FEM for 1D problems

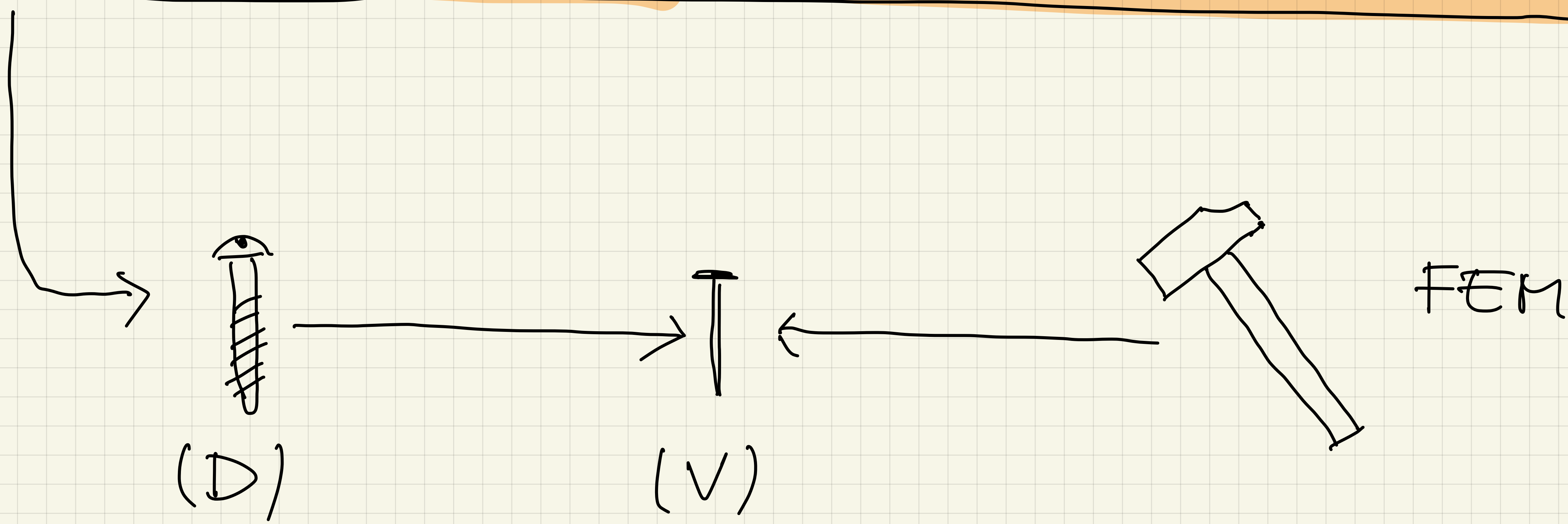
Weak form (variational problem)

Boundary value problem (D): Find a function $u : [0,L] \to \mathbb{R}$ which satisfies the differential equation

$$\boxed{EA\,u''(x) - c \cdot u(x) = -n}$$

and the boundary conditions

$$\boxed{EA\,u'(0) = -F} \qquad \boxed{EA\,u'(L) + S \cdot u(L) = 0}$$



Ingredients of finite element solution

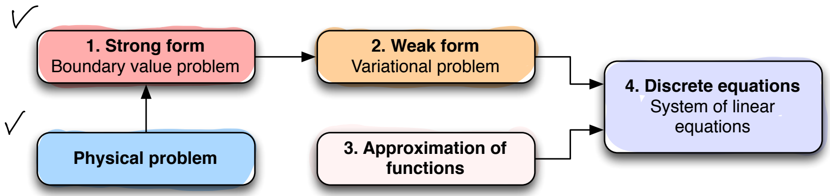


Diagram according to Fish and Belytschko, 2007

Strong form: Mathematical model of real world process, differential equation and boundary conditions

Weak form: Basis for finite element solution

Approximation of functions: Construct approximate solution by combining predefined functions

Discrete equations: Inserting predefined functions into weak form yields linear system of equations

It's only math once the boundary value problem has been formulated!

Reminder: $\int_a^b u'(x)\cdot v(x)\,dx = \left[u(x)\cdot v(x)\right]_a^b - \int_a^b u(x)\cdot v'(x)\,dx$

Derivation of weak form

$$EA\,u''(x) - C\cdot u(x) = -n \qquad \Big|\cdot \delta u(x)$$

$$EA\,u''(x)\cdot\delta u(x) - C\cdot u(x)\cdot\delta u(x) = -n\cdot\delta u(x) \qquad \Big|\int_0^l \cdot\, dx$$

Function $\delta u: [0,L] \to \mathbb{R}$ is called test function or virtual displacement in mechanics.

$$\int_0^L \left(EA\,u''(x)\cdot\delta u(x) - C\cdot u(x)\cdot\delta u(x)\right) dx = -\int_0^L n\cdot\delta u(x)\,dx$$

$$EA\int_0^L u''\cdot\delta u\,dx - C\int_0^l u\cdot\delta u\,dx = -n\cdot\int_0^l \delta u\,dx$$

!

Integration by parts

$$EA\int_0^l u''\cdot\delta u\,dx = EA\left[u'\cdot\delta u\right]_0^l - EA\int_0^l u'\cdot\delta u'\,dx$$

u' $\quad$ v

$$= \underbrace{EA\cdot u'(l)}_{-S\cdot u(l)}\cdot\delta u(l) - \underbrace{EA\,u'(0)}_{-F}\cdot\delta u(0) - EA\int_0^l u'\cdot\delta u'\,dx$$

$$EA\int_0^l u''\cdot\delta u\,dx = -S\cdot u(l)\cdot\delta u(l) + F\cdot\delta u(0) - EA\int_0^l u'\cdot\delta u'\,dx$$

$$-S\cdot u(l)\cdot\delta u(l) + F\cdot\delta u(0) - EA\int_0^l u'\cdot\delta u'\,dx$$

$$-c\int_0^l u\cdot\delta u\,dx = -n\cdot\int_0^l\delta u\,dx$$

$$EA\int_0^l u'\cdot\delta u'\,dx + c\cdot\int_0^l u\cdot\delta u\,dx + S\cdot u(l)\cdot\delta u(l)$$

$$= n\cdot\int_0^l\delta u\,dx + F\cdot\delta u(0)$$

Variational problem (V): Find function $u:[0,L]\to\mathbb{R}$ such that

$$EA\cdot\int_0^L u'\cdot\delta u'\,dx + C\cdot\int_0^L u\cdot\delta u\,dx + S\cdot u(L)\cdot\delta u(L) = n\cdot\int_0^L \delta u\,dx + F\cdot\delta u(0)$$

for all (admissible) test functions δu.

If u solves (D), then u solves (V) (trivial)

If u solves (V), then u solves (D) (not trivial)

$$(D) \iff (V)$$

About strong and weak forms

Comparison of problems

Strong form Find a function which satisfies an equation at each point in the considered domain

Weak form Find a function, for which a scalar valued equation holds for any test function

The following terms have the same meaning

- **Boundary value problem** and **strong form**
- **Variational problem** and **weak form**

Why the names strong form and weak form?

Strong form: Requirements on u

- fulfill the differential equation pointwise (strongly) for each $x \in [0, l]$
- two times differentiable

Weak form: Requirements on u

- differential equation fulfilled in an integral sense
- one time differentiable (Fish and Belytschko 2007, p. 49)