Stochastic Processes, Final, 2025 Fall

Solution and Grading

- Duration: 120 minutes
- Weight: 35% of final grade
- Closed material, No calculator

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- Write legibly.
- Justification is necessary unless stated otherwise.

#1. Let $X = \{X(t) : t \geq 0\}$ be a continuous time Markov chain with state space $\{1, 2, 3\}$ with a rate matrix G of the following:

$$G = \begin{pmatrix} -4 & ? & 1 \\ 1 & ? & 2 \\ 1 & ? & -2 \end{pmatrix}$$

Let us assume we have computed e^G as follows:

$$e^G = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}.$$

(a) Find the stationary distribution. `[5pts]`

(b) Find $\mathbb{P}\big[X(1) = 2,\ X(3) = 3 \bigm| X(0) = 2\big]$. `[5pts]`

(a) First to fill the missing element in G using property that each row must sum up to 0. Then, solving $\pi G = 0$ gives $\pi = (4/20\ 7/20\ 9/20)$.

(b) First to note that e^{1G} serves for one-step transition matrix. $\mathbb{P}\big[X(1) = 2,\ X(3) = 3 \bigm| X(0) = 2\big] = \mathbb{P}\big[X(1) = 2 \bigm| X(0) = 2\big]\mathbb{P}\big[X(3) = 3 \bigm| X(1) = 2\big] = 0.4(0.3 \cdot 0.5 + 0.4 \cdot 0.4 + 0.4 \cdot 0.4) = 0.188$

Difficulty: Medium
Amount of work: 30%
Partial Grading: (a) From the three tasks: i) fill the missing elements, ii) set $\pi G = 0$, and iii) carry out the rest of calculation, if only one task has an error, then 2.5 pts were given. (b) If only one mistake were made, then 2.5 pts were given.

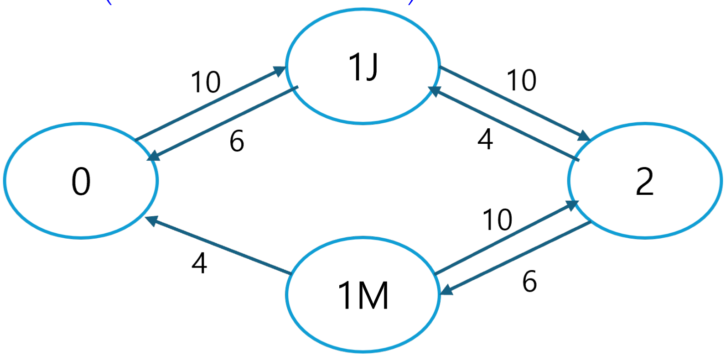
Note that e^G has an error in the second row. Since this matrix is transition matrix, the sum of row must be equal to one. The second row should corrected, for example, as (0.3 0.4 0.3).

#2. Customers arrive at a two-server system according to a Poisson process with rate $\lambda = 10$ per hour. An arrival finding server 1 (John) free will begin his service with server 1. An arrival finding server 1 busy and server 2 (Mary) free will join server 2. An arrival finding both servers busy goes away. Once a customer is served by either server, he departs the system. The service times at both servers are exponential random variables. Assume that the service rate of the first server is 6 per hour and the service rate of the second server is 4 per hour. Describe a continuous-time Markov chain to model the system and give the rate transition diagram.`[5pts]`

Let the state space be $\mathcal{S} = \{0,\, 1J,\, 1M,\, 2\}$ where:

- 0: both idle
- $1J$: only server 1 (John) busy
- $1M$: only server 2 (Mary) busy
- 2: both servers busy

$$\begin{array}{c} 0 \\ 1J \\ 1M \\ 2 \end{array} \begin{pmatrix} -10 & 10 & 0 & 0 \\ 6 & -16 & 0 & 10 \\ 4 & 0 & -14 & 10 \\ 0 & 4 & 6 & -10 \end{pmatrix}$$



Difficulty: Medium
Amount of work: 10%
Partial Grading: The state definition must be correct to receive any credit. If state definition is correct, but one mistake was made in describing rates, then 2.5pts were given.

#3. An AI data center receives queries arriving according to a Poisson process with an average rate of 2500 queries per hour. Processing each query consumes 4 Wh of energy.

(a) What is the expected hourly energy consumption for processing the queries? [5pts]

(b) Approximate the 95% quantile of the hourly energy consumption using a normal approximation. (Hint: What is the standard deviation for hourly energy consumption?) [5pts]

(a) Let N be the counting process for arriving query. Then, N is $PP(2500/hr)$. Since each query consumes 4 Wh, let $Y := 4N$ be the amount of energy consumption. $\mathbb{E}[Y] = 4 \times 2500 = 10000$ Wh.

Since N is Poissont process, $E(N) = Var(N) = 2500$. Thus, $Var(Y) = 16 \times Var(N) = 40000$, so $\sigma_Y = 200Wh$. 95% quantile of Y is $\approx 10000 + 1.645 \times 200 = 10329Wh$.

Difficulty: Medium
Amount of work: 30%
Partial Grading: For (a), no partial point was given. For (b), if solution quality is generally good, but with one minor mistake, then 2.5 pts was given.

#4. Consider a DTMC with the following transition matrix.

$$P = \begin{pmatrix} 0.25 & 0.5 & 0.25 & 0 & 0 & 0 \\ 0.5 & 0.25 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 0 & 0.7 & 0.3 \end{pmatrix}$$

What is P^{100}? `[10pts]`

$$\mathbf{P}^{100} \approx \lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} 0 & 0 & 0.3 & 0.3 & 0.2 & 0.2 \\ 0 & 0 & 0.2 & 0.2 & 0.3 & 0.3 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

Difficulty: Medium
Amount of work: 30%
Partial Grading: If one error is made, then 5pts were given. If two errors are made, then 2.5pts were given.