Chapter 4. Vector Spaces (1/2)

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4.1. Vector Spaces and Subspaces

Vector spaces

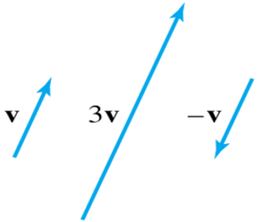
- **Definition:** A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (*real numbers*), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.
  1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
  2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
  3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
  4. There is a zero vector 0 in V such that $\mathbf{u}+0=\mathbf{u}$
  5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=0$
  6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
  7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
  8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
  9. $c(d\mathbf{u})=(cd)\mathbf{u}$
  10. $1\mathbf{u}=\mathbf{u}$

- Using these axioms, we can show that
  - the zero vector in Axiom 4 is unique, and
  - the vector $-\mathbf{u}$, called the **negative** of $\mathbf{u}$, in Axiom 5 is unique for each $\mathbf{u}$ in V.
  - *The identity and inverse with respect to vector addition are unique.*

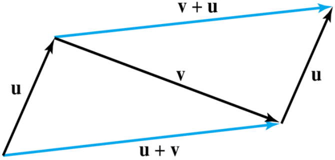
- For each $\mathbf{u}$ in V and scalar c,

$$\begin{aligned} 0\mathbf{u} &= 0 \\ c0 &= 0 \\ -\mathbf{u} &= (-1)\mathbf{u} \end{aligned}$$

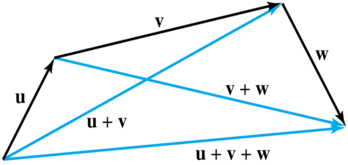
- **Example 2:**
  - Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction.
  - Define addition by the parallelogram rule, and for each $\mathbf{v}$ in V.
  - Define $c\mathbf{v}$ to be the arrow whose length is $|c|$ times the length of $\mathbf{v}$, pointing in the same direction as $\mathbf{v}$ if $c \geq 0$ and otherwise pointing in the opposite direction.
  - See the figure below. Show that V is a vector space.



- **Solution:**
  - The definition of V is geometric, using concepts of length and direction. No x y z-coordinate system is involved. An arrow of zero length is a single point and represents the zero vector.
  - The negative of **v** is $(-1)\mathbf{v}$.
  - So Axioms 1, 4, 5, 6, and 10 are evident. See the following figures.



$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.



$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.

Subspaces

- **Definition**: A **subspace** of a vector space V is a subset H of V that has three properties:
  a. The zero vector of V is in H.
  b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
  c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

- **Remark**
  - Properties (a), (b), and (c) guarantee that a subspace H of V is itself a vector space, under the vector space operations already defined in V.
  - Every subspace is a vector space.
  - Conversely, every vector space is a subspace (of itself and possibly of other larger spaces).

A subspace spanned by a set

- The set consisting of only the zero vector in a vector space V is a subspace of V, called the **zero subspace** and written as $\{0\}$.
- As the term **linear combination** refers to any sum of scalar multiples of vectors, and $Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ denotes the set of all vectors that can be written as linear combinations of $\mathbf{v}_1, ..., \mathbf{v}_p$.

- **Example 10**: Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = Span\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V.

- **Solution**
  1. The *zero vector* is in H, since $0 = 0\mathbf{v}_1 + 0\mathbf{v}_2$.
  2. To show that H is *closed under vector addition*, take two arbitrary vectors in H, say. $\mathbf{u} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2$ and $\mathbf{w} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$. By Axioms 2, 3, and 8 for the vector space V,
  $$\mathbf{u} + \mathbf{w} = (s_1\mathbf{v}_1 + s_2\mathbf{v}_2) + (t_1\mathbf{v}_1 + t_2\mathbf{v}_2) = (s_1 + t_1)\mathbf{v}_1 + (s_2 + t_2)\mathbf{v}_2$$
  So, $\mathbf{u} + \mathbf{w}$ is in H.
  3. Furthermore, if c is any scalar, then by Axioms 7 and 9,
  $$c\mathbf{u} = c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2) = (cs_1)\mathbf{v}_1 + (cs_2)\mathbf{v}_2$$
  which shows that $c\mathbf{u}$ is in H and H is *closed under scalar multiplication*.
  - Thus, H is a subspace of V.

- **Theorem 1**: If $\mathbf{v}_1, ..., \mathbf{v}_p$ are in a vector space V, then $Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ is a subspace of V.
- We call $Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ **the subspace spanned** (or **generated**) by $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$.
- Given any subspace H of V, a **spanning** (or **generating**) set for H is a set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ in H such that $H = \{\mathbf{v}_1, ..., \mathbf{v}_p\}$.

Suggested Exercise

- 4.1.13

4.2 Null spaces, Colum spaces, and Linear transformation

Null space of matrix

- **Definition:** The **null space** of an $m \times n$ matrix A, written as $Nul\,A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = 0$. In set notation,

$$Nul\,A = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^n \text{ and } A\mathbf{x} = 0\}$$

- **Theorem 2**: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$

- **Proof**:
  - $Nul\ A$ is a subset of $\mathbb{R}^n$ because A has n columns.
  - We need to show that $Nul\ A$ satisfies the three properties of a subspace.
    1. 0 is in $Nul\ A$ (trivial solution)
    2. Next, let $\mathbf{u}$ and $\mathbf{v}$ represent any two vectors in $Nul\ A$. Then, $A\mathbf{u} = 0$ and $A\mathbf{u} = 0$. To show that $\mathbf{u} + \mathbf{v}$ is in $Nul\ A$, we must show that $A(\mathbf{u} + \mathbf{v}) = 0$. Using a property of matrix multiplication, we have $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = 0 + 0 = 0$. Thus, $\mathbf{u} + \mathbf{v}$ is in $Nul\ A$, and $Nul\ A$ is closed under vector addition.
    3. Finally, if c is any scalar, then $A(c\mathbf{u}) = c(Au) = c(0) = 0$, which shows that $c\mathbf{u}$ is in $Nul\ A$.
    - Thus, $Nul\ A$ is a subspace of $\mathbb{R}^n$

- **An Explicit Description of $Nul\ A$**
  - There is no obvious relation between vectors in $Nul\ A$ and the entries in A.
  - We say that $Nul\ A$ is defined *implicitly*, because it is defined by a condition that must be checked.
  - No explicit list or description of the elements in $Nul\ A$ is given.
  - *Solving* the equation $A\mathbf{x} = 0$ amounts to producing an *explicit* description of $Nul\ A$

- **Example 3:** Fine a spanning set for the null space of the matrix

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

1. The first step is to find the general solution of $A\mathbf{x} = 0$ in terms of free variables. Row reduce the augmented matrix $[A \mid 0]$ to *reduce* echelon form in order to write the basic variables in terms of the free variables:

$$A = \begin{pmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

2. The general solution is
   - $x_1 = 2x_2 + x_4 - 3x_5$
   - $x_3 = -2x_4 + 2x_5$
   - x_2, x_4, x_5 free.

3. Next, decompose the vector giving the general solution into a linear combination of *vectors where the weights are the free variables*. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$
$$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$$

4. Every linear combination of **u**, **v**, and **w** is an element of $Nul\ A$. Thus, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for $Nul\ A$. $\square$

- Remark
  1. The spanning set produced by the method in **Example 3** is automatically linearly independent because the free variables are the weights on the spanning vectors.
  2. When $Nul\ A$ contains nonzero vectors, the number of vectors in the spanning set for $Nul\ A$ equals the number of free variables in the equation $A\mathbf{x} = 0$.

Column space of matrix

- **Definition:** The **column space** of an $m \times n$ matrix A, written as $Col\ A$, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1, ..., \mathbf{a}_n]$, then

$$Col\ A = Span\{\mathbf{a}_1, ..., \mathbf{a}_n\}$$

- **Theorem 3**: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$

- Remark
    - A typical vector in $Col\ A$ can be written as $A\mathbf{x}$ for some $\mathbf{x}$ because the notation $A\mathbf{x}$ stands for a linear combination of the columns of A. That is,

    $$Col\ A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$$

    - The notation $A\mathbf{x}$ for vectors in $Col\ A$ also shows that $Col\ A$ is the *range* of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.
    - The column space of an $m \times n$ matrix A is all of $\mathbb{R}^n$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

- **Example 7:** Let

$$A = \begin{bmatrix} 2 & -4 & -2 & -1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \ u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \ v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

a. Determine if **u** is in $Nul\ A$. Could **u** be in $Col\ A$?
b. Determine if **v** is in $Col\ A$. Could **v** be in $Nul\ A$?

- **Solution to (a)**
  - An explicit description of $Nul\ A$ is not needed here. Simply compute the product $A\mathbf{u}$.

$$Au = \begin{bmatrix} 2 & -4 & -2 & -1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

u is not a solution of $A\mathbf{x} = 0$, so **u** is not in $Nul\ A$. Also, with four entries, **u** could not possibly be in $Col\ A$, since $Col\ A$ is a subspace of $\mathbb{R}^3$.

- **Solution to (b)**
  - Reduce $[A \mid \mathbf{v}]$ to an echelon form.

$$[A \quad v] = \begin{bmatrix} 2 & -4 & -2 & -1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & -4 & -2 & -1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{v}$ is consistent, so v is in $Col\ A$. With only three entries, $\mathbf{v}$ could not possibly be in $Nul\ A$, since $Nul\ A$ is a subspace of $\mathbb{R}^4$.

Kernel and range of linear transformation

- Subspaces of vector spaces other than $\mathbb{R}^n$ are often described in terms of a linear transformation instead of a matrix.

- **Definition**: A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that
  1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in V, and
  2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ and V and all scalars c.

- **Definition**:
  - The **kernel** (or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that $T(u) = 0$ (the zero vector in W).
  - The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

- **Remark**:
  - The kernel of T is a subspace of V.
  - The range of T is a subspace of W.

Contrast between $Nul\ A$ and $Col\ A$ for an $m \times n$ matrix A

	$Nul\ A$	$Col\ A$
1	$Nul\ A$ is a subspace of $\mathbb{R}^n$	$Col\ A$ is a subspace of $\mathbb{R}^m$
2	$Nul\ A$ is implicitly defined, i.e., you are given only a condition ($A\mathbf{x} = 0$) that vectors in $Nul\ A$ must satisfy.	$Col\ A$ is explicitly defined, *i.e.*, you are told how to build vectors in $Col\ A$
3	It takes time to find vectors in $Nul\ A$. Row operation on $[A \mid 0]$ are required.	It is easy to find vectors in $Col\ A$. The columns of A are displayed; others are formed from them.
4	There is no obvious relation between $Nul\ A$ and the entries in A.	There is an obvious relation between $Col\ A$ and the entries in A, since each column of A is in $Col\ A$.

(*continued*)

	$Nul\ A$	$Col\ A$
5	A typical vector $\mathbf{v}$ in $Nul\ A$ has the property that $A\mathbf{v} = 0$.	A typical vector $\mathbf{v}$ in $Col\ A$ has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6	Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in $Nul\ A$. Just compare $A\mathbf{v}$.	Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in $Col\ A$. Row operation on $[A \mid \mathbf{v}]$ are required.
7	$Nul\ A = \{0\}$ if and only if the equation $A\mathbf{x} = 0$ has only the trivial solution.	$Col\ A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8	$Nul\ A = \{0\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is *one-to-one*.	$Col\ A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Suggested Excercises

- 4.2.5
- 4.2.17

4.3 Linearly independent sets; Bases

Linearly independent sets; Bases

- An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = 0 \tag{1}$$
has *only* the trivial solution, $c_1 = 0, \dots, c_p = 0$.

- The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if (1) has a nontrivial solution, *i.e.*, if there are some weights, $c_1, \dots, c_p$, *not all zero*, such that (1) holds. In such a case, (1) is called a **linear dependence relation** among $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

- **Theorem 4:** An indexed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq 0$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

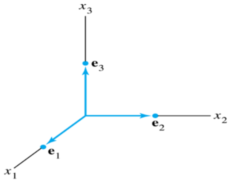
- **Definition:** Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a basis for H if
  - *i)* $\mathcal{B}$ is a linearly independent set, and
  - *ii)* The subspace spanned by $\mathcal{B}$ coincides with H; that is, $H = Span\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$
- **Remark:**
  - The definition of a basis applies to the case when $H = V$, because any vector space is a subspace of itself. Thus, a basis of V is a linearly independent set that spans V.
  - When $H \neq V$, condition ii) includes the requirement that each of the vectors $\mathbf{b}_1, \dots, \mathbf{b}_p$ must belong to H, because $Span\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ contains $\mathbf{b}_1, \dots, \mathbf{b}_p$.

Standard basis

- Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the columns of the $n \times n$ matrix, I_n.
- That is,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$. See the following figure.



The standard basis for $\mathbb{R}^3$.

The spanning set theorem

- **Theorem 5:** Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a set in V, and let $H = Span\{\mathbf{v}_1, \dots, \mathbf{v}_p$.
  a. If one of the vectors in S − say, $\mathbf{v}_k$ − is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.
  b. If $H \neq \{0\}$, some subset of S is a basis for H.

- **Proof for** a.
  - By rearranging the list of vectors in S, if necessary, we may suppose that $\mathbf{v}_p$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$ − say,

$$v_p = a_1 v_1 + \cdots + a_{p-1} v_{p-1} \tag{3}$$

  - Given any $\mathbf{x}$ in H, we may write

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_{p-1}\mathbf{v}_{p-1} + c_p\mathbf{v}_p \tag{4}$$

    for suitable scalars $c_1, c_2, \dots, c_p$.
  - Substituting the expression for $\mathbf{v}_p$ from (3) into (4), it is easy to see that $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$.
  - Thus, $\{\mathbf{v}_1, \dots, \mathbf{v}_{p-1}\}$ spans H, because $\mathbf{x}$ was an arbitrary element of H.

- **Proof for** b.
  - If the original spanning set S is linearly independent, then it is already a basis for H.
  - Otherwise, one of the vectors in S depends on the others and can be deleted, by part (a).
  - So long as there are two or more vectors in the spanning set, we can repeat this process until the spanning set is linearly independent and hence is a basis for H.
  - If the spanning set is eventually reduced to one vector, that vector will be nonzero (and hence linearly independent) because $H \neq \{0\}$.

- **Example 7:** Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$$

and $H = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and show that $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}$. Then find a basis for the subspace H.

1. **Proof for** $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = Span\{\mathbf{v}_1, \mathbf{v}_2\}$:
   - 1.1 $Span\{\mathbf{v}_1, \mathbf{v}_2\} \subset Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
     - Every vector in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to H because $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3$.
   - 1.2 $Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset Span\{\mathbf{v}_1, \mathbf{v}_2\}$
     - Now let $\mathbf{x}$ be any vector in H – say, $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$.
     - Since $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, we may substitute

$$\begin{aligned} x &= c_1v_1 + c_2v_2 + c_3v_3 \\ &= c_1v_1 + c_2v_2 + c_3(5v_1 + 3v_2) \end{aligned}$$

     - Thus $\mathbf{x}$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$, so every vector in H already belongs to $Span\{\mathbf{v}_1, \mathbf{v}_2\}$
   - We conclude that H and $Span\{\mathbf{v}_1, \mathbf{v}_2\}$ are actually the same set of vectors.

2. **Find a basis** for the subspace H:
   - It follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of H since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Basis for $Col\ B$

- **Example 8:** Find a basis for $Col\ B$, where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Solution**
  - Each non-pivot column of B is a linear combination of the pivot columns. In fact, $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_2 - \mathbf{b}_3$. By the Spanning Set Theorem, we may discard $\mathbf{b}_2$ and $\mathbf{b}_4$, and $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span $Col\ B$.
  - Let

$$S = \{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\} = \{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\}$$

  - Since $\mathbf{b}_1 \neq 0$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent. (Theorem 4). Thus, S is a basis for $Col\ B$.

Bases for $Nul\ A$ and $Col\ A$

- **Theorem 6:** The pivot columns of a matrix A form a basis for Col A.
- **Proof:**
  - Let B be the reduced echelon form of A. The set of pivot columns of B is linearly independent, for no vector in the set is a linear combination of the vectors that precede it.
  - Since A is row equivalent to B, the pivot columns of A are linearly independent as well, because any linear dependence relation among the columns of A corresponds to a linear dependence relation among the columns of B.
  - For this reason, every non-pivot column of A is a linear combination of the pivot columns of A.
  - Thus the non-pivot columns of a may be discarded from the spanning set for $Col\ A$, by the Spanning Set Theorem.
  - This leaves the pivot columns of A as a basis for $Col\ A$.
- **Warning:** The pivot columns of a matrix A are evident when A has been reduced only to echelon form. But, be careful to use the pivot columns of A itself for the basis of $Col\ A$. Row operations can change the column space of a matrix. The columns of an echelon form B of A are often not in the column space of A.

Two Views of a Basis

- When the Spanning Set Theorem is used, the deletion of vectors from a spanning set must stop when the set becomes linearly independent.
- If an additional vector is deleted, it will not be a linear combination of the remaining vectors, and hence the smaller set will no longer span V.
- Thus a basis is a spanning set that is as small as possible.
- A basis is also a linearly independent set that is as large as possible.
- If S is a basis for V, and if S is enlarged by one vector — say, $\mathbf{w}$ — from V, then the new set cannot be linearly independent, because S spans V, and $\mathbf{w}$ is therefore a linear combination of the elements in S.
- Sim: **"Basis is small enough to be linearly independent to each other, but basis is large enough to span the space."**

Acknowledgement

- This lecture note is based on the instructor's lecture notes (formatted as ppt files) provided by the publisher (Pearson Education) and the textbook authors (David Lay and others)
- The pdf conversion project for this chapter was possible thanks to the hard work by Jaemin Park (ITM 17'). Professor Sim recruited Mr. Park after his outstanding performance at Engineering Math. In the Applied Probability Lab, he is growing an intelligent reinforcement learning agent that optimizes energy storage systems (ESS).