International Baccalaureate®
Baccalauréat International
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Diploma Programme

# Mathematics: analysis and approaches formula booklet

For use during the course and in the examinations
First examinations 2021

Version 1.0

# Contents

# Prior learning – SL and HL

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|---|---|
| Area of a parallelogram | $A = bh$ , where $b$ is the base, $h$ is the height |
| Area of a triangle | $A = \frac{1}{2}(bh)$ , where $b$ is the base, $h$ is the height |
| Area of a trapezoid | $A = \frac{1}{2}(a+b)h$ , where $a$ and $b$ are the parallel sides, $h$ is the height |
| Area of a circle | $A = \pi r^2$ , where $r$ is the radius |
| Circumference of a circle | $C = 2\pi r$ , where $r$ is the radius |
| Volume of a cuboid | $V = lwh$ , where $l$ is the length, $w$ is the width, $h$ is the height |
| Volume of a cylinder | $V = \pi r^2 h$ , where $r$ is the radius, $h$ is the height |
| Volume of a prism | $V = Ah$ , where $A$ is the area of cross-section, $h$ is the height |
| Area of the curved surface of a cylinder | $A = 2\pi rh$ , where $r$ is the radius, $h$ is the height |
| Distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ | $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ |
| Coordinates of the midpoint of a line segment with endpoints $(x_1, y_1)$ and $(x_2, y_2)$ | $\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$ |

# Topic 1: Number and algebra – SL and HL

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|---|---|---|
| **SL 1.2** | The $n$th term of an arithmetic sequence<br><br>The sum of $n$ terms of an arithmetic sequence | $u_n = u_1 + (n-1)d$<br><br>$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$; $S_n = \frac{n}{2}(u_1 + u_n)$ |
| **SL 1.3** | The $n$th term of a geometric sequence<br><br>The sum of $n$ terms of a finite geometric sequence | $u_n = u_1 r^{n-1}$<br><br>$S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{1-r}$, $r \neq 1$ |
| **SL 1.4** | Compound interest | $FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$, where $FV$ is the future value, $PV$ is the present value, $n$ is the number of years, $k$ is the number of compounding periods per year, $r\%$ is the nominal annual rate of interest |
| **SL 1.5** | Exponents and logarithms | $a^x = b \Leftrightarrow x = \log_a b$, where $a > 0, b > 0, a \neq 1$ |
| **SL 1.7** | Exponents and logarithms | $\log_a xy = \log_a x + \log_a y$<br>$\log_a \frac{x}{y} = \log_a x - \log_a y$<br>$\log_a x^m = m \log_a x$<br><br>$\log_a x = \frac{\log_b x}{\log_b a}$ |
| **SL 1.8** | The sum of an infinite geometric sequence | $S_\infty = \frac{u_1}{r-1}$, $\lvert r \rvert < 1$ |
| **SL 1.9** | Binomial theorem | $(a+b)^n = a^n + {}^n\mathrm{C}_1\, a^{n-1}b + \ldots + {}^n\mathrm{C}_r\, a^{n-r}b^r + \ldots + b^n$<br><br>${}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}$ |

# Topic 1: Number and algebra – HL only

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|---|---|---|
| **AHL 1.10** | Combinations | $^{n}\mathrm{C}_{r} = \frac{n!}{r!(n-r)!}$ |
| | Permutations | $^{n}\mathrm{P}_{r} = \frac{n!}{(n-r)!}$ |
| **AHL 1.12** | Complex numbers | $z = a + b\mathrm{i}$ |
| **AHL 1.13** | Modulus-argument (polar) and exponential (Euler) form | $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta} = r\,\mathrm{cis}\,\theta$ |
| **AHL 1.14** | De Moivre's theorem | $\left[r(\cos\theta + \mathrm{i}\sin\theta)\right]^{n} = r^{n}(\cos n\theta + \mathrm{i}\sin n\theta) = r^{n}\mathrm{e}^{\mathrm{i}n\theta} = r^{n}\,\mathrm{cis}\,n\theta$ |

# Topic 2: Functions – SL and HL

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| **SL 2.1** | Equations of a straight line<br><br>Gradient formula | $y = mx + c$; $ax + by + d = 0$; $y - y_1 = m(x - x_1)$<br><br>$m = \frac{y_2 - y_1}{x_2 - x_1}$ |
| **SL 2.6** | Axis of symmetry of the graph of a quadratic function | $f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry is $x = -\frac{b}{2a}$ |
| **SL 2.7** | Solutions of a quadratic equation<br><br>Discriminant | $ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \ a \neq 0$<br><br>$\Delta = b^2 - 4ac$ |
| **SL 2.9** | Exponential and logarithmic functions | $a^x = e^{x \ln a}$; $\log_a a^x = x = a^{\log_a x}$ where $a, x > 0, a \neq 1$ |

# Topic 2: Functions – HL only

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| **AHL 2.12** | Sum and product of the roots of polynomial equations of the form<br>$\sum_{r=0}^{n} a_r x^r = 0$ | Sum is $\frac{-a_{n-1}}{a_n}$; product is $\frac{(-1)^n a_0}{a_n}$ |

# Topic 3: Geometry and trigonometry – SL and HL

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| **SL 3.1** | Distance between two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ | $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ |
| | Coordinates of the midpoint of a line segment with endpoints $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ | $\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$ |
| | Volume of a right-pyramid | $V = \frac{1}{3}Ah$, where $A$ is the area of the base, $h$ is the height |
| | Volume of a right cone | $V = \frac{1}{3}\pi r^2 h$, where $r$ is the radius, $h$ is the height |
| | Area of the curved surface of a cone | $A = \pi r l$, where $r$ is the radius, $l$ is the slant height |
| | Volume of a sphere | $V = \frac{4}{3}\pi r^3$, where $r$ is the radius |
| | Surface area of a sphere | $A = 4\pi r^2$, where $r$ is the radius |
| **SL 3.2** | Sine rule | $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ |
| | Cosine rule | $c^2 = a^2 + b^2 - 2ab\cos C$; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ |
| | Area of a triangle | $A = \frac{1}{2}ab\sin C$ |
| **SL 3.4** | Length of an arc | $l = r\theta$, where $r$ is the radius, $\theta$ is the angle measured in radians |
| | Area of a sector | $A = \frac{1}{2}r^2\theta$, where $r$ is the radius, $\theta$ is the angle measured in radians |

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| **SL 3.5** | Identity for $\tan\theta$ | $\tan\theta = \frac{\sin\theta}{\cos\theta}$ |
| **SL 3.6** | Pythagorean identity<br><br>Double angle identities | $\cos^2\theta + \sin^2\theta = 1$<br><br>$\sin 2\theta = 2\sin\theta\cos\theta$<br><br>$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ |

# Topic 3: Geometry and trigonometry – HL only

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| **AHL 3.9** | Reciprocal trigonometric identities<br><br><br><br>Pythagorean identities | $\sec\theta = \frac{1}{\cos\theta}$<br><br>$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$<br><br>$1 + \tan^2\theta = \sec^2\theta$<br>$1 + \cot^2\theta = \operatorname{cosec}^2\theta$ |
| **AHL 3.10** | Compound angle identities<br><br><br><br><br><br>Double angle identity for tan | $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$<br><br>$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$<br><br>$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$<br><br>$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$ |
| **AHL 3.12** | Magnitude of a vector | $\lvert \boldsymbol{v} \rvert = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ |

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| **AHL 3.13** | Scalar product | $\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$<br><br>$\boldsymbol{v} \cdot \boldsymbol{w} = \left\| \boldsymbol{v} \right\| \left\| \boldsymbol{w} \right\| \cos\theta$, where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ |
| | Angle between two vectors | $\cos\theta = \dfrac{v_1 w_1 + v_2 w_2 + v_3 w_3}{\left\| \boldsymbol{v} \right\| \left\| \boldsymbol{w} \right\|}$ |
| **AHL 3.14** | Vector equation of a line | $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ |
| | Parametric form of the equation of a line | $x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$ |
| | Cartesian equations of a line | $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$ |
| **AHL 3.16** | Vector product | $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$<br><br>$\left\| \boldsymbol{v} \times \boldsymbol{w} \right\| = \left\| \boldsymbol{v} \right\| \left\| \boldsymbol{w} \right\| \sin\theta$, where $\theta$ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ |
| | Area of a parallelogram | $A = \left\| \boldsymbol{v} \times \boldsymbol{w} \right\|$ where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two adjacent sides of a parallelogram |
| **AHL 3.17** | Vector equation of a plane | $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$ |
| | Equation of a plane (using the normal vector) | $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ |
| | Cartesian equation of a plane | $ax + by + cz = d$ |

# Topic 4: Statistics and probability – SL and HL

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| **SL 4.2** | Interquartile range | $\text{IQR} = Q_3 - Q_1$ |
| **SL 4.3** | Mean, $\overline{x}$, of a set of data | $\overline{x} = \frac{\sum_{i=1}^{k} f_i x_i}{n}$, where $n = \sum_{i=1}^{k} f_i$ |
| **SL 4.5** | Probability of an event $A$ | $\text{P}(A) = \frac{n(A)}{n(U)}$ |
| | Complementary events | $\text{P}(A) + \text{P}(A') = 1$ |
| **SL 4.6** | Combined events | $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$ |
| | Mutually exclusive events | $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$ |
| | Conditional probability | $\text{P}(A \mid B) = \frac{\text{P}(A \cap B)}{\text{P}(B)}$ |
| | Independent events | $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$ |
| **SL 4.7** | Expected value of a discrete random variable $X$ | $\text{E}(X) = \sum x\,\text{P}(X = x)$ |
| **SL 4.8** | Binomial distribution $X \sim \text{B}(n, p)$ | |
| | Mean | $\text{E}(X) = np$ |
| | Variance | $\text{Var}(X) = np(1-p)$ |
| **SL 4.12** | Standardized normal variable | $z = \frac{x - \mu}{\sigma}$ |

# Topic 4: Statistics and probability – HL only

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| **AHL 4.13** | Bayes' theorem | $\mathrm{P}(B \mid A) = \dfrac{\mathrm{P}(B)\ \mathrm{P}(A \mid B)}{\mathrm{P}(B)\ \mathrm{P}(A \mid B) + \mathrm{P}(B')\ \mathrm{P}(A \mid B')}$<br><br>$\mathrm{P}(B_i \mid A) = \dfrac{\mathrm{P}(B_i)\mathrm{P}(A \mid B_i)}{\mathrm{P}(B_1)\mathrm{P}(A \mid B_1) + \mathrm{P}(B_2)\mathrm{P}(A \mid B_2) + \mathrm{P}(B_3)\mathrm{P}(A \mid B_3)}$ |
| **AHL 4.14** | Variance $\sigma^2$ | $\sigma^2 = \dfrac{\sum_{i=1}^{k} f_i \left(x_i - \mu\right)^2}{n} = \dfrac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$ |
| | Standard deviation $\sigma$ | $\sigma = \sqrt{\dfrac{\sum_{i=1}^{k} f_i \left(x_i - \mu\right)^2}{n}}$ |
| | Linear transformation of a single random variable | $\mathrm{E}\left(aX + b\right) = a\mathrm{E}(X) + b$<br>$\mathrm{Var}\left(aX + b\right) = a^2\ \mathrm{Var}(X)$ |
| | Expected value of a continuous random variable $X$ | $\mathrm{E}(X) = \mu = \int_{-\infty}^{\infty} x\, f(x)\, \mathrm{d}x$ |
| | Variance | $\mathrm{Var}(X) = \mathrm{E}(X - \mu)^2 = \mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2$ |
| | Variance of a discrete random variable $X$ | $\mathrm{Var}(X) = \sum (x - \mu)^2\ \mathrm{P}(X = x) = \sum x^2\ \mathrm{P}(X = x) - \mu^2$ |
| | Variance of a continuous random variable $X$ | $\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, \mathrm{d}x = \int_{-\infty}^{\infty} x^2 f(x)\, \mathrm{d}x - \mu^2$ |

# Topic 5: Calculus – SL and HL

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| **SL 5.3** | Derivative of $x^n$ | $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ |
| **SL 5.5** | Integral of $x^n$ | $\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$ |
| | Area between a curve $y = f(x)$ and the $x$-axis, where $f(x) > 0$ | $A = \int_a^b y \, \mathrm{d}x$ |
| **SL 5.6** | Derivative of $\sin x$ | $f(x) = \sin x \Rightarrow f'(x) = \cos x$ |
| | Derivative of $\cos x$ | $f(x) = \cos x \Rightarrow f'(x) = -\sin x$ |
| | Derivative of $\mathrm{e}^x$ | $f(x) = \mathrm{e}^x \Rightarrow f'(x) = \mathrm{e}^x$ |
| | Derivative of $\ln x$ | $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$ |
| | Chain rule | $y = g(u)$, where $u = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$ |
| | Product rule | $y = uv \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$ |
| | Quotient rule | $y = \frac{u}{v} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$ |
| **SL 5.9** | Acceleration | $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ |
| | Distance travelled from $t_1$ to $t_2$ | distance $= \int_{t_1}^{t_2} \lvert v(t) \rvert \, \mathrm{d}t$ |
| | Displacement from $t_1$ to $t_2$ | displacement $= \int_{t_1}^{t_2} v(t) \mathrm{d}t$ |

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| **SL 5.10** | Standard integrals | $\int \frac{1}{x}\,\mathrm{d}x = \ln\lvert x\rvert + C$<br>$\int \sin x\,\mathrm{d}x = -\cos x + C$<br>$\int \cos x\,\mathrm{d}x = \sin x + C$<br>$\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + C$ |
| **SL 5.11** | Area of region enclosed by a curve and $x$-axis | $A = \int_a^b \lvert y\rvert\,\mathrm{d}x$ |

# Topic 5: Calculus – HL only

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| **AHL 5.12** | Derivative of $f(x)$ from first principles | $y = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$ |
| **AHL 5.15** | Standard derivatives | |
| | $\tan x$ | $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$ |
| | $\sec x$ | $f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$ |
| | $\operatorname{cosec} x$ | $f(x) = \operatorname{cosec} x \Rightarrow f'(x) = -\operatorname{cosec} x \cot x$ |
| | $\cot x$ | $f(x) = \cot x \Rightarrow f'(x) = -\operatorname{cosec}^2 x$ |
| | $a^x$ | $f(x) = a^x \Rightarrow f'(x) = a^x(\ln a)$ |
| | $\log_a x$ | $f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x \ln a}$ |
| | $\arcsin x$ | $f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$ |
| | $\arccos x$ | $f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1-x^2}}$ |
| | $\arctan x$ | $f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$ |

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| **AHL 5.15** | Standard integrals | $\int a^x \,\mathrm{d}x = \frac{1}{\ln a} a^x + C$<br>$\int \frac{1}{a^2 + x^2} \mathrm{d}x = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$<br>$\int \frac{1}{\sqrt{a^2 - x^2}} \mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + C, \quad \lvert x \rvert < a$ |
| **AHL 5.16** | Integration by parts | $\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x$ or $\int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u$ |
| **AHL 5.17** | Area of region enclosed by a curve and $y$-axis | $A = \int_a^b \lvert x \rvert \mathrm{d}y$ |
| | Volume of revolution about the $x$ or $y$-axes | $V = \int_a^b \pi y^2 \mathrm{d}x$ or $V = \int_a^b \pi x^2 \mathrm{d}y$ |
| **AHL 5.18** | Euler's method | $y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$, where $h$ is a constant (step length) |
| | Integrating factor for $y' + P(x)y = Q(x)$ | $\mathrm{e}^{\int P(x) \mathrm{d}x}$ |
| **AHL 5.19** | Maclaurin series | $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$ |
| | Maclaurin series for special functions | $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \ldots$<br>$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$<br>$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$<br>$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$<br>$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ |