FEM for 1D problems

Weak form (variational problem)

- Properties

Strong form of pile foundation problem

Boundary value problem (D): Find a function $u : [0,l] \to \mathbb{R}$ which satisfies the differential equation

$$\boxed{EAu''(x) - c \cdot u(x) = -n}$$

and the boundary conditions

$$\boxed{EAu'(0) = -F} \qquad \boxed{EAu'(l) + S \cdot u(l) = 0}$$

Weak form of pile foundation problem

Variational problem (V): Find function $u:[0,L]\to\mathbb{R}$ such that

$$EA\cdot\int_0^L u'\cdot\delta u'\,dx + C\cdot\int_0^L u\cdot\delta u\,dx + S\cdot u(L)\cdot\delta u(L) = n\cdot\int_0^L \delta u\,dx + F\cdot\delta u(0)$$

for all (admissible) test functions δu.

It can be shown that if u solves

- (D), then u solves (V) (trivial)
- (V), then u solves (D) (not trivial)

Both formulations are equivalent

$$(D) \iff (V)$$

(if certain requirements are fulfilled)

Ingredients and properties of weak form

Variational problem (V): Find function $u : [0, L] \to \mathbb{R}$ such that

$$EA \cdot \int_0^L u' \cdot \delta u' \, dx + C \cdot \int_0^L u \cdot \delta u \, dx + S \cdot u(L) \cdot \delta u(L) = n \cdot \int_0^L \delta u \, dx + F \cdot \delta u(0)$$

for all (admissible) test functions δu.

Example: Insert

$$u(x) = 1 - x^2 \quad , \quad u'(x) = -2x$$

$$\delta u(x) = x \quad , \quad \delta u'(x) = 1$$

into integrals of weak form

$$EA \int_0^L -2x \cdot 1 \, dx + C \cdot \int_0^L (1 - x^2) \cdot x \, dx + S \cdot (1 - L^2) \cdot L = n \cdot \int_0^L x \, dx + F \cdot 0$$

$$-EA \left[x^2 \right]_0^L + C \cdot \left[\frac{1}{2} x^2 - \frac{1}{4} x^4 \right]_0^L + S \cdot (L - L^3) \qquad = n \cdot \left[\frac{1}{2} x^2 \right]_0^L$$

$$\underbrace{-EA \cdot L^2 + C \cdot \left(\frac{1}{2} L^2 - \frac{1}{4} L^4 \right) + S \cdot (L - L^3)}_{\text{Number}} \qquad = \underbrace{\frac{1}{2} \cdot n \cdot L^2}_{\text{Number}} \qquad \Longleftrightarrow \quad F$$

Observation: The function $u(x) = 1 - x^2$ is not a solution of (V)

$u(x) = 1 - x^2$ $\qquad$ $\delta u(x) = x$

$$\boxed{EA\cdot\int_0^l u'\cdot\delta u'\,dx + C\cdot\int_0^l u\cdot\delta u\,dx + S\cdot u(l)\cdot\delta u(l)} = \boxed{n\cdot\int_0^l \delta u\,dx + F\cdot\delta u(0)}$$

Number $\qquad$ Number

Left hand side: Two functions are mapped on a number

$$(u,\delta u) \mapsto EA\int_0^l u'\delta u'\,dx + C\cdot\int_0^l u\cdot\delta u\,dx + S\,\delta u(l)\cdot u(l)$$

Right hand side: One function is mapped on a number

$$\delta u \mapsto n\int_0^l \delta u\,dx + F\cdot\delta u(0)$$

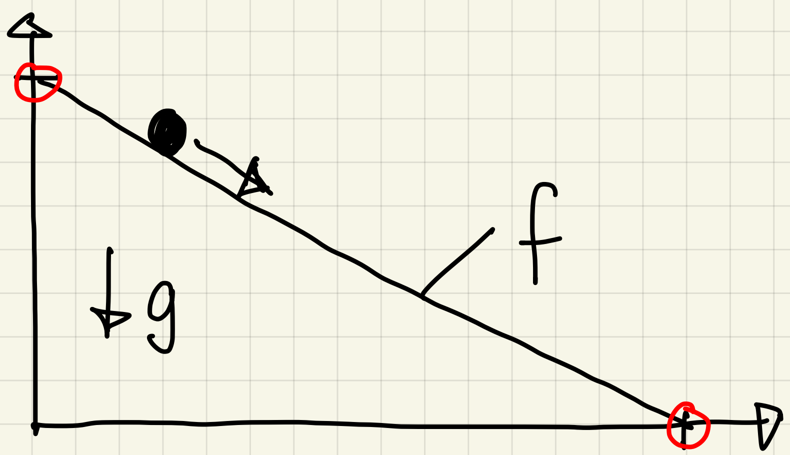
Definition (Functional): A mapping which assigns to one or more functions a number is called a functional

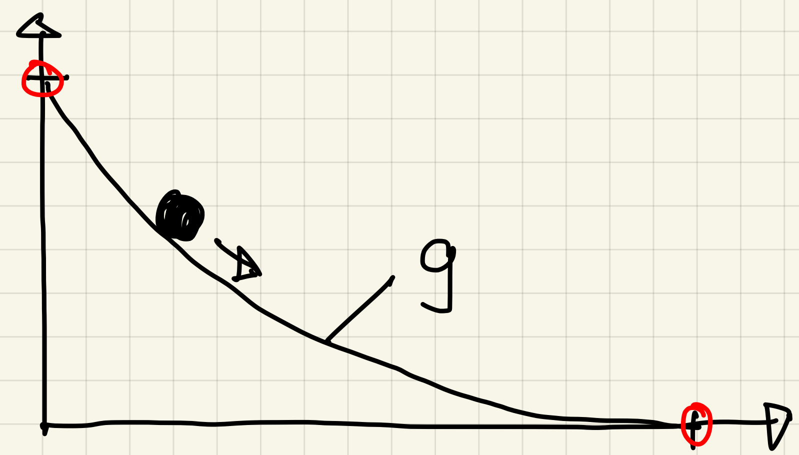
Functions and functionals

$f(x) = x^2$ $\quad x \in \mathbb{R} \longrightarrow \boxed{f} \longrightarrow f(x) \in \mathbb{R}$

$b(u) = \int_0^1 u(x)\,dx$ $\quad u \text{ function} \longrightarrow \boxed{b} \longrightarrow b(u) \in \mathbb{R}$

Famous example: Brachistochrone problem





$t = t(f)$

Find function f such that the marble takes the least time

$\rightarrow$ Youtube

Pile foundation problem

Functional on left hand side

$$a(u,\delta u) = EA \int_0^l u' \cdot \delta u' \, dx + C \cdot \int_0^l u \cdot \delta u \, dx + S \cdot u(l) \cdot \delta u(l)$$

Functional on right hand side

$$b(\delta u) = n \cdot \int_0^l \delta u \, dx + F \cdot \delta u(0)$$