DRP: Fourier Analysis on Groups

Adithya Ganesh

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1 Introduction to Finite Markov Chains

A finite Markov chain is a process which moves among the elements of a finite set Ω in the following manner: when at $x \in \Omega$, the next position is chosen according to a fixed probability distribution $P(x, \cdot)$.

(More formal definition.)

Theorem.

Definition. *For $x \in \Omega$, define the* **hitting time** *for x to be*

$$\tau_x := \min \{t \geq 0, X_t = x\},$$

that is, the first time at which the chain visits state x.

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$s_0, \ldots, s_n$ family of random variables. $\mathcal{F}_i$ is the filtration at time i, which is equal to $\sigma(s_0, s_1, \ldots, s_i)$, the signal algebra generated by $s_0, \ldots, s_i$.

A probability space is a given of three things:

$$(\Omega, \mathcal{F}, p),$$

where Ω is the state space, $\mathcal{F}$ is a sigma-algebra, and p is a prob measures

When $\mathcal{F}$ is the power set in the discrete case. The σ algebra is a collection C of subsets such that

- $\emptyset \in C$.
- $A \in C \implies A^C \in C$.
- $A_1, A_2, \ldots, A_n \in C \implies \bigcup A_i \in C$.

A probability measure is a function $p : \mathcal{F} \to [0, 1]$ such that

- $p(\emptyset) = 0$

- $p(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ *if* $A_i \cap A_j \neq \emptyset$.

Why sigma algebras? Consider a sequence of n *coin tosses. We can write*

$$(\Omega, \mathcal{F}, p) = \left(\{0,1\}^n, 2^{\{0,1\}^n}, p = \textit{uniform}\right).$$

In the continuous case, $(\Omega, \mathcal{F}, p)$, $\Omega = [0,1]$, *where* $p =$ *uniform measure,* $p\left((a,b)\right) = b - a$.

Definition. *Martingales.*

Definition. *If is a* σ*-algebra,* X *is an RV, then*

$$\mathbb{E}[X|]$$

is the unique rv s.t.

$$\mathbb{E}[X 1_A] = \mathbb{E}\left[\mathbb{E}\left[x|\right] 1_A\right],$$

for all $A \in$.

Let $T =$ stopping time, which is the property that

$$\{T \leq n\} \in \mathcal{F}_n = \sigma(S_0, \dots, S_n).$$

The optimal stopping theorem states that

$$\mathbb{E}\left[S_T\right] = 0,$$

if $S_{\min(N,T}$ is bounded.

The Gambler's problem is a martingale.

$$\mathbb{E}\left[S_T\right] = 0,$$

so

$$P(S_T = -b)(-b) + P(S_T = a)a = 0,$$

so $p = \frac{b}{a+b}$.

Exercise. Gambler ruin with bias. Suppose $S_n = X_1 + X_2 + \cdots + X_n$, $\mathbb{E}[X_i] = p < 0$.