The NP-Completeness of Edge-Colouring

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Abstract. We show that it is NP-complete to determine the chromatic index of an arbitrary graph. The problem remains NP-complete even for cubic graphs.

Key words. computational complexity, NP-complete problems, chromatic index, edge-coloring

1. Introduction. The chromatic index of a graph is the number of colors required to color the edges of the graph in such a way that no two adjacent edges have the same color. By Vizing's theorem [1], the chromatic index is either d or $d+1$, where d is the maximum vertex degree.

We prove the conjecture (Garey and Johnson [2, p268]) that it is NP-complete to determine the chromatic index of an arbitrary graph. In fact, we prove the stronger result that it is NP-complete to determine whether the chromatic index of a cubic graph is 3 or 4. Thus this problem probably has no polynomial time algorithm.

The terminology and results of NP-completeness are given in [2]. It is clear that the chromatic index problem is in the class NP. To prove that the problem is NP-complete, we exhibit a polynomial reduction from the known NP-complete problem 3SAT which is defined as follows. A set of clauses $C = \{C_1, C_2, \ldots, C_r\}$ in variables $u_1, u_2, \ldots, u_s$ is given, each clause C_i consisting of three literals $l_{i,1}, l_{i,2}, l_{i,3}$, where a literal $l_{i,j}$ is either a variable u_k or its negation $\overline{u}_k$. The problem is to determine whether C is satisfiable, that is, whether there is a truth assignment to the variables which simultaneously satisfies all the clauses in C. A clause is satisifed if one or more of its literals has value "true".

2. The parity condition. We will use the following lemma given in Isaacs [3].

LEMMA *Let G be a cubic, 3-edge-colored graph and $V' \subseteq V(G)$ a set of vertices of G. Let $E' \subseteq E(G)$ be the set of edges of G which connect V' to the remainder of the graph. If the number of edges of color i in E' is k_i $(i = 1, 2, 3)$, then*

$$k_1 \equiv k_2 \equiv k_3 \pmod 2$$

Proof. If E_{12} is the set of edges of G which are colored with color 1 or 2, then E_{12} consists of a collection of cycles. Thus E_{12} meets E' in an even number of edges, and so $k_1 + k_2 \equiv 0 \pmod 2$ which gives $k_1 \equiv k_2 \pmod 2$. Similarly $k_2 \equiv k_3 \pmod 2$. □

3. The components used in the construction. Given an instance C of the problem 3SAT, we will show how to construct a cubic graph G which is 3-edge-colorable if and only if C is satisfiable. The graph G will be put together from pieces or "components" which carry out specific tasks. Information will be carried between components by pairs of edges. In a 3-edge-coloring of G, such a pair of edges is said to represent the value T ("true") if the edges have the same color, and to represent F ("false") if the edges have distinct colors.

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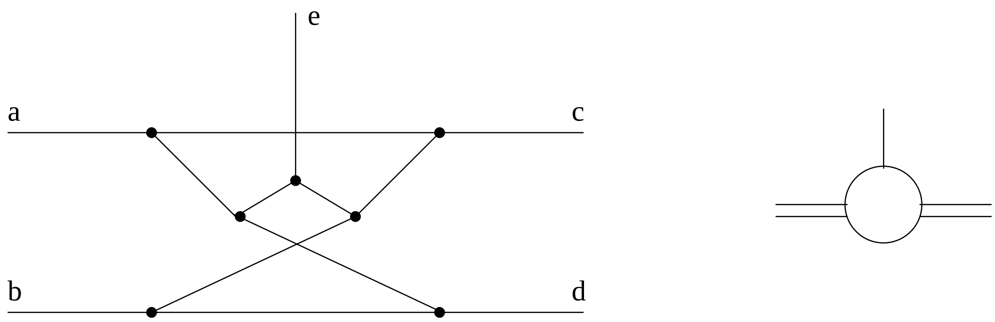

Figure 1: *The inverting component and its symbolic representation*

The inverting component is shown with its symbol in Fig. 1. It was used by Loupekine (see [4]) to construct a large family of cubic graphs with chromatic index 4. Using the parity condition above, it may be checked that if this component is 3-edge-colored, one of the pairs of connecting edges marked a, b or c, d must have equal colors and the remaining 3 edges must have distinct colors. There is no further restriction on the possible colors of the five connecting edges. Regarding the pair of edges a, b as the input and the pair c, d as the output, the component changes a representation of T to one of F and vice versa.

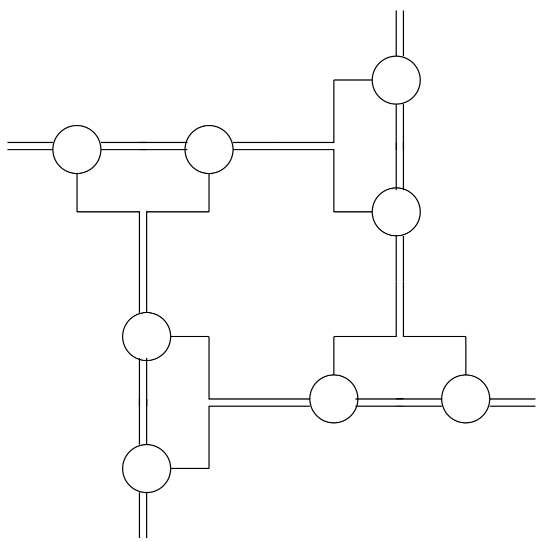

Figure 2: *The variable-setting component made from 8 inverting components and having 4 output pairs of edges. More generally, it is made from 2n inverting components and has n output pairs.*

The truth or falsity of each variable u_i will be represented by a variable-setting component such as that shown in Fig. 2. The component shown has 4 pairs of output edges, but in general the component representing u_i should have as many output pairs as there are appearances of u_i or $\overline{u}_i$ among the clauses of C. It may be checked that in any 3-edge-coloring of a variable-setting component, all the output pairs must represent the same value.

The truth of each clause c_j will be tested by a satisfaction-testing component as shown in Fig. 3. This component can be 3-edge-colored if and only if the three input pairs of edges do not all represent F. The remaining connecting edges will be discussed later.

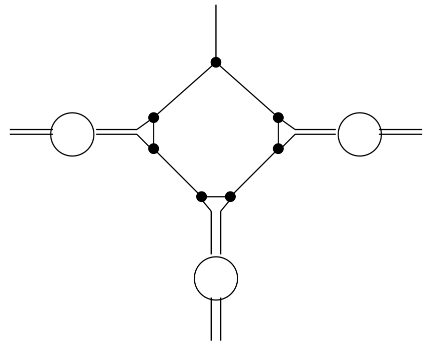

Figure 3: *The satisfaction testing component.*

4. The main theorem. We are now in a position to prove the following theorem.

THEOREM. *It is NP-complete to determine whether the chromatic index of a cubic graph is 3 or 4.*

Proof. The problem is clearly in the class NP. We exhibit a polynomial reduction from the problem 3SAT. Consider an instance C of 3SAT and construct from it a graph G as follows.

For each variable u_i take a variable-setting component U_i with one output pair of edges associated with each appearance of u_i or $\overline{u}_i$ among the clauses of C. Take also a satisfaction-testing component C_j for each clause c_j. Suppose literal $l_{j,k}$ in clause c_j is the variable u_i. Then identify the kth input pair of C_j with the associated output pair of U_i. If, on the other hand, $l_{j,k}$ is $\overline{u}_i$, then insert an inverting component between the kth input pair of C_j and the associated output pair of U_i. The resulting graph H still has some connecting edges unaccounted for. The cubic graph G is formed from two copies of H by identifying the remaining connected edges in corresponding pairs.

The graph G has a 3-edge-coloring if and only if the collection C of clauses is satisfiable, as can be verified using the properties of the components developed above. Moreover, the graph G can be produced from C using a polynomial time algorithm, so we have the result. $\square$

5. Comments. The above theorem may give some insight into the difficulty in classifying graphs according to their chromatic index. At any rate, it probably excludes the possibility of a polynomially checkable criterion, and it indicates that the restriction to cubic graphs is no easier.

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