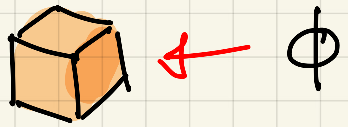
FEM for 2D problems

Instationary heat conduction

Strong form of problem

Energy conservation

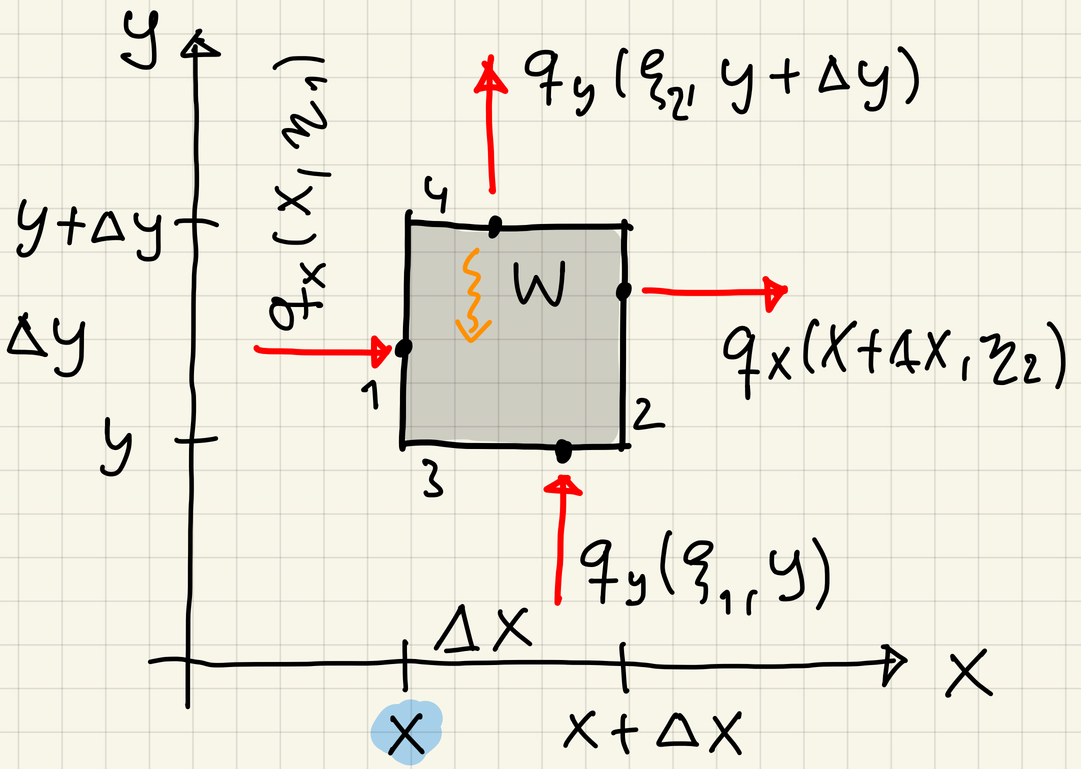


$\dot{\theta} = \frac{1}{m \cdot c} \cdot \phi$ now not equal to zero!

$\Leftrightarrow \quad m \cdot c \cdot \dot{\theta} = \phi$

c : Specific heat capacity [J/(kg·K)]

1. Conservation of energy



Basic relation

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + W \cdot \Delta x \cdot \Delta y \cdot 1 = \underbrace{\rho \cdot \Delta x \cdot \Delta y \cdot 1}_{m} \cdot c \cdot \dot{\theta}$$

Heat flux over boundary 3

$$\Phi_3 = \int_x^{x+\Delta x} 1 \cdot q_y(\xi, y)\, d\xi = q_y(\xi, y) \cdot \Delta x$$

thickness — Appendix

$$\left(q_x(x, \eta_1) - q_x(x+\Delta x, \eta_2)\right) \cdot \Delta y + \left(q_y(\xi_1, y) - q_y(\xi_2, y+\Delta y)\right) \cdot \Delta x + W \cdot \Delta x \cdot \Delta y = \rho \cdot c\, \Delta x \cdot \Delta y \cdot \dot{\theta} \quad \Big| : -\Delta x \cdot \Delta y$$

$$\frac{q_x(x+\Delta x, \eta_2) - q_x(x, \eta_1)}{\Delta x} + \frac{q_y(\xi_2, y+\Delta y) - q_y(\xi_1, y)}{\Delta y} - W = -\rho \cdot c \cdot \dot{\theta} \quad \Big| \Delta x, \Delta y \to 0$$

$$q_{x,x}(x, y) + q_{y,y}(x, y) = -\rho \cdot c \cdot \dot{\theta} + W$$

Note: Now, $\underline{q}$ also depends on time, i.e. $\underline{q} = \underline{q}(\underline{x}, t)$ $\quad \underline{x} = (x, y)$

omitted here!

$$\boxed{\rho \cdot c \cdot \dot{\theta} + \operatorname{div} \underline{q} = W}$$

2. Fourier's law

$$\underline{q} = -\lambda \cdot \nabla \theta$$

$$\boxed{\rho \cdot c \cdot \dot{\theta} - \lambda \cdot \operatorname{div} \nabla \theta = w}$$

Alternative form for $w = 0$

$$\dot{\theta} = \alpha \cdot \Delta \theta \qquad \text{where} \qquad \alpha = \frac{\lambda}{\rho \cdot c} \qquad \text{and} \qquad \Delta \theta = \operatorname{div} \nabla \theta$$

→ Classical form of heat equation

Initial Boundary value problem for heat conduction (D):

Find temperature distribution $\theta : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ with

$$\rho \cdot c \cdot \dot{\theta}(\underline{x},t) - \lambda \cdot \operatorname{div} \nabla \theta(\underline{x},t) = w(t) \qquad \text{for} \quad \underline{x} \in \Omega$$

and

$$\nabla \theta(\underline{x},t) \cdot \underline{n}(\underline{x}) = \frac{h}{\lambda}\left(\theta^*(t) - \theta(\underline{x},t)\right) \qquad \text{for} \quad \underline{x} \in \Gamma_R$$

$$\nabla \theta(\underline{x},t) \cdot \underline{n}(\underline{x}) = 0 \qquad \text{for} \quad \underline{x} \in \Gamma_N$$

$$\theta(\underline{x},t) = 0 \qquad \text{for} \quad \underline{x} \in \Gamma_D$$

$$\theta(\underline{x},0) = \theta_0(\underline{x}) \qquad \text{initial condition}$$

$\underline{x} = (x,y)$

Weak formulation

Differential equation

$$\rho\cdot c\cdot\dot{\theta} \quad -\lambda\cdot \operatorname{div}\nabla\theta \quad = w$$

$\delta\theta$

Multiplication with test function and integration over Ω

$$\int_{\Omega}\rho\cdot c\cdot\dot{\theta}\cdot\delta\theta\,dA-\int_{\Omega}\lambda\cdot\operatorname{div}\nabla\theta\cdot\delta\theta\,dA=\int_{\Omega}w\cdot\delta\theta\,dA$$

Integration by parts

$$\int_{\Omega}\rho\cdot c\cdot\dot{\theta}\cdot\delta\theta\,dA+\int_{\Omega}\lambda\cdot\nabla\theta\cdot\nabla\delta\theta\,dA+\int_{\Gamma_R}h\cdot\theta\cdot\delta\theta\,ds$$

$$=\int_{\Omega}w\cdot\delta\theta\,dA+\int_{\Gamma_R}h\cdot\theta^{*}\cdot\delta\theta\,ds$$

We skip the statement of the weak form!

Approximate Solution

$\hat{\theta}_i : \mathbb{R}_+ \to \mathbb{R}$

$$\theta_h(\underline{x}, t) = \sum_{i=1}^{N} \varphi_i(\underline{x}) \cdot \hat{\theta}_i(t)$$

$$\rightsquigarrow \quad \dot{\theta}_h(\underline{x}, t) = \sum_{i=1}^{N} \varphi_i(\underline{x}) \cdot \dot{\hat{\theta}}_i(t)$$

Insert into integral equation

Initial value problem

$$\underline{M}\,\dot{\underline{\theta}}(t) + \underline{K}\,\underline{\theta}(t) = \underline{\Gamma}(t) \iff \underbrace{\dot{\underline{\theta}}(t) = \underline{M}^{-1}\left(-\underline{K}\,\underline{\theta}(t) + \underline{\Gamma}(t)\right)}_{\text{Can be solved with built-in Matlab functions}}$$

With

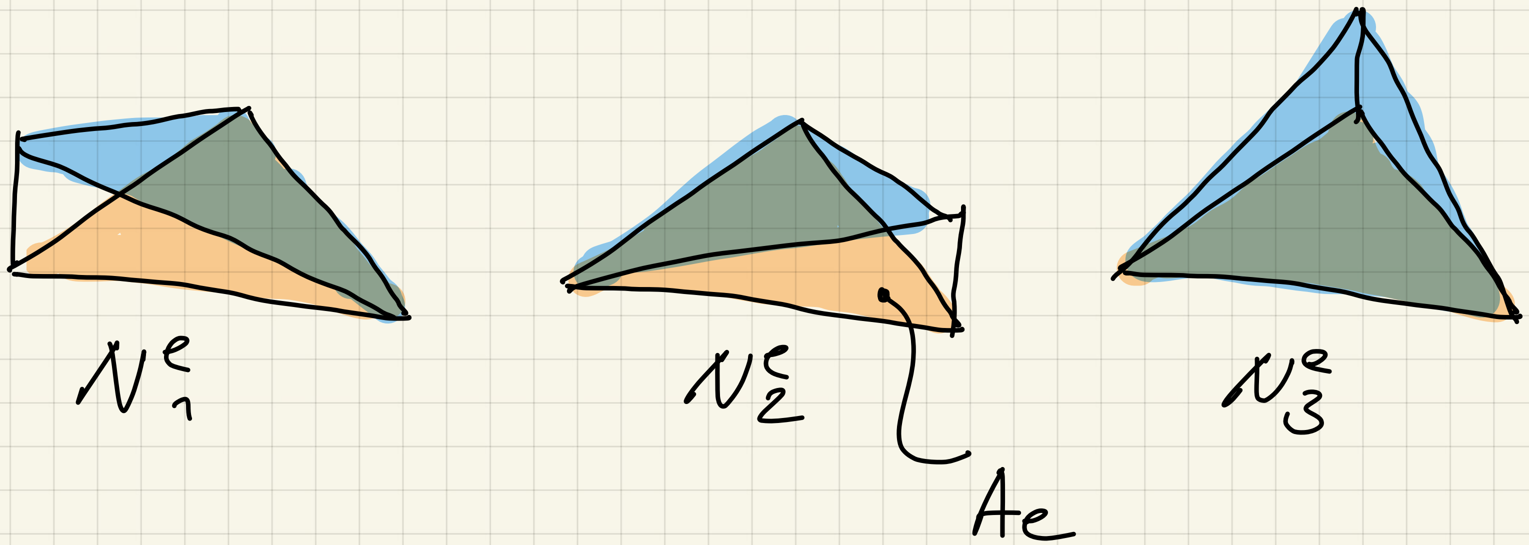
$$m_{ij} = \int_{\Omega} \rho \cdot c \cdot \varphi_i \cdot \varphi_j \, dA$$

$\underline{M}$: Heat storage matrix (mass matrix)

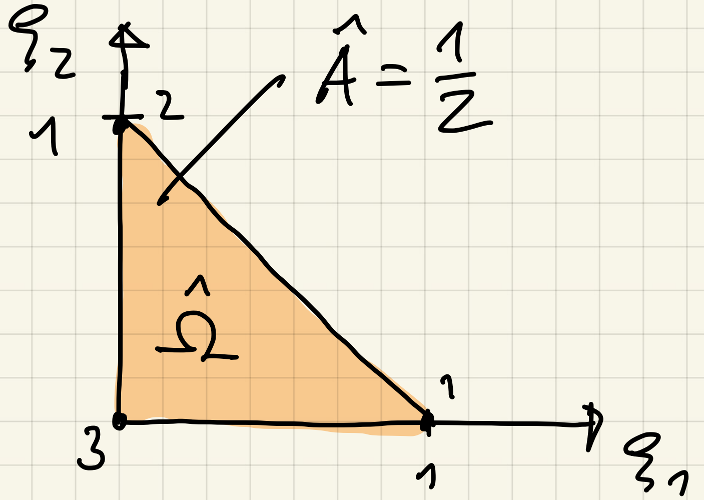
$\underline{K}$: Heat conduction matrix (Stiffness matrix)

Element mass matrix

$$m^e_{ij} = \int_{\Omega^e} \rho \cdot c \cdot \underbrace{N^e_i \cdot N^e_j}_{\text{quadratic function}} \, dA$$



Integration over reference element



$$\hat{N}_1(\xi_1, \xi_2) = \xi_1$$

$$\hat{N}_2(\xi_1, \xi_2) = \xi_2$$

$$\hat{N}_3(\xi_1, \xi_2) = 1 - \xi_1 - \xi_2$$

It can be shown that

$$\int_{\Omega^e} N^e_i \cdot N^e_j \, dA = 2 \cdot A_e \cdot \int_{\hat{\Omega}} \hat{N}_i \cdot \hat{N}_j \, dA$$

$$2A_e = \frac{A_e}{\hat{A}}$$