\hat{\mathbf{y}} - \mathbf{v}\|^2$$

- (See the colored right triangle in the figure. The length of each side is labeled.)
- Now $\|\mathbf{y} - \mathbf{v}\|^2 > 0$ because $\mathbf{y} - \mathbf{v} \neq 0$, and so inequality (3) follows immediately.

- **Example 4:** The distance from a point \mathbf{y} in \mathbb{R}^n to a subspace W is defined as the distance from \mathbf{y} to the nearest point in W . Find the distance from \mathbf{y} to $Span\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Solution :

- By the Best Approximation Theorem, the distance from \mathbf{y} to W is $\|\mathbf{y} - \hat{\mathbf{y}}\|$, where $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$.
- Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for W ,

$$\hat{\mathbf{y}} = \frac{15}{30}\mathbf{u}_1 + \frac{-21}{6}\mathbf{u}_2 = \frac{1}{2} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + -\frac{7}{2} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}$$

$$\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

$$\|\mathbf{y} - \hat{\mathbf{y}}\|^2 = 3^2 + 6^2 = 45$$

- The distance from \mathbf{y} to W is $\sqrt{45} = 3\sqrt{5}$

• Theorem 10:

- If $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of \mathbb{R}^n , then

$$\text{proj}_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y} \cdot \mathbf{u}_p)\mathbf{u}_p \quad (4)$$

- If $U = [\mathbf{u}_1 \mathbf{u}_2 \dots \mathbf{u}_p]$, then for all \mathbf{y} in \mathbb{R}^n ,

$$\text{proj}_W \mathbf{y} = \mathbf{U}\mathbf{U}^T \mathbf{y} \quad (5)$$

• Proof:

- Formula (4) follows immediately from (2) in Theorem 8.
- Also, (4) shows that $\text{proj}_W \mathbf{y}$ is a linear combination of the columns of U using the weights $\mathbf{y} \cdot \mathbf{u}_1, \mathbf{y} \cdot \mathbf{u}_2, \dots, \mathbf{y} \cdot \mathbf{u}_p$.
- The weights can be written as $\mathbf{u}_1^T \mathbf{y}, \mathbf{u}_2^T \mathbf{y}, \dots, \mathbf{u}_p^T \mathbf{y}$, showing that they are the entries in $\mathbf{U}^T \mathbf{y}$ and justifying (5).

6.4 Gram-Schmidt process

Gram-Schmidt process

• Theorem 11: The Gram-Schmidt Process

- Given a basis $\{x_1, \dots, x_p\}$ for a nonzero subspace W of \mathbb{R}^n , define

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}$$

- Then $\{v_1, \dots, v_p\}$ is an orthogonal basis for W .
- In addition $\text{Span}\{v_1, \dots, v_k\} = \text{Span}\{x_1, \dots, x_k\}$ for $1 \leq k \leq p$.

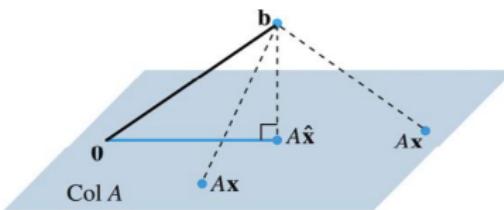
6.5 Least-Squares Problems

Least-Squares Problems

- **Definition:** If A is $m \times n$ and \mathbf{b} is in \mathbb{R}^m , a **least-squares solution** of $A\mathbf{x} = \mathbf{b}$ is an \mathbf{x} in \mathbb{R}^n such that

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| \leq \|\mathbf{b} - A\mathbf{x}\|$$

for all \mathbf{x} in \mathbb{R}^n .



The vector \mathbf{b} is closer to $A\hat{\mathbf{x}}$ than to $A\mathbf{x}$ for other \mathbf{x} .

• Example 1



- **Theorem 14:** Let A be an $m \times n$ matrix. The following statements are logically equivalent:
 - a. The equation $Ax = b$ has a unique least-squares solution for each b in \mathbb{R}^m .
 - b. The columns of A are linearly independent.
 - c. The matrix $A^T A$ is invertible.
 - d. The normal equation has a unique solution.

• Example 4

Suggested Exercises

- 6.5.2
- 6.8.1

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