

Chapter 7. Symmetric Matrices and Quadratic Forms

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1 7.1. Diagonalization of Symmetric Matrices

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Symmetric Matrix

- A **symmetric matrix** is a matrix A such that $A^T = A$.
- Such a matrix is necessarily square.
- Its main diagonal entries are arbitrary, but its other entries occur in pairs—on opposite sides of the main diagonal.
- **Theorem 1:** If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.
- **Theorem 2:** An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric matrix.

- **Example 3:** Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, whose characteristic equation is $-\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2) = 0$

- **Solution:**

- The usual calculations produce bases for the eigenspaces:

$$\lambda = 7 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}; \lambda = -2 : \mathbf{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

- Although \mathbf{v}_1 and \mathbf{v}_2 are linearly independent, they are not orthogonal. The projection of \mathbf{v}_2 onto \mathbf{v}_1 is $\frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1$. The component of \mathbf{v}_2 orthogonal to \mathbf{v}_1 is

$$\mathbf{z}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

- Then $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an orthogonal set in the eigenspace for $\lambda = 7$
- (Note that \mathbf{z}_2 is linear combination of the eigenvectors \mathbf{v}_1 and \mathbf{v}_2 , so \mathbf{z}_2 is in the eigenspace.)

● (solution continued:)

- Since the eigenspace is two-dimensional (with basis $\mathbf{v}_1, \mathbf{v}_2$), the orthogonal set $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an *orthogonal basis* for the eigenspace, by the Basis Theorem.
- Normalize \mathbf{v}_1 and \mathbf{z}_2 to obtain the following orthonormal basis for the eigenspace for $\lambda = 7$:

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

- An orthonormal basis for the eigenspace for $\lambda = -2$ is

$$\mathbf{u}_3 = \frac{1}{\|2\mathbf{v}_3\|} 2\mathbf{v}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

- By Theorem 1, \mathbf{u}_3 is orthogonal to the other eigenvectors \mathbf{u}_1 and \mathbf{u}_2 . Hence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.
- Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}$. Then P orthogonally diagonalizes A , and $A = PDP^{-1}$.

Suggested Exercises

- 7.5.1
- 7.5.3
- 7.6 (p.450)

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