ITM426, Final, 2024 Fall

Solution and Grading

- ITM 426 Engineering Mathematics 24F
- Dec 13, 2024
- Duration: 120 minutes
- Weights: 30%
- 5 Questions

- Name: ______________________
- Student ID: ______________________
- E-mail: ____________ `@seoultech.ac.kr`

- Write legibly.
- Justification is necessary unless stated otherwise.
- Partial points are given only sparingly for the most problems because you are expected to 1) carry out proper sanity check and 2) correct your mistake by doing so.

1	20
2	10
3	10
4	10
5	10
Total	60

#1. Answer the following short questions. `[Each 5pt]`

(a) State the key advantage and disadvantage of the Newton Method compared to the Fixed Point Method. (Unnecessary or incorrect statements will be penalized.)

(b) State the key difference between Newton method and Secant method. (Unnecessary or incorrect statements will be penalized.)

(c) Consider two vectors: $\mathbf{x} = (1, 1, 3, 1)$ and $\mathbf{y} = (3, 1, 1, 1)$. Let θ be the angle between these two vectors. What is the value of $cos\,\theta$?

(d) Mark True or False. No justification is necessary. (`TRUE` / `FALSE`)

> For an $m \times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space A.

Difficulty: Medium
Amount of work: 20%
Solution:

- Newton method is faster but requires an analytic derivative available.
- Secant method relies on the numerical derivative instead of the analytic derivative.
- Because $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|cos\,\theta$, we have $cos\,\theta = \frac{8}{\sqrt{12}\sqrt{12}} = 2/3$
- True. As it is stated.

Partial credit: In (c), for each minor computation mistake, -2.5pt.

#2. Consider $\mathbf{x}_1 = [1\ 1\ 1\ 1]^t$, $\mathbf{x}_1 = [0\ 1\ 1\ 1]^t$, and $\mathbf{x}_3 = [0\ 0\ 1\ 1]^t$. What is the distance between $Span\{\mathbf{x}_1, \mathbf{x}_2\}$ and $\mathbf{x}_3$?`[10pt]`

Difficulty: Medium-Hard
Amount of work: 20%
Solution:
Step 1) First to perform Gram-Schimdt process to generate $\mathbf{v}_1 = [1\ 1\ 1\ 1]^t$ and $\mathbf{v}_2 = [-3\ 1\ 1\ 1]^t$. So far, $Span\{\mathbf{x}_1, \mathbf{x}_2\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal basis.

Step 2) Projection of $\mathbf{x}_3$ onto $\mathbf{v}_1$ is $[1/2\ 1/2\ 1/2\ 1/2]^t$ and projection of $\mathbf{x}_3$ onto $\mathbf{v}_2$ is $[-1/2\ 1/6\ 1/6\ 1/6]^t$. Thus, $Proj_{Span\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{x}_3 = Proj_{\mathbf{v}_1}\mathbf{x}_3 + Proj_{\mathbf{v}_1}\mathbf{x}_3 = [0\ 2/3\ 2/3\ 2/3]^t$.

Step 3) It follows that the distance between $\mathbf{x}_3$ and $Proj_{Span\{\mathbf{x}_1,\mathbf{x}_2\}}\mathbf{x}_3$ is therefore $\|[0\ 0\ 1\ 1]^t - [0\ 2/3\ 2/3\ 2/3]^t\| = \|[0\ \ -2/3\ 1/3\ 1/3]\| = \sqrt{0^2 + (-2/3)^2 + (1/3)^2 + (1/3)^2} = \sqrt{2/3}$.

Alternative approach for the Step 1 and Step 2 is to use $Proj_{Span\{\mathbf{x}_1,\mathbf{x}_2\}}\mathbf{x}_3 = X(X^tX)^{-1}X^t\mathbf{x}_3$, where X is 4 by 2 matrix whose column vectors are $\mathbf{x}_1$ and $\mathbf{x}_2$

Partial credit: The most common mistake is missing the Gram-Schimdt process at the beginning. One must make original vectors into a orthogonal basis before making a projection. Step 1 counts for 5pts, Step 2 counts for 2.5pts, and Step 3 counts for 2.5 pts. For each minor calculation mistake, -2.5pts.

#3. Construct the normal equations for $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$$

and find the solution $\hat{\mathbf{x}}$. [10pt]

Difficulty: Medium
Amount of work: 20%
Solution:

To find the normal equations and to find $\hat{\mathbf{x}}$, compute

$$A^T A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}; \ A^T \mathbf{b} = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}.$$

a. The normal equations are $(A^T A)\mathbf{x} = A^T \mathbf{b}$: $\begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -24 \\ -2 \end{bmatrix}$.

b. Compute $\hat{\mathrm{x}} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}^{-1} \begin{bmatrix} -24 \\ -2 \end{bmatrix} = \frac{1}{56} \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix} \begin{bmatrix} -24 \\ -2 \end{bmatrix} = \frac{1}{56} \begin{bmatrix} -224 \\ 168 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

Partial credit: For each minor calculation mistake, -2.5pts.

#4. Perform a Cholesky decomposition for a matrix A.`[10pt]`

$$A = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix}$$

Difficulty: Medium
Amount of work: 20 %
Suggested answer:
$A = LL^t$, where $L = \begin{bmatrix} \sqrt{3} & 0 \\ 3\sqrt{3} & 2\sqrt{2} \end{bmatrix}$

#5. Let A be an $m \times n$ matrix such that $A^T A$ is invertible. Prove that the columns of A are linearly independent.[1] `[10pt]`

Difficulty: Hard
Amount of work: 20 %
Suggested answer:
Suppose $A\mathbf{x} = 0$. If the only solution to this is $\mathbf{x} = 0$, then the columns of A are linearly independent by definition. Indeed, this is the case by the following argument.

All $\mathbf{x}$ that solves $A\mathbf{x} = 0$ must solve $A^T A\mathbf{x} = 0$. In other words, the null space of A is a subset of the null space of $A^T A$. The null space of $A^t A$ has only one element, namely, $A\mathbf{x} = 0$ because $A^t A$ is invertible. Therefore, the null space of A has only one element, $\mathbf{x} = 0$ as well. From the statement at the beginning, the columns of A are therefore linearly independent.

[1] Warning: You may not assume that A is invertible. It may not even be square.

(detachable)