Chapter 4. Vector Spaces (2/2)

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4.5 The dimension of a vector space

Dimension of a vector space

- **Theorem 9:** If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.
- *(Proof skipped)*

- **Remark:** Theorem 9 implies that if a vector space V has a basis $B = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$, then each linearly independent set in V has no more than n vectors.

- **Theorem 10:** If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.
- *(Proof skipped)*

Dimension of a vector space

- **Definition:**
  - If V is spanned by a finite set, then V is said to be **finite-dimensional**, and
  - the **dimension** of V, written as dim V, is the number of vectors in a basis for V.
  - The dimension of the zero vector space $\{0\}$ is defined to be zero.
  - If V is not spanned by a finite set, then V is said to be **infinite-dimensional.**

The basis theorem

- **Theorem 12:** Let V be a p-dimensional vector space, $p \geq 1$.
  - Any linearly independent set of exactly p elements in V is automatically a basis for V.
  - Any set of exactly p elements that spans V is automatically a basis for V.

The dimensions of $Nul\ A$ and $Col\ A$.

- Let A be an $m \times n$ matrix, and suppose the equation $A\mathbf{x} = 0$ has k free variables.
  - # of var: n
    - # of free var.: k
    - # of pivot var.: $n - k$
  - $dim\, Nul\ A = k$
  - $dim\, Col\ A = n - k$

- A spanning set for $Nul\ A$ will produce exactly k linearly independent vectors — say,$\mathbf{u}_1, \ldots, \mathbf{u}_k$ — one for each free variable.

- So $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a basis for $Nul\ A$, and the number of free variables determines the size of the basis.

- Thus, the dimension of $Nul\ A$ is the number of free variables in the equation $A\mathbf{x} = 0$, and the dimension of $Col\ A$ is the number of pivot columns in A.

- **Example 5:** Find the dimensions of the null space and the column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

- **Solution:**
    - Row reduce the augmented matrix $[A\,0]$ to echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

    - There are three free variable: x_2, x_4 and x_5. Hence the dimension of $Nul\ A$ is 3.
    - Also $dim\,Col\,A = 2$ because A has two pivot columns.

Suggested Exercises

- 4.5.13
- 4.5.19

4.6 Rank

The row space

- If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$
- The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by $Row\ A$.
- Each row has n entries, so $Row\ A$ is a subspace of $\mathbb{R}^n$.
- Since the rows of A are identified with the columns of A^T, we could also write $Col\ A^T$ in place of $Row\ A$.
- **Theorem 13:** If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

- **Example 2:** Find bases for the row space, the column space, and the null space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & -5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

- **Solution for row space:**
  - To find bases for the row space and the column space, row reduce A to an echelon form:

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - By Theorem 13, the first three rows of B form a basis for the row space of A (as well as for the row space of B). Thus,

Basis for $Row\, A : \{(1, 3, -5, 1, 5),\ (0, 1, -2, 2, -7),\ (0, 0, 0, -4, 20)\}$

- **Solution for column space:**
  - For the column space, observe from B that the pivots are in columns 1, 2, and 4. Hence, columns 1, 2, and 4 of A (not B) form a basis for $Col\ A$:

$$\text{Basis for } Col\ A = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

- Notice that any echelon form of A provides (in its nonzero rows) a basis for $Row\ A$ and also identifies the pivot columns of A for $Col\ A$.

- **Solution for null space:**
  - However, for $Nul\ A$, we need the *reduced echelon form*. Further row operations on B yield

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - The equation $A\mathbf{x} = 0$ is equivalent to $C\mathbf{x} = 0$, that is,

$$\begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 - 2x_3 + 3x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

    So, $x_1 = -x_3 - x_5,\ \ x_2 = 2x_3 - 3x_5,\ \ x_4 = 5x_5$, with x_3 and x_5 free variables.
  - The calculation shows that

$$\text{Basis for } Nul\ A = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

  - Observe that, unlike the basis for $Col\ A$, the bases for $Row\ A$ and $Nul\ A$ have no simple connection with the entries in A itself.

The rank theorem

- **Definition:** The rank of A is the dimension of the column space of A.
- **Remark**
  - Since $Row\ A$ is the same as $Col A^T$, the dimension of the row space of A is the rank of A^T.
  - The dimension of the null space ($dim\ Nul\ A$) is sometimes called the nullity of A.

- **Theorem 14:** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$rank\, A + dim\, Nul\, A = n$$

$$\left\{\begin{array}{c}\text{number of}\\ \text{pivot columns}\end{array}\right\} + \left\{\begin{array}{c}\text{number of}\\ \text{nonpivot columns}\end{array}\right\} = \left\{\begin{array}{c}\text{number of}\\ \text{columns}\end{array}\right\}$$

- **Example 3:**
  - *a.* If A is a 7 x 9 matrix with a two-dimensional null space, what is the rank of A?
  - *b.* Could a 6 x 9 matrix have a two-dimensional null space?
- **Solution:**
  - *a.* Since A has 9 columns, $rank\, A + 2 = 9$, and hence $rank\, A = 7$.
  - *b.* No. If a 6 x 9 matrix, call it B, has a two-dimensional null space, it would have to have rank 7, by the Rank Theorem. But the columns of B are vectors in $\mathbb{R}^6$, and so the dimension of $Col\, B$ cannot exceed 6; that is, $rank\, B$ cannot exceed 6.

The invertible matrix theorem (continued)

- **Theorem:** Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.
  - m. The columns of A form a basis of $\mathbb{R}^n$
  - n. $Col\,A = \mathbb{R}^n$
  - o. $dim\,Col\,A = n$
  - p. $rank\,A = n$
  - q. $Nul\,A = \{0\}$
  - r. $dim\,Nul\,A = 0$

Suggested excercises

- 4.6.3
- 4.6.11

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