ITM426, Quiz 2, 2024 Fall

Solution and Grading

- ITM 426 Engineering Mathematics 24F
- Oct 25, 2024
- Duration: 90 minutes
- Weights: 25%
- 5 Questions

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- Write legibly.
- Justification is necessary unless stated otherwise.
- Partial points are given only sparingly for the most problems because you are expected to 1) carry out proper sanity check and 2) correct your mistake by doing so.

1	8
2	12
3	10
4	10
5	10
Total	50

#1. Provide the clear definition by completing the the following sentences. `[Each 4pt]`

- A mapping $T : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if ( ).
- A mapping $T : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ is said to be **one-to-one** if ( ).

Difficulty: Medium
Amount of work: 15%
Solution: This question asks for the *definition* and does not assume that T is a linear mapping. The provided *equivalent conditions (necessary and sufficient conditions)* below are under the assumption that T is a linear mapping. For this year, I consider the equivalent conditions below as correct answers only because all mappings considered in this course are a linear mapping. I must advise you that, though *equivalent conditions* may often be considered as correct answers, it is advisible to provide definitions when definition is asked.

- onto
  - Definition (The definition is not limited to linear mapping. The statements should not even use A.)
    - (textbook definition) each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.
    - (general definition) Image(T) = $\mathbb{R}^m$
    - (general definition) Image(T) = Codomain
    - (general definition) Image of T covers all of the Codomain.
  - Equivalent conditions assuming T is a linear mapping. Hereafter, A is the standard matrix corresponding to the linear mapping T.
    - for each $\mathbf{b}$ in $\mathbb{R}^m$, there exists a vector $\mathbf{x}$ in $\mathbb{R}^n$ that satisfies $A\mathbf{x} = \mathbf{b}$.
    - for each $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has at least one solution.
    - all rows of A have a pivot. (all rows are pivot rows.)
    - $m = r$ (where r is the number of pivots)
    - column vectors of A spans $\mathbb{R}^m$.
- one-to-one
  - Definition (The definition is not limited to linear mapping. The statements should not even use A.)
    - (textbook definition) each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.
    - (general definition) $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.
    - (general definition) each input has a unique output.
  - Equivalent conditions assuming T is a linear mapping. Hereafter, A is the standard matrix corresponding to the linear mapping T.
    - for each $\mathbf{b}$ in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ has at most one solution.
    - columns of A are linearly independent.
    - $Ker(T) = 0$ has only trivial solution.
    - $n = r$ (where r is the number of pivots)
    - all columns of A have a pivot. (all variables are pivot variables; all columns are pivot columns; there exists no free variable)

A few students made statements regarding one-to-one as "$A\mathbf{x} = \mathbf{b}$ has a unique solution". The statements of "having a unique solution" and "the solution is unique" are two different statements. The former statement implies that the number of solution is 1. The latter statment implies that *if a solution exists*, then the solution is unique. One-to-one does not guarantee the existence of solution, but the statement of "$A\mathbf{x} = \mathbf{b}$ has a unique solution" implies the existence of solution as well. Thus, "$A\mathbf{x} = \mathbf{b}$ has a unique solution" is an incorrect description of one-to-one. In this case, the half credit was given this year. Followings are few examples:

- *onto* but not *one-to-one* $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$
- *one-to-one* but not *onto* $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
- not *onto* and not *one-to-one* $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$
- onto and *one-to-one* $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

#2. Mark True or False for each statement. No justification is necessary. `[Each 4pt]`

- When two linear transformations are performed one after another, the combined effect may not always be a linear transformation. (**TRUE** / **FALSE**)
- If A is 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to one. (**TRUE** / **FALSE**)
- If a system $A\mathbf{x} = \mathbf{b}$ has more than one solution, then so does the system $A\mathbf{x} = 0$ (**TRUE** / **FALSE**)

Difficulty: Medium
Amount of work: 15%
Solution:

- False. See the paragraph before Table 1 in Section 1.9.
- False. See Example 5 in Section 1.9.
- True. Theorem 6 in Section 1.5 essentially says that when $A\mathbf{x} = \mathbf{b}$ is consistent, the solution sets of the nonhomogeneous equation and the homogeneous equation are translates of each other. In this case, the two equations have the same number of solutions.

#3. Find an LU factorization of the following matrix [10pt]

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}$$

Difficulty: Easy
Amount of work: 20%
Suggested answer:

$$\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 & 1 \\ & & 1 & 2 \\ & & & \end{bmatrix}$$

Some students presented $\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 3 & 3 & 0 \end{bmatrix}$ as L. The algorithm provided in the textbook leads to this matrix. But this lower triangular matrix does not have 1's on the diagonal. Thus, this matrix is not *unit lower triangular matrix* and not invertible. Some students seemed to have made additional correction afterward, to make L to be a proper unit lower triangular matrix. Since this matrix is obtained after performing the textbook algorithm and LU still recovers the original matrix, the full credit is given this year. If only U is correct, then 3 pts are given as partial credit.

#4. Consider a 3 by 3 matrix A. The matrix reduces to the identity matrix after the following three row operations in order.

- Subtract 4 $\times$ (R1) from (R2),
- Subtract 3 $\times$ (R1) from (R3),
- Subtract (R3) from (R2),

where each row vector of matrix A is denoted as $(R1)$, $(R2)$, and $(R3)$, respectively. What is the original matrix A and its inverse matrix A^{-1}? `[10pt]`

Difficulty: Medium
Amount of work: 25%
Solution: $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$. One can start from $AE_1E_2E_3 = I$ where the elementary matrices are identified, then find A^{-1} and A accordingly. After A and A^{-1} are found, students are expected to check if i) going through the three steps indeed generates I from A and ii) $AA^{-1} = I$. Partial credit of 3pts is given this year if i) A is correct or ii) presented A and A^{-1} are indeed inverse.

#5. Consider a 3 by 4 matrix A that depends on the value of c.

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 3 & c & 2 & 8 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

For each value of c, find the solution of $A\mathbf{x} = 0$ by describing in parametric vector form.`[10pt]`

Difficulty: Medium
Amount of work: 25%
Solution:

- If $c \neq 3$, $x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$
- If $c = 3$, $x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

In order to solve this problem, students first need to identify that whether or not c being equal to 3 bifurcates the solution set. If $c \neq 3$, then the solution set is 1D space and there are one free variable. If $c = 3$, then the solution set is 2D space and there are one free variable. Partial credit is given as:

- If a student identified the bifurcation properly but presented the wrong vector with only one minor error, then 5pts.
- If a student presented only one vector $[-2\ 0\ \ -1\ 1]^t$ that works for all values of c, then the presented solution set is only proper subset of solution set in the case of $c = 3$. In this case, 3pts were given.
- In this problem, suggested sanity check is to multiply original matrix by the obtained vector, then see if it is indeed a zero vector.

(detachable)