University of Saskatchewan
Department of Geography and Planning
Centre for Hydrology

Process-Based Hydrological Modelling
(GEOG 825)

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ASSIGNMENT 01: MODEL MECHANICS
Due: 9:00 a.m. CST - Thursday, February 2, 2023

1 Introduction

In this assignment, you will create a simple hydrological model “from scratch”, test the model with some simple synthetic data, and critically reflect on the trade-offs between accuracy, realism, and computational speed.

Multiple options to obtain extra credit are specified at the end of this assignment. These extra exercises are not mandatory, and full marks can be obtained without completing them. However, these exercises are more difficult and provide an additional opportunity to work with the provided material. You can select them based on their utility for your own work, future projects, or personal interest.

2 Learning objectives

After successfully completing this exercise, students are expected to:

1. Understand the connection between conceptual understanding of rainfall-runoff processes and the expression of these processes as ordinary differential equations (ODEs);
2. Be able to use a numerical approximation scheme to find solutions to the ODEs;
3. Be able to synthesize their knowledge of hydrological processes into a functioning computer model.

3 Background: Model description

3.1 The perceptual model

You have been studying rainfall-runoff processes in a humid, forested catchment in a region of complex topography. Your catchment has impermeable bedrock and shallow soils. During precipitation events, the soils in your catchment become saturated in areas of topographic

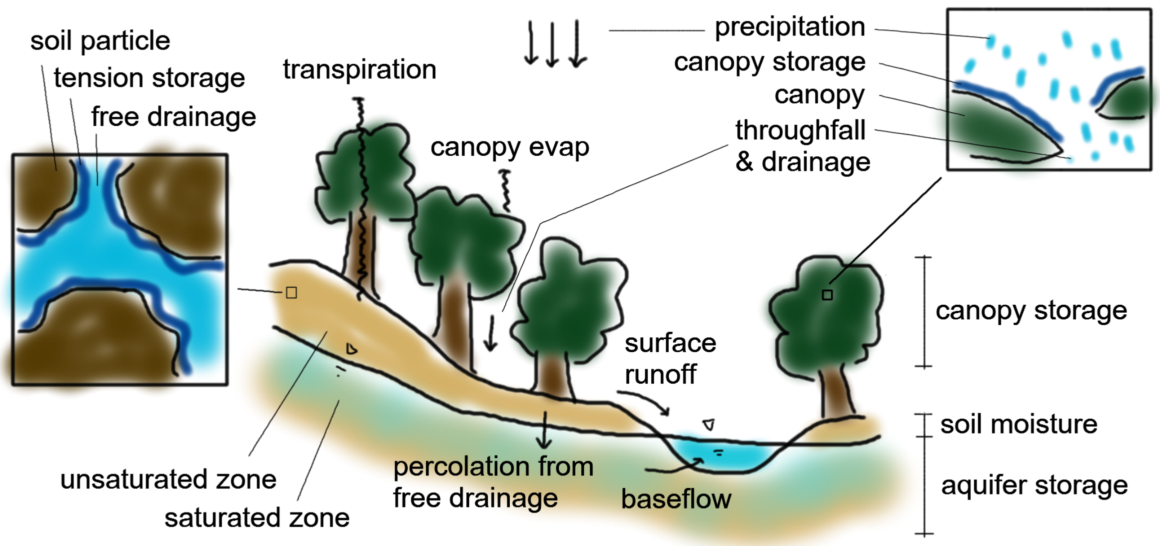

Figure 1: Perceptual model of catchment behavior

convergence. Based on observations, you have developed the following perceptual model of your catchment's behaviour during an intense precipitation event (see Figure 1):

- **Vegetation Canopy** ...
  - *System description*: Precipitation is initially intercepted by the vegetation canopy. The water stored in the vegetation canopy empties through evaporation and drainage; precipitation can fall through the precipitation canopy to the ground surface.
  - *Fluxes of water*: Canopy evaporation depends on potential evaporation rate and the wetted area of the vegetation canopy; the wetted area of the vegetation canopy is a non-linear function of canopy storage. Throughfall is a linear function of the storage of water in the canopy. Canopy drainage only occurs if the canopy storage is above the maximum storage capacity of the vegetation canopy.
- **Unsaturated zone** ...
  - *System description*: The vertical fluxes of water from the vegetation canopy (throughfall of precipitation and drainage of water from the vegetation canopy) reach the soil where they can infiltrate into the unsaturated zone. The vertical water flux from the vegetation canopy that falls on saturated areas of the catchment leaves the catchment as saturation-excess runoff. Water can percolate from the unsaturated zone to the saturated zone. Water is also used for transpiration by vegetation. The canopy is dense enough that bare soil evaporation is negligible.
  - *Fluxes of water*: The saturated area of a catchment is a function of the water storage in the unsaturated zone, i.e., the variable source area for saturation-excess runoff expands and contracts as water storage in the unsaturated zone increases

and decreases. Saturation excess runoff is the vertical water flux from the vegetation canopy that falls on saturated areas of the catchment. Transpiration occurs at the potential rate if unsaturated storage is above field capacity, and decreases linearly as storage drops below this point. Transpiration only occurs over the dry fraction of the canopy. Percolation to the saturated zone only occurs if storage in the unsaturated zone exceeds field capacity and is a non-linear function of storage above field capacity. There is no lateral flow in the unsaturated zone.

- **Saturated zone** ...
  - *System description*: The saturated zone is recharged by percolation from the unsaturated zone. The saturated zone drains laterally to the stream (the baseflow flux).
  - *Fluxes of water*: The baseflow from the saturated zone is parameterized as a linear function of the storage in the saturated zone.
- **Catchment routing** ...
  - The exercises in this assignment assume instantaneous routing. Students are welcome to add a Unit Hydrograph to the model as an option for extra credit (see the section on additional exercises).

3.2 Model equations

3.2.1 State equations

Given the conceptual diagram in Figure 2, the changes in storage over time can be described using a set of coupled ordinary differential equations for the vegetation canopy, the unsaturated zone, and the saturated zone.

In the following equations, we only show the dependence of the fluxes on the model state variables, understanding that the fluxes also depend on the model forcing data and the model parameters. The state equations are:

$$\frac{dS_c}{dt} = P - E_c(S_c) - Q_t(S_c) - Q_c(S_c), \tag{1}$$

$$\frac{dS_{uz}}{dt} = P_e(S_c) - E_{uz}(S_c, S_{uz}) - Q_o(S_c, S_{uz}) - Q_p(S_{uz}), \tag{2}$$

$$\frac{dS_{sz}}{dt} = Q_p(S_{uz}) - Q_b(S_{sz}), \tag{3}$$

where eqs. (1) to (3) respectively define the ordinary differential equations for the vegetation subdomain, the unsaturated zone subdomain, and the saturated zone subdomain. In eqs. (1) to (3), P is the precipitation flux, $E_c(S_c)$ is evaporation from interception storage, $Q_t(S_c)$ is throughfall, $Q_c(S_c)$ is canopy drainage, $P_e(S_c)$ is the sum of throughfall and canopy

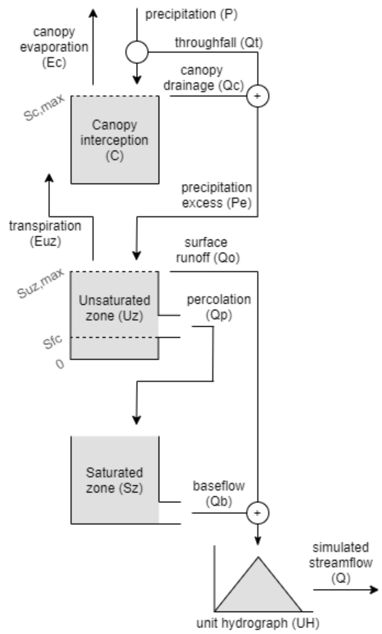

Figure 2: Conceptual model of catchment behavior

drainage, called precipitation excess, $E_{uz}(S_c, S_{uz})$ is evaporation from the unsaturated zone, $Q_o(S_c, S_{uz})$ is overland flow, $Q_p(S_{uz})$ is percolation, and $Q_b(S_{sz})$ is baseflow.

Equations (1) to (3) can also be written as

$$\frac{dS_c}{dt} = f_c(S_c), \tag{4}$$

$$\frac{dS_{uz}}{dt} = f_{uz}(S_c, S_{uz}), \tag{5}$$

$$\frac{dS_{sz}}{dt} = f_{sz}(S_{uz}, S_{sz}), \tag{6}$$

or

$$\frac{d\mathbf{S}}{dt} = f(\mathbf{S}), \tag{7}$$

where $\mathbf{S} = (S_c, S_{uz}, S_{sz})$, with S_c, S_{uz}, and S_{sz} is the storage in the interception zone (canopy), storage in the unsaturated zone, and storage in the saturated zone, respectively.

3.2.2 Meteorological input data

The model is driven (forced) by time series of meteorological data (in a modelling context, the meteorological input data is known as forcing data). In this basic model, we use two primary meteorological inputs: Precipitation, P (mm/day), defining the available water; and potential evapotranspiration, E_p (mm/day), defining the available energy.

3.2.3 Flux parameterizations

As defined earlier, the model fluxes define the time evolution of model states. This subsection summarizes the model fluxes for the vegetation subdomain, the unsaturated zone subdomain, and the saturated zone subdomain.

Possible parametrizations for the fluxes in the vegetation canopy are:

$$E_c\left(S_c\right) = E_p F_{wet} \tag{8}$$

$$Q_t\left(S_c\right) = \begin{cases} P\frac{S_c}{S_{c,max}}, & S_c < S_{c,max}, \\ P, & otherwise \end{cases} \tag{9}$$

$$Q_c\left(S_c\right) = \begin{cases} 0, & S_c < S_{c,max}, \\ k_{can}\left(S_c - S_{c,max}\right), & otherwise, \end{cases} \tag{10}$$

$$P_e = Q_t + Q_c, \tag{11}$$

where F_{wet} is the wetted fraction of the canopy, defined as

$$F_{wet}\left(S_c\right) = \begin{cases} \left(\frac{S_c}{S_{c,max}}\right)^{\gamma}, & S_c < S_{c,max}, \\ 1, & otherwise, \end{cases} \tag{12}$$

Possible parametrizations for the fluxes in the unsaturated zone are:

$$E_{uz}\left(S_c, S_{uz}\right) = \begin{cases} E_p\frac{S_{uz}}{S_{fc}}\left(1 - F_{wet}\right), & S_{uz} < S_{fc}, \\ E_p\left(1 - F_{wet}\right), & otherwise, \end{cases} \tag{13}$$

$$Q_o\left(S_c, S_{uz}\right) = P_e F_{sat}, \tag{14}$$

$$Q_p\left(S_{uz}\right) = \begin{cases} 0, & S_{uz} < S_{fc}, \\ k_{sat}\left(\frac{S_{uz} - S_{fc}}{S_{uz,max} - S_{fc}}\right)^{\beta}, & otherwise, \end{cases} \tag{15}$$

where F_{sat} is the saturated fraction, defined as

$$F_{sat} = \begin{cases} 1 - \left(1 - \frac{S_{uz}}{S_{uz,max}}\right)^{\alpha}, & S_{uz} < S_{uz,max}, \\ 1, & otherwise, \end{cases} \tag{16}$$

Possible parametrizations for the fluxes in the saturated zone are:

$$Q_b = k_{sz} S_{sz}, \tag{17}$$

All parameters as specified in Table 1.

3.2.4 Model parameters

The model parameters define the characteristic properties of the system that are assumed to be "time-invariant" (remain constant over the time duration of interest). In this context, the model parameters can be considered the adjustable coefficients in the flux parameterizations.

Table 1: Model parameters

Parameter	**Description**	**Unit**
γ	Non-linearity in the wetted fraction of the canopy	[–]
$S_{c,max}$	Maximum non-drainable interception storage	[mm]
k_{can}	Canopy drainage coefficient	[day^{-1}]
S_{fc}	Field capacity	[mm]
$S_{uz,max}$	Maximum storage in the unsaturated zone	[mm]
α	Non-linearity in the variable source area	[–]
k_{sat}	Maximum percolation rate	[mm day^{-1}]
β	Percolation non-linearity	[–]
k_{sz}	Runoff coefficient for the saturated zone	[day^{-1}]

4 Exercises

In this assignment you will code up a basic rainfall-runoff model "from scratch". This may seem overwhelming (at least at first), especially for those without model development experience. However, the process is straightforward.

We will build this model from the inside out. We start by computing the variables that are used in the flux parameterizations (the wetted area of the vegetation canopy and the saturated area of the catchment). We then calculate all of the fluxes used in the ODEs. Next, we implement a basic time stepping scheme for each state equation individually using temporally constant forcing. Finally, we run the coupled model using time-varying forcing.

The exercises will use the parameters in Table 2 (see Table 1 for parameter definitions).

Table 2: Model parameters used in the exercises.

Parameter	**Value**	**Unit**
γ	2/3	–
$S_{c,max}$	3.00	mm
k_{can}	0.50	day^{-1}
S_{fc}	25.00	mm
$S_{uz,max}$	100.00	mm
α	0.25	–
k_{sat}	500.00	mm day^{-1}
β	3.00	–
k_{sz}	0.10	day^{-1}

4.1 Compute the diagnostic variables used in the flux parameterizations

Our first step is to write functions to compute the diagnostic variables that are used in the flux parameterizations. These are the wetted area of the vegetation canopy, eq. (12), and the saturated area of the catchment, eq. (16).

The specific tasks are:

1. Define functions to calculate F_{wet}, eq. (12), and F_{sat}, eq. (16). Your functions should have state variables and parameters as arguments.

2. Plot F_{wet} and F_{sat} as a function of canopy storage and unsaturated zone storage respectively.

An example solution to this exercise is in fig. 3 using the parameters in Table 2.

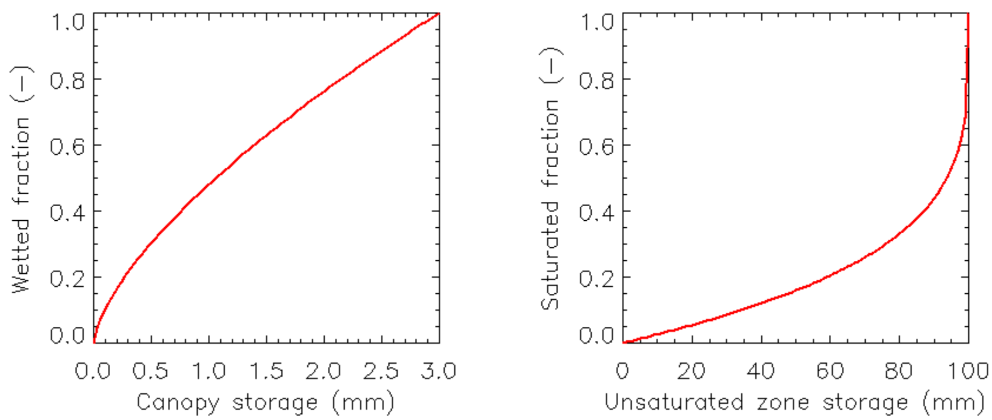

Figure 3: Diagnostic variables used in the flux calculations

4.2 Compute the model fluxes

Our next step is to write functions to compute the model fluxes, i.e., eqs. (8) to (10) for the vegetation canopy, eqs. (13) to (15) for the unsaturated zone, and eq. (17) for the saturated zone.

The specific tasks are similar to the previous exercise:

1. Define functions to calculate the fluxes in eqs. (8) to (10), (13) to (15) and (17). The forcing data should be as follows:

   - The precipitation flux: $P = 1$ mm day^{-1}.
   - The potential ET flux: $E_p = 1$ mm day^{-1}

   To check the functional behaviour of the flux parameterizations in the unsaturated zone, it is useful to exclude the impact of the vegetation canopy. As such, in eq. (13) we set $F_{wet} = 0$ and in eq. (14) we set $P_e = P$. Note in eq. (14) that $F_{sat} = f\left(S_{uz}\right)$, so, combining eqs. (14) and (16), $Q_o\left(S_c, S_{uz}\right) = P_e\left(1 - \left(1 - \frac{S_{uz}}{S_{uz,max}}\right)^{\alpha}\right)$ for the case where $S_{uz} < S_{uz,max}$.

   Your functions should have state variables, forcing data, and parameters as arguments.

2. Plot the fluxes as a function of the state variables.

Note: It will take some time to complete this exercise. However, if the tasks in this exersise are done well (i.e., separate functions for each flux, using state variables, forcing data, and parameters as arguments), the rest of the assignment will be straightforward.

An example solution to this exercise is in fig. 4 using the parameters in Table 2. Note that fig. 3 does not include a plot for eq. (17) (baseflow) because the expected behaviour of this flux is straightforward: there should be a linear relationship between saturated storage S_{sz} and baseflow Q_b. It's still a good idea to ensure your code produces the desired behaviour for this flux.

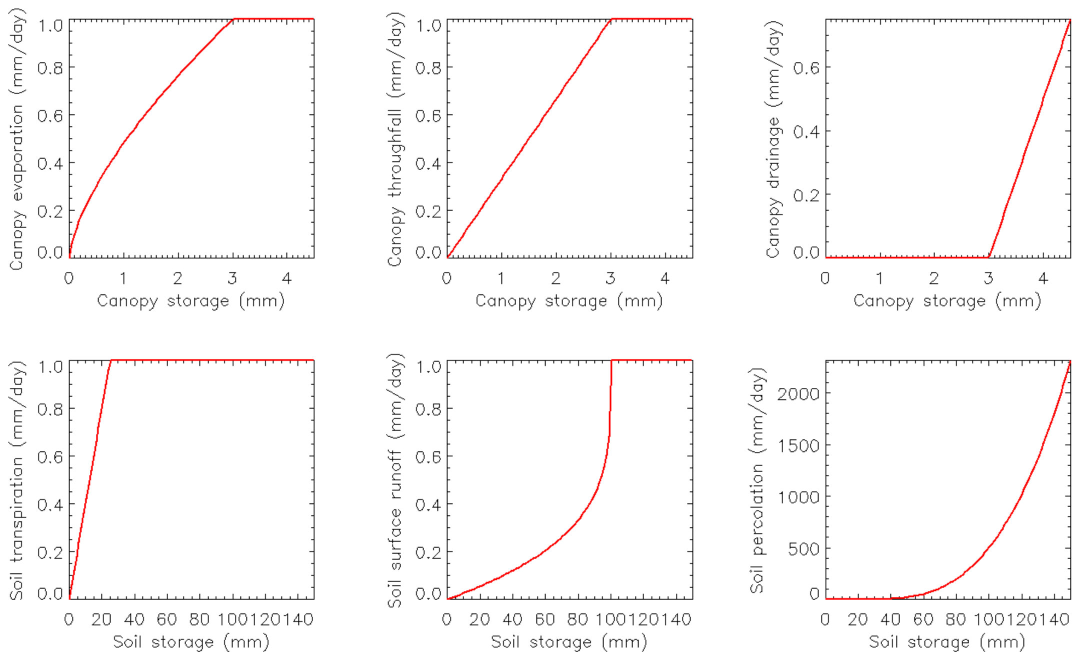

Figure 4: Model fluxes for the vegetation and unsaturated zone subdomains

4.3 Solve individual ODEs

In this task we implement a basic time stepping scheme for each state equation individually using temporally constant forcing.

The specific tasks are as follows:

1. Write a function to compute the right-hand-side of the ODEs defined in eqs. (1) to (3) in order to formulate the state equations eqs. (4) to (6). Note: if the previous exercise is done well, this task should require just 4-5 lines of code for each model sub-domain (the vegetation canopy, the unsaturated zone, and the saturated zone); the work required here is to call the functions already written to compute each flux, and compute the right-hand-side of eqs. (1) to (3), i.e., the fluxes at the boundaries of each control volume.

2. Temporally integrate eq. (4) with constant forcing for a time period of 10 days. Use the explicit Euler method

$$(S_c)^{n+1} = (S_c)^n + \left(\frac{dS_c}{dt}\right)^n \Delta t \tag{18}$$

   where the superscript n defines the index of the time step and Δt defines the length of the time step. Complete three separate model simulations using $\Delta t = 1$ day, $\Delta t =$ 1 hour, and $\Delta t = 5$ minutes. Use temporally constant forcing of

   - The precipitation flux: $P = 100$ mm day^{-1}.

- The potential ET flux: $E_p = 5$ mm day^{-1}

Note that your forcing vectors will have different lengths for each of these three cases to account for the different number of time steps within the 10-day period. Use the initial condition $S_c\,(t = 0) = 0$; use the parameters defined in Table 2. Plot time series of S_c from simulations using the different time steps.

3. Temporally integrate eq. (5) with constant forcing for a time period of 10 days. Use the explicit Euler method

$$(S_{uz})^{n+1} = (S_{uz})^{n} + \left(\frac{dS_{uz}}{dt}\right)^{n} \Delta t \tag{19}$$

and, as in task 2, complete three separate model simulations using $\Delta t = 1$ day, $\Delta t = 1$ hour, and $\Delta t = 5$ minutes. Use the same temporally constant forcing as in task 2:

- The *effective* precipitation flux: $P_e = 100$ mm day^{-1}.
- The potential ET flux: $E_p = 5$ mm day^{-1}

where P_e is prescribed, instead of computed from the canopy model, in order to restrict attention to the unsaturated zone. Moreover, since the unsaturated zone is run in isolation, set $F_{wet} = 0$. Use the initial condition $S_{uz}\,(t = 0) = 0$; use the parameters defined in Table 2. Plot the time series of S_{uz} from simulations using the different time steps.

4. Repeat task 3 with the parameter $S_{uz,max} = 10$ mm and $S_{fc} = 2.5$ mm. Use the parameters defined in Table 2. Plot the time series of S_{uz} from simulations using the different time steps.

It will be necessary to implement non-negativity constraints (i.e., the state variables S_c, S_{uz}, and S_{sz} cannot be negative). For the purpose of this exercise it is acceptable to simply set a state variable to zero if the explicit Euler solution produces a negative value, understanding that this approach will not preserve the water balance, i.e., the change in state over the time interval will not be consistent with the fluxes at the boundaries of the control volume. In many hydrological models the fluxes are added to the state variables sequentially – i.e., a form of operator splitting – and the constraints are imposed on each flux individually. This approach intermingles the physics with the numerical solution and makes it difficult to implement more robust time stepping schemes. The preferred approach to impose non-negativity constraints is to modify the fluxes based on the relative proportion of each flux to the total flux. Note that such corrections are not needed in the *implicit* Euler solution – in the *implicit* Euler solution, non-negativity constraints are naturally handled as part of the solution without the need for *post hoc* adjustments to the states and fluxes.

Note: The modularity in task 1 effectively separates the physics from the numerical solution. Numerical analysts will not have to "worry" about all of the physics in the flux equations; they just need to solve eqs. (4) to (6). This allows much easier collaboration on model development – the hydrologist does not focus primarily the numerical solution, and the numerical analyst can work on the numerical problems without needing a detailed understanding of the hydrological processes.

4.4 Solve the coupled ODE system

Our final step is to run the coupled model using time-varing forcing (and still using the basic explicit Euler time stepping scheme).

The specific tasks are as follows:

1. Define synthetic time-varying precipitation forcing at five minute time intervals for a ten day time period using

$$P(t) = P_{\max} \exp\left[-\left(\frac{\bar{t}-t}{\sigma_t}\right)^2\right] \tag{20}$$

   where $P_{max} = 100$ mm day^{-1} is the maximum precipitation rate, t is the vector of time steps in days, $\bar{t} = 2$ days is the time of the precipitation peak, and $\sigma_t = 0.24$ is the scale factor that affects the duration of the precipitation peak. The scale factor $\sigma_t = 0.24$ produces precipitation event with duration of approximately one day. Plot the synthetic precipitation forcing.

2. Run the full coupled model, i.e., eq. (7) for a ten day period using the explicit Euler time stepping scheme with time steps of five minutes. These simulations should use the initial conditions $S_c(t=0) = 0$ mm, $S_{uz}(t=0) = 10$ mm, and $S_{sz}(t=0) = 1$ mm. These simulations should use the precipitation forcing generated using eq. (20) and temporally constant potential ET flux of $E_p = 5$ mm day^{-1}. These simulations should use the parameters defined in Table 2. Plot time series of the state variables S_c, S_{uz}, and S_{sz}. Also plot time series of total runoff $Q_{tot} = Q_o + Q_b$.

An example code structure is illustrated in fig. 5

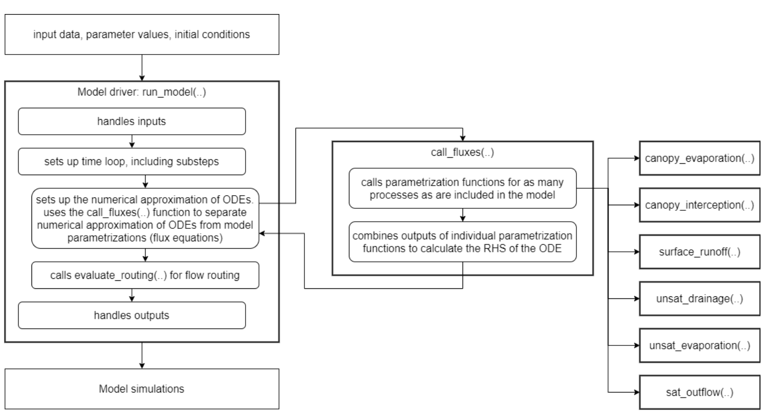

Figure 5: Schematic of code design, outlining how modular code can be constructed. Note that this schematic includes the option for substepping which is useful in cases where the temporal resolution of your forcing data is coarse but smaller model steps are still needed to limit the impact of numerical errors. Note that your model code for this assignment does not need to include substepping because the forcing data can easily be generated at a time resolution where numerical errors are minimal.

Note: This exercise is a modest extension of the previous exercise. Given the modular construction of the physics routines (the flux parameterizations), this exercise should only require a few lines of code in the time loop.

4.5 Reflection

You have completed a simple hydrologic model. Reflect critically on the benefits and drawbacks of the structure of this program and your selected numerical approximation scheme. 250 words maximum.

5 Additional Exercises

These exercises are entirely optional. Successful completion will provide extra credit that counts towards your grade for this assignment. Full marks can be obtained without completing any of these extra exercises. Complete any or all of the following:

- Implement a different numerical scheme, such as (a) implicit Euler; or (b) explicit Heun with numerical error control and adaptive substepping.
- Assess the impact of using different parametrizations, parameter values, and initial conditions.
- Implement a unit hydrograph model to simulate the time delay due to routing.
- Configure/calibrate the model to simulate hydrological processes in a catchment of your choice.

6 Results to be submitted

Submit the following results via email to `martyn.clark@usask.ca`:

1. A typeset document (e.g., Word, LaTeX) containing:
   - answers to each of the exercises (text and figures)
   - a critical reflection on the assumptions and choices made during the numerical implementation of your model.
2. A commented copy of your computer code. Please use extensive comments to define variables (including units) and to describe sections of the source code; strive for modularity.

High marks will be given when students clearly understand how to solve the problem; when the student's source code is well organized, well documented, and is easy to run by others; and when the student's figures convey clear conclusions. Minor mistakes can appear so long as they do not indicate a lack of conceptual understanding.

If you have questions, then please feel free to contact Martyn Clark (`martyn.clark@usask.ca`) or Vahid Aghaie (`vahid.aghaie@usask.ca`).