ITM426, Long Quiz 2, 2025 Fall

Solution and Grading

- ITM 426 Engineering Mathematics 2025 F
- Duration: 90 minutes
- Weights: 30%
- 5 Questions

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- Write legibly.
- Justification is necessary unless stated otherwise.
- Partial points are given only sparingly for the most problems because you are expected to 1) carry out proper sanity check and 2) correct your mistake by doing so.

1	10
2	10
3	10
4	10
5	10
Total	50

#1. Mark True or False. No justification is necessary. `[Each 2.5pt]`

- If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution to $A\mathbf{x} = \mathbf{b}$ is unique. (**TRUE** / **FALSE**)
- If one row in an echelon form of an augmented matrix is [0 0 0 5 0], then the associated linear system is inconsistent. (**TRUE** / **FALSE**)
- In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations. (**TRUE** / **FALSE**)
- Let A be a 3×2 matrix. The equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$.
(**TRUE** / **FALSE**)

Difficulty: Hard
Amount of work: 20 %
Suggested answer:

- True
(Geometric argument using Theorem 6.) Since $A\mathbf{x} = \mathbf{b}$ is consistent, its solution set is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, by Theorem 6. So the solution set of $A\mathbf{x} = \mathbf{b}$ is a single vector if and only if the solution set of $A\mathbf{x} = \mathbf{0}$ is a single vector, and that happens if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(Proof using free variables.) If $A\mathbf{x} = \mathbf{b}$ has a solution, then the solution is unique if and only if there are no free variables in the corresponding system of equations, that is, if and only if every column of A is a pivot column. This happens if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- False. The row shown corresponds to the equation $5x_4 = 0$, which does not by itself lead to a contradiction. So the system might be consistent or it might be inconsistent.
- False. See Chapter 1, Theorem 1.
- True. A 3×2 matrix has three rows and two columns. With only two columns, A can have at most two pivot columns, and so A has at most two pivot positions, which is not enough to fill all three rows. By Theorem 4, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$. Generally, if A is an m×n matrix with $m > n$, then A can have at most n pivot positions, which is not enough to fill all m rows. Thus, the equation $A\mathbf{x} = \mathbf{b}$ cannot be consistent for all $\mathbf{b}$ in $\mathbb{R}^3$.

#2. Find an LU factorization of the following matrix `[10pt]`

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 1 & 8 \end{bmatrix}$$

Difficulty: Medium
Amount of work: 20 %
Suggested answer:

$$\begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ & -1/2 & 1/2 \\ & & 1 \end{bmatrix}$$

#3. Find an inverse of the following matrix `[10pt]`

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 1 & 8 \end{bmatrix}$$

Difficulty: Medium
Amount of work: 20 %
Suggested answer:

$$\begin{bmatrix} 2 & 5 & -2 \\ 0 & -4 & 1 \\ -1 & -2 & 1 \end{bmatrix}$$

#4. Provide the clear definition by completing the the following sentences. (Hint: This question asks for the *definition* and does not assume that T is a linear mapping. Your statement should not even use A, which is defined as a standard matrix under the assumption of T being linear mapping.) `[Each 5pts]`

- A mapping $T : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if ( ).
- A mapping $T : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is said to be **one-to-one** if ( ).

Difficulty: Medium
Amount of work: 20%
Solution:

- onto
  - (textbook definition) each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.
  - (general definition) Image(T) = $\mathbb{R}^m$
  - (general definition) Image(T) = Codomain
  - (general definition) Image of T covers all of the Codomain.
- one-to-one
  - (textbook definition) each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.
  - (general definition) $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.
  - (general definition) each input has a unique output.

#5. Consider a linear mapping $T : \mathbb{R}^n \Rightarrow \mathbb{R}^m$. For each case, provide an example of standard matrix A.`[Each 2.5pts]`

- T is *onto* but not *one-to-one*
- T is *one-to-one* but not *onto*
- T is not *onto* and not *one-to-one*
- T is *onto* and *one-to-one*

- *onto* but not *one-to-one* $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$
- *one-to-one* but not *onto* $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
- not *onto* and not *one-to-one* $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$
- *onto* and *one-to-one* $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

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Write your name before detaching this page. Your Name: ____________________