Chapter 4. Vector Spaces (2/2)

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4.5 The dimension of a vector space

Dimension of a vector space

- **Theorem 9:** If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.
- (*Proof skipped*)

- **Remark:** Theorem 9 implies that if a vector space V has a basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$, then each linearly independent set in V has no more than n vectors.

- **Theorem 10:** If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.
- (*Proof skipped*)

Dimension of a vector space

- **Definition:**
  - If V is spanned by a finite set, then V is said to be **finite-dimensional**, and
  - the **dimension** of V, written as dim V, is the number of vectors in a basis for V.
  - The dimension of the zero vector space $\{0\}$ is defined to be zero.
  - If V is not spanned by a finite set, then V is said to be **infinite-dimensional.**

The basis theorem

- **Theorem 12:** Let V be a p-dimensional vector space, $p \geq 1$.
  - Any linearly independent set of exactly p elements in V is automatically a basis for V.
  - Any set of exactly p elements that spans V is automatically a basis for V.

The dimensions of $Nul\ A$ and $Col\ A$.

- Let A be an $m \times n$ matrix, and suppose the equation $A\mathbf{x} = 0$ has k free variables.
  - # of var: n
    - # of free var.: k
    - # of pivot var.: $n - k$
  - $dim\, Nul\ A = k$
  - $dim\, Col\ A = n - k$
- A spanning set for $Nul\ A$ will produce exactly k linearly independent vectors — say,$\mathbf{u}_1, \ldots, \mathbf{u}_k$ — one for each free variable.
- So $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis for $Nul\ A$, and the number of free variables determines the size of the basis.
- Thus, the dimension of $Nul\ A$ is the number of free variables in the equation $A\mathbf{x} = 0$, and the dimension of $Col\ A$ is the number of pivot columns in A.

- **Example 5:** Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

- **Solution:**
  - Row reduce the augmented matrix $[A\,0]$ to echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - There are three free variable: x_2, x_4 and x_5. Hence the dimension of $Nul\,A$ is 3.
  - Also $dim\,Col\,A = 2$ because A has two pivot columns.

Suggested Excercises

- 4.5.13
- 4.5.19

4.6 Rank

The row space

- If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$
- The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by $Row\ A$.
- Each row has n entries, so $Row\ A$ is a subspace of $\mathbb{R}^n$.
- Since the rows of A are identified with the columns of A^T, we could also write $Col\ A^T$ in place of $Row\ A$.
- **Theorem 13:** If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

- **Example 2:** Find bases for the row space, the column space, and the null space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & -5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

- **Solution for row space:**
  - To find bases for the row space and the column space, row reduce A to an echelon form:

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - By Theorem 13, the first three rows of B form a basis for the row space of A (as well as for the row space of B). Thus,

Basis for $Row\ A : \{(1, 3, -5, 1, 5),\ (0, 1, -2, 2, -7),\ (0, 0, 0, -4, 20)\}$

- **Solution for column space:**
  - For the column space, observe from B that the pivots are in columns 1, 2, and 4. Hence, columns 1, 2, and 4 of A (not B) form a basis for $Col\ A$:

$$\text{Basis for } Col\ A = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

- Notice that any echelon form of A provides (in its nonzero rows) a basis for $Row\ A$ and also identifies the pivot columns of A for $Col\ A$.

- **Solution for null space:**
  - However, for $Nul\ A$, we need the *reduced echelon form*. Further row operations on *B* yield

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - The equation $A\mathbf{x} = 0$ is equivalent to $C\mathbf{x} = 0$, that is,

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 - 2x_3 + 3x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

  So, $x_1 = -x_3 - x_5,\ \ x_2 = 2x_3 - 3x_5,\ \ x_4 = 5x_5$, with x_3 and x_5 free variables.
  - The calculation shows that

$$\text{Basis for } Nul\ A = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

  - Observe that, unlike the basis for $Col\ A$, the bases for $Row\ A$ and $Nul\ A$ have no simple connection with the entries in A itself.

The rank theorem

- **Definition:** The rank of A is the dimension of the column space of A.
- **Remark**
  - Since $Row\, A$ is the same as $Col A^T$, the dimension of the row space of A is the rank of A^T.
  - The dimension of the null space ($dim\, Nul\, A$) is sometimes called the nullity of A.

- **Theorem 14:** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$rank\, A + dim\, Nul\, A = n$$

$$\left\{\begin{array}{c}\text{number of}\\ \text{pivot columns}\end{array}\right\} + \left\{\begin{array}{c}\text{number of}\\ \text{nonpivot columns}\end{array}\right\} = \left\{\begin{array}{c}\text{number of}\\ \text{columns}\end{array}\right\}$$

- **Example 3:**
  - *a.* If A is a 7 x 9 matrix with a two-dimensional null space, what is the rank of A?
  - *b.* Could a 6 x 9 matrix have a two-dimensional null space?
- **Solution:**
  - *a.* Since A has 9 columns, $rank\, A + 2 = 9$, and hence $rank\, A = 7$.
  - *b.* No. If a 6 x 9 matrix, call it B, has a two-dimensional null space, it would have to have rank 7, by the Rank Theorem. But the columns of B are vectors in $\mathbb{R}^6$, and so the dimension of $Col\, B$ cannot exceed 6; that is, $rank\, B$ cannot exceed 6.

The invertible matrix theorem (continued)

- **Theorem:** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.
  m. The columns of A form a basis of $\mathbb{R}^n$
  n. $Col\,A = \mathbb{R}^n$
  o. $dim\,Col\,A = n$
  p. $rank\,A = n$
  q. $Nul\,A = \{0\}$
  r. $dim\,Nul\,A = 0$

Suggested excercises

- 4.6.3
- 4.6.11

Acknowledgement

- This lecture note is based on the instructor's lecture notes (formatted as ppt files) provided by the publisher (Pearson Education) and the textbook authors (David Lay and others)
- The pdf conversion project for this chapter was possible thanks to the hard work by Jaemin Park (ITM 17'). Professor Sim recruited Mr. Park after his outstanding performance at Engineering Math. In the Applied Probability Lab, he is growing an intelligent reinforcement learning agent that optimizes energy storage systems (ESS).