ITM426, Quiz 3, 2022 Fall

Solution and Grading

- ITM 426 Engineering Mathematics
- Nov 25, 2022
- Duration: 90 minutes
- Weights: 25% or 30% depending on other quiz scores
- 6 Questions

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- Write legibly.
- Justification is necessary unless stated otherwise.
- Partial points are given only sparingly for the most problems because you are expected to 1) carry out proper sanity check and 2) correct your mistake by doing so.

1	20
2	10
3	20
4	20
5	15
6	15
Total	100

#1. Answer the following questions. `[Each 10pt]`

- If the subspace of all solutions of $A\mathbf{x} = \mathbf{0}$ has a basis consisting of three vectors and if A is 5×7 matrix, what is the rank of A?

- Construct a 4×3 matrix with rank 1.

Difficulty: Easy
Amount of work: 20 %
Suggested answer:

- The fact that the solution space of $A\mathbf{x} = \mathbf{0}$ has a basis of three vectors means that $dim\, Nul\ A = 3$. Since a 5×7 matrix A has 7 columns, the Rank Theorem shows that $rank\, A = 7 - dim\, Nul\ A = 4$.

- A rank 1 matrix has a one-dimensional column space. Every column is a multiple of some fixed vector. To construct a 4×3 matrix, choose any nonzero vector in $\mathbb{R}^4$, and use it for one column. Choose any multiples of the vector for the other two columns.

#2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$. Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why or why not? [10pt]

Difficulty: Easy
Amount of work: 10 %
Suggested answer:
The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution. Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span full 3-dimensional space. $\mathbf{w}$ cannot be expressed as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

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#3. Assume that A is row equivalent to B. Find bases for $Nul\,A$ and $Col\,A$.[Each 10pt]

$$A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Difficulty: Easy
Amount of work: 20 %
Suggested answer:

The information $A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 0 & 1 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ shows that columns 1, 2, and 4 of A form a basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -7 \\ 11 \end{bmatrix}$. For Nul A,

$$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & -9 & 0 & 5 & 0 \\ 0 & 1 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \begin{aligned} x_1 \quad - 9x_3 \quad + 5x_5 &= 0 \\ x_2 + 2x_3 \quad - 3x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned}$$

x_3 and x_5 are free variables

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 9x_3 - 5x_5 \\ -2x_3 + 3x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}. \text{ Basis for Nul } A: \begin{bmatrix} 9 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

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#4. For the following matrix, one eigenvalue is 5 and one eigenvector is $(-2, 1, 2)$. Identify all eigenvalues and their corresponding eigenvector. Then, perform a diagonalization. [20pt]

$$A = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}$$

Difficulty: Medium
Amount of work: 20 %
Suggested answer:

An eigenvalue of A is given to be 5; an eigenvector $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is also given. To find the eigenvalue corresponding to $\mathbf{v}_1$, compute $A\mathbf{v}_1 = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = -3\mathbf{v}_1$. Thus the eigenvalue in question is -3.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -12 & -16 & 4 \\ 6 & 8 & -2 \\ 12 & 16 & -4 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-5I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 4/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2\begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$, and a nice basis for the eigenspace is $\{\mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 1 \\ 1 & 3 & 0 \\ 2 & 0 & 3 \end{bmatrix}$. Then set $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively. Note that this answer differs from the text. There, $P = \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1 \end{bmatrix}$ and the entries in D are rearranged to match the new order of the eigenvectors. According to the Diagonalization Theorem, both answers are correct.

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#5. Compute the determinants of the following matrix `[15pt]`

$$A = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$$

Difficulty: Easy
Amount of work: 15 %
Suggested answer:

$$\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -15 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 10 \end{vmatrix} = -10$$

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#6. Show that if A is both diagonalizable and invertible, then A^{-1} is also diagonalizable. `[15pt]`

Difficulty: Hard
Amount of work: 15 %
Suggested answer:

If A is diagonalizable, then $A = PDP^{-1}$ for some invertible P and diagonal D. Since A is invertible, 0 is not an eigenvalue of A. So the diagonal entries in D (which are eigenvalues of A) are not zero, and D is invertible. By the theorem on the inverse of a product, $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$. Since D^{-1} is obviously diagonal, A^{-1} is diagonalizable.

Write your name before detaching this page. Your Name: ______________________