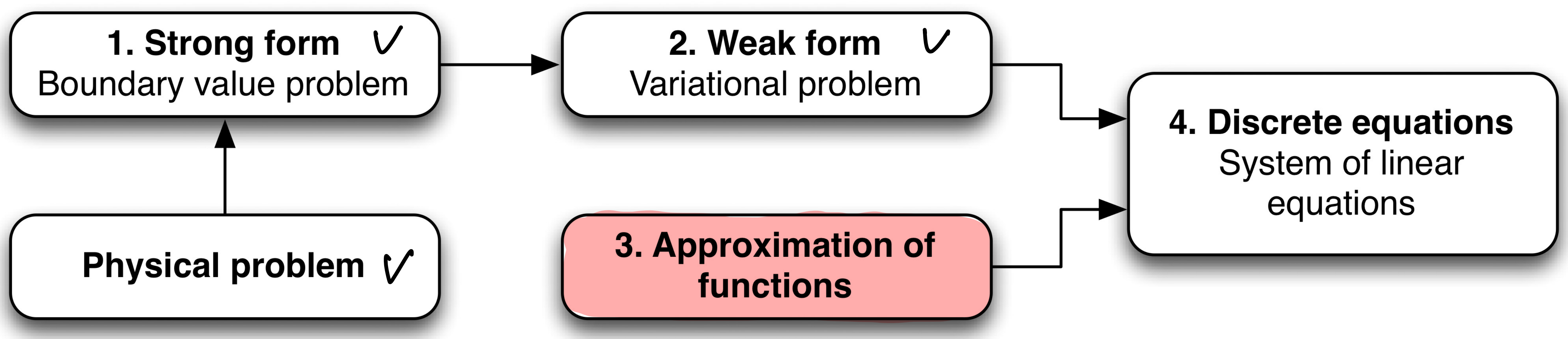
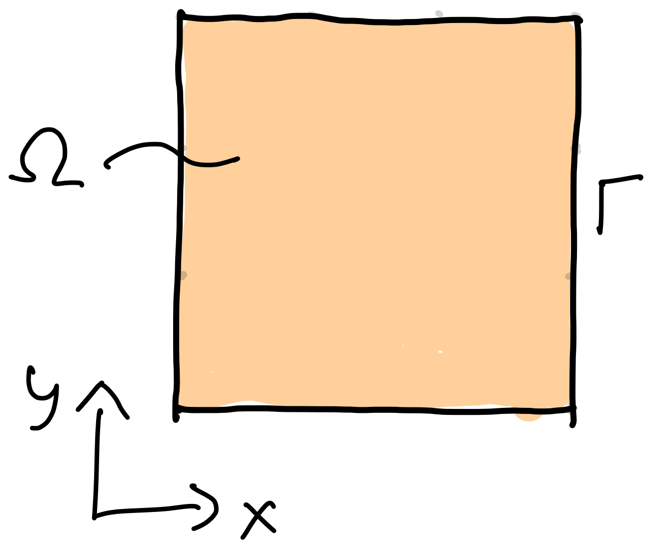
FEM for 2D problems

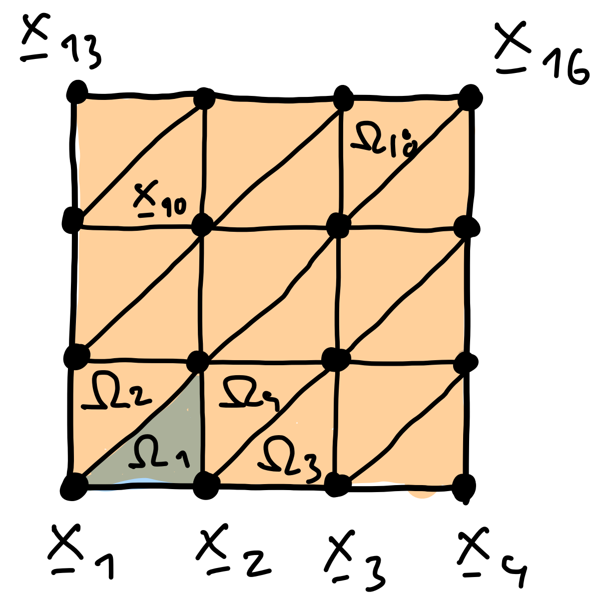
Heat conduction: Finite element formulation


1. Strong form √
Boundary value problem
2. Weak form √
Variational problem
4. Discrete equations
System of linear equations
Physical problem √
3. Approximation of functions

Global basis functions

Simplest approach: Triangle mesh and piecewise linear functions





Nodes $\underline{x}_n$, $n = 1, \dots, N_n$

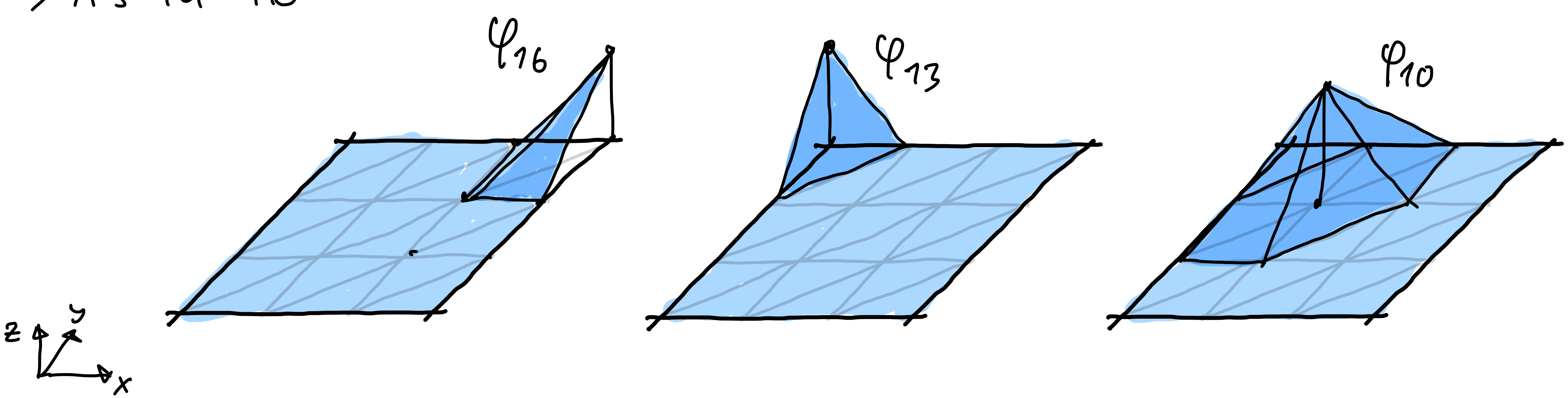
Elements Ω_e , $e = 1, \dots, N_e$

$\Omega_e \subset \Omega$, $\underline{x}_n \in \Omega$

Basis functions: $\varphi_i : \Omega \to \mathbb{R}$, $i = 1, \dots, N$

- Piecewise linear
- Function value one at node i and zero for all other nodes $\to \varphi_i(\underline{x}_j) = \delta_{ij}$

$\to$ As in 1D



Linear system

$$a(\theta,\delta\theta) = \int_\Omega \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta \cdot \delta\theta \, ds$$

$$b(\delta\theta) = \int_\Omega w \cdot \delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta^* \cdot \delta\theta \, ds$$

Approximate solution

$$\theta_h = \sum_{i=1}^{N} \varphi_i \cdot \hat{\theta}_i \quad , \quad \hat{\theta}_i \in \mathbb{R}$$

As before

$$\underline{K}\,\underline{\hat{\theta}} = \underline{r}$$

where

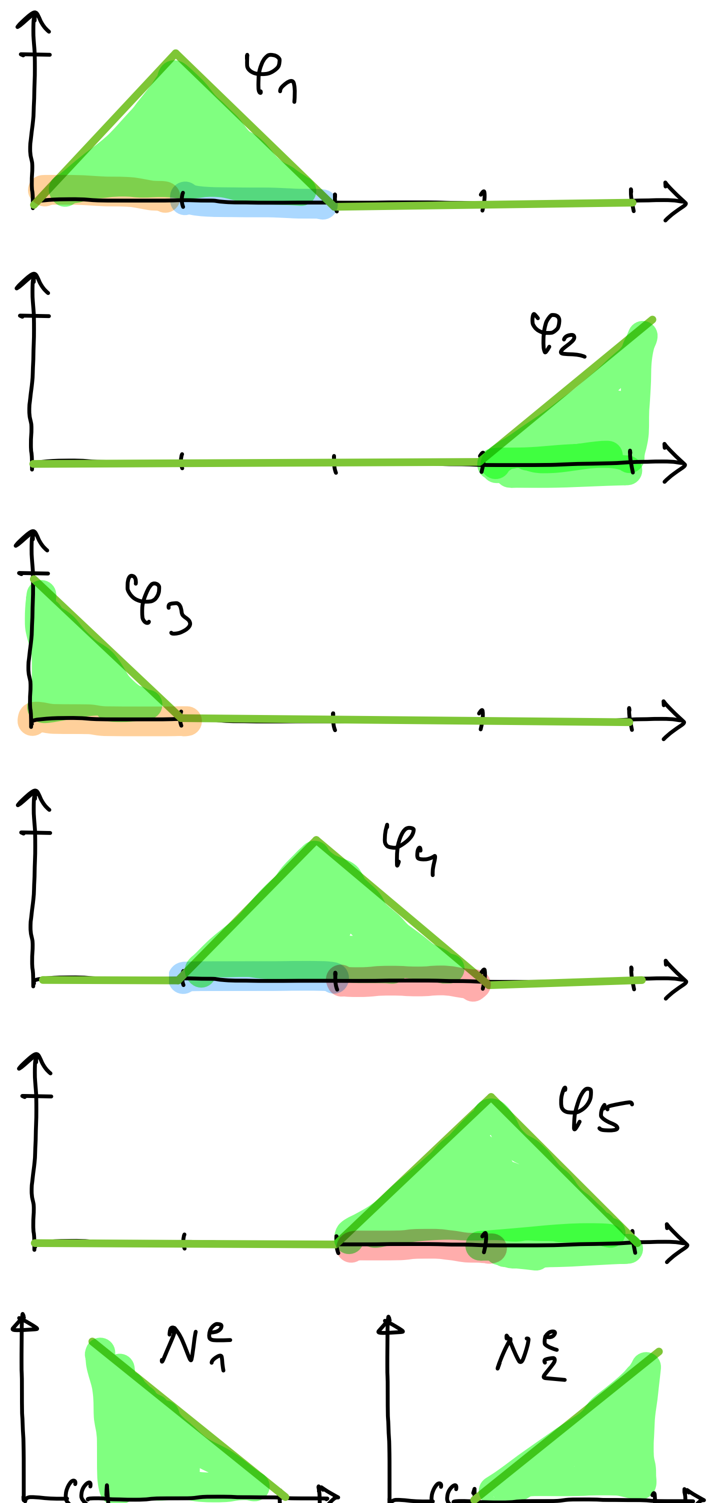
$$K_{ij} = a(\varphi_i, \varphi_j) = \int_\Omega \lambda \cdot \nabla\varphi_i \cdot \nabla\varphi_j \, dA + \int_{\Gamma_R} h \cdot \varphi_i \cdot \varphi_j \, ds$$

$$r_i = b(\varphi_i) = \int_\Omega w \cdot \varphi_i \, dA + \int_{\Gamma_R} h \cdot \theta^* \cdot \varphi_i \, ds$$

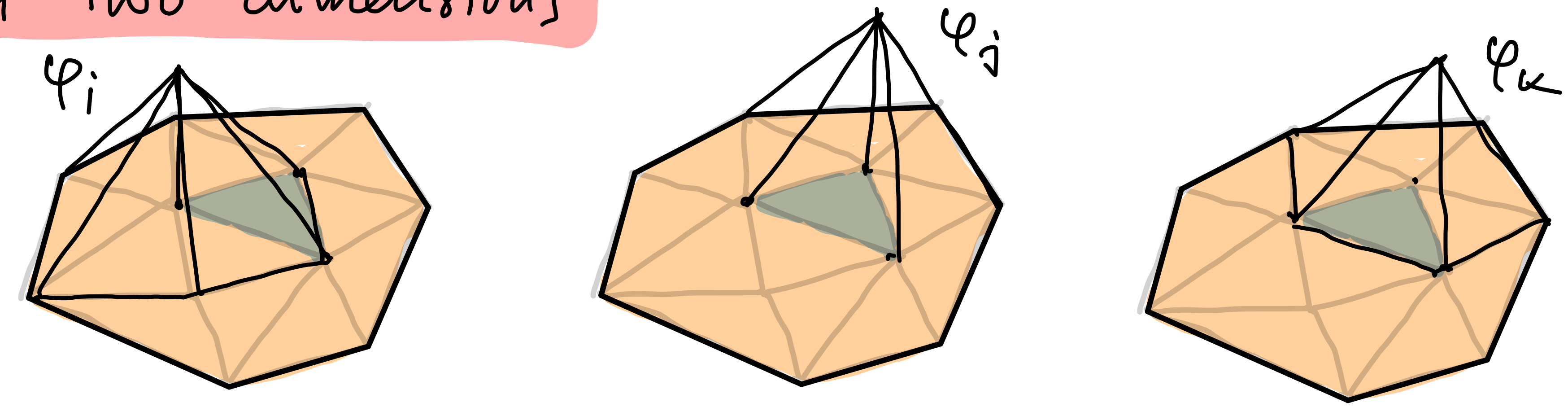
$\rightarrow$ Compute element-wise

Linear element functions on triangle

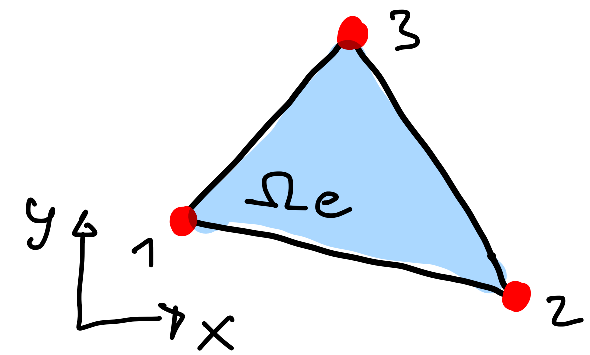
In one dimension



In two dimensions



Element $\Omega_e \subset \Omega$

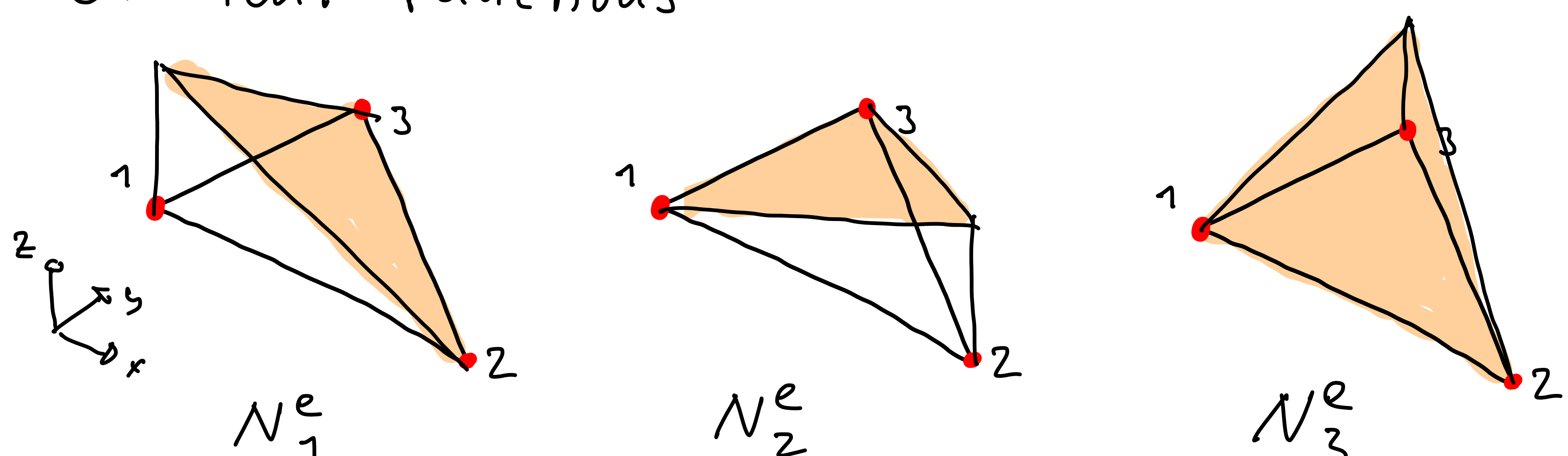


Nodes $\underline{x}_1^e, \underline{x}_2^e, \underline{x}_3^e$

Area A_e

We omit e when we consider a single element

Element functions



$N_i^e : \Omega_e \to \mathbb{R}$ with $N_i^e(\underline{X}_j^e) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$, $i,j = 1,2,3$

N_i^e are functions of planes

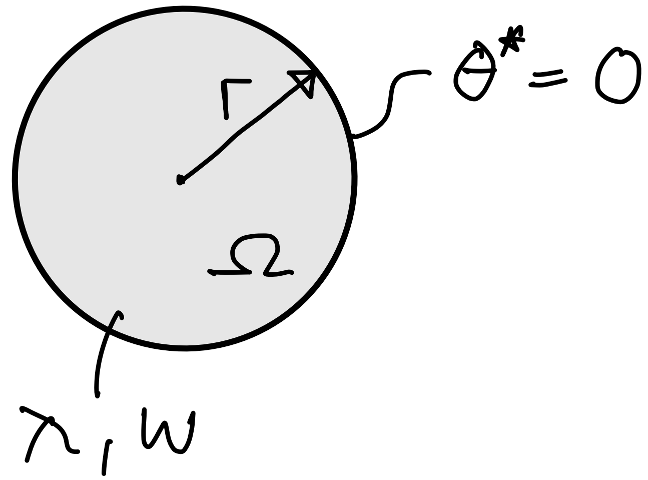
$$N_i^e(x,y) = a_i + b_i \cdot x + c_i y$$

$$\nabla N_i^e(x,y) = \begin{pmatrix} b_i \\ c_i \end{pmatrix}$$

$\to$ Gradient is element-wise constant

Milestones

1. Verification with analytical solution



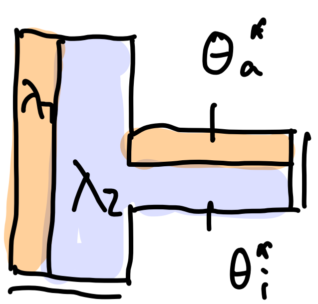
$$\theta(x_1, x_2) = \frac{w}{4 \cdot \lambda} \left(r^2 - x_1^2 - x_2^2 \right)$$

$$\theta_{max} = \frac{w \cdot r^2}{4 \cdot \lambda}$$

Required

- Element conductivity matrix $\underline{K}^e$ and source vector $\underline{r}^e$
- Dirichlet BCs
- Mesh with triangle elements

2. Application to practical problem



Required

- Robin and Neumann BCs
- Varying values for λ

Element conduction matrix and source vector

Global bilinear and linear forms

$$a(\theta,\delta\theta) = \int_{\Omega} \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta \cdot \delta\theta \, ds$$

$$b(\delta\theta) = \int_{\Omega} w \cdot \delta\theta \, dA + \int_{\Gamma_R} h \cdot \theta^{*} \cdot \delta\theta \, ds$$

later

Element bilinear and linear forms

$$a^{e}(\theta,\delta\theta) = \int_{\Omega_e} \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA$$

$$b^{e}(\delta\theta) = \int_{\Omega_e} w \cdot \delta\theta \, dA$$

Element stiffness matrix

$$a^e(\theta,\delta\theta) = \int_{\Omega_e} \lambda \cdot \nabla\theta \cdot \nabla\delta\theta \, dA$$

$$K_{ij}^e = a^e(N_i^e, N_j^e) = \int_{\Omega_e} \lambda \cdot \nabla N_i^e \cdot \nabla N_j^e \, dA$$

For linear triangle

$$\stackrel{\downarrow}{=} \lambda \cdot \nabla N_i^e \cdot \nabla N_j^e \int_{\Omega_e} 1 \, dA$$

$$= \lambda \cdot A_e \cdot \nabla N_i^e \cdot \nabla N_j^e$$

$$\underline{K}^e = \lambda \cdot A_e \cdot \begin{pmatrix} \nabla N_1^e \cdot \nabla N_1^e & \nabla N_1^e \cdot \nabla N_2^e & \nabla N_1^e \cdot \nabla N_3^e \\ & \nabla N_2^e \cdot \nabla N_2^e & \nabla N_2^e \cdot \nabla N_3^e \\ \text{Sym} & & \nabla N_3^e \cdot \nabla N_3^e \end{pmatrix}$$

Element matrix with the B-matrix

$\underline{N}^e = (N_1^e \;\; N_2^e \;\; N_3^e)$, Vector of shape functions

$\underline{D} = \nabla = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \end{pmatrix}$, Differential-operator

Exercise

$\underline{B}^e = \underline{D}\,\underline{N}^e = (\nabla N_1^e \;\; \nabla N_2^e \;\; \nabla N_3^e)$

$\underline{K}^e = A_e \cdot \lambda \cdot \underline{B}^{eT} \underline{B}^e$