Chapter 5. Eigenvalues and Eigenvectors

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5.1. Eigenvectors and Eigenvalues

Eigenvectors and Eigenvalues

- **Definition**
  - An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.
  - A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution x of $A\mathbf{x} = \lambda\mathbf{x}$, such an x is called an *eigenvector corresponding to* λ.
- Remark
  - λ is an eigenvalue of an $n \times n$ matrix A if and only if the following equation has nontrivial solution.
$$(A - \lambda I)\mathbf{x} = 0 \tag{3}$$
  - The set of all solutions of (3) is just the null space of the matrix $A - \lambda I$.
  - So this set is a subspace of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ.
  - The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

- **Example 3:** Show that 7 is an eigenvalue of matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

- **Solution:** (proof that 7 is eigenvalue)
  - The scalar 7 is an eigenvalue of A if and only if the equation has a nontrivial solution.

$$A\mathbf{x} = 7\mathbf{x} \tag{1}$$

  - But (1) is equivalent to $A\mathbf{x} - 7\mathbf{x} = 0$ or

$$(A - 7I)\mathbf{x} = 0 \tag{2}$$

  - To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

  - The columns of $A - 7I$ are obviously linearly dependent, so (2) has nontrivial solutions.

- **Solution:** (finding its corresponding eigenvector)
  - To find the corresponding eigenvectors, use row operations:

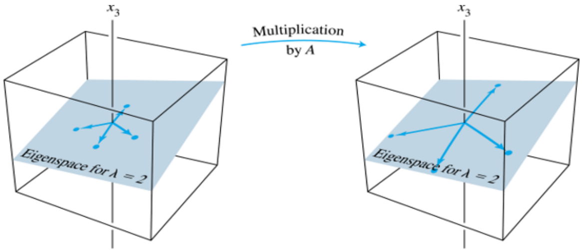
$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  - The general solution has the form $x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  - Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$.
  - (*If you multiply a constant to an eigenvector, it is again an eigenvector*.)

- **Example 4:** Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.
- **Solution:**
  - Form
    $$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$
    and row reduce the augmented matrix for $(A - 2I)\mathbf{x} = 0$.
    $$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
  - At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A - 2I)\mathbf{x} = 0$ has free variables.
  - The general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, x_2 and x_3 free.

- (*Solution continued*)
  - The eigenspace, shown in the following figure, is a two-dimensional subspace of $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$



A acts as a dilation on the eigenspace.

- **Theorem 1:** The eigenvalues of a triangular matrix are the entries on its main diagonal.

- **Proof:**
  - For simplicity, consider the 3×3 case. If A is upper triangular, the $A - \lambda I$ has the form

$$A-2I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$$

  - The scalar λ is an eigenvalue of A if and only if the equation $(A - \lambda I)\mathbf{x} = 0$ has a nontrivial solution, that is, if and only if the equation has a free variable.
  - Because of the zero entries in $A - \lambda I$, it is easy to see that $(A - \lambda I)\mathbf{x} = 0$ has a free variable if and only if at least one of the entries on the diagonal of $A - \lambda I$ is zero.
  - This happens if and only if λ equals one of the entries a_{11}, a_{22}, a_{33} in A.
  - Sim: "The equation ($(A - \lambda I)\mathbf{x} = 0$)" has a nontrivial solution if and only if $|A - \lambda I| = 0$.
  - Sim: This equation is also called as *characteristic polynomial*.

- **Theorem 2:** If $\mathbf{v_1}, ..., \mathbf{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, ..., \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v_1}, ..., \mathbf{v_r}\}$ is linearly independent.

- **Proof:**
  - Suppose $\{\mathbf{v_1}, ..., \mathbf{v_r}\}$ is linearly dependent. Since $\mathbf{v_1}$ is nonzero, Theorem 7 in Section 1.7 says that one of the vectors in the set is a linear combination of the preceding vectors.
  - Let p be the least index such that $\mathbf{v}_{p+1}$ is a linear combination of the preceding (linearly independent) vectors.
  - Then, there exist scalars $c_1, ..., c_p$ such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{v}_{p+1} \tag{5}$$

  - Multiplying both sides of (5) by A and using the fact that $Av_k = \lambda_k v_k$ for each k, we obtain $c_1 A\mathbf{v}_1 + \cdots + c_p A\mathbf{v}_p = A\mathbf{v}_{p+1}$

$$c_1\lambda_1\mathbf{v}_1 + ...c_p\lambda_p\mathbf{v}_p = \lambda_{p+1}\mathbf{v}_{p+1} \tag{6}$$

  - Multiplying both sides of (5) by λ_{p+1} and subtracting the result from (6), we have

$$c_1(\lambda_1\lambda_{p+1})\mathbf{v}_1 + ...c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p = 0 \tag{7}$$

  - Since $\mathbf{v_1}, ..., \mathbf{v_r}$ is linearly independent, the weights in (6) are all zero.

- (*Proof continued*)
  - But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct. Hence, $c_i = 0$ for $i = 1, ..., p$.
  - But then (4) says that $\mathbf{v}_{p+1} = 0$, which is impossible.
  - Hence $\{\mathbf{v_1}, ..., \mathbf{v_r}\}$ cannot be linearly dependent and therefore must be linearly independent.
  - If A is an $n \times n$ matrix, then (8) is a recursive description of a sequence $\{x_k\}$ in $\mathbb{R}^n$.

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad (k = 0, 1, 2...) \tag{8}$$

  - A **solution** of (8) is an explicit description of $\{x_k\}$ whose formula for each x_k does not depend directly on A or on the preceding terms in the sequence other than the initial term $\mathbf{x_0}$.
  - The simplest way to build a solution of (8) is to take an eigenvector $\mathbf{x}_0$ and its corresponding eigenvalue λ and let

$$\mathbf{x}_k = \lambda^k \mathbf{x}_0 \quad (k = 1, 2...) \tag{9}$$

  - This sequence is a solution because

$$A\mathbf{x}_k = A(\lambda^k \mathbf{x}_0) = \lambda^k (A\mathbf{x}_0) = \lambda^k (\lambda \mathbf{x}_0) = \lambda^{k+1}\mathbf{x}_0 = \mathbf{x}_{k+1}$$

$\square$

Suggeted Exercises

- 5.1.3
- 5.1.6
- 5.1.15

5.2. The Characteristic Equation

Determinants (review of)

- Let A be an $n \times n$ matrix, let U be any echelon form obtained from A by row replacements and row interchanges (without scaling), and let r be the number of such row interchanges.
- Then the **determinant** of A, written as $det\ A$, is $(-1)^r$ times the product of the diagonal entries $u_{11}, ..., u_{nn}$ in U.
- If A is invertible, then $u_{11}, ..., u_{nn}$ are all pivots (because $A \sim I_n$ and the u_{ii} have not been scaled to 1's).
- Otherwise, at least u_{nn} is zero, and the product $u_{11}, ..., u_{nn}$ is zero.
- Thus,

$$\det A = \begin{cases} (-1)^r \cdot (\text{product of pivots in } U), & \text{when A is invertible} \\ 0, & \text{when A is not invertible} \end{cases}$$

- **Example 1:** Compute det A for $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

- **Solution:**
  - The following row reduction uses one row interchange:

$$A \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = U_1$$

  - So $detA$ equals $(-1)^1(1)(-2)(-1) = -2$.
  - The following alternative row reduction avoids the row interchange and produces a different echelon form.
  - The last step adds $-1/3$ times row 2 to row 3:

$$A \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & 0 & 1/3 \end{bmatrix} = U_2$$

  - This time $detA$ is $(-1)^0(1)(-6)(1/3) = -2$, the same as before.

Theorems

- **The invertible matrix theorem (continued):** Let A be an $n \times n$ matrix. Then A is invertible if and only if:
  - s. The number 0 is not an eigenvalue of A.
  - t. The determinant of A is *not* zero.

- **Theorem 3: (Properties of Determinants)** Let A and B be $n \times n$ matrices.
  - a. A is invertible if and only if $det\ A \neq 0$.
  - b. $det\ AB = (det\ A)(det\ B)$.
  - c. $det\ A^T = det A$.
  - d. If A is triangular, then $det A$ is the product of the entries on the main diagonal of A.
  - e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

- Remark
  - Theorem 3(a) shows how to determine when a matrix of the form $A - \lambda I$ is not invertible.

The characteristic equation

- The scalar equation $det(A - \lambda I) = 0$ is called the characteristic equation (or, characteristic polynomial) of A.
- A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$det(A - \lambda I) = 0$$

- **Example 3:** Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Solution:**
  - Form $A - \lambda I$, and use Theorem 3(d):

$$\begin{aligned} det(A - \lambda I) &= det \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} \\ &= (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) \end{aligned}$$

  - The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0 \text{ or } (\lambda-5)^2(\lambda-3)(\lambda-1) = 0$$

  - Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

- **Remark**
  - If A is an $n \times n$ matrix, then $det(A - \lambda I)$ is a polynomial of degree n called the **characteristic polynomial** of A.
  - The eigenvalue 5 in Example 3 is said to have *multiplicity* 2 because $(\lambda - 5)$occurs two times as a factor of the characteristic polynomial.
  - In general, the **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Similarity

- If A and B are $n \times n$ matrices, then A **is similar to** B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently, $A = PBP^{-1}$.
- Writing Q for P^{-1}, we have $Q^{-1}BQ = A$.
- So B is also similar to A, and we say simply that A and B **are similar.**
- Changing A into $P^{-1}AP$ is called a **similarity transformation.**

- **Theorem 4:** If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).
- **Proof:**
  - If $B = P^{-1}AP$ then,
    $B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$
  - Using the multiplicative property in Theorem 3(b), we compute
    $det(B - \lambda I) = det[P^{-1}(A - \lambda I)P] = det(P^{-1}) \cdot det(A - \lambda I) \cdot det(P)$
  - Since $det(P^{-1}) \cdot det(P) = det(P^{-1}P) = det\, I = 1$, we see from the previous equation that $det(B - \lambda I) = det(A - \lambda I)$.

- **Warnings:**
  1. The matrices are not similar even though they have the same eigenvalues.

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

  2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E). Row operations on a matrix usually change its eigenvalues.

Suggested Exercises

- 5.2.3

5.3. Diagonalization

Diagonalization (its benefit and definition)

- **Example 2:** Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$
- **Solution:**
  - The standard formula for the inverse of a 2×2 matrix yields $P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$
  - Then, by associativity of matrix multiplication,

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1} = PDDP^{-1} \\ &= PD^2P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \end{aligned}$$

  - Again,

$$A^3 = (PDP^{-1})A^2 = (PD\underbrace{P^{-1})P}_{I}D^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

- (*solution continued*)
  - In general, for $k \geq 1$,

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix}$$

The diagonalization theorem

- **Definition:** A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal, matrix D.

- **Theorem 5 (diagonalization theorem):**
  a. An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
  b. In other words, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A.
  c. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

- **Remark:** In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenvector basis** of $\mathbb{R}^n$.

- **Proof** (*only if part of a & b; and the statement c*)
  - First, observe that if P is any $n \times n$ matrix with columns $\mathbf{v_1}, \cdots, \mathbf{v_n}$, and if D is any diagonal matrix with diagonal entries $\lambda_1, \cdots, \lambda_n$, then

$$AP = A[\mathbf{v_1}\ \mathbf{v_2}\ \cdots\ \mathbf{v_n}] = [A\mathbf{v_1}\ A\mathbf{v_2}\ \cdots\ A\mathbf{v_n}] \tag{1}$$

  while

$$PD = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1\mathbf{v_1}\ \lambda_2\mathbf{v_2}\ \cdots\ \lambda_n\mathbf{v_n}] \tag{2}$$

  - Now suppose A is diagonalizable and $A = PDP^{-1}$. Then right-multiplying this relation by P, we have $AP = PD$.
  - In this case, equations (1) and (2) imply that

$$[A\mathbf{v_1}\ A\mathbf{v_2}\ \cdots\ A\mathbf{v_n}] = [\lambda_1\mathbf{v_1}\ \lambda_2\mathbf{v_2}\ \cdots\ \lambda_n\mathbf{v_n}] \tag{3}$$

  - Equating columns, we find that

$$A\mathbf{v_1} = \lambda_1\mathbf{v_1}, A\mathbf{v_2} = \lambda_2\mathbf{v_2}, ..., A\mathbf{v_n} = \lambda_n\mathbf{v_n} \tag{4}$$

- (*proof continued*)
  - Since P is invertible, its columns $\mathbf{v_1}, \cdots, \mathbf{v_n}$ must be linearly independent.
  - Also, since these columns are nonzero, the equations in (4) show that $\lambda_1, \cdots, \lambda_n$ are eigenvalues and $\mathbf{v_1}, \cdots, \mathbf{v_n}$ are corresponding eigenvectors.
  - This argument proves the 'only if' parts of the first and second statements, along with the third statement, of the theorem.

- **Proof** (*if part of a & b*)
  - Finally, given any n eigenvectors $\mathbf{v_1}, \cdots, \mathbf{v_n}$, use them to construct the columns of P and use corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$ to construct D.
  - By equations (1)–(3), $AP = PD$.
  - This is true without any condition on the eigenvectors.
  - If, in fact, the eigenvectors are linearly independent, then P is invertible (by the Invertible Matrix Theorem), and $AP = PD$ implies that $A = PDP^{-1}$. $\square$

Diagonalizing Matrices

- **Example 3:** Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

- **Solution:** There are four steps to implement the description in Theorem 5.
  - **Step 1. Find the eigenvalues of A.**
  - Here, the characteristic equation turns out to involve a cubic polynomial that can be factored: $0 = det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$
  - The eigenvalues are $\lambda = 1$ and $\lambda = -2$.
  - **Step 2. Find three linearly independent eigenvectors of A.**
  - *Three* vectors are needed because A is a 3×3matrix.
  - This is a critical step. If it fails, then Theorem 5 says that A cannot be diagonalized.

- (*solution continued*)
  - Basis for $\lambda = 1$: $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
  - Basis for $\lambda = -2$: $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
  - You can check that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly independent set.
  - **Step 3. Construct P from the vectors in step 2.**
  - The order of the vectors is unimportant. Using the order chosen in step 2, form
    $P = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}] = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
  - **Step 4. Construct D from the corresponding eigenvalues.**
  - In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P.
  - Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to
    $\lambda = -2$: $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

- (*solution continued - sanity check*)
  - To avoid computing P^{-1}, simply verify that $AP = PD$.
  - Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

- **Theorem 6:** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

- **Proof:**
  - Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A.
  - Then $\{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is linearly independent, by Theorem 2 in Section 5.1.
  - Hence A is diagonalizable, by Theorem 5.

Materices whose eigenvalues are not distinct

- **Remarks**
  - It is not *necessary* for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable.
  - The 3×3 matrix in Example 3 is diagonalizable even though it has only two distinct eigenvalues.
  - If an $n \times n$ matrix A has n distinct eigenvalues, with corresponding eigenvectors $\mathbf{v_1}, ..., \mathbf{v_n}$, and if $P = [\mathbf{v_1}, ..., \mathbf{v_n}]$, then P is automatically invertible because its columns are linearly independent, by Theorem 2.
  - When A is diagonalizable but has fewer than n distinct eigenvalues, it is still possible to build P in a way that makes P automatically invertible, as the next theorem shows.

Suggested Exercises

- 5.3.12
- 5.3.14
- 5.3.21

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