FEM for 1D problems

Approximation of functions

Ingredients of finite element solution

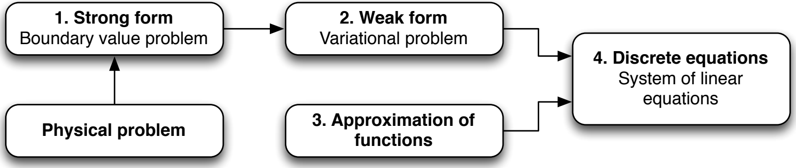


Diagram according to Fish and Belytschko, 2007

Strong form: Mathematical model of real world process, differential equation and boundary conditions

Weak form: Basis for finite element solution

Approximation of functions: Construct approximate solution by combining predefined functions

Discrete equations: Inserting predefined functions into weak form yields linear system of equations

It's only math once the boundary value problem has been formulated!

Finite element method is based on two simple ideas

1. Construct approximate solution by combining predefined functions
2. Define functions piecewise on so called elements

But as always: The devil is in the details

"I'm here about the details."

- How to determine best combination?
- What does it mean to combine functions?
- How to consider (inhomogenous) Dirichlet boundary conditions?
- How to define the piecewise functions?

First main idea of FEM: Combine functions

Choose some basis functions $\varphi_1, \varphi_2, \dots, \varphi_N$ and approximate the solution u by the function

$$u_h(x) = \varphi_1(x)\cdot \hat{u}_1 + \varphi_2(x)\cdot \hat{u}_2 + \dots + \varphi_N(x)\cdot \hat{u}_N, \quad \hat{u}_i \in \mathbb{R}$$

By that, the original problem of finding a function is replaced by the problem of finding real numbers $\hat{u}_i$.

Taylor-series

$$\varphi_k(x) = (x-x_0)^k, \quad f(x) = \sum_{k=0}^{\infty} \underbrace{\frac{f^{(k)}(x_0)}{k!}}_{\hat{u}_i} \cdot \underbrace{(x-x_0)^k}_{\varphi_i(x)}$$

Fourier series

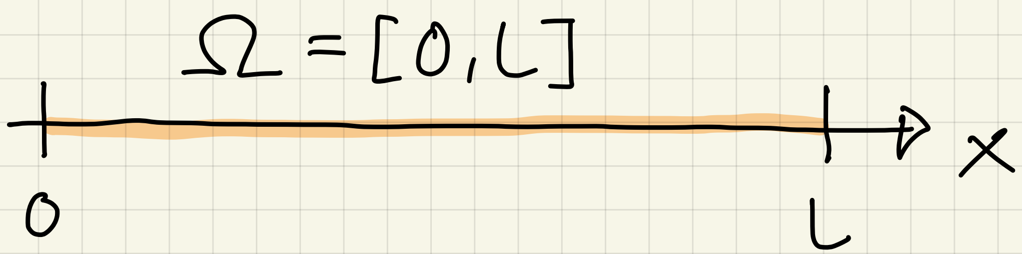
$$\varphi_k(x) = e^{ikx}, \quad f(x) = \sum_{k=0}^{\infty} c_k \cdot e^{ikx}$$

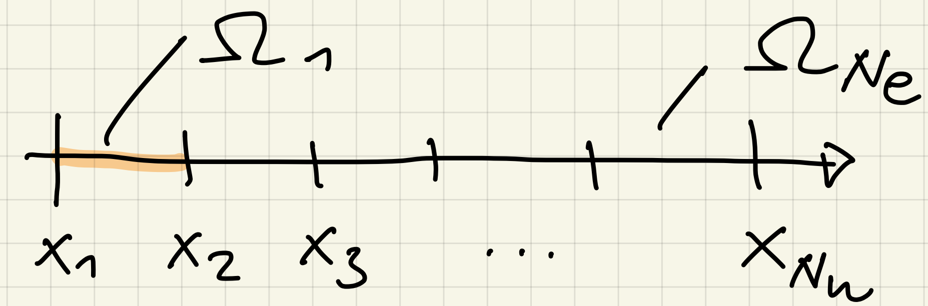
First applications of this idea

- Ritz (1878-1909)
- Galerkin (1871-1945)
- Gander + Wanner (2012)

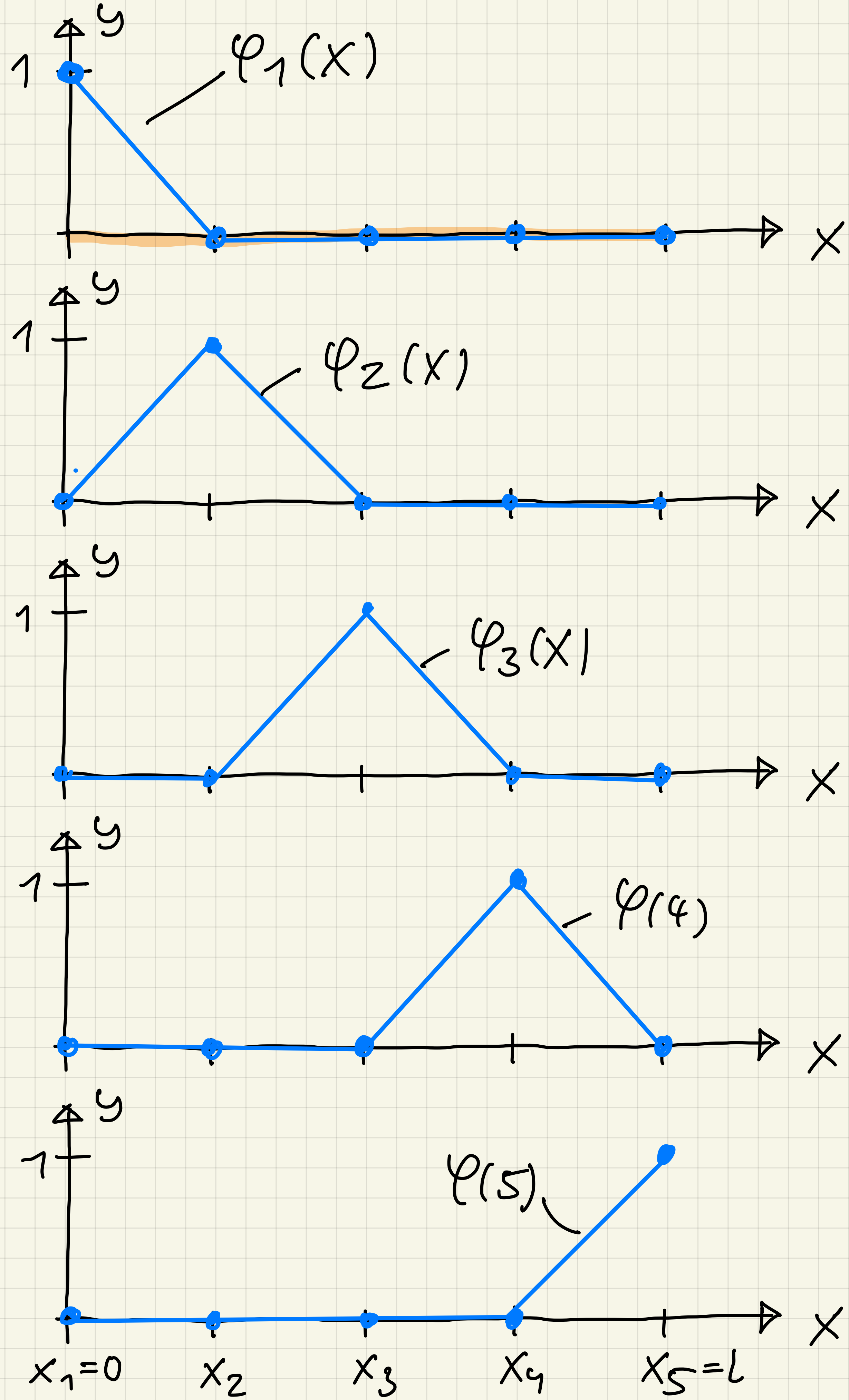
Second main idea of FEM: Define functions piecewise

Piecewise linear basis functions





Divide domain $\Omega = [0, L]$ into elements Ω_e, $e = 1, \dots, N_e$. The elements are connected at nodes x_n, $n = 1, \dots, N_n$ such that $\Omega_e = [x_e, x_{e+1}]$ (in the simplest case). The maximum length of an element is called h.



The functions φ_i are piecewise linear with

$$\varphi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad i,j = 1, \dots, N$$

Approximate solution

$$u_h(x) = \varphi_1(x) \cdot \hat{u}_1 + \varphi_2(x) \cdot \hat{u}_2 + \dots + \varphi_5(x) \cdot \hat{u}_5$$

The numbers $\hat{u}_i \in \mathbb{R}$ are called degrees of freedom

Example: $\underline{\hat{u}} = (1, 0.75, 1, 0.5, 0.5)^T$