FEM for 1D problems

Numerical solution

- Pile foundation problem

$\boxed{\begin{array}{l}\text{Variational problem (V): Find function } u:[0,L]\to\mathbb{R} \text{ such that} \\ \quad EA\cdot\int_0^L u'\cdot\delta u'\,dx + C\cdot\int_0^L u\cdot\delta u\,dx + S\cdot u(L)\cdot\delta u(L) = n\cdot\int_0^L \delta u\,dx + F\cdot\delta u(0) \\ \text{for all (admissible) test functions } \delta u.\end{array}}$

We have in this case

$$a(u,\delta u) = EA\cdot\int_0^L u'\cdot\delta u'\,dx + C\cdot\int_0^L u\cdot\delta u\,dx + S\cdot u(L)\cdot\delta u(L)$$

$$b(\delta u) = n\cdot\int_0^L \delta u\,dx + F\cdot\delta u(0)$$

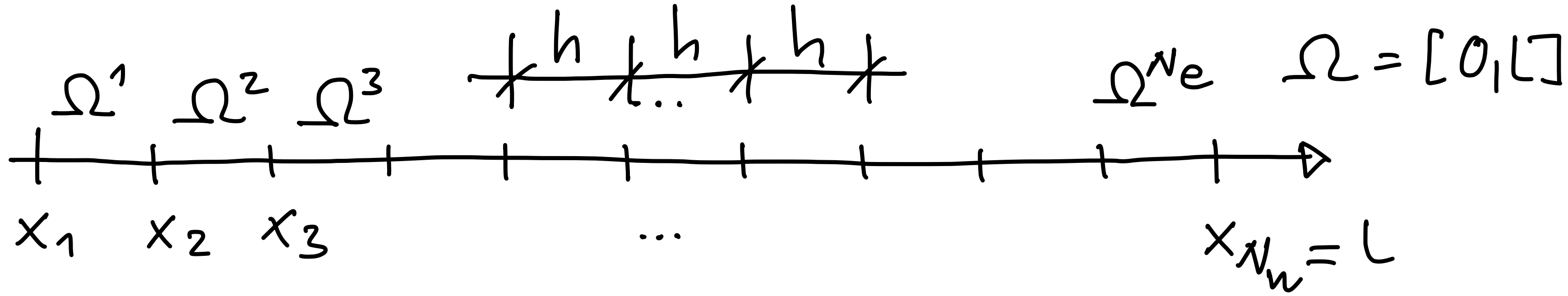
Choice of basis functions

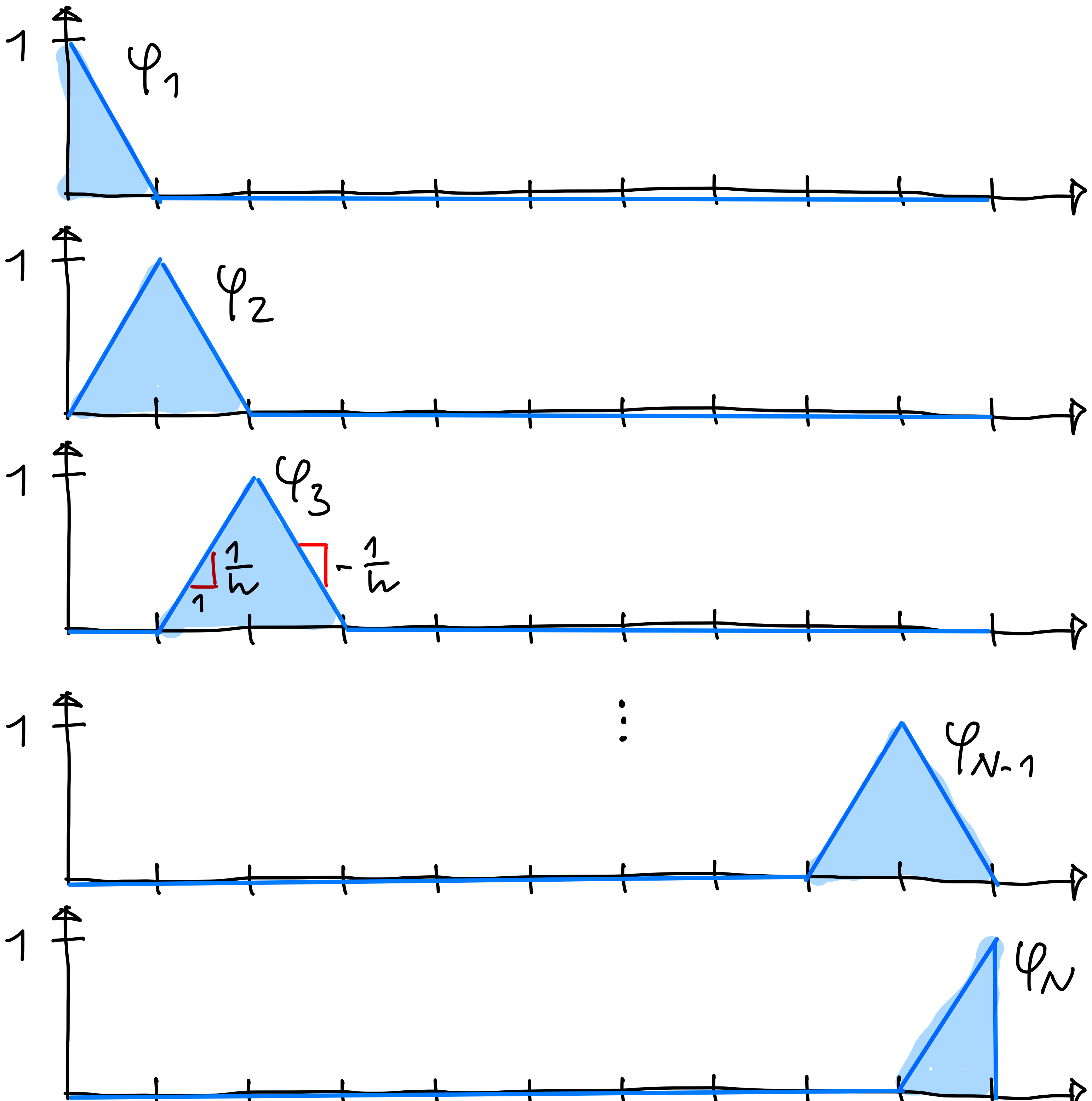
N_n – Number of nodes

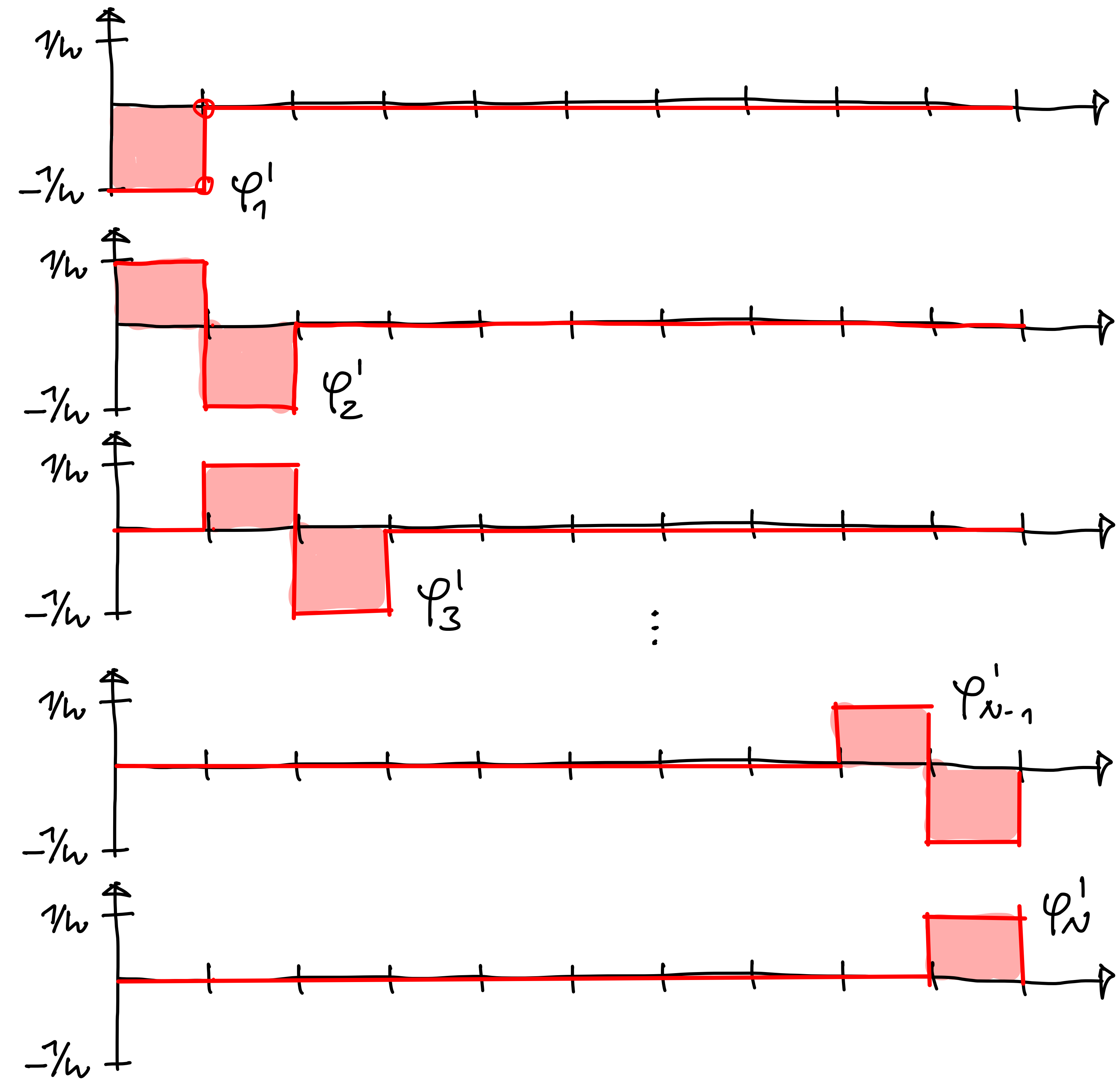
N_e – Number of elements

N – Number of basis functions

h – Element size







Stiffness matrix $\underline{K}$ with $k_{ij} = a(\varphi_i, \varphi_j)$

Split contributions to matrix $\underline{K}$

$$K_{ij} = a(\varphi_i, \varphi_j) = \underbrace{EA \cdot \int_0^L \varphi_i' \cdot \varphi_j' \, dx}_{} + \underbrace{C \cdot \int_0^l \varphi_i \cdot \varphi_j \, dx}_{} + \underbrace{S \cdot \varphi_i(L) \cdot \varphi_j(L)}_{}$$

$$= \quad K_{ij}^{EA} \quad + \quad K_{ij}^{C} \quad + \quad K_{ij}^{S}$$

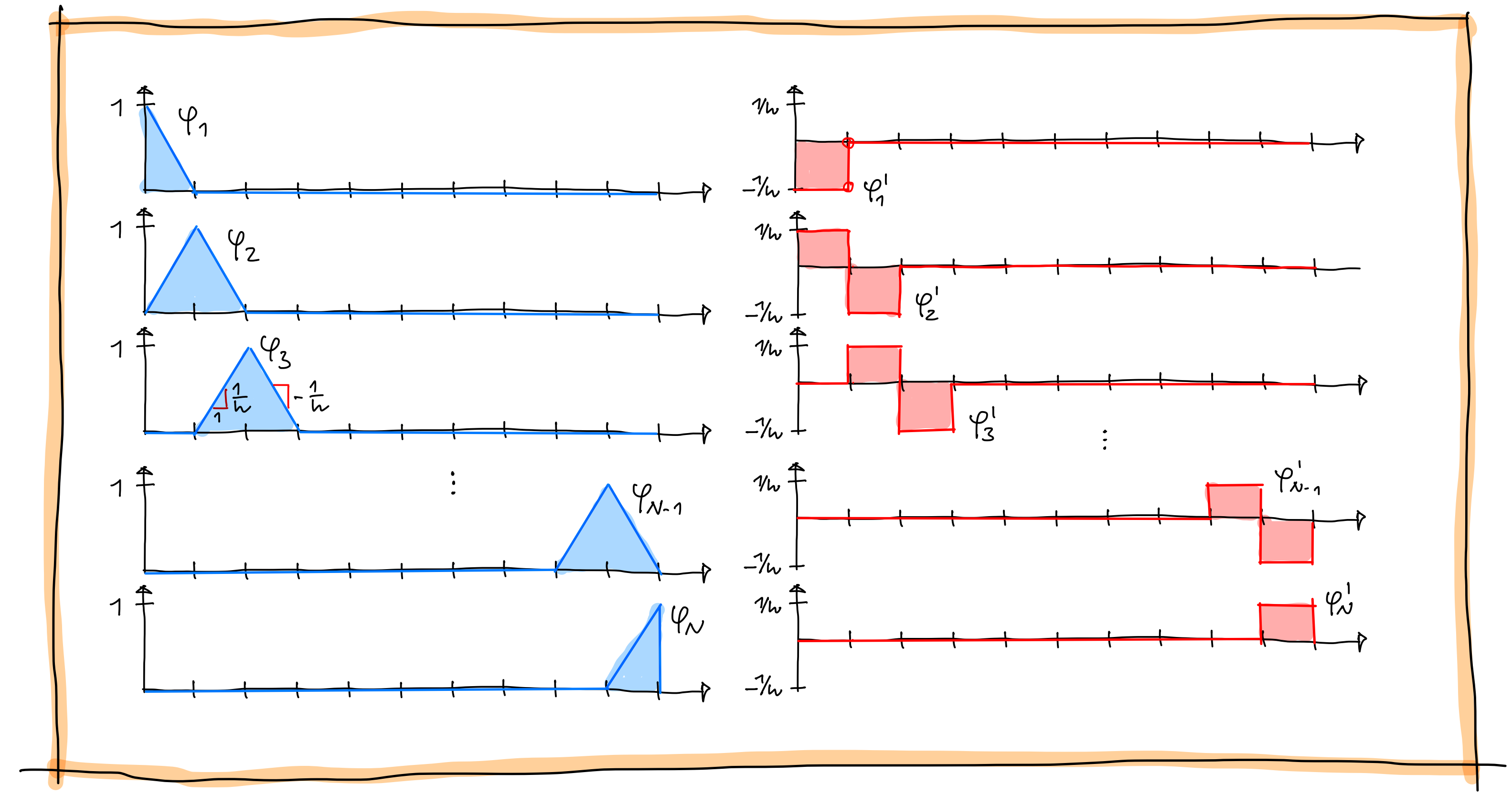
$$a^{EA}(u, \delta u) = EA \cdot \int_0^l u' \cdot \delta u' \, dx \rightsquigarrow K_{ij}^{EA} = a^{EA}(\varphi_i, \varphi_j) = EA \cdot \int_0^l \varphi_i' \cdot \varphi_j' \, dx$$

$$a^{C}(u, \delta u) = C \cdot \int_0^l u \cdot \delta u \, dx \rightsquigarrow K_{ij}^{C} = a^{C}(\varphi_i, \varphi_j) = C \cdot \int_0^l \varphi_i \cdot \varphi_j \, dx$$

$$a^{S}(u, \delta u) = S \cdot u(L) \cdot \delta u(L) \rightsquigarrow K_{ij}^{S} = a^{S}(\varphi_i, \varphi_j) = S \cdot \varphi_i(L) \cdot \varphi_j(L)$$

Contribution of EA

$$K_{ij}^{EA} = a^{EA}(\varphi_i, \varphi_j) = EA \cdot \int_0^l \varphi_i' \cdot \varphi_j' \, dx$$



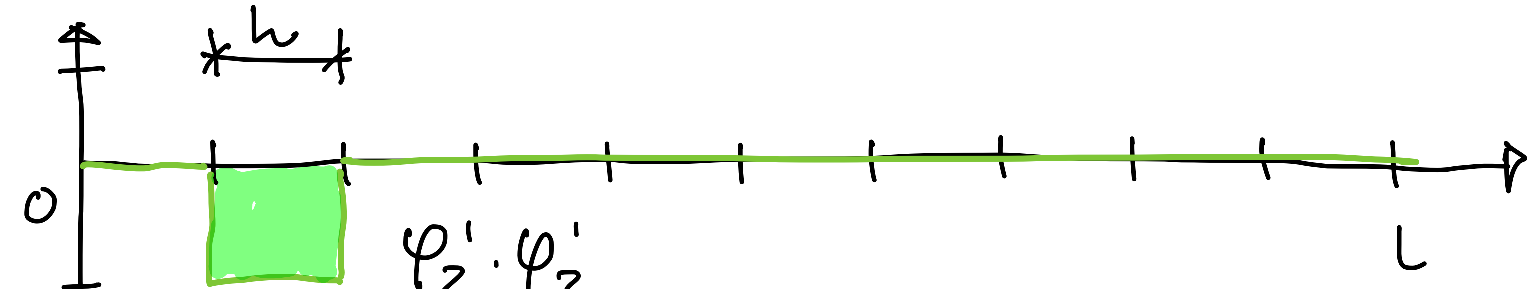
Row 1, i=1

$$k_{11}^{EA} = EA \cdot \int_0^l \varphi_1' \cdot \varphi_1' \, dx = EA \cdot h \cdot \frac{1}{h^2} = \frac{EA}{h}$$

$$k_{12}^{EA} = EA \cdot \int_0^l \varphi_1' \cdot \varphi_2' \, dx = EA \cdot h \cdot \frac{-1}{h^2} = -\frac{EA}{h}$$

$$k_{13}^{EA} = EA \cdot \int_0^l \varphi_1' \cdot \varphi_3' \, dx = 0$$

$$k_{14}^{EA} = \quad \dots \quad = K_{1N}^{EA} = 0$$



Row 2, i=2

$$k_{21}^{EA} = k_{12}^{EA} = -\frac{EA}{h}$$

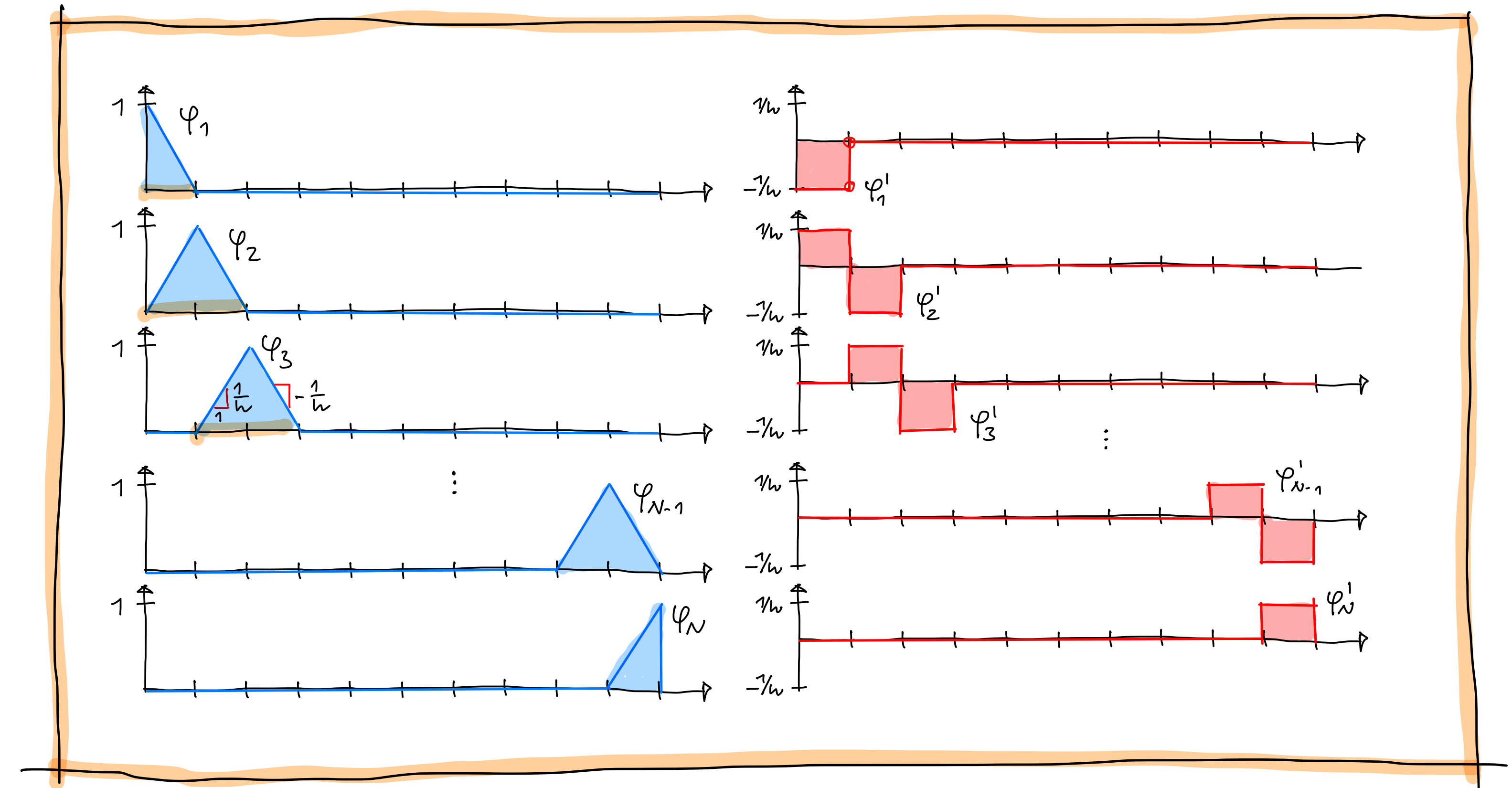
$$k_{22}^{EA} = EA \cdot \int_0^l \varphi_2' \cdot \varphi_2' \, dx = EA \cdot 2 \cdot h \cdot \frac{1}{h^2} = \frac{2EA}{h}$$

$$K_{23}^{EA} = EA \cdot \int_0^l \varphi_2' \cdot \varphi_3' \, dx = EA \cdot h \cdot \frac{-1}{h^2} = -\frac{EA}{h} \quad , \quad k_{34}^{EA} = \dots = K_{3N}^{EA} = 0$$

Row 3

$$k_{31}^{EA} = 0 \, , \quad k_{32}^{EA} = -\frac{EA}{h} \, , \quad k_{33}^{EA} = \frac{2EA}{h} \, , \quad k_{34}^{EA} = -\frac{EA}{h}$$

$$\underline{K}^{EA} = \frac{EA}{h}\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$



Sparse and symmetric