Stochastic Processes, Mid-term, 2025 Fall

Solution and Grading

- Duration: 90 minutes
- Weight: 30% of final grade
- Closed material, No calculator

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- Write legibly.
- Justification is necessary unless stated otherwise.

#1. The selling price of lettuce salad is \$6, the buying price of one unit of lettuce is \$1. Of course, leftover lettuce of a day cannot be used for future salad and you have to pay 50 cents per unit of lettuce for disposal. The demand for the lettuce salad is given as discrete uniform distribution between 21 and 30.

(a) What is the optimal order quantity for lettuce salad? `[5pts]`

(b) What is the expected profit if 25 units of lettuce salad is ordered? `[5pts]`

(a) $c_o = 1.5, c_u = 5, \frac{c_u}{c_o+c_u} = \frac{5}{1.5+5} \approx 0.77$. Since $F(27) = 0.7$ and $F(28) = 0.8$, the optimal order quantity is 28.

(b)

$$\begin{aligned}
\mathbb{E}[Profit(25)] &= \mathbb{E}[\text{Revenue}] - \mathbb{E}[\text{Cost}] \\
&= \mathbb{E}[\text{Sales Revenue}] + \mathbb{E}[\text{Salvage Revenue}] - \mathbb{E}[\text{Cost}] \\
&= 6\mathbb{E}[min(25, D)] - 0.5\mathbb{E}[(25 - D)^+] - 1 \cdot 25 \\
&= 6 \cdot (1/10)[21 + 22 + 23 + 24 + 25 + 25 + 25 + 25 + 25 + 25] \\
&\quad -0.5 \cdot (1/10)[4 + 3 + 2 + 1 + 0 + 0 + 0 + 0 + 0 + 0] - 25 \\
&= 6 \cdot (1/10) \cdot 240 - 0.5 \cdot (1/10) \cdot 10 - 25 \\
&= 144 - 0.5 - 25 = 118.5
\end{aligned}$$

The distribution was given as *discrete uniform distribution*, but some of students instead took it as *continuous uniform distribution*. In such case, the optimal order quantity would be $x = 363/13$ or $x = 27.93$ because $F(x) = \frac{x-21}{9} = 0.77$. The expected profit when 25 units are ordered would be:

$$\begin{aligned}
\mathbb{E}[Profit(25)] &= 6\mathbb{E}[min(25, D)] - 0.5\mathbb{E}[(25 - D)^+] - 1 \cdot 25 \\
&= 6 \int_{21}^{30} \frac{1}{9} min(25, y)\, dy - 0.5 \int_{21}^{25} \frac{1}{9}(25 - y)\, dy - 25 \\
&= 434/3 - 4/9 - 25 = 1073/9 \approx 119.2
\end{aligned}$$

Difficulty: Medium
Amount of work: 30%
Partial Grading:
-2.5pts for using discrete uniform distribution.
-1pts for each minor computation mistake.

#2. A small bank is staffed by a single server. It has been observed that, during a normal business day, the inter-arrival times of customers to the bank are iid having exponential distribution with mean 3 minutes. Also, the processing times of customers are iid having the following distribution (in minutes):

$$P(X=1)=\frac{1}{4},\quad P(X=2)=\frac{1}{2},\quad P(X=3)=\frac{1}{4}.$$

An arrival finding the server busy joins the queue. The waiting space is infinite.

(a) What is long-run average waiting time of each customer in the queue? `[5pts]`

(b) What is long-run average number of customers in the bank? `[5pts]`

(a)
$\mathbb{E}[U]=3$ min, $\qquad \lambda=\frac{1}{3}, \qquad c_a^2=1.$

$\mathbb{E}[V]=1\cdot\frac{1}{4}+2\cdot\frac{1}{2}+3\cdot\frac{1}{4}=\frac{1+4+3}{4}=2$ min, $\qquad \mu=\frac{1}{2}.$

$\mathbb{E}[V^2]=1^2\cdot\frac{1}{4}+2^2\cdot\frac{1}{2}+3^2\cdot\frac{1}{4}=\frac{1+8+9}{4}=\frac{9}{2}.$

$\mathrm{Var}(V)=\mathbb{E}[V^2]-(\mathbb{E}[V])^2=\frac{9}{2}-4=\frac{1}{2},$

$$c_s^2=\frac{(\mathrm{sd}(V))^2}{(\mathbb{E}[V])^2}=\frac{\frac{1}{2}}{4}=\frac{1}{8}.$$

$$\rho=\frac{\lambda}{\mu}=\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{2}{3}.$$

$$\mathbb{E}[W_q]=\mathbb{E}[V]\cdot\frac{\rho}{1-\rho}\cdot\frac{c_a^2+c_s^2}{2}=2\cdot\frac{\frac{2}{3}}{1-\frac{2}{3}}\cdot\frac{1+\frac{1}{8}}{2}=\frac{9}{4}\text{ min.}$$

(b)
$L_q=\lambda\,\mathbb{E}[W_q]=\frac{1}{3}\cdot\frac{9}{4}=\frac{3}{4},$
$L_s=\lambda\,\mathbb{E}[V]=\frac{1}{3}\cdot 2=\frac{2}{3},$
$L_{\text{bank}}=L_q+L_s=\frac{3}{4}+\frac{2}{3}=\frac{17}{12}\approx 1.42.$

Difficulty: Medium
Amount of work: 30%
Partial Grading: -2.5pts for each error

#3. A company operates under an (S, s) inventory policy with parameters $S = 5$ and $s = 2$. That is, at the end of each day, the inventory level is checked. If the inventory is less than or equal to 2 units, an order is placed to raise the stock level to 5 units. Orders are replenished before the beginning of the next day. The daily demand follows a discrete uniform distribution between 1 and 4 units (inclusive).

(a) Any item of inventory at the end of the day counts toward the holding cost. If the holding cost is \$10 per unit per day, what is the average daily holding cost? `[5pts]`

(b) Over a 30-day month, how many times is an order placed on average? `[5pts]`

(a) Let the state be the *starting inventory* each day. Possible states are $\{3, 4, 5\}$, since $s = 2$ triggers replenishment to 5. From demand outcomes:

$$\begin{aligned}
&\text{From state 5: } D = 1, 2, 3, 4 \ \Rightarrow\ \text{next start} = 4, 3, 5, 5,\\
&\text{From state 4: } D = 1, 2, 3, 4 \ \Rightarrow\ \text{next start} = 3, 5, 5, 5,\\
&\text{From state 3: } D = 1, 2, 3, 4 \ \Rightarrow\ \text{next start} = 5, 5, 5, 5.
\end{aligned}$$

The transition matrix is therefore

$$P = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

This DTMC has a stationary distribution of $\pi = (0.20, 0.16, 0.64)$. Conditioned on the beginning count, the following is the expected number of items at the end.

$$\begin{aligned}
E[I_{\text{end}}|5] &= \frac{4+3+2+1}{4} = 2.5,\\
E[I_{\text{end}}|4] &= \frac{3+2+1+0}{4} = 1.5,\\
E[I_{\text{end}}|3] &= \frac{2+1+0+0}{4} = 0.75.
\end{aligned}$$

Weighted by π, the expected number of items at the end of the day is $0.20(0.75)+0.16(1.5)+0.64(2.5) = 1.99$. This leads to the holding cost per day of \$19.90 per day.

(b) Order occurs if the end of the day inventory is less than or equal to 2. Probability of ordering given start state is

$$P(\text{order}|5) = \frac{1}{2}, \quad P(\text{order}|4) = \frac{3}{4}, \quad P(\text{order}|3) = 1.$$

Thus,

$$E[\text{orders/day}] = 0.64\left(\frac{1}{2}\right) + 0.16\left(\frac{3}{4}\right) + 0.20(1) = 0.64.$$

Over 30 days is 19.2 orders.

Difficulty: Medium
Amount of work: 40%