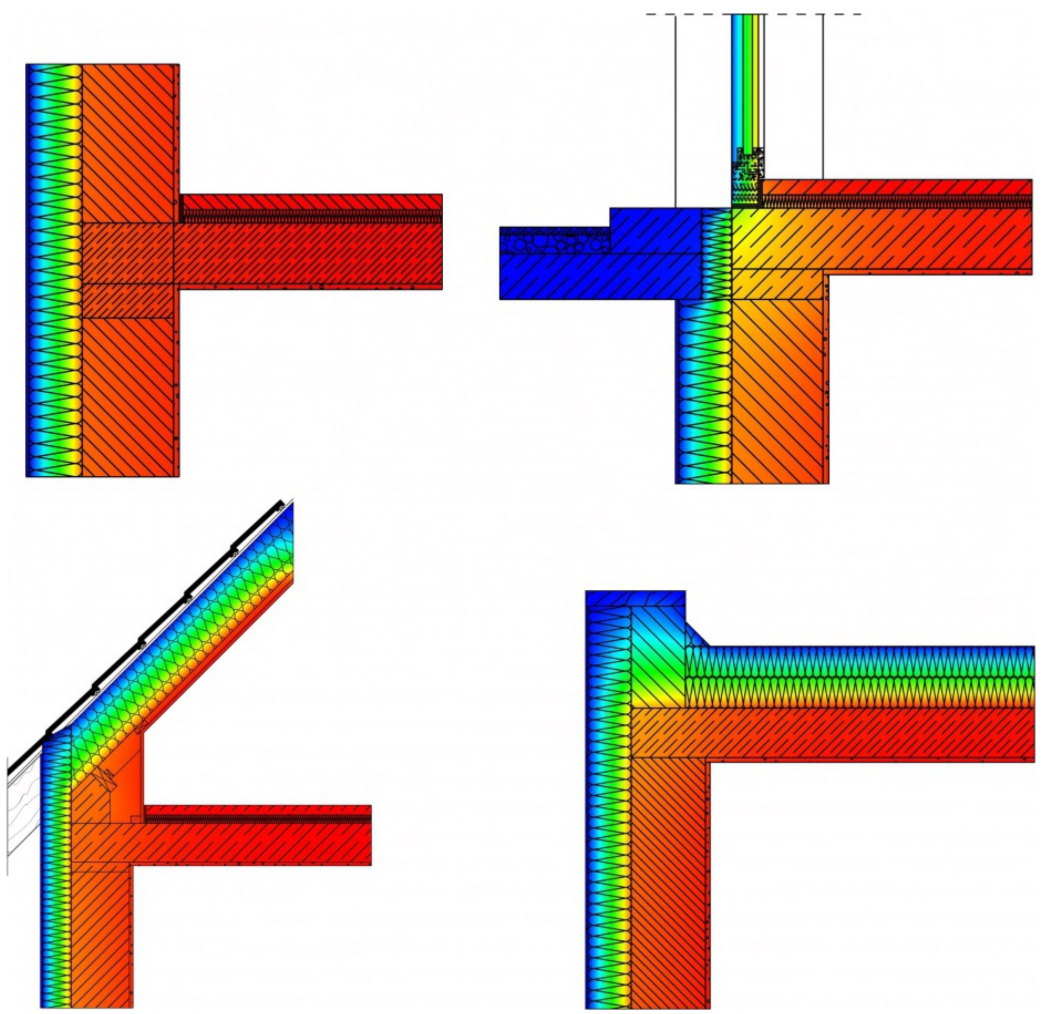
FEM for 2D problems

Heat conduction: Boundary value problem

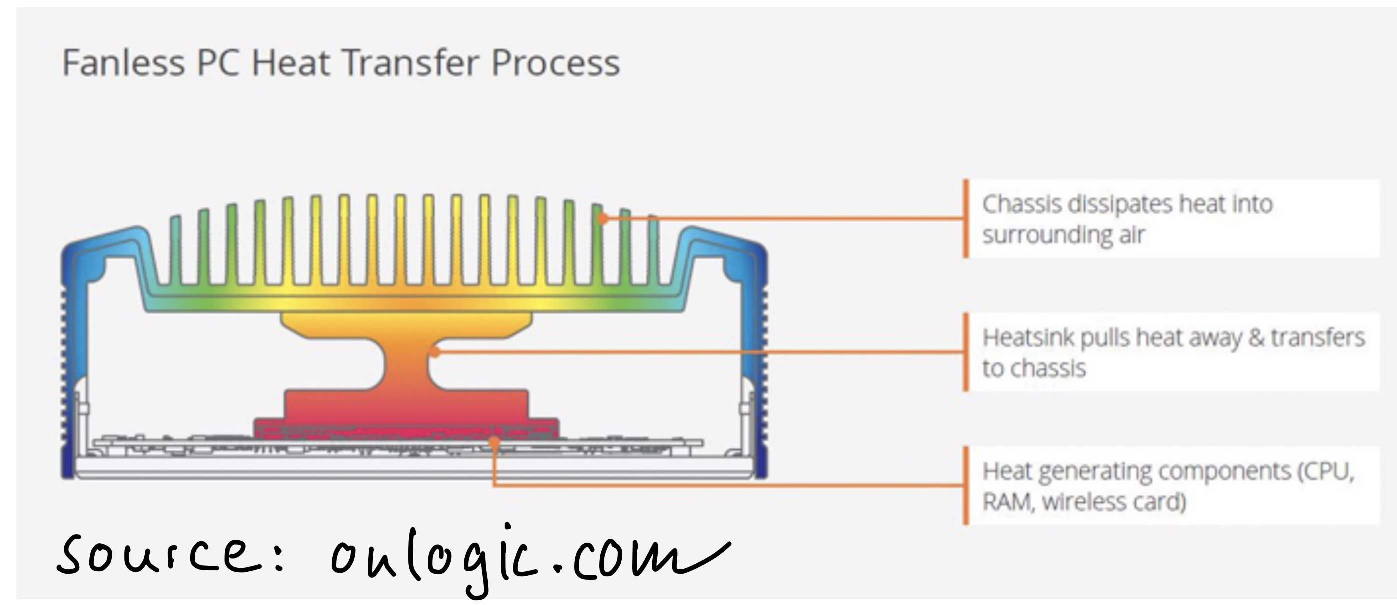
Applications

Insulation



source: waermebruecken-online.de

Cooling



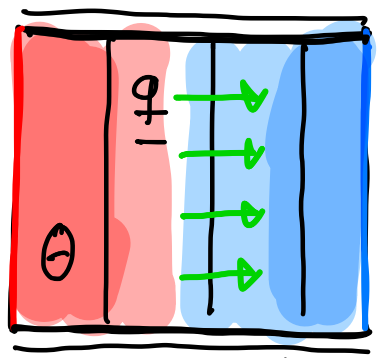
source: onlogic.com

We need to understand what happens

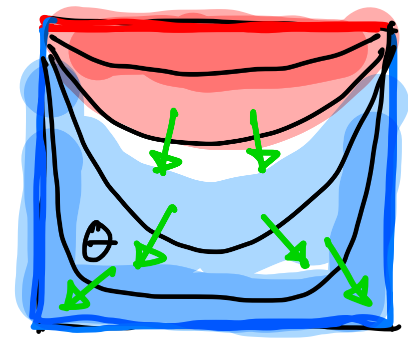
- inside the body
- on the boundary

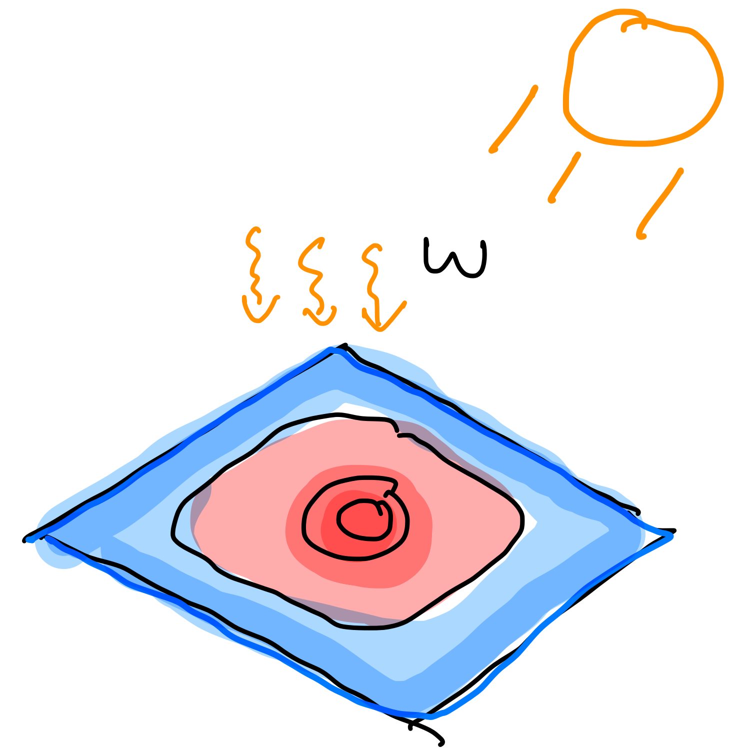
Physical background

Examples



└ isolated





Quantities

θ – Temperature $[K/°C]$, $\theta : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$

w – Heat source $[W/m^3]$, $w \in \mathbb{R}$

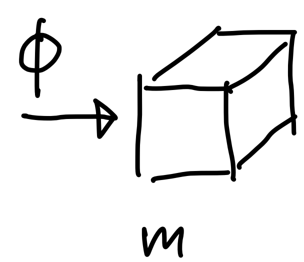
$\underline{q}$ – Heat flux density $[W/m^2]$, $\underline{q} : \Omega \to \mathbb{R}^2$

Physical laws

Fouriers law

$$\underline{q} = -\lambda \cdot \nabla \theta$$

Conservation of energy



$$\dot{\theta} = \frac{1}{c \cdot m} \phi \underset{\uparrow}{=} 0$$

stationary

Thermal conductivity

λ $[W/(m \cdot K)]$

Specific heat capacity

c $[J/(K \cdot kg)]$

Heat flow

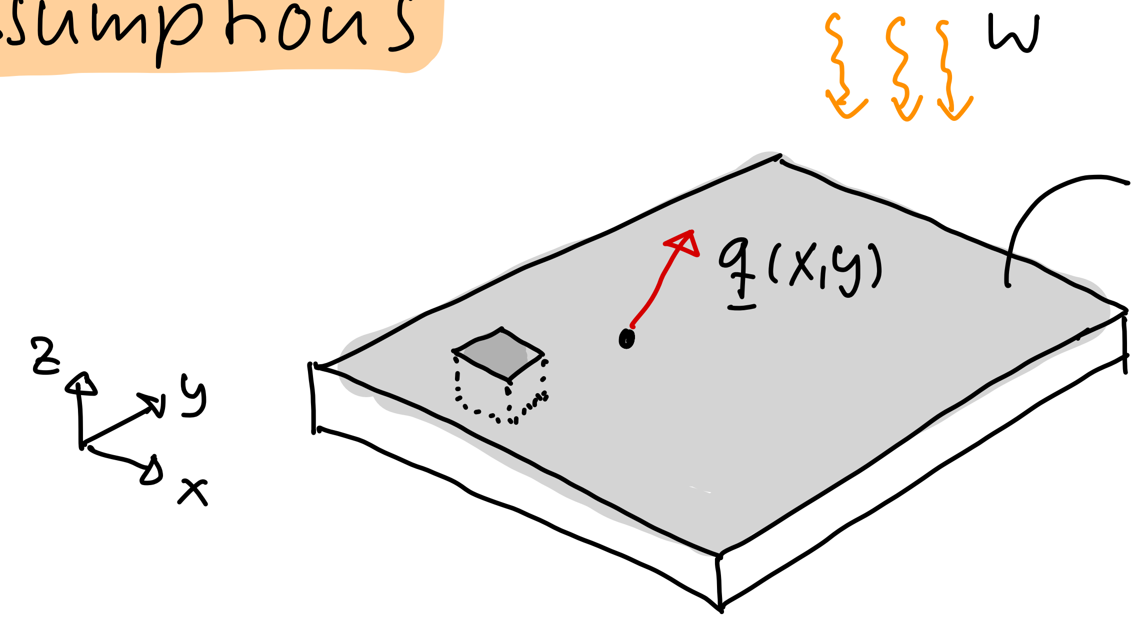
ϕ $[W]$

Boundary value problem (D)

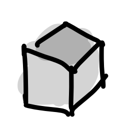
- Differential equation
- Boundary conditions

Differential equation

Assumptions

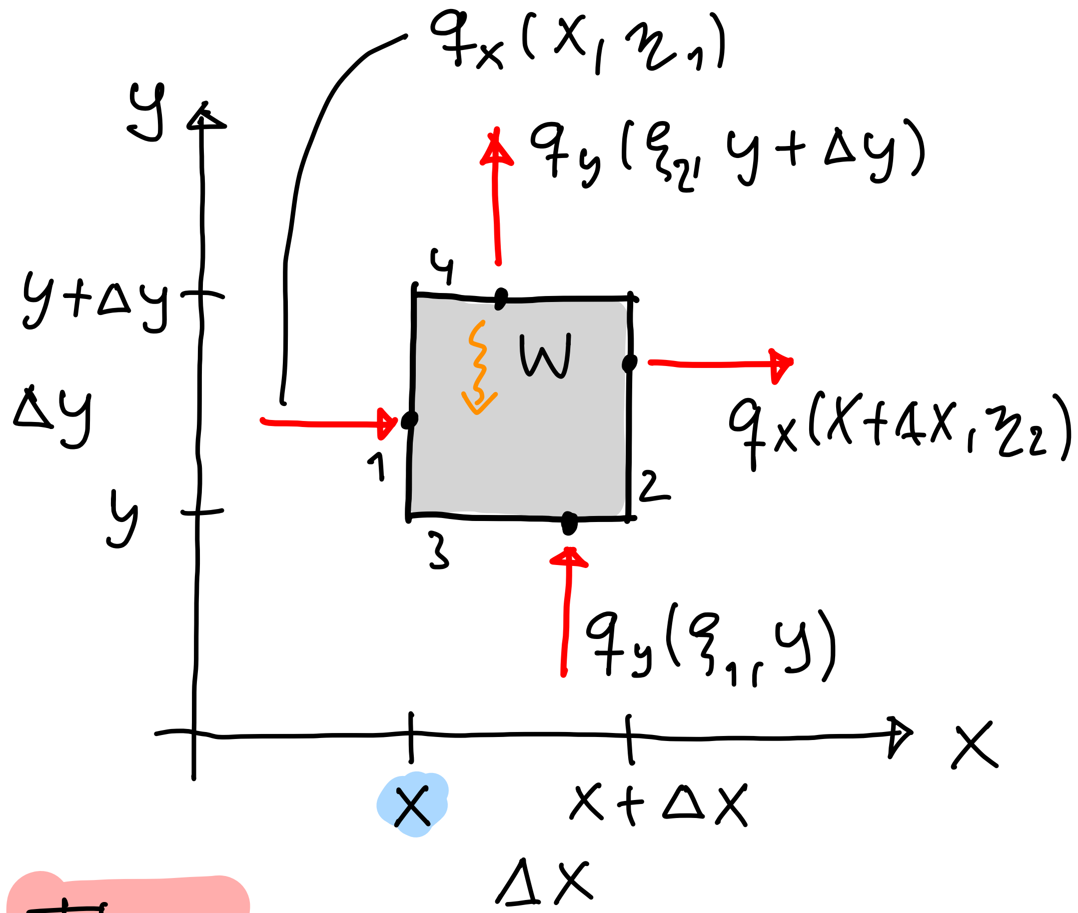


$\updownarrow$ $t = 1$

- Plate of thicknes 1
- Quantities $\theta, \underline{q}$ independent of z
- Stationary process with $\dot{\theta} = 0$
- For each cut out part  $\leftarrow \phi$

$$\dot{\theta} = \frac{1}{m \cdot c} \cdot \phi \overset{!}{=} 0 \iff \underbrace{\phi = 0}_{\text{Energy conservation}}$$

1. Conservation of energy



Basic relation

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + w\cdot\Delta x\cdot\Delta y\cdot 1 \overset{!}{=} 0$$

Heat flux over boundary 3

$$\Phi_3 = \int_x^{x+\Delta x} 1\cdot q_y(\xi, y)\, d\xi = q_y(\xi_1, y)\cdot\Delta x$$

thickness

Appendix

Thus

$$\big(q_x(x, z_1) - q_x(x+\Delta x, z_2)\big)\cdot\Delta y + \big(q_y(\xi_1, y) - q_y(\xi_2, y+\Delta y)\big)\cdot\Delta x + w\cdot\Delta x\cdot\Delta y = 0 \quad \Big|: -\Delta x\cdot\Delta y$$

$$\frac{q_x(x+\Delta x, z_2) - q_x(x, z_1)}{\Delta x} + \frac{q_y(\xi_2, y+\Delta y) - q_y(\xi_1, y)}{\Delta y} - w = 0 \quad \Big| \Delta x, \Delta y \to 0$$

$$q_{x,x}(x, y) + q_{y,y}(x, y) = w$$

Note for $\Delta x, \Delta y \to 0$, we have
$z_1, z_2 \to y$, $\xi_1, \xi_2 \to x$

$$\boxed{\operatorname{div} \underline{q}(x, y) = w}$$

2. Fourir's law

Insert $\underline{q}(x,y) = -\lambda \nabla \theta(x,y)$ into conservation law

$$\operatorname{div}\left(-\lambda \nabla \theta(x,y)\right) = \omega$$

Heat equation (stationary, with source)

$$\boxed{-\lambda \cdot \operatorname{div} \nabla \theta(x,y) = \omega}$$

Remark

Laplace-operator

$$\operatorname{div} \nabla \theta = \operatorname{div} \begin{pmatrix} \theta_x \\ \theta_y \end{pmatrix} = \theta_{xx} + \theta_{yy} =: \Delta \theta$$

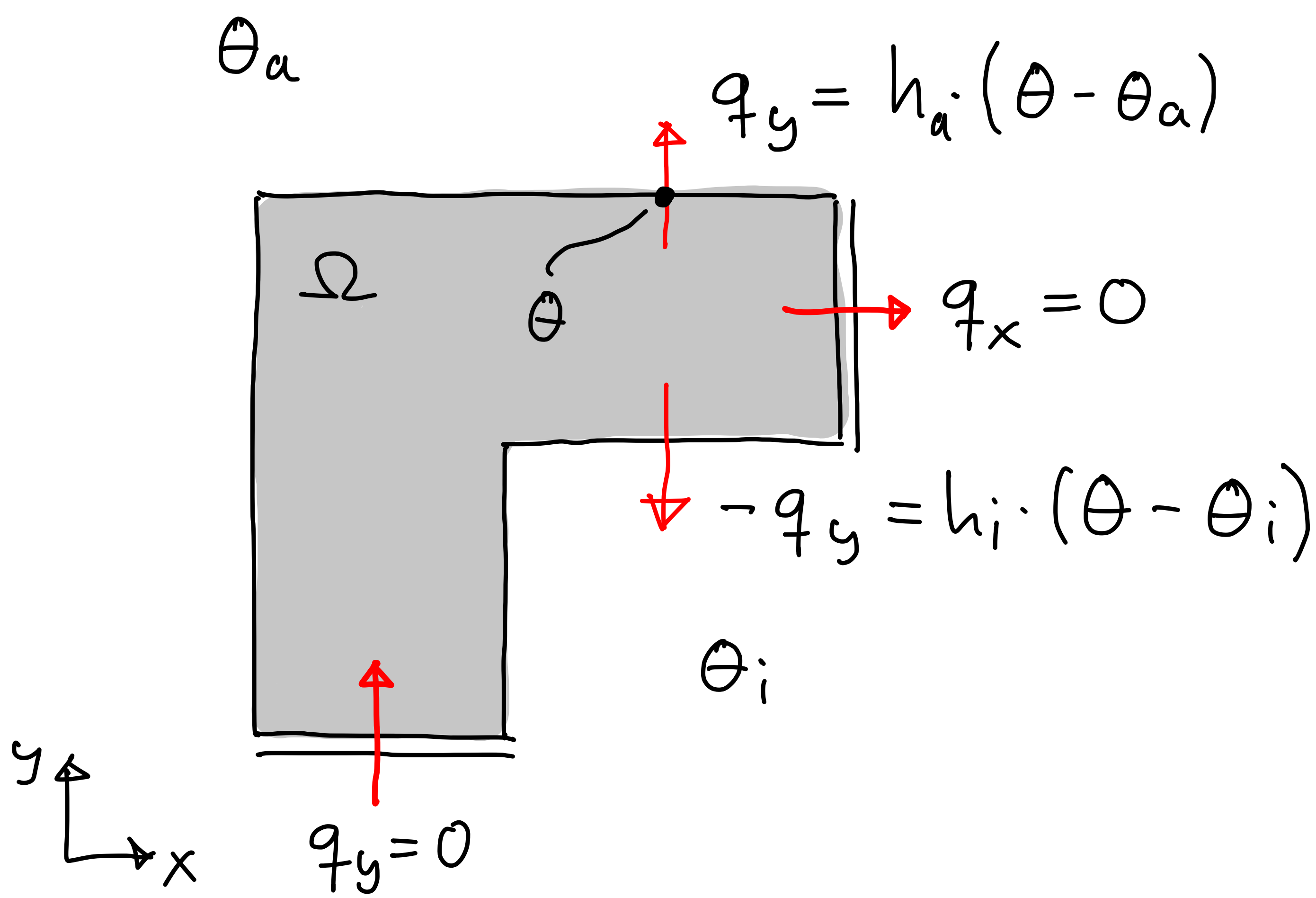
$\longrightarrow$ Partial differential equation!

Boundary value problem (D)

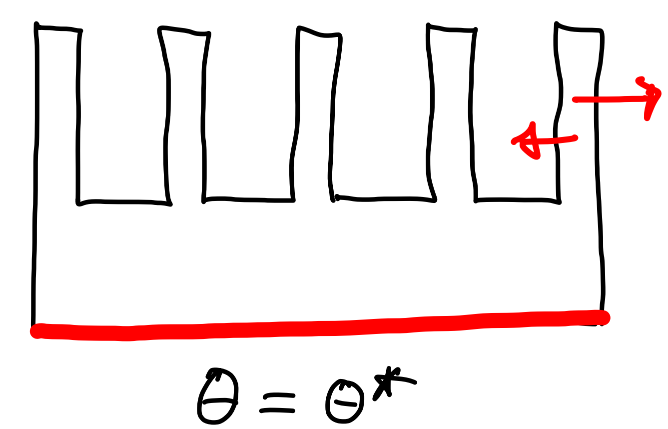
- Differential equation
- Boundary conditions

Boundary conditions

Outward corner of a wall



Cooling device



Heat transfer coefficient

$$h_i = 1/R_{si} = 1/0.25$$

$$h_a = 1/R_{sa} = 1/0.04$$

Types of boundary conditions

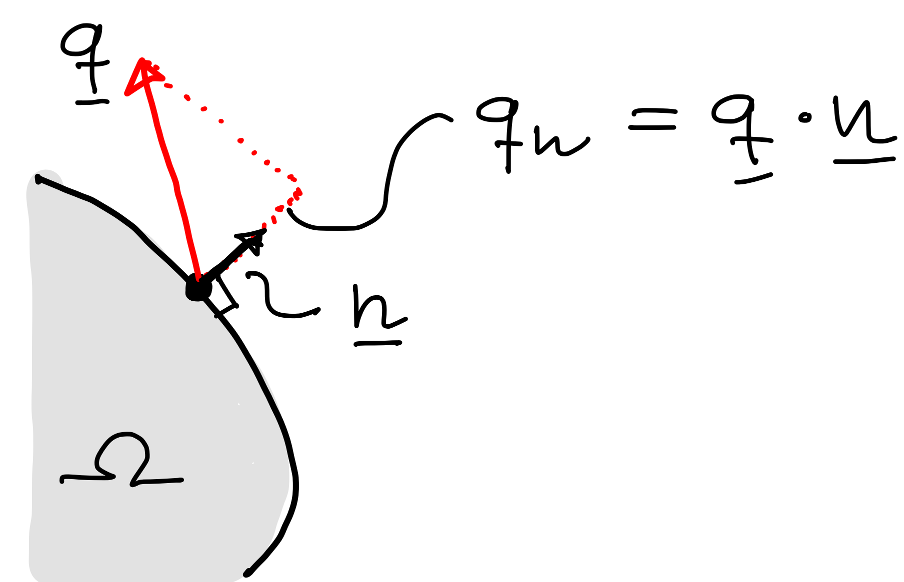
$\theta = \theta^*$ – Prescribed temperature (Dirichlet BC)

$q_n = q^*$ – Prescribed heat flux (Neumann BC)

$q_n = h \cdot (\theta - \theta^*)$ – Flux depends on $\Delta\theta$ (Robin BC)

Heat flux over boundary

$$q_n = \underline{q} \cdot \underline{n} = -\lambda \nabla\theta \cdot \underline{n}$$

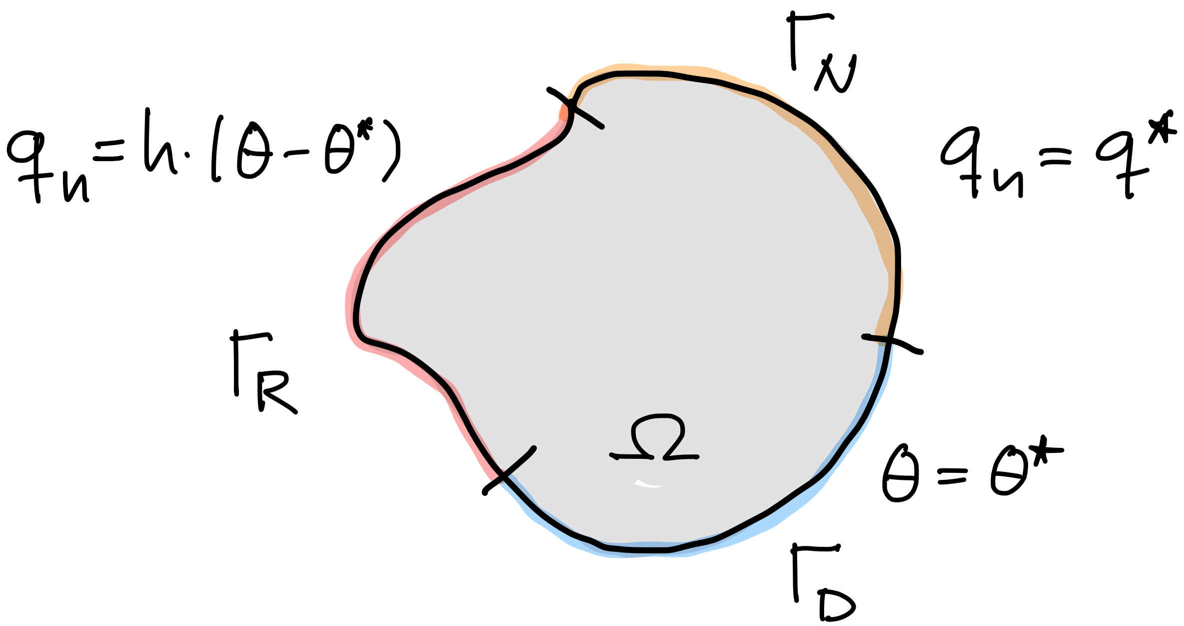


$|\underline{n}| = 1$

Boundary value problem (D)

- Differential equation
- Boundary conditions

Boundary value problem



$\Omega \in \mathbb{R}^2$: Computational domain

$\Gamma \in \mathbb{R}^2$: Boundary of Ω

$$\Gamma = \underset{\text{Dirichlet}}{\Gamma_D} \cup \underset{\text{Neumann}}{\Gamma_N} \cup \underset{\text{Robin}}{\Gamma_R}$$

Simplification (we won't need that anyway)

$$q_n^* = 0 \quad \text{and} \quad \theta^* = 0$$

Formulation of BCs with $q_n = -\lambda \cdot \nabla\theta \cdot \underline{n}$ and $q_n^* = 0$ and $\theta^* = 0$

$$\Gamma_R : \; -\lambda \cdot \nabla\theta \cdot \underline{n} = h \cdot (\theta - \theta^*) \iff \nabla\theta \cdot \underline{n} = \frac{h}{\lambda}(\theta^* - \theta)$$

$$\Gamma_N : \; -\lambda \cdot \nabla\theta \cdot \underline{n} = 0 \iff \nabla\theta \cdot \underline{n} = 0$$

$$\Gamma_D : \qquad \theta = 0$$

Boundary value problem for heat conduction (D): Find temperature distribution $\theta : \Omega \to \mathbb{R}$ with

$$-\lambda \cdot \operatorname{div} \nabla \theta(x,y) = w \qquad \text{for } (x,y) \in \Omega$$

and

$$\nabla \theta(x,y) \cdot \underline{n}(x,y) = \frac{h}{\lambda}\left(\theta^* - \theta(x,y)\right) \qquad \text{for } (x,y) \in \Gamma_R$$

$$\nabla \theta(x,y) \cdot \underline{n}(x,y) = 0 \qquad \text{for } (x,y) \in \Gamma_N$$

$$\theta(x,y) = 0 \qquad \text{for } (x,y) \in \Gamma_D$$

Remark: Works for $\Omega \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{R}^3$.

Appendix

Heat flux over boundary in Fish & Belytschko

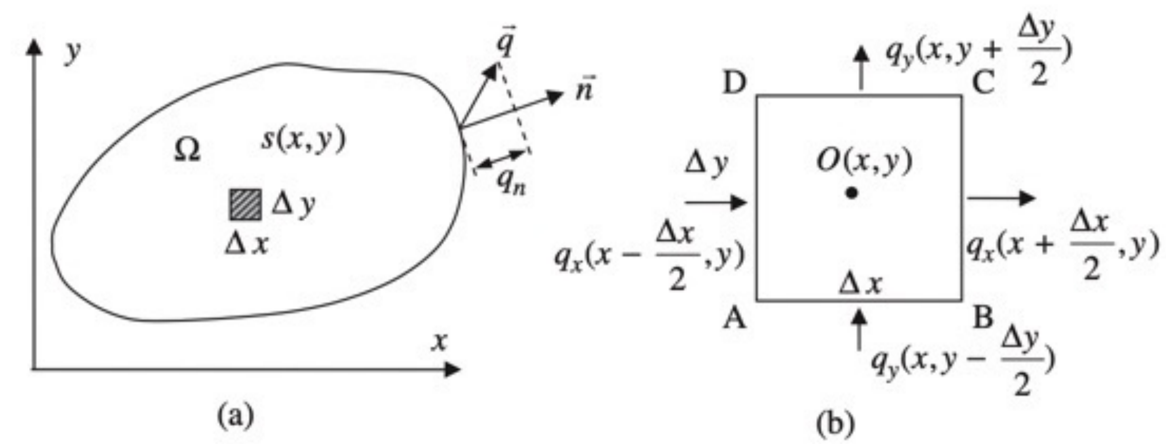
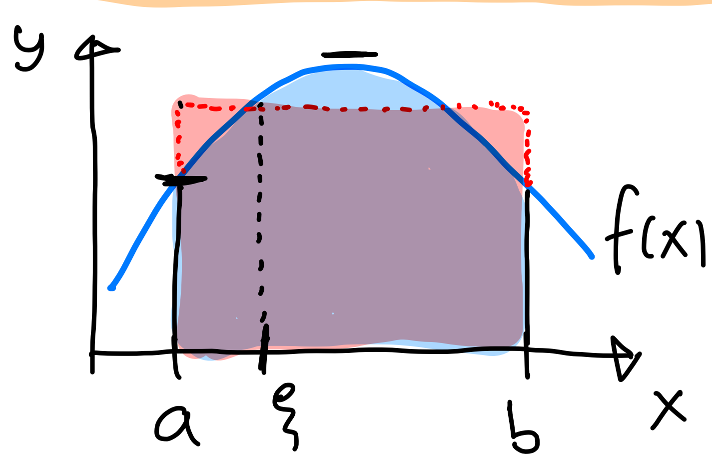


Figure 6.5 Problem definition: (a) domain of a plate with a control volume shaded and (b) heat fluxes in and out of the control volume.

$$q_x\left(x-\frac{\Delta x}{2},y\right)\Delta y - q_x\left(x+\frac{\Delta x}{2},y\right)\Delta y$$
$$+\,q_y\left(x,y-\frac{\Delta y}{2}\right)\Delta x - q_y\left(x,y+\frac{\Delta y}{2}\right)\Delta x + s(x,y)\Delta x\Delta y = 0.$$

Mean value theorem for definite integrals



There exists a value $\xi \in [a,b]$ s.t.

$$\int_a^b f(x)\,dx = (b-a)\cdot f(\xi)$$

(Mathematik 2)