Pólya-Burnside Enumeration in Combinatorics

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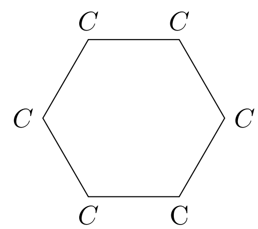
1 A Class of Problems

The following problem in *chemistry* is historically significant, as G. Pólya originally popularized his theory through applications in chemical enumeration. How many different chemical compounds can be made by attaching H, CH_3, or OH radicals to each of the carbon atoms in the benzene ring pictured below?



G. Pólya, R. C. Read (1987). *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds.* New York: Springer-Verlag.

Here are other problems that can be approached using Pólya-Burnside.

1. In how many ways can an $n \times n$ tablecloth be colored with k colors?

2. How many different necklaces can be made with n beads and k colors?

3. How many ways can the faces of a polyhedron be colored using at most n colors?

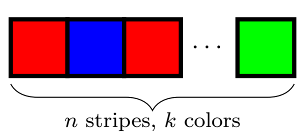
4. Find the number of simple graphs with n vertices, up to isomorphism.

One can observe a common theme of enumerating the number of objects with some equivalence under *symmetry*.

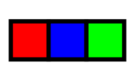
2 An Example: Coloring a Flag

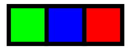
Problem:

How many ways are there to color a flag with n stripes lined side by side with k colors?



Do not count as different flags with colors "flipped." The following two flags would be considered the same:

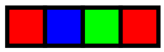


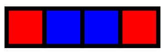


2.1 Solving it with Standard Methods

Let's take the simple case when $n = 4$ and $k = 2$.

Assume we count the number of patterns normally, without accounting for reflection. $N = 2^4$. Let N_r denote the number of distinct colorings under reflection. $N_r \neq \frac{2^4}{2}$, as one might think! We need to separately handle **symmetric patterns** and **asymmetric patterns**.

An asymmetric pattern like  yields a new pattern that we don't want to double count , , when it is reflected. Should be divided by 2.

A symmetric pattern like  when reflected does not create a new pattern. We don't need to divide by 2 here.

Let A be the number of asymmetric patterns not accounting for reflection, and S be the number of symmetric patterns. (Note that $A + S = 2^4$.)

The number of patternings accounting for reflection, N_r, is given by

$$N_r = \frac{A}{2} + S = \frac{2^4 - S}{2} + S$$

$S = 2^2$, since picking the first two squares uniquely defines the last two. Hence $N_r = \frac{2^4 - 2^2}{2} + 2^2 = 10$.

Exercise. Show that in the general case, $N_r = \frac{k^n + k^{\lfloor (n+1)/2 \rfloor}}{2}$.

2.2 Applying Pólya's Theory

We will first apply Pólya's method without explaining how it works.

Solution.

Cycle Notation

Any permutation can be expressed as the product of *cycles*. For instance, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix} = (1\ 5\ 2)(3\ 4)$.

Denote the flag patterning as a 4-letter string of colors *abcd*.

There are two symmetries:

1. the trivial identity permutation that maps $abcd \to abcd$, specifically (a)(b)(c)(d),
2. reflection permutation that maps $abcd \to dcba$, specifically $\begin{pmatrix} a & b & c & d \\ d & c & b & a \end{pmatrix}$, or the cycle product

(a d)(b c).

The first is a product of four 1-cycles, and the second is the product of two 2-cycles. (An n-cycle is a cycle of length n).

Cycle index polynomial

$$P = \frac{1 \cdot f_1^4 + 1 \cdot f_2^2}{2}$$

We make the following substitution: $f_n = x^n + y^n$. We use two terms since we are considering two colors.

P becomes

$$P(x, y) = \frac{(x+y)^4 + (x^2+y^2)^2}{2}$$

To find the answer from before, we add the coefficients of this polynomial. This is equivalent to taking $P(1,1)$, which gives $\frac{2^4+2^2}{2} = \boxed{10}$, as from before. Importantly, not only is the sum equal, but the constituents of the sum are similar as well: this is a hint at some sort of combinatorial equivalence between the two processes.

2.3 P is a Generating Function for Colorings

Even more remarkably, P is a **generating function** for each coloring!

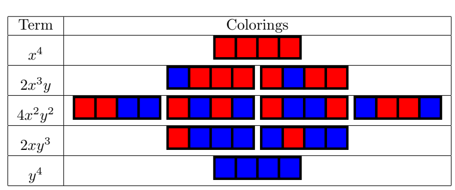
$$P = x^4 + 2x^3y + 4x^2y^2 + 2xy^3 + y^4$$

The coefficient of x^jy^k gives the number of patterns with j squares colored with color 1 and k squares colored with color 2.

Let color 1 be red and color 2 be blue.

$$P = x^4 + 2x^3y + 4x^2y^2 + 2xy^3 + y^4$$

The coefficient of x^jy^k gives the number of patterns with j squares colored with color 1 and k squares colored with color 2.



3 Counting Necklaces: PuMAC 2009

2009 PUMaC Combinatorics A10: Taotao wants to buy a bracelet. The bracelets have 7 different beads on them, arranged in a circle. Two bracelets are the same if one can be rotated or flipped to get the other. If

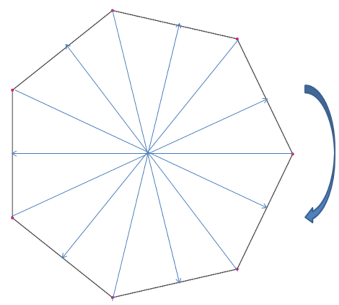


Figure 1: Symmetries of a 7-gon

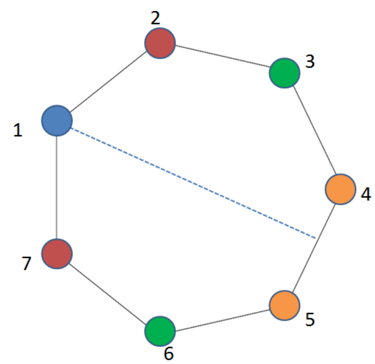


Figure 2: Reflectional symmetry: vertices that get mapped to each other are the same color

she can choose the colors and placement of the beads, and the beads come in orange, white, and black, how many possible bracelets can she buy?

Solution. Imagine the 7 beads at the vertices of a regular heptagon. See Figure 1.

7-gon Symmetries

7 reflections through a vertex and midpoint of opposite side

7 rotations of $n\left(\frac{360}{7}\right)^\circ$ for $n \in 1, 2 \cdots 7$. ($n = 7$: identity case: 360 degree rotation).

Permutation Cycle Structure: Reflections

All the 7 reflections have the same cycle structure, by symmetry. This corresponds to the permutation structure (1)(7 2)(6 3)(5 4): see Figure 2.

In the cycle index polynomial, this is $7f_1f_2^3$, since we have one 1-cycle and three 2-cycles, and 7 such reflections, since we can take a reflection through any vertex.

Permutation Cycle Structure: Rotations

All the 6 nonidentity rotations are 7-cycles, since 7 is a prime number. This contributes $6 \cdot f_7$.

To understand why this is true, ,we look at a case when n, the number of sides of the polygon, is composite. In a hexagon, (6 sides, and 6 is composite) a rotation of $360/3^\circ$ yields (1 3 5)(2 4 6), which would correspond to f_3^2 in the cycle index polynomial: see Figure 3.

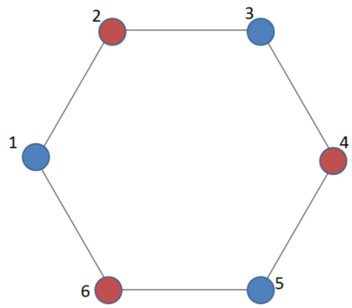


Figure 3: 6 is composite, so we can have disjoint cycles

The Identity The identity is trivially a product of seven 1-cycles, contribute $1 \cdot f_1^7$.

Cycle Index Polynomial

The cycle index polynomial is thus

$$\frac{7 \cdot f_1 f_2^3 + 6 \cdot f_7 + 1 \cdot f_1^7}{14}$$

Substituting $f_n = x^n + y^n + z^n$ (since we have three colors), this is

$$f(x,y,z) = \frac{\substack{7 \cdot (x+y+z)(x^2+y^2+z^2)^3 + 6(x^7+y^7+z^7) \\ +(x+y+z)^7}}{14}$$

The sum of the coefficients is the desired answer.

Plugging in 1, we find

$$f(1,1,1) = \frac{7 \cdot 3^4 + 6 \cdot 3 + 3^7}{14} = 198$$

3.1 Multinomial Theorem: Generalizing the Binomial Theorem

Recall: coefficient of $x^i y^j z^k$ corresponds to the number of necklaces with i of the first color, j of the second color, and k of the third.

(You can replace necklaces with whatever entity you desire, and color with whatever assignment you desire, such as chemical radicals.)

The binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{\substack{i+j=n \\ i,j \geq 0}} \frac{n!}{i!j!} x^i y^j$$

Generalizing,

$$(x+y+z)^n = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \frac{n!}{i!j!k!} x^i y^j z^k$$

$$(w+x+y+z)^n = \sum_{\substack{i+j+k+l=n \\ i,j,k,l \geq 0}} \frac{n!}{i!j!k!l!} w^i x^j y^k z^l$$

3.2 Calculating the number of necklaces of a type: Finding a coefficient of the generating function

Let's say we wanted to find the number of necklaces with 2 red beads, 2 orange beads, and 3 yellow beads: $x^2y^2z^3$. We don't want to expand the generating function!

$$f(x,y,z) = \frac{\substack{7\cdot(x+y+z)(x^2+y^2+z^2)^3+6(x^7+y^7+z^7) \\ +(x+y+z)^7}}{14}$$

To obtain $x^2y^2z^3$, in the first term, $7\cdot(x+y+z)(x^2+y^2+z^2)^3$ we must take a factor of z from $(x+y+z)$, and a factor of x^2 from the first $(x^2+y^2+z^2)$, y^2 from the second, and z^2 from the third (or any possible ordering).

There are $3! = 6$ to pick the ordering of x^2, y^2 and z^2, so the number of terms from here is $7\cdot 3! = 42$.

$$f(x,y,z) = \frac{\substack{7\cdot(x+y+z)(x^2+y^2+z^2)^3+6(x^7+y^7+z^7) \\ +(x+y+z)^7}}{14}$$

In the middle term, $6(x^7+y^7+z^7)$, we will not obtain any terms of our form.

In the last term, $(x+y+z)^7$ we can just apply the multinomial theorem to find that the coefficient is $nom72,2,3 = \frac{7!}{2!2!3!} = 210$.

Thus the coefficient we want is

$$\frac{7\cdot 3! + \binom{7}{2,2,3}}{14} = 18$$

There are 18 necklaces with 2 red beads, 2 orange beads, and 3 yellow beads.

3.3 A Symmetric Generating Function

$$f(x,y,z) = \substack{(x^7+y^7+z^7) \\ +(x^6y+xy^6+x^6z+yz^6+xz^6+y^6z) \\ +(3x^5y^2+3x^5z^2+3y^2z^5+3x^2y^5+3x^2z^5+3y^5z^2) \\ +(3x^5yz+3xy^5z+3xyz^5) \\ +(4x^4y^3+4y^4z^3+4y^3z^4+4x^4z^3+4x^3y^4+4x^3z^4) \\ +(9x^4y^2z+9x^4yz^2+9x^2y^4z+9x^2yz^4+9xy^4z^2+9xy^2z^4) \\ +(10x^3y^3z+10x^3yz^3+10xy^3z^3) \\ +(18x^3y^2z^2+18x^2y^3z^2+18x^2y^2z^3)}$$

Why is the polynomial symmetric?

4 Computational Utility

Just as in problem 1, casework is in principle possible.

Computuational utility: seen when we increase the number of beads or colors even modestly.

Suppose we have a necklace with 17 beads and 4 colors.

$$P = \frac{f_1^{17} + 16f_{17} + 17f_1 f_2^8}{34}$$

Substituting,

$$P = \frac{\substack{(w+x+y+z)^{17}+16(w^{17}+x^{17}+y^{17}+z^{17}) \\ +17(w+x+y+z)(w^2+x^2+y^2+z^2)^8}}{34}$$

$$P(1,1,1,1) = 5054421344$$

You can use the multinomial theorem similarly to find specific coefficients. Number of cases increase fast, but only **3** different permutation structures exist, making Pólya-Burnside easy to apply!

5 Introduction to Group Theory

Applications of Abstract Algebra/Group Theory

- Matrix groups to study the symmetric groups of 3-D solids, various problems in physics, and crystallographic groups.

- Extension fields for geometrical constructions, including the classical impossibility of duplicating the cube, trisecting an angle, and squaring a circle.

  (If n is a positive integer such that the regular n-gon is constructible with ruler and compass, then $n = 2^k \prod_{i=1}^{k} p_i$, where $k \geq 0$ and the p_i are distinct Fermat primes, that is, primes of the form $2^{2^m}+1$.)

- Combinatorial enumeration via group action on sets and Burnside 's Lemma (subject of this talk).

Definition of a Group

A group $(G, *)$ contains a set G of elements and a binary operation $*$.

- $*$ is closed on G. That is, if $g, h \in G$, then $g * h \in G$.
- $*$ is associative. That is, if $a, b, c \in G$, then $a * (b * c) = (a * b) * c$.
- There exists a (unique) identity element $e \in G$ such that for all $g \in G$, we have $g * e = e * g = g$.
- For all $g \in G$, there exists an inverse, denoted g^{-1}, such that $g * g^{-1} = g^{-1} * g = e$

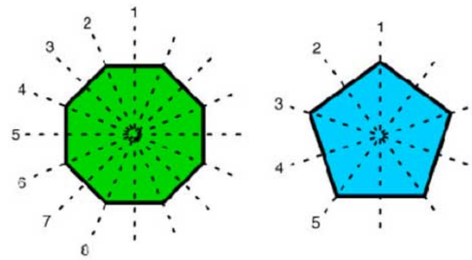


Figure 4: Dihedral Group in Odd and Even Cases

Basic Examples of Groups

- $(\mathbb{C}, +)$, $(\mathbb{R}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{Z}, +)$
- The set of symmetries of a rectangle, the Klein 4-group:
- The group of all permutations of three elements, S_3
- The example most relevant to us: **the dihedral group**, D_n, the group all symmetries (rotational and reflectional) of a regular n-sided polygon, with $2n$ elements.

The Dihedral Group, D_n

The **dihedral group** with $2n$ elements consists of the symmetries of an n-gon. We have two cases, n odd, and n even.

When n odd, reflections are through a vertex and the midpoint of the opposite side. When n even, reflections are through midpoints of opposite sides.

6 Group Action and Burnside's Lemma

6.1 Group Action on Sets

A group $(G, *)$ acts on the set X if there is a function that takes pairs of elements in G and elements in X – (g, x) – to new elements in X.

In our case, X will be the set of objects **without accounting for symmetry.**

More formally, the group $(G, *)$ acts on the set X if there is a function

$$f : G \times X \to X$$

such that when we denote $f(g, x)$ as $g(x)$, we have

$(g_1 g_2)(x) = g_1(g_2(x))$ for all $g_1, g_2 \in G, x \in X$

$e(x) = x$ if e is the identity of the group and $x \in X$

The Orbit and Stabilizer

If G acts on a set X and $x \in X$, then the **stabilizer** of x is defined to be the set

$$\text{Stab } x = \{g \in G \mid g(x) = x\}$$

that is, the set of elements in the group that take the element x to itself.

Similarly, let Fix g denote the number of elements of X fixed by g, that is the set $\{x \in X \mid g(x) = x\}$.

The set of all outputs of an element $x \in X$ under group action is called the orbit defined as the set

$$\text{Orb } x = \{g(x) \mid g \in G\}$$

The Orbit-Stabilizer Theorem

If a finite group G acts on a set X, for each $x \in X$, we have

$$|G| = |\text{Stab } x| \, |\text{Orb } x| \, .$$

where $|G|$ denotes the number of elements in the group.

Intuition: First, recall that there are 24 rotational symmetries of a cube. There are 8 places one vertex can go, and 3 places you can put one of its neighbors, yielding $8 \cdot 3 = 24$.

- Fix one face. You can move the cube 4 ways (you can only rotate it). These are the stabilizers.
- There are 6 faces you can pick. This is the orbit of the face.

Hence $4 \cdot 6 = 24$, the order of the group of cube symmetries, as expected.

6.2 Burnside's Lemma

Burnside's Lemma

If G is a finite group that acts on the elements of a finite set X, and N is the number of orbits of X under G, then

$$N = \frac{1}{|G|} \sum_{g \in G} |\text{Fix } g|$$

The orbit of an element $x \in X$ refers to all possible colorings you can obtains by some rotation or reflection on some coloring.

If we count the number of orbits, we are counting the number of colorings that are distinct under rotation or reflection!

Proof of Burnside's Lemma

Consider $\sum_{g \in G} |\text{Fix } g|$.

But this is also

$$|S| = \sum_{g \in G} |\text{Fix } g| = \sum_{x \in X} |\text{Stab } x|$$

Representative elements from each orbit of X under G, $x_1, x_2, \cdots, x_N$.

If an element x is in the same orbit as x_i, then Orb x = Orb x_i, and by the orbit-stabilizer theorem, $|\text{Stab } x| = |\text{Stab } x_i|$.

We have

$$\sum_{g\in G} |\text{Fix } g| = \sum_{i=1}^{N} \sum_{x\in \text{Orb } x_i} |\text{Stab } x| = \sum_{i=1}^{N} |\text{Orb } x_i|\,|\text{Stab } x_i|$$

Which implies

$$\sum_{g\in G} |\text{Fix } g| = \sum_{i=1}^{N} |\text{Orb } x_i|\,|\text{Stab } x_i|$$

By the orbit-stabilizer theorem, each of the summands equals $|G|$.

Hence

$$\sum_{g\in G} |\text{Fix } g| = \sum_{i=1}^{N} |\text{Orb } x_i|\,|\text{Stab } x_i| = N \cdot |G|$$

Burnside's Lemma follows:

$$N = \frac{1}{|G|} \sum_{g\in G} |\text{Fix } g|$$

7 Intuition: Why Pólya-Burnside Enumeration Works

Plugging in 1 Yields Burnside's Lemma!

Recall the generating functions from the previous examples:

Problem 1: Number of different flag colorings

$$f(x,y) = \frac{(x+y)^4 + (x^2+y^2)^2}{2}$$

2^4: number of elements fixed by the identity.

2^2: number of elements fixed by reflection across middle

2: order of $|D_2|$.

Problem 2: Number of different bracelets

$$g(x,y,z) = \frac{\substack{7\cdot(x+y+z)(x^2+y^2+z^2)^3+6(x^7+y^7+z^7)\\ +(x+y+z)^7}}{14}$$

$7 \cdot 3^4$: number of elements fixed by reflections

$6 \cdot 3$: number of elements fixed by the six nonidentity rotations

3^7: number of elements fixed by identity

Note that plugging in 1 for all the variables gives you $N = \frac{1}{|G|}\sum_{g\in G} |\text{Fix } g|$!

$$f(1,1) = \frac{2^4+2^2}{2}; g(1,1) = \frac{7\cdot 3^4 + 6\cdot 3 + 3^7}{14}$$

Why the Generating Function Substitution Works

Recall:

If an object that can be colored with k colors has a symmetry as follows: A permutation with p_1 cycles of length 1, p_2 of length 2, p_n of length n ($p_i = 0$ allowed) contributes

$$f_1^{p_1} f_2^{p_2} \cdots f_n^{p_n}$$

to the cycle index polynomial. If you have k colors, substitute $f_i = (c_1^i + c_2^i + \cdots + c_k^i)$.

Intuition. To count fixed elements, ($|\text{Fix } g|$), all entities in the respective cycles must be the same color!

In the generating function, $c_1^i c_2^j c_3^k$ represents i, j, k instances of c_1, c_2, c_3 respectively.

To be the same, we can have substitute $c_n^{p_n}$ for **any color**, since it doesn't matter what color we pick, so we substitute $f_i = (c_1^i + c_2^i + \cdots + c_k^i)$.

7.1 Examples: Why the Substitution Works

Reflection in the four color flag example.

Let flag patterning by 4-letter string of colors $abcd$.

Permutation structure: $abcd \to dcba$: (a d)(b c)

Term in cycle index polynomial: f_2^2.

Need to substitute $f_2 = x^2 + y^2$.

For a pattern to be fixed, $a = d$ and $b = c$, as in 

If $x^i y^j$ represents i, j instances of colors x, y, then $x^2 + y^2$ provide two terms: we can either have both parts of the cycle be red (x), or blue (y).

Rotation of a Hexagon: D_6

In a hexagon, a rotation of $360/3°$ yields (1 3 5)(2 4 6)

In this case, we must have $1 = 3$, $2 = 4$, and $5 = 6$.

Hence the cycle index term is f_3^2.

If we have two colors, substitute $f_3^2 = (x^3 + y^3)^2$ to give all possible colorings.

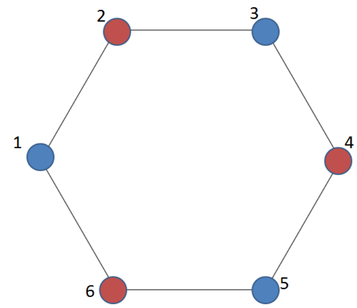


Figure 5: 6 is composite, so we can have disjoint cycles

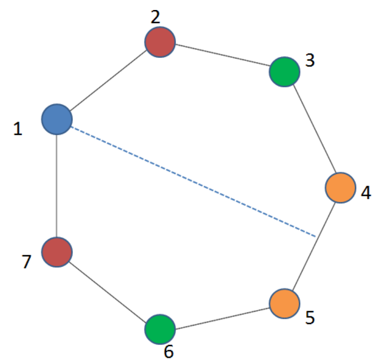


Figure 6: Reflectional symmetry: vertices that get mapped to each other are the same color

Reflection of a Heptagon: D_7

Suppose we have a heptagon.

Notice that the permutation maps (1)(7 2)(6 3)(5 4).

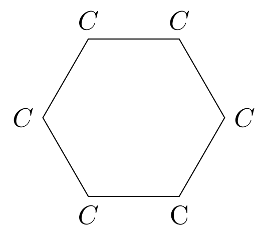
This corresponds to the cycle structure $f_2^3 f_1$.

We substitute $f_n = x^n + y^n$ giving $f_2^3 f_1 = (x^2 + y^2)^3(x + y)$.

The reds, oranges, and greens have to be equal, so we must have either x^2 or y^2 three times (so we cube). Finally, the blue can be either color, x or y.

8 Chemical Isomer Enumeration

How many different chemical compounds can be made by attaching H, CH_3, or OH radicals to each of the carbon atoms in the benzene ring pictured below?



With your help, if time and volunteers!

9 Additional Problems

With the exception of #1, these are an assortment of problems in which it isn't immediately clear that Burnside's Lemma can be applied.

Source: Art of Problem Solving Forums

1. Two of the squares of a 7×7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible? (AIME 1996, #7)

2. Find the number of second-degree polynomials $f(x)$ with integer coefficients and integer zeros for which $f(0) = 2010$. (AIME 2010, #10)

3. Two quadrilaterals are considered the same if one can be obtained from the other by a rotation and a translation. How many different convex cyclic quadrilaterals are there with integer sides and perimeter equal to 32? (AMC 12A 2010, #25)

4. How many subsets $\{x, y, z, t\} \subset \mathbb{N}$ are there that satisfy the following conditions?

$$12 \leq x < y < z < t$$
$$x + y + z + t = 2011$$

5. Prove that, for all positive integers n and k, we have

$$n | \sum_{i=0}^{n-1} k^{\gcd(i,n)}$$

   where $a|b$ means that a divides b.