Numerical Methods for Partial Differential Equations

1. Finite element solution of 1D problems

Ingredients of finite element solution

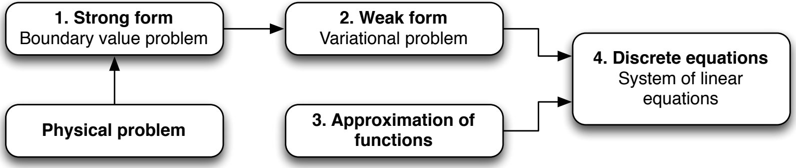


Diagram according to Fish and Belytschko, 2007

Strong form: Mathematical model of real world process, differential equation and boundary conditions

Weak form: Basis for finite element solution

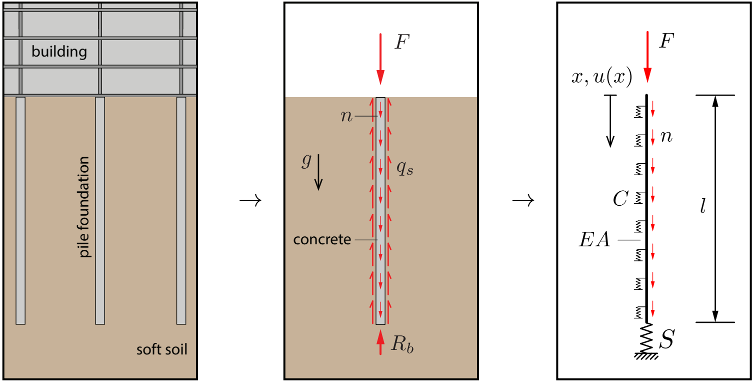
Approximation of functions: Construct approximate solution by combining predefined functions

Discrete equations: Inserting predefined functions into weak form yields linear system of equations

It's only math once the boundary value problem has been formulated!

Modelling pile foundations . . .

Mechanical model



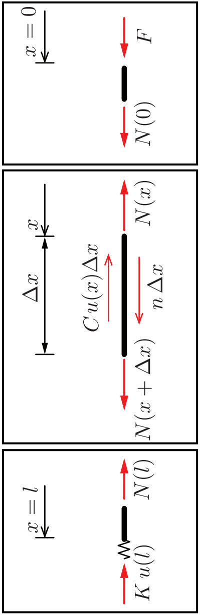
Loads and resistances

- F Imposed load building
- n Dead weight pile $n = g\rho A$
- q_s Mantle resistance
- R_b Tip resistance

Mechanical model

- $u(x)$ Vertical displacement
- $N(x)$ Axial force $N(x) = EAu'(x)$
- C Distributed spring $q_s(x) = C\,u(x)$
- S Spring at tip $R_b = S\,u(l)$

Mathematical model: Differential equation and boundary conditions

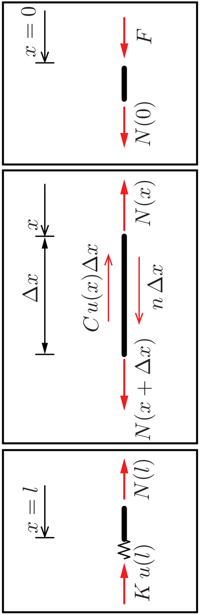


Differential equation

Balance of forces for piece of length Δx

$$N(x+\Delta x) - N(x) + n\Delta x - C\,u(x)\Delta x = 0$$

Mathematical model: Differential equation and boundary conditions



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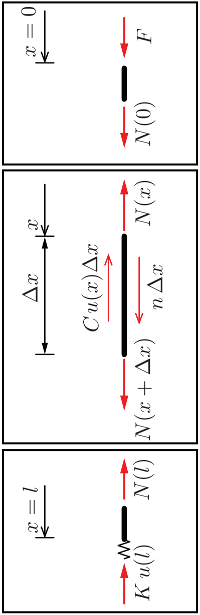

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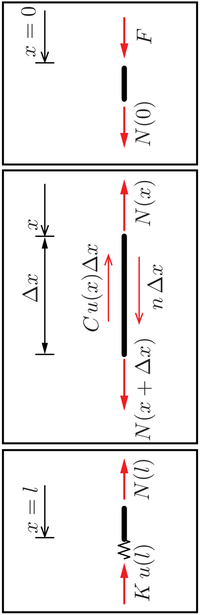
$$N(x+\Delta x) - N(x) + n\Delta x - C\,u(x)\Delta x = 0 \qquad |:\Delta x$$

$$\frac{N(x+\Delta x) - N(x)}{\Delta x} - C\,u(x) = -n$$

Mathematical model: Differential equation and boundary conditions



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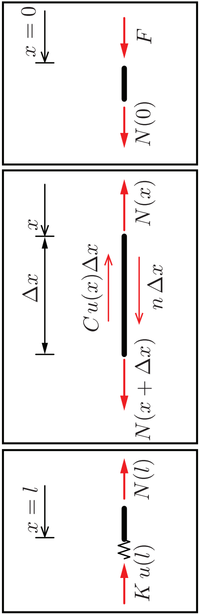

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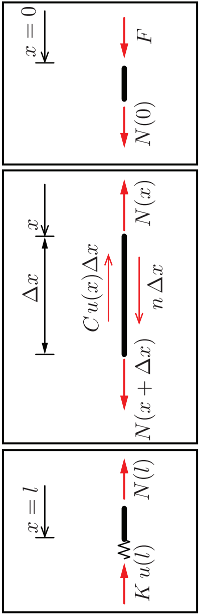
$$N'(x) - C\,u(x) = -n$$

Mathematical model: Differential equation and boundary conditions



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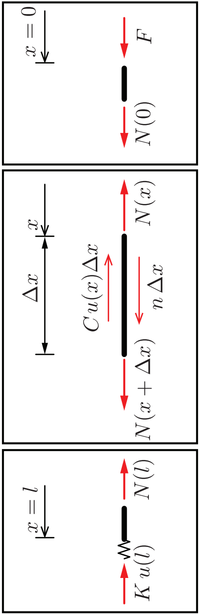
$$N'(x) - C\,u(x) = -n$$

Using $N(x) = EAu'(x)$ we obtain the differential equation

$$\boxed{EA\,u''(x) - C\,u(x) = -n}$$

Mathematical model: Differential equation and boundary conditions



Differential equation

Balance of forces for piece of length Δx

$$
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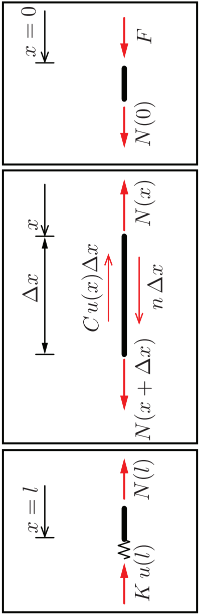

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Using $N(x) = EAu'(x)$ we obtain the differential equation

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Boundary conditions

Sum of forces at top and bottom end:

$$N(0) + F = 0 \quad \text{and} \quad Su(l) + N(l) = 0$$

Inserting $N(x) = EAu'(x)$ gives the boundary conditions

$$\boxed{EA\,u'(0) = -F} \quad \text{und} \quad \boxed{EA\,u'(l) + S\,u(l) = 0}$$

1.1 Strong form . . .

Boundary value problem or strong form of the problem

Boundary value problem (D): Find a function $u : [0, l] \to \mathbb{R}$ which satisfies the differential equation

$$EA\, u''(x) - C\, u(x) = -n$$

and the boundary conditions

$$EA\, u'(0) = -F \quad \text{and} \quad EA\, u'(l) + S\, u(l) = 0$$

(D) is also called **strong form** of the problem (explanation of name later)

Types and names of boundary conditions (BCs)

Classification

Types of boundary conditions are associated with names (of mathematicians):

$$u(x_0) = c \qquad \text{(Function value – Dirichlet)}$$

$$u'(x_0) = c \qquad \text{(Derivative – Neumann)}$$

$$a\,u(x_0) + b\,u'(x_0) = c \qquad \text{(Mixed – Robin)}$$

where

x_0: Point on boundary (left or right)

a, b, c: Constants (prescribed according to problem)

In the case of $c = 0$, a boundary condition is called *homogeneous* or *natural*, otherwise *inhomogeneous*

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For pile foundation problem

$x = 0$: Inhomogeneous Neumann boundary condition

$x = l$: Homogeneous Robin boundary condition

1.2 Weak form . . .

Derivation of weak form 1/2

Multiply differential equation by test function $\delta u : [0, l] \to \mathbb{R}$ and integrate:

$$EA\, u''(x) - C\, u(x) = -n \qquad \Big| \cdot \delta u(x) \quad (1)$$

$$EA\, u''(x)\delta u(x) - C\, u(x)\delta u(x) = -n\delta u(x) \qquad \Big| \int \cdot \ \mathrm{d}x \quad (2)$$

$$\int_0^l \big(EA\, u''(x)\delta u(x) - C\, u(x)\delta u(x)\big)\ \mathrm{d}x = \int_0^l -n\, \delta u(x)\ \mathrm{d}x \quad (3)$$

$$EA\int_0^l u''(x)\delta u(x)\ \mathrm{d}x - C\int_0^l u(x)\delta u(x)\ \mathrm{d}x = -n\int_0^l \delta u(x)\ \mathrm{d}x \quad (4)$$

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Key idea

Require equations (2) – (4) to hold for any test function δu

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Key idea

Require equations (2) – (4) to hold for any test function δu

Not so nice

- First integral in (4) not symmetric, second derivatives of u
- Solution: Integration by parts

Derivation of weak form 2/2

Integrate by parts and insert boundary conditions of (D)

$$EA\int_0^l u''(x)\delta u(x)\,\mathrm{d}x = EA\Big[u'(x)\delta u(x)\Big]_0^l - EA\int_0^l u'(x)\delta u'(x)\,\mathrm{d}x$$

$$= \underbrace{EAu'(l)}_{-S\,u(l)}\delta u(l) - \underbrace{EAu'(0)}_{-F}\delta u(0) - EA\int_0^l u'(x)\delta u'(x)\,\mathrm{d}x$$

$$= -S\,u(l)\delta u(l) + F\delta u(0) - EA\int_0^l u'(x)\delta u'(x)\,\mathrm{d}x$$

Insert into (4)

$$-S\,u(l)\delta u(l) + F\delta u(0) - EA\int_0^l u'(x)\delta u'(x)\,\mathrm{d}x - C\int_0^l u(x)\delta u(x)\,\mathrm{d}x = -n\int_0^l \delta u(x)\,\mathrm{d}x$$

and rearrange to

$$EA\int_0^l u'(x)\delta u'(x)\,\mathrm{d}x + C\int_0^l u(x)\delta u(x)\,\mathrm{d}x + S\,u(l)\delta u(l) = n\int_0^l \delta u(x)\,\mathrm{d}x + F\delta u(0)$$

$\rightarrow$ Basic equation for weak form

Variational or weak form of boundary value problem

Variational problem (V): Find a function $u : [0, l] \to \mathbb{R}$ such that

$$EA \int_0^l u'(x) \delta u'(x) \, \mathrm{d}x + C \int_0^l u(x) \delta u(x) \, \mathrm{d}x + S \, u(l) \delta u(l) = n \int_0^l \delta u(x) \, \mathrm{d}x + F \delta u(0)$$

for all (admissible) test functions δu

(V) is also called **weak form** or principle of virtual work in structural mechanics where δu is called virtual displacement

About strong and weak forms

Comparison of problems

Strong form Find a function which satisfies an equation at each point in the considered domain

Weak form Find a function, for which a scalar valued equation holds for any test function

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The following terms have the same meaning

- **Boundary value problem** and **strong form**
- **Variational problem** and **weak form**

About strong and weak forms

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- **Boundary value problem** and **strong form**
- **Variational problem** and **weak form**

Why the names strong form and weak form?

Strong form: Requirements on u

- fulfill the differential equation point wise (strongly) for each $x \in [0, l]$
- two times differentiable

Weak form: Requirements on u

- differential equation fulfilled in an integral sense
- one time differentiable (Fish and Belytschko 2007, p. 49)

Relation between strong form (D) and weak form (V)

Propositions

1. Solution of boundary value problem solves variational problem

$$(D) \implies (V)$$

2. Solution of variational problem solves boundary value problem

$$(V) \implies (D)$$

Relation between strong form (D) and weak form (V)

Propositions

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Proofs

1. Obvious, (V) derived by manipulating the differential equation and by inserting the boundary conditions from (D)
2. Not obvious, integration possibly not equivalence preserving

A closer look at the proposition $(V) \implies (D)$

Start with weak form

$$EA\int_0^l u'(x)\delta u'(x)\,\mathrm{d}x + C\int_0^l u(x)\delta u(x)\,\mathrm{d}x + S\,u(l)\delta u(l)$$
$$= n\int_0^l \delta u(x)\,\mathrm{d}x + F\delta u(0)$$

Integrate by parts (back again)

$$EA\Big[u'(x)\delta u(x)\Big]_0^l - EA\int_0^l u''(x)\delta u(x)\,\mathrm{d}x + C\int_0^l u(x)\delta u(x)\,\mathrm{d}x + S\,u(l)\delta u(l)$$
$$= n\int_0^l \delta u(x)\,\mathrm{d}x + F\delta u(0)$$

Rearrange to

$$\int_0^l \big(EAu''(x) - Cu(x) + n\big)\delta u(x)\,\mathrm{d}x$$
$$+ \big((F - EAu'(0)\big)\delta u(0) + \big(S\,u(l) + u'(l)\big)\delta u(l) = 0 \quad (5)$$

Task now: Show that (5) holds for arbitrary functions δu only if

$$EAu''(x) - Cu(x) + n = 0 \qquad \text{(differential equation)}$$
$$F - EAu'(0) = 0 \qquad \text{(left BC)}$$
$$S\,u(l) + u'(l) = 0 \qquad \text{(right BC)}$$

Step 1: Differential equation

Consider only test functions with $\delta u(0) = \delta u(l) = 0$ (boundary terms go away)

$$\int_0^l \big(EAu''(x) - Cu(x) + n\big)\delta u(x)\,\mathrm{d}x = 0$$

Residual function (continuous since u has to be twice differentiable)

$$r(x) = EAu''(x) - Cu(x) + n$$

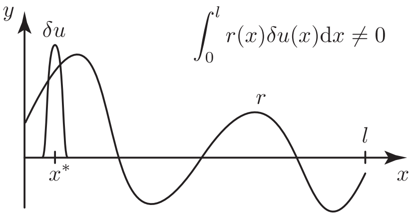
Idea: For continuous functions r, the relation

$$\int_0^l r(x)\delta u(x)\,\mathrm{d}x = 0$$

holds for arbitrary functions δu with $\delta u(0) = \delta u(l) = 0$ only if $r(x) = 0$. Why? If $r(x) \neq 0$ for some x^*, then there exists a neighborhood of x^* where $r(x)$ is strictly positive or strictly negative (since r is continuous). Using a function δu which is positive in that neighborhood and 0 everywhere else, we can always achieve that the integral is not equal to 0. This is basically the *fundamental lemma of variational calculus.* Alternative proof in (Hughes, 1987, p. 5).

Result: If u solves (V), then it fulfills the differential equation of (D)

Illustration of fundamental lemma of variational calculus



If $r(x) \neq 0$ zero somewhere, it is always possible to find a function δu such that the integral does not vanish

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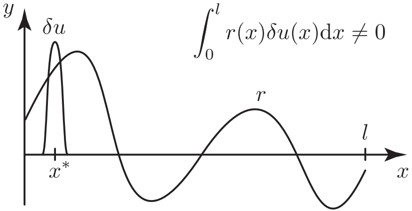

If $r(x) \neq 0$ zero somewhere, it is always possible to find a function δu such that the integral does not vanish

Remark

In order to show that the differential equation is fulfilled if a function u solves (V), we restricted the choice of test functions by imposing $\delta u(0) = \delta u(l) = 0$. Of course, this is still true that restriction is removed: If the differential equation has to be fulfilled for some functions with a certain property, it has also to be fulfilled if we consider a larger class of functions

Step 2: Boundary conditions

With the result of Step 1, equation (5) reduces to

$$\big((F - EAu'(0)\big)\delta u(0) + \big(S\,u(l) + u'(l)\big)\delta u(l) = 0$$

Insert test function δu with $\delta u(0) = 1$ and $\delta u(l) = 0$ such that

$$F - EAu'(0) = 0$$

Insert test function δu with $\delta u(0) = 0$ and $\delta u(l) = 1$ such that

$$S\,u(l) + u'(l) = 0$$

Result: If u solves (V), then it fulfills the boundary conditions in (D)

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Result: If u solves (V), then it fulfills the boundary conditions in (D)

Conclusion (the good news)

We have shown that

- if u solves (D) it also solves (V)
- if u solves (V) and is twice differentiable, then it solves (D)

The problems (D) and (V) are equivalent! An approximate solution to (V) is (in some sense) also an approximate solution to (D)

Ingredients of weak form: Blackboard...

1.3 Approximation of functions. . .

Simple and not so simple

The finite element method is based on two very simple ideas

1. Construct an approximate solution by combining given functions
2. Define these given functions element-wise

Simple and not so simple

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But as always: The devil lies in the detail!

- How to determine the best combination of functions?
- What does it mean to combine functions?
- How to consider Dirichlet boundary conditions?
- How to define functions element-wise?

The two main ideas of FEM: 1. Combine functions

Choose some functions $\varphi_1, \varphi_2, \ldots, \varphi_N$ and **approximate** the solution u by the function

$$u_h(x) = \varphi_1(x) \cdot \hat{u}_1 + \varphi_2(x) \cdot \hat{u}_2 + \cdots + \varphi_N(x) \cdot \hat{u}_N$$

By that, the problem to find a function (very hard) is replaced by the problem to find some numbers $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_N$ (by far not as hard).

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Remarks

1. Combining functions is not really new
   - Taylor-polynomial: $\varphi_i(x) = x^{i-1}$
   - discrete Fourier-transformation: $\sin$ and $\cos$ functions
2. First application of this idea to solve
   - minimization problems: Walter Ritz (1878 – 1909)
   - differential equations: Boris Galerkin (1871 – 1945)

   $\rightarrow$ Gander and Wanner (2012) give a comprehensive overview

The two main ideas of FEM: 2. Define functions element-wise

Divide the domain of the problem into subdomains (the elements) and define the functions **element-wise**.

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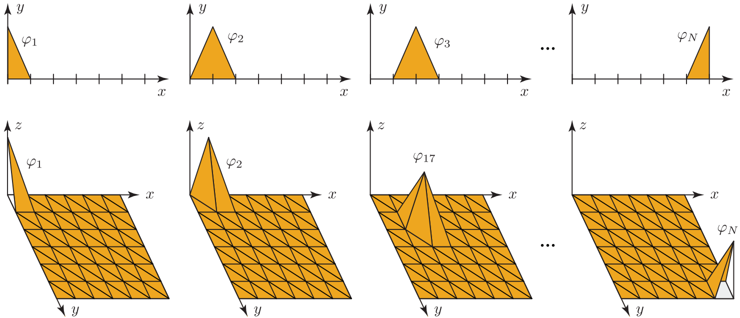
Simplest case: Element-wise linear functions in 1D and 2D



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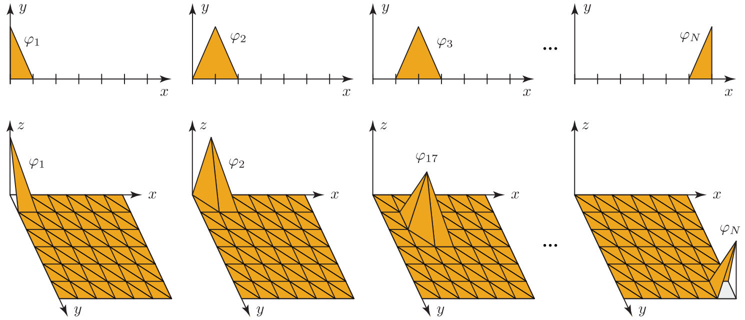

Remark

- FEM is a special Ritz-Galerkin method
- Many other options than element-wise linear functions exist

Blackboard. . .

What do we do when approximating functions like that? 1/3

We know how to add functions and how to multiply functions with a number. Thus, instead of

$$u_h(x) = \varphi_1(x)\hat{u}_1 + \varphi_2(x)\hat{u}_2 + \cdots + \varphi_N(x)\hat{u}_N = \sum_{i=1}^{N} \varphi_i(x)\hat{u}_i$$

we can also write

$$u_h = \varphi_1\hat{u}_1 + \varphi_2\hat{u}_2 + \cdots + \varphi_N\hat{u}_N = \sum_{i=1}^{N} \varphi_i\hat{u}_i$$

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Definition (linear combination): A sum of the form

$$\varphi_1\hat{u}_1 + \varphi_2\hat{u}_2 + \cdots + \varphi_N\hat{u}_N = \sum_{i=1}^{N} \varphi_i\hat{u}_i$$

where φ_i are elements of a linear space and $\hat{u}_i$ are numbers is called linear combination.

Finite dimensional subspace V_h

- We already introduced the space

$$V = \text{'The set of all (nice) functions on } [0, l]\text{'}$$

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- We now introduce the new space

$$V_h = \text{'The set of all possible linear combinations of } \varphi_1, \varphi_2, \dots, \varphi_N\text{'}$$

and say that V_h is the space which is spanned by the basis functions $\varphi_1, \varphi_2, \dots, \varphi_N$

What do we do when approximating functions like that? 2/3

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  and say that V_h is the space which is spanned by the basis functions $\varphi_1, \varphi_2, \ldots, \varphi_N$

- The number N is the dimension of the space V_h

What do we do when approximating functions like that?

Finite dimensional subspace V_h

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- The number N is the dimension of the space V_h
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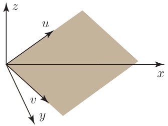
- Simple example: Plane in $3D$ space is spanned by two vectors



Finite element method

- Solve variational problem approximately by searching in a finite dimensional subspace V_h of all possible solutions
- The space V_h is spanned by functions which are defined element-wise

Numerical solution . . .

Blackboard…

Galerkin- and Ritz methods (Johnson 1986, p. 20)

Galerkin method: Solve the

Abstract discrete variational problem (ADV): Find a function $u_h \in V_h$ such that

$$a(u_h, \delta u_h) = b(\delta u_h)$$

for all $\delta u_h \in V_h$.

Ritz method: Solve the

Abstract discrete minimization problem (ADM): Find a function $u_h \in V_h$ such that

$$F(u_h) \leq F(\delta u_h)$$

for all $\delta u_h \in V_h$.

The finite element method is a special version of one of these two methods where V_h contains element-wise defined functions.

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