

## *Chapter 7. Symmetric Matrices and Quadratic Forms*

Sim, Min Kyu, Ph.D., [mksim@seoultech.ac.kr](mailto:mksim@seoultech.ac.kr)



## 1 7.1. Diagonalization of Symmetric Matrices

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## Symmetric Matrix

- A **symmetric matrix** is a matrix  $A$  such that  $A^T = A$ .
- Such a matrix is necessarily square.
- Its main diagonal entries are arbitrary, but its other entries occur in pairs—on opposite sides of the main diagonal.
- **Theorem 1:** If  $A$  is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.
- **Theorem 2:** An  $n \times n$  matrix  $A$  is orthogonally diagonalizable if and only if  $A$  is symmetric matrix.

- **Example 3:** Orthogonally diagonalize the matrix  $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ , whose characteristic equation is  $-\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2) = 0$

### • Solution:

- The usual calculations produce bases for the eigenspaces:

$$\lambda = 7 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = -2 : \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

- Although  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent, they are not orthogonal. The projection of  $\mathbf{v}_2$  onto  $\mathbf{v}_1$  is  $\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$ . The component of  $\mathbf{v}_2$  orthogonal to  $\mathbf{v}_1$  is

$$\mathbf{z}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

- Then  $\{\mathbf{v}_1, \mathbf{z}_2\}$  is an orthogonal set in the eigenspace for  $\lambda = 7$
- (Note that  $\mathbf{z}_2$  is linear combination of the eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , so  $\mathbf{z}_2$  is in the eigenspace.)

• (solution continued:)

- Since the eigenspace is two-dimensional (with basis  $\mathbf{v}_1, \mathbf{v}_2$ ), the orthogonal set  $\{\mathbf{v}_1, \mathbf{z}_2\}$  is an *orthogonal basis* for the eigenspace, by the Basis Theorem.
- Normalize  $\mathbf{v}_1$  and  $\mathbf{z}_2$  to obtain the following orthonormal basis for the eigenspace for  $\lambda = 7$ :

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

- An orthonormal basis for the eigenspace for  $\lambda = -2$  is

$$\mathbf{u}_3 = \frac{1}{\|2\mathbf{v}_3\|} 2\mathbf{v}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

- By Theorem 1,  $\mathbf{u}_3$  is orthogonal to the other eigenvectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$ . Hence  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  is an orthonormal set.

- Let  $P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}$ . Then  $P$  orthogonally diagonalizes  $A$ , and  $A = PDP^{-1}$ .



## Suggested Exercises

- 7.5.1
- 7.5.3
- 7.6 (p.450)



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